Original citation:
Grzesik, Andrzej, Král', Daniel and Lovasz, Laszlo Miklos. (2017) Extremal graph theory and finite forcibility. Electronic Notes in Discrete Mathematics, 61. pp. 541-547.

Permanent WRAP URL:
http://wrap.warwick.ac.uk/88890

Extremal graph theory and finite forcibility

Andrzej Grzesik[*] Daniel Král'[†] László Miklós Lovász[‡]

Abstract

We study the uniqueness of optimal solutions to extremal graph theory problems. Our main result is a counterexample to the following conjecture of Lovász, which is often referred to as saying that "every extremal graph theory problem has a finitely forcible optimum": every finite feasible set of subgraph density constraints can be extended further by a finite set of density constraints such that the resulting set is satisfied by an asymptotically unique graph.

1 Introduction

Many problems in extremal graph theory do not have asymptotically unique solutions. As an example, we consider the problem of minimizing the sum of the induced subgraph densities of K_3 and its complement. It can be shown that this sum is minimized by any n-vertex graph where all vertices have degrees close to $n/2$. For example, the complete bipartite graph $K_{n/2,n/2}$, the union of two $(n/2)$-vertex complete graphs, or (with high probability) an Erdős-Rényi random graph $G_{n,1/2}$ all minimize the sum. However, the structure of an optimal solution can be made unique by adding additional density constraints. In our example, setting the triangle density to be zero forces the structure to be that of the complete bipartite graph with parts of equal sizes. Alternatively, fixing the density of cycles of length four forces the structure to be that of a quasirandom graph.

[*]Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Prof. St. Lojasiewicza 6, 30-348 Cracow, Poland. E-mail: Andrzej.Grzesik@tcs.uj.edu.pl.

[†]Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK. E-mail: d.kral@warwick.ac.uk. The work of this author was supported by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 648509) and by the Engineering and Physical Sciences Research Council Standard Grant number EP/M025365/1. This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.

[‡]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, US. E-mail: lmlovasz@math.mit.edu.

Lovász conjectured (Conjecture 1 below) that this is a general phenomenon for a large class of problems in extremal graph theory. We disprove this conjecture.

We treat Conjecture 1 in the language of the theory of graph limits. This theory has offered analytic tools to represent and analyze large graphs, and led to new tools and views on various problems in mathematics and computer science. It is also closely related to the flag algebra method of Razborov [14], which changed the landscape of extremal combinatorics [15]. We refer the reader to a monograph by Lovász [10] for a detailed introduction to this theory.

We now introduce several concepts from the theory of graph limits, so that we may state Conjecture 1 in that language. The *density* of a k-vertex graph H in G, denoted by $d(H, G)$, is the probability that k randomly chosen vertices of G induce a subgraph isomorphic to H; if G has less than k vertices, we set $d(H, G) = 0$. A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the sequence $(d(H, G_n))_{n\in\mathbb{N}}$ converges for every graph H. In this paper, we only consider convergent sequences of graphs where the number of vertices tends to infinity.

A convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is *finitely forcible* if there exist graphs $H_1, \ldots, H_\ell$ with the following property: if $(G'_n)_{n\in\mathbb{N}}$ is another convergent sequence of graphs such that

$$\lim_{n\to\infty} d(H_i, G_n) = \lim_{n\to\infty} d(H_i, G'_n)$$

for every $i = 1, \ldots, \ell$, then

$$\lim_{n\to\infty} d(H, G_n) = \lim_{n\to\infty} d(H, G'_n)$$

for every graph H. For example, a classical result on quasirandom graphs [2,16,17] is equivalent to saying that the sequence of Erdős-Rényi random graphs is finitely forcible (by densities of 4-vertex subgraphs) with probability one. Lovász and Sós [11] generalized this result to graph limits corresponding to stochastic block models (which are represented by so-called step graphons). Additional examples can be found, e.g., in [12].

Our main result (Theorem 3) implies that the following conjecture of Lovász, which is often referred to as saying "Every extremal problem has a finitely forcible optimum.", is false. The conjecture has appeared, e.g., in [8, Conjecture 3], [9, Conjecture 9.12], [10, Conjecture 16.45], and [12, Conjecture 7].

Conjecture 1. *Let $H_1, \ldots, H_k$ be graphs and $\delta_1, \ldots, \delta_k$ reals. If there exists a convergent sequence of graphs with the limit density of H_i equal to δ_i, $i = 1, \ldots, k$, then there exists such a finitely forcible sequence.*

2 Main result

In this section, we state our main result and present the main ideas behind its proof. To do so, we first recall an analytic object used to represent a convergent sequence of graphs. A *graphon* is a symmetric measurable function $W : [0,1]^2 \to [0,1]$, where symmetric means that $W(x,y) = W(y,x)$ for all $x, y \in [0,1]$. Given a graphon W, a *W-random graph* with n vertices is a graph obtained by sampling n points $v_1, v_2, \ldots, v_n \in [0,1]$ independently and uniformly at random and joining vertices v_i and v_j by an edge with probability $W(v_i, v_j)$ for all $i, j \in [n]$. The density of an n-vertex graph H in a graphon W, denoted by $d(H,W)$, is the probability that a W-random graph with n vertices is isomorphic to H. A graphon W is a limit of a convergent sequence $(G_n)_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty} d(H, G_n) = d(H, W)$$

for every graph H. It is not hard to show that a sequence of W-random graphs with increasing number of vertices is convergent with probability one and the graphon W is its limit. Lovász and Szegedy [13] showed that every convergent sequence of graphs has a limiting graphon. This graphon need not be unique, but it is unique up to a certain type of measure preserving transformation [1], which in some sense corresponds to permutations of the vertices of a graph; two graphons equivalent in this sense are called *weakly isomorphic*. See [1, 10] for more details.

A graphon is *finitely forcible* if it is a limit of a finitely forcible convergent sequence of graphs; such graphons are determined by finitely many densities (up to weak isomorphism). It was believed that every finitely forcible graphon has a simple structure [12, Conjectures 9 and 10]. However, this is not the case in quite a strong sense [5], also see [4, 6, 7].

Theorem 1. *For every graphon W_F, there exists a finitely forcible graphon W such that W_F is a subgraphon of W, and the subgraphon is formed by a $1/13$ fraction of the vertices of W.*

An equivalent characterization of finitely forcible graphons is the following.

Proposition 2. *A graphon W is finitely forcible if and only if there exist graphs $H_1, \ldots, H_k$ and reals $\alpha_1, \ldots, \alpha_k$ such that the graphon W is the unique (up to weak isomorphism) minimizer of the sum*

$$\sum_{i=1}^{k} \alpha_i d(H_i, W) \; .$$

Proposition 2 says that finitely forcible graphons are unique optimal solutions to a certain kind of extremal graph theory problem, and Conjecture 1 can be understood as saying that the converse is also true.

Our main result is the following theorem, which implies that Conjecture 1 is false.

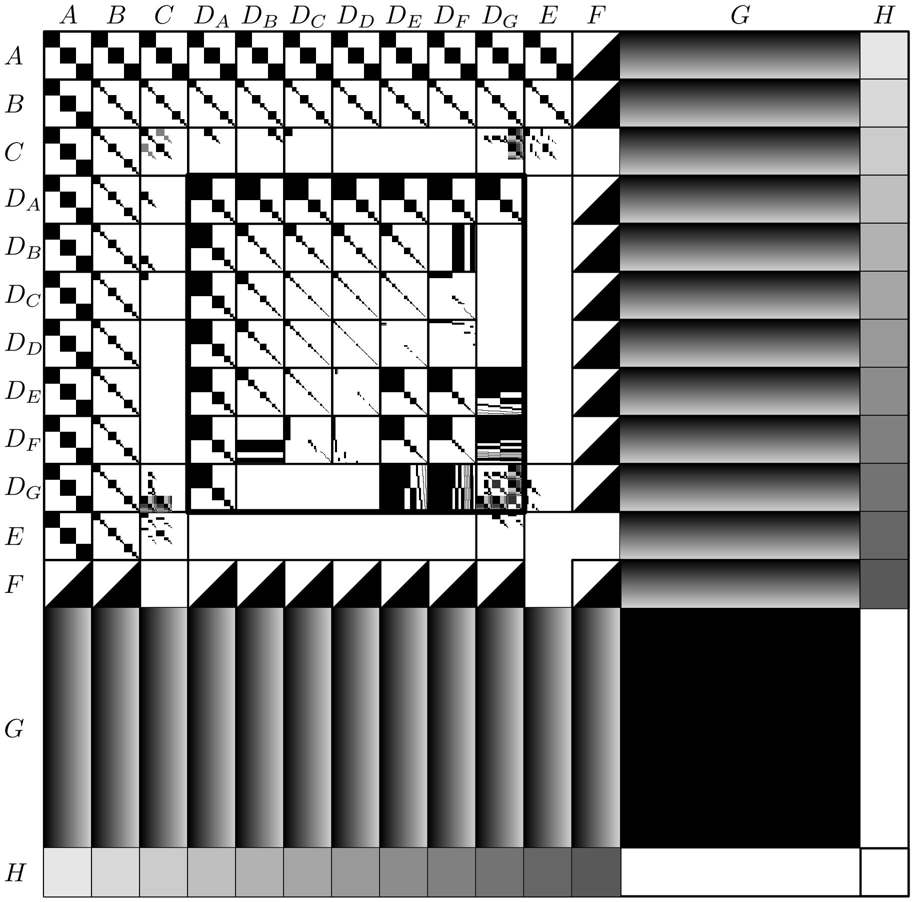

Figure 1: The structure of the graphon $W_P(z_1, z_2, \ldots)$. The parts of the graphon corresponding to the graphon W from Theorem 1 are framed by a black box.

Theorem 3. *There exist a family of graphons $W_P(z_1, z_2, \ldots)$ parametrized by $z_i \in [0,1]$, $i \in \mathbb{N}$, a set $Z \subseteq [0,1]^{\mathbb{N}}$, and finitely many graphs $H_1, \ldots, H_k$ with reals $\delta_1, \ldots, \delta_k$ such that a graphon W is weakly isomorphic to a graphon $W_P(z_1, z_2, \ldots)$ for some $(z_1, z_2, \ldots) \in Z$ if and only if $d(H, W) = \delta_i$ for every $i = 1, \ldots, k$.*

In addition, there exists a bijection $f : [0,1]^{\mathbb{N}} \to Z$ such that if $x = (x_1, x_2, \ldots)$ and $x' = (x'_1, x'_2, \ldots)$ agree on the first m coordinates, then the graphons $W_P(f(x))$ and $W_P(f(x'))$ have the same density of all graphs up to m vertices.

We now show that Theorem 3 indeed implies that Conjecture 1 is false. Consider the graphs $H_1, \ldots, H_k$ and their densities $\delta_1, \ldots, \delta_k$ from Theorem 3, and assume that there exists a finitely forcible graphon W such that $d(H_i, W) = \delta_i$ for all $i = 1, \ldots, k$. Further, let m be the maximum number of vertices of a graph in a set that witnesses the finite forcibility of W. By Theorem 3, there exist z_i, $i \in \mathbb{N}$, such that W and $W_P(z_1, z_2, \ldots)$ are weakly isomorphic. Further, let $x = f^{-1}(z_1, z_2, \ldots)$ where f is the bijection from Theorem 3. Choose any

$x' \in [0,1]^{\mathbb{N}}$ such that $x \neq x'$ but x and x' agree on the first m coordinates; set $W' = W_P(f(x'))$. By Theorem 3, the graphon W and W' are not weakly isomorphic (since $x \neq x'$) but they have the same density of all graphs up to m vertices. This contradicts the assumption that there is a set of graphs with at most m vertices that witnesses the finite forcibility of W.

We finish with giving a high level overview of the proof of Theorem 3. The proof uses the method of decorated constraints developed in [6, 7]. We start by describing the structure of $W_P(z_1, z_2, \ldots)$, which is also visualized in Figure 1. The structure of $W_P(z_1, z_2, \ldots)$ depends on the parameters $z_i \in [0,1]$, $i \in \mathbb{N}$, and a countable set P of polynomials, each in a finite set of z_i's. Each graphon W_P has several parts, and each part has vertices with the same degree; this degree uniquely determines the part. Each parameter z_i is represented by the density of a square in the part $C \times C$. The parts $D_A, \ldots, D_G$ of the graphon W_P induce a part of the graphon W from Theorem 1, which allows embedding any graphon in the part $D_G \times D_G$. The particular graphon that is embedded in the part $D_G \times D_G$ is a graphon that contains an encoding of the coefficients of the polynomials from P; this encoding also appears in the part $C \times D_G$. The values of the polynomials in P appear as widths of rectangles in the part $C \times E$ and are used to force that $p(z_1, z_2, \ldots) \geq 0$ for every $p \in P$.

The particular set P of polynomials such that $W_P(z_1, z_2, \ldots)$ satisfies Theorem 3 is constructed iteratively. We fix an enumeration $H_1, H_2, \ldots$ of all graphs. In the k-th step, we find an integer $m_k > m_{k-1}$, a subset $Z_k \subseteq [0,1]^{m_k}$ with positive measure and numbers b_i, $i > m_k$, such that the density of H_ℓ, $\ell < k$, in the graphon $W_P(z_1, \ldots, z_{m_k}, b_{k+1}, b_{k+2}, \ldots)$ is independent of the values of the variables $z_{m_\ell+1}, \ldots, z_{m_k}$. The values of b_i change only by a small amount in each step. We approximate the characteristic function of Z_k by a polynomial p in $z_1, \ldots, z_{m_k}$ and add the polynomial $p - 1$ to P, getting a set P' such that the cut norm between W_P and $W_{P'}$ is very small (for the same values of z_i). We eventually set Z to be the intersection of $Z_k \times [0,1]^{\mathbb{N}}$ over all $k \in \mathbb{N}$. The existence of the subsets Z_k follows from an application of the Implicit Function Theorem. During the proof, we need to prevent the intersection of $Z_k \times [0,1]^{\mathbb{N}}$ from becoming degenerate (to guarantee the existence of the bijective function f in Theorem 3).

References

[1] Borgs, C., J. T. Chayes and L. Lovász, *Moments of two-variable functions and the uniqueness of graph limits*, Geom. Funct. Anal. **19** (2010), 1597–1619.

[2] Chung, F. R. K., R. L. Graham and R. M. Wilson, *Quasi-random graphs*, Combinatorica **9** (1989), 345–362.

[3] Conlon, D., and J. Fox, *Bounds for graph regularity and removal lemmas*, Geom. Funct. Anal. **22** (2012), 1191–1256.

[4] Cooper, J. W., T. Kaiser, D. Král' and J. A. Noel, *Weak regularity and finitely forcible graph limits*, to appear in Trans. Amer. Math. Soc.

[5] Cooper, J. W., D. Král' and T. Martins, *Finitely forcible graph limits are universal*, preprint available as arXiv:1701.03846.

[6] Glebov, R., T. Klimošová and D. Král', *Infinite dimensional finitely forcible graphon*, preprint available as arXiv:1404.2743.

[7] Glebov, R., D. Král' and J. Volec, *Compactness and finite forcibility of graphons*, preprint available as arXiv:1309.6695.

[8] Lovász, L., *Graph homomorphisms: Open problems*, manuscript, 2008, available as `http://www.cs.elte.hu/~lovasz/problems.pdf`.

[9] Lovász, L., *Very large graphs*, Curr. Dev. Math. **2008** (2009), 67–128.

[10] Lovász, L., "Large networks and graph limits", AMS, Providence, RI, 2012.

[11] Lovász, L., and V. T. Sós, *Generalized quasirandom graphs*, J. Combin. Theory Ser. B **98** (2008), 146–163.

[12] Lovász, L., and B. Szegedy, *Finitely forcible graphons*, J. Combin. Theory Ser. B **101** (2011), 269–301.

[13] Lovász, L., and B. Szegedy, *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), 933–957.

[14] Razborov, A., *Flag algebras*, J. Symbolic Logic **72** (2007), 1239–1282.

[15] Razborov, A., *Flag algebras: an interim report*, in Graham, R.,, J. Nešetřil, S. Butler (eds.), The mathematics of Paul Erdős II, Springer, 2013, pp. 207–232.

[16] Thomason, A., *Pseudo-random graphs*, Ann. Discrete Math. **33** (1987), 307–331.

[17] Thomason, A., *Random graphs, strongly regular graphs and pseudorandom graphs*, in: C. Whitehead (ed.), "Surveys in Combinatorics 1987", London Mathematical Society Lecture Note Series, 123, Cambridge Univ. Press, Cambridge (1987), 173-195.