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Real and strongly real classes in $\mathrm{PGL}_n(q)$ and quasi-simple covers of $\mathrm{PSL}_n(q)$

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Abstract. We classify the real and strongly real conjugacy classes in $\mathrm{PGL}_n(q)$, $\mathrm{PSL}_n(q)$, and all quasi-simple covers of $\mathrm{PSL}_n(q)$. In each case we give a formula for the number of real, and the number of strongly real, conjugacy classes.

This is a companion paper to [4] in which we classified the real and strongly real conjugacy classes in $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$.

1 Introduction

Let G be a group. An element g of G is called *real* if there exists $h \in G$ such that $hgh^{-1} = g^{-1}$. If h can be chosen to be an involution (i.e. $h^2 = 1$) then we say that g is *strongly real*. In all cases we say that h is a *reversing element* for g. If g is real (resp. strongly real) then all conjugates of g are real (resp. strongly real), hence we talk about *real classes* and *strongly real classes* in G.

In [4] we classified the real and strongly real conjugacy classes in $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$. In this paper we extend this classification to cover the groups $\mathrm{PGL}_n(q)$, $\mathrm{PSL}_n(q)$ and the quasi-simple covers of $\mathrm{PSL}_n(q)$. We use the notation and methods established in [4]. In particular we do not repeat definitions from [4].

The analysis in this paper is of a slightly different flavour to that of [4] as the groups of interest are no longer subgroups of $\mathrm{GL}_n(q)$, but quotients of subgroups. In particular, to understand reality in $\mathrm{PGL}_n(q)$ and $\mathrm{PSL}_n(q)$ we need to understand the *ζ-real elements* in $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$; these elements were introduced in [4, §2], and were studied in parallel with real elements throughout the rest of [4]. Their significance is explained by Lemma 2.4.

An understanding of reality in $\mathrm{PGL}_n(q), \mathrm{PSL}_n(q)$ and the remaining quasi-simple covers of $\mathrm{PSL}_n(q)$ requires that we understand how conjugacy is affected when we factor out the centre of a group. This is discussed in the first half of Section 2; that discussion sets the scene for what follows in Sections 2 to 6, each of which includes a theorem near the end summarizing its main results.

Section 7 covers some exceptional quasi-simple covers of $\mathrm{PSL}_n(q)$ that require different techniques, and thereby completes our analysis of real and strongly real classes in the quasi-simple covers of $\mathrm{PSL}_n(q)$. Section 8 outlines possible areas of future research.

As far as we know a complete classification of the real and strongly elements for any of the families $\mathrm{PGL}_n(q)$, $\mathrm{PSL}_n(q)$, and the quasi-simple covers of $\mathrm{PSL}_n(q)$ does not exist in the literature. However Gow has communicated with us concerning work on real classes in $\mathrm{PGL}_n(q)$; so, although this has not been published, some of the results are already known.

The preprint version of this paper [3] contains explicit calculations for small rank groups (cf. Section 13). Formulae are given there for the number of real and strongly real classes in all relevant groups with rank at most 6.

2 $\mathrm{PGL}_n(q)$

First, some notation: consider two groups, X and Y, such that $Y \leqslant Z(X)$. We say that an element $h \in X/Y$ *lifts* to an element g in X (or, equivalently, *g projects onto h*) if $h = gY$. Now suppose that $W < X$. We say that *g projects into W/Y* if there exists $y \in Y$ such that $gy \in W$. We continue the practice established in [4] so that, for $g \in \mathrm{GL}_n(q)$ and $\eta \in \mathbb{F}_q^*$, we will abuse notation and write ηg for $g(\eta I)$.

2.1 Conjugacy in $\mathrm{PGL}_n(q)$. Set $Z = Z(\mathrm{GL}_n(q))$; then $\mathrm{PGL}_n(q) = \mathrm{GL}_n(q)/Z$. Our first job is to understand how conjugacy in $\mathrm{PGL}_n(q)$ works. Let g be an element of C, a conjugacy class of $\mathrm{GL}_n(q)$, represented by $u = (u_1(t), u_2(t), \ldots)$ corresponding to a partition $\nu = 1^{n_1}2^{n_2}\ldots$. Macdonald's result asserts that the conjugacy class of ηg is represented by $(u_1(\eta t), u_2(\eta t), \ldots)$; see [12, p. 30]. Thus all elements in $\mathrm{GL}_n(q)$ which project onto an element $gZ \in \mathrm{PGL}_n(q)$ are of type ν; we therefore refer to gZ as being *of type* ν.

Suppose that g projects onto gZ which is real in $\mathrm{PGL}_n(q)$. Then we want to calculate how many real and ζ-real elements in $\mathrm{GL}_n(q)$ project onto gZ. We call two sequences of self-reciprocal polynomials (resp. ζ-self-reciprocal polynomials), $u = (u_1(t), u_2(t), \ldots)$ and $v = (v_1(t), v_2(t), \ldots)$ *equivalent* if, for some $\eta \in \mathbb{F}_q^*$, $v_i(t) = u_i(\eta t)$ for all i.

To understand what this means for reality in $\mathrm{PGL}_n(q)$ we need to return to the study of self-reciprocal and ζ-self-reciprocal polynomials that was started in [4, §2].

2.2 An action of $\mathbb{F}_q^*$ on polynomials. We define an action of $\mathbb{F}_q^*$ on the set of degree n polynomials by $\eta.f(t) = f(\eta t)$ for $\eta \in \mathbb{F}_q^*$. Recall the definition of sets T_n and S_n given in [4, §2]. We are interested in classifying the orbits of $\mathbb{F}_q^*$ intersected with T_n and S_n. That is to say, we wish to determine the size of the sets

$$[f]_T = \{f(\eta t) \in T_n \mid \eta \in \mathbb{F}_q^*\} \quad \text{and} \quad [f]_S = \{f(\eta t) \in S_n \mid \eta \in \mathbb{F}_q^*\},$$

for a degree n polynomial f in $\mathbb{F}_q[t]$.

In what follows we write $|k|_2$ for the largest power of 2 which divides an integer k. We begin with a lemma from [4]:

Lemma 2.1 ([4, Lemma 4.3]). *Let $\mathbb{F}_q$ be a finite field with q odd. Then there exists $\alpha \in \mathbb{F}_q^*$ with $\alpha^n = -1$ if and only if $|n|_2 < |q-1|_2$.*

Lemma 2.2. *If q is even then $[f]_T$ and $[f]_S$ contain at most one element. If q is odd then $[f]_T$ and $[f]_S$ contain at most two elements.*

Proof. Write X to mean either T or S. Take $f(t) \in X_n$ such that $f(\eta t) \in X_n$; then $\eta^n = \pm 1$. Since $f(\eta t) \in X_n$, for any coefficient $a_k \neq 0$ of $f(t)$, we must have $a_k \eta^{n-k} = \pm a_k \eta^k$. Thus, if the order of η is denoted by e, then $e \mid 2k$.

If q is even this implies that $e|k$ and $f(t) \in \mathbb{F}_q[t^e]$; thus $f(\eta t) = f(t)$ as required. Suppose that q is odd. If e is odd then $e|k$ and, again, $f(\eta t) = f(t)$. So assume that e is even. We must have $f(t) \in \mathbb{F}_q[t^{e/2}]$ and so $f(\eta^2 t) = f(t)$.

Now suppose that $f(\varepsilon t) \in X_n$ for some $\varepsilon \in \mathbb{F}_q^*$. To avoid redundancy, let us assume that $f(\eta t) \neq f(t) \neq f(\varepsilon t)$. Clearly $|d|_2 = |e|_2$, where d is the order of ε, and $f(t) \in \mathbb{F}_q[t^{e/2}] \backslash \mathbb{F}_q[t^e]$. But then ε and η are doing the same thing to coefficients of $f(t)$, namely swapping the sign of coefficients of $t^{ae/2}$ for odd a. Thus $f(\eta t) = f(\varepsilon t)$ as required. $\square$

Note that, when q is odd, we have effectively shown that if $f \in X_n$ and $[f]_X$ has two elements then $[f]_X = \{f(t), f(\eta t)\}$ and the order of η is a power of 2. We continue our analysis for q odd.

Lemma 2.3. *Let q be odd.*

(1) *If n is odd then S_n is empty and, for $f(t) \in T_n$, we have $|[f]_T| = 2$.*

(2) *Suppose that n is even.*
   (a) *If $f(t) \notin \mathbb{F}_q[t^{|q-1|_2}]$ and $f(t) \in T_n$ (resp. $f(t) \in S_n$) we have $|[f]_T| = 2$ (resp. $|[f]_S| = 2$). Moreover if $[f]_T$ is non-empty (resp. $[f]_S$ is non-empty), then $[f]_S$ is empty (resp. $[f]_T$ is empty).*
   (b) *If $f(t) \in \mathbb{F}_q[t^{|q-1|_2}]$ and $f(t) \in T_n$ (or $f(t) \in S_n$) then $|[f]_S| = |[f]_T| = 1$.*

Proof. First let n be odd. There are no ζ-self-reciprocal polynomials in this case, hence S_n is empty. Moreover, $T_n = F_n \cup G_n$, and we have a bijection from F_n to G_n given by $f(t) \mapsto f(-t)$. Thus, for $f \in T_n$, $[f]_T = \{f(t), f(-t)\}$ as required.

Now let n be even, so $|n|_2 \geqslant 2$. Again write X for either T or S and suppose that $f(t) \in X_n$. We know that $[f]_X = \{f(t), f(\eta t)\}$ where η has order a power of 2. If $f(t) \in \mathbb{F}_q[t^{|q-1|_2}]$ this implies that $f(\eta t) = f(t)$ and so $|[f]_X| = 1$. On the other hand if $f(t) \notin \mathbb{F}_q[t^{|q-1|_2}]$ let e be the smallest power of 2 such that $f(t) \notin \mathbb{F}_q[t^e]$. Taking η of order e we check easily that $f(\eta t) \in X_n$ and $f(\eta t) \neq f(t)$.

Suppose that $f(t) \in S_n$ (i.e. $f(t)$ is ζ-self-reciprocal) and $f(t) \notin \mathbb{F}_q[t^{|q-1|_2}]$. We check if $f(\lambda t)$ is self-reciprocal for some $\lambda \in \mathbb{F}_q$. Then this implies that, whenever $a_k \neq 0$,

$$a_k \frac{\lambda^{n-k}}{\zeta^{n/2-k}} = \pm a_k \lambda^k.$$

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This implies that $\lambda^{n-k} = \pm\zeta^{n/2}$. Setting $k = 0$ we find that $|q-1|_2$ divides $|n|_2$ and so $\lambda^{2k} = \pm\zeta^k$ for $k > 0$. Since ζ is a non-square this is impossible. Thus, if $[f]_S$ is non-empty, then $[f]_T$ is empty. A similar argument shows that, if $[f]_T$ is non-empty, then $[f]_S$ is empty.

Finally consider the case when $f(t) \in \mathbb{F}_q[t^{|q-1|_2}] \cap T_n$. Let η be a non-square of order $|q-1|_2$. Set κ to be an element in $\mathbb{F}_q^*$ which satisfies $\kappa^2 = \eta/\zeta$. Then one can check that

$$f(\kappa t) = \pm(\kappa t)^n + a_{|q-1|_2}(\kappa t)^{n-|q-1|_2} \pm a_{2|q-1|_2}(\kappa t)^{n-2|q-1|_2} + \cdots \pm a_{|q-1|_2}(\kappa t)^{|q-1|_2} + 1$$

lies in S_n. Similarly if $f(t) \in \mathbb{F}_q[t^{|q-1|_2}] \cap S_n$ then $f(\theta t) \in T_n$ where θ is an element of $\mathbb{F}_q^*$ which satisfies $\theta^2 = \eta\zeta$. Thus, in both cases, $|[f]_S| = |[f]_T| = 1$. $\square$

2.3 Real classes in $\mathbf{PGL}_n(q)$. We wish to use our results concerning reality in $\mathrm{GL}_n(q)$ to classify reality in $\mathrm{PGL}_n(q)$. In general, when converting our results from a group X to X/Y where $Y \leqslant Z(X)$, we are faced with the following problem: if g is real (resp. strongly real) in X then gY is real (resp. strongly real) in X/Y, however the converse does not hold.

When $X = \mathrm{GL}_n(q)$ we are able to give a partial converse. Write Z for the centre of $\mathrm{GL}_n(q)$ and recall that ζ is a fixed non-square in $\mathbb{F}_q$.

Lemma 2.4. *Suppose that gZ is real in $\mathrm{PGL}_n(q)$. Then gZ lifts to a real or a ζ-real element in $\mathrm{GL}_n(q)$.*

Proof. Clearly gZ lifts to an element g which is conjugate in $\mathrm{GL}_n(q)$ to $g^{-1}(\eta^{-1}I)$ for some $\eta \in \mathbb{F}_q^*$, i.e. there exists $h \in \mathrm{GL}_n(q)$ such that $hgh^{-1} = \eta^{-1}g^{-1}$. If η is a square, $\eta = \lambda^2$ say, we get $h(\lambda g)h^{-1} = \lambda^{-1}g^{-1}$. That is, gZ lifts to a real element λg in $\mathrm{GL}_n(q)$. When η is not a square, we write $\eta = \zeta^{-1}\lambda^2$ and we have $h(\lambda g)h^{-1} = \zeta(\lambda g)^{-1}$. In this case gZ lifts to a ζ-real element λg in $\mathrm{GL}_n(q)$. $\square$

Recall that we write gl_ν for the number of real conjugacy classes of type ν in $\mathrm{GL}_n(q)$. Then [4, Theorem 3.8] states that $gl_\nu = \prod_{n_i>0} n_{q,n_i}$ where n_{q,n_i} is defined in [4, Lemma 2.1]. Now the number of real classes in $\mathrm{PGL}_n(q)$ is equal to $\sum_{|\nu|=n} pgl_\nu$, where pgl_ν is the number of real conjugacy classes in $\mathrm{PGL}_n(q)$ of type ν. Lemma 2.4 implies that pgl_ν is equal to the number of equivalence classes in the set of sequences of self-reciprocal and ζ-self-reciprocal polynomials associated with ν (or, equivalently, the number of equivalence classes of real and ζ-real conjugacy classes in $\mathrm{GL}_n(q)$ of type ν).

For the rest of this section we calculate pgl_ν for various values of ν, n and q.

2.3.1 *q* is even. There are no ζ-real conjugacy classes in $\mathrm{GL}_n(q)$ in this case. What is more, Lemma 2.2 implies that all equivalence classes of real conjugacy classes in $\mathrm{GL}_n(q)$ are of size 1; hence $pgl_\nu = gl_\nu = \prod_{n_i>0} n_{q,n_i}$.

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2.3.2 *q* is odd.

Lemma 2.5. *If C is a real (resp. ζ-real) class in $\mathrm{GL}_n(q)$ then C is equivalent to at most one other real (resp. ζ-real) class in $\mathrm{GL}_n(q)$.*

Proof. The proof is very similar to the proof of Lemma 2.2. Suppose that C is real (resp. ζ-real) and corresponds to the sequence $u = (u_1(t), u_2(t), \ldots)$. Consider conjugacy classes C_η corresponding to $u = (u_1(\eta t), u_2(\eta t), \ldots)$, for some $\eta \in \mathbb{F}_q^*$, and assume that C_η is real.

Let e be the largest power of 2 such that all elements u_i are in $\mathbb{F}_q[t^{e/2}]$. Let f be the order of η. If $|f|_2 < e$ then $C_\eta = C$. If $|f|_2 = e$ then C_η is the conjugacy class corresponding to $v = (v_1(t), v_2(t), \ldots)$, where $v_i(t)$ is the same as $u_i(t)$ except that the coefficient of t^{ae} has reversed sign for odd a. If $|f|_2 > e$ then C_η is not real (resp. ζ-real) which is a contradiction.

Thus there is at most one other real (resp. ζ-real) conjugacy class in $\mathrm{GL}_n(q)$ which is equivalent to C. $\square$

Lemma 2.6. *Let C be a conjugacy class in $\mathrm{GL}_n(q)$ of type v with corresponding sequence $u = (u_1(t), u_2(t), \ldots)$. Let $d = |(n_1, n_2, \ldots)|_2$.*

(1) *If $d = 1$, then the set of real conjugacy classes in $\mathrm{GL}_n(q)$ is partitioned into equivalence classes of size 2, and there are no ζ-real classes.*

(2) *Suppose that $d \geqslant 2$.*
   (a) *If C is a real (resp. ζ-real) class then C is equivalent to one other real (resp. ζ-real) class provided that at least one u_i is not in $\mathbb{F}_q[t^{|q-1|_2}]$. Moreover, if C is a real class such that not all u_i lie in $\mathbb{F}_q[t^{|q-1|_2}]$ then C is not equivalent to any ζ-real class.*
   (b) *If C is a real (resp. ζ-real) class, and all u_i lie in $\mathbb{F}_q[t^{|q-1|_2}]$, then C is equivalent to exactly one ζ-real (resp. real) class; moreover C is not equivalent to any other real (resp. ζ-real) class.*

Proof. Suppose first that $d = 1$. Then n_i is odd for some i and, in particular, there are no ζ-real classes of type v. Now suppose that C is real. Then, since n_i is odd, we have two distinct equivalent sequences which both correspond to a real class in $\mathrm{GL}_n(q)$:

$$(u_1(t), u_2(t), \ldots) \quad \text{and} \quad (u_1(-t), u_2(-t), \ldots).$$

Thus C is equivalent to a distinct real class in $\mathrm{GL}_n(q)$ as required.

Now suppose that $d \geqslant 2$, so that n_i is even for all i. Let C be a real (resp. ζ-real) class such that at least one u_i is not in $\mathbb{F}_q[t^{|q-1|_2}]$. Let e be the largest power of 2 such that all of the u_i are contained in $\mathbb{F}_q[t^{e/2}]$ and take $\eta \in \mathbb{F}_q^*$ of order e. Then, once again, we have two distinct equivalent sequences which both correspond to a real (resp. ζ-real) class in $\mathrm{GL}_n(q)$:

$$(u_1(t), u_2(t), \ldots) \quad \text{and} \quad (u_1(\eta t), u_2(\eta t), \ldots).$$

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Thus C is equivalent to one other real (resp. ζ-real) class as required. Lemma 2.3 implies that C cannot be equivalent to a real (resp. ζ-real) class.

Now suppose that $d \geqslant 2$ that all of the u_i lie in $\mathbb{F}_q[t^{|q-1|_2}]$. If C is real (resp. ζ-real) then Lemma 2.3 implies that C is not equivalent to any other real (resp. ζ-real) class. If C is real (resp. ζ-real) then define κ (resp. θ) just as in the proof of Lemma 2.3; the class corresponding to κu (resp. θu) is ζ-real (resp. real). $\square$

This lemma allows us to write down a formula for pgl_ν valid whenever q is odd.

Corollary 2.7. *Let $\nu = 1^{n_1}2^{n_2}\dots$ be a partition of n. If $d = 1$ then $pgl_\nu = \frac{1}{2}gl_\nu$. If $d > 1$ then $pgl_\nu = gl_\nu$.*

2.3.3 Conclusion. We summarize our results for both odd and even characteristic in the following theorem:

Theorem 2.8. *The number of real conjugacy classes in $\mathrm{PGL}_n(q)$ is given by*

$$\sum_{|\nu|=n} \frac{1}{2^{\sigma_\nu}} \prod_{n_i>0} n_{q,n_i}.$$

Here we set $d = |(n_1, n_2, \dots)|_2$ and define σ_ν to equal 0 if $|dq|_2 > 1$ and to equal 1 otherwise. All real conjugacy classes in $\mathrm{PGL}_n(q)$ are strongly real.

We have not proved the statement about strong reality, however this follows from the work of Vinroot [17, Theorem 3]. Note too that, for q even or n odd, the number of $\mathrm{GL}_n(q)$-classes of $\mathrm{SL}_n(q)$-real elements in $\mathrm{SL}_n(q)$ is the same as the number of $\mathrm{PGL}_n(q)$-real classes. This is reminiscent of an observation of Lehrer [11] that the total number of conjugacy classes in $\mathrm{PGL}_n(q)$ is the same as the total number of $\mathrm{GL}_n(q)$-classes in $\mathrm{SL}_n(q)$.

3 $\mathrm{PSL}_n(q)$, q is even or $|n|_2 \neq |q-1|_2$

In this section we begin work on a classification of the real conjugacy classes in $\mathrm{PSL}_n(q)$. We think of $\mathrm{PSL}_n(q)$ as the image of $\mathrm{SL}_n(q)$ under the quotient map $\mathrm{GL}_n(q) \to \mathrm{PGL}_n(q) = \mathrm{GL}_n(q)/Z$, hence the elements of $\mathrm{PSL}_n(q)$ will be written as gZ where $g \in \mathrm{SL}_n(q)$.

Recall that we may describe gZ as being of type ν, for some partition ν, since all elements in $\mathrm{GL}_n(q)$ to which gZ lifts are of the same type ν. Our first result holds for any q and for any cover of $\mathrm{PSL}_n(q)$.

Proposition 3.1. *Let $Y \leqslant Z(\mathrm{GL}_n(q))$ and set G to equal $\mathrm{GL}_n(q)/Y$ and H to equal $\mathrm{SL}_n(q)/(\mathrm{SL}_n(q) \cap Y)$. Let C be a G-conjugacy class in H which is associated with a partition $\nu = (\nu_1, \nu_2, \dots)$. Then C splits into $h_\nu = (q-1, \nu_1, \nu_2, \dots)$ H-conjugacy classes.*

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Proof. We use the methods of [18]. Let $C = C_1 \cup \cdots \cup C_h$, where $C_1, \ldots, C_h$ are H-conjugacy classes; to prove the proposition we must calculate the value of h. Suppose that g projects into C_1 and let

$$X = \{h \in \mathrm{GL}_n(q) : hgh^{-1} = yg \text{ for some } y \in Y\}.$$

Set D_Y to be the group $\det X$ which lies in $\mathbb{F}_q^*$.

The class C_1 is stabilized in G by XH. Now there is an isomorphism $G/(XH) \cong \mathbb{F}_q^*/D_Y$. Thus $h = (q-1)/|D_Y|$.

We must now calculate $|D_Y|$. Suppose that $Y_1 \leqslant Y_2 \leqslant Z(\mathrm{GL}_n(q))$. It is clear that $D_{Y_1} \leqslant D_{Y_2}$ if and only if $Y_1 \leqslant Y_2$. But now [18, Theorem 4] states that $D_{Y_1} = D_{Y_2}$ for $Y_1 = \{1\}$ and $Y = Z(\mathrm{GL}_n(q))$. Hence $D_{Y_1} = D_{Y_2}$.

When $Y = \{1\}$, we can use [12, (3.1)] to calculate D_Y. Macdonald proved that a $\mathrm{GL}_n(q)$ conjugacy class of type $v = \{v_1, \ldots, v_r\}$ contained in $\mathrm{SL}_n(q)$ is the union of h_v conjugacy classes for $\mathrm{SL}_n(q)$ where $h_v = (q-1, v_1, \ldots, v_r)$. Thus, for any $Y \leqslant Z(\mathrm{GL}_n(q))$, D_Y is the subgroup of $\mathbb{F}_q^*$ of index $h_v = (q-1, v_1, v_2, \ldots)$; the result follows. $\square$

Write psl_v for the number of $\mathrm{PGL}_n(q)$-real $\mathrm{PGL}_n(q)$-conjugacy classes of type v contained in $\mathrm{PSL}_n(q)$. Proposition 3.1, together with [4, Corollary 4.5] implies that, for $|n|_2 \neq |q-1|_2$, the number of real conjugacy classes in $\mathrm{PSL}_n(q)$ is equal to

$$\sum_{|v|=n} h_v psl_v.$$

Thus, for the remainder of this section (and the next), we will calculate psl_v for differing v, q and n.

3.1 *q* is even. We know that all real elements in $\mathrm{PGL}_n(q)$ lift to real elements in $\mathrm{GL}_n(q)$. Since there are no equivalences for q even, there is a 1–1 correspondence of real conjugacy classes between the two groups; indeed the same holds for $\mathrm{GL}_n(q)/Y$ where Y is any subgroup of $Z(\mathrm{GL}_n(q))$.

Now all real elements in $\mathrm{GL}_n(q)$ are in $\mathrm{SL}_n(q)$. What is more, these elements are real, in fact strongly real, in $\mathrm{SL}_n(q)$. Hence there is also a 1–1 correspondence between real elements in $\mathrm{SL}_n(q)$ and those in $\mathrm{PSL}_n(q)$. We conclude that $psl_v = sl_v = \prod_{n_i > 0} n_{q,n_i}$.

3.2 q is odd and $|n|_2 < |q-1|_2$.

Lemma 3.2. *Suppose that* $2 \leqslant |n|_2 < |q-1|_2$. *If* g *is* ζ*-real in* $\mathrm{GL}_n(q)$ *then* g *does not project into* $\mathrm{PSL}_n(q)$.

Proof. Suppose that $hgh^{-1} = \zeta g$. Then $\det g = \pm\zeta^{n/2}$. Now take $\alpha \in \mathbb{F}_q^*$ and observe that $\det g(\alpha I) = \pm\zeta^{n/2}\alpha^n$. We may suppose without loss of generality that ζ generates $\mathbb{F}_q^*$ and suppose that $\alpha = \zeta^a$ for some integer a. Then $\det(\alpha g) = \zeta^{n/2+an}$ or

$\zeta^{n/2+an+(q-1)/2}$. If g projects down to $\mathrm{PSL}_n(q)$ then we must have $\det(\alpha g) = 1$ for some $\alpha \in \mathbb{F}_q^*$. But $q-1$ divides $\left(\frac{1}{2}n + an\right)$ or $\left(\frac{1}{2}n + an + \frac{1}{2}(q-1)\right)$ which is impossible. $\square$

Lemma 3.3. *If* $|n|_2 < |q-1|_2$ *then all real elements in* $\mathrm{GL}_n(q)$ *project into* $\mathrm{PSL}_n(q)$. *These projections are strongly real in* $\mathrm{PSL}_n(q)$.

Proof. Lemma 2.1 implies that there exists $\alpha \in \mathbb{F}_q$ such that $\alpha^n = -1$, thereby implying that $\det(\alpha I) = -1$. Now take g real in $\mathrm{GL}_n(q)$ so that, in particular, $\det g = \pm 1$. If $\det g = -1$ then $\det(\alpha g) = 1$ and αg is conjugate to αg^{-1}. Thus $\alpha g Z$ lies in $\mathrm{PSL}_n(q)$ as required.

Take $h \in \mathrm{GL}_n(q)$ such that $hgh^{-1} = g^{-1}$. From [4, Corollary 4.5], all real elements in $\mathrm{GL}_n(q)$ are strongly real in $\langle \mathrm{SL}_n(q), (\alpha I) \rangle$, hence we may assume that $\det h = \pm 1$ and $h^2 = 1$. If $\det h = 1$ then $hZ \in \mathrm{PSL}_n(q)$ and so gZ is strongly real in $\mathrm{PSL}_n(q)$. If $\det h = -1$ then $\alpha hZ \in \mathrm{PSL}_n(q)$ and, once more, gZ is strongly real in $\mathrm{PSL}_n(q)$. $\square$

These two lemmas imply that the number of real classes in $\mathrm{PSL}_n(q)$ is equal to the number of equivalence classes of real elements in $\mathrm{GL}_n(q)$. We conclude that, in this case, $psl_v = \frac{1}{2} gl_v = \frac{1}{2} \prod_{n_i > 0} n_{q,n_i}$.

3.3 q is odd and $|n|_2 > |q-1|_2$.

Lemma 3.4. *Suppose that* q *is odd and* $|n|_2 \geqslant |q-1|_2$. *If* g *and* ηg *are both real* (*resp. both* ζ*-real*) *then* $\det g = \det(\eta g)$.

Proof. Observe that $\det(\eta g) = \eta^n \det g$. Now if g and ηg are both real (or both ζ-real) then $\det(\eta g) = \pm \det g$ and so $\eta^n = \pm 1$. Since $|n|_2 \geqslant |q-1|_2$, Lemma 2.1 implies that $\det h = \det g$. $\square$

Lemma 3.5. *Suppose that* q *is odd and* $|n|_2 \geqslant |q-1|_2$. *A* $\mathrm{PGL}_n(q)$*-real conjugacy class* $gZ \in \mathrm{PGL}_n(q)$ *is contained in* $\mathrm{PSL}_n(q)$ *if and only if*

(1) *it lifts to a real element in* $\mathrm{GL}_n(q)$ *which is contained in* $\mathrm{SL}_n(q)$, *or*

(2) *it lifts to a* ζ*-real element in* $\mathrm{GL}_n(q)$ *of determinant* $\zeta^{n/2}$ *in the case where* $|n|_2 > |q-1|_2$, *and of determinant* $-\zeta^{n/2}$ *otherwise.*

Proof. Suppose that gZ lies in $\mathrm{PGL}_n(q)$ with $g \in \mathrm{GL}_n(q)$. If g has determinant -1 in $\mathrm{GL}_n(q)$ then $\det(\eta g) = \eta^n \det g = -\eta^n$, for $\eta \in \mathbb{F}_q^*$. Lemma 2.1 implies that $\det(\eta g) \neq 1$ and so g does not project into $\mathrm{PSL}_n(q)$. Thus if gZ is $\mathrm{PGL}_n(q)$-real and contained in $\mathrm{PSL}_n(q)$ then there are two possibilities:

(1) g is real in $\mathrm{GL}_n(q)$ with $\det g = 1$;

(2) g is ζ-real in $\mathrm{GL}_n(q)$.

In the second case, we can take ζ to be any non-square in $\mathbb{F}_q$; in particular we assume that ζ is a generator of the cyclic group $\mathbb{F}_q^*$. Now g is conjugate to ζg^{-1} and so $(\det g)^2 = \zeta^n$. In particular $\det g = \zeta^{n/2}$ or $\zeta^{(n+q-1)/2}$. Now g projects into $\mathrm{PSL}_n(q)$ only if there exists $\alpha \in \mathbb{F}_q^*$ such that $\alpha g \in \mathrm{SL}_n(q)$. Write $\alpha = \zeta^b$ for some integer b. Then such an α exists provided one of the following equations has a solution:

$$\frac{n}{2} + bn \equiv 0 \pmod{q-1}, \quad \frac{n+q-1}{2} + bn \equiv 0 \pmod{q-1}.$$

These equations translate into two cases:

(2a) If $|n|_2 > |q-1|_2$ then only the first solution is possible. This corresponds to the situation where $\det g = \zeta^{n/2}$.

(2b) If $|n|_2 = |q-1|_2$ then only the second solution is possible. This corresponds to the situation where $\det g = -\zeta^{n/2}$. $\square$

For $v = 1^{n_1} 2^{n_2} \ldots$, set $d = |(n_1, n_2, \ldots)|_2$ and, as before, define σ_v to equal 0 if $|dq|_2 > 1$ and to equal 1 otherwise.

If $|n|_2 > |q-1|_2$ and $d > 1$ then the number of ζ-real classes which project into $\mathrm{PSL}_n(q)$ is the same as the number of real classes which project into $\mathrm{PSL}_n(q)$; hence $psl_v = sl_v$. If $d = 1$ then there are no ζ-real classes in $\mathrm{GL}_n(q)$ and so $psl_v = \frac{1}{2} sl_v$. Furthermore by [4, Corollary 4.5] we know that all of these conjugacy classes are strongly real in $\mathrm{PSL}_n(q)$.

3.4 Conclusion. We summarize our findings in the following theorem:

Theorem 3.6. *Suppose that q is even or $|n|_2 \neq |q-1|_2$. Let $d = |(n_1, n_2, \ldots)|_2$. Then the number of real classes in $\mathrm{PSL}_n(q)$ of type v is equal to $h_v psl_v$ where*

$$psl_v = \begin{cases} \frac{1}{(2,q-1)} \prod_{n_i > 0} n_{q,n_i}, & |n|_2 < |q-1|_2 \text{ or } q \text{ is even}; \\ sl_v, & |n|_2 > |q-1|_2,\ d > 1 \text{ and } q \text{ is odd}. \\ \frac{1}{2} sl_v, & |n|_2 > |q-1|_2,\ d = 1 \text{ and } q \text{ is odd}. \end{cases}$$

What is more, all real classes in $\mathrm{PSL}_n(q)$ are strongly real.

Note that, when q is even, Theorem 3.6 also holds for $\mathrm{SL}_n(q)/(\mathrm{SL}_n(q) \cap Y)$ where Y is any subgroup of $Z(\mathrm{GL}_n(q))$.

4 $\mathrm{PSL}_n(q)$, q is odd and $|n|_2 = |q-1|_2$

As before psl_v denotes the number of $\mathrm{PGL}_n(q)$-real $\mathrm{PGL}_n(q)$-conjugacy classes of type v contained in $\mathrm{PSL}_n(q)$. We start by calculating psl_v for various cases. Note that both of the lemmas from Section 3.3 apply here.

First set $v = (1^{n_1} 2^{n_2} \ldots)$, $d = |(n_1, n_2, \ldots)|_2$. We will use the methods and notation of [4, Proposition 4.1], and consider various cases. In particular suppose that C is a real class (resp. a ζ-real class) of type v in $\mathrm{GL}_n(q)$. Then C is associated with a sequence of polynomials $(u_1(t), u_2(t), \dots)$ that are self-reciprocal (resp. ζ-self-reciprocal). Let a_i be the leading term in $u_i(t)$; then, for $g \in C$, $\det g = (-1)^n \prod_{n_i>0} a_i^i = \prod_{n_i>0} a_i^i$. We know that a_i is equal to ± 1 (resp. $\pm\zeta^{-n_i/2}$) when C is real (resp. C is ζ-real).

For this section it will help to choose ζ to be a non-square which satisfies $\zeta^{n/2} = -1$; this means that a ζ-real element, like a real element, will have determinant ± 1. Then, for g real or ζ-real, Lemma 2.1 implies that, if $\det g = -1$, then g does not project into $\mathrm{PSL}_n(q)$.

(P1) Suppose that $d = 1$; thus there are no ζ-real elements. We must have $\det g = 1$ and so Lemma 2.6 implies that C is equivalent to one other real class in $\mathrm{GL}_n(q)$; Lemma 3.4 implies that this class consists of elements of determinant 1. Thus $psl_v = \frac{1}{2} sl_v$.

(P2) Suppose that $d > 1$ and that $n_i = 0$ for all odd i; in particular, $d < |n|_2$. If C is real then Lemma 2.6 implies that C is equivalent to one other real class in $\mathrm{GL}_n(q)$; Lemma 3.4 implies that this class consists of elements of determinant 1.

If C is ζ-real then we must have $d > 1$ and

$$\det g = \prod_{n_i>0} a_i^i = \prod_{n_i>0} \left(\pm \frac{1}{\zeta^{n_i/2}}\right)^i = \frac{1}{\zeta^{n/2}}.$$

Thanks to our choice of ζ we have $\det g = \zeta^{n/2} = -1$ and so, as we have already observed, g does not project into $\mathrm{PSL}_n(q)$. Hence once again we have $psl_v = \frac{1}{2} sl_v$.

(P3) Suppose that $d > 1$ and that there exists i odd for which $n_i > 0$. The number of real classes which lie in $\mathrm{SL}_n(q)$ is given by [4, Proposition 4.1] and is equal to

$$f_v(q)\left(\prod_{i \text{ odd}, n_i>0} q^{n_i/2-1}\right)\left(\prod_{i \text{ even}, n_i>0} n_{q,n_i}\right).$$

We also need to count the number of ζ-real classes for which the determinant is equal to $-\zeta^{-n/2} = 1$. The same methods as in [4, Proposition 4.1] yield that the number of such classes is equal to

$$g_v(q)\left(\prod_{i \text{ odd}, n_i>0} q^{n_i/2-1}\right)\left(\prod_{i \text{ even}, n_i>0} n_{q,n_i}\right)$$

where $g_v(q) = \frac{1}{2}((q+1)^r - (q-1)^r)$ with r the number of odd values of i for which $n_i > 0$.

The total number of these two types of conjugacy class is $\prod_{n_i>0} n_{q,n_i}$. Lemma 2.6 implies that these conjugacy classes partition into $psl_v = \frac{1}{2}\prod_{n_i>0} n_{q,n_i}$ equivalence classes.

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Note that if $4|n$ then [4, Proposition 4.4] implies that all of the above classes are strongly real in $\mathrm{PSL}_n(q)$. Hence, using Proposition 3.1, we have the following:

Proposition 4.1. *Suppose that $|n|_2 = |q-1|_2$ and $4|n$. Then the total number of real conjugacy classes in $\mathrm{PSL}_n(q)$ is the same as the total number of strongly real conjugacy classes and is given by*

$$\sum_{|v|=n} h_v psl_v$$

where the values for psl_v are as given above.

4.1 $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$. This is the only case left to consider for $\mathrm{PSL}_n(q)$. In the five points above we have calculated the number of $\mathrm{PGL}_n(q)$-classes of $\mathrm{PGL}_n(q)$-classes lying in $\mathrm{PSL}_n(q)$. But in this case we do not know if all of these classes will remain real in $\mathrm{PSL}_n(q)$.

Proposition 4.2. *Suppose that $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$. Then the total number of real conjugacy classes in $\mathrm{PSL}_n(q)$ is given by*

$$\sum_{|v|=n} h_v psl_v$$

where psl_v is non-zero exactly when $n_i > 0$ for some odd i. In this case the values for psl_v are given by (P1), (P2) *and* (P3).

Proof. If $n_i > 0$ for some odd i, then [4, Proposition 5.5] implies that a real (or ζ-real) conjugacy class of $\mathrm{GL}_n(q)$ contained in $\mathrm{SL}_n(q)$ is real (or ζ-real) within $\mathrm{SL}_n(q)$. Hence we only need to deal with the situations of (P1) and (P2) where $n_i = 0$ for all odd i. In fact (P2) cannot occur for $n \equiv 2 \pmod 4$.

Thus we are left with the case (P1) only and there are no ζ-real elements. Furthermore, for a real element, [4, Lemma 5.1 and Proposition 5.5] imply that all reversing elements have non-square determinant in $\mathbb{F}_q$. But such elements do not project into $\mathrm{PSL}_n(q)$ (see Lemma 2.1), hence this situation does not yield real elements in $\mathrm{PSL}_n(q)$. □

4.2 Conclusion. We summarize our results in the following theorem.

Theorem 4.3. *Suppose that $|n|_2 = |q-1|_2$ and let $d = |(n_1, n_2, \ldots)|_2$. Then the number of real classes in $\mathrm{PSL}_n(q)$ is given by*

$$\sum_{v} h_v psl_v.$$

If $4|n$ *then*

$$psl_v = \begin{cases} \frac{1}{2}\prod_{n_i>0} n_{q,n_i}, & \text{if } d > 1 \text{ and } n_i > 0 \text{ for some odd } i, \\ \frac{1}{2} sl_v, & \text{otherwise}. \end{cases}$$

If $4 \nmid n$ *then*

$$psl_v = \begin{cases} \frac{1}{2}\prod_{n_i>0} n_{q,n_i}, & \text{if } d > 1 \text{ and } n_i > 0 \text{ for some odd } i, \\ \frac{1}{2} sl_v, & \text{if } d = 1 \text{ and } n_i > 0 \text{ for some odd } i, \\ 0, & \text{otherwise}. \end{cases}$$

5 Strongly real classes in $\mathrm{PSL}_n(q)$

We have shown that if $n \not\equiv 2 \pmod 4$ or $q \not\equiv 3 \pmod 4$ then reality and strong reality coincide in $\mathrm{PSL}_n(q)$. Throughout this section we examine the strongly real classes in $\mathrm{PSL}_n(q)$ when $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$.

Lemma 5.1. *Suppose that* $n \equiv 2 \pmod 4$ *and* $q \equiv 3 \pmod 4$. *An element* gZ *is strongly real in* $\mathrm{PSL}_n(q)$ *if and only if* gZ *lifts to an element* g *in* $\mathrm{GL}_n(q)$ *for which there is an element* h *satisfying*

(1) $hgh^{-1} = g^{-1}$ (*or* ζg^{-1});

(2) $h^2 \in Z(\mathrm{GL}_n(q))$;

(3) $\det h$ *is a square*.

Proof. Suppose that such an element h exists. Then h^c has determinant 1 for some odd integer c. Furthermore $(h^c)^2 \in Z(\mathrm{GL}_n(q))$ and $(h^c)g(h^c)^{-1} = g^{-1}$ (or ζg^{-1}). Thus gZ is strongly real in $\mathrm{PSL}_n(q)$.

On the other hand if gZ is strongly real in $\mathrm{PSL}_n(q)$ then, by definition, an element h exists in $\mathrm{GL}_n(q)$ satisfying the first two criteria given. What is more h projects into $\mathrm{PSL}_n(q)$; in other words ηh has determinant 1 for some scalar η. This means that $\det h = \eta^{-n}$ which is a square since n is even. □

Take gZ real in $\mathrm{PSL}_n(q)$. Then g is of type v where $n_i > 0$ for some odd i. Let g be a real or ζ-real element in $\mathrm{GL}_n(q)$ and let V be the module associated with g. Let h be a reversing element for g in $\mathrm{GL}_n(q)$ which satisfies $h^2 \in Z(\mathrm{GL}_n(q))$.

Now h permutes the minimal cyclic submodules of V with orbits of size 2 (in the proof of [4, Proposition 5.5] we called these orbits *h-minimal* submodules of V). This fact allows us to break the general situation into smaller subcases which we deal with in the next two lemmas.

Lemma 5.2. *Suppose that* $V = W_p \oplus W_q$ *where* W_p *and* W_q *are cyclic modules with irreducible characteristic polynomials* $p(t)^a$ *and* $\tilde{p}(t)^a$ (*resp.* $\check{p}(t)^a$). *Furthermore assume that* h *swaps* W_p *and* W_q. *Set the degree of* $p(t)$ (*and* $q(t)$) *to be* d. *Then* $\det h$ *can be a square or a non-square if* ad *is odd*; *otherwise* $\det h$ *is a square*.

Proof. We proceed as per the proof of [4, Lemma 5.4]; in particular, we can take g to equal

$$\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & 0 \\ 0 & \zeta B^{-1} \end{pmatrix}$$

for some $B \in \mathrm{GL}_{ad}(q)$. This means that

$$h = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

where X and Y centralize B in $\mathrm{GL}_{ad}(q)$. Then

$$\det h = (-1)^{ad}(\det X)(\det Y) = (-1)^{ad}(\det XY).$$

Since $h^2 \in Z(\mathrm{GL}_n(q))$ we must have $XY \in Z(\mathrm{GL}_{ad}(q))$ and so

$$\det h = (-1)^{ad}\alpha^{ad} = (-\alpha)^{ad}$$

where $\alpha \in \mathbb{F}_q$. Thus if ad is even this determinant is a square. On the other hand if ad is odd then we can let $X = Y = I$ and $\det h$ is a non-square, or take $X = I = -Y$ and $\det h$ is a square. □

Lemma 5.3. *Suppose that V is a cyclic module W_p with irreducible characteristic polynomial $p(t)^a$; suppose furthermore that $p(t)$ is self-reciprocal (resp. ζ-self-reciprocal). Set $d = \deg(p(t))$.*

(1) *If d is odd then* $\det h$ *can be chosen to be a square or a non-square if a is odd; otherwise* $\det h$ *is a square for* $a \equiv 0 \pmod 4$ *and* $\det h$ *is a non-square for* $a \equiv 2 \pmod 4$.

(2) *If d is even and a is even, then* $\det h$ *is a square.*

(3) *If d is even and a is odd, then* $\det h$ *can be chosen to be a square or a non-square if* $d \equiv 2 \pmod 4$; *otherwise* $\det h$ *is a square for* $d \equiv 0 \pmod 4$.

Proof. Let us examine the relevant cases. We will use [4, Lemmas 5.2 and 5.3]; these give conditions for $\det h$ to be a square, but they do not assume that $h^2 \in Z(\mathrm{GL}_n(q))$. In the cases where these lemmas allow for $\det h$ to be a square or a non-square we need to check the situation under this extra assumption.

Suppose that d is odd. Then $p(t) = t \pm 1$ and we refer to [4, Lemma 5.2] and observe that the conclusions given there apply here also. The only thing we have to check is that the element h satisfies $h^2 \in Z(\mathrm{GL}_n(q))$. But the h which we exhibit in the proof is an involution so we are done.

Suppose that d is even and a is even. Then [4, Lemma 5.3] implies that $\det h$ is a square.

Suppose that d is even and a is odd. Let $g = g_s g_u$ be the Jordan decomposition of g in $\mathrm{GL}_{ad}(q)$. Then g_s is centralized by $\mathrm{GL}_a(q^d)$ and so the centralizer of g must lie in $\mathrm{GL}_a(q^d)$. Furthermore a reversing element for g must be a reversing element for g_s and hence must normalize $C_G(g_s)$. Thus this element must act as a field involution of $\mathrm{GL}_a(q^d)$.

Suppose that $d \equiv 2 \pmod 4$. Then [4, Corollary 4.5] implies that there exists a reversing involution h_0. Since a is odd, we can choose $z \in Z(\mathrm{GL}_a(q^d)) = Z(C_G(g_s))$ such that $\det z$ is a non-square. Now h_0 acts as a field automorphism on $C_G(g_s)$ hence

$$(zh_0)^2 = zz^{h_0}h_0^2 = z^{q^{d/2}+1}.$$

Clearly h_0 and zh_0 are reversing elements with different determinant. Now write $(zh_0)^2$ as an element of $\mathrm{GL}_a(q^d)$: $(zh_0)^2 = \beta I$ for some $\beta \in \mathbb{F}_{q^d}$. For this to lie in $Z(\mathrm{GL}_{ad}(q))$ we must have $\beta^{(q^{d/2}+1)(q-1)} = 1$.

Since $|q^{d/2} - 1|_2 = |q - 1|_2$ we can take an odd power of z, z^c say, such that $(z^c h_0)^2 \in Z(\mathrm{GL}_{ad}(q))$. Clearly $z^c h_0$ is a reversing element for g and $\det(z^c h_0)$ is a square if and only if $\det(zh_0)$ is a square. We conclude that, in this situation, we can take h to have determinant a square or a non-square.

Suppose that $d \equiv 0 \pmod 4$. By [4, Corollary 4.5], a reversing involution h_0 exists which acts as a field automorphism on $\mathrm{GL}_a(q^d)$ and has determinant a square. Then any other reversing element must have form zh_0 where z centralizes g. The form of z is given (as an element in $\mathrm{GL}_a(q^d)$) by

$$z = \begin{pmatrix} \beta_1 & \beta_2 & & \iddots \\ & \ddots & \ddots & \\ & & \ddots & \beta_2 \\ & & & \beta_1 \end{pmatrix}.$$

Then

$$(zh_0)^2 = zz^{h_0}h_0^2 = zz^{h_0} = \begin{pmatrix} \beta_1^{q^{d/2}+1} & & \iddots \\ & \ddots & \\ & & \beta_1^{q^{d/2}+1} \end{pmatrix}.$$

For $(zh_0)^2$ to lie in $Z(\mathrm{GL}_{ad}(q))$, we must have $\beta_1^{(q^{d/2}+1)(q-1)} = 1$. But this means that β_1 must be a square in $\mathbb{F}_{q^d}$. Thus $\det zh_0$ is a square in all cases. $\square$

It is now just a matter of summing up what we have proved so far, and converting our result into the language of Macdonald.

Theorem 5.4. *Let $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$. Let gZ be real in $\mathrm{PSL}_n(q)$ of type $v = 1^{n_1}2^{n_2}\dots$, and suppose that g can be taken to be self-reciprocal* (*resp. ζ-self-reciprocal*). *Then gZ fails to be strongly real in $\mathrm{PSL}_n(q)$ if and only if the following conditions hold for all odd i such that $n_i > 0$*:

(1) *all factors of* $u_i(t)$ *have even degree*;

(2) *all self-reciprocal* (*resp.* ζ-*self-reciprocal*) *factors of* $u_i(t)$ *have degree equivalent to* $0 \pmod 4$.

Note that Theorem 4.3 implies that an odd i exists for which $n_i > 0$.

Proof. Let V be the module associated with g. By Lemma 5.1 we need to show that any reversing element h, which satisfies $h^2 \in Z(\mathrm{GL}_n(q))$, has $\det h$ a non-square.

Break V up into h-minimal submodules, W, as in the proof of [4, Proposition 5.5]. Suppose that $g|_W$ is reversible by an element h_0 which satisfies $h^2 \in Z(\mathrm{GL}(W))$ and for which we can choose $\det h_0$ to be square or non-square. Then clearly we can choose h to be a square or a non-square; Lemmas 5.2 and 5.3 give the conditions under which this is possible. These conditions are precisely the ones excluded by the statement of the theorem.

Thus the conditions given in the theorem ensure that, for every h-minimal submodule W, the reversing elements of $g|_W$ in $\mathrm{GL}(W)$ either all have determinant a square or all have determinant a non-square; in fact Lemmas 5.2 and 5.3 imply that the determinant will be a square if and only if the dimension of W is divisible by 4. Since $n \equiv 2 \pmod 4$ we conclude that $\det h$ must have determinant a non-square as required. □

6 Quotients of $\mathrm{SL}_n(q)$

We examine the real and strongly-real classes in $\mathrm{SL}_n(q)/Y$ where Y is some subgroup of $Z(\mathrm{SL}_n(q))$. We have noted already that Theorem 3.6 holds for $\mathrm{SL}_n(q)/Y$ where q is even. In fact, if $|Y|$ is odd, then the number of real (resp. strongly real) classes will equal the number of real (resp. strongly real) classes in $\mathrm{SL}_n(q)$. Similarly if $|Y|_2 = |(n, q-1)|_2$ then the number of such classes will be the same as in $\mathrm{PSL}_n(q)$. Hence, in this section, we assume that $1 < |Y|_2 < |(n, q-1)|_2$; in particular we assume that

$$q \equiv 1 \pmod 4, \quad \text{and} \quad n \equiv 0 \pmod 4.$$

In what follows we will think of Y as being a subgroup of $\mathrm{SL}_n(q)$, $\mathrm{GL}_n(q)$ or $\mathbb{F}_q^*$ depending on the context. We need two new concepts that mirror our treatment of projective groups from Section 2.

Firstly we say that elements g_1 and g_2 of $\mathrm{GL}_n(q)$ are Y-*equivalent* if they project onto the same element of $\mathrm{GL}_n(q)/Y$; so $g_2 = g_1 y$ for some $y \in Y$. This notion can be extended to conjugacy classes of $\mathrm{GL}_n(q)$ and $\mathrm{GL}_n(q)/Y$.

Secondly we generalize the idea of a ζ-real element. Let ζ_Y be an element of Y such that $\zeta_Y \neq \alpha^2$ for all $\alpha \in Y$ (ζ_Y is a *non-square* in Y); we say that g is ζ_Y-*real* in $\mathrm{GL}_n(q)$ if there exists $h \in \mathrm{GL}_n(q)$ such that $hgh^{-1} = \zeta_Y g^{-1}$. It is easy to see that all real elements in $\mathrm{SL}_n(q)/Y$ will lift to a real element or a ζ_Y-real element in $\mathrm{GL}_n(q)$ (cf. Lemma 2.4).

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For ease of calculation we will set ζ_Y to be an element which satisfies $\zeta_Y^{|Y|_2} = 1$. In particular this means that all ζ_Y-real elements, like all real elements, have determinant ± 1. Since $|Y|_2 < |n|_2$, we know that only elements of determinant 1 project into $\mathrm{SL}_n(q)/Y$ (cf. Lemma 2.1).

Now [4, Proposition 4.4] states that all $\mathrm{GL}_n(q)$-real elements in $\mathrm{SL}_n(q)$ are strongly real in $\mathrm{SL}_n(q)$. It is easy enough to modify the proof to show that all $\mathrm{GL}_n(q)$-ζ_Y-real elements in $\mathrm{SL}_n(q)$ are strongly ζ_Y-real in $\mathrm{SL}_n(q)$ (where *strongly* ζ_Y*-real* has the obvious definition).

Finally Proposition 3.1 implies that if a $\mathrm{GL}_n(q)/Y$-class is of type v then the class will split into h_v classes in $\mathrm{SL}_n(q)/Y$. This combines to give the following proposition:

Proposition 6.1. *Let Y be a subset of $Z(\mathrm{SL}_n(q))$ such that $1 < |Y|_2 < |(n, q-1)|_2$. The total number of real classes in $\mathrm{SL}_n(q)/Y$ is the same as the number of strongly real classes in $\mathrm{SL}_n(q)/Y$ and is equal to*

$$\sum_{|v|=n} h_v \mathrm{sl}y_v.$$

Here $\mathrm{sl}y_v$ is the number of Y-equivalence classes in the set of all real and ζ_Y-real conjugacy classes of type v and determinant 1 in $\mathrm{GL}_n(q)$.

It remains to calculate the value of $\mathrm{sl}y_v$ for differing v, Y, q and n. Recall that we defined sl_v to be the total number of $\mathrm{GL}_n(q)$-real $\mathrm{GL}_n(q)$-conjugacy classes of type v contained in $\mathrm{SL}_n(q)$. Now [4, Proposition 4.1] established that

$$sl_v = \begin{cases} \prod_{n_i>0} n_{q,n_i}, & \text{if } q \text{ is even or } n_i \text{ is zero for } i \text{ odd;} \\ \frac{1}{2}\prod_{n_i>0} n_{q,n_i}, & \text{if } q \text{ is odd and } in_i \text{ is odd for some } i; \\ f_v(q) \prod_{i \text{ odd}, n_i>0} q^{n_i/2-1} \prod_{i \text{ even}, n_i>0} n_{q,n_i}, & \text{otherwise.} \end{cases}$$

From here things are easy. The number of real classes and the number of ζ_Y-real conjugacy classes will be the same (cf. [4, Lemma 2.2] and note that ζ_Y-self-reciprocal polynomials exist with odd degree). These will be partitioned into sets of size 2 as described in Lemma 2.6. Hence $\mathrm{sl}y_v = sl_v$. We summarize our results as follows.

Theorem 6.2. *Let Y be a subset of $Z(\mathrm{SL}_n(q))$ such that $1 < |Y|_2 < |(n, q-1)|_2$. The number of real classes in $\mathrm{SL}_n(q)/Y$ is equal to the number of strongly real classes, and is given by*

$$\sum_{|v|=n} h_v sl_v.$$

This is the same as the number of real classes in $\mathrm{SL}_n(q)$ (see [4, Theorem 4.6]).

7 Some exceptional cases

In order to complete our classification of real classes in all quasi-simple covers of $\mathrm{PSL}_n(q)$ we must deal with some exceptional situations, namely quasi-simple covers of $\mathrm{PSL}_n(q)$ which are not quotients of $\mathrm{SL}_n(q)$. There are five situations where this may occur: $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_2(9)$, $\mathrm{PSL}_3(4)$ and $\mathrm{PSL}_4(2)$; see [10, Theorem 5.1.4].

Write $M(G)$ for the Schur multiplier of a simple group G. If $G = \mathrm{PSL}_2(4)$ (resp. $\mathrm{PSL}_3(2)$) then $|M(G)| = 2$ and the double cover of G is isomorphic to $\mathrm{SL}_2(5)$ (resp. $\mathrm{SL}_2(7)$). We have already analysed the real classes in these groups. The remaining three groups need to be analysed in turn; we start by recording some information about each (see [10, Proposition 2.9.1 and Theorem 5.1.4]):

G	isomorphism	$M(G)$
$\mathrm{PSL}_2(9)$	A_6	C_6
$\mathrm{PSL}_4(2)$	A_8	C_2
$\mathrm{PSL}_3(4)$		$C_4 \times C_{12}$

Here C_n is the cyclic group of order n, and the middle column lists groups to which G is isomorphic. Information about real conjugacy classes can, for quasi-simple groups with cyclic centre, be found in [2]; we will need to do extra work to understand those groups that do not have cyclic centre, and to classify the strongly real conjugacy classes.

Our approach in this section is, in some sense, the reverse of that in the rest of the paper. We have complete information about (strongly) real classes in G, and we wish to deduce information about (strongly) real classes in quasi-simple covers of G. We start with a lemma which applies to this situation in some generality.

Lemma 7.1. *Let G, H be groups such that $H/Z \cong G$ where Z is an odd-order central subgroup of H. Let C be a real class in G containing elements of order n; then C lifts to a unique real class C_H in H and this class consists of elements of order n. What is more if C is strongly real than C_H is strongly real.*

Proof. Let χ be a real-valued irreducible complex character (or *rvicc*) of H. Let $g \in Z$; then $\chi(g) = 1$ for $g \neq 1$. This implies that χ is an rvicc of G. Since every rvicc of G is an rvicc of H, we conclude that G and H have the same number of rvicc's. Thus G and H have the same number of real classes.

Now suppose that h and gh lie in different conjugacy classes of H, with $g \in Z$, $g \neq 1$. Since h and hg are not conjugate in H, there exists an irreducible complex representation Φ with character χ such that $\chi(gh) \neq \chi(h)$. Now $\Phi(g) = \eta I$ and so $\chi(gh) = \eta\chi(h)$; in particular, $\eta \neq 1$ and $\chi(h) \neq 0$. Since g has odd order it follows that η is not real. Thus gh and h cannot be both real, and we conclude that every real class of G lifts to a unique real class in H.

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Suppose next that h is real in H and hZ is real in G of order m. Then $h^m \in Z$, and the only element in Z that is real in H is the identity. Thus $h^m = 1$ as required.

Finally suppose that C_H is a real class in H such that $h \in C_H$ and hZ lies in C, a strongly real class in G. Then there exists $fZ \in G$ such that

$$(fZ)(hZ)(fZ)^{-1} = h^{-1}Z \quad \text{and} \quad (fZ)^2 = Z \in G.$$

Since Z has odd order we can assume that $f^2 = 1 \in H$. Then $fhf^{-1} = h^{-1}g$ for some $g \in Z$. Since Z has odd order, $g = g_1^{-2}$ for some $g_1 \in Z$. Then $f(hg_1)f^{-1} = (hg_1)^{-1}$. Thus hg^{-1} is strongly real in H and projects onto hZ in G. Since C_H is the unique real class to which C lifts, we conclude that C_H is strongly real. □

A consequence of this lemma is that G and H have the same number of real (resp. strongly real) classes. If $G \cong \mathrm{PSL}_2(9)$ or $G \cong \mathrm{PSL}_2(4)$, then $M(G)$ contains a unique involution; thus G has a unique double cover, $2.G$. Lemma 7.1 reduces the study of real and strongly real classes in the covers of G to the study of real and strongly real classes in G and $2.G$. The case of $\mathrm{PSL}_3(4)$ is more difficult.

Throughout what follows, G is a simple group, and H a quasi-simple group with centre $Z = Z(H)$ such that $G \cong H/Z$.

7.1 Covers of PSL$_2$(9). The group $\mathrm{PSL}_2(9) \cong A_6$ has seven conjugacy classes (with elements of order 1, 2, 3, 3, 4, 5 and 5), all of which are real (see Theorem 3.6 or [16]). Lemma 7.1 implies that $3.\mathrm{PSL}_2(9)$ has seven real conjugacy classes with elements of the same orders. Theorem 3.6 implies that all conjugacy classes in $\mathrm{PSL}_2(9)$ are strongly real, hence the same is true of $3.\mathrm{PSL}_2(9)$.

Table 1. Real and strongly real classes in J_8.

Line	$gZ \in A_8$	$g \in J_8$	order in J_8	real	strongly real
1	(1)	± 1	$1, 2$	yes	yes
2	(12)(34)	$\pm s_1 s_3$	4	yes	no
3	(123)	$\pm s_1 s_2$	$3, 6$	yes	no
4	(123)(456)	$\pm s_1 s_2 s_4 s_5$	$3, 6$	yes	yes
5	(1234)(56)	$\pm s_1 s_2 s_3 s_5$	8	yes	no
6	(12345)	$\pm s_1 s_2 s_3 s_4$	$5, 10$	yes	no
7	(12)(34)(56)(78)	$\pm s_1 s_3 s_5 s_7$	2	yes	yes
8	(1234)(5678)	$\pm s_1 s_2 s_3 s_5 s_6 s_7$	4	yes	yes
9	(123)(45)(67)	$\pm s_1 s_2 s_4 s_6$	12	yes	no
10	(123456)(78)	$\pm s_1 s_2 s_3 s_4 s_5 s_7$	$6, 6$	no	no

Similarly $\mathrm{SL}_2(9) \cong 2.\mathrm{PSL}_2(9)$ has 13 conjugacy classes (with elements of orders 1, 2, 3, 3, 4, 5, 5, 6, 6, 8, 8, 10, and 10) all of which are real. Lemma 7.1 implies that $6.\mathrm{PSL}_2(9)$ has 13 real conjugacy classes with elements of the same orders.

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The groups $\mathrm{SL}_2(9) \cong 2.\mathrm{PSL}_2(9)$ and $6.\mathrm{PSL}_2(9)$ contain a single involution (in the centre); thus both groups contain precisely two strongly real classes.

7.2 Covers of $\mathbf{PSL_4(2)}$. Consider the group H_{2n} given by the presentation

$$H_{2n} = \langle s_1, \ldots, s_{2n-1} \mid s_k^2, -(s_k s_{k+1})^3, -(s_k s_j)^2 \rangle,$$

where $j, k = 1, \ldots, 2n-1$, $|j-k| > 1$, and -1 is defined to be a central element. The group H_{2n} is a double cover of S_{2n} with center $Z = \{1, -1\}$; see [1, p. 175]; the projection map is given by

$$\pi : H_{2n} \mapsto S_{2n}, \quad s_k \mapsto (k \quad k+1).$$

Then H_{2n} has a subgroup J_{2n} of index 2 which is the double cover of A_{2n}; this is the group of interest here. Clearly J_{2n} consists of all elements $\pm x$ where x is the product of an even number of the elements s_i.

Now let g be a real element of J_{2n}; then gZ is a real element of A_{2n}. What is more, if g is (strongly) real in J_{2n} then $-g$ is also (strongly) real (since $hgh^{-1} = g^{-1}$ implies that $h(-g)h^{-1} = -g^{-1} = (-g)^{-1}$).

With this in mind we list the real and strongly real classes in $J_8 \cong 2.\mathrm{PSL}_4(2)$ in Table 1. We need to explain the columns of this table: The first column records the line number. The second column lists representatives from all real classes in A_8. The third column lists the two elements in J_8 that project onto the given representative in A_8. The fourth column gives the order of elements in J_8 which project onto gZ in A_8; the presence of two numbers in this column means that there are two different conjugacy classes of elements in J_8 that project onto the same conjugacy class in A_8. The final two columns state whether or not the elements $\pm g$ are (strongly) real.

Proposition 7.2. *Table* 1 *is correct*.

Proof. The first five columns follow immediately from [2] (see also the explicit calculations given in [3]).

Now consider the final column. Obviously involutions and the identity are strongly real. Recall also that, since g is strongly real if and only if $-g$ is strongly real, we need only prove the result for any h projecting onto gZ.

The only non-central involutions in J_8 correspond to 4-transpositions in A_8 (line 7 of the table). We can use this to rule out some cases: observe that

$$R_{A_8}((123)) \cong (\langle (123) \rangle \times A_5) : \langle (12)(45) \rangle,$$

hence $R_{A_8}((123))$ contains no 4-transpositions. We conclude that lines 3 and 9 do not correspond to strongly real classes. Similarly

$$R_{A_8}((12345)) \cong (\langle (12345) \rangle \times \langle (678) \rangle) : \langle (25)(34) \rangle,$$

and, again, this contains no 4-transpositions. Hence line 6 does not correspond to strongly real classes.

Now the group $R_{A_8}((123)(456))$ contains a 4-transposition, $(16)(25)(34)(78)$, which reverses $(123)(456)$. Since $(123)(456)$ lifts to elements of different orders, we conclude that they must be strongly real. In other words, line 4 corresponds to strongly real classes.

Next consider line 2 and take $g \in J_8$ which projects onto $gZ = (12)(34)$. Let H be the group of even permutations of $\{5, 6, 7, 8\}$ (so $H \cong A_4$). Then

$$C_{J_8}(g)/Z = \{(1), (12)(34), (13)(24), (14)(23)\} \times H,$$
$$C_{A_8}(gZ) = C_{J_8}(g)/Z.\langle (12)(56) \rangle.$$

Any element that reverses g must centralize gZ. However all 4-transpositions in $C_{A_8}(gZ)$ are contained in $C_{J_8}(g)/Z$; hence g is not strongly real.

We move on to line 5 and take $g \in J_8$ which projects onto $gZ = (1234)(56)$. Then

$$C_{J_8}(g)/Z = \langle (1234)(56) \rangle, \quad C_{A_8}(gZ) = \langle (1234)(56), (1234)(78) \rangle,$$
$$R_{A_8}(gZ) = \langle (1234)(56), (1234)(78), (14)(23) \rangle.$$

There are four cosets of $C_{J_8}(g)/Z$ in $R_{A_8}(gZ)$, two of which reverse g. Only one of these cosets contains a 4-transposition. Now observe that

$$(s_1 s_5 s_2 s_1 s_2 s_3 s_2 s_1)(s_1 s_2 s_3 s_5)(s_1 s_5 s_2 s_1 s_2 s_3 s_2 s_1)^{-1} = -(s_1 s_2 s_3 s_5)^{-1}.$$

Thus the coset containing a 4-transposition does not reverse g, and we conclude that line 5 does not correspond to a real class in J_8.

Finally consider line 8 and take $g \in J_8$ which projects onto $gZ = (1234)(5678)$ in A_8. Observe that

$$\begin{aligned} C_{J_8}(g)/Z &= \langle (1234)(5678) \rangle, \\ C_{A_8}(gZ) &= \langle (1234)(5678), (1234)(8765) \rangle : \langle (15)(26)(37)(48) \rangle, \\ R_{A_8}(gZ) &= (C_{A_8}(gZ)) : \langle (15)(26)(37)(48) \rangle. \end{aligned}$$

Set $H = \langle (1234)(5678) \rangle$; then H has four cosets in $R_{A_8}(gZ)$, all containing 4-transpositions. One coset must lift to the set $\{h : hgh^{-1} = g^{-1}\}$, and we conclude that line 8 does correspond to a strongly real class in J_8. $\square$

7.3 Covers of PSL₃(4). Let $G = \mathrm{PSL}_3(4)$, and let H be a quasi-simple cover of G. Observe that G has a single conjugacy class of involutions; also [5, Proposition 6.4.1] implies that an involution $g \in G$ lifts to an involution $h \in H$.

Using our work above, we calculate that G contains eight real classes (with elements of order 1, 2, 3, 4, 4, 4, 5 and 5) and they are all strongly real. Furthermore

$Z(H) \leqslant C_4 \times C_{12}$ and Lemma 7.1 allows us to assume that $Z(H)$ is a non-trivial subgroup of $C_4 \times C_4$. Thus there are seven covers of $L_3(4)$ to be discussed:

$$\mathrm{PSL}_3(4), \quad 2.\mathrm{PSL}_3(4), \quad E_4.\mathrm{PSL}_3(4), \quad 4_1.\mathrm{PSL}_3(4), \quad 4_2.\mathrm{PSL}_3(4),$$
$$(E_4 4_1).\mathrm{PSL}_3(4), \quad (C_4 \times C_4).\mathrm{PSL}_3(4).$$

By E_4 we mean an elementary abelian group of order 4; by 4_1 and 4_2 we mean quotients of $M(G)$ by cyclic groups of order 4 that lie in $M(G)$ and are not in the same orbit of $\mathrm{Out}(\mathrm{PSL}_3(4))$. That this list of covers is comprehensive follows easily from [5, Theorem 6.3.1] and [6, Lemma 2.3 (i), p. 463].

Before we proceed with our analysis we establish some notation. Let P be a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$; observe that P is isomorphic to

$$P_1 \cong \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4 \right\}.$$

We will identify P with P_1, allowing us to write elements of P as matrices. Write P_H for the Sylow 2-subgroup of H that projects onto P. For $h \in H$, define

$$Z_h = \{ z \in Z \mid h_1 h h_1^{-1} = hz \text{ for some } h_1 \in P_H \}.$$

Observe that this is a subgroup of Z and, that $|Z_h| = |C_P(g) : C_{P_H}(h)/Z|$.

Proposition 7.3. *Let H be a quasi-simple cover of $G = \mathrm{PSL}_3(4)$ with centre Z, a 2-group. Suppose that $g = hZ \in G = H/Z$, with g real in G of order d.*

(1) *If d is odd, then h is real if and only if the order of h is d or $2d$. What is more h is strongly real if and only if h is real.*

(2) *If $d = 2$, then h is strongly real.*

Note that we are not dealing with the case when $d = 4$. We address this situation in the next proposition.

Proof. Case 1. Suppose that d is odd. Then the set hZ generates a cyclic subgroup of H, and we may take the order of h to equal d. If $d = 1$ then h is central and is (strongly) real if and only if $h^2 = 1$.

Now suppose that $d > 1$. Let $g_1 \in G$ satisfy $g_1 g g_1^{-1} = g^{-1}$ and $g_1 = h_1 Z$ for $h \in H$. Then $h_1 h h_1^{-1} = h^{-1} z$ for some $z \in Z$. Since h has odd order, we conclude that $z = 1$. In other words $h_1 h h_1^{-1} = h^{-1}$ and h is real. More generally this implies that

$$h_1 (hz) h_1^{-1} = h^{-1} z.$$

Thus hz is real if and only if $z^2 = 1$. Furthermore, since g is strongly real we may take g_1 to be an involution. By [5, Proposition 6.4.1] we may therefore take h_1 to be an involution. Thus if hz is real then hz is strongly real; we have proved (1).

Case 2. Suppose that $d = 2$. We take h to have order 2, which we may do by [5, Proposition 6.4.1]. We take

$$g = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(P),$$

where $x \in \mathbb{F}_4^*$.

We may assume that $H \cong (C_4 \times C_4).\mathrm{PSL}_3(4)$ and that $Z(H) = \langle z_1, z_2 \rangle$, so z_1 and z_2 are elements of order 4. Now [6, Lemma 2.3(c), p. 463] implies that $|C_P(g) : C_{P_H}(h)/Z| = 4$ and so $|Z_h| = 4$. Since h has order 2, this implies that $Z_h = \{1, z_1^2, z_2^2, z_1^2 z_2^2\}$. The elements of hZ can therefore be written in subsets of conjugate elements as follows:

$$\begin{gathered} \{h, hz_1^2, hz_2^2, hz_1^2 z_2^2\}, \quad \{hz_1, hz_1^3, hz_1 z_2^2, hz_1^3 z_2^2\}, \\ \{hz_2, hz_1^2 z_2, hz_2^3, hz_1^2 z_2^3\}, \quad \{hz_1 z_2, hz_1^3 z_2, hz_1 z_2^3, hz_1^3 z_2^3\}. \end{gathered}$$

In particular these elements are all real. We need to establish that they are, in fact, strongly real. Observe first that if $h_1^2 = hz$, with h of order 2 and z central, then

$$hh_1 = h_1^{-1} z = h_1 h.$$

In other words, an element h_1 satisfying $h_1^2 = hz$ commutes with h. This, along with [6, Lemma 2.3(c), p. 463], implies that $C_{P_H}(h)/Z$ is isomorphic to the group

$$\begin{aligned} C &= \left\langle \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4, ac = x \right\rangle \\ &= \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4, ac = x \text{ or } a = c = 0 \right\} \cong C_2 \times Q_8. \end{aligned}$$

Now let g_1 be some element of P such that $\langle C, g_1 \rangle$ is a degree 2 extension of $C_{P_H}(h)/Z$. Then

$$g_1 = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for some } a, b, c \in \mathbb{F}_4, \text{ with } (a, c) \neq (0, 0).$$

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If a or c is equal to 0 then $g_1^2 = 1$ and this extension is split. If $a \neq 0 \neq c$, then there exists $g_0 \in C$ such that

$$g_0 = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a^{-1}x \\ 0 & 0 & 1 \end{pmatrix},$$

and $g_0 g_1$ is an involution; thus, again, the extension is split. Thus any degree 2 extension of $C_{P_H}(h)/Z$ in P is split.

Since $C_{P_H}(hz)/Z = C_{P_H}(h)/Z$ for any $z \in Z$, [5, Proposition 6.4.1] implies that $R_P(hz)$ is a split extension of $C_P(hz)$. In other words, hz is strongly real, as required. □

We must now examine those elements $h \in H$ for which $g = hZ$ is an element of order 4 in $\mathrm{PSL}_3(4)$. There are three conjugacy classes of elements of order 4 in $\mathrm{PSL}_3(4)$. They are fused by an outer automorphism of $\mathrm{PSL}_3(4)$ and intersect P in the following sets:

$$C_k = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4, ac^{-1} = k \right\} \quad \text{with } k \in \mathbb{F}_4^*.$$

If $Z(H) = E_4$ or $Z(H) = C_4 \times C_4$ then the set of conjugacy classes in H that project onto C_k is mapped, via an outer automorphism of H, to the set of conjugacy classes in H that project onto $C_{k'}$ for $k' \neq k$; see [5, Table 6.3.1].

Proposition 7.4. *Let H be a quasi-simple cover of $G = \mathrm{PSL}_3(4)$ with centre Z, a 2-group. Suppose that $g = hZ \in G = H/Z$, with g real in G of order 4. Suppose that g lies in the set C_k for some $k \in \mathbb{F}_4^*$.*

(1) *If $Z(H) = C_2$ or $Z(H) = 4_1$, then all elements in hZ are real.*

(2) *If $Z(H) = 4_2$, then the number of real elements in hZ depends on k. For two values of k, every element in hZ is real; for the third, precisely half of the elements in hZ are real.*

(3) *If $Z(H) = C_4 \times C_4$, then precisely half of the elements in hZ are real.*

(4) *If $Z(H) = E_4 4_1$, then the number of real elements in hZ depends on k. For two values of k, every element in hZ is real; for the third, precisely half of the elements in hZ are real.*

(5) *If $Z(H) = E_4$, then all elements in hZ are real.*

(6) *If $Z(H)$ is non-cyclic, then no elements in hZ are strongly real.*

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(7) *If $Z(H) = C_4$, then the number of real elements in hZ depends on k. For two values of k, no elements in hZ are strongly real; for the third, precisely half of the elements in hZ are strongly real.*

(8) *If $Z(H) = C_2$, then the number of real elements in hZ depends on k. For two values of k, no elements in hZ are strongly real; for the third, all elements in hZ are strongly real.*

Proof. Statements (1) and (2) follow immediately from [2, p. 28]; we therefore start with (3). Throughout the proof we will refer to the *universal* 2-*cover* as $H_U \cong (C_4 \times C_4).\mathrm{PSL}_3(4)$. To begin we take $H = H_U$, $x \in \mathbb{F}_4^*$, and set

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}; \quad \text{thus} \quad g^2 = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This implies that

$$C_P(g) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle|\, a, b, c \in \mathbb{F}_4, c = ax \right\} \cong C_4 \times C_4.$$

Clearly $C_{P_H}(h) \geqslant \langle h \rangle Z(H)$ and so $C_{P_H}(h).Z$ has index at most 4 in $C_P(g)$. Now one of the classes of order 4 in $\mathrm{PSL}_3(4)$ lifts to four separate conjugacy classes in $4_2.\mathrm{PSL}_3(4)$; see [2, p. 28]. We conclude that $C_{P_H}(h)$ has index at least 4 in $C_P(g)$; thus $C_{P_H}(h)/Z = \langle h \rangle Z/Z$, and $|C_P(g) : C_{P_H}(h)/Z| = 4 = |Z_h|$.

Now write $Z = \langle z_1 \rangle \times \langle z_2 \rangle$. Suppose that Z_h is elementary abelian; then

$$Z_h = \{1, z_1^2, z_2^2, z_1^2 z_2^2\}.$$

Now there is an element in $C_{P_H}(h^2)/Z$ that conjugates g to g^{-1}; hence it must map h to an element of $h^{-1}Z_h$ (since $h^{-1}Z_h$ contains all elements of $h^{-1}Z$ whose square is equal to h^2). Since $Z_{h^{-1}}$ contains h^{-1} we conclude that h is real; indeed, all elements in hZ are real. Since covers with cyclic centre are epimorphic images of covers with non-cyclic centre, the same conclusion will follow if $Z(H)$ is cyclic. This contradicts statements (1) and (2).

We conclude that Z_h is cyclic; write $Z_h = \{1, z_1, z_1^2, z_1^3\}$. Now there is an element $h_1 \in C_{P_H}(h^2)/Z$ that conjugates g to g^{-1}; hence $Z_h = Z_{h^{-1}}$. We have several cases to consider:

(1) If h_1 conjugates hZ_h to $h^{-1}z_2^3 Z_h$, then relabel so that z_2^3 becomes z_2; then we lie in the next case.

(2) If h_1 conjugates hZ_h to $h^{-1}z_2 Z_h$, then $hZ \cup h^{-1}Z$ splits into four sets of conjugate elements, with elements from distinct sets non-conjugate:

$$(hZ_h \cup h^{-1}z_2 Z_h),\ z_2(hZ_h \cup h^{-1}z_2 Z_h),\ z_2^2(hZ_h \cup h^{-1}z_2 Z_h),\ z_2^3(hZ_h \cup h^{-1}z_2 Z_h).$$

We conclude that none of these elements are real. Moreover, for $h_1 \in hZ$, we find that $h_1\langle z_1\rangle$ is not real in $H/\langle z_1\rangle$. However $H/\langle z_1\rangle$ is a cover of $\mathrm{PSL}_3(4)$ with cyclic center. This contradicts statements (1) and (2).

(3) If h_1 conjugates hZ_h to $h^{-1}z_2^2Z_h$, then the set of conjugates of h in $h^{-1}Z$ is equal to $\{h^{-1}z_2^2, h^{-1}z_1z_2^2, h^{-1}z_1^2z_2^2, h^{-1}z_1^3z_2^2\}$. In this case relabel so that h becomes hz_2; then we lie in the next case.

(4) The set of conjugates of h in $h^{-1}Z$ is equal to $\{h^{-1}, h^{-1}z_1, h^{-1}z_1^2, h^{-1}z_1^3\}$.

Thus, provided we label appropriately, the following elements are all conjugate:

$$hZ_h \cup h^{-1}Z_h = \{h, hz_1, hz_1^2, hz_1^3, h^{-1}, h^{-1}z_1, h^{-1}z_1^2, h^{-1}z_1^3\}.$$

Similarly the following sets consist of conjugate elements:

$$z_2(hZ_h \cup h^{-1}Z_h), \quad z_2^2(hZ_h \cup h^{-1}Z_h), \quad z_2^3(hZ_h \cup h^{-1}Z_h).$$

Thus, of all elements in hZ, precisely the elements in the sets hZ_h, $z_2^2hZ_h$ are real in H.

We have proved (3). To prove (4) and (5) we examine the following sets of conjugate elements in H_U:

$$hZ_h \cup h^{-1}Z_h, \quad z_2(hZ_h \cup h^{-1}Z_h), \quad z_2^2(hZ_h \cup h^{-1}Z_h), \quad z_2^3(hZ_h \cup h^{-1}Z_h).$$

Consider $H = H_U/Z_1$ where Z_1 is a central subgroup of H_U. The following table lists those elements $h_1 \in hZ$ for which h_1Z_1 is real in H:

Z_1	H	real elements
$\langle z_1^2\rangle$	$(E_44_1).\mathrm{PSL}_3(4)$	Z_h, $z_2^2Z_h$
$\langle z_2^2\rangle$	$(E_44_1).\mathrm{PSL}_3(4)$	Z_h, z_2Z_h, $z_2^2Z_h$, $z_2^3Z_h$
$\langle z_1^2z_2^2\rangle$	$(E_44_1).\mathrm{PSL}_3(4)$	Z_h, z_2Z_h, $z_2^2Z_h$, $z_2^3Z_h$
$\langle z_1^2, z_2^2\rangle$	$E_4.\mathrm{PSL}_3(4)$	Z_h, z_2Z_h, $z_2^2Z_h$, $z_2^3Z_h$

This yields (4) and (5). We have three entries for $H = (E_44_1).\mathrm{PSL}_3(4)$ as the order 3 automorphisms of $\mathrm{PSL}_3(4)$ do not lift to this group.

To prove the remaining statements we must determine when h is strongly real. Suppose first that Z is cyclic and non-trivial. We start by considering the 4-covers of $\mathrm{PSL}_3(4)$; let $Z = \langle z\rangle$ and let Y be the pre-image of $Z(P)$ in P_H; then [6, Lemma 2.2, p. 463] implies that $Y = X \times Z$ where $X \cong C_2 \times C_2$. Furthermore [6, Lemma 2.3(e), p. 463] implies that $C_{P_H}(Y)/Z \cong C_4 \times C_4$ and so contains an element h such that hZ is an element of order 4 in $\mathrm{PSL}_3(4)$. In particular $C_{P_H}(h) > Y$.

Now suppose that $H \cong 4_1.\mathrm{PSL}_3(4)$; then $C_{P_H}(h)/Z$ is a proper subgroup of $C_P(g)$ (otherwise, hZ intersects four distinct conjugacy classes of $4_1.\mathrm{PSL}_3(4)$, and $Z_h = \{1\}$;

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this is impossible by [2, p. 24]). Thus $C_{P_h}(h) = \langle h \rangle Y$, which has index 2 in $C_P(g)$. This implies, firstly, that hZ intersects two conjugacy classes, call them C_1 and C_2, in H. It implies, secondly, that $Z_h = \{1, z^2\} = Z_{h^{-1}}$ and, since h is real, the elements h, h^{-1}, hz^2 and $h^{-1}z^2$ are all conjugate.

Now the set of elements in P that reverse g is equal to

$$R = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & (a+1)x \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{F}_4 \right\}.$$

In addition observe that $C = C_{P_H}(h)/Z \ = \langle g, Z(P) \rangle$. Consider degree 2 extensions of C of the form $\langle C, r \rangle$ for some $r \in R$. There are two such extensions, one split (when the element r has $a = 1$ or $a = 0$ in the matrix form given above) and the other non-split.

Thus h is mapped by an involution to precisely one of either h^{-1} or $h^{-1}z^2$. This implies that h is strongly real if and only if hz is not strongly real. Thus we conclude that precisely two of the elements in hZ are strongly real in $4_1.\mathrm{PSL}_3(4)$. In particular not all real elements are strongly real in $4_1.\mathrm{PSL}_3(4)$.

We return to the situation where $H = H_U$ and write $Z(H) = \langle z_1, z_2 \rangle$. As before we choose h so that $Z_h = \{1, z_1, z_1^2, z_1^3\}$. Again the following sets consist of conjugate elements:

$$hZ_h \cup hZ_{h^{-1}}, \quad z_2(hZ_h \cup hZ_{h^{-1}}), \quad z_2^2(hZ_h \cup hZ_{h^{-1}}), \quad z_2^3(hZ_h \cup hZ_{h^{-1}}).$$

Thus, in $H/\langle z_1 z_2^2 \rangle \cong 4_1.\mathrm{PSL}_3(4)$, the set hZ splits into two conjugacy classes; then [2, p. 24] implies that these conjugacy classes must coincide with C_1 and C_2 described above.

Now define groups $S < R < P$ as follows:

$$R = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4, a \in \{0, 1\} \ c \in \{0, x\} \right\} \cong C_4 \times C_4;$$

$$S = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_4, (a, c) \in \{(0, 0), (1, x)\} \right\} \cong C_4 \times C_2.$$

Let R_H (resp. S_H) be the pre-image of R (resp. S) in H; then R_H is a degree 4 extension of $C_{P_H}(h)$. In addition, firstly, all of the involutions that reverse g are contained in R. Secondly, S_H is a subgroup of R_H of index 2, and S_H/Z centralizes g. Thus

$$\{z \in Z \mid h_1 h h_1^{-1} = hz \text{ for some } h_1 \in S_H\},$$

is a subgroup of Z_h of size 2; it must equal $\{1, z_1^2\}$.

The R-conjugates of h have two possible forms; the first possibility is that the R-conjugacy class of h is $h^R = \{h, hz_1^2, h^{-1}, h^{-1}z_1^2\}$. The strongly real elements of hZ are then all elements hz such that the set zh^R contains $(hz)^{-1}$. A quick calculation demonstrates that the elements satisfying this requirement are precisely the real elements in hZ. Thus all real elements in H are strongly real. Clearly the same result applies to all epimorphic images of H, which contradicts our earlier calculations in $4_1.\mathrm{PSL}_3(4)$.

The second possibility is that the R-conjugates of h are h, hz_1^2, $h^{-1}z_1$, $h^{-1}z_1^3$. In this case half of the elements in hZ are strongly real in $H/\langle z_1z_2^2\rangle$, which is consistent with our calculations above. It immediately follows that none of the elements in hZ are strongly real in H_U.

To complete our analysis we consider, as before, $H = H_U/Z_1$, where Z_1 is a central subgroup of H_U. The following table lists those choices of Z_1 for which hZ contains any strongly real elements; in each case the table lists those elements $h_1 \in hZ$ for which h_1Z_1 is strongly real in H; we write Y_h for the set $\{1, z_1^2\}$:

Z_1	H	strongly real elements
$\langle z_1z_2^2\rangle$	$4_1.\mathrm{PSL}_3(4)$	hz_2Y_h, $hz_1z_2Y_h$, $hz_2^3Y_h$, $hz_1z_2^3Y_h$
$\langle z_1\rangle$	$4_2.\mathrm{PSL}_3(4)$	hY_h, hz_1Y_h, hz_2Y_h, $hz_1z_2^2Y_h$
$\langle z_1, z_2^2\rangle$	$2.\mathrm{PSL}_3(4)$	hZ

Statements (6), (7) and (8) follow immediately from the table. □

8 Further work

It is natural to ask if the real and strongly real classes can be counted in other families of finite groups of Lie type. The work of Macdonald extends (as he explains in [12]) to the unitary groups, so this is the natural next step.

In counting real conjugacy classes for a finite group G we are also, of course, counting real irreducible representations for G. The question arises whether we now construct these representations. For the case of $\mathrm{GL}_n(q)$ we hope to use Green's classical method [9]. For other finite groups of Lie type this is likely to be very difficult, and to require the Deligne–Lusztig theory.

Real irreducible characters come from two different kinds of irreducible representation, the orthogonal and the symplectic ones. It is not clear whether there is any such division for the number of real conjugacy classes; however we can make some observations. For instance, note that for $\mathrm{GL}_n(q)$, and for $\mathrm{SL}_n(q)$ with $n \not\equiv 2 \pmod 4$, all real conjugacy classes are strongly real; it turns out that in these cases the self-dual representations are orthogonal, i.e., the real characters actually come from orthogonal representations (cf. [8], [13] where it is shown that the Schur index for the complex characters of the above mentioned groups is always 1).

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This correspondence between strongly real classes and orthogonal representations has been studied from a variety of angles; see Gow's work on 2-regular structure [7], and Prasad's work on groups of Lie type and p-adic groups [14], [15]. Nonetheless, although the correspondence can be seen to hold in particular cases (and not in others), it is unclear how general a phenomenon it really is.

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