PROBABILISTIC SOLUTIONS TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF GENERALIZED CAPUTO AND RIEMANN-LIOUVILLE TYPE

M. E. HERNÁNDEZ-HERNÁNDEZ*, V. N. KOLOKOLTSOV

Abstract. This paper provides well-posedness and integral representations of the solutions to nonlinear equations involving generalized Caputo and Riemann-Liouville type fractional derivatives. As particular cases, we study the linear equation with non constant coefficients and the generalized composite fractional relaxation equation. Our approach relies on the probabilistic representation of the solution to the generalized linear problem recently obtained by the authors. These results encompass some known cases in the context of classical fractional derivatives, as well as their far reaching extensions including various mixed derivatives.

1. Introduction

Over the last decades, the theory of fractional differential equations has been actively studied due to its vast applications in different fields of science (see, e.g., [3, 14, 25, 26, 30], and references therein). The use of fractional ordinary differential equations (FODE's) and fractional partial differential equations (FPDE's) for modeling relaxation phenomena, viscoelastic systems, anomalous diffusions, and continuous time random walks (CTRW's), have been addressed, e.g., in [2, 15, 19, 21, 23, 30].

To solve this type of equations a variety of numerical and analytical approaches have been investigated. Amongst them, *the Laplace, the Mellin* and *the Fourier transform* techniques play an important role (see, e.g., [4, 14, 25, 26]). In the context of probability theory, some connections between the solutions of certain fractional differential equations and stochastic processes can be found in the literature (see, e.g., [9, 15, 16, 23, 24, 28]). A more recent connection between fractional equations and stochastic analysis was given in [11, 17].

This work focuses on the well-posedness for *the generalized nonlinear fractional equation*

$$-\tilde{D}^{(\nu)}u(t) = -f(t,u(t)), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{1}$$

and *the generalized composite fractional relaxation equation*

$$-\tilde{D}^{(\nu)}u(t) - \gamma(t)u'(t) - \lambda u(t) = -f(t,u(t)), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{2}$$

2010 *Mathematics Subject Classification.* 60H30; 34A08; 35S15; 34G20.

Key words and phrases. Fractional differential equation; Composite fractional relaxation equation; Generalized Caputo type derivative; Generalized Riemann-Liouville type derivative; Boundary point; Stable subordinator; Feller process.

*Corresponding author. Email: M.E.Hernandez-Hernandez@warwick.ac.uk.

for some given functions f and γ, and $\lambda \geq 0$. Notation $-\tilde{D}^{(\nu)}$ refers either to the generalized RL type operator $-D^{(\nu)}_{a+}$ or to the Caputo type operator $-D^{(\nu)}_{a+*}$ (defined below). These operators were introduced in [17] as natural extensions (from a probabilistic point of view) of the Caputo and RL derivatives of order $\beta \in (0,1)$. They can be thought of as the generators of decreasing Feller processes *interrupted* on an attempt to cross a boundary.

Some particular examples of equation (1) include the initial value problem for the nonlinear equation with the classical Caputo derivative D^{β}_{0+*}:

$$D^{\beta}_{0+*}u(t) = f(t,u(t)), \quad t \in (0,b], \quad u(0) = u_0, \quad \beta \in (0,1), \tag{3}$$

and the *fully mixed* (or multi-term) fractional equation

$$\sum_{i=1}^{d} \omega_i(t) D^{\beta_i(t)}_{0+*}u(t) = f(t,u(t)), \quad t \in (0,b], \quad u(0) = u_0, \quad \beta_i \in (0,1), \tag{4}$$

for a given continuous function f and nonnegative functions $\omega_i(\cdot)$, $i \in \{1,\ldots,d\}$. The existence and uniqueness results for the fractional equation (3) have been proved by transforming this equation into a Volterra type equation and then by using fixed point arguments (see, e.g., Theorem 5.1 and Theorem 6.1 in [4] for the RL and the Caputo case, respectively).

Our method to prove well-posedness for the generalized problem in (1) is also based on transforming the equation into an integral equation. However, the integral equation used here is taken from the probabilistic solution to the corresponding linear problem which was recently obtained in [11].

We also study the linear equation with non constant coefficients:

$$-\tilde{D}^{(\nu)}u(t) = \lambda(t)u(t) - g(t), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \tag{5}$$

for given functions λ and g. For this case we give an explicit solution in terms of the transition probabilities of the underlying stochastic process. One of the reasons to deal with this case separately is due to the fact that the probabilistic representation of its solution has an explicit form as a Feynman-Kac type formula.

The generalized equation (5) encompasses the initial value problem for the linear equation with non constant coefficients involving the classical Caputo derivative:

$$D^{\beta}_{0+*}u(t) = \lambda(t)u(t) + g(t), \quad t \in (0,b], \quad u(0) = u_0, \tag{6}$$

for $\beta \in (0,1)$. It was proved by analytical methods that if $g \in C[0,b]$, then equation (6) has a unique solution $u \in C[0,b]$ given by (see, e.g., [4], Theorem 7.10)

$$u(t) = T(t) + \int_0^t R(t,r)T(r)dr, \quad t \in (0,b], \tag{7}$$

where

$$T(t) := u_0 + I^{\beta}_{0+}g(t), \quad R(t,r) := \sum_{j=1}^{\infty} k_j(t,r),$$

I^{β}_{0+} denotes the Riemann-Liouville integral operator of order β,

$$k_1(t,r) := k(t,r) = \frac{1}{\Gamma(\beta)}(t-r)^{\beta-1}\lambda(r),$$

and

$$k_j(t,r) := \int_r^t k(t,s)k_{j-1}(s,r)ds, \ (j = 2,3,\ldots).$$

The probabilistic approach used here provides a different representation of the solution in (7) when λ is a positive function. This representation is given in terms of path functionals and can also be written explicitly in terms of the transition probabilities of the underlying decreasing process.

The last part of this paper addresses the nonlinear equation (2). Some particular cases have been studied in the literature, for instance, the initial value problem for *the composite fractional relaxation equation* [18] (also called *the generalized Basset equation* [19]):

$$c_1 D^{\beta}_{0+*}u(t) + c_2 \frac{d}{dt}u(t) = -u(t) + g(t), \quad t \in (0, b], \quad u(0) = u_0, \tag{8}$$

for $\beta \in (0, 1)$, $c_1 > 0$, $c_2 = 1$ and g a continuous function, was solved in [18] via the Laplace transform method. The explicit solution in terms of the *fundamental solution* $\phi(t)$ and the so-called *impulse-response solution* $-\phi'(t)$ is

$$u(t) = u_0\phi(t) - \int_0^t g(t-r)\phi'(r)dr; \tag{9}$$

where

$$\phi(t) = \int_0^\infty e^{-rt} H^{(1)}_{\beta,0}(r; c_1)dr, \tag{10}$$

and

$$H^{(1)}_{\beta,0}(r; c_1) = \frac{1}{\pi} \frac{c_1 r^{\beta-1}\sin(\beta\pi)}{(1-r)^2 + c_1^2 r^{2\beta} + 2(1-r)c_1 r^{\alpha}\cos(\alpha\pi)}. \tag{11}$$

The results obtained here extend the ones known for the equation (8). Firstly, by considering the nonlinear version, and secondly, by allowing the parameters c_1 and c_2 being more general (functions instead of constants). The generalized equation (2) is also an extension of the linear case studied in [11], where the well-posedness was treated but without the drift term.

Further, as was done in [11] for the linear case, we study the existence of two types of solutions: *solutions in the domain of the generator* and *generalized solutions* (see definitions later). For some specific cases (which encompass the classical fractional operators), we also investigate the existence of *smooth solutions*.

The main contribution of this work lies on displaying the use of stochastic analysis as a valuable approach for the study of fractional differential equations as well as their generalizations. Since this probabilistic method allows us to obtain explicit solutions in terms of mathematical expectations, it also leads to many interesting potential applications, e.g., by providing new numerical approaches to obtain approximating solutions to a variety of equations arising in fractional modelling.

The paper is organized as follows. The next section sets standard notation and gives a quick review about generalized Caputo and RL type operators. Section 3 summarizes some important properties and results obtained in [11] concerning the generalized fractional operators. Then, the well-posedness results for the equation (1) is addressed in Section 4. In Section 5 the equation (5) is analyzed, whilst Section 6 focuses on the *generalized composite fractional relaxation equation* given in (2). Section 7 gives some general comments in connection with the classical fractional framework. Finally, for the sake of clarity, the lengthy proofs of Theorem 2.1 and Proposition 4.1 are presented in Section 8.

2. Preliminaries

2.1. **Notation.** Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the real line, respectively. For any open set A, the standard notation $B(A)$, $C(A)$ and $C_b(A)$ denotes the set of bounded Borel measurable functions, continuous functions and bounded continuous functions defined on A, respectively. Notation $||\cdot||$ stands for the sup-norm $||h|| = \sup_{x\in A}|h(x)|$ for $h \in B(A)$. The space of continuous functions on A with continuous derivative on A (first order partial derivatives if $A \subset \mathbb{R}^d$) shall be denoted by $C^1(A)$. This space is equipped with the norm $||h||_{C^1} := ||h|| + ||h'||$. Similar notation is used for the corresponding spaces of functions defined on the closure $\bar{A}$ of A. In this case, $C^1(\bar{A})$ indicates the continuously differentiable functions up to the boundary. Furthermore, notation $C_a[a,b]$ and $C_a^1[a,b]$ refers to the space of continuous functions vanishing at a and the space $C_a[a,b] \cap C^1[a,b]$, respectively.

Letters $\mathbf{P}$ and $\mathbf{E}$ mean the probability and the mathematical expectation, respectively. Notation $w_\beta(\cdot;\sigma,l)$ stands for the density function of a $\beta-$stable random variable (r.v.) with scaling parameter σ, skewness parameter l and location parameter zero. For a given Feller semigroup $\{S_s\}_{s\geq 0}$ on $C_b(A)$, its *resolvent operator* R_λ for $\lambda > 0$ is defined as the Bochner integral (see, e.g., [7])

$$R_\lambda g := \int_0^\infty e^{-\lambda s} S_s g \, ds, \quad g \in C_b(S). \tag{12}$$

By taking $\lambda = 0$ in (12), one obtains the *potential operator* denoted by $R_0 g$ (whenever it exists). Additional superscripts will be used to differentiate amongst different resolvent and potential operators. When referring to the generator of a Feller process, say L with domain D_L, we will use the short notation (L, D_L).

Notation $\Gamma(z)$ and $B(\alpha,\beta)$ stands for the Gamma and the Beta function, respectively. For all $\alpha, \beta > 0$, the Beta function is defined by

$$B(\alpha,\beta) := \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du. \tag{13}$$

We shall use the following rather standard identities

$$\Gamma(z+1) = z\Gamma(z), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{14}$$

and the inequality

$$\Gamma(na) > (n-1)! a^{2(n-1)} \big(\Gamma(a)\big)^n, \tag{15}$$

for $n \in \mathbb{N}$ and $a > 0$.

Finally, letters t and r are mainly used as space variables, and the letter s is reserved for the time variable.

2.2. **Generalized fractional operators of Caputo and RL type.** This section is a quick summary about the generalized fractional operators introduced in [17], as well as some related definitions and results obtained in [11].

Let $\nu(t,r)$ be a function satisfying the condition:

(H0) The function $\nu(t,r)$ is continuous as a function of two variables and continuously differentiable in the first variable. Furthermore,

$$\sup_t \int (r \wedge 1)\nu(t,r) dr < \infty, \quad \sup_t \int (r\wedge 1)\Big|\frac{\partial}{\partial t}\nu(t,r)\Big| dr < \infty,$$

and
$$\lim_{\delta\to 0}\sup_t \int_{r\leq\delta} r\nu(t,r)dr = 0.$$

Definition 2.1. *Let $a, b \in \mathbb{R}$, $a < b$, and let h be a function on $[a, b]$. For any function ν satisfying condition (H0), the operator $-D^{(\nu)}_{a+*}$ defined by*
$$-D^{(\nu)}_{a+*}h(t) = \int_0^{t-a} (h(t-r) - h(t))\nu(t,r)dr + (h(a) - h(t))\int_{t-a}^{\infty} \nu(t,r)dr, \quad (16)$$
is called the generalized Caputo type operator; and the operator $-D^{(\nu)}_{a+}$ defined by
$$-D^{(\nu)}_{a+}h(t) = \int_0^{t-a} (h(t-r) - h(t))\nu(t,r)dr - h(t)\int_{t-a}^{\infty} \nu(t,r)dr, \quad (17)$$
is called the generalized RL type operator.

Remark 2.1. *The sign $-$ in the notation $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$ is introduced to comply with the standard notation of fractional derivatives. Note also that these operators are well-defined at least for continuously differentiable functions h. Moreover, they can be defined (respectively) for functions on $[a, +\infty)$ and $(-\infty, b]$, but for our purposes it is enough to work on the closed interval $[a, b]$.*

The operators (16) and (17) can be thought of as the generators of *interrupted* Feller processes which are forced to land exactly at $t = a$ on the first attempt to cross this barrier point a. More precisely, if ν satisfies condition (H0) and $(-D^{(\nu)}, \mathfrak{D}^{(\nu)})$ is the generator of a decreasing Feller process on $(-\infty, b]$ given by
$$-D^{(\nu)}h(t) = \int_0^{\infty} (h(t-r) - h(t))\,\nu(t,r)dr, \quad t \leq b, \quad h \in \mathfrak{D}^{(\nu)}, \quad (18)$$
then the corresponding process interrupted at the barrier point $t = a$ (for any $a < b$) is a Feller process on $[a, b]$ with the generator $(-D^{(\nu)}_{a+*}, \mathfrak{D}^{(\nu)}_{a+*})$. This generator has a domain $\mathfrak{D}^{(\nu)}_{a+*} \subset C[a, b]$ and $C^1[a, b] \subset \mathfrak{D}^{(\nu)}_{a+*}$.

Remark 2.2. *Since the process generated by $(-D^{+(\nu)}, \mathfrak{D}^{(\nu)})$ is decreasing, the interruption procedure effectively means stopping the process at the boundary point $t = a$.*

Moreover, if the process is also killed at the barrier point $t = a$ (meaning analytically to set $h(a) = 0$), then the corresponding Feller (sub-Markov) process on $(a, b]$ has the generator $(-D^{(\nu)}_{a+}, \mathfrak{D}^{(\nu)}_{a+})$. This generator has a domain $\mathfrak{D}^{(\nu)}_{a+} \subset C_a[a, b]$ and $C^1_a[a, b] \subset \mathfrak{D}^{(\nu)}_{a+}$.

Thus, the operators $-D^{(\nu)}_{a+*}$ (resp. $-D^{(\nu)}_{a+}$) arise as generators of decreasing Feller processes *stopped* (resp. *killed*) on an attempt to cross a given barrier point.

The previous discussion is formalized in the following theorem (see also Theorem 4.1 in [17]).

Theorem 2.1. *Let ν be a function satisfying assumption (H0). Then,*

(i) *The operator $(-D^{(\nu)}_{a+}, \mathfrak{D}^{(\nu)}_{a+})$ generates a Feller semigroup $\{S_s\}_{s\geq 0}$ on $C_a[a, b]$, where the domain $\mathfrak{D}^{(\nu)}_{a+}$ contains the space $C^1_a[a, b]$.*

(ii) *If additionally $\nu(t,r)$ is twice continuously differentiable in the first variable, and*

$$\sup_t \int (r\wedge 1)\left|\frac{\partial^2}{\partial t^2}\nu(t,r)\right|dr<\infty,\quad \lim_{\delta\to 0}\sup_t\int_{r\le\delta} r\left|\frac{\partial}{\partial t}\nu(t,r)\right|dr=0, \tag{19}$$

then the semigroup $\{S_s\}_{s\ge 0}$ is also strongly continuous on the space

$$\left\{f\in C_a^1[a,b]\,:\,f'(a)=0\right\}.$$

(iii) *The operator $(-D_{a+*}^{(\nu)},\mathfrak{D}_{a+*}^{(\nu)})$ generates a Feller semigroup $\{S_s^*\}_{s\ge 0}$ on $C[a,b]$, where the domain $\mathfrak{D}_{a+*}^{(\nu)}$ contains the space $C^1[a,b]$.*

Proof. See proof in Section 8. □

2.2.1. *Particular cases.* (*i*) *The Caputo and RL fractional derivatives of order $\beta\in(0,1)$.* The classical fractional derivatives are particular cases of the previous interruption procedure applied to $\beta-$stable subordinators. Namely, on regular enough functions h,

$$\text{if}\quad \nu(t,r)=-\frac{1}{\Gamma(-\beta)r^{1+\beta}},\quad \beta\in(0,1),\quad \text{then}\quad \left\{\begin{array}{lcl} -D_{a+*}^{(\nu)}h(t) &=& -D_{a+*}^{\beta}h(t),\\ -D_{a+}^{(\nu)}h(t) &=& -D_{a+}^{\beta}h(t),\end{array}\right. \tag{20}$$

where D_{a+*}^{β} and D_{a+}^{β} stand for the (left-sided) Caputo and the (left-sided) Riemann-Liouville fractional derivatives of order $\beta\in(0,1)$, respectively. Hence,

$$D_{a+*}^{\beta}h(t)=\frac{1}{\Gamma(-\beta)}\int_0^{t-a}\frac{h(t-r)-h(t)}{r^{1+\beta}}dr+\frac{h(t)-h(a)}{\Gamma(1-\beta)(t-a)^{\beta}},\quad t\in(a,b], \tag{21}$$

and

$$D_{a+}^{\beta}h(t)=\frac{1}{\Gamma(-\beta)}\int_0^{t-a}\frac{h(t-r)-h(t)}{r^{1+\beta}}dr+\frac{h(t)}{\Gamma(1-\beta)(t-a)^{\beta}},\quad t\in(a,b]. \tag{22}$$

Let us recall that *the Riemann-Liouville approach* defines the Caputo and the RL fractional operators, respectively, by

$$D_{a+*}^{\beta}h(t):=I_{a+}^{m-\beta}D^m h(t),\quad \beta>0,\ \beta\notin\mathbb{N},\quad t>a,\ m=\lceil\beta\rceil, \tag{23}$$

and

$$D_{a+}^{\beta}h(t):=D^m I_{a+}^{m-\beta}h(t),\quad \beta>0,\ \beta\notin\mathbb{N},\ t>a,\ m=\lceil\beta\rceil, \tag{24}$$

where D^m denotes the classical mth derivative for $m\in\mathbb{N}$. Notation $\lceil\cdot\rceil$ means the ceiling function and I_{a+}^{α} is the *Riemann-Liouville integral operator* defined by

$$I_{a+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,\quad t>a,$$

for any $\alpha>0$ and $a\in\mathbb{R}\cup\{-\infty\}$. For convention, I_{a+}^0 refers to the identity operator.

For $\beta\in(0,1)$ and smooth enough functions h, the expressions in (23) and (24) coincide with (21) and (22), respectively (see, e.g., Appendix in [17]). For a detailed treatment about the Riemann-Liouville approach we refer, e.g., to [4, 25, 26] and references therein.

(*ii*) *Fractional derivatives of variable order.* Let $\beta : \mathbb{R} \to (0,1)$ be a continuously differentiable function with values in a compact subset of $(0,1)$. Then, the function

$$\nu(t,r) = -\frac{1}{\Gamma(-\beta(t))r^{1+\beta(t)}} \tag{25}$$

defines the *Caputo and RL type operators of variable order*, denoted by $-D^{(\nu)}_{a+*} \equiv -D^{\beta(\cdot)}_{a+*}$ and $-D^{(\nu)}_{a+} \equiv -D^{\beta(\cdot)}_{a+}$, respectively. They can be thought of as the generators of inverted stable-like processes (see, e.g., [1],[16]) with the jump density (25) which are stopped (resp. killed) on an attempt to cross the boundary point $t = a$.

(*iii*) *Multi-term fractional operators.* These are operators of the form

$$-D^{(\nu)}_{a+*}h(t) = -\sum_{i=1}^{d} \omega_i(t) D^{\beta_i}_{a+*}h(t), \quad \beta_i \in (0,1) \tag{26}$$

with nonnegative functions $\omega_i(\cdot) \geq 0$, where $\nu(t,r) = -\sum_{i=1}^{d} \omega_i(t) \frac{1}{\Gamma(-\beta_i)r^{1+\beta_i}}$. Even more generally, it includes the *generalized distributed order fractional derivatives:*

$$-D^{(\nu)}_{a+*}h(t) = -\int_{-\infty}^{\infty} \omega(s,t) D^{\beta(s,t)}_{a+*}h(t)\, \mu(ds), \tag{27}$$

with

$$\nu(t,r) = -\int_{-\infty}^{\infty} \omega(s,t) \frac{\mu(ds)}{\Gamma(-\beta(s,t))r^{1+\beta(s,t)}}$$

satisfying condition (H0). In the context of standard fractional derivatives, special cases of (27) have been studied, e.g., in [20, 10].

3. Properties of the underlying stochastic processes

This section summarizes some properties and results related to the stochastic processes generated by the operators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$. For the sake of clarity we will retain some notation from [11], wherein details can be found.

For a given function ν satisfying condition (H0) and for $t \in (a,b]$, the following notation will be used hereafter: $T^{+(\nu)}_t = \{T^{+(\nu)}_t(s)\}_{s\geq 0}$ denotes the *underlying* (decreasing) Feller process (started at t) generated by $(-D^{(\nu)}, \mathfrak{D}^{(\nu)})$ as given in (18); $T^{a+*(\nu)}_t = \{T^{a+*(\nu)}_t(s)\}_{s\geq 0}$ stands for the *interrupted* Feller process generated by $(-D^{(\nu)}_{a+*}, \mathfrak{D}^{(\nu)}_{a+*})$; and $T^{a+(\nu)}_t = \{T^{a+(\nu)}_t(s)\}_{s\geq 0}$ refers to the Feller (sub-Markov) process generated by $(-D^{(\nu)}_{a+}, \mathfrak{D}^{(\nu)}_{a+})$.

For $t \in [a,b]$, notation $\tau^{(\nu)}_a(t)$ denotes the first time the process $T^{+(\nu)}_t$ (or the process $T^{a+*(\nu)}_t$) leaves $(a,b]$, i.e.

$$\tau^{(\nu)}_a(t) = \inf\{s \geq 0 \,:\, T^{+(\nu)}_t(s) \notin (a,b]\,\} = \inf\{s \geq 0 \,:\, T^{a+*(\nu)}_t(s) \notin (a,b]\,\},$$

and, of course, $\tau^{(\nu)}_a(a) = 0$. Note that $\tau^{(\nu)}_a(t)$ is a stopping time with respect to the natural filtration generated by the process $T^{+(\nu)}_t$. Further, $\tau^{(\nu)}_a(t)$ is also the first exit time from $(a,b]$ of the killed process $T^{a+(\nu)}_t$ for $t > a$.

Notation $p^{+(\nu)}_s(r,E)$, $p^{a+(\nu)}_s(r,E)$ and $p^{a+*(\nu)}_s(r,E)$ denote the transition probabilities (from the state r to a Borel set E with s being the time variable) of the processes $T^{+(\nu)}$, $T^{a+(\nu)}$ and $T^{a+*(\nu)}$, respectively.

Sometimes we will use the following additional assumptions concerning the function ν and the transition probabilities of the process $T^{+(\nu)}$:

(H1) There exist $\epsilon > 0$ and $\delta > 0$, such that the function ν satisfies $\nu(t,r) \geq \delta > 0$ for all t and $|r| < \epsilon$.

(H2) The transition probabilities of the process $T_t^{+(\nu)}$ are absolutely continuous with respect to the Lebesgue measure (the transition densities are denoted by $p_s^{+(\nu)}(r,y)$).

(H3) The transition density function $p_s^{+(\nu)}(r,y)$ is continuously differentiable in the variable s with bounded derivative.

The following (rather simple) facts hold ([11], Section 4):

(1) If $E \subset \mathcal{B}(a,b]$ and $r \in (a,b]$, then $p_s^{+(\nu)}(r,E) = p_s^{a+*(\nu)}(r,E) = p_s^{a+(\nu)}(r,E)$. Moreover,
$$p_s^{a+*(\nu)}(r,\{a\}) = p_s^{+(\nu)}(r,(-\infty,a]), \quad r \in (a,b].$$

(2) Under the assumptions (H0)-(H1), the point a is regular in expectation (i.e., $\mathbf{E}[\tau_a^{(\nu)}(t)] \to 0$ as $t \downarrow a$) for both operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$, see [17, Theorem]. Moreover, $\mathbf{E}\left[\tau_a^{(\nu)}(t)\right] < +\infty$ uniformly on $t \in (a,b]$; and if additionally (H2)-(H3) hold, then
$$\mathbf{E}\left[\tau_a^{(\nu)}(t)\right] = \int_0^\infty \mathbf{P}\left[\tau_a^{(\nu)}(t) > s\right] ds = \int_0^\infty \int_a^t p_s^{+(\nu)}(t,r)\, dr\, ds; \tag{28}$$
and the distribution law of $\tau_a^{(\nu)}(t)$ has the density
$$\mu_a^{t,(\nu)}(s) := -\frac{\partial}{\partial s}\int_a^t p_s^{+(\nu)}(t,r)dr, \quad t \in (a,b]. \tag{29}$$

By the standard theory of Feller processes, it is known that the domains of the generators $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ and $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$ coincide with the images of their corresponding resolvent operators, denoted (for any $\lambda > 0$) by $R_\lambda^{a+*(\nu)}$ and $R_\lambda^{a+(\nu)}$, respectively. Namely, $u \in \mathfrak{D}_{a+*}^{(\nu)}$ if, and only if, there exists $g \in C[a,b]$ such that $u(t) = R_\lambda^{a+*(\nu)}g(t)$. Analogously, $w \in \mathfrak{D}_{a+}^{(\nu)}$ if, and only if, there exists $g \in C_a[a,b]$ such that $w(t) = R_\lambda^{a+(\nu)}g(t)$. Moreover, the functions u and w solve the so-called resolvent equations:
$$-D_{a+*}^{(\nu)}u(t) = \lambda u(t) - g(t), \quad g \in C[a,b],$$
and
$$-D_{a+}^{(\nu)}w(t) = \lambda w(t) - g(t), \quad g \in C_a[a,b],$$
respectively.

Hereafter, notation $-\tilde{D}^{(\nu)}$ stands for either the RL type operator $-D_{a+}^{(\nu)}$ or the Caputo type operator $-D_{a+*}^{(\nu)}$. Analogously, notation $\tilde{\mathfrak{D}}^{(\nu)}$ and $\tilde{R}_\lambda^{(\nu)}$ will denote, respectively, the domain of the generator and the resolvent (or potential operator if $\lambda = 0$) associated with the operator $-\tilde{D}^{(\nu)}$. The space wherein the semigroup generated by the operator $-\tilde{D}^{(\nu)}$ is strongly continuous shall be denoted by $\tilde{C}[a,b]$, meaning $C_a[a,b]$ or $C[a,b]$ whether the operator refers to the RL or the Caputo type operator, respectively. Similarly, $\tilde{u}_a \in \mathbb{R}$ will mean $\tilde{u}_a = 0$ for RL type equations, and any real number for Caputo type equations.

Notation $(-\tilde{D}^{(\nu)}, \lambda, g, \tilde{u}_a)$ is used to represent the linear problem

$$-\tilde{D}^{(\nu)}u(t) = \lambda u(t) - g(t), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{30}$$

for any $\lambda \geq 0$.

Let us now define the different notions of solutions we shall be interested in (see Definition 5.1 and Definition 5.3 in [11]).

Definition 3.1. *Let $g \in B[a,b]$ and $\lambda \geq 0$. A function u on $[a,b]$ is said to solve the linear equation $(-\tilde{D}^{(\nu)}, \lambda, g, \tilde{u}_a)$ as*

(i) *a solution in the domain of the generator if u satisfies (30) and u belongs to the domain of the generator $(-\tilde{D}^{(\nu)}, \tilde{\mathfrak{D}}^{(\nu)})$;*

(ii) *a generalized solution if for all sequence of functions $g_n \in C_a[a,b]$ such that $\sup_n ||g_n|| < \infty$ and $\lim_{n\to\infty} g_n \to g$ a.e., it holds that $u(t) = \tilde{u}_a + \lim_{n\to\infty} w_n(t)$ for all $t \in [a,b]$, where w_n is the solution (in the domain of the generator $(-\tilde{D}^{(\nu)}, \tilde{\mathfrak{D}}^{(\nu)})$) to the RL type problem $(-D^{(\nu)}_{a+}, \lambda, g_n - \lambda\tilde{u}_a, 0)$;*

(iii) *a smooth classical solution if u is a generalized solution belonging to $\tilde{C}[a,b] \cap C^1(a,b]$.*

For the existence results we will use the following preliminary result taken from Theorem 5.2 and Theorem 5.4 in [11].

Lemma 3.1. *Let ν be a function satisfying conditions (H0)-(H1). Assume that $g \in B[a,b]$ and $\tilde{u}_a \in \mathbb{R}$. Then,*

(i) *the unique generalized solution u to the linear problem $(-\tilde{D}^{(\nu)}, 0, g, \tilde{u}_a)$ is given by*

$$u(t) = \tilde{u}_a + \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(t)} g\left(T_t^{+(\nu)}(s)\right) ds\right]. \tag{31}$$

Moreover, if ν also satisfies conditions (H2)-(H3), then the solution rewrites

$$u(t) = \tilde{u}_a + \int_0^\infty \int_a^t g(r) p_s^{+(\nu)}(t,r)\, dr\, ds.$$

(ii) *If $g \in C_a[a,b]$ and $\tilde{u}_a = 0$, then the solution in (31) is the unique solution in the domain of the generator $(-\tilde{D}^{(\nu)}, \tilde{\mathfrak{D}}^{(\nu)})$.*

4. Nonlinear equations involving RL and Caputo type operators

This section is concerned with the well-posedness results for the nonlinear equation

$$-\tilde{D}^{(\nu)}u(t) = -f(\,t,\, u(t)\,), \quad t \in (a,b], \qquad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{32}$$

where f is a given bounded function $f : G \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition 4.1. *Let $f \in B(G)$, $G \subset \mathbb{R}^2$ and let ν be a function satisfying (H0). A function u on $[a,b]$ is called a solution (*generalized, smooth classical *or* in the domain of the generator*) to the nonlinear equation (32) if u is a solution (generalized, smooth classical or in the domain of the generator, respectively) to the linear equation*

$$-\tilde{D}^{(\nu)}u(t) = -g(t), \quad t \in (a,b]; \qquad u(a) = \tilde{u}_a, \tag{33}$$

where $g(t) := f(t, u(t))$ for all $t \in [a,b]$.

Lemma 4.1. *Let ν be a function satisfying conditions (H0)-(H3). Suppose that $f : G \subset \mathbb{R}^2 \to \mathbb{R}$ is a function in $B(G)$. Then, a function u on $[a,b]$ is a generalized solution to the problem (32) if, and only if, u solves the nonlinear integral equation*

$$u(t) = \tilde{u}_a + \int_0^\infty \int_a^t f(\,r,\, u(r))p_s^{+(\nu)}(t,r)\, dr\, ds. \tag{34}$$

Proof. By Definition 4.1, u on $[a,b]$ is a generalized solution to (32) if, and only if, u is a generalized solution to (33) with $g(t) := f(t, u(t))$. Note that if $u \in B[a,b]$, then g is a bounded measurable function. Under the assumptions (H2)-(H3), Lemma 3.1 provides the integral equation (34). □

Remark 4.1. *Definition 4.1 and Lemma 4.1 can be extended to the RL type equation*

$$-D_{a+}^{(\nu)} u(t) = \lambda u(t) - f(\,t,\, u(t)\,), \quad t \in (a,b], \qquad u(a) = 0, \tag{35}$$

for any $\lambda > 0$. In this case, the equation (33) should be replaced with the equation in (30), whilst the integral equation

$$u(t) = \int_0^\infty \int_a^t e^{-\lambda s} f(\,r,\, u(r))p_s^{+(\nu)}(t,r)\, dr\, ds, \tag{36}$$

will replace the one in (34) (see Theorem 5.1 in [11]*). Moreover, to study the corresponding Caputo type problem, an additional term will appear in the integral equation (see Theorem 5.3 in* [11]*).*

Let us now see that the integral equation (34) possesses a unique solution under the following assumptions:

(H4) There exists $\beta \in (0,1)$ such that the function ν satisfies that $\nu(t,r) \geq Cr^{-1-\beta}$ for some constant $C > 0$.

(H5) For $K > 0$ and $\tilde{u}_a \in \mathbb{R}$, the function f belongs to $B(G_K)$ where

$$G_K := \left\{(t,x) \in \mathbb{R}^2 \,:\, t \in [a,b] \text{ and } x \in [\tilde{u}_a - K, \tilde{u}_a + K]\right\}.$$

Moreover, f fulfills a Lipschitz condition with respect to the second variable, i.e., for all $(t,x), (t,y) \in G_K$

$$|f(t,x) - f(t,y)| < L_f|x-y|, \tag{37}$$

for a constant $L_f > 0$ (independent of t).

Remark 4.2. *Condition $(H4)$ ensures the regularity in expectation of the boundary point a. This follows from* [17, Theorem 4.1]*.*

Let us also consider a positive constant κ satisfying

$$\int_0^\infty y^{-1/\beta} w_\beta(y^{-1/\beta}; 1, 1) dy < \kappa. \tag{38}$$

Recall that w_β represents a β-stable density (see Preliminaries). The existence of κ can be obtained by splitting the integral (38) into two regions, over the sets $\{y \leq 1\}$ and $\{y \geq 1\}$. Then, the upper bounds for the β-stable densities in each region (see, e.g., Theorem 7.3.1 in [16]) provide the required bound.

Proposition 4.1. *Let $K > 0$, $a, b \in \mathbb{R}$ and $\tilde{u}_a \in \mathbb{R}$. Let ν be a function satisfying conditions (H0) and (H2)-(H4). Assume that $f : G_K \subset \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying condition (H5). Define M_K and b^* by setting*

$$M_K = \sup\{\, |f(t,x)| \,:\, (t,x) \in G_K\}, \quad b^* = \min\left\{ b, \left(\frac{K\beta}{\kappa M_K}\right)^{1/\beta} + a \right\}.$$

Then, the integral equation (34) has a unique solution $u \in \tilde{C}[a, b^*]$.

Proof. See Section 8. □

Observe that the previous result ensures the existence of a solution to the integral equation only in a subinterval $[a, b^*] \subset [a, b]$. A solution in the whole interval can be guaranteed with an additional assumption, as shown below.

Corollary 4.1. *Let $a, b \in \mathbb{R}$ and $\tilde{u}_a \in \mathbb{R}$. Let ν be a function satisfying conditions (H0) and (H2)-(H4). Assume that f belongs to $B([a, b] \times \mathbb{R})$ and it satisfies the Lipschitz condition (37). Then, the integral equation (34) has a unique solution $u \in \tilde{C}[a, b]$.*

Proof. It follows directly from Proposition 4.1 by taking the constant K such that $(b-a)^{\beta}\kappa M < K$ with $M := ||f||$. □

Theorem 4.1. *Suppose that the assumptions in Corollary 4.1 hold. Then,*

(i) *There exists a unique generalized solution $u \in \tilde{C}[a, b]$ to the nonlinear problem in (32).*
(ii) *If additionally the function f is continuous and satisfies that $f(a, \tilde{u}_a) = 0$ with $\tilde{u}_a = 0$, then there exists a unique solution in the domain of the generator.*

Proof. (i) According to Lemma 4.1, the existence of a generalized solution to (32) is equivalent to the existence of a solution to the integral equation (34). The latter follows by Corollary 4.1.

(ii) Setting $g(t) := f(t, u(t))$, the assertion (ii) in Lemma 3.1 implies that u belongs to *the domain of the generator* whenever $g(a) = 0$ and $\tilde{u}_a = 0$, i.e., when $f(a, 0) = 0$ and $\tilde{u}_a = 0$. □

Theorem 4.2. *Suppose that the assumptions in Corollary 4.1 hold. Consider the equation*

$$-\tilde{D}^{(\nu)}u(t) = \lambda u(t) - f(t, u(t)), \quad t \in (a, b], \quad u(a) = \tilde{u}_a, \tag{39}$$

for any $\lambda > 0$ and $\tilde{u}_a \in \mathbb{R}$. Then,

(i) *There exists a unique generalized solution $u \in \tilde{C}[a, b]$ to the nonlinear equation (39).*
(ii) *If additionally the function f is continuous satisfying $f(a, \tilde{u}_a) = \lambda \tilde{u}_a$, then there exists a unique solution in the domain of the generator.*

Proof. By Remark 4.1, the proof of these assertions is quite similar to the case $\lambda = 0$, so that the details are omitted. □

Remark 4.3. *Since the function $f(t, u) = \lambda(t)u + g(t)$ (with bounded functions λ and g) is not bounded in $[a, b] \times \mathbb{R}$, Theorem 4.1 can only guarantee the well-posedness for the linear equation with non constant coefficients on the interval $[a, b^*]$ for some $b^* \leq b$. In the next section we shall analyze this case in a different way via purely probabilistic arguments.*

Remark 4.4. *By imposing additional assumptions on the function ν, it is possible to extend Theorem 4.1 and Theorem 4.2 to the case of possibly unbounded functions $f(t,u)$. This case will be studied in a separate paper.*

4.1. **Smoothness of solutions.** To finish this section, let us now consider the existence of smooth solutions for some specific cases. We will start with the linear equation whose smoothness was not studied in [11].

Theorem 4.3. ***(Linear case)*** *Let $\nu(t,r)$ be a function satisfying the assumptions (H0)-(H3). Let $\lambda > 0$ and $g \in C^1[a,b]$. Suppose that ν is twice continuously differentiable in the first variable and satisfies (19).*

(i) If $g(a) = 0 = g'(a)$, then there exists a unique solution u in the domain of the generator to the RL type problem $(-D^{(\nu)}_{a+}, \lambda, g, 0)$ such that $u \in C^1_a[a,b]$.
(ii) If $g(a) = \lambda u_a$ and $g'(a) = 0$, then there exists a unique generalized solution in $C^1[a,b]$ to the Caputo type problem $(-D^{(\nu)}_{a+*}, \lambda, g, u_a)$.

Proof. (i) This follows from Theorem 2.1 which guarantees that under assumption (H0)-(H1) and (19), the semigroup of the process $T^{a+(\nu)}_t$ is strongly continuous on the space $\{f \in C^1_a[a,b] : f'(a) = 0\}$. Consequently, the resolvent operator $R^{a+(\nu)}_\lambda$ associated with the operator $-D^{(\nu)}_{a+}$ maps the latter space into itself. Therefore, for any $g \in \{f \in C^1_a[a,b] : f'(a) = 0\}$ the function $u(t) = R^{a+(\nu)}_\lambda g(t)$ solves $(-D^{(\nu)}_{a+}, \lambda, g, 0)$ and belongs to $C^1_a[a,b]$, as required.
(ii) By definition, the solution to the Caputo type problem is given by $u(t) = u_a + w(t)$ (see Definition 5.5), where $w(t)$ is the solution to the RL type problem $(-D^{(\nu)}_{a+}, \lambda, g - \lambda u_a, 0)$. Hence, $u \in C^1[a,b]$ whenever $w \in C^1[a,b]$, but this follows from assertion (i) because $g(a) - \lambda u_a = 0$ and $g'(a) = 0$. □

To avoid technicalities in the nonlinear case, we only study the existence of smooth solution for the Lévy case, i.e., for functions $\nu(t,r)$ independent of the variable t.

Theorem 4.4. ***(Nonlinear Lévy case)*** *Let $a, b \in \mathbb{R}$ and $\tilde{u}_a \in \mathbb{R}$. Suppose that $\nu(t,r)$ is a function independent of the variable t satisfying assumptions (H0) and $(H2)-(H4)$. Assume $f \in C^1_b([a,b] \times \mathbb{R})$.*

- (i) *If $f(a, \tilde{u}_a) = 0$ and $\tilde{u}_a = 0$, then there exists a unique solution (in the domain of the generator) $u \in C^1_a[a,b]$ to the nonlinear RL type equation in (32).*
- (ii) *If $f(a, \tilde{u}_a) = 0$, then there exists a unique generalized solution $u \in C^1[a,b]$ to the Caputo type equation (32).*

Proof. The existence of a unique continuous solution u (in both the RL and Caputo case) is ensured by Theorem 4.1). It remains to prove that its derivative exists and is continuous.

(i) Since the function ν is independent of t, then the transition density function of the underlying (inverted) Lévy subordinator $T^{+(\nu)}_t$ satisfies $p^{+(\nu)}_s(t,r) = \psi(s, t-r)$

for some function ψ depending on the variable s and the difference $t-r$. Consequently, $u'(t)$ (if exists) should satisfy

$$u'(t) = \int_0^\infty \int_0^{t-a} \left(\frac{\partial}{\partial t} f(\,t-r,\, u(t-r)) + \frac{\partial}{\partial u} f(t-r,\, u(t-r))u'(t-r) \right) p_s^{+(\nu)}(t, t-r)\, dr\, ds + \\ + f(a, u(a)) \int_0^\infty p_s^{+(\nu)}(t, a)\, ds.$$

Assumption $f(a, u(a)) = 0$ leads us to define the operator

$$\tilde{\Psi} u'(t) := \int_0^\infty \int_0^{t-a} \left(\frac{\partial}{\partial t} f(\,t-r,\, u(t-r)) + \frac{\partial}{\partial u} f(t-r,\, u(t-r))u'(t-r) \right) p_s^{+(\nu)}(t, t-r)\, dr\, ds. \tag{40}$$

Since

$$|\tilde{\Psi} u'(t) - \tilde{\Psi} v'(t)| \le \tilde{L}_f \int_0^\infty \int_0^{t-a} |u'(t-r) - v'(t-r)| p_s^{+(\nu)}(t, t-r)\, dr\, ds,$$

where $\tilde{L}_f := ||f||_{C^1}$, the same arguments used in the proofs of Proposition 4.1 and Corollary 4.1 imply the existence of a unique fixed point in $C[a, b]$ for the operator $\tilde{\Psi}$. Thus, u' exists and belongs to $C[a, b]$, as required.

(ii) Since the Caputo type equation can be written in terms of the RL type operator, its solution equals $u(t) = \tilde{u}_a + w(t)$, where $w(t)$ is the unique solution (in the domain of the generator of the RL operator) solving

$$w(t) = \int_a^t \int_0^\infty f(t, w(t) + \tilde{u}_a) p_s^{+(\nu)}(t, r)\, ds\, dr.$$

Define $\tilde{f}(t, w) := f(t, w(t) + \tilde{u}_a)$, then assertion (i) and assumption $f(a, \tilde{u}_a) = 0$ imply the existence of a unique solution $w \in C_a^1[a, b]$, which in turn yields the smoothness for the generalized solution u. □

Remark 4.5. *Notice that if the function f in the previous result is continuously differentiable in a smaller region $[a, b] \times [u_a - K, u_a + K]$ for some constant $K > 0$ instead of $[a, b] \times \mathbb{R}$, then the procedure above can only guarantee the existence of a solution in $C_a^1[a, b^*]$ for some subinterval $[a, b^*] \subset [a, b]$.*

5. Linear equations with non constant coefficients

This section provides probabilistic solutions to linear equations with non constant coefficients involving generalized fractional derivatives. These solutions are given in terms of (stationary) Feynman-Kac type formulas.

5.1. **Auxiliary results.** Let us start with some preliminary results. Let λ be a nonnegative function in $C_a[a, b]$. Define

$$p_{s,\lambda}^{a+(\nu)}(t, E) := \mathbf{E} \left[\mathbf{1}_E \left(T_t^{a+(\nu)}(s) \right) \exp\left\{ - \int_0^s \lambda \left(T_t^{a+(\nu)}(\gamma) \right) d\gamma \right\} \right],$$

and $S_{s,\lambda}^{a+(\nu)} g(t) := \int g(y) p_{s,\lambda}^{a+(\nu)}(t, dy)$ for any $g \in B[a, b]$ such that $g(a) = 0$. Then

$$S_{s,\lambda}^{a+(\nu)} g(t) = \mathbf{E} \left[g \left(T_t^{a+(\nu)}(s) \right) \exp\left\{ - \int_0^s \lambda \left(T_t^{a+(\nu)}(\gamma) \right) d\gamma \right\} \right].$$

Lemma 5 in [8] states that for $\lambda \in C_a[a,b]$, $g \in C_a[a,b]$ and $\delta > 0$, the Laplace transform at $\delta > 0$ of $S^{a+(\nu)}_{s,\lambda} g(t)$ (as a function of s), denoted by $R^{a+(\nu)}_{\delta,\lambda} g(t)$, solves the equation

$$R^{a+(\nu)}_{\delta,\lambda} g(t) = R^{a+(\nu)}_{\delta} g(t) - R^{a+(\nu)}_{\delta}\left[\lambda(\cdot) R^{a+(\nu)}_{\delta,\lambda} g(\cdot)\right](t), \quad t \in [a,b],$$

where $R^{a+(\nu)}_{\delta}$ is the resolvent operator (for $\delta > 0$) for the process $T^{a+(\nu)}_t$.

Equivalently (see Theorem 4.3.1 in [16]), the function $w(t) = R^{a+(\nu)}_{\delta,\lambda} g(t)$ is the unique solution in the domain of the generator $(-D^{(\nu)}_{a+}, \mathfrak{D}^{(\nu)}_{a+})$ solving

$$-D^{(\nu)}_{a+} w(t) = (\lambda(t) + \delta) w(t) - g(t), \quad t \in [a,b]. \tag{41}$$

Remark 5.1. *The function $p^{a+(\nu)}_{s,\lambda}(t,E)$ defines a transition probability function (from t to E with s as the time variable) for a Feller (sub-Markov) process with semigroup $S^{a+(\nu)}_{s,\lambda}$ and generator $-D^{(\nu)}_{a+} - \lambda(\cdot)$ (see* [8]*, Chapter II, Section 5). Moreover, the resolvent of this process (for $\delta > 0$) coincides with $R^{a+(\nu)}_{\delta,\lambda} g$.*

Let us now define

$$M^{a+(\nu)}_{\delta,\lambda} g(t) := \mathbf{E}\left[\int_0^{\tau^{(\nu)}_a(t)} \exp\left\{-\delta\, s - \int_0^s \lambda\left(T^{+(\nu)}_t(\gamma)\right) d\gamma\right\} g\left(T^{+(\nu)}_t(s)\right) ds\right],$$

for any $g \in B[a,b]$, $t \in (a,b]$ and $\lambda \in C[a,b]$, with λ being a nonnegative function.

Notice that $M^{a+(\nu)}_{\delta,\lambda} g$ coincides with the solution (in the domain of the generator) to (41) only when $g \in C_a[a,b]$. This function will appear in the *generalized solution* to the nonlinear equation with non constant coefficients for any $g \in B[a,b]$. In order to write it down explicitly, we will need the following auxiliary results.

Set $Y(0) := 0$ and $Y(\xi) := \int_0^\xi \lambda\left(T^{+(\nu)}_t(\gamma)\right) d\gamma$ for any $\xi > 0$, where $\lambda \in C[a,b]$ is a nonnegative function and $T^{+(\nu)}_t$ is the Feller process generated by the operator $(-D^{(\nu)}, \mathfrak{D}^{(\nu)})$ (see (18)). Define the pair process

$$(Y,Z) = \{(\, Y(\xi), Z(\xi)\,) \,:\, \xi \geq 0\},$$

where

$$\left\{\begin{array}{lcl} Y(\xi) & = & \int_0^\xi \lambda\,(\, Z(\gamma)\,)\, d\gamma \\ Z(\xi) & = & T^{+(\nu)}_t(\xi). \end{array}\right. \tag{42}$$

Then (42) is the solution to the *Langevin type equation*:

$$dY = \lambda(Z) d\xi, \quad dZ = dT^{+(\nu)}_t(\xi),$$

with initial condition $(Y(0), Z(t)) = (0,t)$ (see, e.g., [1, 16]). The process (Y,Z) is a Markov process on $\mathbb{R}_+ \times (-\infty, b]$ with initial state $(0,t)$.

For any $(y_1,t_1),(y_2,t_2) \in \mathbb{R}_+ \times (-\infty,b]$, denote by $p_\xi(y_1,t_1;y_2,t_2)$ the transition density function from (y_1,t_1) to (y_2,t_2) with ξ being the time variable.

Remark 5.2. *If ν is the Lévy kernel in (20), then the process in (42) is the solution to a stable noise driven Langevin equation, see, e.g.,* [1, 12, 16].

Lemma 5.1. *Let ν be a function satisfying conditions (H0)-(H3) and let $\lambda \in C[a,b]$ be a nonnegative function. Assume that the process (Y,Z) has transition densities $p_s(y_1,t_1;y_2,t_2)$ which are continuously differentiable in the variable s (with bounded derivative). Then, for fixed $\xi \geq 0$ and for all $y \geq 0$, the distribution law of the random vector $\big(Y(\xi), \tau_a^{(\nu)}(t)\big)$ has the density $\phi_{\xi,a}^{t,\lambda}(y,\xi)$ at point (y,ξ) given by*

$$\phi_{\xi,a}^{t,\lambda}(y,\xi) = -\frac{\partial}{\partial \xi} \int_a^t p_\xi\big(0,t;y,r\big)dr.$$

Proof. Since the r.v.'s $Y(\xi)$ and $\tau_a^{(\nu)}(t)$ are not independent, to compute the distribution of the pair $(Y(\xi), \tau_a^{(\nu)}(t))$ we use the next equivalence

$$\big\{\, Y(\xi) > y,\, \tau_a^{(\nu)}(t) > \xi \,\big\} \equiv \big\{\, Y(\xi) > y,\, T_t^{+(\nu)}(\xi) > a \,\big\},$$

to obtain

$$\phi_{\xi,a}^{t,\lambda}(y,\xi) = \frac{\partial^2}{\partial y\, \partial \xi} \int_y^\infty \int_a^t p_\xi\big(0,t;w,r\big)dr\,dw = -\frac{\partial}{\partial \xi}\int_a^t p_\xi\big(0,t;y,r\big)dr,$$

as required. $\square$

Lemma 5.2. *Under the assumptions of Lemma 5.1, the distribution law of the random vector $\big(Y(s),\, T_t^{+(\nu)}(s), \tau_a^{(\nu)}(t)\big)$ has the density $\psi_{s,a}^{t,\lambda}(y,r,\xi)$ at point (y,r,ξ) given by*

$$\psi_{s,a}^{t,\lambda}(y,r,\xi) = -\,p_s(0,t;y,r)\,\frac{\partial}{\partial \xi}\int_a^r p_{\xi-s}^{+(\nu)}(r,z)\,dz,$$

for all $(y,r,\xi) \in \mathbb{R}_+ \times (a,t] \times [s,\infty)$.

Proof. The next equivalence between events

$$\big\{\, Y(s) > y, T_t^{+(\nu)}(s) > r, \tau_a^{(\nu)}(t) > \xi \,\big\} \equiv \big\{\, Y(s) > y, T_t^{+(\nu)}(s) > r, T_t^{+(\nu)}(\xi) > a \,\big\},$$

implies that, if $s < \xi$, then

$$\mathbf{P}\left[\, Y(s) > y, T_t^{+(\nu)}(s) > r, T_t^{+(\nu)}(\xi) > a \,\right] = \int_y^\infty \int_r^t p_s(\,0,t;\gamma,w)\,\left(\int_a^w p_{\xi-s}^{+(\nu)}(w,z)dz\right)\,dw\,d\gamma,$$

where $p_s(\,\cdot,\cdot;\cdot,\cdot)$ and $p_s^{+(\nu)}(\cdot,\cdot)$ denote the transition density function of the pair process $(Y, T_t^{+(\nu)})$ and of the process $T_t^{+(\nu)}$, respectively. The result follows by differentiating the last expression with respect to the variables y, r and ξ. $\square$

Lemma 5.3. *Let $\lambda \in C[a,b]$ be a nonnegative function. Let $\delta > 0$ and $g \in B[a,b]$. Suppose that conditions of Lemma 5.1 hold. Then*

$$\mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{(\nu)}(t)} \lambda\left(T_t^{+(\nu)}(s)\right)ds\right\}\right] = \int_0^\infty \int_0^\infty \exp\{-y\}\phi_{\xi,a}^{t,\lambda}(y,\xi)\,dy\,d\xi; \quad (43)$$

and

$$M_{\delta,\lambda}^{a+(\nu)}g(t) = \int_0^{t-a} g(t-r)\int_0^\infty \int_0^\infty \exp\{-\delta s - y\}\;\, p_s(0,t;y,t-r)\,dy\,ds\,dr, \quad (44)$$

where $\phi_{\xi,a}^{t,\lambda}(y,\xi)$ stands for the density function at point (y,ξ) of the random vector $\big(Y(\xi), \tau_a^{(\nu)}(t)\big)$.

Proof. Equality (43) follows by conditioning on the r.v. $\tau_a^{(\nu)}(t)$ and then by using the joint density $\phi_{\xi,a}^{t,\lambda}(\cdot)$ of the random vector $(Y(\xi), \tau_a^{(\nu)}(t))$ as given in Lemma 5.1. To prove (44), Fubini's theorem and the definition of Y yield

$$M_{\delta,\lambda}^{a+(\nu)} = \int_0^\infty \mathbf{E}\left[1_{\left\{\tau_a^{(\nu)}(t)>s\right\}} \exp\{-\delta\, s - Y(s)\}\, g\left(T_t^{+(\nu)}(s)\right)\right] ds.$$

Then, Lemma 5.2 provides the density $\psi_{s,a}^{t,\lambda}(y,r,\xi)$ (at the point (y,r,ξ)) of the random vector $(Y(s), T_t^{+(\nu)}(s), \tau_a^{(\nu)}(t))$, yielding

$$\begin{aligned}
M_{\delta,\lambda}^{a+(\nu)} &= \int_0^\infty \int_0^\infty \int_a^t \int_s^\infty \exp\{-\delta s - y\}\, g(r) \psi_{s,a}^{t,\lambda}(y,r,\xi)\, d\xi\, dr\, dy\, ds,\\
&= \int_a^t g(r) \int_0^\infty \int_0^\infty \exp\{-\delta s - y\}\;\; p_s(0,t;y,r) \int_s^\infty \left(-\frac{\partial}{\partial \xi}\int_a^r p_{\xi-s}^{+(\nu)}(r,z)\, dz\right) d\xi\;\, dy\, ds\, dr,\\
&= \int_a^t g(r) \int_0^\infty \int_0^\infty \exp\{-\delta s - y\}\;\; p_s(0,t;y,r) \left(\int_0^\infty \mu_a^{r,(\nu)}(\tilde{\xi})\, d\tilde{\xi}\right) dy\, ds\, dr,\\
&= \int_a^t g(r) \int_0^\infty \int_0^\infty \exp\{-\delta s - y\}\;\; p_s(0,t;y,r)\, dy\, ds\, dr,
\end{aligned}$$

where we have also used that $\mu_a^{r,(\nu)}(\cdot)$ is the density of the r.v. $\tau_a^{(\nu)}(r)$ defined in (29). □

5.2. **Explicit solutions: Feynman-Kac type formulas.** Consider the problem of finding a function $w \in C_a[a,b]$ satisfying

$$-D_{a+}^{(\nu)} w(t) = \lambda(t) w(t) - g(t), \quad t \in (a,b], \qquad w(a) = w_a, \tag{45}$$

for a given nonnegative function $\lambda \in C[a,b]$, $g \in B[a,b]$ and $w_a = 0$. Hereafter, we shall refer to (45) as the problem $(-D_{a+}^{(\nu)}, \lambda(\cdot), g, w_a)$. Similar notation will be used for the corresponding problem with the Caputo type operator.

Case 1: RL type operator

Theorem 5.1. *Let ν be a function satisfying conditions (H0)-(H1). Suppose that λ is a nonnegative function in $C[a,b]$ such that $\inf_{t\in[a,b]} \lambda(t) = \delta > 0$.*

(i) *If $g \in C_a[a,b]$, then the unique solution (in the domain of the generator) to the problem $(-D_{a+}^{(\nu)}, \lambda(\cdot), g, 0)$ is given by formula (46) below.*

(ii) *For any $g \in B[a,b]$, the linear problem $(-D_{a+}^{(\nu)}, \lambda(\cdot), g, 0)$ has a unique generalized solution. This solution is given by the Feynman-Kac type formula*

$$w(t) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(t)} \exp\left\{-\int_0^s \lambda\left(T_t^{+(\nu)}(\gamma)\right) d\gamma\right\} g\left(T_t^{+(\nu)}(s)\right) ds\right]. \tag{46}$$

Moreover, if ν also satisfies conditions (H2)-(H3) and assumptions in Lemma 5.1 hold, then (46) rewrites

$$w(t) = \int_0^{t-a} g(t-r) \int_0^\infty \int_0^\infty \exp\{-y\}\;\; p_s(0,t;y,t-r)\, dy\, ds\, dr, \quad t \in (a,b], \tag{47}$$

where $p_s(\cdot,\cdot;\cdot,\cdot)$ denotes the transition densities of the pair process (Y,Z) defined in (42).

Proof. *(i)* Let $\delta > 0$ be as in the statement. Rewrite (45) as

$$-D_{a+}^{(\nu)} w(t) = \hat{\lambda}(t) w(t) + \delta w(t) - g(t), \quad t \in (a, b], \qquad w(a) = 0, \tag{48}$$

where $\hat{\lambda}(t) := \lambda(t) - \delta$.

If $g \in C_a[a, b]$, then Theorem 4.3.1 in [16] states the existence of a solution $w \in \mathfrak{D}_{a+}^{(\nu)}$ (in the domain of the generator) to equation (48). Such a solution is given by the stationary Feynman-Kac (FK) formula

$$w(t) = \mathbf{E}\left[\int_0^\infty \exp\left\{-\delta s - \int_0^s \tilde{\lambda}\left(T_t^{a+(\nu)}(\gamma)\right) d\gamma\right\} g\left(T_t^{a+(\nu)}(s) ds\right)\right],$$

where $T_t^{a+\nu}$ is the process generated by $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$. Note that this solution coincides with (46) due to the fact that $g(a) = 0$ and $\mathbf{E}\left[\tau_a^{(\nu)}(t)\right] < \infty$. Moreover, the positive maximum principle (see, e.g., [16]) implies the uniqueness of the solution.

(ii) For the general case $g \in B[a, b]$, the stationary FK formula no longer provides a solution. However, by definition, the generalized solution can be obtained as a limit of solutions in the domain of the generator. More precisely, take a sequence of functions $\{g_n\}_{n \geq 1}$ satisfying $g_n \to g$ a.e., $g_n \in C_a[a, b]$ and $\sup_n ||g_n|| < +\infty$, then the generalized solution is given by $w = \lim_{n\to\infty} w_n$, where for $n \geq 1$, w_n is the unique solution (in the domain of the generator) to the problem

$$-D_{a+}^{(\nu)} w_n(t) = \lambda(t) w_n(t) - g_n(t), \quad t \in (a, b], \qquad w_n(a) = 0.$$

For $n > 0$, the previous case provides the solution $w_n(t) = M_{\delta,\lambda}^{a+(\nu)} g_n(t)$. Hence, assumption (H1) and the dominated convergence theorem imply that the generalized solution is $w(t) = M_{\delta,\lambda}^{a+(\nu)} g(t)$, as required. Representation (47) follows directly from Lemma 5.3.

□

Case 2: Caputo type operator

Theorem 5.2. *Suppose the assumptions of Theorem 5.1 hold.*

(i) *If $g \in C[a, b]$ and $g(a) = u_a \lambda(a)$, then there exists a unique solution in the domain of the generator.*

(ii) *For any $g \in B[a, b]$ and $u_a \in \mathbb{R}$, the linear problem $(-D_{a+*}^{(\nu)}, \lambda(\cdot), g, u_a)$ has a unique generalized solution given by the Feynman-Kac type formula*

$$\begin{aligned} u(t) &= u_a\, \mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{(\nu)}(t)} \lambda(T_t^{+(\nu)}(\gamma)) d\gamma\right\}\right] + \\ &+ \mathbf{E}\left[\int_0^{\tau^{(\nu)}(t)} g(T_t^{+(\nu)}(s)) \exp\left\{-\int_0^s \lambda(T_t^{+(\nu)}(\gamma)) d\gamma\right\} ds\right]. \end{aligned} \tag{49}$$

Moreover, if ν also satisfies conditions (H2)-(H3) and assumptions in Lemma 5.1 hold, then the solution u rewrites

$$\begin{aligned} u(t) &= u_a \int_0^\infty \int_0^\infty \exp\{-y\} \phi_{\xi,a}^{t,\lambda}(y, \xi)\, dy\, d\xi \\ &+ \int_0^{t-a} g(t - r) \int_0^\infty \int_0^\infty \exp\{-y\}\ p_s(0, t; y, t - r)\, dy\, ds\, dr, \end{aligned} \tag{50}$$

where $p_s(\cdot,\cdot;\cdot,\cdot)$ denotes the transition densities of the pair process (Y,Z) defined in (42), and $\phi^{t,\lambda}_{\xi,a}(\cdot,\cdot)$ is the density function of the random vector $(Y(\xi),\tau_a^{(\nu)}(t))$.

Proof. (i) Define $v(t) := u(t) - u_a$ for $t \in [a,b]$. Using that the Caputo type derivative of a constant function is zero yields

$$-D^{(\nu)}_{a+*}v(t) = \lambda(t)u(t) - g(t) = \lambda(t)v(t) - [g(t) - \lambda(t)u_a] =: \lambda(t)v(t) - \tilde{g}(t). \quad (51)$$

Further, note that $-D^{(\nu)}_{a+*} = -D^{(\nu)}_{a+}$ whenever $v(a) = 0$. Consequently, Theorem 5.1 gives

$$v(t) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(t)} \left(g(T_t^{+(\nu)}(s)) - \lambda(T_t^{+(\nu)}(s))u_a\right) \exp\left\{-\int_0^s \lambda(T_t^{+(\nu)}(\gamma))d\gamma\right\} ds\right] \quad (52)$$

as the unique generalized solution to (51) for any $g \in B[a,b]$. Since (by Leibniz's formula)

$$\int_0^{\tau_a^{(\nu)}(t)} \lambda(T_t^{+(\nu)}(s)) \exp\left\{-\int_0^s \lambda(T_t^{+(\nu)}(\gamma))d\gamma\right\} ds = 1 - \exp\left\{-\int_0^{\tau_a^{(\nu)}(t)} \lambda(T_t^{+(\nu)}(\gamma))d\gamma\right\},$$

the equation (52) becomes

$$v(t) = -\,u_a + u_a\mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{(\nu)}(t)} \lambda(T_t^{+(\nu)}(s))\right\} ds\right] +$$
$$+\,\mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(t)} g(T_t^{+(\nu)}(s)) \exp\left\{-\int_0^s \lambda(T_t^{+(\nu)}(\gamma))d\gamma\right\} ds\right].$$

Equality $u(t) = v(t) + u_a$ then implies the result in (49). Finally, Lemma 5.3 implies directly (50).

(i) Follows from the previous case and the first assertion in Theorem 5.1. □

Remark 5.3. *A stochastic representation similar to (49) is a standard tool for studying parabolic PDE's (see* [13]*, Proposition 7.2).*

Remark 5.4. *The explicit representations (47) and (50) can be obtained in terms of the transition probabilities instead of the transition densities, whose existence was assumed for simplicity.*

6. Composite fractional relaxation equation of Caputo and RL type

Let us now consider the equation

$$-\tilde{D}^{(\nu)}u(t) - \gamma(t)\frac{d}{dt}u(t) - \lambda u(t) = -g(t), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \quad (53)$$

with $\lambda \geq 0$ and some given functions g and γ. This equation is the generalized version of *the composite fractional relaxation equation* introduced in [18],[19].

To prove its well-posedness we will use the next result which is an immediate consequence of Theorem 4.1 in [17].

Lemma 6.1. *Let ν be a function satisfying assumption (H0) and suppose $\gamma \in C^1[a,b]$. Then, (i) If either $\gamma(a) = 0 = \gamma(b)$ or γ is a nonnegative function, then the operator $-\tilde{D}^{(\nu,\gamma)} := -\tilde{D}^{(\nu)} - \gamma(\cdot)\frac{d}{dt}$ generates a Feller process $\tilde{T}_t^{(\nu,\gamma)}$ on $\tilde{C}[a,b]$ with a domain containing the space $\tilde{C}[a,b] \cap C^1[a,b]$; (ii) If γ is a nonnegative function and $\gamma(a) > 0$, then the boundary point $t = a$ is regular in expectation.*

Using the same notation introduced in Section 3, the operator $-\tilde{D}^{(\nu,\gamma)}$ should be understood as either the generator $-D_{a+}^{(\nu,\gamma)} := -D_{a+}^{(\nu)} - \gamma(\cdot)\frac{d}{dt}$, or the generator $-D_{a+*}^{(\nu,\gamma)} := -D_{a+*}^{(\nu)} - \gamma(\cdot)\frac{d}{dt}$ depending on $-\tilde{D}^{(\nu)}$. We will denote by $T_t^{a,(\nu,\gamma)}$ and $T_t^{a*(\nu,\gamma)}$ the corresponding Feller processes.

The probabilistic interpretation of the operator $-\tilde{D}^{(\nu,\gamma)}$ as the generator of an *interrupted* Feller process still holds. If $T_t^{(\nu,\gamma)}$ is the Feller process (started at t) generated by $-D^{+(\nu)} - \gamma(\cdot)\frac{d}{dt}$ (the sum of the decreasing process in (18) and a drift term), then $T_t^{a,(\nu,\gamma)}$ (resp. $T_t^{a*(\nu,\gamma)}$) can be obtained by interrupting (resp. killing) the process $T_t^{(\nu,\gamma)}$ on an attempt to cross the boundary point a.

Remark 6.1. *The three notions of solutions introduced in Section 2 (generalized, smooth, and in the domain of the generator) are extended to the linear problem (53) and the corresponding nonlinear problem with $g(t) := f(t, u(t))$. This is done by replacing $-\tilde{D}^{(\nu)}$ with the operator $-\tilde{D}^{(\nu,\gamma)}$ in Definition 3.1 and Definition 4.1, respectively.*

Well-posedness results (nonnegative γ)

The following result is the extension to Lemma 3.1 for the new operator $-\tilde{D}^{(\nu,\gamma)}$ considering $\lambda \geq 0$.

Theorem 6.1. *(**Linear case**) Let ν be a function satisfying assumption (H0). Suppose that γ is a nonnegative function in $C^1[a,b]$ and $\gamma(a) > 0$.*

(i) *If $g \in C[a,b]$ and $g(a) = \lambda\tilde{u}_a$, then there exists a unique solution $u \in \tilde{C}[a,b]$ in the domain of the generator to (53) given by $u(t) = R_\lambda^{(\nu,\gamma)} g(t)$, the resolvent operator of the semigroup generated by $-\tilde{D}^{(\nu,\gamma)}$.*

(ii) *For any $g \in B[a,b]$, the equation (53) has a unique generalized solution u which admits the stochastic representation*

$$u(t) = u_a \mathbf{E}\left[e^{-\lambda\tau_a^{(\nu,\gamma)}(t)}\right] + \mathbf{E}\left[\int_0^{\tau_a^{(\nu,\gamma)}(t)} e^{-\lambda s} g\left(T_t^{(\nu,\gamma)}(s)\right) ds\right], \tag{54}$$

where $\tau_a^{(\nu,\gamma)}(t)$ denotes the first time the process $T_t^{(\nu,\gamma)}$ leaves the interval $(a,b]$. Moreover, if additionally ν satisfies conditions (H2)-(H3), then the solution takes the form

$$u(t) = \tilde{u}_a \int_0^\infty e^{-\lambda s}\mu_a^{t,(\nu,\gamma)}(s) ds + \int_a^t g(r) \int_0^\infty e^{-\lambda s} p_s^{(\nu,\gamma)}(t,r)\, ds\, dr, \tag{55}$$

where $\mu_a^{t,(\nu,\gamma)}(s)$ and $p_s^{(\nu,\gamma)}(t,r)$ are the density function of the r.v. $\tau_a^{(\nu,\gamma)}(t)$ and the transition densities of the process $T_t^{(\nu,\gamma)}$, respectively.

Proof. (i) Since γ is a nonnegative function, the process generated by $-\tilde{D}^{(\nu,\gamma)}$ is a decreasing process, Theorem 1.1 in [6] and Lemma 6.1 imply the result. (ii) Holds by using the definition of a generalized solution (see Remark 6.1) and the case (i) above. Details have been omitted as they are quite similar to those used in [11] for the operator $-\tilde{D}^{(\nu)}$. □

The following theorem is the analogue to Theorem 4.1 and Theorem 4.2 for the nonlinear case with the operator $-\tilde{D}^{(\nu,\gamma)}$.

Theorem 6.2. ***(Nonlinear case)*** *Let ν be a function satisfying conditions (H0) and (H2)-(H4). Suppose that $\gamma \in C^1[a,b]$ is a nonnegative function and $\gamma(a) > 0$. If f is a function satisfying condition (H5), then*

(i) *there exists a unique generalized solution $u \in \tilde{C}[a,b]$ to the nonlinear equation*

$$-\tilde{D}^{(\nu)}u(t) - \gamma(t)u'(t) - \lambda u(t) = -f(t,u(t)), \quad t \in (a,b], \quad u(a) = \tilde{u}_a, \tag{56}$$

(ii) *If, additionally, f is continuous and satisfies $f(a,\tilde{u}_a) = \lambda \tilde{u}_a$, then there is a unique solution in the domain of the generator.*

Proof. Since the drift term γ is nonnegative and the assumption $\nu(x,y) > Cy^{-1-\beta}$ holds, the process $T_t^{+(\nu,\gamma)}$ is decreasing and *dominates* the inverted $\beta-$stable subordinator $T_t^{+\beta}$ (see proof of Proposition 4.1 above for the notion of this concept). Hence, all the arguments and calculations used in the proof of Proposition 4.1 and Theorem 4.1 can be carried out similarly, details are then omitted. □

Another interesting case arises when the function γ is assumed to take negative values as well. In this case, since the process $\tilde{T}_t^{a,(\nu,\gamma)}$ is no longer decreasing, the condition $\gamma(a) > 0$ is not sufficient to guarantee that the boundary point a is regular in expectation.

Proposition 6.1. *Let ν be a function satisfying (H0) and let $\gamma \in C^1[a,b]$.*

(i) *If condition (H4) holds and γ also satisfies that $\gamma(a) = 0$, then the point a is regular in expectation for the operator $-\tilde{D}^{(\nu,\gamma)}$.*
(ii) *if $\gamma(b) > 0$, then the boundary point b is regular in expectation for the operator $-\tilde{D}^{(\nu,\gamma)}$, whilst if $\gamma(b) = 0$ then the point b is unattainable.*

Proof. Statements (i) and (ii) follow by the Lyapunov method [16, Proposition 6.3.2] using the Lyapunov functions $f_\omega(t) = (t-a)^\omega$ and $h_\omega(t) = (b-t)^\omega$, for an appropriate $\omega \in (0,1)$, respectively. □

Theorem 6.3. ***(Linear case)*** *Let ν be a function satisfying assumptions (H0) and (H2)-(H4). Assume that γ is a function in $C^1[a,b]$ such that $\gamma(a) = 0$ and $\gamma(b) = 0$. Then, the assertions $(i)-(ii)$ in Theorem 6.1 (except for the equation (55)) hold for the linear problem (53).*

Proof. Since Proposition 6.1 ensures that a is regular in expectation and b is not attainable, the same arguments used in [11] with the operator $-\tilde{D}^{(\nu)}$ remain valid for the operator $-\tilde{D}^{(\nu,\gamma)}$. □

Remark 6.2. *It is worth noting that the explicit equation (55) does not hold when γ also takes negative values because the join distribution of $(T_t^{(\nu,\gamma)}(s), \tau_a^{(\nu)}(t))$ cannot be obtained as was done in Section 4,* [11] *wherein the monotonicity played the main role in the calculations.*

Remark 6.3. *If the process generated by $-\tilde{D}^{(\nu,\gamma)}$ is extended to $\mathbb{R}$ (see* [17, Section 3]*) instead of restricting it to the interval $[a,b]$ as we did before, we would drop the* *assumption $\gamma(b) = 0$. However, under assumption, e.g., $\gamma(b) > 0$, the boundary point b will be also regular in expectation, therefore the uniqueness of solutions is no longer satisfied for the equation (53) for such a γ. Thus, infinitely many solutions can be found unless one imposes conditions on the boundary point b, that is, the uniqueness can be obtained for the equation*

$$-\tilde{D}^{(\nu)}u(t) - \gamma(t)u'(t) = -g(t), \quad t \in (a,b], \quad u(a) = \tilde{u}_a,\ u(b) = u_b, \tag{57}$$

for some $u_b, \tilde{u}_a \in \mathbb{R}$. Since the points a and b are regular in expectation in this case, the arguments used in the previous results can be extended by replacing the stopping time $\tau_a^{t,(\nu)}$ with the corresponding r.v.

$$\tau_{a,b}^{t,(\nu)} := \inf\{\, s \geq 0 \,:\, T_t^{(\nu,\gamma)}(s) \notin (a,b)\},$$

which denotes the first time the process $T_t^{(\nu,\gamma)}$ leaves the interval (a,b).

Remark 6.4. *The nonlinear version of equation (53) cannot be obtained (as was done for the nonnegative γ) under the assumptions $(H4)$ and $(H5)$ when γ is allowed to take negative values. The reason is that, as the process $T_t^{(\nu,\gamma)}$ is not decreasing, condition $(H4)$ no longer provides the upper bounds needed for the fixed point arguments.*

7. Remarks on the classical fractional setting

Since the generalized operators include the classical RL and Caputo derivatives, all the results presented above apply to the classical fractional setting and to their generalizations. This section highlights some important points in this context.

(1) Lemma 4.1 applied to the fractional case states the equivalence between the fractional nonlinear equation

$$\tilde{D}^{\beta}u(t) = f(t, u(t)), \quad t \in (a,b], \qquad u(a) = \tilde{u}_a, \tag{58}$$

and the integral equation

$$u(t) = \tilde{u}_a + \int_a^t f(r, u(r))(t-r)^{\beta-1} \int_0^\infty s^{-1/\beta} w_\beta \left(s^{-1/\beta}; 1, 1\right) ds\, dr, \tag{59}$$

where w_β denotes the $\beta-$stable density (see Preliminaries) and $\tilde{D}^\beta$ stands for either the RL classical fractional derivatives D_{a+}^{β} or the Caputo derivate D_{a+*}^{β}, for $\beta \in (0,1)$. By comparing the integral equation (59) with the Volterra integral equation

$$u(t) = \tilde{u}_a + I_{a+}^{\beta} f(t, u(t)), \tag{60}$$

one can conclude (by uniqueness) that

$$\int_0^\infty s^{-1/\beta} w_\beta \left(s^{-1/\beta}; 1, 1\right) ds = \frac{1}{\Gamma(\beta)}. \tag{61}$$

The Volterra equation (60) is the one commonly used in fractional calculus to prove the well-posedness for fractional differential equations (see, e.g., [4]) The equivalence between (60) and the RL equation (58) has been proved on a space of functions similar to the space F_K defined in (68) (see, e.g., [4], [14]). This equivalence also holds for more general (possibly unbounded) continuous functions f on $(a,b] \times [-K,K]$ with some $K > 0$ such that

$(t-a)^\sigma f(t,u(t)) \in C([a,b]\times[-K,K])$ with $0 \le \sigma < \beta < 1$, (see, e.g., [14], [29]).

(2) Theorem 4.1 provides the well-posedness for fractional nonlinear equations as well as for nonlinear equations involving fully mixed (multi-term) fractional derivatives (see Section 2.2.).

(3) In the fractional setting, Theorem 4.3 implies the next result.

Corollary 7.1. *Assume that $g \in C^1[a,b]$ and $\beta \in (0,1)$. If $g(a) = 0 = g'(a)$, then there is a unique solution $u \in C^1_a[a,b]$ to the problem $(-D^\beta_{a+}, \lambda, g, 0)$ for any $\lambda > 0$. Moreover, if $g(a) = \lambda u_a$, then there is a unique solution $u \in C^1[a,b]$ for the Caputo type problem $(-D^\beta_{a+*}, \lambda, g, u_a)$.*

Notice that if $g(a) \neq 0$, the derivative u' is continuous but unbounded as $t \to a$. This can be seen by differentiating the solution

$$u(t) = \int_0^{t-a}\int_0^\infty g(t-r)e^{-\lambda s}p_s^{+\beta}(t,t-r)\,ds\,dr\,,$$

to obtain

$$\begin{aligned} u'(t) = &\int_0^{t-a} g'(t-r)r^{\beta-1}\int_0^\infty \exp\{-\lambda u r^\beta\}u^{-1/\beta}w_\beta(u^{-1/\beta};1,1)\,du\,dr \\ &+ g(a)(t-a)^{\beta-1}\int_0^\infty \exp\{-\lambda u(t-a)^\beta\}u^{-1/\beta}w_\beta(u^{-1/\beta};1,1)\,du. \end{aligned} \tag{62}$$

As for the nonlinear case, the existence of a smooth solution in the closed interval $[a,b]$ follows by Theorem 4.4 under the assumption $f \in C^1_b([a,b]\times\mathbb{R})$ and $f(a,\tilde{u}_a) = 0$.

(4) Theorem 6.1 implies that the solution to the *composite fractional relaxation equation* given in (9)-(11) can be rewritten as

$$u(t) = u_0\int_0^\infty e^{-s}\mu_0^{t,(c_1,\beta,c_2)}(s)ds + \int_0^t g(t-r)\int_0^\infty e^{-s}p_s^{+(c_1,\beta,c_2)}(t,t-r)\,ds\,dr, \tag{63}$$

with $c_1, c_2 > 0$, $g \in C[0,b]$. Notation $\mu_0^{t,(c_1,\beta,c_2)}(s)$ denotes the density function of the first exit time and $p_s^{+(c_1,\beta,c_2)}(t,r)$ refers to the transition density function of the Feller process generated by $-c_1 D^\beta_{0+*} - c_2\frac{d}{dt}u(t)$, respectively.

(5) By uniqueness of solutions, for any $g \in C[0,b]$ and any strictly positive function $\lambda \in C[0,b]$, Theorem 6.2 provides another integral representation of the solution to the fractional linear equation with non constant coefficients given in (7) which was obtained by analytical methods.

8. Proofs

Proof. (of Theorem 2.1)
(i) This can be proved by approximation arguments and perturbation theory as was done in [17, Theorem 4.1]. Namely, we work with a family of bounded operators $\{L_h := -D^{(\nu_h)}_{a+}\}_{h\in(0,1]}$ which approximates the operator $-D^{(\nu)}_{a+}$ as $h \to 0$, where $\nu_h(t,r) = \mathbf{1}_{r>h}\nu(t,r)$. For each h the operator L_h is bounded in $C_a[a,b]$ (due to assumption (H0)) and is conditionally positive, hence it generates a Feller semigroup

S_s^h on $C_a[a,b]$. This semigroup is the unique (bounded) solution to the evolution equation

$$\frac{d}{ds}f_s(t) = L_h f_s(t), \quad f_0(t) = f(t). \tag{64}$$

Notice that due to the smoothness of ν the operator L_h is also bounded in $C_a^1[a,b]$, hence the semigroup S_s^h has the latter space as an invariant space.

We will prove that $\{S_s^h\}_{h\in(0,1)}$ is a Cauchy sequence. To do so, let us first observe that $L_h S_s^h f \in C_a^1[a,b]$ because $S_s^h f \in C_a^1[a,b]$ whenever $f \in C_a^1[a,b]$. Differentiating (64) with respect to t yields the evolution equation for $g_s(t) = f_s'(t)$ given by

$$\frac{d}{ds}g_s(t) = L_h^{(1)} g_s(t), \quad g_0(t) = g(t) = f'(t),$$

where

$$L_h^{(1)} g(t) := L_h g(t) + A^{(\partial_t \nu_h)} g(t),$$

with the operator $A^{(\mu)}$ defined by

$$A^{(\mu)} g(t) = -\int_0^{t-a}\int_{t-r}^t g(z)dz\mu(t,r)dr - \int_a^t g(z)dz\int_{t-a}^\infty \mu(t,r)dr,$$

for functions μ satisfying the uniform bounds given in (H0). Notice the use of notation $\partial_t\nu$ for partial derivatives. Let us stress that we have used the fact that $f(a) = 0$ to rewrite the derivative of $(L_h f)(t)$ by means of the operator $A^{(\partial_t\nu)}$ and without the term $f(a)\nu(t, t-a)$.

Since the operator $L_h^{(1)}$ decomposes as the sum of the generator L_h perturbed by the bounded operator $A^{\partial_t\nu_h}$ on $C_a[a,b]$, perturbation theory (see, e.g., [16, Theorem 1.9.2]) implies that $L_h^{(1)}$ generates a strongly continuous semigroup on $C_a[a,b]$, which we denote by $S_s^{h,(1)}$. Due to the invariance of the space $C_a^1[a,b]$, it follows that $(S_s^h f)' = S_s^{h,(1)} f'$ for $f \in C_a^1[a,b]$. Moreover, the perturbation series representation for the semigroup $S_s^{h,(1)}$ [16, p.52] implies

$$||S_s^{h,(1)} f'|| \le ||f'|| + \sum_{m=1}^\infty \frac{(s||A^{(\partial_t\nu_h)}||)^m}{m!}||f'||.$$

Since $A^{(\partial_t\nu_h)}$ is uniformly bounded in h due to assumption (H0), we obtain that the derivative $\frac{d}{dt}(S_s^h f)(t)$ is uniformly bounded in h whenever $f \in C_a^1[a,b]$.

Take $0 < h_2 \le h_1 < 1$ and $f \in C_a^1[a,b]$. By rewriting

$$(S_s^{h_1} - S_s^{h_2})f = \int_0^s S_{s-u}^{h_2}(L_{h_1} - L_{h_2})S_u^{h_1} f du.$$

and using that $S_s^{h_1} f$ is differentiable (with derivative uniformly bounded in h), we can estimate (by mean value theorem)

$$\begin{aligned}
\left|(L_{h_1} - L_{h_2})S_u^{h_1} f\right| &\le \int_{h_2\le|r|\le h_1} |S_u^{h_1} f(t-r) - S_u^{h_1} f(t)|\nu(t,r)dr \\
&\le \int_{h_2\le|r|\le h_1} ||(S_s^{h_1} f)'||r\nu(t,r)dr \\
&= o(1)||f||_{C^1}, \quad h_1 \to 0.
\end{aligned}$$

The last equality due to the tightness property of ν given by assumption (H0). Therefore,

$$||(S_s^{h_1} - S_s^{h_2} f)|| = o(1)s||f||_{C^1}.$$

Thus, the family $\{S_s^h f\}_h$ converges to a limiting family, say $S_s f$, as $h \to 0$, for any $f \in C_a^1[a,b]$. Using that

$$\frac{S_s f - f}{s} = \frac{S_s f - S_s^h f}{s} + \frac{S_s^h f - f}{s},$$

we conclude that $C_a^1[a,b]$ belongs to the domain of the generator and that the generator is given by $-D_{a+}^{(\nu)}$. Finally, by standard approximation arguments, we obtain that the limiting family forms also a strongly continuous semigroup of contractions on $C_a[a,b]$, as required.

(ii) From the previous approximation procedure, we have a family of Feller semigroups $\{S_s^h\}_{h\in(0,1]}$ on the space $C_a[a,b]$. We need to prove that this family forms a Cauchy sequence in the C^1-norm. Hence, from previous calculations we already have the estimate for $||(S_s^{h_1} - S_s^{h_2})f||$, so that it remains to estimate $||(S_s^{h_1} f)' - (S_s^{h_2} f)'||$. Let us first observe that under the additional assumption (19), the operator L_h is bounded in $\hat{C} := \{f \in C_a^2[a,b] : f'(a) = 0\}$. Hence, S_s^h is invariant under the latter space and thus $L_h S_s^h f \in \hat{C}$ whenever $f \in \hat{C}$. Proceeding as before, we now take $f \in \hat{C}$ and we differentiate twice the evolution equation (64) with respect to t. We obtain then the evolution equation for $k_s(t) = f_s''(t)$:

$$\frac{d}{ds} k_s(t) = L_h^{(2)} k_s(t), \quad k_0(t) = k(t) = f''(t),$$

where

$$L_h^{(2)} k(t) := L_h k(t) + 2A^{(\partial_t \nu_h)} k(t) + B^{(\partial_t^2 \nu_h)} k(t)$$

with the operator $A^{(\mu)}$ as defined above and $B^{(\mu)}$ given by

$$B^{(\mu)} g(t) = -\int_0^{t-a} \int_{t-r}^{t} \int_a^z g(y) dy dz \mu(t,r) dr - \int_a^t \int_a^z g(y) dy dz \int_{t-a}^{\infty} \mu(t,r) dr.$$

As before, it is worth noting that the previous holds because f is such that $f(a) = 0 = f'(a)$.

Since the operators $A^{(\partial_t \nu_h)}$ and $B^{(\partial_t^2 \nu_h)}$ are bounded in $\hat{C}$ (also uniformly bounded in h), perturbation theory implies again that $L_h^{(2)}$ generates a strongly continuous semigroup, denoted by $S_s^{h,(2)}$, on $C_a[a,b]$ and, further, $(S_s^h f)'' = S_s^{h,(2)} f''$ for $f \in \hat{C}$. The latter due to the invariance of $\hat{C}$. Again, the series representation for $S_s^{h,(1)}$ implies that the second derivative $(S_s^h f)''$ is uniformly bounded in h for $f \in \hat{C}$.

As before, rewrite

$$(S_s^{h_1,(1)} - S_s^{h_2,(1)}) f' = \int_0^s S_{s-u}^{h_2,(1)} (L_{h_1}^{(1)} - L_{h_2}^{(1)}) S_u^{h_1,(1)} f' du. \tag{65}$$

Notice that if $g \in \hat{C}$, then $A^{(\partial_t \nu_h)} g'(t)$ coincides with $\tilde{L}_h g(t) := -D_{a+}^{(\partial_t \nu_h)} g(t)$. Thus, since $(S_s^h f)' = S_s^{h,(1)} f'$, we obtain that $A^{(\partial_t \nu_h)} S_s^{h,(1)} f'(t) = \tilde{L}_h S_s^h f(t)$ yielding (by definition of the operator $L_h^{(1)}$)

$$(L_{h_1}^{(1)} - L_{h_2}^{(1)}) S_u^{h_1,(1)} f'(t) = (L_{h_1} - L_{h_2}) S_u^{h_1,(1)} f'(t) + (\tilde{L}_{h_1} - \tilde{L}_{h_2}) S_u^{h_1} f(t).$$

Due to the tightness property for $\partial_t \nu$ given in (19), similar arguments as in the proof of statement (i) yield

$$\left|(\tilde{L}_{h_1} - \tilde{L}_{h_2}) S_u^{h_1} f\right| \le o(1) ||f||_{C^2}, \quad h_1 \to 0, \tag{66}$$

whereas

$$\begin{aligned}\big|(L_{h_1}-L_{h_2})S_u^{h_1,(1)}f'\big| &\le \int_{h_2\le|r|\le h_1}|S_u^{h_1,(1)}f'(t-r)-S_u^{h_1,(1)}f'(t)|\nu(t,r)dr\\ &\le \int_{h_2\le|r|\le h_1}||(S_s^{h_1}f)''||r\nu(t,r)dr\\ &= o(1)||f||_{C^2}, \quad h_1\to 0, \end{aligned} \tag{67}$$

where we have used that the derivative $(S_s^{h_1}f)''$ is uniformly bounded in h. Plugging (66) and (67) into (65), we obtain

$$||(S_s^{h_1}f)'-(S_s^{h_2}f)'|| = ||(S_s^{h_1,(1)}-S_s^{h_2,(1)})f'|| = o(1)s||f||_{C^2}.$$

We obtain then the convergence of $\{S_s^h\}_h$ as $h\to 0$ in the C^1-norm. Proceeding as before, we can conclude that the limiting semigroup is also strongly continuous on $\{f\in C_a^1[a,b] : f'(a)=0\}$, as required.

(iii) The statement for the Caputo type operator follow similar arguments and thus we omit the details. □

Proof. (of Proposition 4.1) To prove the existence of a unique solution to (34) we rewrite it as a fixed point problem $u(t)=(\Psi u)(t)$ for a certain operator Ψ.

Step a) Defining the operator Ψ. Let us consider the space F_K given by

$$F_K=\Big\{u\in\tilde{C}[a,b^*]\ :\ ||u-\tilde{u}_a||_{\tilde{C}[a,b^*]}\le K\Big\}. \tag{68}$$

Note that F_K is a closed subset of the space $\tilde{C}[a,b^*]$, the latter space endowed with the supnorm denoted by $||\cdot||_{\tilde{C}[a,b^*]}$. Hence, $(F_K,||\cdot||_{\tilde{C}[a,b^*]})$ is a complete metric space. Next, define the operator Ψ on F_K by

$$(\Psi u)(t):=\tilde{u}_a+\int_0^\infty\int_a^t f(r,\,u(r))p_s^{+(\nu)}(t,r)\,dr\,ds, \qquad t\in[a,b^*].$$

Note that if $u\in F_K$, then $\Psi u\in\tilde{C}[a,b^*]$. Further,

$$\begin{aligned}|\Psi u(t)-\tilde{u}_a| &= \Big|\int_0^\infty\int_a^t f(r,\,u(r))p_s^{+(\nu)}(t,r)\,dr\,ds\Big|\\ &\le \int_0^\infty\int_a^t \sup_{y\le r}|f(y,\,u(y))|p_s^{+(\nu)}(t,r)\,dr\,ds.\end{aligned}$$

Since for any ν satisfying $(H0)$ the underlying process is decreasing, assumption $(H4)$ implies that the process $T_t^{+(\nu)}$ *dominates* the inverted β-stable subordinator $T_t^{+\beta}$ in the sense that $\mathbf{P}[T_t^{+(\nu)}(s)>r]\le\mathbf{P}[T_t^{+\beta}(s)>r]$, for all $r\le b^*$ and for all $s\ge 0$ (or, equivalently, $\mathbf{P}[T_t^{+\beta}(s)\le r]\le\mathbf{P}[T_t^{+(\nu)}(s)\le r]$). Therefore, $\mathbf{E}\left[g\left(T_t^{+(\nu)}(s)\right)\right]\le\mathbf{E}\left[g\left(T_t^{+\beta}(s)\right)\right]$ for any increasing function g. Hence, using the function $g(r)=\mathbf{1}_{[a,t]}(r)\sup_{y\le r}|f(y,u(y))|$ we obtain

$$\begin{aligned}|\Psi u(t)-\tilde{u}_a| &\le \int_0^\infty\int_a^t \sup_{y\le r}|f(y,\,u(y))|p_s^{+\beta}(t,r)\,dr\,ds\\ &\le M_K\int_0^\infty\int_a^t p_s^{+\beta}(t,r)\,dr\,ds,\end{aligned}$$

where $p_s^{+\beta}(t,r)$ stands for the transition densities of the inverted β-stable subordinator $T_t^{+\beta}$. The scaling property and the stationary increments of the process $T_t^{+\beta}$ imply $p_s^{+\beta}(t,r) = s^{-1/\beta} w_\beta(s^{-1/\beta}(t-r);1,1)$ (see, e.g., [27]). Hence

$$\begin{aligned}
|\Psi u(t) - \tilde{u}_a| &\leq M_K \int_0^\infty \int_a^t s^{-1/\beta} w_\beta(s^{-1/\beta}(t-r);1,1)\,dr\,ds \\
&\leq M_K \int_a^t (t-r)^{\beta-1} \int_0^\infty y^{-1/\beta} w_\beta(y^{-1/\beta};1,1)\,dy\,dr \\
&\leq \kappa M_K \frac{1}{\beta}(t-a)^\beta \leq \kappa M_K \frac{1}{\beta}(b^*-a)^\beta \leq K.
\end{aligned}$$

In the second inequality we have used Fubini's theorem, and then the change of variable $y = s(t-r)^{-\beta}$. Third inequality follows from (38) and the last inequality holds by definition of b^*. Therefore, we proved that $\Psi : F_K \to F_K$.

Step b) Let Ψ^n denote the n-fold iteration of the operator Ψ for $n \geq 0$. For convention Ψ^0 denotes the identity operator. We will prove that for any $t \in [a, b^*]$,

$$|\Psi^n u(t) - \Psi^n v(t)| \leq \left(\kappa L_f (t-a)^\beta\right)^n ||u-v||_t \prod_{k=0}^{n-1} B(k\beta+1,\beta), \qquad n \geq 1, \quad (69)$$

where

$$||u-v||_t := \sup_{z \leq t} |u(z)-v(z)|, \quad t \in [a, b^*],$$

L_f is the Lipschitz constant of the function f, notation $B(\cdot,\cdot)$ refers to the Beta function and κ is as before.

To prove (69), let us proceed by induction. For $n=1$, the definition of the operator Ψ and the Lipschitz condition yield

$$\begin{aligned}
|\Psi u(t) - \Psi v(t)| &\leq L_f \int_0^\infty \int_a^t |u(r)-v(r)| p_s^{+(\nu)}(t,r)\,dr\,ds \\
&\leq L_f \int_0^\infty \int_a^t ||u-v||_r p_s^{+(\nu)}(t,r)\,dr\,ds \\
&\leq L_f \int_0^\infty \int_a^t ||u-v||_r p_s^{+\beta}(t,r)\,dr\,ds \\
&\leq ||u-v||_t L_f \int_0^\infty \int_a^t p_s^{+\beta}(t,r)\,dr\,ds \\
&\leq \kappa L_f ||u-v||_t \frac{1}{\beta}(t-a)^\beta,
\end{aligned}$$

which implies the result for $n=1$ as $B(1,\beta) = 1/\beta$. Note that the third inequality is justified as before: we use that assumption $(H4)$ implies that $\mathbf{E}\left[g\left(T_t^{+(\nu)}(s)\right)\right] \leq \mathbf{E}\left[g\left(T_t^{+\beta}(s)\right)\right]$ for the function $g(r) = \mathbf{1}_{[a,t]}(r)||u-v||_r$. The last inequality follows from the previous calculations using the β-stable transition densities $p_s^{+\beta}(t,r)$.

Now let us assume that the inequality (69) holds for $n-1$. Then

$$
\begin{aligned}
|\Psi^n u(t) - \Psi^n v(t)| &\le L_f \int_0^\infty \int_a^t \left|\Psi^{n-1}u(r) - \Psi^{n-1}v(r)\right| p_s^{+(\nu)}(t,r)\,dr\,ds \\
&\le L_f \int_0^\infty \int_a^t \sup_{z\le r} \left|\Psi^{n-1}u(z) - \Psi^{n-1}v(z)\right| p_s^{+(\nu)}(t,r)\,dr\,ds \\
&\le L_f \int_0^\infty \int_a^t \sup_{z\le r} \left|\Psi^{n-1}u(z) - \Psi^{n-1}v(z)\right| p_s^{+\beta}(t,r)\,dr\,ds \\
&\le \kappa^{n-1} L_f^n ||u-v||_t \prod_{k=0}^{n-2} B(k\beta+1,\beta) \int_0^\infty \int_a^t (r-a)^{(n-1)\beta} p_s^{+\beta}(t,r)\,dr\,ds \\
&\le \kappa^n L_f^n ||u-v||_t \prod_{k=0}^{n-2} B(k\beta+1,\beta) \int_a^t (r-a)^{(n-1)\beta}(t-r)^{\beta-1}dr\,,
\end{aligned}
\tag{70}
$$

where the first, third and fourth inequalities hold due to the Lipschitz condition, condition (H4) and the induction hypothesis, respectively.

For the integral in (70), the change of variable $z = (r-a)/(t-a)$ yields

$$
\begin{aligned}
\int_a^t (r-a)^{(n-1)\beta}(t-r)^{\beta-1}dr &= (t-a)^{n\beta} \int_0^1 z^{(n-1)\beta}(1-z)^{\beta-1}dz \\
&= (t-a)^{n\beta} B((n-1)\beta+1,\beta),
\end{aligned}
$$

which implies inequality (69), as required.

Step c) To conclude that Ψ has a fixed point, we will apply the Weissenger fixed point theorem. Hence, we shall prove that

$$
||\Psi^n u - \Psi^n v||_{C[a,b^*]} \le \alpha_n ||u-v||_{C[a,b^*]}, \tag{71}
$$

for every $n \ge 0$ and every $u, v \in F_K$, where $\alpha_n \ge 0$ and $\sum_{n=0}^\infty \alpha_n$ converges (see, e.g., Appendix in [4]).

A proof by induction (using the identities in (14)) yields

$$
\prod_{k=0}^{n-1} B(k\beta+1,\beta) = \frac{(\ \Gamma(\beta)\ )^n}{n\beta\Gamma(n\beta)}, \quad n \in \mathbb{N}.
$$

Moreover, the inequality (15) implies

$$
\frac{(\,\Gamma(\beta)\,)^n}{n\beta\Gamma(n\beta)} \le \frac{(\ \Gamma(\beta)\ )^n}{n\beta(n-1)!\beta^{2(n-1)}\,(\,\Gamma(\beta)\,)^n} \le \frac{1}{n!\beta^{2n}}.
$$

Therefore

$$
\begin{aligned}
|\Psi^n u(t) - \Psi^n v(t)| &\le \kappa^n L_f^n ||u-v||_t (t-a)^{n\beta} \frac{1}{n!\beta^{2n}} \\
&\le \kappa^n L_f^n ||u-v||_{C[a,b^*]} (b^*-a)^{n\beta} \frac{1}{n!\beta^{2n}},
\end{aligned}
$$

implying the inequality (71) with $\alpha_n := \left(\beta^{-2}\kappa L_f (b^*-a)^\beta\right)^n / n!$.

Since $\sum_{n=0}^\infty \alpha_n = \exp\{\beta^{-2}\kappa L_f (b^*-a)^\beta\}$, the Weissinger fixed point theorem guarantees the existence of a unique fixed point $u^* \in F_K$, as required. $\square$

Acknowledgements

The first author is supported by Chancellor International Scholarship through University of Warwick, UK

References

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd ed., Cambridge Stud. Adv. Math. **116**, Cambridge Univ. Press, Cambridge, 2009.

[2] J. P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: statistical mechanism, models and physical applications*, Phys. Rep., **195** (1990), pp. 127-293.

[3] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, CISM International Centre for Mechanical Sciences, Springer Verlag, Wien-New York, 1997.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations, An application-oriented exposition using differential operators of Caputo Type*, Lecture Notes in Math., Springer, 2010.

[5] S.S. Dragomir, R.P. Agarwal and N. S. Barnett, *Inequalities for Beta and Gamma functions via some classical and new integral inequalities*, J. Inequalities and Applications, **5**, 2000.

[6] E.B. Dynkin, *Markov processes*, Vol. I, Springer-Verlag, 1965.

[7] S.N. Ethier and T.G. Kurtz, *Markov processes. Characterization and Convergence*, Wiley Ser. Probab. Math. Statist. Wiley, New York, 2010.

[8] I.I. Gihman and A.V. Skorohod, *The Theory of Stochastic Processes II*, Springer-Verlag Berlin Heidelberg New York, 1975.

[9] Gorenflo, R. and Mainardi, F. *Fractional calculus and stable probability distributions*, Arch. Mech., **50** (3), (1998), pp. 377-388.

[10] R. Gorenflo, Y. Luchko, and M. Stojanovic, *Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density*, Fract. Calc. Appl. Anal., **16** (2), (2013) pp. 297-316.

[11] M.E. Hernández-Hernández and V. N. Kolokoltsov, *On the probabilistic approach to the solution of generalized fractional differential equations of Caputo and Riemann-Liouville type*, J Fractional Calc. & Appl. **7**, No 1 (2016), Article No 14, 147-175.

[12] L. Huang and S. Menozzi, *A parametrix approach for some degenerate stable driven SDEs*, (2015) available at arXiv:1402.3997v2

[13] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus,* Grad. Texts in Math., Vol. **113**, 2nd ed. Springer, 1998.

[14] A. A. Kilbas, H.M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., **204**, 2006.

[15] V. N. Kolokoltsov, *Generalized continuous-time random walks (CTRW), Subordination by Hitting times and fractional dynamics.* Theory Probab. Appl. Vol. **53** (4), (2009) pp. 549-609.

[16] V. N. Kolokoltsov, *Markov processes, Semigroups and Generators.* DeGruyter Studies in Mathematics, Book 38, 2009.

[17] V. N. Kolokoltsov, *On fully mixed and multidimensional extensions of the Caputo and Riemann-Liouville derivatives, related Markov processes and fractional differential equations*, Fract. Calc. Appl. Anal. 18:4 (2015) pp. 1039-1073.

[18] R. Gorenflo, and F. Mainardi, *Fractional calculus: Integral and Differential Equations of Fractional Order*, arXiv:0805.3823v1, 2008.

[19] F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*, http://arxiv.org/abs/1201.0863v1, 2001.

[20] F. Mainardi, A. Mura, G. Pagnini, and R. Gorenflo, *Time-fractional diffusion of distributed order*, J. Vib. Control **14** (2008), pp. 1267-1290.

[21] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An introduction to Mathematical Models*, Imperial College Press., 2010.

[22] M. Meerschaert, E. Nane, and P. Vellaisamy, *Fractional Cauchy problems on bounded domains.* Ann. Probab., **37** (3), (2009) pp. 979-1007.

[23] M.M. Meerschaert, and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter Studies in Mathematics **43**, 2012.

[24] T. F. Nonnenmacher, *Fractional integral an differential equations for a class of Lévy-type probability densities*, J. Phys. A: Math. Gen. **23** (14), 1990.

[25] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Mathematics in Science and Engineering, v. 198. Academic Press, Inc., San Diego, 1999.

[26] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers S. A., 1993.

[27] Samorodnitsky, G. and Taqqu, M. (1994) *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall (New York).

[28] E. Scalas, *A class of CTRW's: compound fractional Poisson processes*, Fractional dynamics, World Sci. Publ., Hackensack, NJ, (2012) pp. 353-374.

[29] F. Wang, *Existence and uniqueness of solutions for a nonlinear fractional differential equation*, J. Appl. Math. Comput., (2012), **39**:53-67.

[30] G. M. Zaslavsky, *Chaos, fractional kinetics, and anomalous transport*, Phys. Rep., **371** (6), (2002) pp. 461-580.

M. E. Hernández-Hernández

Department of Statistics, University of Warwick, Coventry, United Kingdom.

E-mail address: M.E.Hernandez-Hernandez@warwick.ac.uk

V. N. Kolokoltsov

Department of Statistics, University of Warwick, Coventry, United Kingdom,
And Associate member of Institute of Informatics Problems, FRC CSC RAS.

E-mail address: V.Kolokoltsov@warwick.ac.uk