On the probabilistic approach to the solution of generalized fractional differential equations of Caputo and Riemann-Liouville type

by

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Dedicatoria

Con todo mi amor y agradecimiento
dedico este trabajo
A mis padres, Enedina y Ascención.
Porque su amor, cariño y confianza
son mi fortaleza para seguir adelante.
Porque sin su apoyo difícilmente
habría podido llevar a término
este proyecto.
Infinitas gracias por estar al pendiente de mí
a pesar de la distancia.

Elena

Contents

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Declarations

1. A paper containing the results of Chapters 2-3 has been published in *Journal of Fractional Calculus and Applications*:
(with V. N. Kolokoltsov (PhD supervisor)) *On the probabilistic approach to the solution of generalized fractional differential equations of Caputo and Riemann-Liouville type*, Vol. 7 (1), Jan. 2016, pp. 147-175.

2. A paper containing the results of Chapter 4 has been submitted to *Stochastics, An International Journal of Probability and Stochastic Processes*:
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3. A paper containing the results of Chapter 5 has been accepted for publication in the journal *Fractional Calculus and Applied Analysis* (FCAA):
(with V. N. Kolokoltsov (PhD supervisor)) *On the solution of two-sided fractional ordinary differential equations of Caputo type.* To appear in FCAA, Vol. 19 (6), Dec. 2016.

4. A preprint containing results of Chapter 6 is in preparation for submission:
(with V. N. Kolokoltsov (PhD supervisor)) *Well-posedness results for generalized fractional evolution equations of Caputo type.*

Abstract

This dissertation focuses on the study of generalized fractional differential equations involving a general class of non-local operators which are referred to as *the generalized fractional derivatives of Caputo and Riemann-Liouville (RL) type.* These operators were introduced recently as a probabilistic extension of the classical fractional Caputo and Riemann-Liouville derivatives of order $\beta \in (0,1)$ (when acting on regular enough functions).

The equations studied here include, as particular cases, some fractional differential equations well analyzed in the literature, as well as their far reaching extensions including various mixed derivatives. They encompass, for example, two sided equations of the form

$$\omega_1(t) D^{\beta_1(t)}_{a+*} u(t) + \omega_2(t) D^{\beta_2(t)}_{b-*} u(t) = \lambda u(t) + \gamma(t) u'(t) + \alpha(t) u''(t), \quad t \in (a,b),$$
$$u(a) = u_a, \quad u(b) = u_b,$$

as well as (nonhomogeneous) fractional evolution equations

$$D^{\beta(t)}_{0+*} u(t,x) = -(-\Delta_x)^{\alpha/2} u(t,x) + g(t,x), \quad \beta \in (0,1), \quad \alpha \in (1,2),$$

where $D^{\beta(\cdot)}_{a+*}$ (resp. $D^{\beta(\cdot)}_{b-*}$) is the left-sided Caputo (resp. the right-sided Caputo) derivative of variable order $\beta(\cdot) \in (0,1)$.
The results presented in this work cover the following aspects:

(i) *Well-posedness.* Existence and uniqueness of generalized solutions (and, in some cases, of smooth solutions). The well-posedness is proved via a probabilistic method based on the properties of the resolvent (or potential) operator of the underlying stochastic process. In the last chapter, we also appeal to analytical methods to prove the well-posedness for generalized fractional evolution equations.

(ii) *Stochastic representations for the solutions.* These are obtained by resorting to the probabilistic interpretation of the generalized operators as generators of Feller processes. Hence, standard results from probability theory (Dynkin's martingale and Doob's stopping theorem) allow us to rewrite the solutions as mathematical expectations related to the underlying stochastic processes. Furthermore, for some particular cases we also provide series representations for the solutions.

The main contribution of this work lies in displaying the use of stochastic analysis as a valuable approach for the study of fractional differential equations and their generalizations. The stochastic representations presented here also lead to many interesting potential applications, e.g., by providing new numerical approaches to approximate solutions to equations for which an explicit solution is not available.

Notation and Terminology

Abbreviations

a.s.	almost surely
a.e.	almost everywhere
FODE	Fractional ordinary differential equation
FPDE	Fractional partial differential equation
ODE	Ordinary differential equation
PDE	Partial differential equation
RL	Riemann-Liouville
r.v.	random variable

Symbols

$\mathbb{N}$	$\{1, 2, 3, ...\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}^d$	d–dimensional Euclidean space, $d \in \mathbb{N}$, with $\mathbb{R}^1 \equiv \mathbb{R}$
$\mathbb{C}$	the complex space
$:=$	equality by definition
∂A	boundary of a set A
$\bar{A}$	closure of a set A, i.e. $\bar{A} = A \cup \partial A$
$a \wedge b$	$\min(a, b)$
$\blacksquare$	end of a proof
$\mathfrak{B}(G)$	Borel σ–algebra of a set $G \subset \mathbb{R}^d$

Δ_x	$\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, the Laplacian operator, $x \in \mathbb{R}^d$
$-(-\Delta)^{\alpha/2}$	the fractional Laplacian
$\mathbf{1}_A$	indicator function of the set A defined as $\mathbf{1}_A(x) := 1$ if $x \in A$, and zero otherwise
$\mathbf{1}$	the identity operator
$\lceil \cdot \rceil$	the ceiling function, i.e. $\lceil x \rceil$ means the smallest integer greater than x
$E_\beta(\cdot)$	the Mittag-Leffler function of order $\beta > 0$
$E_{\beta_1,\beta_2}(\cdot)$	the *two-parameter Mittag-Leffler function* of order $\beta_1, \beta_2 \geq 0$
$\Gamma(\cdot)$	the Gamma function
$B(\alpha, \beta)$	the Beta function for $\alpha, \beta > 0$
$\delta(x - x_0)$	the Dirac delta function either as a measure (assigning mass one at x_0, and zero mass elsewhere), or as a distribution (generalized function)

Spaces of real-valued functions Let G be a complete separable metric space.

$B(G)$	the space of bounded measurable functions defined on G
$C(G)$	the space of continuous functions on G
$C_b(G)$	the space of bounded continuous functions on G
$C_c(G) :=$	continuous functions with compact support
$C_\infty(\mathbb{R}^n)$	the space of continuous functions vanishing at infinity, i.e. $f \in C(\mathbb{R}^n)$ such that $\lim_{\lvert x\rvert\to\infty} f(x) = 0$

Remark. All these spaces are equipped with the usual sup norm $\lVert f\rVert := \sup_{x\in G} \lvert f(x)\rvert$.

Differentiable functions. For any open subset $A \subset \mathbb{R}^d$:

$C_c^\infty(\mathbb{R}^d)$	the space of infinitely differentiable functions with compact support
$C^k(A)$	the space of continuous functions on A with continuous derivatives up to and including order k (partial derivatives if $A \subset \mathbb{R}^d$, $d \geq 2$)

$C^k(\bar{A})$	the space of k times continuously differentiable functions on A such that the derivatives $f^{(n)}$ has a continuous extension to the closure $\bar{A}$ for each $n \in \{1, \dots, k\}$

Remark. Notation $d^k/dx^k f(x)$ and $f^{(k)}$ will be used interchangebly to denote the kth-derivative of the function f. For $k \le 3$, we shall also use prime notation.

Other spaces of functions

$A^m[a,b]$	functions whose derivatives of order $m-1$ are absolutely continuous
$C_0[a,b]$	the space of continuous functions on $[a,b]$ vanishing at the boundary points a and b
$C_0^k[a,b] :=$	$C_0[a,b] \cap C^k[a,b]$

Remark. If we replace the subscript 0 by a (resp. by b) in the previous notation, then we define the corresponding spaces of functions vanishing only at the boundary a (resp. only at the boundary b).

For any Banach space $(\mathsf{B}, \|\cdot\|_{\mathsf{B}})$,

$C([a,b];\mathsf{B})$	the space of continuous functions on $[a,b]$ with values on the Banach space B endowed with the sup norm $\|f\|_{C_{\mathsf{B}}} := \sup_{t\in[a,b]} \|f(t)\|_{\mathsf{B}}$

Probability theory

$(\Omega, \mathcal{F}, \mathbf{P})$ denotes a probability space where Ω is the sample space space, $\mathcal{F}$ is its Borel σ-algebra and $\mathbf{P}$ is a probability measure on $(\Omega, \mathcal{F})$. For a given stochastic process $X = \{X(s)\}_{s\ge0}$, we use notation $p_s(x,y)$ to denote its transition density from x to y, where s is the time variable. Notation $\mathcal{F}_t^X$ means the natural filtration generated by the stochastic process X, i.e. $\sigma(X_r, 0 \le r \le t)$ for each $t \ge 0$. Letters $\mathbf{P}$ and $\mathbf{E}$ are reserved for the probability and the mathematical expectation, respectively.

Some operators

D^m	the classical mth derivative of integer-order $m \in \mathbb{N}$
I^{β}_{a+}	the Riemann-Liouville integral operator of order $\beta > 0$
$I^{(\nu)}_{a+}$	the generalized fractional integral associated with a function ν
D^{β}_{a+*}	left-sided Caputo derivative of order β and terminal a
D^{β}_{b-*}	right-sided Caputo derivative of order β and terminal b
D^{β}_{a+}	left-sided Riemann-Liouville derivative of order β and terminal a
D^{β}_{b-}	right-sided Riemann-Liouville derivative of order β and terminal b
$-D^{(\nu)}_{a+*}$	left-sided generalized fractional operator of Caputo type
$-D^{(\nu)}_{a+}$	left-sided generalized fractional operator of RL type
$-D^{(\nu)}_{b-*}$	right-sided generalized fractional operator of Caputo type
$-D^{(\nu)}_{b-}$	right-sided generalized fractional operator of RL type
$S := \{S_s\}_{s\geq 0}$	semigroup for a given operator
R_λ	the resolvent operator (defined for $\lambda > 0$)
R_0	the potential operator

Remark. Additional superscripts will be used to differentiate amongst different stochastic processes, resolvent and potential operators.

Chapter 1

Introduction

Over the last decades, the theory of fractional differential equations has been actively studied due to its vast applications for modeling a variety of physical phenomena arising in different fields of science. Their numerous applications include areas such as engineering, physics, biophysics, continuum and statistical mechanics, finance, control processing, econophysics, probability, and so on. Their successful use to provide more accurate models to describe, for example, relaxation phenomena, processes of oscillation, viscoelastic systems, diffusions in disordered media (also called *anomalous diffusions*) and continuous time random walks (CTRW's) among others, has promoted an increasing research on the fields of fractional ordinary differential equations (FODE's) and fractional partial differential equations (FPDE's). We refer, e.g., to [10], [11], [48], [62], [65], [45], [53], [67], [73], [76], [86] (and references cited therein) for an account of historical notes, theory and applications of fractional calculus, as well as different analytical and numerical methods to address both fractional ordinary differential equations (FODE's) and fractional partial differential equations (FPDE's).

To solve this type of equation various numerical and analytical approaches have been investigated. The standard analytical methods to solve fractional differential equations include, among others, *the Laplace transform, the Mellin transform* and *the Fourier transform* techniques [15], [45], [73], [72], [76], as well as the *operational calculus* method [32], [37], [60]. Regarding the numerical approaches, one can mention the *fractional difference method*, the *quadrature formula approach*, the *predictor-corrector approach* as well as some numerical approximations using the *short memory principle*, amongst others (see, e.g., [12],[13], [15], [18], [43], [68], [73] and references therein).

In a probabilistic framework, the remarkable connection between stochastic analysis and probability theory allows one to solve classical differential equations by relating them with boundary value problems of diffusion processes. In the fractional setting some connections between probability and FPDE's have also been explored [26], [52], [53], [67], [69], [71], [78]. For instance, the probabilistic interpretations of the Green (or fundamental) solution to the *time-fractional diffusion equation* and *the time-space fractional diffusion equation* are already known (see references above). It is precisely in the probabilistic setting wherein the topic of this dissertation relies on.

The generalized fractional operator of Riemann-Liouville (RL) type and Caputo type considered in this work provide a powerful link between stochastic analysis and the solutions to classical fractional equations (and their probabilistic generalizations). The RL and Caputo type operators (hereafter denoted by $-D_{a+}^{(\nu)}$ and $-D_{a+*}^{(\nu)}$, respectively) can be thought of as the generators of Feller processes interrupted on the first attempt to cross certain boundary point. These operators were introduced in [55] as generalizations (from a probabilistic point of view) of the classical Riemann-Liouville and Caputo derivatives of order $\beta \in (0,1)$ when applied to regular enough functions. This fact allows one to solve classical fractional equations as particular cases of more general equations involving operators of the type $-D_{a+}^{(\nu)}$ and $-D_{a+*}^{(\nu)}$ (see the precise definitions later).

Therefore, in this thesis we study boundary value problems corresponding to generalized linear equations, nonlinear equations and two-sided equations. These equations can be seen as the counterpart of some fractional *ordinary* differential equations with derivatives of order $\beta \in (0,1)$. We also investigate *generalized fractional evolution equations* which can be thought of as the counterpart of standard (or fractional) *evolution equations.*

For the *ordinary* case (Chapters 3-5) we employ similar probabilistic arguments to those used in standard differential equations. Namely, using the probabilistic interpretation of the operators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$, we transform the original problem into a Dirichlet type problem for the corresponding generator. We prove then the existence and uniqueness of generalized solutions. The notion of generalized solution in these chapters is understood as a limit of approximating solutions taken from the domain of the operators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$ (seen as generators of Feller processes). Regarding the generalized evolution equations studied in Chapter 6, the technique relies on the concept of *generalized solution* which is based on the notion of the *Green's function* for the operator $-D^{(\nu)}_{a+*}$. In analogy with standard analytical methods to solve classical evolution equations, to study the Caputo type evolution equations we transform them into an *abstract* linear equation (of the type studied in Chapter 3) but on a suitable Banach space.

Let us now mention that, as happens not only in the fractional setting but even in the classical differential setting, explicit analytical solutions are generally not available. In this respect, the probabilistic arguments employed in this work allow us to derive stochastic representations (as mathematical expectations) for the corresponding solutions. In some cases (linear case, for instance) we provide explicit solutions in terms of transition probabilities. The analytical method in Chapter 6 also provides series representations for the solutions to some specific cases.

The main contribution of this dissertation lies on displaying the use of stochastic analysis as a valuable approach for the study of fractional differential equations as well as their numerous generalizations. The results established here encompass and extend many results very well-known in the theory of classical fractional differential equations. Further, the stochastic representations provided here also lead to many interesting potential applications, e.g., by providing new numerical approaches to approximate solutions to those fractional equations for which explicit solutions are unknown.

Outline of the dissertation.

The main content of this work is organized in 5 chapters. For clarity and convenience to the reader, each chapter contains an introduction with a short survey of the relevant literature related to the topic in consideration.

Chapter 2 starts with a quick review of basic definitions related to classical Caputo and Riemann-Liouville derivatives. Then, it continues with the definition of the generalized fractional operators $-D^{(\nu)}_{a+}$ and $-D^{(\nu)}_{a+*}$. In particular, the study is restricted to generalized fractional operators that are the counterpart of classical fractional derivatives of order $\beta \in (0,1)$. Their probabilistic interpretation, as well as some important properties of the underlying stochastic processes are provided.

Chapter 3 studies the well-posedness for boundary value problems for the linear equation with constant coefficients

$$-D^{(\nu)}_{a+*}u(x) = \lambda u(x) - g(x), \quad x \in (a,b],$$

as well as for the generalized *mixed fractional linear equation* (see details later). The main ideas behind the probabilistic approach used in Chapters 3-5 are introduced in this section.

Chapter 4 focuses on the well-posedness for the generalized nonlinear equation

$$-D^{(\nu)}_{a+*}u(x) = f(x, u(x)), \quad x \in (a, b].$$

It also studies the linear equation with nonconstant coefficients

$$-D^{(\nu)}_{a+*}u(x) = \lambda(x)u(x) - g(x), \quad x \in (a, b].$$

Moreover, a stochastic representation of Feynman-Kac type is obtained for the solution to the latter equation.

Chapter 5 is devoted to the study of the well-posedness for two-sided equations (i.e. equations involving left- and right-sided operators both acting on the same variable):

$$-D^{(\nu)}_{a+*}u(x) - D^{(\nu)}_{b-*}u(x) + \lambda(x)u'(x) + \gamma(x)u''(x) = \lambda u(x) - g(x), \quad x \in (a, b].$$

Existence, uniqueness and stochastic representations for the solutions are established via the same probabilistic arguments used in Chapters 3-4.

Chapter 6 establishes the well-posedness for the nonhomogeneous generalized fractional evolution equation

$$\begin{aligned} -{}_tD^{\nu}_{a+*}u(t,x) &= A_x u(t,x) - g(t,x,u), && t \in (a,b], x \in \mathbb{R}^d \\ u(a,x) &= \phi_a(x), && x \in \mathbb{R}^d, \end{aligned}$$

where $-A_x$ is the generator of a Feller process in $\mathbb{R}^d$. The approach used in this chapter is based on analytical methods via the notion of generalized solution given in terms of a Green's function.

Appendix includes some standard definitions related to Feller processes and β−stable subordinators, as well as the definitions of relevant functions connected with fractional calculus.

Chapter 2

Generalized fractional operators

This chapter provides the definition of the generalized Caputo and Riemann-Liouville type operators as introduced in [55], along with some properties and related definitions.

2.1 Preliminaries

Since the generalized fractional operators considered in this work are a probabilistic extension of the classical fractional derivatives of order $\beta \in (0,1)$, this section provides a quick summary of basic definitions concerning the classical Riemann-Liouville and Caputo fractional operators. For a detailed treatment refer, e.g., to [15], [45], [73], [76] and references therein.

Caputo and Riemann-Liouville derivatives

In the theory of fractional calculus, the Caputo and the RL fractional operators play an important role amongst the different notions of fractional derivatives known in the literature. The so-called *Riemann-Liouville approach* defines the classical *fractional differential operators* in terms of two operators: the standard differential operator of integer order (hereafter denoted by D^m, $m \in \mathbb{N}$) and the integral operator of fractional order, I^{α}_{a+}.

The integral operator I_{a+}^{α} is defined as the generalization of the Cauchy integral for n-fold integration [15, Lemma 1.1], wherein the integer n and the factorial function are replaced by a real number α and the Gamma function, respectively.

Definition 2.1.1. *The Riemann-Liouville integral operator I_{a+}^{α} of order $\alpha > 0$ acting on functions from $L_1[a,b]$ is defined by*

$$I_{a+}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}h(y)dy, \tag{2.1.1}$$

for any $a \in \mathbb{R} \cup \{-\infty\}$. For convention, I_{a+}^{0} refers to the identity operator, hereafter denoted by $\mathbf{1}$.

The *left-sided Riemann-Liouville (RL)* derivative is then defined as the left-inverse of the RL integral operator I_{a+}^{α}, that is $D_{a+}^{\beta} \circ I_{a+}^{\beta} = \mathbf{1}$, $\beta > 0$ (see, e.g., [15], [76]).

Definition 2.1.2. *Let $\beta \in \mathbb{R}^{+}$ and $m = \lceil \beta \rceil$ ($\lceil \cdot \rceil$ denoting the ceiling function). The (left-sided) Riemann-Liouville derivative D_{a+}^{β} of order β and terminal a is defined by*

$$D_{a+}^{\beta}h(x) := D^m I_{a+}^{m-\beta}h(x), \quad \beta > 0,\ \beta \notin \mathbb{N},\ x > a. \tag{2.1.2}$$

where D^m denotes the classical mth derivative of integer-order $m \in \mathbb{N}$.

An alternative fractional differential operator is the *left-sided Caputo operator* defined as follows (see, e.g., [15, Chapter 3] for details).

Definition 2.1.3. *The Caputo derivative D_{a+*}^{β} of order β is defined by*

$$D_{a+*}^{\beta}h(x) := I_{a+}^{m-\beta}D^m h(x), \quad \beta > 0,\ \beta \notin \mathbb{N}, \quad x > a,\ m = \lceil \beta \rceil, \tag{2.1.3}$$

A sufficient condition for $D_{a+}^{\beta}h$ to be well-defined is to assume that $h \in A^m[a,b]$, i.e., its derivatives of order $m-1$ are absolutely continuous (see, e.g., [15, Lemma 2.12]), whereas the Caputo operator is well-defined at least on functions h with absolute integrability of its derivatives of order $m = \lceil \beta \rceil$. It can be proved (see, e.g., [15, Theorem 3.1]) that, for $h \in A^m[a,b]$ and any non-integer $\beta > 0$, both fractional differential operators are related by the equality

$$D^{\beta}_{a+*}h(x) = D^{\beta}_{a+}[h - T_{m-1}[h;a]](x), \tag{2.1.4}$$

where $T_{m-1}[h;a]$ denotes the Taylor expansion of order $m-1$, centered at a, for the function h. Hence, in general

$$D^{\beta}_{a+}h(x) := D^m I^{m-\beta}_{a+}h(x) \quad \neq \quad I^{m-\beta}_{a+}D^m h(x) =: D^{\beta}_{a+*}h(x),$$

unless the function $h(x)$ along with its first $m-1$ derivatives vanish at $a+$ (or as $x \to -\infty$ whenever $a = -\infty$).

Remark 2.1.1. *When $a = 0$, the integral operator defined in (2.1.1) is equivalent to Riemann's definition [76] and is referred to by some authors as the Riemann integral operator. The case $a = -\infty$ is also known as the Liouville's definition, so that it is sometimes referred to as the Liouville operator or the Weyl's integral operator.*

Remark 2.1.2. *Other different notions of fractional derivatives known in the literature include the Grunwald-Letnikov, the Riesz, the Weyl, the Marchaud, and the Miller-Ross (or sequential) fractional derivatives (see references cited above). Moreover, numerous generalizations (mostly from an analytical point of view) have been proposed by many authors, we refer, e.g., to [2], [36], [41], [46] [47], [75].*

Remark 2.1.3. *The left-sided derivatives have a direct counterpart to the right-sided versions (see previous references for details).*

Special case: $\beta \in (0,1)$

Throughout this work, we are mostly interested in the generalizations of fractional derivatives of order $\beta \in (0,1)$. In this case, equations (2.1.2) and (2.1.3) become

$$D^{\beta}_{a+}h(x) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\left(\int_a^x (x-y)^{-\beta}h(y)dy\right), \quad x > a,$$

and

$$D^{\beta}_{a+*}h(x) = \frac{1}{\Gamma(1-\beta)}\int_a^x (x-y)^{-\beta}h'(y)dy, \quad x > a,$$

respectively. Further, for smooth enough functions h (e.g., h in the Schwartz space), integration by parts allows us to derive the following expressions [55, Appendix])

$$D^{\beta}_{a+}h(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a} \frac{h(x-y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \quad x > a,$$

and

$$D^{\beta}_{a+*}h(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a} \frac{h(x-y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)-h(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \quad x > a.$$

Here we use $\Gamma(-\beta) = -\Gamma(1-\beta)/\beta$ for $\beta \in (0,1)$. Thus, for $\beta \in (0,1)$, the relationship between the Caputo and the RL derivates given in (2.1.4) translates to

$$D^{\beta}_{a+*}h(x) = D^{\beta}_{a+}[\,h-h(a)](x) = D^{\beta}_{a+}h(x) - \frac{h(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \quad x > a.$$

For smooth bounded integrable functions or functions that vanish at $x = a$ (or as $x \to -\infty$ whenever $a = -\infty$), the previous equality implies that the Caputo derivative and the RL derivative coincide. Its common value for $a = -\infty$, denoted also by d^{β}/dx^{β}, is sometimes called *the generator form of the fractional derivative of order* $\beta \in (0,1)$, [67] and is given by

$$\frac{d^{\beta}}{dx^{\beta}}h(x) := D^{\beta}_{-\infty+}h(x) = D^{\beta}_{-\infty+*}h(x) = \frac{1}{\Gamma(-\beta)}\int_0^{\infty} \frac{h(x-y)-h(x)}{y^{1+\beta}}dy. \qquad (2.1.5)$$

2.2 Definition of generalized fractional operators

The generalized fractional operators studied here can be thought of as an extension (from a probabilistic point of view) of the classical Caputo and Riemann-Liouville fractional derivatives when applied to sufficiently regular functions. They can be obtained as the infinitesimal generators of Feller processes interrupted on an attempt to cross a boundary point.

These operators are defined in terms of a function ν that, probabilistically, plays the role of a jump density. Namely, let $\nu : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^+$ be a function of two variables. The next condition we will be always assumed when dealing with generalized fractional operators.

(H0): The function $\nu(x, y)$ is continuous as a function of two variables and continuously differentiable in the first variable. Furthermore,

$$\sup_x \int \min\{1, |y|\}\nu(x, y)dy < \infty, \quad \sup_x \int \min\{1, |y|\}\Big|\frac{\partial}{\partial x}\nu(x, y)\Big|dy < \infty,$$

and

$$\lim_{\delta \to 0} \sup_x \int_{|y| \le \delta} |y|\nu(x, y)dy = 0.$$

Remark 2.2.1. *The uniform boundedness, the tightness property and the regularity conditions on ν as stated in assumption (H0) are technical conditions which allow us to guarantee the existence of the corresponding jump-type process, as well as to guarantee that continuously differentiable functions form a core for the generator, see, e.g., [53, Theorem 5.1.1].*

Definition 2.2.1. *Let $a, b \in \mathbb{R}$ with $a < b$. For any function ν satisfying the condition (H0), the operators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{b-*}$ defined by*

$$\left(-D^{(\nu)}_{a+*}h\right)(x) = \int_0^{x-a} (h(x-y) - h(x))\nu(x, y)dy + (h(a) - h(x)) \int_{x-a}^{\infty} \nu(x, y)dy, \tag{2.2.1}$$

for functions $h : [a, \infty) \to \mathbb{R}$, and by

$$\left(-D^{(\nu)}_{b-*}h\right)(x) = \int_0^{b-x} (h(x+y) - h(x))\nu(x, y)dy + (h(b) - h(x)) \int_{b-x}^{\infty} \nu(x, y)dy, \tag{2.2.2}$$

for functions $h : (-\infty, b] \to \mathbb{R}$, are called the generalized left-sided Caputo type operator and the generalized right-sided Caputo type operator, respectively.
The operators $-D^{(\nu)}_{a+}$ and $-D^{(\nu)}_{b-}$ defined by

$$\left(-D^{(\nu)}_{a+}h\right)(x) = \int_0^{x-a} (h(x-y) - h(x))\nu(x,y)dy - h(x)\int_{x-a}^{\infty} \nu(x,y)dy, \quad (2.2.3)$$

for functions $h : [a, \infty) \to \mathbb{R}$, and by

$$\left(-D^{(\nu)}_{b-}h\right)(x) = \int_0^{b-x} (h(x+y) - h(x))\nu(x,y)dy - h(x)\int_{b-x}^{\infty} \nu(x,y)dy, \quad (2.2.4)$$

for functions $h : (-\infty, b] \to \mathbb{R}$, are called the generalized left-sided Riemann-Liouville type derivative and the generalized right-sided Riemann-Liouville type derivative, respectively. The values a and b will be referred to as the terminals of the generalized fractional derivatives.

Due to assumption (H0), the operators (2.2.1) and (2.2.3) are well defined at least on the space of continuously differentiable functions (with bounded derivative) on $[a, \infty)$, whereas (2.2.2) and (2.2.4) are well-defined on continuously differentiable functions (with bounded derivative) on $(-\infty, b]$.

Remark 2.2.2. *The sign $-$ appearing in the notation of the generalized fractional operators is introduced to comply with the standard notation of fractional derivatives. Notice also that to define the operators (2.2.1)-(2.2.4) the function $\nu(x, \cdot)$ needs to be defined only on $\mathbb{R}^+ \setminus \{0\}$ rather than $\mathbb{R} \setminus \{0\}$. The latter case will be used in Chapter 5 for the definition of two-sided operators*

Observe that the left-sided (resp. the right-sided) RL and Caputo type derivatives with the same terminals coincide on functions h vanishing at a (resp. at b) yielding the relations

$$\left(-D^{(\nu)}_{a+*}h\right)(x) = -D^{(\nu)}_{a+}[h - h(a)](x) = \left(-D^{(\nu)}_{a+}h\right)(x) + h(a)\int_{x-a}^{\infty} \nu(x,y)dy, \quad (2.2.5)$$

and

$$\left(-D^{(\nu)}_{b-*}h\right)(x) = -D^{(\nu)}_{b+}[h - h(b)](x) = \left(-D^{(\nu)}_{b-}h\right)(x) + h(b)\int_{b-x}^{\infty}\nu(x,y)dy. \quad (2.2.6)$$

Moreover, if $h(x)$ is the constant function, say equal to $K \in \mathbb{R}$, then

$$-D^{(\nu)}_{a+*}K = 0, \quad \text{and } -D^{(\nu)}_{a+}K = -K\int_{t-a}^{\infty}\nu(x,y)dy,$$

and

$$-D^{(\nu)}_{b-*}K = 0, \quad \text{and } -D^{(\nu)}_{b-}K = -K\int_{b-t}^{\infty}\nu(x,y)dy.$$

Remark 2.2.3. *If the terminal $a = -\infty$ (resp. $b = +\infty$), then the operators $-D^{(\nu)}_{-\infty+*}$ and $-D^{(\nu)}_{-\infty+}$ (resp. $-D^{(\nu)}_{+\infty-*}$ and $-D^{(\nu)}_{+\infty-}$) coincide on functions vanishing at infinity. Moreover, the operator $-D^{(\nu)}_{-\infty+*}$ (resp. $-D^{(\nu)}_{+\infty-*}$) takes the form of the generator of a jump-type process on $\mathbb{R}$ with only negative jumps (resp. with only positive jumps). These operators can be seen as the left- and right-sided generalizations of the Marchaud derivatives [76, Formulas 5.57-5.58], which are also referred to as the generator form of fractional derivatives (see equation 2.1.5).*

2.3 Probabilistic interpretation

Probabilistically, the generalized Caputo and RL type operators can be seen, respectively, as the generators of stopped and killed Feller processes. Namely, let $x, a \in \mathbb{R}$ and suppose $x > a$. Take a function ν satisfying (H0) and consider the *decreasing* Feller process $X^{+(\nu)}_x = \{X^{+(\nu)}_x(s)\}_{s\geq 0}$ starting at $x \in (a, b]$ and generated by the operator $(-G^{(\nu)}_+, \mathfrak{D}^{(\nu)}_G)$ defined by

$$-G^{(\nu)}_+h(x) = \int_0^{\infty}\left(h(x-y) - h(x)\right)\nu(x,y)dy, \quad f \in \mathfrak{D}^{(\nu)}_G, \quad (2.3.1)$$

where $\mathfrak{D}^{(\nu)}_G$ stands for the domain of the operator $-G^{(\nu)}_+$. If the natural motion of the (underlying) process $X^{+(\nu)}_x$ is interrupted in such a way that the process is forced to land exactly at the point a on its first attempt to leave the interval $(a,+\infty)$ (i.e., jumps aimed to land outside $[a,\infty)$ are forced to land exactly at a), then the corresponding *interrupted* process, say $X_x^{a+*} = \{X_x^{a+*}(s)\}_{s\geq 0}$, is a Feller process on $[a,\infty)$ and has the generator $-D_{a+*}^{(\nu)}$ with a domain, hereafter denoted by $\mathfrak{D}_{a+*}^{(\nu)}$, satisfying

$$\mathfrak{D}_{a+*}^{(\nu)} \subset \left\{ f \in C_\infty[a,\infty) : \left(-D_{a+*}^{(\nu)} f\right)(a) = 0 \right\}.$$

Furthermore, the generator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ has the space $C_\infty^1[a,+\infty) \subset \mathfrak{D}_{a+*}^{(\nu)}$ as an invariant core [55, Theorem 4.1].

Remark 2.3.1. *Since the process generated by $-G_+^{(\nu)}$ is decreasing, the interruption procedure effectively means stopping the process at the boundary point $x = a$. The point a can be seen as a barrier point for the underlying process $X_x^{+(\nu)}$ and this point coincides with the terminal of the generalized operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$.*

On the other hand, if the process is killed on crossing the barrier point a (meaning analytically to set $h(a) = 0$), then the corresponding (sub-Markov) process takes values on (a,∞) and has the generator $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$, where

$$\mathfrak{D}_{a+}^{(\nu)} \subset \left\{ f \in C_\infty[a,\infty) : f(a) = 0 \right\}.$$

Further, the space $C_\infty^1[a,\infty)$ is an invariant core for the generator $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$. Analogously, one can obtain the probabilistic interpretation of the right-sided operators (2.2.2) and (2.2.4), wherein the underlying (*increasing*) process has the generator $(-G_-^{(\nu)}, \mathfrak{D}_G^{-(\nu)})$ given by

$$-G_-^{(\nu)} h(x) = \int_0^\infty \left(h(x+y) - h(x)\right) \nu(x,y) dy, \quad f \in \mathfrak{D}_G^{-(\nu)}, \tag{2.3.2}$$

and the barrier point is taken to be $x = b$.

Thus, the Caputo type operators (resp. the RL type operators) arise as generators of decreasing Feller processes *stopped* (resp. *killed*) on an attempt to cross a given barrier point determined by the terminals of the operators.

Remark 2.3.2. *Note that* $-D_{a+}^{(\nu)} f = -D_{a+*}^{(\nu)} f$ *whenever* $f(a) = 0$*, which implies that* $\mathfrak{D}_{a+}^{(\nu)} \subset \mathfrak{D}_{a+*}^{(\nu)}$.

By definition of the generator of a Feller process, if $S_s^{a+*(\nu)}$ is the semigroup of the process $X_x^{a+*(\nu)}$, then $u \in \mathfrak{D}_{a+*}^{(\nu)}$ if, and only if,

$$-D_{a+*}^{(\nu)} u = \lim_{s \downarrow 0} \frac{S_s^{a+*(\nu)} u - u}{s},$$

where the limit is in the sense of the norm in $C[a,b]$. Analogously, if $S_s^{a+(\nu)}$ is the semigroup of the killed process $X_x^{a+(\nu)}$, then $u \in \mathfrak{D}_{a+}^{(\nu)}$ if, and only if,

$$-D_{a+}^{(\nu)} u = \lim_{s \downarrow 0} \frac{S_s^{a+(\nu)} u - u}{s},$$

where the limit is in the sense of the norm in $C_a[a,b]$. The latter denoting the space of continuous functions on $[a,b]$ vanishing at a.

On the other hand, by the standard theory of Feller processes ([40, Theorem 4]), the domains of the generators $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ and $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$ coincide with the images of their corresponding resolvent operators, denoted (for any $\lambda > 0$) by $R_\lambda^{a+*(\nu)}$ and $R_\lambda^{a+(\nu)}$, respectively (see Appendix, definition (A.1.2)). Namely,

$$\mathfrak{D}_{a+*}^{(\nu)} = \left\{ u_g : u_g(x) = R_\lambda^{a+*(\nu)} g(x),\ \ g \in C[a,b] \right\},$$

and

$$\mathfrak{D}_{a+}^{(\nu)} = \left\{ u_g : u_g(x) = R_\lambda^{a+(\nu)} g(x),\ g \in C_a[a,b] \right\}.$$

Moreover, the images of the resolvent operators are independent of λ (see, e.g., [17], [40, Theorem 1]). Therefore, $u \in \mathfrak{D}_{a+*}^{(\nu)}$ if, and only if, there exists $g \in C[a,b]$ such that $u(x) = R_\lambda^{a+*(\nu)} g(x)$. Analogously, $w \in \mathfrak{D}_{a+}^{(\nu)}$ if, and only if, there exists $g \in C_a[a,b]$ such that $w(x) = R_\lambda^{a+(\nu)} g(x)$. Hence, the functions u and w are the unique *solution in the domain of the generator* to the *resolvent equation* (see Theorem A.1.1 in

Appendix)

$$-D_{a+*}^{(\nu)}u(x) = \lambda u(x) - g(x), \quad g \in C[a,b],$$

and

$$-D_{a+}^{(\nu)}w(x) = \lambda w(x) - g(x), \quad g \in C_a[a,b],$$

respectively.

Remark 2.3.3. *Since we are interested in the solutions to generalized fractional differential equations on finite intervals, we will only consider the operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$ acting on functions defined on the interval $[a,b]$ instead of $[a,\infty)$ as was done originally in [55, Theorem 4.1].*

2.4 Particular cases

2.4.1 Caputo and RL fractional derivatives of order $\beta \in (0,1)$.

The classical fractional derivatives are particular cases of the operators (2.2.1)-(2.2.4). Namely, on smooth enough functions h,

$$\text{if} \quad \nu(x,y) = -\frac{1}{\Gamma(-\beta)y^{1+\beta}}, \quad \beta \in (0,1), \quad \text{then} \quad \begin{cases} -D_{a+*}^{(\nu)}h &= -D_{a+*}^{\beta}h, \\ -D_{a+}^{(\nu)}h &= -D_{a+}^{\beta}h, \\ -D_{b-*}^{(\nu)}h &= -D_{b-*}^{\beta}h, \\ -D_{b-}^{(\nu)}h &= -D_{b-}^{\beta}h, \end{cases} \tag{2.4.1}$$

where D_{a+*}^{β} and D_{a+}^{β} stand for the *left-sided Caputo derivative* and the *left-sided RL derivative* of order $\beta \in (0,1)$, respectively. Notation D_{b-*}^{β} and D_{b-}^{β} denote the corresponding right-sided versions. Hence,

$$\left(D_{a+*}^{\beta}h\right)(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{h(x-y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)-h(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{2.4.2}$$

$$\left(D_{a+}^{\beta}h\right)(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{h(x-y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{2.4.3}$$

and

$$\left(D_{b-*}^{\beta}h\right)(x) = \frac{1}{\Gamma(-\beta)}\int_0^{b-x}\frac{h(x+y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)-h(b)}{\Gamma(1-\beta)(b-x)^{\beta}}, \tag{2.4.4}$$

$$\left(D_{b-}^{\beta}h\right)(x) = \frac{1}{\Gamma(-\beta)}\int_0^{b-x}\frac{h(x+y)-h(x)}{y^{1+\beta}}dy + \frac{h(x)}{\Gamma(1-\beta)(b-x)^{\beta}}. \tag{2.4.5}$$

Here we use $\Gamma(-\beta) = -\Gamma(1-\beta)/\beta$ for $\beta \in (0,1)$.

As mentioned before, for smooth enough functions h, the expressions in (2.4.2)-(2.4.5) can be obtained from the standard analytical definitions (2.1.2)-(2.1.3) (see Section 2.1).

Therefore, the classical fractional derivatives are particular cases of the previous interruption procedure applied to $\beta-$stable subordinators. More precisely, the Caputo fractional derivative $-D_{a+*}^{\beta}$ can be seen as the generator of a Feller process on $[a,b]$ which is obtained by stopping an inverted β-stable subordinator $X^{+\beta}$ (see Appendix for the definition) on an attempt to cross the boundary point $x = a$. Similarly, the RL derivative $-D_{a+}^{\beta}$ can be thought of as the generator of a Feller (sub-Markov) process obtained by killing $X^{+\beta}$ upon leaving $(a,+\infty)$.

Remark 2.4.1. *The probabilistic extensions of fractional derivatives of order* $\beta \in (1,2)$ *were also introduced in [55], but this case is not considered in this work.*

2.4.2 Fractional derivatives of variable order

For a given function $\beta : \mathbb{R} \to (0,1)$, define

$$\nu(x,y) = -\frac{1}{\Gamma(-\beta(x))y^{1+\beta(x)}}. \tag{2.4.6}$$

Lemma 2.4.2. *If* $\beta : \mathbb{R} \to (0,1)$ *is a continuously differentiable function with values on a compact subset of* $(0,1)$*, then the function defined in (2.4.6) satisfies condition (H0).*

Proof. Follows by the smoothness of the function β in a compact set of $(0,1)$ and the fact that the Lévy density (2.4.1) satisfies (H0). ∎

Lemma 2.4.2 allows us to define the *Caputo and RL type operators of variable order*, denoted by $-D^{(\nu)}_{a+*} \equiv -D^{\beta(x)}_{a+*}$ and $-D^{(\nu)}_{a+} \equiv -D^{\beta(x)}_{a+}$, respectively. They can be thought of as the generators of inverted stable-like processes (see, e.g., [3],[53]) with the jump density (2.4.6) which are stopped (resp. killed) on an attempt to cross the boundary point $x = a$.

2.4.3 Multi-term fractional derivatives

Other particular cases of the generalized fractional derivatives include the *multi-term fractional operators*:

$$-D^{(\nu)}_{a+*}h(x) = -\sum_{i=1}^{d} \omega_i(x) D^{\beta_i}_{a+*}h(x), \tag{2.4.7}$$

with nonnegative functions $\omega_i(\cdot) \geq 0$ for $i \in \{1, \ldots, d\}$, where

$$\nu(x,y) = -\sum_{i=1}^{d} \omega_i(x) \frac{1}{\Gamma(-\beta_i) y^{1+\beta_i}}.$$

Even more generally, they include the *generalized distributed order fractional derivatives*:

$$\left(-D^{(\nu)}_{a+*}h\right)(x) := \int_{-\infty}^{\infty} \omega(s,x) \left(-D^{\beta(s,x)}_{a+*}h\right)(x)\, \mu(ds), \quad \beta(\cdot,\cdot) \in (0,1). \tag{2.4.8}$$

where

$$\nu(x,y) = -\int_{-\infty}^{\infty} \omega(s,x) \frac{\mu(ds)}{\Gamma(-\beta(s,x)) y^{1+\beta(s,x)}}.$$

Special cases of (2.4.8) have been studied by analytical methods, e.g., in [64], [31].

Observe that, according to Definition 2.2.1, the operators in (2.4.7)-(2.4.8) are well defined as generalized fractional operators as long as the functions $\omega_i(\cdot)$, $\omega(\cdot,\cdot)$, $\mu(\cdot)$

and $\beta(\cdot,\cdot)$ are such that the corresponding function ν satisfies condition (H0).

We can also define the left-sided (or right-sided) version of the *tempered fractional derivatives* [75] by taking the function ν as

$$\nu(x,y) \equiv \nu(y) = \frac{\beta}{\Gamma(1-\beta)} e^{-\lambda y} y^{-\beta-1}, \quad \beta \in (0,1).$$

In this case, the operator $-D_{a+*}^{(\nu)}$ can be thought of as the generator of a *tempered stable process* [5] interrupted on an attempt to cross the boundary point a.

Remark 2.4.3. *Some other extensions can be considered by taking, for instance, a function ν of the form:*

$$\nu(t,r;x) = \frac{\beta(t,x)}{\Gamma(1-\beta(t,x)) r^{1+\beta(t,x)}}, \tag{2.4.9}$$

for some "external" variable $x \in \mathbb{R}^d$. This type of function allows us to deal with operators of the form

$$\left(-{}_t\tilde{D}_a^{\beta(t,x)} - A_x^{(t)} \right) f(t,x), \quad t \geq a, \ x \in \mathbb{R}^d, \tag{2.4.10}$$

where $-{}_t\tilde{D}_a^{\beta(t,x)}$ denotes either the Caputo or the RL type derivative acting on the variable t and depending on the variable x as a parameter; and $-A_x^{(t)}$ denotes the generator of a Feller process acting on the variable x and depending on the variable t as a parameter. This type of operator is not analyzed in this work.

2.5 Properties of the underlying stochastic processes

In this section we study the underlying stochastic processes generated by the operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$. These results shall be used in the following chapters to obtain the explicit solutions to equations involving these operators.

For a given function ν satisfying condition (H0) and for $x \in (a,b]$, the following notation will be used hereafter:

(i) $X_x^{+(\nu)} = \{X_x^{+(\nu)}(s) : s \geq 0\}$ denotes the *underlying* decreasing Feller process (started at x) generated by the operator $(-G_+^{(\nu)}, \mathfrak{D}_G^{(\nu)})$ as given in (2.3.1).

(ii) $X_x^{a+*(\nu)} = \{X_x^{a+*(\nu)}(s) : s \geq 0\}$ stands for the (*interrupted*) Feller process generated by the operator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ with the invariant core $C^1[a,b]$.

(iii) $X_x^{a+(\nu)} = \{X_x^{a+(\nu)}(s) : s \geq 0\}$ denotes the Feller sub-Markov process generated by the operator $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$ with the invariant core $C_a^1[a,b]$.

(iv) For $x \in [a,b]$, notation $\tau_a^{(\nu)}(x)$ refers to the first time the underlying process $X_x^{+(\nu)}$ leaves the interval $(a,+\infty)$, i.e.

$$\tau_a^{(\nu)}(x) := \inf\left\{s \geq 0 : X_x^{+(\nu)}(s) \notin (a,+\infty)\right\},$$

and, of course, $\tau_a^{(\nu)}(a) = 0$.

(v) Notation $p_s^{+(\nu)}(x,E)$, $p_s^{a+(\nu)}(x,E)$ and $p_s^{a+*(\nu)}(x,E)$ denote the transition probabilities (from the state x to a Borel set E) of the processes $X^{+(\nu)}$, $X^{a+(\nu)}$ and $X^{a+*(\nu)}$, respectively.

The following (rather simple) facts hold:

1. If $E \subset \mathfrak{B}(a,b]$ and $x \in (a,b]$, then $p_s^{+(\nu)}(x,E) = p_s^{a+*(\nu)}(x,E) = p_s^{a+(\nu)}(x,E)$. Moreover,

$$p_s^{a+*(\nu)}(x,\{a\}) = p_s^{+(\nu)}(x,(-\infty,a]), \quad x \in (a,b].$$

and

$$p_s^{a+(\nu)}(x,(a,x]) = p_s^{+(\nu)}(x,(a,x]) = 1 - p_s^{+(\nu)}(x,(-\infty,a]) \leq 1.$$

2. The r.v. $\tau_a^{(\nu)}(x)$ is a stopping time with respect to the natural filtration $\mathcal{F}_s^{X_x^{+(\nu)}}$. Furthermore, the distribution of $\tau_a^{(\nu)}(x)$ coincides with the distribu-

tion of the first exit time from $(a,b]$ of both the killed process $X_x^{a+(\nu)}$ for $x > a$ and the process $X_x^{a+*(\nu)}$ for $x \geq a$. Therefore, we will use the same notation interchangeably.

Sometimes we will use the following additional assumptions concerning the function ν and the transition probabilities of the underlying process $X^{+(\nu)}$:

(H1): There exist $C > 0$ and $q \in (0,1)$ such that

$$\int_0^\infty \min\{y,\epsilon\}\nu(a,y)dy > C\epsilon^q. \tag{2.5.1}$$

(H2): The transition probabilities of the process $X^{+(\nu)}$ are absolutely continuous with respect to the Lebesgue measure (the transition densities are denoted by $p_s^{+(\nu)}(x,y)$).

(H3): The transition density function $p_s^{+(\nu)}(x,y)$ is continuously differentiable in the variable s with bounded derivative.

Remark 2.5.1. *Some important comments about the previous assumptions:*

- *Assumption (H1) is a technical condition to ensure the regularity of the boundary point "a" for a given interval $[a,b]$, see Definition 2.5.1 and Lemma 2.5.2 below.*

- *There exist several criteria that ensure the validity of assumptions (H2) and (H3). If the process $X^{+(\nu)}$ is a Lévy process, then the conditions to guarantee the existence and smoothness of transition densities are well-known. See, e.g., reference [50] for a quick account of conditions stated on the characteristic exponent of the corresponding process. For certain Lévy-type processes conditions on the function ν have been studied via Malliavin calculus as can be seen, e.g., in reference [38].*

- *Condition (H3) is a technical condition to ensure the existence of a density function for the random variable $\tau_a^{(\nu)}(x)$, as well as for joint distributions* *including it. See, e.g., Proposition 2.5.4 and Proposition 2.5.5 below.*

Let us also introduce the notion of *regularity* needed for the boundary points (see, e.g., [53, Chapter 6]).

Definition 2.5.1. *For a domain $D \subset \mathbb{R}$ with boundary ∂D, a point $x_0 \in \partial D$ is said to be regular in expectation for a Markov process X (or for its generator) if*

$$\mathbf{E}\left[\tau_D(x)\right] \to 0, \quad x \to x_0,\ x \in D,$$

where $\tau_D(x)$ is the first exit time from D of the process X starting at $x \in D$, i.e. $\tau_D(x) := \inf\{s \geq 0 : X_x(s) \notin D\}$, with the usual convention that $\inf\{\varnothing\} = \infty$.

Lemma 2.5.2. *Suppose that the conditions (H0)-(H1) hold for a function ν. Then, (i) the stopping time $\tau_a^{(\nu)}(x)$ is finite a.s., the point a is regular in expectation for both operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$ and $\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] < +\infty$ uniformly on $x \in (a,b]$. (ii) the expectation of $\tau_a^{(\nu)}(x)$ is given by*

$$\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] = \int_0^\infty \int_a^x p_s^{+(\nu)}(x,y)\, dy\, ds, \quad x \in (a,b]. \tag{2.5.2}$$

Proof. (i) The regularity in expectation of $\tau_a^{+(\nu)}(x)$ and the fact that $\mathbf{E}\left[\ \tau_a^{+(\nu)}(x)\right]$ is finite for all $x \in (a,b]$ is a direct consequence of Theorem 4.1 in [55]. We will repeat here the proof of this statement in order to illustrate the use of Lyapunov method for proving regularity of boundary points as this method will be used again in this work to prove similar results.

By Proposition 6.3.2 in reference [53], to prove the regularity of a for the generator $-D_{a+*}^{(\nu)}$ (equivalently, for the process $X_x^{a+*(\nu)}$) it is sufficient to find a continuous function f on $[a,b]$ and a neighborhood of a, say V_a, such that f is differentiable on (a,b), $f(a) = 0$, and $f(x) > 0$ implies $-D_{a+*}^{(\nu)} f(x) < -c$ for all $x \in (a,b) \cap V_a$ and some positive constant c. As such function we can take $f_\omega(x) = (x-a)^\omega$ for some $\omega \in (0,1)$. This function is differentiable on (a,b), vanishes at the boundary point a and it is positive on (a,b). Hence, to conclude the regularity of the point a, we only need to ensure that $-D^{(\nu)}_{a+*} f_\omega(x) < -c$ for $x \in (a,b) \cap V_a$ and for some positive constant c. To prove the latter condition, observe that for x approaching a from the right, $-D^{(\nu)}_{a+*} f(x)$ is of order

$$-\omega\,(x-a)^{\omega-1} \int_0^\infty \min\{y, x-a\}\nu(x,y)dy. \tag{2.5.3}$$

Thus, taking $\omega = 1-q$, where q is as given in Condition (H1) for $\epsilon = x-a$, we obtain that the term (2.5.3) is bounded away from 0 as $x \to a$. Notice that passing to the limit in the previous term is justified by condition (H0), which also guarantees the continuity of the function ν. The previous implies that $-D^{(\nu)}_{a+*} f(x) < -c$ for all x in a neighborhood of a and some positive constant c, as required.

(ii) It follows from the equality

$$\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] = \int_0^\infty \mathbf{P}\left[\ \tau_a^{(\nu)}(x) > s\right] ds,$$

and the equivalence between the events $\left\{\tau_a^{(\nu)}(x) > s\right\}$ and $\left\{X_x^{+(\nu)}(s) > a\right\}$ for $x \in (a,b]$ and all $s > 0$ (due to the monotonicity of the process $X_x^{+(\nu)}$), implying

$$\mathbf{P}\left[\tau_a^{(\nu)}(x) > s\right] = \mathbf{P}\left[X_x^{+(\nu)}(s) > a\right] = \int_a^x p_s^{+(\nu)}(x,y)dy = 1 - \int_{-\infty}^a p_s^{+(\nu)}(x,y)dy.$$

■

Remark 2.5.3. *An important consequence of assumption (H1) and the regularity in expectation of the boundary point a is the following upper bound for the expectation of $\tau_a^{(\nu)}(x)$, which is given in terms of the function f_ω (see Proposition 6.3.2 in reference [53]):*

$$\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] \;<\; C f_\omega(x) \;=\; C(x-a)^\omega, \tag{2.5.4}$$

for some constant $C > 0$ and some $\omega \in (0,1)$.

The calculations in the proof of the statement (ii) of Lemma 2.5.2 yield the following result.

Proposition 2.5.4. *Suppose that the conditions (H0)-(H3) hold. Then, the probability law of $\tau_a^{(\nu)}(x)$, denoted by $\mu_a^{x,(\nu)}(ds)$, is absolutely continuous with respect to the Lebesgue measure for $x \in (a,b]$ and its density $\mu_a^{x,(\nu)}(s)$ is given by*

$$\mu_a^{x,(\nu)}(s) = \frac{\partial}{\partial s}\int_{-\infty}^{a} p_s^{+(\nu)}(x,y)dy = -\frac{\partial}{\partial s}\int_{a}^{x} p_s^{+(\nu)}(x,y)dy. \tag{2.5.5}$$

We will also need the joint distribution of $X_x^{a+*(\nu)}(s)$ and $\tau_a^{(\nu)}(x)$ for any $s \geq 0$. Notice that for any $a \leq y < x$,

$$\mathbf{P}\left[X_x^{a+*(\nu)}(s) > y,\, \tau_a^{(\nu)}(x) > \xi\right] = \mathbf{P}\left[X_x^{a+*(\nu)}(s) > y,\, X_x^{a+*(\nu)}(\xi) > a\right].$$

Moreover, $\xi \leq s$ implies

$$\mathbf{P}\left[X_x^{a+*(\nu)}(s) > y,\, X_x^{a+*(\nu)}(\xi) > a\right] = \mathbf{P}\left[X_x^{a+*(\nu)}(s) > y\right],$$

whilst for $s < \xi$,

$$\begin{aligned}\mathbf{P}\left[X_x^{a+*(\nu)}(s) > y,\, X_x^{a+*(\nu)}(\xi) > a\right] &= \int_y^x p_s^{a+*(\nu)}(x,w)\left(\int_a^w p_{\xi-s}^{a+*(\nu)}(w,z)dz\right)dw \\ &= \int_y^x p_s^{+(\nu)}(x,w)\left(1 - \int_{-\infty}^a p_{\xi-s}^{+(\nu)}(w,z)dz\right)dw.\end{aligned}$$

Therefore, defining

$$\varphi_{s,a}^{x,(\nu)}(y,\xi) := \frac{\partial^2}{\partial\xi\partial y}\mathbf{P}\left[X_x^{a+*(\nu)}(s) \leq y,\, \tau_a^{(\nu)}(x) \leq \xi\right],$$

yields the next result.

Proposition 2.5.5. *Suppose that the conditions (H0)-(H3) hold. Then, for any $s \geq 0$ and $x \in (a,b]$, the joint distribution of the pair $\left(X_x^{a+*(\nu)}(s), \tau_a^{(\nu)}(x)\right)$, denoted by $\varphi_{s,a}^{x,(\nu)}(dy,d\xi)$, has the density $\varphi_{s,a}^{x,(\nu)}(y,\xi)$ given by*

$$\varphi_{s,a}^{x,(\nu)}(y,\xi) = \mathbf{1}_{\{s<\xi\}} p_s^{+(\nu)}(x,y)\frac{\partial}{\partial\xi}\int_{-\infty}^{a} p_{\xi-s}^{+(\nu)}(y,z)dz, \qquad a \leq y < x. \tag{2.5.6}$$

Remark 2.5.6. *Since the processes $X_x^{a+(\nu)}$, $X_x^{a+*(\nu)}$ and $X_x^{+(\nu)}$ coincide before the first exit time $\tau_a^{(\nu)}(x)$, the equation (2.5.6) provides the joint density of the pairs $\left(X_x^{a+(\nu)}(s), \tau_a^{(\nu)}(x)\right)$ and $\left(X_x^{+(\nu)}(s), \tau_a^{(\nu)}(x)\right)$ for any $s \geq 0$ and for any $s < \xi$, respectively.*

Let us now introduce an operator which will play an important role to characterize the domain of the generators $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ and $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$.
For any $\lambda \geq 0$ and for (non constant) functions $g \in B[a,b]$, define

$$M_{a,\lambda}^{+(\nu)} g(x) := \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g\left(X_x^{+(\nu)}(s)\right) ds\right], \quad x \in (a,b], \tag{2.5.7}$$

and for $g(x) \equiv 1(x)$ (the constant function 1) define

$$M_{a,\lambda}^{+(\nu)} 1(x) := \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} ds\right], \quad x \in [a,b]. \tag{2.5.8}$$

Then

$$M_{a,\lambda}^{+(\nu)} 1(x) = \frac{1}{\lambda}\left(1 - \mathbf{E}\left[e^{-\lambda \tau_a^{(\nu)}(x)}\right]\right), \tag{2.5.9}$$

implying

$$\mathbf{E}\left[e^{-\lambda \tau_a^{(\nu)}(x)}\right] = 1 - \lambda M_{a,\lambda}^{+(\nu)} 1(x).$$

Further, the equality

$$M_{a,\lambda}^{+(\nu)} c = c M_{a,\lambda}^{+(\nu)} 1(x), \quad x \in [a,b],$$

holds for any constant function equals to c (we shall use it mainly for the constant $g(a)$). Note also that both the inequality

$$|M_{a,\lambda}^{+(\nu)} g(x)| \leq \|g\| \mathbf{E}\left[\tau_a^{(\nu)}(x)\right], \tag{2.5.10}$$

and the estimate $\mathbf{E}[\tau_a^{(\nu)}(x)] < C f_\omega(x)$ for the continuous function $f_\omega(x) = (x-a)^\omega$ (see Remark 2.5.3) imply that $M_{a,\lambda}^{(\nu)} g(\cdot)$ is continuous on $[a,b]$.

The explicit formulae below will be important in the following chapters.

Lemma 2.5.7. *Suppose that ν satisfies conditions (H0)-(H3). Then*

$$\mathbf{E}\Big[e^{-\lambda\tau_a^{(\nu)}(x)}\Big] = \int_0^\infty e^{-\lambda s}\left(\frac{\partial}{\partial s}\int_{-\infty}^a p_s^{+(\nu)}(x,y)dy\right)ds, \quad x \in (a,b]. \tag{2.5.11}$$

Further, for any $g \in B[a,b]$

$$M_{a,\lambda}^{+(\nu)}g(x) = \int_0^{x-a} g(x-y)\int_0^\infty e^{-\lambda s}p_s^{+(\nu)}(x,x-y)\,ds\,dy, \quad x \in (a,b]. \tag{2.5.12}$$

Proof. Equality (2.5.11) follows directly by using the density function $\mu_a^{x,(\nu)}$ of the r.v. $\tau_a^{(\nu)}(x)$ as given in (2.5.5). To prove (2.5.12), observe that Fubini's theorem allows one to rewrite $M_{a,\lambda}^{+(\nu)}g(x)$ as

$$M_{a,\lambda}^{+(\nu)}g(x) = \int_0^\infty e^{-\lambda s}\mathbf{E}\left[\mathbf{1}_{\{\tau_a^{(\nu)}(x)>s\}}g\left(X_x^{+(\nu)}(s)\right)\right]ds.$$

Using (2.5.6), i.e., the joint density $\varphi_{s,a}^{x,(\nu)}(y,\xi)$ of the process $\left(X_x^{+(\nu)}(s),\tau_a^{(\nu)}(x)\right)$ for $s<\xi$, it follows that

$$\begin{aligned}
M_{a,\lambda}^{+(\nu)}g(x) &= \int_0^\infty e^{-\lambda s}\left[\int_a^x\int_0^\infty \mathbf{1}_{\{\xi>s\}}g(y)\,\varphi_{s,a}^{x,(\nu)}(y,\xi)\,d\xi\,dy\right]ds\\
&= \int_0^\infty e^{-\lambda s}\int_a^x g(y)p_s^{+(\nu)}(x,y)\int_s^\infty\left(\frac{\partial}{\partial\xi}\int_{-\infty}^a p_{\xi-s}^{+(\nu)}(y,z)dz\right)d\xi\,dy\,ds\\
&= \int_0^\infty e^{-\lambda s}\int_a^x g(y)p_s^{+(\nu)}(x,y)\int_0^\infty\left(\frac{\partial}{\partial\gamma}\int_{-\infty}^a p_{\gamma}^{+(\nu)}(y,z)dz\right)d\gamma\,dy\,ds\\
&= \int_0^\infty e^{-\lambda s}\int_a^x g(y)p_s^{+(\nu)}(x,y)\int_0^\infty \mu_a^{y,(\nu)}(\gamma)d\gamma\,dy\,ds\\
&= \int_0^\infty e^{-\lambda s}\int_a^x g(y)p_s^{+(\nu)}(x,y)\,dy\,ds,
\end{aligned}$$

where the last equality holds as $\mu_a^{y,(\nu)}$ is the density function of the r.v. $\tau_a^{(\nu)}(y)$. The result follows then by another interchange in the order of integration and by a change of variable. ∎

Remark 2.5.8. *Using integration by parts we can transfer the derivative in the r.h.s of equality (2.5.11) to the function* $\exp(-\lambda s)$. *Thus, equality (2.5.11) can be* *written as*

$$\mathbf{E}\left[e^{-\lambda\tau_a^{(\nu)}(x)}\right] = \lambda\int_0^{\infty} e^{-\lambda s}\left(\int_{-\infty}^{a} p_s^{+(\nu)}(x,y)dy\right)ds, \quad x \in (a,b]. \tag{2.5.13}$$

Let us now define the space of functions

$$\mathfrak{M}_{\mathfrak{a},\lambda}^{+(\nu)} := \left\{ u : u(x) = cM_{a,\lambda}^{+(\nu)}1(x) + d;\ x \in [a,b],\ c,d \in \mathbb{R}\right\}. \tag{2.5.14}$$

Lemma 2.5.9. *Let ν be a function satisfying conditions (H0)-(H1). If $\lambda > 0$, then*

$$\mathfrak{D}_{a+*}^{(\nu)} = \left\{u_g : u_g(x) = g(a)\frac{1}{\lambda}\left(1 - \lambda M_{a,\lambda}^{+(\nu)}1(x)\right) + M_{a,\lambda}^{+(\nu)}g(x), \quad g \in C[a,b]\right\},$$

and

$$\mathfrak{D}_{a+}^{(\nu)} = \left\{w_g : w_g(x) = M_{a,\lambda}^{+(\nu)}g(x), \quad g \in C_a[a,b]\right\}.$$

Further, if ν also satisfies (H2)-(H3), then equalities (2.5.11) and (2.5.12) give explicit expressions for $\left(1 - \lambda M_{a,\lambda}^{+(\nu)}1(x)\right)$ and $M_{a,\lambda}^{+(\nu)}g(x)$, respectively.

Proof. Let us take any $u \in \mathfrak{D}_{a+*}^{(\nu)}$. Since $-D_{a+*}^{(\nu)}$ is the generator of a Feller process on $C[a,b]$, Theorem A.1.1 implies the existence of a function $g \in C[a,b]$ such that $u = R_\lambda^{a+*(\nu)}g$. By definition of the resolvent operator and by Fubini's theorem

$$u(x) = \mathbf{E}\left[\int_0^{\infty} e^{-\lambda s}g(X_x^{a+*(\nu)}(s))ds\right] = \mathbf{E}\left[\left(\int_0^{\tau_a^{(\nu)}(x)} + \int_{\tau_a^{(\nu)}(x)}^{\infty}\right)e^{-\lambda s}g(X_x^{a+*(\nu)}(s))ds\right],$$

where $\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] < +\infty$ by Lemma 2.5.2.
Since the process $X_x^{a+*(\nu)}$ is absorbed at a by time $\tau_a^{(\nu)}(x)$, the equality $g\left(X_x^{a+*(\nu)}(s)\right) = g(a)$ holds for all $s \geq \tau_a^{(\nu)}(x)$. Moreover, before time $\tau_a^{(\nu)}(x)$ the paths of the processes $X_x^{a+*(\nu)}$ and $X_x^{+(\nu)}$ coincide. Therefore,

$$u(x) = g(a)\mathbf{E}\left[\int_{\tau_a^{(\nu)}(x)}^{\infty} e^{-\lambda s}ds\right] + \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s}g(X_x^{+(\nu)}(s))ds\right]$$

$$= g(a)\left\{\frac{1}{\lambda} - \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} ds\right]\right\} + \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g(X_x^{+(\nu)}(s))ds\right]$$

$$= g(a)\frac{1}{\lambda}\left(1 - \lambda M_{a,\lambda}^{+(\nu)} 1(x)\right) + M_{a,\lambda}^{+(\nu)} g(x), \tag{2.5.15}$$

as required.

The characterization of the domain $\mathfrak{D}_{a+}^{(\nu)}$ is similar to the previous case. Take any $w \in \mathfrak{D}_{a+}^{(\nu)}$, then there exists a function $g \in C_a[a,b]$ such that $w = R_\lambda^{a+(\nu)} g$. Hence, a similar procedure yields (2.5.15) which implies (since $g(a) = 0$)

$$R_\lambda^{a+(\nu)} g(x) = M_{a,\lambda}^{+(\nu)} g(x).$$

Finally, observe that under assumptions (H2)-(H3), Lemma 2.5.7 holds. ∎

Let us now see how the resolvents (and hence the domains) of the processes $X_x^{a+*(\nu)}$ and $X_x^{a+(\nu)}$ are related.

Lemma 2.5.10. *Let ν be a function satisfying condition (H0). Suppose $\lambda > 0$ and $g \in C[a,b]$. Define $\tilde{g}(x) = g(x) - g(a)$, then*

$$R_\lambda^{a+(\nu)} \tilde{g}(x) = R_\lambda^{a+*(\nu)} \tilde{g}(x) = R_\lambda^{a+*(\nu)} g(x) - g(a) R_\lambda^{a+*(\nu)} 1(x),$$

and

$$R_\lambda^{a+(\nu)} \tilde{g}(x) = M_{a,\lambda}^{+(\nu)} g(x) - g(a) M_{a,\lambda}^{+(\nu)} 1(x). \tag{2.5.16}$$

In particular, $R_\lambda^{a+(\nu)} \tilde{g}(x)$ belongs to both domains $\mathfrak{D}_{a+*}^{(\nu)}$ and $\mathfrak{D}_{a+}^{(\nu)}$.

Proof. Follows directly from the linearity of the operators $R_\lambda^{a+*(\nu)}$ and $M_{a,\lambda}^{(\nu)}$, and by using that $\tilde{g}(a) = 0$. ∎

Remark 2.5.11. *Let us stress that Lemma 2.5.10 implies that $M_{a,\lambda}^{+(\nu)} g$ coincides with the resolvent $R_\lambda^{a+(\nu)} g$ only when the function $g(a) = 0$. Hence, only in this case $M_{a,\lambda}^{+(\nu)} g$ belongs to the domain of both generators $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ and $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$.*

Chapter 3

Linear equations of RL and Caputo type

This chapter provides a probabilistic approach to solve linear equations involving Caputo and Riemann-Liouville type derivatives. Using the probabilistic interpretation of these operators as the generators of interrupted Feller processes, we obtain well-posedness results and explicit solutions (in terms of the transition densities of the underlying stochastic processes).

3.1 Introduction

Existence and uniqueness results for fractional ordinary differential equations (FODE's) have been studied for various spaces of functions including Lebesgue integrable functions, continuous functions and continuously differentiable functions. We refer, e.g. to [45, Chapter 3] for a detailed account of the main works on this topic.

Unlike the standard analytical techniques to study FODE's, in this chapter we employ probabilistic arguments to study linear equations involving generalized Caputo and Riemann-Liouville type operators. Namely, the study is based on looking at the given equation as a Dirichlet type problem associated with the operator $-D^{(\nu)}_{a+*}$ seen as the generator of a stochastic process. The equations analyzed in this chapter are:

(i) the linear equation with the Caputo type operator:

$$-D_{a+*}^{(\nu)}u(x) = \lambda u(x) - g(x), \quad x \in (a,b], \quad u(a) = u_a, \tag{3.1.1}$$

for a given $\lambda \geq 0$, a bounded function g and $u_a \in \mathbb{R}$. The relationship between Caputo and RL type operators (see equality (2.2.5)), allows us also to study the corresponding problem with the RL type operator:

$$-D_{a+}^{(\nu)}u(x) = \lambda u(x) - g(x), \quad x \in (a,b], \quad u(a) = 0, \tag{3.1.2}$$

(ii) the generalized *mixed fractional linear equation*

$$-\sum_{i=1}^{d} \tilde{D}^{(\nu_i)}\, u(x_1,\ldots,x_d) = \lambda u(x_1,\ldots,x_d) - g(x_1,\ldots,x_d), \tag{3.1.3}$$

with some prescribed boundary condition, where $-\tilde{D}^{(\nu_i)}$ denotes either the RL type operator $-{}_{x_i}D_{a_i+}^{(\nu_i)}$ or the Caputo type operator $-{}_{x_i}D_{a_i+*}^{(\nu_i)}$. The left subscript x_i indicates that the operator is acting on the variable x_i.

Fractional linear differential equations with Caputo derivatives of order $\beta \in (0,1)$ are particular cases of equation (3.1.1). They have been extensively investigated by means of the Laplace transform method [15], [45], [73],[76]. Hence, it is known that the equation

$$D_{a+*}^{\beta}u(x) = -\lambda u(x) + g(x), \quad u(a) = u_a, \quad \lambda \in \mathbb{R}, \tag{3.1.4}$$

for $\beta \in (0,1)$ and a given continuous function g on $[a,b]$, has the unique solution

$$u(x) = u_a E_\beta\left[-\lambda(x-a)^\beta\right] + \int_a^x g(y)(x-y)^{\beta-1}E_{\beta,\beta}\left(-\lambda(x-y)^\beta\right)dy, \tag{3.1.5}$$

where E_β and $E_{\beta,\beta}$ denote the *Mittag-Leffler functions* (see definitions in Appendix). The solution (3.1.5) can be written in terms of β–stable densities by means of the integral representation of the Mittag-Leffler functions given in (3.2.13). The probabilistic approach introduced here gives this expression directly once one writes down the expectations involved in the general stochastic representation (3.2.11). On the other hand, using the results obtained here and the uniqueness of solutions, we obtain a pure probabilistic proof of the well-known equality in (3.2.13).

Apart from the classical Caputo derivatives, operators $-D^{(\nu)}_{a+*}$ include, as simple particular cases, the multi-term fractional derivatives $\sum_{i=1}^{d}\omega_i(x)D^{\beta_i}_{a_i+*}u(x)$ with non-negative functions ω_i. Hence, as another example of (3.1.1), our approach also applies to the *multi-term fractional equation*

$$\sum_{i=0}^{k}\omega_i(x)D^{\beta_i}_{a+*}u(x) = -\lambda u(x) + g(x), \qquad \beta_i \in (0,1),\ x \in (a,b], \tag{3.1.6}$$

with some given functions g and ω_i, for $i \in \{1,\ldots,k\}$. The explicit solution to (3.1.6) when the functions ω_i are constants and (3.1.6) is a *commensurate equation* (i.e., the quotients β_i/β_j are rational numbers for all i,j), has been analyzed by reducing the equation to either a single- or multi-order fractional differential equation system (see, e.g., [15] and references therein). An approximation for its solution has been also studied, e.g., in, [18]. Our approach encompasses not only the commensurate case with constant coefficients ω_i but also the more general case with nonconstant coefficients $\omega_i(\cdot)$ and, even more generally, functions $\beta_i(x)$.

The fractional counterpart of equation (3.1.3) is the *mixed* fractional equation

$$-\ {}_{x_1}D^{\beta}_{0+}\,u(x_1,x_2) - {}_{x_2}D^{\alpha}_{0+*}\,u(x_1,x_2) = \lambda u(x_1,x_2) - g(x_1,x_2), \quad \beta,\alpha \in (0,1), \tag{3.1.7}$$

subject to some boundary condition, where g is a given function on $[0,b_1]\times[0,b_2]$. The probabilistic approach presented here provides the explicit solution in terms of $\beta-$ and $\alpha-$stable densities. To the best of our knowledge, mixed fractional equations of the type in (3.1.7) involving classical fractional Caputo and RL derivatives of order in $(0,1)$ have not been explored explicitly in the literature. The same arguments also apply to study the well-posedness for the d–dimensional case

$$-\sum_{i=1}^{d} \tilde{D}^{\beta_i} u(x_1,\ldots,x_d) = \lambda u(x_1,\ldots,x_d) - g(x_1,\ldots,x_d), \tag{3.1.8}$$

with $\tilde{D}^{\beta_i}$ being either the RL or the Caputo derivative.

It is worth mentioning that equations involving Caputo and RL type operators do not usually have solutions in the domain of the operators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$ as generators of Feller processes. The existence of such solutions is restricted to a specific value in the boundary condition. Thus, as usual in classical stochastic analysis, by introducing the concept of a *generalized solution* we are able to study the well-posedness (in a generalized sense) for these equations. To illustrate this concept, consider the very-well known ordinary differential equation (ODE)

$$u'(x) = \lambda u(x) + g(x), \quad x \in (0,b], \quad u(0) = u_0,$$

where $b > 0$ and $g \in C[0,b]$, whose solution is

$$u(x) = u_0 e^{\lambda x} + \int_0^x \exp\{\lambda(x-y)\} g(y)dy, \quad x \in (0,b]. \tag{3.1.9}$$

Probabilistically, this problem can be thought of as the boundary value problem associated with the deterministic linear motion on $(-\infty,b]$ which is stopped at reaching the boundary point $x = 0$. In this case, the semigroup $\{S_s\}_{s\geq 0}$ of the deterministic process is given by $S_s f(x) = f(x-s)$ for any $x \in (-\infty,b]$ and $f \in C_\infty(-\infty,b]$, whilst the semigroup $\{S_s^{0+*}\}_{s\geq 0}$ of the stopped process corresponds to $S_s^{0+*} f(x) = f(\max\{0, x-s\})$ for any $x \in [0,b]$ and $f \in C[0,b]$. Hence, the resolvent operator of the semigroup S_s^{0+*} provides the function (3.1.9) as the unique *solution in the domain of the generator* (the space $C^1[0,b]$) if, and only if, $u_0 = \frac{1}{\lambda} g(0)$. Otherwise this solution is (in our terminology) only a *generalized solution* as it can be obtained as a limit of solutions taken from the domain of the generator. Moreover, in this case the generalized solution is also a *classical (smooth) solution* lying on $C[0,b] \cap C^1(0,b]$

instead of $C^1[0,b]$. Similar situations occur when considering fractional differential equations. We will see that the solutions found in the literature are usually solutions in the generalized sense, as they generally do not belong to the domain of $-D^{\beta}_{a+*}$ or $-D^{\beta}_{a+}$ as generators of Feller processes.

The main contribution of this chapter relies on providing well-posedness results and explicit integral representations for the solutions to linear equations with Caputo and RL type operators. We deal with the existence of two types of solutions in a probabilistic framework: *solutions in the domain of the generator* and *generalized solutions*. The latter concept defined for rather general, even not continuous, functions g in (3.1.1). Moreover, all solutions are given in terms of expectations of functionals of Markov processes. From the point of view of numerical analysis, this representation can be exploited to obtain numerical solutions to a variety of problems by performing Monte Carlo techniques. Simulation methods have been effectively used for classical differential equations and, in recent years, different methods for evaluating path functionals of Lévy processes have been actively researched (see, e.g., [22], [23], [56]).

3.2 Linear equations involving generalized fractional operators

The probabilistic representation of the solutions (in the generalized sense) to linear equations involving RL and Caputo type operators are studied in this section.

3.2.1 Linear equation of RL type

Consider the problem of finding a continuous function u on $[a,b]$ satisfying

$$-D^{(\nu)}_{a+}w(x) = \lambda w(x) - g(x), \quad x \in [a,b], \qquad w(a) = w_a, \tag{3.2.1}$$

for $\lambda \geq 0$, $g \in B[a,b]$ and $w_a = 0$. Hereafter, we refer to (3.2.1) as the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g, w_a)$ for which we always assume $w_a = 0$. Similar notation shall be used for the linear equation of Caputo type: $(-D_{a+*}^{(\nu)}, \lambda, g, w_a)$ for any $w_a \in \mathbb{R}$. Let us also recall that notation $C_a[a,b]$ denotes the space of continuous functions on $[a,b]$ vanishing at a.

Definition 3.2.1. *Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $w \in C_a[a,b]$ is said to solve the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ as*

(i) a solution in the domain of the generator *if w satisfies (3.2.1) and belongs to the domain of the generator $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$;*

(ii) a generalized solution *if for all sequence of functions $g_n \in C_a[a,b]$ such that $\sup_n \|g_n\| < \infty$ uniformly on n and $\lim_{n\to\infty} g_n \to g$ a.e., it holds that $w(x) = \lim_{n\to\infty} w_n(x)$ for all $x \in [a,b]$, where w_n is the solution (in the domain of the generator) to the RL problem $(-D_{a+}^{(\nu)}, \lambda, g_n, 0)$;*

(iii) a smooth solution *if u is a generalized solution belonging to $C_a[a,b] \cap C^1(a,b)$.*

Remark 3.2.1. *From this definition it follows that if there exists a generalized solution, then the solution is unique.*

Definition 3.2.2. *The RL type equation (3.2.1) is well-posed in the generalized sense if it has a unique generalized solution.*

Well-posedness results for the RL type linear equation.

Theorem 3.2.2. *(**Case** $\lambda > 0$) Let ν be a function satisfying conditions (H0)-(H1) and assume $\lambda > 0$.*

(i) If $g \in C_a[a,b]$, then the linear problem $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ has a unique solution in the domain of the generator given by $w = R_\lambda^{a+(\nu)} g$ (the resolvent operator at λ associated with the generator $-D_{a+}^{(\nu)}$).

(ii) For any $g \in B[a,b]$, the linear equation $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$w(x) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g\left(X_x^{+(\nu)}(s) ds\right)\right]. \tag{3.2.2}$$

Moreover, if additionally ν satisfies conditions (H2)-(H3), then

$$w(x) = \int_0^{x-a} g(x-y) \int_0^{\infty} e^{-\lambda s} p_s^{+(\nu)}(x, x-y)\, ds\, dy. \tag{3.2.3}$$

(iii) If $g \in C[a,b]$, then the solution to (3.2.1) belongs to $\mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{+(\nu)}$, the direct sum of the domain of the generator $-D_{a+}^{(\nu)}$ and the space defined in (2.5.14).

Proof. (i) Take $g \in C_a[a,b]$. Using the conditions $g(a) = 0$, $w(a) = 0$ and $\lambda > 0$ together with the fact that the operator $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$ is the generator of a Feller process on $C_a[a,b]$, Theorem A.1.1 implies directly that $w(x) = R_\lambda^{a+(\nu)} g(x)$ is the unique solution to (3.2.1) belonging to the domain of the generator. Moreover, Lemma 2.5.9 implies

$$w(x) = M_{a,\lambda}^{+(\nu)} g(x) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g\left(X_x^{+(\nu)}(s)\right) ds\right].$$

(ii) Let us now take any function $g \in B[a,b]$. Since g does not necessarily belong to $C_a[a,b]$, the resolvent operator no longer provides a solution to (3.2.1). However, using Definition 3.2.1 we will see that there exists a unique *generalized solution*. To do this, take any sequence $g_n \in C_a[a,b]$ such that $\lim_{n\to\infty} g_n = g$ a.e. and $\sup_n \|g_n\| < \infty$. The procedure consists in finding the generalized solution as a limit of solutions to the equations

$$-D_{a+}^{(\nu)} w_n(x) = \lambda w_n(x) - g_n(x), \quad x \in (a,b], \qquad w_n(a) = 0.$$

Since each $g_n \in C_a[a,b]$, the previous case guarantees the existence of a unique solution $w_n \in \mathfrak{D}_{a+}^{(\nu)}$ given by

$$w_n(x) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g_n\left(X_x^{+(\nu)}(s)\right) ds\right].$$

Using that $\|g_n\|$ is uniformly bounded, the dominated convergence theorem (DCT) implies

$$\lim_{n\to\infty} w_n(x) = \mathbb{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g\left(X_x^{+(\nu)}(s)\right) ds\right] =: w(x).$$

The continuity of w follows from the fact that $w(\cdot) = M_{a,\lambda}^{+(\nu)} g(\cdot)$ and $M_{a,\lambda}^{+(\nu)} g(\cdot)$ is continuous on $[a,b]$ under assumption (H1) (due to the inequality (2.5.10), the regularity in expectation of a and the upper bound of the expectation of $\tau_a^{(\nu)}$ in (2.5.4)). Therefore, $w \in C_a[a,b]$ is a generalized solution to the linear equation (3.2.1). Finally, the representation in (3.2.3) follows directly from Lemma 2.5.7.

(iii) To prove that $w \in \mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{(\nu)}$ whenever $g \in C[a,b]$, we use the equality (2.5.16) in Lemma 2.5.10 to obtain

$$M_{a,\lambda}^{+(\nu)} g(x) = R_\lambda^{a+(\nu)} \hat{g}(x) + g(a) M_{a,\lambda}^{+(\nu)} 1(x),$$

where $\hat{g}(x) = g(x) - g(a)$, implying the result. ∎

Theorem 3.2.3. ***(Case*** $\lambda = 0$***)*** *Theorem 3.2.2 is valid for* $\lambda = 0$*, i.e. for the equation*

$$-D_{a+}^{(\nu)} w(x) = -g(x), \quad x \in (a,b]; \qquad w(a) = 0. \tag{3.2.4}$$

Proof. The proof follows similar arguments to those used for $\lambda > 0$, so that we skip the details. Let us just notice that, since the potential operator $R_0^{a+(\nu)}$ associated with $-D_{a+}^{(\nu)}$ satisfies

$$|R_0^{a+(\nu)} g(x)| \le \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} |g(X_x^{a+(\nu)}(s))| ds\right] \le \|g\| \sup_{x\in(a,b]} \mathbf{E}\left[\tau_a^{(\nu)}(x)\right],$$

Lemma 2.5.2 implies the boundedness of $R_0^{a+(\nu)}$. Thus, Theorem A.1.2 ensures that the proof of Theorem 3.2.2 remains true for $\lambda = 0$ if one replaces the resolvent operator $R_\lambda^{a+(\nu)}$ by the potential operator $R_0^{a+(\nu)}$. Also observe that $w \in \mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,0}^{+(\nu)}$ whenever $g \in C[a,b]$ since w can be written as

$$w(x) = R_0^{a+(\nu)}\tilde{g}(x) + g(a)\mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} ds\right],$$

where $\tilde{g}(x) := g(x) - g(a)$, for all $x \in [a,b]$. ■

3.2.2 Linear equation of Caputo type

Let $a \in \mathbb{R}$ and $\lambda \geq 0$. Consider the problem of finding a function $u \in C[a,b]$ solving

$$-D_{a+*}^{(\nu)}u(x) = \lambda u(x) - g(x), \quad x \in (a,b], \quad u(a) = u_a, \tag{3.2.5}$$

for a given function g on $[a,b]$.

Let us observe that the linear equation (3.2.5) can be written in terms of the RL type operator $D_{a+}^{(\nu)}$ as follows. Define $w(x) := u(x) - u_a$ for all $x \in [a,b]$, then $-D_{a+*}^{(\nu)}w(x) = -D_{a+*}^{(\nu)}u(x)$ as $-D_{a+*}^{(\nu)}u_a = 0$. Setting $\tilde{g}(x) := g(x) - \lambda u_a$, it follows that

$$-D_{a+}^{(\nu)}w(x) = \lambda w(x) - \tilde{g}(x), \quad x \in (a,b] \quad w(a) = 0. \tag{3.2.6}$$

Hence, $u(x) = w(x) + u_a$ is a solution to the original problem if, and only if, w solves (3.2.6). The previous discussion motivates the following definition.

Definition 3.2.3. *Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $u \in C[a,b]$ is said to solve the Caputo type equation $(-D_{a+*}^{(\nu)}, \lambda, g, u_a)$ as*

(i) a solution in the domain of the generator *if w satisfies (3.2.5) and belongs to the domain of the generator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$;*

(ii) a generalized solution if $u(x) = u_a + w(x)$ for all $x \in [a,b]$, where w is the (possibly generalized) solution to the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g - \lambda u_a, 0)$;

(iii) a smooth solution *if u is a generalized solution belonging to $C[a,b] \cap C^1(a,b]$.*

Remark 3.2.4. *The concept of a generalized solution in Definition 3.2.3 is given in terms of a RL type solution. Equivalently, one can define a generalized solution for Caputo type equations in terms of approximating solutions taken from the domain of the (Caputo type) generator* $(-D^{(\nu)}_{a+*}, \mathfrak{D}^{(\nu)}_{a+*})$ *for the case* $\lambda > 0$.

Definition 3.2.4. *The Caputo type equation (3.2.5) is well-posed in the generalized sense if it has a unique generalized solution depending continuously on the initial condition.*

Well-posedness results for the Caputo type linear equation.

Theorem 3.2.5. ***(Case*** $\lambda > 0$***)*** *Let* ν *be a function satisfying conditions (H0)-(H1) and suppose* $\lambda > 0$.

(i) If $g \in C[a,b]$ *and* $g(a) = \lambda u_a$*, then the linear equation* $(-D^{(\nu)}_{a+*}, \lambda, g, u_a)$ *has a unique solution in the domain of the generator given by* $u = R^{a+*(\nu)}_{\lambda} g$ *(the resolvent operator at* λ *associated with the operator* $-D^{(\nu)}_{a+*}$*).*

(ii) For any $g \in B[a,b]$ *and* $u_a \in \mathbb{R}$*, the linear equation* $(-D^{(\nu)}_{a+*}, \lambda, g, u_a)$ *is well-posed in the generalized sense and the solution admits the stochastic representation*

$$u(x) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^{(\nu)}(x)}\right] + \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} e^{-\lambda s} g\left(X_x^{+(\nu)}(s) ds\right)\right]. \tag{3.2.7}$$

Moreover, if additionally ν *satisfies conditions (H2)-(H3), then*

$$u(x) = u_a \int_0^\infty e^{-\lambda s} \mu_a^{x,(\nu)}(s) ds + \int_0^{x-a} g(x-y) \int_0^\infty e^{-\lambda s} p_s^{+(\nu)}(x, x-y)\, ds\, dy, \tag{3.2.8}$$

where $\mu_a^{x,(\nu)}(s)$ *denotes the density function of the r.v.* $\tau_a^{(\nu)}(x)$ *as given in* (2.5.5).

(iii) If $g \in C[a,b]$*, then the solution to (3.2.5) belongs to* $\mathfrak{D}^{(\nu)}_{a+*} \oplus \mathfrak{M}^{(\nu)}_{a,\lambda}$*, the direct sum of the domain of the generator* $-D^{(\nu)}_{a+*}$ *and the space defined in (2.5.14).*

Proof. (i) Since $(-D^{(\nu)}_{a+*}, \mathfrak{D}^{(\nu)}_{a+})$ is the generator of a Feller process on $C[a,b]$, for any $g \in C[a,b]$ the function $u(x) = R^{a+*(\nu)}_{\lambda} g(x)$ is the unique solution (in the domain of the generator) to the resolvent equation

$$-D^{(\nu)}_{a+*} u = \lambda u - g.$$

A simple calculation shows that $u(a) = R^{a+*(\nu)}_{\lambda} g(a) = g(a)/\lambda$. Hence, condition $g(a) = \lambda u_a$ ensures that u satisfies the boundary condition $u(a) = u_a$, as required.

(ii) By Definition 3.2.3, $u(x) = w(x) + u_a$ is the generalized solution to (3.2.5), where w is the solution to the RL type problem $(-D^{(\nu)}_{a+}, \lambda, g(x) - \lambda u_a, 0)$ whose well-posedness is guaranteed by Theorem 3.2.2.
The solution is then given by (3.2.2) which can be rewritten in terms of the operator $M^{+(\nu)}_{a,\lambda}$ (see equality (2.5.7)) as $w = M^{+(\nu)}_{a,\lambda}[g - \lambda u_a]$. Thus, the linearity of $M^{+(\nu)}_{a,\lambda}$ yields

$$w(x) = M^{+(\nu)}_{a,\lambda} g(x) - \lambda u_a M^{+(\nu)}_{a,\lambda} 1(x) = M^{+(\nu)}_{a,\lambda} g(x) - u_a \left(1 - \mathbf{E}\left[e^{-\lambda \tau^{(\nu)}_a (x)}\right]\right),$$

where the last equality holds due to equation (2.5.9). Thus, (3.2.7) is obtained by plugging the previous expression into $u(x) = w(x) + u_a$. This representation implies directly the continuity on the initial condition u_a, as required for the well-posedness. Finally, the explicit solution (3.2.8) follows from Lemma 2.5.7.

(iii) Assume now that $g \in C[a,b]$. Since $u(x) = M^{+(\nu)}_{a,\lambda} g(x) - \lambda u_a M^{(\nu)}_{a,\lambda} 1(x) + u_a$, by linearity one can rewrite it as

$$\begin{aligned} u(x) &= M^{+(\nu)}_{a,\lambda}[g - g(a) + g(a)](x) - \lambda u_a M^{+(\nu)}_{a,\lambda} 1(x) + u_a \\ &= R^{a+*(\nu)}_{\lambda}[g - g(a)](x) + [g(a) - \lambda u_a] M^{+(\nu)}_{a,\lambda} 1(x) + u_a. \end{aligned}$$

We then conclude that $u \in \mathfrak{D}^{(\nu)}_{a+*} \oplus \mathfrak{M}^{+(\nu)}_{a,\lambda}$ since

$$R^{a+*(\nu)}_{\lambda}[g - g(a)] \;\in\; \mathfrak{D}^{(\nu)}_{a+*},$$

and

$$[g(a) + \lambda u_a] M^{+(\nu)}_{a,\lambda} 1(x) + u_a \;\in\; \mathfrak{M}^{+(\nu)}_{a,\lambda}.$$

■

Theorem 3.2.6. *(**Case** $\lambda = 0$) Theorem 3.2.5 remains valid with $\lambda = 0$, i.e., it holds for the equation*

$$-D^{(\nu)}_{a+*} u(x) = -g(x), \quad x \in (a,b], \qquad u(a) = u_a. \tag{3.2.9}$$

Proof. Since the problem (3.2.9) can be rewritten

$$-D^{(\nu)}_{a+} w(x) = -\tilde{g}(x), \quad x \in (a,b], \qquad w(a) = 0, \tag{3.2.10}$$

with $w(x) = u(x) - u_a$ and $\tilde{g}(x) = g(x)$, Theorem 3.2.3 gives the potential operator $R^{a+(\nu)}_0 \tilde{g}(x)$ as the solution to (3.2.10) for any $\tilde{g} \in C_a[a,b]$. Hence, the unique generalized solution to (3.2.9) is given by $u(x) = u_a + \lim_{n\to\infty} R^{a+(\nu)}_0 \tilde{g}_n(x)$ for any sequence $\tilde{g}_n$ satisfying the conditions given in Definition 3.2.3. Consequently, the same arguments used for $\lambda > 0$ remain valid. ■

3.2.3 Examples: classical fractional setting

Since Caputo derivatives are particular cases of the generalized fractional operators $-D^{(\nu)}_{a+*}$, the solution to fractional linear equations with the Caputo derivative D^{β}_{a+*}, for $\beta \in (0,1)$, is obtained by a direct application of the previous results. Namely, Theorem 3.2.5 implies that, for any $g \in B[a,b]$ and $\lambda > 0$, the problem

$$D^{\beta}_{a+*} u(x) = -\lambda u(x) + g(x), \quad x \in (a,b], \quad u(a) = u_a \in \mathbb{R},$$

has a unique generalized solution given by

$$u(x) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^\beta(x)}\right] + \mathbf{E}\left[\int_0^{\tau_a^\beta(x)} e^{-\lambda s} g(X_x^{+\beta}(s))ds\right], \tag{3.2.11}$$

where $X_x^{+\beta}$ is the inverted β−stable subordinator started at $x \in (a,b]$ (see definition in Appendix A.2). Moreover, using the transition densities of $X_x^{+\beta}$, which are given in terms of the density function $w_\beta(\cdot;1,1)$ of a standard β-stable random variable (see (A.2.1) and (A.2.4) in Appendix), formula (3.2.8) yields

$$\begin{aligned} u(x) = u_a \frac{1}{\beta}(x-a) \int_0^\infty e^{-\lambda s} \left(s^{-\frac{1}{\beta}-1} w_\beta\left((x-a)s^{-1/\beta};1,1\right)\right) ds + \\ + \int_0^{x-a} g(x-y) \left(y^{\beta-1} \int_0^\infty \exp\left\{-\lambda s y^\beta\right\} s^{-1/\beta} w_\beta(s^{-1/\beta};1,1) ds\right) dy. \end{aligned} \tag{3.2.12}$$

Further, if $g(a) = \lambda u_a$ and $g \in C[a,b]$, then u belongs to the domain of the generator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$.

The previous calculations imply the following new relationship between the Mittag-Leffler function $E_\beta(\cdot)$ and the Laplace transform of the first exit time $\tau_a^\beta(x)$ for $\beta \in (0,1)$.

Corollary 3.2.7. *Let $x \in (a,b]$ and $\lambda > 0$. Then the Laplace transform of the first exit time from $(a,b]$ for the inverted β−stable subordinator started at x is given by*

$$\boldsymbol{E}[e^{-\lambda \tau_a^\beta(x)}] = E_\beta(-\lambda(x-a)^\beta),$$

with E_β denoting the Mittag-Leffler function (see Appendix A.3). Further,

$$E_\beta(-\lambda(x-a)^\beta) = \frac{1}{\beta}(x-a) \int_0^\infty \exp(-\lambda s)\, s^{-\frac{1}{\beta}-1} w_\beta\left((x-a)s^{-1/\beta};1,1\right) ds.$$

Proof. By uniqueness, it follows as a consequence of formulas (3.1.5) and (3.2.12) with $g \equiv 0$ and $u_a = 1$. ∎

Remark 3.2.8. *Alternatively, Corollary 3.2.7 can also be obtained by using the identity (see [88], Theorem 2.10.2)*

$$\beta E_\beta(-z) = \int_0^\infty \exp(-zy) y^{-1-1/\beta} w_\beta(y^{-1/\beta}; 1, 1) dy. \tag{3.2.13}$$

Moreover, this identity also shows that (3.2.12) coincides with the well known solution given in (3.1.5).

3.3 Mixed linear equations

In this section we study linear equations involving both the RL type and the Caputo type operators, but each one acting on different variables. The general setting will be explained first in $\mathbb{R}^d$, and then we shall restrict ourselves to the simplest 2-dimensional case. This is done to avoid cumbersome calculations which nevertheless can be extended straightforward from the simple case analyzed here.

Let $\mathbf{a} = (a_1, \dots, a_d), \mathbf{b} = (b_1, \dots, b_d) \in \mathbb{R}^d$ such that $\mathbf{a} < \mathbf{b}$. The Euclidean space $\mathbb{R}^d$ is assumed to be equipped with its natural partial order, the Pareto order, i.e. $\mathbf{a} < \mathbf{b}$ means $a_i < b_i$ for all $i \in \{1, \dots, d\}$. Notation $[\mathbf{a}, \mathbf{b}]$ denotes the Cartesian product $[a_1, b_1] \times \cdots \times [a_d, b_d]$ and $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ means $x_i \in [a_i, b_i]$ for all $i \in \{1, \dots, d\}$. Let us denote by $B[\mathbf{a}, \mathbf{b}]$ and $C[\mathbf{a}, \mathbf{b}]$ the space of bounded Borel measurable functions and continuous functions on $[\mathbf{a}, \mathbf{b}]$, respectively, and by $C^1[\mathbf{a}, \mathbf{b}]$ the space of continuous functions on $[\mathbf{a}, \mathbf{b}]$ with continuous first order partial derivatives up to the boundary on $[\mathbf{a}, \mathbf{b}]$. Similar notation is used for $(\mathbf{a}, \mathbf{b}]$ and $(-\infty, \mathbf{b}]$.
Notation $\mathbf{y}^{a_i}$ means a vector $\mathbf{y} = (y_1, \dots, y_d) \in [\mathbf{a}, \mathbf{b}]$ having $y_i = a_i$ as its ith-coordinate. Since all the processes considered in this section have decreasing sample paths, we are only interested in the boundary of $(\mathbf{a}, \mathbf{b}]$ given by $\mathbf{y}^{a_i}$ for all $i \in \{1, \dots, d\}$. This subset is denoted by

$$\partial_{\mathbf{a}}(\mathbf{a}, \mathbf{b}] := \bigcup_{i=1}^d \{\mathbf{y} \in [\mathbf{a}, \mathbf{b}] : \mathbf{y} = \mathbf{y}^{a_i}\}.$$

The space of continuous functions on $[\mathbf{a},\mathbf{b}]$ vanishing at the boundary $\partial_{\mathbf{a}}(\mathbf{a},\mathbf{b}]$ is denoted by $C_{\mathbf{a}}[\mathbf{a},\mathbf{b}]$.

For $i \in \{1,\dots,d\}$, let ν_i be a function satisfying conditions (H0)-(H1) and let $\mathbf{x} \in (\mathbf{a},\mathbf{b}]$. The operator $-{}_{x_i}D^{(\nu_i)}_{a_i+}$ stands for the RL type operator defined by ν_i acting (independently of the other operators) on the variable x_i. For notational convenience set $\nu = (\nu_1,\dots,\nu_d)$ and define the *mixed RL type operator* associated with the vector ν by

$$-\mathbf{D}^{(\nu)}_{\mathbf{a}+} := -\sum_{i=1}^{d} {}_{x_i}D^{(\nu_i)}_{a_i+}, \tag{3.3.1}$$

Hence, the operator $-\mathbf{D}^{(\nu)}_{\mathbf{a}+}$ is a sum of RL type operators each one acting on a different variable. Analogously, we define the mixed operators $-\mathbf{G}^{(\nu)}_{+}$ and $-\mathbf{D}^{(\nu)}_{\mathbf{a}+*}$ by using $-{}_{x_i}G^{(\nu_i)}_{+}$ (see definition in (2.3.1)) and $-{}_{x_i}D^{(\nu_i)}_{a_i+*}$, respectively.

Consider the RL type linear equation

$$\begin{aligned} -\mathbf{D}^{(\nu)}_{\mathbf{a}+} w(\mathbf{x}) &= \lambda w(\mathbf{x}) - g(\mathbf{x}), && \mathbf{x} \in (\mathbf{a},\mathbf{b}], \\ w(\mathbf{x}) &= 0, && \mathbf{x} \in \partial_{\mathbf{a}}(\mathbf{a},\mathbf{b}], \end{aligned} \tag{3.3.2}$$

for a given function $g \in B[\mathbf{a},\mathbf{b}]$ and $\lambda \geq 0$.

The operator $-\mathbf{D}^{(\nu)}_{\mathbf{a}+}$ can be thought of as the generator $(-\mathbf{D}^{(\nu)}_{\mathbf{a}+}, \hat{\mathfrak{D}}^{(\nu)}_{\mathbf{a}+})$ (with a suitable domain $\hat{\mathfrak{D}}^{(\nu)}_{\mathbf{a}+}$) of a Feller process on $(\mathbf{a},\mathbf{b}]$ obtained by killing the process

$$\mathbf{X}^{+(\nu)}_{\mathbf{x}} = \left(X^{+(\nu_1)}_{x_1},\dots,X^{+(\nu_d)}_{x_d}\right),$$

on an attempt to cross the boundary $\partial_{\mathbf{a}}(\mathbf{a},\mathbf{b}]$, where $X^{+(\nu_i)}_{x_i}$ is the Feller process generated by $(-{}_{x_i}G^{(\nu_i)}_{+}, \mathfrak{D}^{(\nu_i)}_{G})$. Hence, $\mathbf{X}^{+(\nu)}_{\mathbf{x}}$ is the process generated by $-\mathbf{G}^{(\nu)}_{+}$ for each $i \in \{1,\dots,d\}$. The killed process (started at $\mathbf{x}$) shall be denoted by $\mathbf{X}^{\mathbf{a}+(\nu)}_{\mathbf{x}} = \left\{\mathbf{X}^{\mathbf{a}+(\nu)}_{\mathbf{x}}(s) : s \geq 0\right\}$ and is defined by

$$\mathbf{X}^{\mathbf{a}+(\nu)}_{\mathbf{x}}(s) := \mathbf{X}^{+(\nu)}_{\mathbf{x}}(s), \quad \text{for all} \quad s < \tau^{(\nu)}_{\mathbf{a}}(\mathbf{x}),$$

where

$$\tau_{\mathbf{a}}^{(\nu)}(\mathbf{x}) := \inf\{s \geq 0 : \mathbf{X}_{\mathbf{x}}^{+(\nu)}(s) \notin (\mathbf{a}, +\infty)\}.$$

Since the first exit time from $(\mathbf{a}, +\infty)$ occurs when one of the coordinate processes $X_{x_i}^{+(\nu_i)}$ leaves the interval $(a_i, +\infty)$, we have

$$\tau_{\mathbf{a}}^{(\nu)}(\mathbf{x}) = \min\left\{\tau_{a_i}^{(\nu_i)}(x_i),\ i \in \{1, \dots, d\}\right\}.$$

Hence,

$$\mathbf{X}_{\mathbf{x}}^{\mathbf{a}+(\nu)}(s) = \left(X_{x_1}^{a_1+(\nu_1)}(s), \dots, X_{x_d}^{a_d+(\nu_d)}(s)\right), \qquad s < \tau_{\mathbf{a}}^{(\nu)}(\mathbf{x})$$

wherein each coordinate $X_{x_i}^{a_i+(\nu_i)}$ is the Feller (sub-Markov) process generated by $(-{}_{x_i}D_{a_i+}^{(\nu_i)}, \mathfrak{D}_{a_i+}^{(\nu_i)})$. Hence, the process $\mathbf{X}_{\mathbf{x}}^{a+(\nu)}$ is also sub-Markov with a Feller semi-group $\mathbf{S}_s^{\mathbf{a}+(\nu)}$ on $C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$. Moreover, u belongs to the domain of the generator $\hat{\mathfrak{D}}_{\mathbf{a}+}^{(\nu)}$ if, and only if, the limit

$$-\mathbf{D}_{\mathbf{a}+}^{(\nu)} u(\mathbf{x}) = \lim_{s \to 0} \frac{\mathbf{S}_s^{\mathbf{a}+(\nu)} u(\mathbf{x}) - u(\mathbf{x})}{s},$$

exists in the norm of $C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$.

To solve (3.3.2), let us introduce some definitions which extend those used in the one-dimensional case.

Definition 3.3.1. *Let $g \in B[\boldsymbol{a}, \boldsymbol{b}]$, and $\lambda \geq 0$. A function $w \in C_{\boldsymbol{a}}[\boldsymbol{a}, \boldsymbol{b}]$ is said to solve the RL type problem $(-\boldsymbol{D}_{\mathbf{a}+}^{(\nu)}, \lambda, g, 0)$ as*

- *(i)* a solution in the domain of the generator *if w satisfies (3.3.2) and belongs to the domain of the generator $(-\mathbf{D}_{\boldsymbol{a}+}^{(\nu)}, \hat{\mathfrak{D}}_{\boldsymbol{a}+}^{(\nu)})$;*
- *(ii)* a generalized solution *if for all sequence of functions $g_n \in C_{\boldsymbol{a}}[\boldsymbol{a}, \boldsymbol{b}]$ such that $\sup_n \|g_n\| < \infty$ uniformly on n and $g_n \to g$ a.e., it holds that $w(\boldsymbol{x}) = \lim_{n\to\infty} w_n(\boldsymbol{x})$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, where w_n is the solution (in the domain of the generator) to the RL type problem $(-\boldsymbol{D}_{\mathbf{a}+}^{(\nu)}, \lambda, g_n, 0)$.*

Remark 3.3.1. *By definition, if there exists a generalized solution, then this is unique.*

For the sake of transparency, hereafter we restrict ourselves to the analysis for $d = 2$ and $\mathbf{a} = \mathbf{0}$. Namely, let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ with $\mathbf{x} \in [\mathbf{0}, \mathbf{b}]$. Consider the equation

$$\begin{aligned} -\ {}_{x_1}D_{0+}^{(\nu_1)} w(x_1, x_2) -\ {}_{x_2}D_{0+}^{(\nu_2)} w(x_1, x_2) &= \lambda w(x_1, x_2) - g(x_1, x_2), \\ w(0, x_2) = w(x_1, 0) &= 0, \end{aligned}$$

where $x_i \in (0, b_i]$ for $i \in \{1, 2\}$.
Let $p_s^{+(\nu_i)}(x_i, y)$ (resp. $p_s^{0+(\nu_i)}(x_i, y)$) denote the transition density function of the process $X_{x_i}^{+(\nu_i)}$ (resp. $X_{x_i}^{0+(\nu_i)}$). If $\tau_0^{(\nu_i)}(x_i)$ is the first exit time from $(0, b_i]$ of the process $X_{x_i}^{+(\nu_i)}$ (started at x_i), then the first exit time from $(\mathbf{0}, \mathbf{b}] = (0, b_1] \times (0, b_2]$ of the process $\mathbf{X}_{\mathbf{x}}^{+(\nu)}$, denoted by $\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})$, is given by

$$\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x}) = \min\left\{ \tau_0^{(\nu_i)}(x_i) \ : \ i \in \{1, 2\} \right\}.$$

Due to the independence between the coordinates of the process $\mathbf{X}_{\mathbf{x}}^{+(\nu)}$, its transition density function, denoted by $\mathbf{p}_s^{+(\nu)}(\mathbf{x}, \mathbf{y})$, satisfies

$$\mathbf{p}_s^{+(\nu)}(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{2} p_s^{+(\nu_i)}(x_i, y_i), \qquad \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2),$$

yielding the following result.

Lemma 3.3.2. *Let $\boldsymbol{x} = (x_1, x_2) \in (0, b_1] \times (0, b_2]$. Suppose (H0)-(H1) hold for both functions ν_1 and ν_2. Then,*

(i) The boundary points $(0, x_2) \in \mathbb{R}^2$ for all $x_2 \in [0, b_2)$, and $(x_1, 0) \in \mathbb{R}^2$ for all $x_1 \in [0, b_1)$, are regular in expectation for both operators $-\boldsymbol{D}_{\boldsymbol{0}+}^{(\nu)}$ and $-\boldsymbol{D}_{\boldsymbol{0}+*}^{(\nu)}$. Moreover, $\mathbf{E}\left[\tau_{\boldsymbol{0}}^{(\nu)}(\boldsymbol{x})\right] < +\infty$ uniformly on $\boldsymbol{x}$.

(ii) If additionally each ν_i satisfies assumptions (H2)-(H3) and $\mu_{\boldsymbol{0}}^{\boldsymbol{x},(\nu)}(ds)$ denotes

the probability law of $\tau_{\boldsymbol{0}}^{(\nu)}(\boldsymbol{x})$*, then its density function* $\mu_{\boldsymbol{0}}^{\boldsymbol{x},(\nu)}(s)$ *is given by*

$$\mu_{\boldsymbol{0}}^{\boldsymbol{x},(\nu)}(s) = \mu_0^{x_1,(\nu_1)}(s)\int_0^{x_2} p_s^{+(\nu_2)}(x_2,y)+\mu_0^{x_2,(\nu_2)}(s)\int_0^{x_1} p_s^{+(\nu_1)}(x_1,y), \quad s \geq 0.$$

(iii) Further, assuming again that each ν_i *also satisfies (H2)-(H3), the joint distribution* $\varphi_{s,\boldsymbol{a}}^{\boldsymbol{x},(\nu)}(d\boldsymbol{y},d\xi)$ *of the pair* $\left(\boldsymbol{X}_{\boldsymbol{x}}^{\boldsymbol{0}+(\nu)}(s),\tau_{\boldsymbol{0}}^{(\nu)}(\boldsymbol{x})\right)$ *has the density*

$$\varphi_{s,\boldsymbol{0}}^{\boldsymbol{x},(\nu)}(\boldsymbol{y},\xi) = \varphi_{s,0}^{x_2,(\nu_2)}(y_2,\xi)p_s^{+(\nu_1)}(x_1,y_1)\int_0^{y_1} p_{\xi-s}^{+(\nu_1)}(y_1,z)dz + \\ + \varphi_{s,0}^{x_1,(\nu_1)}(y_1,\xi)p_s^{+(\nu_2)}(x_2,y_2)\int_0^{y_2} p_{\xi-s}^{+(\nu_2)}(y_2,z)dz,$$

for $0 \leq s < \xi$ *and* $\boldsymbol{y} = (y_1,y_2)$, $\boldsymbol{y} \in (\boldsymbol{0},\boldsymbol{x}]$.

Proof. (i) The regularity in expectation of the boundary $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$ is a consequence of assumption (H1) and the Lyapunov method applied to the Lyapunov function

$$h_\omega(x_1,x_2) = x_1^{\omega_1}x_2^{\omega_2}, \quad \omega = (\omega_1,\omega_2), \quad \omega_1,\omega_2 \in (0,1).$$

More precisely, using Proposition 6.3.2, (ii) in [53, p. 280] it is enough to prove that $h_\omega \in C^1(\mathbf{0},\mathbf{b})$, $h_\omega(\mathbf{x}) = 0$ for all $\mathbf{x}$ belonging to the boundary $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$, and for each $\mathbf{x} \in \partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$ there exists a neighborhood $V_{\mathbf{x}}$ of $\mathbf{x}$ such that $\left(-\mathbf{D}_{\mathbf{0}+}^{(\nu)}h_\omega\right)(\mathbf{y}) < -c$ for $\mathbf{y} \in V_{\mathbf{x}} \cap (\mathbf{0},\mathbf{b}]$ and some positive constant c whenever $h_\omega(\mathbf{y}) > 0$. Since the function h_ω is differentiable on $(\mathbf{0},\mathbf{b}]$ and vanishes on the boundary points $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$, we need to see that $\left(-\mathbf{D}_{\mathbf{0}+}^{(\nu)}h_\omega\right)(\mathbf{y}) < -c$ for some positive constant c. However, the latter inequality follows from Condition (H1) by taking $\omega_1 = 1 - q_1$ and $\omega_2 = 1 - q_2$, where $q_1, q_2 \in (0,1)$ are given by Condition (H1), that is

$$\int_0^\infty \min\{y,\epsilon\}\nu_1(0,y)dy > C_1\epsilon^{q_1}$$

and

$$\int_0^\infty \min\{y,\epsilon\}\nu_2(0,y)dy > C_2\epsilon^{q_2},$$

for some positive constants C_1 and C_2. See also the proof of Lemma 2.5.2. Notice also that the finite expectation of $\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})$ is a consequence of the finite expectation of each $\tau_0^{(\nu_i)}(x_i)$.

(ii) This is a generalization of Proposition 2.5.4 and follows directly by differentiating

$$\mathbf{P}\left[\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x}) > s\right] = \mathbf{P}\left[\tau_0^{(\nu_1)}(x_1) > s\right]\mathbf{P}\left[\tau_0^{(\nu_2)}(x_2) > s\right],$$

with respect to s. Notice the use of the independence assumption in the previous equality.

(iii) This is a generalization of Proposition 2.5.5 and is obtained by differentiating

$$\mathbf{P}\left[\ \mathbf{X}_{\mathbf{x}}^{\mathbf{0}+(\nu)}(s) > \mathbf{y}, \tau_{\mathbf{0}}^{(\nu)}(\mathbf{x}) > \xi\right] = \prod_{i=1}^{2}\mathbf{P}\left[\ X_{x_i}^{0+(\nu_i)}(s) > y_i, X_{x_i}^{0+(\nu_i)}(\xi) > 0\right],$$

with respect to y_1, y_2 and ξ. ■

Let us now generalize the definitions given in (2.5.7) and (2.5.8). For $\lambda \geq 0$ and $g \in B[\mathbf{0}, \mathbf{b}]$ define

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\mathbf{x}) := \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})} e^{-\lambda s}g(\mathbf{X}_{\mathbf{x}}^{+(\nu)}(s))ds\right], \quad \mathbf{x} \in (\mathbf{0}, \mathbf{b}],$$

and

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}1(\mathbf{x}) := \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})} e^{-\lambda s}ds\right], \quad \mathbf{x} \in [\mathbf{0}, \mathbf{b}].$$

Note that $\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\cdot)$ is continuous on $(\mathbf{0}, \mathbf{b}]$ and

$$|\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\mathbf{x})| \leq \|g\| \sup_{\mathbf{x}\in[\mathbf{0},\mathbf{b}]} \mathbf{E}\left[\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})\right].$$

Moreover,

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}1(\mathbf{x}) = \frac{1}{\lambda}\left(1 - \mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})}\right]\right),$$

implying

$$\mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})}\right] = 1 - \lambda\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}1(\mathbf{x}),$$

and yielding the next generalization of Lemma 2.5.7.

Lemma 3.3.3. *Let $\boldsymbol{x} = (x_1, x_2) \in (\boldsymbol{0}, \boldsymbol{b}]$ and $\lambda > 0$. Suppose that ν_i satisfies conditions (H0)-(H1) for $i \in \{1,2\}$. Then,*

$$\mathbf{E}\left[e^{-\lambda\tau_{\boldsymbol{0}}^{(\nu)}(\boldsymbol{x})}\right] = \mathbf{E}\left[e^{-\lambda\tau_0^{(\nu_1)}(x_1)}1_{\left\{\tau_0^{(\nu_1)}(x_1)<\tau_0^{(\nu_2)}(x_2)\right\}}\right] + \mathbf{E}\left[e^{-\lambda\tau_0^{(\nu_2)}(x_2)}1_{\left\{\tau_0^{(\nu_2)}(x_2)<\tau_0^{(\nu_1)}(x_1)\right\}}\right].$$

If additionally ν_i satisfies (H2)-(H3) for $i \in \{1,2\}$, then

$$\mathbf{E}\left[e^{-\lambda\tau_{\boldsymbol{0}}^{(\nu)}(\boldsymbol{x})}\right] = \int_0^\infty e^{-\lambda s}\left(\mu_0^{x_1,(\nu_1)}(s)\int_0^{x_2} p_s^{+(\nu_2)}(x_2,y)dy\right)ds + \\ + \int_0^\infty e^{-\lambda s}\left(\mu_0^{x_2,(\nu_2)}(s)\int_0^{x_1} p_s^{+(\nu_1)}(x_1,y)dy\right)ds.$$

Further,

$$\boldsymbol{M}_{\boldsymbol{0},\lambda}^{+(\nu)}g(\boldsymbol{x}) = \int_0^{x_1}\int_0^{x_2} g(x_1-y_1,\, x_2-y_2)\int_0^\infty e^{-\lambda s}p_s^{+(\nu_1)}(x_1,x_1-y_1)p_s^{+(\nu_2)}(x_2,x_2-y_2)\,ds\,dy_2\,dy_1. \tag{3.3.3}$$

Proof. Similar to the proof of Lemma 2.5.7 but using the density function of the r.v. $\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})$ and the joint distribution of the pair $\left(\mathbf{X}_{\mathbf{x}}^{\mathbf{0}+(\nu)}(s), \tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})\right)$ both given in Lemma 3.3.2. ∎

Well-posedness result for the RL type linear equation.

Theorem 3.3.4. *(**Case** $\lambda > 0$) Let $\nu = (\nu_1, \nu_2)$ be a vector such that each ν_i is a function satisfying conditions (H0)-(H1). Suppose that $\lambda > 0$ and $\boldsymbol{x} \in [\boldsymbol{0}, \boldsymbol{b}]$ with $\boldsymbol{x} = (x_1, x_2)$ and $[\boldsymbol{0}, \boldsymbol{b}] = [0, b_1] \times [0, b_2]$.*

(i) If $g \in C_{\boldsymbol{0}}[\boldsymbol{0}, \boldsymbol{b}]$, then the equation $(-\,{}_{x_1}D_{0+}^{(\nu_1)} - {}_{x_2}D_{0+}^{(\nu_2)}, \lambda, g, 0)$ has a unique solution in the domain of the generator given by $w = \boldsymbol{R}_\lambda^{\boldsymbol{0}+(\nu)}g$, the resolvent operator of the process $\boldsymbol{X}_{\boldsymbol{x}}^{\boldsymbol{0}+(\nu)}$.

(ii) For any $g \in B[\boldsymbol{0}, \boldsymbol{b}]$, the equation $(- {}_{x_1}D^{(\nu_1)}_{0+} - {}_{x_2}D^{(\nu_2)}_{0+}, \lambda, g, 0)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$w(x_1,x_2) = \mathbf{E}\left[\int_0^{\tau_{\boldsymbol{0}}^{(\nu)}((x_1,x_2))} e^{-\lambda s} g\left(X^{0+(\nu_1)}_{x_1}(s)\ ,\ X^{0+(\nu_2)}_{x_2}(s)\right) ds \right]. \quad (3.3.4)$$

Moreover, if additionally ν_i satisfies conditions (H2)-(H3) for $i \in \{1,2\}$, then $w(x_1,x_2)$ takes the explicit form in (3.3.3).

Proof. (i) Follows from Theorem A.1.1 as in the one-dimensional case.

(ii) If $g \in B[\mathbf{0}, \mathbf{b}]$, the solution is obtained as a limit of solutions $\mathbf{R}^{\mathbf{0}+(\nu)}_{\lambda} g_n(\mathbf{x})$ in the domain of the generator $-\mathbf{D}^{(\nu)}_{\mathbf{0}+}$, where the sequence of functions $\{g_n\}_{n\geq 1}$ satisfies the conditions of Definition 3.3.1. Finally, Lemma 3.3.3 provides the explicit representation of the solution w in terms of the transition densities. ■

Theorem 3.3.5. *(**Case** $\lambda = 0$) All assertions in Theorem 3.3.4 are valid for $\lambda = 0$.*

Proof. Since the potential operator is bounded due to the finite expectation of $\tau^{(\nu)}_{\mathbf{0}}(\mathbf{x})$, the arguments used in the proof of Theorem 3.3.4 remain valid for the case $\lambda = 0$ by replacing the resolvent $\mathbf{R}^{\mathbf{0}+(\nu)}_{\lambda}$ with the corresponding potential operator $\mathbf{R}^{\mathbf{0}+(\nu)}_{0}$. ■

Finally, we analyze the mixed linear equation which involves both the RL type and the Caputo type operator:

$$\begin{aligned} - {}_{x_1}D^{(\nu_1)}_{0+} u(x_1,x_2) - {}_{x_2}D^{(\nu_2)}_{0+*} u(x_1,x_2) &= \lambda u(x_1,x_2) - g(x_1,x_2), && (x_1,x_2) \in (0,b_1] \times (0,b_2], \\ u(0,x_2) &= 0, && x_2 \in [0,b_2] \\ u(x_1,0) &= \phi(x_1) && x_1 \in (0,b_1], \end{aligned} \quad (3.3.5)$$

for a given function $\phi \in C_0[0,b_1]$. This equation will be referred to as the *mixed linear problem* $(- {}_{x_1}D^{(\nu_1)}_{0+} - {}_{x_2}D^{(\nu_2)}_{0+*}, \lambda, g, \phi)$.

Denote by $\mathbf{X}^{\mathbf{0}+(\nu)*}_{\mathbf{x}} := \left(X^{0+(\nu_1)}_{x_1}, X^{0+*(\nu_2)}_{x_2}\right)$ the Feller process (with values on $(0,b_1] \times [0,b_2]$) generated by the operator $- {}_{x_1}D^{(\nu_1)}_{0+} - {}_{x_2}D^{(\nu_2)}_{0+*}$. This process is obtained from a process $\mathbf{X}^{+(\nu)}_{\mathbf{x}} := \left(X^{+(\nu_1)}_{x_1}\ ,\ X^{+(\nu_2)}_{x_2}\right)$ by either killing it whether the first coordinate attempts to cross the boundary point $x_1 = 0$, or by stopping it if the second coordinate does the same with the boundary point $x_2 = 0$. As before, $\tau_{\mathbf{0}}^{(\nu)}(\mathbf{x})$ denotes the first exit time from $(0, b_1] \times (0, b_2]$.

In order to solve the mixed equation (3.3.5), we rewrite it as a linear equation involving only RL type operators. Namely, let $\psi \in C([0, b_1] \times [0, b_2])$ be a function satisfying the boundary conditions in (3.3.5). Define $w(\mathbf{x}) := u(\mathbf{x}) - \psi(\mathbf{x})$ for any $\mathbf{x} = (x_1, x_2) \in [\mathbf{0}, \mathbf{b}]$. Observe that, by definition, w vanishes at the boundary $\partial_{\mathbf{0}}[\mathbf{0}, \mathbf{b}]$.

If u and ψ belong to the domain of the generator $-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}$, then

$$\left(-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}\right) w = \left(-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}\right) u + \left(\ {}_{x_1}D_{0+}^{(\nu_1)} +\ {}_{x_2}D_{0+*}^{(\nu_2)}\right) \psi$$
$$= \lambda u - \left[g - \left(\ {}_{x_1}D_{0+}^{(\nu_1)} +\ {}_{x_2}D_{0+*}^{(\nu_2)}\right) \psi \right] =: \lambda w - \tilde{g},$$

with $\tilde{g} := g - \lambda\psi -\ {}_{x_1}D_{0+}^{(\nu_1)}\psi -\ {}_{x_2}D_{0+*}^{(\nu_2)}\psi$. Due to the properties satisfied by ψ, the function w satisfies $-\ {}_{x_2}D_{0+*}^{(\nu_2)} w(x) = -\ {}_{x_2}D_{0+}^{(\nu_2)} w(x) = 0$ on the boundary $\partial_{\mathbf{0}}(\mathbf{0}, \mathbf{b}]$. Consequently, the solution u to (3.3.5) can be written as $u = w + \psi$, where w is the solution to the corresponding RL type equation. This motivates the next definition.

Definition 3.3.2. *Let $g \in B[\mathbf{0}, \mathbf{b}]$, $\lambda \geq 0$, and $\phi \in C_0[0, b_1]$. A function $u \in C[\mathbf{0}, \mathbf{b}]$ is said to solve the mixed linear problem $(-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ as*

(i) a solution in the domain of the generator *if u satisfies (3.3.5) and belongs to the domain of the generator $-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}$;*

(ii) a generalized solution *if for any function ψ in the domain of $-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+*}^{(\nu_2)}$ such that $\psi(0, \cdot) = 0$ and $\psi(\cdot, 0) = \phi(\cdot)$, then $u = \omega + \psi$, where ω is a solution (possibly generalized) to the RL type problem*

$$(-\ {}_{x_1}D_{0+}^{(\nu_1)} -\ {}_{x_2}D_{0+}^{(\nu_2)}, \lambda, \tilde{g}, 0),$$

with $\tilde{g} := g - \lambda\psi -\ {}_{x_1}D_{0+}^{(\nu_1)}\psi -\ {}_{x_2}D_{0+*}^{(\nu_2)}\psi$.

Remark 3.3.6. *By definition, it seems that a generalized solution depends on the function ψ, the next result shows that this solution is actually independent of ψ.*

Well-posedness result for the mixed linear equation.

Theorem 3.3.7. *(**Case** $\lambda > 0$) Let $\nu = (\nu_1, \nu_2)$ such that each ν_i is a function satisfying conditions (H0)-(H1). Suppose $\lambda > 0$ and $\phi \in C_0[0, b_1]$.*

(i) If $g \in C[\boldsymbol{0}, \boldsymbol{b}]$ satisfies $g(0, \cdot) \equiv 0$ and $g(\cdot, 0) = \lambda\phi(\cdot)$, then the mixed equation $(-\ {}_{x_1}D_{0+}^{(\nu_1)} - \ {}_{x_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ has a unique solution in the domain of the generator given by $u = \mathfrak{R}_\lambda^{\boldsymbol{0}+(\nu)*} g$, the resolvent operator of the process $\left(X_{x_1}^{0+(\nu_1)},\ X_{x_2}^{0+*(\nu_2)} \right)$.

(ii) For any $g \in B[\boldsymbol{0}, \boldsymbol{b}]$, the mixed linear equation $(-\ {}_{x_1}D_{0+}^{(\nu_1)} - \ {}_{x_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$
\begin{aligned}
u(x_1, x_2) = \mathbf{E}\left[e^{-\lambda \tau_0^{(\nu_2)}(x_2)}\ \phi\left(X_{x_1}^{0+(\nu_1)} \left(\tau_0^{(\nu_2)}(x_2) \right) \right)\ 1_{\left\{ \tau_0^{(\nu_2)}(x_2) < \tau_0^{(\nu_1)}(x_1) \right\}} \right] \\
+ \mathbf{E}\left[\int_0^{\tau_0^{(\nu)}(\boldsymbol{x})} e^{-\lambda s} g\left(X_{x_1}^{0+(\nu_1)}(s),\ X_{x_2}^{0+*(\nu_2)}(s) \right) \right].
\end{aligned}
\tag{3.3.6}
$$

Moreover, if additionally each ν_i for $i \in \{1, 2\}$, satisfies conditions (H2)-(H3), then the solution can be rewritten

$$
\begin{aligned}
u(x_1, x_2) &= \int_0^{x_1} \phi(x_1 - y) \int_0^\infty e^{-\lambda s} \mu_0^{x_2, (\nu_2)}(s) p_s^{+(\nu_1)}(x_1, x_1 - y)\, ds\, dy + \\
&+ \int_0^{x_1} \int_0^{x_2} g(x_1 - y_1, x_2 - y_2) \int_0^\infty e^{-\lambda s} p_s^{+(\nu_1)}(x_1, x_1 - y_1) p_s^{+(\nu_2)}(x_2, x_2 - y_2)\, ds\, dy_2\, dy_1.
\end{aligned}
\tag{3.3.7}
$$

Proof. (i) As before, we apply Theorem A.1.1 to the generator $-\ {}_{x_1}D_{0+}^{(\nu_1)} - \ {}_{x_2}D_{0+*}^{(\nu_2)}$. Therefore, if g is a continuous function on $[0, b_1] \times [0, b_2]$ such that $g(0, \cdot) \equiv 0$, then the function $u(x_1, x_2) = \mathfrak{R}_\lambda^{\boldsymbol{0}+(\nu)*} g(x_1, x_2)$ solves the equation

$$
-\ {}_{x_1}D_{0+}^{(\nu_1)} u - \ {}_{x_2}D_{0+*}^{(\nu_2)} u = \lambda u - g.
$$

Further, a simple calculation shows that

$$u(x_1,0) = \mathfrak{R}_\lambda^{\mathbf{0}+(\nu)*} g(x_1,0) = g(x_1,0)/\lambda, \qquad \text{and} \qquad u(0,x_2) = 0,$$

which implies that, under condition $g(\,\cdot\,,0) = \lambda\phi(\cdot)$, the function u solves the problem (3.3.5).

(ii) For the general case, $g \in B[\mathbf{0},\mathbf{b}]$, take a function ψ satisfying the conditions of Definition 3.3.2 and set $w := u - \psi$. Since w vanishes at the boundary $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$, Theorem 3.3.4 yields

$$w(x_1,x_2) = \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)} e^{-\lambda s}\tilde{g}\left(X_{x_1}^{0+(\nu_1)}(s)\ ,\ X_{x_2}^{0+(\nu_2)}(s)\right) ds\right],$$

with $\tilde{g} = g - \lambda\psi - (\ {}_{x_1}D_{0+}^{(\nu_1)} + \ {}_{x_2}D_{0+*}^{(\nu_2)})\,\psi$. Hence $w(x) = I - II$, where

$$I := \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)} e^{-\lambda s} g\left(X_{x_1}^{0+(\nu_1)}(s)\ ,\ X_{x_2}^{0+(\nu_2)}(s)\right) ds\right]$$
$$II := \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)} e^{-\lambda s}(\lambda + \ {}_{x_1}D_{0+}^{(\nu_1)} + \ {}_{x_2}D_{0+*}^{(\nu_2)})\,\psi\left(X_{x_1}^{0+(\nu_1)}(s)\ ,\ X_{x_2}^{0+(\nu_2)}(s)\right) ds\right].$$

Using that ψ belongs to the domain of the generator $-\ {}_{x_1}D_{0+}^{(\nu_1)} - \ {}_{x_2}D_{0+*}^{(\nu_2)}$, Theorem A.1.4 in Appendix implies that

$$\begin{aligned} Y_r := {} & e^{-\lambda r}\psi\left(X_{x_1}^{0+(\nu_1)}(r)\ ,\ X_{x_2}^{0+*(\nu_2)}(r)\right) + \\ & + \int_0^r e^{-\lambda s}\left(\lambda + \ {}_{x_1}D_0^{(\nu_1)} + \ {}_{x_2}D_{0+*}^{(\nu_2)}\right)\psi\left(X_{x_1}^{0+(\nu_1)}(s)\ ,\ X_{x_2}^{0+*(\nu_2)}(s)\right) ds \end{aligned} \tag{3.3.8}$$

is a martingale for all $\lambda > 0$. Furthermore, since $\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)$ has finite expectation, Doob's stopping theorem [53, Theorem 3.10.1, p. 142] applied to the martingale (3.3.8) implies that

$$\psi(x_1,x_2) = \mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)}\psi\left(X_{x_1}^{0+(\nu_1)}(\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2))\ ,\ X_{x_2}^{0+(\nu_2)}(\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2))\right)\right] +$$

$$+\ \mathbf{E}\Bigg[\ \int_0^{\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)} e^{-\lambda s}(\lambda +\ {}_{x_1}D_{0+}^{(\nu_1)} +\ {}_{x_2}D_{0+*}^{(\nu_2)})\,\psi\left(X_{x_1}^{0+(\nu_1)}(s)\ ,\ X_{x_2}^{0+(\nu_2)}(s)\right)ds\Bigg].$$
(3.3.9)

Therefore,

$$II\ =\ \psi(x_1,x_2) - \mathbf{E}\Big[e^{-\lambda\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2)}\psi\left(X_{x_1}^{0+(\nu_1)}(\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2))\ ,\ X_{x_2}^{0+(\nu_2)}(\tau_{\mathbf{0}}^{(\nu)}(x_1,x_2))\right)\Big],$$

which in turn yields (3.3.6) as $u = w + \psi$ and $\psi(\cdot,0) = \phi(\cdot)$.

Finally, the second term in (3.3.7) is a consequence of Lemma 3.3.3, whilst the first term is obtained by conditioning first on $\tau_0^{(\nu_2)}(x_2)$ and then by using the joint density of the pair $(X_{x_1}^{0+(\nu)}(s), \tau_0^{(\nu_1)}(x_1))$. ■

Theorem 3.3.8. ***(Case*** $\lambda = 0$***)*** *All the assertions in Theorem 3.3.7 are valid for the case* $\lambda = 0$.

Proof. For functions $g \in C_{\mathbf{0}}[\mathbf{0},\mathbf{b}]$, the arguments in the proof of Theorem 3.3.7 remain valid using the potential operator $\mathfrak{R}_0^{\mathbf{0}+(\nu)*}$ instead of the resolvent operator $\mathfrak{R}_\lambda^{\mathbf{0}+(\nu)*}$. In case of general $g \in B[\mathbf{0},\mathbf{b}]$, the martingale (3.3.8) should be replaced by the corresponding martingale with $\lambda = 0$. ■

Remark 3.3.9. *As an application of Theorem 3.3.7, one obtains that for* $\boldsymbol{x} = (x_1,x_2)$ *the function*

$$u(x_1,x_2) = \frac{1}{\alpha}x_2\int_0^{x_1}\phi(x_1-y)\int_0^\infty e^{-\lambda s}\ s^{-\frac{1}{\alpha}-\frac{1}{\beta}-1}w_\alpha\left(x_2 s^{-1/\alpha};1,1\right)w_\beta\left(ys^{-1/\beta};1,1\right)\ ds\,dy +$$
$$+\int_0^{x_1}\int_0^{x_2} g(x_1-y_1,x_2-y_2)\int_0^\infty e^{-\lambda s}s^{-\frac{1}{\beta}-\frac{1}{\alpha}}w_\beta\left(y_1 s^{-1/\beta};1,1\right)w_\alpha\left(y_2 s^{-1/\alpha};1,1\right)ds\,dy_2\,dy_1$$

is the generalized solution to the mixed fractional linear equation

$$\begin{aligned}
-\ {}_{x_1}D_{0+}^{\beta}u(x_1,x_2) -\ {}_{x_2}D_{0+*}^{\alpha}u(x_1,x_2) &= \lambda u(x_1,x_2) - g(x_1,x_2), \quad (x_1,x_2)\in(\boldsymbol{0},\boldsymbol{b}],\\
u(0,x_2) &= 0, \qquad x_2 \in [0,b_2],\\
u(x_1,0) &= \phi(x_1) \qquad x_1 \in (0,b_1],
\end{aligned}$$

for a given function $\phi \in C_0[0,b_1]$ *and* $\beta,\alpha \in (0,1)$. *Let us recall that* $-\ {}_{x_1}D_{0+}^{\beta}$

and $-\ {}_{x_2}D^{\alpha}_{0+*}$ *stand for the classical RL and Caputo derivatives of order* β *and* α*, respectively; and* w_β *and* w_α *denote* $\beta-$ *and* $\alpha-$*stable densities, respectively (see Appendix). Further, the solution* u *belongs to the domain of the generator only when* $g \in C[\boldsymbol{0}, \boldsymbol{b}]$ *satisfying* $g(\cdot, 0) = \lambda\phi(\cdot)$ *and* $g(0, \cdot) \equiv 0$.

Chapter 4

Nonlinear equations of RL and Caputo type

This chapter establishes well-posedness for *nonlinear* equations involving generalized Caputo and Riemann-Liouville type derivatives. We also study the generalized versions of both the linear equation with nonconstant coefficients and the composite fractional relaxation equation. The approach used here relies on the use of the explicit solution to the linear equation studied in Chapter 3.

4.1 Introduction

In the classical fractional setting, the study of nonlinear equations usually require the use of analytical techniques that are different to those used in the linear case. For example, the Laplace transform method, which is very powerful for linear equations with constant coefficients, it is useless for solving fractional linear equations with variable coefficients, and even more, for the study of nonlinear equations.

In this chapter we establish the well-posedness for *the generalized nonlinear frac-*

tional equation

$$-\tilde{D}^{(\nu)}u(x) = -f(x,u(x)), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{4.1.1}$$

and for *the generalized composite fractional relaxation equation*

$$-\tilde{D}^{(\nu)}u(x) - \gamma(x)u'(x) - \lambda u(x) = -f(x,u(x)), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{4.1.2}$$

for some given functions f and γ, and $\lambda \geq 0$. Notation $-\tilde{D}^{(\nu)}$ refers to either the generalized RL type operator $-D^{(\nu)}_{a+}$ or the Caputo type operator $-D^{(\nu)}_{a+*}$.
Some particular examples of equation (4.1.1) include the initial value problem for the nonlinear equation with the classical Caputo derivative D^{β}_{0+*}:

$$D^{\beta}_{0+*}u(x) = f(x,u(x)), \quad x \in (0,b], \quad u(0) = u_0, \quad \beta \in (0,1), \tag{4.1.3}$$

and the *fully mixed* (or multi-term) fractional equation

$$\sum_{i=1}^{d} \omega_i(x) D^{\beta_i(x)}_{0+*}u(x) = f(x,u(x)), \quad x \in (0,b], \quad u(0) = u_0, \quad \beta_i \in (0,1), \tag{4.1.4}$$

for a given continuous function f and nonnegative functions $\omega_i(\cdot)$, $i \in \{1,\ldots,d\}$.
The existence and uniqueness results for the fractional equation (4.1.3) have been proved, for example, by transforming (4.1.3) into a Volterra type equation and then by using fixed point arguments (see, e.g., Theorem 5.1 and Theorem 6.1 in [15] for the RL and the Caputo case, respectively).
The method we use to prove the well-posedness for the generalized problem in (4.1.1) is also based on transforming (4.1.1) into an integral equation. However, the integral equation used here is taken from the probabilistic solution to the corresponding linear problem obtained in Chapter 3.

Another particular case of (4.1.1) is the linear equation with nonconstant coefficients

$$-\tilde{D}^{(\nu)}u(x) = \lambda(x)u(x) - g(x), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \tag{4.1.5}$$

for given functions λ and g. For this case an explicit solution in terms of the transition probabilities of the underlying stochastic processes is given. We deal with this case separately due to the fact that, unlike the general case $f(x,u(x))$, the probabilistic representation of its solution has an explicit form as a Feynman-Kac type formula whilst for the general case we are only able to prove existence and uniqueness of solutions.

The generalized equation (4.1.5) encompasses the initial value problem for the linear equation with nonconstant coefficients involving the classical Caputo derivative:

$$D^{\beta}_{0+*}u(x) = \lambda(x)u(x) + g(x), \quad x \in (0,b], \quad u(0) = u_0, \tag{4.1.6}$$

for $\beta \in (0,1)$. It was proved by analytical methods that if $g \in C[0,b]$, then equation (4.1.6) has a unique solution $u \in C[0,b]$ given by (see, e.g., [15], Theorem 7.10)

$$u(x) = T(x) + \int_0^x R(x,y)T(y)dy, \quad x \in (0,b], \tag{4.1.7}$$

where

$$T(x) := u_0 + I^{\beta}_{0+}g(x), \quad R(x,y) := \sum_{j=1}^{\infty} k_j(x,y),$$

I^{β}_{0+} denotes the Riemann-Liouville integral operator of order β,

$$k_1(x,y) := k(x,y) = \frac{1}{\Gamma(\beta)}(x-y)^{\beta-1}\lambda(y),$$

and

$$k_j(x,y) := \int_y^x k(x,s)k_{j-1}(s,y)ds, \ (j = 2,3,\ldots).$$

The probabilistic approach used here provides a different representation of the solution in (4.1.7) when λ is a positive function. This representation is given in terms of path functionals and can also be written explicitly in terms of the transition probabilities of the underlying decreasing process.

The last part of this chapter addresses the nonlinear equation (4.1.2). Some particular cases have been studied in the literature; for instance, the initial value problem for *the composite fractional relaxation equation* [28] (also called *the generalized Basset equation* [62]):

$$c_1 D^{\beta}_{0+*} u(x) + c_2 \frac{d}{dx} u(x) = -u(x) + g(x), \quad x \in (0, b], \quad u(0) = u_0, \tag{4.1.8}$$

for $\beta \in (0,1)$, $c_1 > 0$, $c_2 = 1$ and g a continuous function, was solved in [28] via the Laplace transform method. The explicit solution in terms of the *fundamental solution* $\phi(x)$ and the so-called *impulse-response solution* $-\phi'(x)$ is

$$u(x) = u_0 \phi(x) - \int_0^x g(x-y)\phi'(y)dy; \tag{4.1.9}$$

where

$$\phi(x) = \int_0^\infty e^{-yx} H^{(1)}_{\beta,0}(y; c_1) dy, \tag{4.1.10}$$

and

$$H^{(1)}_{\beta,0}(y; c_1) = \frac{1}{\pi} \frac{c_1 y^{\beta-1} \sin(\beta\pi)}{(1-y)^2 + c_1^2 y^{2\beta} + 2(1-y) c_1 y^{\alpha} \cos(\alpha\pi)}. \tag{4.1.11}$$

The results presented in this chapter extend the ones known for the equation (4.1.8). Firstly, by considering the nonlinear version, and secondly, by allowing the parameters c_1 and c_2 being more general (functions instead of constants). The generalized equation (4.1.2) is also an extension of the linear case studied in the previous chaper, wherein the well-posedness was treated but without the drift term.

Further, as was done in the preceding chapter, we study the existence of two types of solutions: *solutions in the domain of the generator* and *generalized solutions.*

For some specific cases (which encompass the classical fractional operators), we also investigate the existence of *smooth solutions*.

4.2 Preliminaries

Hereafter, notation $-\tilde{D}^{(\nu)}$ stands for either the RL type operator $-D_{a+}^{(\nu)}$ or the Caputo type operator $-D_{a+*}^{(\nu)}$. Analogously, $\tilde{R}_\lambda^{(\nu)}$ will denote the resolvent (or potential operator if $\lambda = 0$) associated with the operator $-\tilde{D}^{(\nu)}$. The space wherein the semigroup generated by the operator $-\tilde{D}^{(\nu)}$ is strongly continuous shall be denoted by $\tilde{C}[a,b]$, meaning $C_a[a,b]$ or $C[a,b]$ whether the operator refers to the RL or the Caputo type operator, respectively. Similarly, $\tilde{u}_a \in \mathbb{R}$ will mean $\tilde{u}_a = 0$ for RL type equations, and any real number for Caputo type equations.

Notation $(-\tilde{D}^{(\nu)}, \lambda, g, \tilde{u}_a)$ is used to represent the linear problem

$$-\tilde{D}^{(\nu)}u(x) = \lambda u(x) - g(x), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{4.2.1}$$

for any $\lambda \geq 0$.

For the existence results we will use the following preliminary result taken from Theorem 3.2.3 and Theorem 3.2.6.

Lemma 4.2.1. *Let ν be a function satisfying conditions (H0)-(H1). Assume that $g \in B[a,b]$ and $\tilde{u}_a \in \mathbb{R}$. Then,*

(i) the unique generalized solution $u \in \tilde{C}[a,b]$ to the linear problem $(-\tilde{D}^{(\nu)}, 0, g, \tilde{u}_a)$ is given by

$$u(x) = \tilde{u}_a + \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} g\left(X_x^{+(\nu)}(s)\right) ds\right], \tag{4.2.2}$$

where $X_x^{+(\nu)}$ is the underlying process generated by $(G_+^{(\nu)}, \mathfrak{D}_G)$, see definition in (2.3.1).

Moreover, if ν also satisfies conditions (H2)-(H3), then the solution rewrites

$$u(x) = \tilde{u}_a + \int_0^{\infty} \int_a^x g(y) p_s^{+(\nu)}(x,y)\, dy\, ds,$$

where $p_s^{+(\nu)}(x,y)$ are the transition densities of the process $X_x^{+(\nu)}$.

(ii) If $g \in C_a[a,b]$ and $\tilde{u}_a = 0$, then the solution in (4.2.2) is the unique solution in the domain of the generator.

4.3 Nonlinear equations involving RL and Caputo type operators

This section is concerned with the well-posedness results for the nonlinear equation

$$-\tilde{D}^{(\nu)} u(x) = -f(\,x,\, u(x)\,), \quad x \in (a,b], \qquad u(a) = \tilde{u}_a, \quad \tilde{u}_a \in \mathbb{R}, \tag{4.3.1}$$

for a given bounded function $f : G \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition 4.3.1. *Let $f \in B(G)$ and $G \subset \mathbb{R}^2$. Assume that ν satisfies condition (H0). A function $u \in \tilde{C}[a,b]$ is called a solution (*generalized, classical *or* in the domain of the generator*) to the nonlinear equation (4.3.1) if u is a solution (generalized, classical or in the domain of the generator, respectively) to the linear equation*

$$-\tilde{D}^{(\nu)} u(x) = -g(x), \quad x \in (a,b]; \qquad u(a) = \tilde{u}_a, \tag{4.3.2}$$

where $g(x) := f(x, u(x))$ for all $x \in [a,b]$.

Lemma 4.3.1. *Let ν be a function satisfying conditions (H0)-(H3). Suppose that $f : G \subset \mathbb{R}^2 \to \mathbb{R}$ is a function in $B(G)$. Then, a function $u \in \tilde{C}[a,b]$ is a generalized solution to the problem (4.3.1) if, and only if, u solves the nonlinear integral equation*

$$u(x) = \tilde{u}_a + \int_0^{\infty} \int_a^x f(\,y,\, u(y)) p_s^{+(\nu)}(x,y)\, dy\, ds. \tag{4.3.3}$$

Proof. By Definition 4.3.1, $u \in \tilde{C}[a,b]$ is a generalized solution to (4.3.1) if, and only if, u is a generalized solution to (4.3.2) with $g(x) := f(x, u(x))$. Note that if $u \in \tilde{C}[a,b]$, then g is a bounded measurable function. Under the assumptions (H2)-(H3), Lemma 4.2.1 provides the integral equation (4.3.3). ∎

Remark 4.3.2. *Definition 4.3.1 and Lemma 4.3.1 can be extended to the RL type equation for any* $\lambda > 0$*:*

$$-D_{a+}^{(\nu)} u(x) = \lambda u(x) - f(\, x, \, u(x)\,), \quad x \in (a, b], \qquad u(a) = 0. \tag{4.3.4}$$

In this case, the equation (4.3.2) should be replaced with the equation in (4.2.1), whilst the integral equation

$$u(x) = \int_0^\infty \int_a^x e^{-\lambda s} f(\, y, \, u(y)) p_s^{+(\nu)}(x, y)\, dy\, ds, \tag{4.3.5}$$

will replace the one in (4.3.3) (see Theorem 3.2.2). Moreover, to study the corresponding Caputo type problem, an additional term will appear in the integral equation (see Theorem 3.2.5).

Let us now see that the integral equation (4.3.3) possesses a unique solution under the additional assumptions:

(H4): There exist $\epsilon > 0$ and $\beta \in (0,1)$ such that the function ν satisfies that $\nu(x,y) \geq Cy^{-1-\beta}$ for some constant $C > 0$ and $0 < y < \epsilon$.

(H5): For $K > 0$ and $\tilde{u}_a \in \mathbb{R}$, the function f belongs to $B(G_K)$ where

$$G_K := \left\{(x,y) \in \mathbb{R}^2 \, : \, x \in [a,b] \text{ and } y \in [\tilde{u}_a - K, \tilde{u}_a + K]\right\}.$$

Moreover, f fulfills a Lipschitz condition with respect to the second variable, i.e., for all $(x,y), (x,z) \in G_K$

$$|f(x,y) - f(x,z)| < L_f |y - z|, \tag{4.3.6}$$

for a constant $L_f > 0$ (independent of x).

Remark 4.3.3. • *Condition* $(H4)$ *ensures the regularity in expectation of the point* a *(by Lemma 2.5.2), as well as the existence of a positive constant* C_1 *such that* $\mathbf{E}[\tau_a^{(\nu)}(x)] < C_1(x-a)^\beta$. *This holds due to the fact that Condition (H4) implies condition (H1) given in (2.5.1). This can be seen as well using Proposition 6.3.2 in [53] and the Lyapunov function* $h(x) = (x-a)^\beta$ *(see proof of Lemma 2.5.2).*

- *Assumptions of the type given in (H5) are standard Lipschitz conditions to prove existence and uniqueness of fixed points. This will be used because the existence and uniqueness of a solution to equation (4.3.3) is equivalent to the existence and uniqueness of a fixed point for the corresponding operator (see definition of* Ψ *in (4.3.8) below).*

Proposition 4.3.4. *Let* $K > 0$, $a, b \in \mathbb{R}$ *and* $\tilde{u}_a \in \mathbb{R}$. *Let* ν *be a function satisfying conditions (H0) and (H2)-(H4). Assume that* $f : G_K \subset \mathbb{R}^2 \to \mathbb{R}$ *is a function satisfying condition (H5). Define* $M_K := \sup\{|f(x,y)| : (x,y) \in G_K\}$ *and* $b^* := \min\left\{b, \frac{K}{C_1 M_K} + a\right\}$. *Then, the integral equation (4.3.3) has a unique solution* $u \in \tilde{C}[a, b^*]$.

Proof. To prove the existence of a unique solution to (4.3.3) we rewrite it as a fixed point problem $u(x) = (\Psi u)(x)$ for a certain operator Ψ.
Step a) Defining the operator Ψ. Let us consider the space F_K given by

$$F_K = \left\{u \in \tilde{C}[a, b^*] \;:\; \|u - \tilde{u}_a\|_{\tilde{C}[a,b^*]} \le K\right\}. \tag{4.3.7}$$

Note that F_K is a closed subset of the space $\tilde{C}[a, b^*]$, the latter space endowed with the supnorm denoted by $\|\cdot\|_{\tilde{C}[a,b^*]}$. Hence, $(F_K, \|\cdot\|_{\tilde{C}[a,b^*]})$ is a complete metric space. Next, define the operator Ψ on F_K by

$$(\Psi u)(x) := \tilde{u}_a + \int_0^\infty \int_a^x f(y, u(y)) p_s^{+(\nu)}(x,y)\, dy\, ds, \qquad x \in [a, b^*]. \tag{4.3.8}$$

Note that if $u \in F_K$, then $\Psi u \in \tilde{C}[a, b^*]$. Further,

$$
\begin{aligned}
|\Psi u(x) - \tilde{u}_a| &= \Big| \int_0^\infty \int_a^x f(y, u(y)) p_s^{+(\nu)}(x, y)\, dy\, ds \Big| \\
&< M_K \int_0^\infty \int_a^x p_s^{+(\nu)}(x, y)\, dy\, ds \\
&\le C_1 M_K (x-a)^\beta \le C_1 M_K (b^* - a) \le K,
\end{aligned}
$$

where the last inequality holds by definition of b^*. Therefore, $\Psi : F_K \to F_K$.

Step b) Let Ψ^n denote the n-fold iteration of the operator Ψ for $n \in \mathbb{N}$. For convention Ψ^0 denotes the identity operator. We will prove that for any $x \in [a, b^*]$,

$$
|\Psi^n u(x) - \Psi^n v(x)| \le \left(\kappa L_f (x-a)^\beta \right)^n \|u - v\|_x \prod_{k=0}^{n-1} B(k\beta + 1, \beta), \qquad n \ge 1, \quad (4.3.9)
$$

where

$$
\|u - v\|_x := \sup_{z \le x} |u(z) - v(z)|, \quad x \in [a, b^*],
$$

L_f is the Lipschitz constant of the function f, notation $B(\cdot, \cdot)$ refers to the Beta function (see Appendix) and κ is a positive constant satisfying

$$
\int_0^\infty y^{-1/\beta} w_\beta(y^{-1/\beta}; 1, 1) dy \le \kappa. \qquad (4.3.10)
$$

Recall that w_β represents a β-stable density (see Appendix). The existence of κ can be obtained by splitting the integral (4.3.10) into two regions, over the sets $\{y \le 1\}$ and $\{y \ge 1\}$. Then, the upper bounds for the β-stable densities in each region (see, e.g., Theorem 7.3.1 in [53]) provide the bound required.

To prove (4.3.9), let us proceed by induction. For $n = 1$, the definition of the operator Ψ and the Lipschitz condition yield

$$
\begin{aligned}
|\Psi u(x) - \Psi v(x)| &\le L_f \int_0^\infty \int_a^x |u(y) - v(y)| p_s^{+(\nu)}(x, y)\, dy\, ds \\
&\le L_f \int_0^\infty \int_a^x \|u - v\|_y p_s^{+(\nu)}(x, y)\, dy\, ds.
\end{aligned}
$$

Since for any ν satisfying (H0) the underlying process is decreasing, assumption (H4) implies that the process $X_x^{+(\nu)}$ *dominates* the inverted β-stable subordinator $X_x^{+\beta}$ in the sense that $\mathbf{P}[X_x^{+(\nu)}(s) > y] \leq \mathbf{P}[X_x^{+\beta}(s) > y]$, for all $y \leq b^*$ and for all $s \geq 0$ (or, equivalently, $\mathbf{P}[X_x^{+\beta}(s) \leq y] \leq \mathbf{P}[X_x^{+(\nu)}(s) \leq y]$). Therefore, $\mathbf{E}\left[g\left(X_x^{+(\nu)}(s)\right)\right] \leq \mathbf{E}\left[g\left(X_x^{+\beta}(s)\right)\right]$ for any non decreasing function g.

Hence, using the function $g(y) = \|u - v\|_y$ we obtain

$$
\begin{aligned}
|\Psi u(x) - \Psi v(x)| &\leq L_f \int_0^\infty \int_a^x \|u - v\|_y p_s^{+\beta}(x, y)\, dy\, ds \\
&\leq \|u - v\|_x L_f \int_0^\infty \int_a^x p_s^{+\beta}(x, y)\, dy\, ds,
\end{aligned}
$$

where $p_s^{+\beta}(x, y)$ stands for the transition densities of the inverted β–stable subordinator $X_x^{+\beta}$. The scaling property and the stationary increments of the process $X_x^{+\beta}$ imply $p_s^{+\beta}(x, y) = s^{-1/\beta} w_\beta(s^{-1/\beta}(x - y); 1, 1)$ (see Appendix). Hence

$$
\begin{aligned}
|\Psi u(x) - \Psi v(x)| &\leq L_f \|u - v\|_x \int_0^\infty \int_a^x s^{-1/\beta} w_\beta(s^{-1/\beta}(x - y); 1, 1)\, dy\, ds \\
&\leq L_f \|u - v\|_x \int_a^x (x - y)^{\beta - 1} \int_0^\infty u^{-1/\beta} w_\beta(u^{-1/\beta}; 1, 1)\, du\, dy \\
&\leq \kappa L_f \|u - v\|_x \frac{1}{\beta}(x - a)^\beta \\
&= \kappa L_f \|u - v\|_x (x - a)^\beta B(1, \beta).
\end{aligned}
$$

In the second inequality we have used Fubini's theorem, and then the change of variable $u = s(x-y)^{-\beta}$. Now let us assume that the inequality (4.3.9) holds for $n-1$. Then

$$
\begin{aligned}
|\Psi^n u(x) - \Psi^n v(x)| &\leq L_f \int_0^\infty \int_a^x \left|\Psi^{n-1} u(y) - \Psi^{n-1} v(y)\right| p_s^{+(\nu)}(x, y)\, dy\, ds \\
&\leq L_f \int_0^\infty \int_a^x \sup_{z \leq y} \left|\Psi^{n-1} u(z) - \Psi^{n-1} v(z)\right| p_s^{+(\nu)}(x, y)\, dy\, ds \\
&\leq L_f \int_0^\infty \int_a^x \sup_{z \leq y} \left|\Psi^{n-1} u(z) - \Psi^{n-1} v(z)\right| p_s^{+\beta}(x, y)\, dy\, ds \\
&\leq \kappa^{n-1} L_f^n \|u - v\|_x \prod_{k=0}^{n-2} B(k\beta + 1, \beta) \int_0^\infty \int_a^x (y - a)^{(n-1)\beta} p_s^{+\beta}(x, y)\, dy\, ds
\end{aligned}
$$

$$\leq \kappa^n L_f^n \|u-v\|_x \prod_{k=0}^{n-2} B(k\beta+1,\beta) \int_a^x (y-a)^{(n-1)\beta}(x-y)^{\beta-1}dy\,, \tag{4.3.11}$$

where the first, third and fourth inequalities hold due to the Lipschitz condition, condition (H4) and the induction hypothesis, respectively.

For the integral in (4.3.11), the change of variable $z = (y-a)/(x-a)$ yields

$$\begin{aligned}\int_a^x (y-a)^{(n-1)\beta}(x-y)^{\beta-1}dy &= (x-a)^{n\beta}\int_0^1 z^{(n-1)\beta}(1-z)^{\beta-1}dz\\ &= (x-a)^{n\beta}B((n-1)\beta+1,\beta),\end{aligned}$$

which implies inequality (4.3.9), as required.

Step c) To conclude that Ψ has a fixed point, we will apply the Weissenger fixed point theorem. Hence, we shall prove that

$$\|\Psi^n u - \Psi^n v\|_{C[a,b^*]} \leq \alpha_n \|u-v\|_{C[a,b^*]}, \tag{4.3.12}$$

for every $n \geq 0$ and every $u, v \in F_K$, where $\alpha_n \geq 0$ and $\sum_{n=0}^{\infty}\alpha_n$ converges (see, e.g., Appendix in [15]).

A proof by induction (using the identities in (A.3.3)) yields

$$\prod_{k=0}^{n-1} B(k\beta+1,\beta) = \frac{(\,\Gamma(\beta)\,)^n}{n\beta\Gamma(n\beta)}, \quad n \in \mathbb{N}.$$

Moreover, the inequality (A.3.4) implies

$$\frac{(\Gamma(\beta)\,)^n}{n\beta\Gamma(n\beta)} \leq \frac{(\,\Gamma(\beta)\,)^n}{n\beta(n-1)!\beta^{2(n-1)}\,(\,\Gamma(\beta)\,)^n} \leq \frac{1}{n!\beta^{2n}}.$$

Therefore

$$\begin{aligned}|\Psi^n u(x) - \Psi^n v(x)| &\leq \kappa^n L_f^n \|u-v\|_x (x-a)^{n\beta}\frac{1}{n!\beta^{2n}}\\ &\leq \kappa^n L_f^n \|u-v\|_{C[a,b^*]}(b^*-a)^{n\beta}\frac{1}{n!\beta^{2n}},\end{aligned}$$

implying the inequality (4.3.12) with $\alpha_n := \left(\beta^{-2}\kappa L_f(b^* - a)^\beta\right)^n / n!$.

Since $\sum_{n=0}^{\infty} \alpha_n = \exp\{\beta^{-2}\kappa L_f(b^* - a)^\beta\}$, the Weissinger fixed point theorem guarantees the existence of a unique fixed point $u^* \in F_K$, as required. ■

Observe that the previous result ensures the existence of a solution to the integral equation only in a subinterval $[a, b^*] \subset [a, b]$. A solution in the whole interval can be guaranteed with an additional assumption, as shown below.

Corollary 4.3.5. *Let $a, b \in \mathbb{R}$ and $\tilde{u}_a \in \mathbb{R}$. Let ν be a function satisfying conditions (H0) and (H2)-(H4). Assume that f belongs to $B([a, b] \times \mathbb{R})$ and it satisfies the Lipschitz condition (6.5.2). Then, the integral equation (4.3.3) has a unique solution $u \in \tilde{C}[a, b]$.*

Proof. It follows directly from Proposition 4.3.4 by taking the constant K such that $(b - a)C_1 M < K$ with $M := \|f\|$. ■

Theorem 4.3.6. *Suppose that the assumptions in Corollary 4.3.5 hold. Then,*

(i) There exists a unique generalized solution $u \in \tilde{C}[a, b]$ to the nonlinear problem in (4.3.1).

(ii) If additionally the function f is continuous satisfying $f(a, \tilde{u}_a) = 0$ and $\tilde{u}_a = 0$, then there exists a unique solution in the domain of the generator.

Proof. (i) According to Lemma 4.3.1, the existence of a generalized solution to (4.3.1) is equivalent to the existence of a solution to the integral equation (4.3.3) which follows by Corollary 4.3.5.

(ii) Setting $g(x) := f(x, u(x))$, the assertion (ii) in Lemma 4.2.1 implies that u belongs to *the domain of the generator* whenever $g(a) = 0$ and $u_a = 0$, i.e., when $f(a, 0) = 0$ and $u_a = 0$, as required. ■

Theorem 4.3.7. *Suppose that the assumptions in Corollary 4.3.5 hold. Consider the equation*

$$-\tilde{D}^{(\nu)} u(x) = \lambda u(x) - f(x, u(x)), \quad x \in (a, b], \quad u(a) = \tilde{u}_a, \tag{4.3.13}$$

for any $\lambda > 0$ and $\tilde{u}_a \in \mathbb{R}$. Then,

(i) There exists a unique generalized solution $u \in \tilde{C}[a,b]$ to the nonlinear equation (4.3.13).

(ii) If additionally the function f is continuous satisfying $f(a,\tilde{u}_a) = \lambda\tilde{u}_a$, then there exists a unique solution in the domain of the generator.

Proof. By Remark 4.3.2, the proof of both statements is quite similar to the case $\lambda = 0$, so that the details are omitted. ∎

Remark 4.3.8. *Since the function $f(x,u) = \lambda(x)u + g(x)$ (with bounded functions λ and g) is not bounded in $[a,b] \times \mathbb{R}$, Theorem 4.3.6 can only guarantee the well-posedness for the linear equation with nonconstant coefficients in $C[a,b^*]$ for some $b^* \le b$. In the next section we shall analyze the equation with nonconstant coefficients in a different way via purely probabilistic arguments.*

Remark 4.3.9. *Requiring additional assumptions on the function ν, it is possible to extend all our previous results to the case of a possibly unbounded function $f(x,u)$. However, these extensions are not included here.*

4.3.1 Smoothness of solutions

To finish this section, let us now consider the existence of smooth solutions for some specific cases. We will start with the linear equation whose smoothness was not studied in the previous chapter.

Theorem 4.3.10. ***(Linear case)*** *Let $\nu(x,y)$ be a function satisfying the assumptions (H0)-(H3) and let $\lambda > 0$ and $g \in C^1[a,b]$. Suppose that ν is twice continuously differentiable in the second variable and*

$$\sup_x \int \min\{1,y\} \left|\frac{\partial^2}{\partial x^2}\nu(x,y)\right| dy < +\infty, \qquad \lim_{\delta\to 0} \sup_x \int_{|y|\le\delta} |y| \left|\frac{\partial}{\partial x}\nu(x,y)\right| dy = 0. \tag{4.3.14}$$

(i) If $g(a) = 0$, then there exists a unique solution u in the domain of the generator to the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ such that $u \in C_a^1[a,b]$.
(ii) If $g(a) = \lambda u_a$, then there exists a unique generalized solution in $C^1[a,b]$ to the Caputo type problem $(-D_{a+*}^{(\nu)}, \lambda, g, u_a)$.

Proof. (i) This statement follows from the fact that under the additional assumption (4.3.14), the semigroup of the process $X_x^{a+(\nu)}$, denoted by $S_s^{a+(\nu)}$, is strongly continuous on the space $C_a^1[a,b]$. This can be proved by approximation arguments and perturbation theory as was done in [55]. Namely, we work with the evolution equation

$$\frac{d}{ds} h_s(x) = -D_{a+}^{(\nu_h)} h_s(x), \quad h_0(x) = h(x), \tag{4.3.15}$$

where $\{-D_{a+}^{(\nu_h)}\}_{h\in(0,1]}$ is a family of bounded operators that approximates the operator $-D_{a+}^{(\nu)}$ as $h \to 0$. We can prove that under assumption (4.3.14), the first and the second derivatives with respect to x of the evolution equation (4.3.15) generate strongly continuous semigroups which are uniformly bounded on h and t (on bounded intervals). Hence, the uniform boundedness of both derivatives allows us to prove that the approximating semigroups, say S_s^h, $h \in (0,1]$ (generated by $-D_{a+}^{(\nu_h)}$) converge in the norm $\|\cdot\|_{C^1}$ to the semigroup $S_s^{a+(\nu)}$. Therefore, $S_s^{a+(\nu)}$ is also strongly continuous on $C_a^1[a,b]$. Consequently, the resolvent operator $R_\lambda^{a+(\nu)}$ associated with the operator $-D_{a+}^{(\nu)}$ maps $C_a^1[a,b]$ into itself, implying that $u(x) = R_\lambda^{a+(\nu)} g(x)$ solves $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ and belongs to $C_a^1[a,b]$ whenever $g \in C_a^1[a,b]$, as required.
(ii) By definition, the solution to the Caputo type problem is given by $u(x) = u_a + w(x)$ (see Definition 3.2.3), where $w(x)$ is the solution to the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g - \lambda u_a, 0)$. Hence, $u \in C^1[a,b]$ whenever $w \in C^1[a,b]$, but this follows from assertion (i) and assumption $g(x) - \lambda u_a = 0$. ∎

To avoid technicalities in the nonlinear case, we only study the existence of smooth solution for the Lévy case, i.e., for functions $\nu(x,y)$ independent of the variable x.

Theorem 4.3.11. ***(Nonlinear Lévy case)*** *Let $a, b \in \mathbb{R}$ and $\tilde{u}_a \in \mathbb{R}$. Suppose that*

$\nu(x,y)$ is a function independent of the variable x satisfying assumptions (H0) and (H2)-(H4). Assume that f is a bounded function belonging to $C^1([a,b]\times\mathbb{R})$.

(i) If $f(a,\tilde{u}_a)=0$ and $\tilde{u}_a=0$, then there exists a unique solution (in the domain of the generator) $u\in C_a^1[a,b]$ to the nonlinear RL type equation in (4.3.1).

(ii) If $f(a,\tilde{u}_a)=0$, then there exists a unique generalized solution $u\in C^1[a,b]$ to the Caputo type equation (4.3.1).

Proof. The existence of a unique continuous solution u (in both the RL and Caputo case) is ensured by Theorem 4.3.6). It remains to prove that its derivative exists and is continuous.

(i) Since the function ν is independent of x, then the transition density function of the underlying Lévy subordinator $X_x^{+(\nu)}$ satisfies $p_s^{+(\nu)}(x,y)=\psi(s,x-y)$ for some function ψ depending on the variable s and the difference $x-y$. Consequently, $u'(x)$ (if exists) should satisfy

$$u'(x)=\int_0^\infty\int_0^{x-a}\left(\frac{\partial}{\partial x}f(x-y,\,u)+\frac{\partial}{\partial u}f(x-y,\,u)u'\right)p_s^{+(\nu)}(x,x-y)\,dy\,ds+$$
$$+f(a,u(a))\int_0^\infty p_s^{+(\nu)}(x,a)\;ds.$$

Assumption $f(a,u(a))=0$ leads us to define the operator

$$\tilde{\Psi}u'(x):=\int_0^\infty\int_0^{x-a}\left(\frac{\partial}{\partial x}f(x-y,\,u)+\frac{\partial}{\partial u}f(x-y,\,u)u'(x)\right)p_s^{+(\nu)}(x,x-y)\,dy\,ds. \tag{4.3.16}$$

Since

$$|\tilde{\Psi}u'(x)-\tilde{\Psi}v'(x)|\le\tilde{L}_f\int_0^\infty\int_0^{x-a}|u'(x-y)-v'(x-y)|p_s^{+(\nu)}(x,x-y)\,dy\,ds,$$

where $\tilde{L}_f:=\|f\|_{C^1}$, the same arguments used in the proofs of Proposition 4.3.4 and Corollary 4.3.5 imply the existence of a unique fixed point in $C[a,b]$ for the operator

$\tilde{\Psi}$. Thus, u' exists and belongs to $C[a,b]$, as required.

(ii) Since the Caputo type equation can be written in terms of the RL type operator, its solution equals $u(x) = \tilde{u}_a + w(x)$, where $w(x)$ is the unique solution (in the domain of the generator of the RL operator) solving

$$w(x) = \int_a^x \int_0^\infty f(x, w(x) + \tilde{u}_a) p_s^{+(\nu)}(x,y)\, ds\, dy.$$

Define $\tilde{f}(x,w) := f(x, w(x) + \tilde{u}_a)$, then assertion (i) and assumption $f(a, \tilde{u}_a) = 0$ imply the existence of a unique solution $w \in C_a^1[a,b]$, which in turn yields the smoothness for the generalized solution u. ■

Remark 4.3.12. *Notice that if the function f in the previous result is continuously differentiable in a smaller region $[a,b] \times [u_a - K, u_a + K]$ for some constant $K > 0$ instead of $[a,b] \times \mathbb{R}$, then the procedure above can only guarantee the existence of a solution in $C_a^1[a,b^*]$ for some subinterval $[a,b^*] \subset [a,b]$.*

4.4 Linear equations with nonconstant coefficients

This section provides probabilistic solutions to linear equations with nonconstant coefficients involving generalized fractional derivatives. These solutions are given in terms of (stationary) Feynman-Kac type formulas.

4.4.1 Auxiliary results

Let us start with some preliminary results. Let λ be a nonnegative function in $C_a[a,b]$. Define

$$p_{s,\lambda}^{a+(\nu)}(x,E) := \mathbf{E}\left[\mathbf{1}_E\left(X_x^{a+(\nu)}(s)\right) \exp\left\{-\int_0^s \lambda\left(X_x^{a+(\nu)}(\gamma)\right) d\gamma\right\}\right],$$

and $S_{s,\lambda}^{a+(\nu)} g(x) := \int g(y) p_{s,\lambda}^{a+(\nu)}(x,dy)$ for any $g \in B[a,b]$ such that $g(a) = 0$. We recall that $X_x^{a+(\nu)}$ is the process generated by $(-D_{a+}^{(\nu)}, \mathfrak{D}_{a+}^{(\nu)})$.

The previous definitions imply

$$S_{s,\lambda}^{a+(\nu)}g(x) = \mathbf{E}\left[g\left(X_x^{a+(\nu)}(s)\right)\exp\left\{-\int_0^s \lambda\left(X_x^{a+(\nu)}(\gamma)\right)d\gamma\right\}\right].$$

Lemma 5 in [25] states that for $\lambda \in C_a[a,b]$, $g \in C_a[a,b]$ and $\delta > 0$, the Laplace transform at $\delta > 0$ of $S_{s,\lambda}^{a+(\nu)}g(x)$ (as a function of s), denoted by $R_{\delta,\lambda}^{a+(\nu)}g(x)$, solves the equation

$$R_{\delta,\lambda}^{a+(\nu)}g(x) = R_{\delta}^{a+(\nu)}g(x) - R_{\delta}^{a+(\nu)}\left[\lambda(\cdot)R_{\delta,\lambda}^{a+(\nu)}g(\cdot)\right](x), \quad x \in [a,b],$$

where $R_{\delta}^{a+(\nu)}$ is the resolvent operator (at $\delta > 0$) for the process $X_x^{a+(\nu)}$.
Equivalently (see Theorem 4.3.1 in [53]), the function $w(x) = R_{\delta,\lambda}^{a+(\nu)}g(x)$ is the unique solution in the domain $\mathfrak{D}_{a+}^{(\nu)}$ solving

$$-D_{a+}^{(\nu)}w(x) = (\lambda(x)+\delta)w(x) - g(x), \quad x \in [a,b]. \tag{4.4.1}$$

Remark 4.4.1. *For a given λ, the function $p_{s,\lambda}^{a+(\nu)}(x,E)$ defines a transition probability function (from x to E with time variable s) for a Feller (sub-Markov) process with semigroup $S_{s,\lambda}^{a+(\nu)}$ and generator $-D_{a+}^{(\nu)} - \lambda(\cdot)$ (see [25], Chapter II, Section 5). Moreover, the resolvent of this process (at $\delta > 0$) coincides with $R_{\delta,\lambda}^{a+(\nu)}g$.*

Let us now define

$$M_{\delta,\lambda}^{a+(\nu)}g(x) := \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} \exp\left\{-\delta s - \int_0^s \lambda\left(X_x^{+(\nu)}(\gamma)\right)d\gamma\right\} g\left(X_x^{+(\nu)}(s)\right)ds\right],$$

for any $g \in B[a,b]$, $x \in (a,b]$ and $\lambda \in C[a,b]$, with λ a nonnegative function.
Notice that $M_{\delta,\lambda}^{a+(\nu)}g$ coincides with the solution (in the domain of the generator) to (4.4.1) only when $g \in C_a[a,b]$. This function will appear in the *generalized solution* to the nonlinear equation with nonconstant coefficients for any $g \in B[a,b]$. In order to write it down explicitly, we will need the following auxiliary results.
Set $Y(0) := 0$ and $Y(\xi) := \int_0^\xi \lambda\left(X_x^{+(\nu)}(\gamma)\right)d\gamma$ for any $\xi > 0$, where $\lambda \in C[a,b]$ is

a nonnegative function and $X_x^{+(\nu)}$ is the Feller process generated by the operator $(-G_+^{(\nu)}, \mathfrak{D}_G)$ in (2.3.1). Define the pair process

$$(Y, Z) = \{ (\, Y(\xi), Z(\xi) \,) : \xi \geq 0 \},$$

where

$$\begin{cases} Y(\xi) &= \int_0^\xi \lambda\,(\, Z(\gamma)\,)\, d\gamma \\ Z(\xi) &= X_x^{+(\nu)}(\xi). \end{cases} \tag{4.4.2}$$

Then (4.4.2) is the solution to the *Langevin type equation*:

$$dY = \lambda(Z) d\xi, \quad dZ = dX_x^{+(\nu)}(\xi),$$

with initial condition $(Y(0), Z(0)) = (0, x)$ (see, e.g., [3, 53]). The process (Y, Z) is a Markov process on $\mathbb{R}_+ \times (-\infty, b]$ with initial state $(0, x)$.

For any $(y_1, z_1), (y_2, z_2) \in \mathbb{R}_+ \times (-\infty, b]$, denote by $p_\xi(y_1, z_1; y_2, z_2)$ the transition density function from (y_1, z_1) to (y_2, z_2) with ξ being the time variable.

Remark 4.4.2. *If ν is the Lévy kernel in (2.4.1), then the process in (4.4.2) is the solution to a stable noise driven Langevin equation, see, e.g., [3, 39, 53].*

Lemma 4.4.3. *Let ν be a function satisfying conditions (H0)-(H3) and let $\lambda \in C[a, b]$ be a nonnegative function. Assume that the process (Y, Z) has transition densities $p_s(y_1, z_1; y_2, z_2)$. Then, for fixed $\xi \geq 0$ and for all $y \geq 0$, the distribution law of the random vector $\big(\, Y(\xi), \tau_a^{(\nu)}(x) \,\big)$ has the density $\phi_{\xi,a}^{x,\lambda}(y, \xi)$ given by*

$$\phi_{\xi,a}^{x,\lambda}(y, \xi) = -\frac{\partial}{\partial \xi} \int_a^x p_\xi\big(\, 0, x; y, r \,\big) dr.$$

Proof. Since the r.v.'s $Y(\xi)$ and $\tau_a^{(\nu)}(x)$ are not independent, to compute the distribution of the pair $(Y(\xi), \tau_a^{(\nu)}(x))$ we use the next equivalence

$$\big\{\, Y(\xi) > y,\, \tau_a^{(\nu)}(x) > \xi \,\big\} \;\equiv\; \big\{\, Y(\xi) > y,\, X_x^{+(\nu)}(\xi) > a \,\big\},$$

to obtain

$$\phi^{x,\lambda}_{\xi,a}(y,\xi) = \frac{\partial^2}{\partial y\,\partial\xi}\int_y^\infty\int_a^x p_\xi\big(0,x;w,r\big)dr\,dw = -\frac{\partial}{\partial\xi}\int_a^x p_\xi\big(0,x;y,r\big)dr,$$

as required. ∎

Lemma 4.4.4. *Under the assumptions of Lemma 4.4.3, the distribution law of the random vector* $\big(Y(s),\,X^{+(\nu)}_x(s),\tau^{(\nu)}_a(x)\big)$ *has the density*

$$\psi^{x,\lambda}_{s,a}(y,r,\xi) = -p_s(0,x;y,r)\,\frac{\partial}{\partial\xi}\int_a^r p^{+(\nu)}_{\xi-s}(r,z)\,dz,$$

for all $(y,r,\xi)\in\mathbb{R}_+\times(a,x]\times[s,\infty)$.

Proof. The equivalence between the events

$$\big\{Y(s)>y,X^{+(\nu)}_x(s)>r,\tau^{(\nu)}_a(x)>\xi\big\}\equiv\big\{Y(s)>y,X^{+(\nu)}_x(s)>r,X^{+(\nu)}_x(\xi)>a\big\},$$

implies that, if $s<\xi$, then

$$\mathbf{P}\big[Y(s)>y,X^{+(\nu)}_x(s)>r,X^{+(\nu)}_x(\xi)>a\big] = \int_y^\infty\int_r^x p_s(0,x;\gamma,w)\left(\int_a^w p^{+(\nu)}_{\xi-s}(w,z)dz\right)dw\,d\gamma,$$

where $p_s(\cdot,\cdot;\cdot,\cdot)$ and $p^{+(\nu)}_s(\cdot,\cdot)$ denote the transition density functions of the pair processes $(Y,X^{+(\nu)}_x)$ and $X^{+(\nu)}_x$, respectively. The result follows by differentiating the last expression with respect to the variables y, r and ξ. ∎

Lemma 4.4.5. *Let* $\lambda\in C[a,b]$ *be a nonnegative function. Let* $\delta>0$ *and* $g\in B[a,b]$. *Suppose condition (H0) holds. Then*

$$\mathbf{E}\left[\exp\left\{-\int_0^{\tau^{(\nu)}_a(x)}\lambda\Big(X^{+(\nu)}_x(s)\Big)\,ds\right\}\right] = \int_0^\infty\int_0^\infty \exp\{-y\}\phi^{x,\lambda}_{\xi,a}(y,\xi)\,dy\,d\xi; \quad (4.4.3)$$

and

$$M^{a+(\nu)}_{\delta,\lambda} g(x) = \int_0^{x-a} g(x-y) \int_0^{\infty} \int_0^{\infty} \exp\{-\delta s - y\}\ p_s(0,x;y,x-r)\, dy\, ds\, dr. \tag{4.4.4}$$

Proof. Equality (4.4.3) follows by conditioning on the r.v. $\tau_a^{(\nu)}(x)$ and then by using the joint density of $(X_x^{+(\nu)}(s), \tau_a^{(\nu)}(x))$ as given in Lemma 4.4.3. To prove (4.4.4), Fubini's theorem and the definition of Y yield

$$M^{a+(\nu)}_{\delta,\lambda} = \int_0^{\infty} \mathbf{E}\left[1_{\left\{\tau_a^{(\nu)}(x)>s\right\}} \exp\{-\delta\, s - Y(s)\}\, g\left(X_x^{+(\nu)}(s)\right)\right] ds.$$

Then, Lemma 4.4.4 implies

$$\begin{aligned}
M^{a+(\nu)}_{\delta,\lambda} &= \int_0^{\infty}\int_0^{\infty}\int_a^{x}\int_s^{\infty} \exp\{-\delta s - y\}\, g(r)\psi^{x,\lambda}_{s,a}(y,r,\xi)\, d\xi\, dr\, dy\, ds,\\
&= \int_a^{x} g(y) \int_0^{\infty}\int_0^{\infty} \exp\{-\delta s - y\}\ p_s(0,t;y,r) \int_s^{\infty}\left(-\frac{\partial}{\partial \xi}\int_a^{r} p^{+(\nu)}_{\xi-s}(r,z)\, dz\right) d\xi\ dy\, ds\, dr,\\
&= \int_a^{x} g(r) \int_0^{\infty}\int_0^{\infty} \exp\{-\delta s - y\}\ p_s(0,x;y,r)\left(\int_0^{\infty} \mu_a^{r,(\nu)}(\tilde{\xi})\, d\tilde{\xi}\right) dy\, ds\, dr,\\
&= \int_a^{x} g(y) \int_0^{\infty}\int_0^{\infty} \exp\{-\delta s - y\}\ p_s(0,x;y,r)\, dy\, ds\, dr,
\end{aligned}$$

where we have used that $\mu_a^{r,(\nu)}(\tilde{\xi})$ is the density of the r.v. $\tau_a^{(\nu)}(r)$ given in (2.5.5).

■

4.4.2 Explicit solutions: Feynman-Kac type formulas.

Consider the problem of finding a function $w \in C_a[a,b]$ satisfying

$$-D^{(\nu)}_{a+} w(x) = \lambda(x) w(x) - g(x), \quad x \in (a,b], \qquad w(a) = w_a, \tag{4.4.5}$$

for a given nonnegative function $\lambda \in C[a,b]$, $g \in B[a,b]$ and $w_a = 0$. Hereafter, we shall refer to (4.4.5) as the problem $(-D^{(\nu)}_{a+}, \lambda(\cdot), g, w_a)$. Similar notation will be used for the corresponding problem with the Caputo type operator.

Case 1: RL type operator

Theorem 4.4.6. *Let ν be a function satisfying conditions (H0)-(H1). Suppose that λ is a nonnegative function in $C[a,b]$ such that $\inf_{x\in[a,b]}\lambda(x) = \delta > 0$.*

(i) If $g \in C_a[a,b]$, then the unique solution (in the domain of the generator) to the problem $(-D^{(\nu)}_{a+}, \lambda(\cdot), g, 0)$ is given by formula (4.4.6).

(ii) For any $g \in B[a,b]$, the linear problem $(-D^{(\nu)}_{a+}, \lambda(\cdot), g, 0)$ has a unique generalized solution. This solution is given by the Feynman-Kac type formula

$$w(x) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} \exp\left\{-\int_0^s \lambda\left(X_x^{+(\nu)}(\gamma)\right)d\gamma\right\} g\left(X_x^{+(\nu)}(s)\right)ds\right]. \quad (4.4.6)$$

Moreover, if ν also satisfies conditions (H2)-(H3), then (4.4.6) rewrites

$$w(x) = \int_0^{x-a} g(x-y)\int_0^\infty\int_0^\infty \exp\{-y\}\ p_s(0,x;y,x-r)\,dy\,ds\,dr, \quad x\in(a,b]. \quad (4.4.7)$$

Proof. *(i)* Let $\delta > 0$ be as in the statement. Rewrite (4.4.5) as

$$-D^{(\nu)}_{a+}w(x) = \hat{\lambda}(x)w(x) + \delta w(x) - g(x), \quad x\in(a,b], \qquad w(a) = 0, \qquad (4.4.8)$$

where $\hat{\lambda}(x) := \lambda(x) - \delta$.
If $g \in C_a[a,b]$, then Theorem 4.3.1 in [53] states the existence of a solution (in the domain of the generator) to (4.4.8) which is given by the stationary Feynman-Kac (FK) formula

$$w(x) = \mathbf{E}\left[\int_0^\infty \exp\left\{-\delta s - \int_0^s \tilde{\lambda}\left(X_x^{a+(\nu)}(\gamma)\right)d\gamma\right\} g\left(X_x^{a+(\nu)}(s)ds\right)\right],$$

where $X_x^{a+\nu}$ is the process generated by $-D^{(\nu)}_{a+}$. Note that this solution coincides with (4.4.6) due to the fact that $g(a) = 0$ and $\mathbf{E}\left[\tau_a^{(\nu)}(x)\right] < \infty$. Moreover, the positive maximum principle (see, e.g., [53]) implies the uniqueness of the solution.
(ii) For the general case $g \in B[a,b]$, the stationary FK formula no longer provides a solution. However, by definition, the generalized solution can be obtained as a

limit of solutions in the domain of the generator. More precisely, take a sequence of functions $\{g_n\}_{n\geq 1}$ satisfying $g_n \to g$ a.e., $g_n \in C_a[a,b]$ and $\{\|g_n\|\}_{n\geq 1}$ being uniformly bounded, then the generalized solution is given by $w = \lim_{n\to\infty} w_n$, where for $n \geq 1$, w_n is the unique solution (in the domain of the generator) to the problem

$$-D^{(\nu)}_{a+} w_n(x) = \lambda(x) w_n(x) - g_n(x), \quad x \in (a,b], \qquad w_n(a) = 0.$$

For $n > 0$, the previous case provides the solution $w_n(x) = M^{a+(\nu)}_{\delta,\lambda} g_n(x)$. Hence, assumption (H1) and the dominated convergence theorem imply that the generalized solution is $w(x) = M^{a+(\nu)}_{\delta,\lambda} g(x)$, as required. Representation (4.4.7) follows directly from Lemma 4.4.5. ∎

Case 2: Caputo type operator

Theorem 4.4.7. *Suppose that the assumptions of Theorem 4.4.6 hold.*

(i) If $g \in C[a,b]$ and $g(a) = u_a \lambda(a)$, then there exists a unique solution in the domain of the generator to the Caputo type equation $(-D^{(\nu)}_{a+*}, \lambda(\cdot), g, u_a)$.

(ii) For any $g \in B[a,b]$ and $u_a \in \mathbb{R}$, the equation $(-D^{(\nu)}_{a+*}, \lambda(\cdot), g, u_a)$ has a unique generalized solution given by the Feynman-Kac type formula

$$\begin{aligned} u(x) &= u_a \, \mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{(\nu)}(x)} \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\}\right] + \\ &+ \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} g(X_x^{+(\nu)}(s)) \exp\left\{-\int_0^s \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\} ds\right]. \quad (4.4.9)\end{aligned}$$

Moreover, if ν also satisfies conditions (H2)-(H3), then the solution can be rewritten

$$\begin{aligned} u(x) &= u_a \int_0^\infty \int_0^\infty \exp\{-y\} \phi^{x,\lambda}_{\xi,a}(y,\xi)\, dy\, d\xi \\ &+ \int_0^{x-a} g(x-r) \int_0^\infty \int_0^\infty \exp\{-y\}\ p_s(0,x;y,x-r)\, dy\, ds\, dr. \end{aligned}$$
(4.4.10)

Proof. (ii) Define $v(x) := u(x) - u_a$ for $x \in [a,b]$. Using that the Caputo type derivative of a constant function is zero, it follows that

$$-D^{(\nu)}_{a+*}v(x) = \lambda(x)u(x) - g(x) = \lambda(x)v(x) - [g(x) - \lambda(x)u_a] =: \lambda(x)v(x) - \tilde{g}(x). \tag{4.4.11}$$

Further, $-D^{(\nu)}_{a+*}v = -D^{(\nu)}_{a+}v$ as $v(a) = 0$. Consequently, Theorem 4.4.6 gives

$$v(x) = \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} \left(g(X_x^{+(\nu)}(s)) - \lambda(X_x^{+(\nu)}(s))u_a\right)\exp\left\{-\int_0^s \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\}ds\right] \tag{4.4.12}$$

as the unique generalized solution to (4.4.11) for any $g \in B[a,b]$. Since (by Leibniz's formula)

$$\int_0^{\tau_a^{(\nu)}(x)} \lambda(X_x^{+(\nu)}(s))\exp\left\{-\int_0^s \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\}ds = 1 - \exp\left\{-\int_0^{\tau_a^{(\nu)}(x)} \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\},$$

the equation (4.4.12) becomes

$$\begin{aligned} v(x) =& -u_a + u_a\mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{(\nu)}(x)} \lambda(X_x^{+(\nu)}(s))\right\}ds\right] + \\ &+ \mathbf{E}\left[\int_0^{\tau_a^{(\nu)}(x)} g(X_x^{+(\nu)}(s))\exp\left\{-\int_0^s \lambda(X_x^{+(\nu)}(\gamma))d\gamma\right\}ds\right]. \end{aligned}$$

Equality $u(x) = v(x) + u_a$ then implies the result in (4.4.9). Finally, Lemma 4.4.5 implies directly (4.4.10).

(i) Follows from the previous case and the first assertion in Theorem 4.4.6. Namely, $u \in \mathfrak{D}^{(\nu)}_{a+*}$ whenever $w \in \mathfrak{D}^{(\nu)}_{a+} \subset \mathfrak{D}^{(\nu)}_{a+*}$, and the latter holds if, and only, if $\tilde{g}(a) = 0$. By definition of $\tilde{g}$ (see (4.4.11)), $\tilde{g}(a) = g(a) - \lambda(a)u_a$, yielding $g(a) = \lambda(a)u_a$, as desired. ∎

Remark 4.4.8. *A stochastic representation similar to (4.4.9) is a standard tool for studying parabolic PDE's (see [44], Proposition 7.2).*

Remark 4.4.9. *The explicit representations (4.4.7) and (4.4.10) can be obtained in terms of the transition probabilities instead of the transition densities, whose existence was assumed for simplicity.*

4.5 Composite fractional relaxation equation of Caputo and RL type

Let us now consider the equation

$$-\tilde{D}^{(\nu)}u(x) - \gamma(x)\frac{d}{dx}u(x) - \lambda u(x) = -g(x), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \qquad (4.5.1)$$

with $\lambda \geq 0$ and some given functions g and γ. This equation is the generalized version of *the composite fractional relaxation equation* introduced in [28],[62].
To prove its well-posedness we will use the next result which is an immediate consequence of Theorem 4.1 in [55].

Lemma 4.5.1. *Let ν be a function satisfying assumption (H0) and suppose $\gamma \in C_0^1[a,b]$. Then, the operator $-\tilde{D}^{(\nu,\gamma)} := -\tilde{D}^{(\nu)} - \gamma(\cdot)\frac{d}{dx}$ generates a Feller process $\tilde{X}_x^{(\nu,\gamma)}$ on $\tilde{C}[a,b]$ with the invariant core $\tilde{C}[a,b] \cap C^1[a,b]$. Moreover, if additionally γ is a nonnegative function and assumption (H4) holds, then the boundary point $x = a$ is regular in expectation.*

The operator $-\tilde{D}^{(\nu,\gamma)}$ should be understood as either the operator

$$-D_{a+}^{(\nu,\gamma)} := -D_{a+}^{(\nu)} - \gamma(\cdot)\frac{d}{dx}$$

or the operator

$$-D_{a+*}^{(\nu,\gamma)} := -D_{a+*}^{(\nu)} - \gamma(\cdot)\frac{d}{dx},$$

depending on $-\tilde{D}^{(\nu)}$. We will denote by $X_x^{a,(\nu,\gamma)}$ and $X_x^{a*(\nu,\gamma)}$ the corresponding Feller processes. Recall that notation $-\tilde{D}^{(\nu)}$ refers either to the Caputo type operator $-D_{a+*}^{(\nu)}$ or the RL type operator $-D_{a+}^{(\nu)}$.

The probabilistic interpretation of the operator $-\tilde{D}^{(\nu,\gamma)}$ as the generator of an *interrupted* Feller process still holds. If $X_x^{(\nu,\gamma)}$ is the Feller process (started at x) generated by $-D^{+(\nu)} - \gamma(\cdot)\frac{d}{dx}$ (the sum of the decreasing process in (2.3.1) and a drift term), then $X_x^{a,(\nu,\gamma)}$ (resp. $X_x^{a*(\nu,\gamma)}$) can be obtained by killing (resp. interrupting) the process $X_x^{(\nu,\gamma)}$ on an attempt to cross the boundary point a.

Remark 4.5.2. *In general, since the Feller process $X_x^{(\nu,\gamma)}$ is not decreasing, the interruption procedure in the Caputo type case does not mean stopping the process unless the drift term γ is nonnegative.*

Remark 4.5.3. *The three notions of solutions (generalized, classical, and in the domain of the generator) considered previously are extended to the linear problem with drift term given in (4.5.1) and to the corresponding nonlinear problem with $g(x) := f(x, u(x))$. This is done by replacing the operator $-\tilde{D}^{(\nu)}$ with the operator $-\tilde{D}^{(\nu,\gamma)}$ in Definition 3.2.1, Definition 3.2.3 and Definition 4.3.1.*

Well-posedness results (nonnegative γ)

The following result is the extension to Lemma 4.2.1 for the new operator $-\tilde{D}^{(\nu,\gamma)}$.

Theorem 4.5.4. ***(Linear case)*** *Let ν be a function satisfying assumption (H0) and (H4). Suppose that γ is a nonnegative function in $C_0^1[a,b]$.*

(i) If $g \in C[a,b]$ and $g(a) = \lambda \tilde{u}_a$, then there exists a unique solution $u \in \tilde{C}[a,b]$ in the domain of the generator to (4.5.1) given by $u(x) = R_\lambda^{(\nu,\gamma)} g(x)$, the resolvent operator of the semigroup generated by $-\tilde{D}^{(\nu,\gamma)}$.

(ii) For any $g \in B[a,b]$, the equation (4.5.1) has a unique generalized solution $u \in \tilde{C}[a,b]$. This solution admits the stochastic representation

$$u(x) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^{(\nu,\gamma)}(x)}\right] + \mathbf{E}\left[\int_0^{\tau_a^{(\nu,\gamma)}(x)} e^{-\lambda s} g\left(X_x^{(\nu,\gamma)}(s)\right) ds\right], \qquad (4.5.2)$$

where $\tau_a^{(\nu,\gamma)}(x)$ denotes the first time the process $X_x^{(\nu,\gamma)}$ leaves the interval $(a,b]$. Moreover, if additionally ν satisfies conditions (H2)-(H3), then the *solution takes the form*

$$u(x) = \tilde{u}_a \int_0^\infty e^{-\lambda s}\mu_a^{x,(\nu,\gamma)}(s)ds + \int_a^x g(y) \int_0^\infty e^{-\lambda s} p_s^{(\nu,\gamma)}(x,y)\, ds\, dy, \quad (4.5.3)$$

where $\mu_a^{x,(\nu,\gamma)}(s)$ *and* $p_s^{(\nu,\gamma)}(x,y)$ *are the density function of the r.v.* $\tau_a^{(\nu,\gamma)}(x)$ *and the transition densities of the process* $X_x^{(\nu,\gamma)}$*, respectively.*

Proof. (i) Since γ is a nonnegative function, the process generated by $-\tilde{D}^{(\nu,\gamma)}$ is a decreasing process, Theorem A.1.1 and Lemma 4.5.1 imply the result. (ii) Holds by using the definition of a generalized solution (see Remark 4.5.3) and the case (i) above. Details have been omitted as they are quite similar to those used in Chapter 3 for the operator $-\tilde{D}^{(\nu)}$. ∎

Theorem 4.5.5. ***(Nonlinear case)*** *Let* ν *be a function satisfying conditions (H0) and (H2)-(H4). Suppose that* $\gamma \in C_0^1[a,b]$ *is a nonnegative function. If* f *is a function satisfying condition (H5), then*

(i) there exists a unique generalized solution $u \in \tilde{C}[a,b]$ *to the nonlinear equation*

$$-\tilde{D}^{(\nu)}u(x) - \gamma(x)u'(x) - \lambda u(x) = -f(x,u(x)), \quad x \in (a,b], \quad u(a) = \tilde{u}_a, \quad (4.5.4)$$

(ii) If, additionally, f *is continuous and satisfies* $f(a,\tilde{u}_a) = \lambda\tilde{u}_a$*, then there is a unique solution in the domain of the generator.*

Proof. Since the drift term γ is nonnegative and the assumption $\nu(x,y) > Cy^{-1-\beta}$ holds, the process $X_x^{+(\nu,\gamma)}$ is decreasing and *dominates* the inverted β–stable subordinator $X_x^{+\beta}$ (see proof of Proposition 4.3.4 above for the notion of this concept). Hence, all the arguments and calculations used in the proof of Proposition 4.3.4 and Theorem 4.3.6 can be carried out similarly, details are then omitted. ∎

4.6 Examples: classical fractional setting

Since the generalized operators include the classical RL and Caputo derivatives, all the results presented above apply to the classical fractional setting and to their generalizations. This section highlights some important points in this context.

1. Lemma 4.3.1 applied to the fractional case states the equivalence between the fractional nonlinear equation

$$\tilde{D}^{\beta} u(x) = f(x, u(x)), \quad x \in (a, b], \qquad u(a) = \tilde{u}_a, \tag{4.6.1}$$

and the integral equation

$$u(x) = \tilde{u}_a + \int_a^x f(y, u(y))(x-y)^{\beta-1} \int_0^{\infty} s^{-1/\beta} w_{\beta}\left(s^{-1/\beta}; 1, 1\right) ds\, dy, \tag{4.6.2}$$

where w_β denotes the β–stable density (see Appendix) and $\tilde{D}^{\beta}$ stands for either the RL classical fractional derivatives D^{β}_{a+} or the Caputo derivate D^{β}_{a+*}, for $\beta \in (0,1)$. By comparing the integral equation (4.6.2) with the Volterra integral equation

$$u(x) = \tilde{u}_a + I^{\beta}_{a+} f(x, u(x)), \tag{4.6.3}$$

one can conclude (by uniqueness of solutions) that

$$\int_0^{\infty} s^{-1/\beta} w_{\beta}\left(s^{-1/\beta}; 1, 1\right) ds = \frac{1}{\Gamma(\beta)}. \tag{4.6.4}$$

The Volterra equation given in (4.6.3) is the integral equation commonly used in fractional calculus to prove the well-posedness for nonlinear fractional differential equations (see, e.g., [15]) The equivalence between (4.6.3) and the RL equation (4.6.1) has been proved on a space of functions similar to the space F_K defined in (4.3.7) (see, e.g., [15], [45]). This equivalence also holds for more general (possibly unbounded) continuous functions f on $(a,b] \times [-K, K]$ with some $K > 0$ such that $(x-a)^{\sigma} f(x, u(x)) \in C([a,b] \times [-K, K])$ with

$0 \le \sigma < \beta < 1$, (see, e.g., [45], [84]).

2. Theorem 4.3.6 provides the well-posedness for fractional nonlinear equations as well as for nonlinear equations involving fully mixed (multi-term) fractional derivatives (see Section 2.2.).

3. In the fractional setting, Theorem 4.3.10 implies the next result.

   Corollary 4.6.1. *Assume that $g \in C^1[a,b]$ and $\beta \in (0,1)$. If $g(a) = 0$, then there is a unique solution $u \in C_a^1[a,b]$ to the problem $(-D_{a+}^{\beta}, \lambda, g, 0)$ for any $\lambda > 0$. Moreover, if $g(a) = \lambda u_a$, then there is a unique solution $u \in C^1[a,b]$ for the Caputo type problem $(-D_{a+*}^{\beta}, \lambda, g, u_a)$.*

   Notice that if $g(a) \ne 0$, the derivative u' is continuous but unbounded as $x \to a$. This can be seen by differentiating the solution

$$u(x) = \int_0^{x-a} \int_0^{\infty} g(x-y) e^{-\lambda s} p_s^{+\beta}(x, x-y)\, ds\, dy\,,$$

   to obtain

$$\begin{aligned} u'(x) = &\int_0^{x-a} g'(x-y) r^{\beta-1} \int_0^{\infty} \exp\{-\lambda u r^{\beta}\} u^{-1/\beta} w_{\beta}(u^{-1/\beta}; 1, 1)\, du\, dy \\ &+ g(a)(x-a)^{\beta-1} \int_0^{\infty} \exp\{-\lambda u (x-a)^{\beta}\} u^{-1/\beta} w_{\beta}(u^{-1/\beta}; 1, 1)\, du. \end{aligned} \tag{4.6.5}$$

   As for the nonlinear case, the existence of a smooth solution in the closed interval $[a,b]$ follows by Theorem 4.3.11 under the assumption that f is a bounded function belonging to $C^1([a,b] \times \mathbb{R})$ satisfying $f(a, \tilde{u}_a) = 0$.

4. Theorem 4.5.4 implies that the solution to the *composite fractional relaxation equation* given in (4.1.9)-(4.1.11) can be rewritten as

$$u(x) = u_0 \int_0^{\infty} e^{-s} \mu_0^{x,(c_1,\beta,c_2)}(s) ds + \int_0^{x} g(x-y) \int_0^{\infty} e^{-s} p_s^{+(c_1,\beta,c_2)}(x, x-y)\, ds\, dy, \tag{4.6.6}$$

with $c_1, c_2 > 0$, $g \in C[0,b]$. Notation $\mu_0^{x,(c_1,\beta,c_2)}(s)$ denotes the density function of the first exit time, whereas $p_s^{+(c_1,\beta,c_2)}(x,y)$ refers to the transition density function of the Feller process generated by $-c_1 D_{0+*}^{\beta} - c_2 \frac{d}{dx}$.

5. By uniqueness of solutions, for any $g \in C[0,b]$ and any strictly positive function $\lambda \in C[0,b]$, Theorem 4.4.7 provides another integral representation of the solution to the fractional linear equation with nonconstant coefficients given in (4.1.7).

Chapter 5

Two-sided generalized equations of RL and Caputo type

This chapter provides well-posedness results for equations involving both the right- and the left-sided generalized fractional operators of Riemann-Liouville and Caputo type. The equations studied here include the two-sided generalized versions of both the fractional ordinary linear equation and the composite fractional relaxation equation. In the context of classical fractional derivatives, the results presented here show the interplay between two-sided fractional differential equations and two-sided exit problems for certain Lévy processes.

5.1 Introduction

We say that a generalized fractional equation is *two-sided* whenever it involves both left-sided and right-sided generalized fractional operators acting on the same variable. This chapter establishes the well-posedness for boundary value problems of *two-sided generalized fractional equations* of the following types:

(i) the *two-sided linear equation* with RL type derivatives $-D_{a+}^{(\nu_+)}$ and $-D_{b-}^{(\nu_-)}$:

$$-D_{a+}^{(\nu_+)}u(x) - D_{b-}^{(\nu_-)}u(x) - Au(x) = \lambda u(x) - g(x), \quad x \in (a,b),$$

$$u(a) = 0, \quad u(b) = 0, \tag{5.1.1}$$

(ii) the *two-sided generalized linear equation* with Caputo type derivatives $-D_{a+*}^{(\nu_+)}$ and $-D_{b-*}^{(\nu_-)}$:

$$-D_{a+*}^{(\nu_+)}u(x) - D_{b-*}^{(\nu_-)}u(x) - Au(x) = \lambda u(x) - g(x), \quad x \in (a,b),$$
$$u(a) = u_a, \quad u(b) = u_b, \tag{5.1.2}$$

where $\lambda \geq 0$, $u_a, u_b \in \mathbb{R}$, and g is a prescribed function on $[a,b]$. Here notation $-A$ denotes the second order differential operator

$$-A := \gamma(\cdot)\frac{d}{dx} + \alpha(\cdot)\frac{d^2}{dx^2}, \tag{5.1.3}$$

for given functions α and γ.

Equations (5.1.1)-(5.1.2) include, as special cases, the fractional differential equations involving left- and right-sided classical (RL or Caputo) fractional derivatives of order $\beta \in (0,1)$. In this setting, a particular case of equation (5.1.2) is the *two-sided fractional differential equation*:

$$D_{a+*}^{\beta_1}u(x) + D_{b-*}^{\beta_2}u(x) = g(x), \quad x \in (a,b), \quad \beta_1, \beta_2 \in (0,1), \tag{5.1.4}$$
$$u(a) = u_a, \quad u(b) = u_b,$$

with $D_{a+*}^{\beta_1}$ and $D_{b-*}^{\beta_2}$ being the left- and the right-sided Caputo derivatives of order β_1 and β_2, respectively.
There are relatively scarce results dealing with classical fractional ordinary differential equations (FODE's) involving both right- and left-sided fractional derivatives.

For example, the two-sided equation involving Riemann-Liouville derivatives

$$D^{\beta}_{0+}u(x) + cD^{\beta}_{b-}u(x) = g(x), \quad \beta = k + \alpha,\ k \in \mathbb{N}_0,\ \alpha \in (0,1), \tag{5.1.5}$$

was analyzed (in the space of distributions) in [80] (see also [81] and references therein). To the best of our knowledge, the explicit (probabilistic) solution to the two-sided fractional equation in (5.1.4) was just recently provided in [55]. In this chapter, we study in more detail the relationship (already mentioned in [55]) between equations (5.1.4)-(5.1.5) and *two-sided exit problems* for certain Lévy processes.

Another special case of equation (5.1.2) is the *two-sided* fractional equation including a drift term $\gamma(\cdot)$:

$$c_1 D^{\beta_1}_{a+*}u(x) + c_2 D^{\beta_2}_{b-*}u(x) + \gamma(x)u'(x) + \lambda u(x) = g(x), \quad x \in (a,b), \tag{5.1.6}$$
$$u(a) = u_a, \qquad u(b) = u_b.$$

If $c_1 > 0,\ c_2 = 0,\ \beta_1 = \frac{1}{2}$ and $\lambda = 1$, then the (one-sided) equation is known as the *Basset equation*, well-studied in the literature (see, e.g. [62] and references therein). The one-sided case for any $\beta_1 \in (0,1)$ (known as the *composite fractional relaxation equation*) was treated via the Laplace transform method in [27, Section 4], whereas the one-sided case with Caputo type and RL type operators was studied for γ nonnegative in the previous chapter.

Some others examples showing the relevance of left- and right-sided derivatives in mathematical modeling appear in the study of fractional partial diffrential equations (FPDE's) on bounded domains, as well as in fractional calculus of variations. For instance, for $c_+(t,x) \geq 0,\ c_-(t,x) \geq 0$ and $\alpha \in [1,2]$, the two-sided space-fractional equation

$$\frac{\partial u(t,x)}{\partial t} = c_+(t,x)D^{\alpha}_{a+}u(t,x) + c_-(t,x)D^{\alpha}_{b-}(t,x) + g(t,x), \quad a < x < b,\ 0 \leq t \leq T,$$

$$u(0,x) = \phi(x), \quad u(t,a) = 0 = u(t,b)$$

with RL fractional derivatives (using our notation), was addressed via numerical methods, e.g., in [43], [68]. The authors in [61] study (via a Fourier transform method) a space-time fractional diffusion equation involving Caputo derivatives and the operator

$$(1-\sigma)D^{\beta}_{-\infty+*} - \beta D^{\beta}_{\infty-*}, \quad 0 < \beta \leq 1, \ \sigma \in \mathbb{R}.$$

In the context of fractional calculus of variations (see, e.g., [1], [8]), compositions of left and right derivatives appear naturally in fractional Euler-Lagrange equations, wherein sequential operators of the form $D^{\alpha}_{b-*}D^{\alpha}_{a+}$ are present (see, e.g., [4], [49], [82]).

The well-posedness for the equations (5.1.1)-(5.1.2) is treated following similar arguments to those used in previous chapters. Roughly speaking, we deal with two types of solutions in a probabilistic framework: *solutions in the domain of the generator* and *generalized solutions*. The first type is understood as a solution u, where u belongs to the domain of the two-sided operator seen as the generator of a Feller process. Since the existence of such a solution is quite restrictive once one imposes boundary conditions, the concept of a generalized solution is introduced as the pointwise limit of approximating solutions taken from the domain of the generator. Additionally, appealing to the relationship between two-sided equations and exit problems for Feller processes (as shown in this work), we provide some explicit solutions to two-sided equations in the context of classical fractional derivatives. Even though exit problems for Lévy processes have been widely studied (see, e.g., [7], [9], [57], [58], [85]), to our knowledge fractional equations of the type in (5.1.4) and their connection with exit problems seem to be novel in the literature. This interplay between probability and fractional operators provides a new approach to approximate solutions of classical fractional equations through the probabilistic solutions presented in this work.

5.2 Two-sided operators of RL type and Caputo type

In this section we introduce new operators and some preliminary results needed to solve two-sided generalized equations via a probabilistic approach.

Given two functions ν_+ and ν_- satisfying condition (H0), we define the function $\nu : \mathbb{R} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ associated with ν_+ and ν_- by setting

$$\nu(x, y) := \nu_+(x, y), \quad y > 0, \qquad \nu(x, y) := \nu_-(x, -y), \quad y < 0. \tag{5.2.1}$$

Define the operators $-L^{(\nu,\gamma,\alpha)}_{[a,b]}$ and $-L^{(\nu,\gamma,\alpha)}_{[a,b]*}$ (acting on functions from $[a, b]$ to $\mathbb{R}$) by

$$\left(-L^{(\nu,\gamma,\alpha)}_{[a,b]} f\right)(x) := \left(-D^{(\nu_+)}_{a+} f\right)(x) + \left(-D^{(\nu_-)}_{b-} f\right)(x) + \left(-A^{(\gamma,\alpha)} f\right)(x), \quad x \in [a, b], \tag{5.2.2}$$

and

$$\left(-L^{(\nu,\gamma,\alpha)}_{[a,b]*} f\right)(x) := \left(-D^{(\nu_+)}_{a+*} f\right)(x) + \left(-D^{(\nu_-)}_{b-*} f\right)(x) + \left(-A^{(\gamma,\alpha)} f\right)(x), \quad x \in [a, b]. \tag{5.2.3}$$

Notation $-A^{(\gamma,\alpha)}$ stands for the second order differential operator with drift term $\gamma(\cdot)$ and diffusion term $\alpha(\cdot)$ as was defined in (5.1.3). Operators (5.2.2) and (5.2.3) will be referred to as the *two-sided operators of RL type* and the *two-sided operators of Caputo type*, respectively.

Remark 5.2.1. *For notational convenience we will also use notation*

$$-L_{[a,b]} \equiv L^{(\nu,\gamma,\alpha)}_{[a,b]}, \quad -L_{[a,b]*} \equiv -L^{(\nu,\gamma,\alpha)}_{[a,b]*}, \quad \textit{and} \quad -A \equiv -A^{(\gamma,\alpha)}.$$

Occasionally, notation $-L^{(\nu)}_{[a,b]}$ and $-L^{(\nu)}_{[a,b]*}$ (resp. $-L^{(\nu,\gamma)}_{[a,b]}$ and $-L^{(\nu,\gamma)}_{[a,b]*}$) will be used in the absence of the operator $-A$ (resp. in the absence of the diffusion term, i.e.

$-A = \gamma(\cdot)\frac{d}{dx}$).

Remark 5.2.2. *In the absence of a diffusion term the aforementioned operators are well-defined on $C^1[a,b]$, otherwise, on the space $C^2[a,b]$.*

Notice now that the equations (5.1.1)-(5.1.2) can be rewritten as

$$-L_{[a,b]}u(x) = \lambda u(x) - g(x) \qquad \text{and} \qquad -L_{[a,b]*}u(x) = \lambda u(x) - g(x),$$
$$u(a) = 0 = u(b) \qquad\qquad u(a) = u_a,\ u(b) = u_b,$$

respectively. Thus, to be able to solve these equations via a probabilistic approach we will need to state conditions to guarantee that both operators $-L_{[a,b]}$ and $-L_{[a,b]*}$ generate Feller processes and that the boundary points $\{a,b\}$ are *regular* enough.

5.2.1 Operators $-L_{[a,b]}$ and $-L_{[a,b]*}$ as generators

We will proceed as was done in [55] for the operator $-L^{(\nu,\gamma)}_{[a,b]*}$ (therein denoted by $A_{[a,b]*}$). We will see that the operator $-L_{[a,b]*}$ can be thought of as the generator of an *interrupted* process on $[a,b]$.

Theorem 5.2.3. *Let ν be a function satisfying assumption (H0). Suppose that $\gamma \in C^3_0[a,b]$, $\alpha \in C^3[a,b]$ with derivative $\alpha' \in C_0[a,b]$ and α being a positive function. Then, the operator $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ on $[a,b]$ with a domain $\hat{\mathfrak{D}}_*$ such that*

$$\left\{f \in C^2[a,b] : f' \in C_0[a,b]\right\} \subset \hat{\mathfrak{D}}_*. \tag{5.2.4}$$

Proof. See proof in Section 5.6. ∎

Particular cases of Theorem 5.2.3 also hold under weaker assumptions.

Theorem 5.2.4. *Assume that the operator $-L_{[a,b]*}$ is such that either $-A$ is not present (i.e. $\gamma(\cdot) = 0 = \alpha(\cdot)$) or $-A = \gamma(\cdot)\frac{d}{dx}$ with $\gamma \in C^1_0[a,b]$. If assumption (H0)* *holds, then the operator* $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ *generates a Feller process* $\hat{X}$ *on* $[a,b]$ *with the space* $C^1[a,b] \subset \hat{\mathfrak{D}}_*$ *as an invariant core.*

Proof. We omit the proof as it follows the same arguments as those used in Section 5.6 for the proof of Theorem 5.2.3. See also [55, Theorem 4.1] for the case $\alpha(\cdot) = 0$. ∎

Stopped and killed processes.

To introduce the notion of solutions we are interested in, we will need the stopped and killed versions of the aforementioned process $\hat{X}$. We will use the concept of regularity for the boundary points $\{a,b\}$ as was given in Definition 2.5.1.

Proposition 5.2.5. *Suppose that the assumptions of Theorem 5.2.3 hold. Then* $\{a,b\}$ *are regular in expectation for the operator* $(-\hat{L}_{[a,b]*}, \hat{\mathfrak{D}}_*)$*. Moreover, the first exit time* $\hat{\tau}_{(a,b)}(x)$ *from the interval* (a,b) *of the process* $\hat{X}_x$*, for* $x \in (a,b)$*, has a finite expectation.*

Proof. See proof in Section 5.6. ∎

Theorem 5.2.6. *Suppose that the assumptions of Theorem 5.2.3 hold. Let* $\hat{X}_x$ *be the process (started at* $x \in (a,b)$*) generated by* $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$*. Then,*

(i) The process $X_x^{[a,b]*}$ *defined by* $X_x^{[a,b]*}(s) := \hat{X}_x(s \wedge \hat{\tau}_{(a,b)}(x))$*, for all* $s \geq 0$ *and* $x \in (a,b)$*, is a Feller process on* $[a,b]$*. If* $(-L_{stop}, \mathfrak{D}^{stop}_{[a,b]*})$ *denotes the generator of* $X^{[a,b]*}$*, then for any* $f \in \hat{\mathfrak{D}}_*$ *such that* $\left(-L_{[a,b]*}f\right)(x) = 0$ *for* $x \in \{a,b\}$*, it follows that* $f \in \mathfrak{D}^{stop}_{[a,b]*}$ *and* $-L_{stop}f = -L_{[a,b]*}f$*.*

(ii) The process $X_x^{[a,b]}$ *defined by* $X_x^{[a,b]}(s) := X_x^{[a,b]*}(s)$ *for* $s < \hat{\tau}_{(a,b)}(x)$ *and* $x \in (a,b)$ *is a Feller (sub-Markov) process on* (a,b)*. If* $(-L_{kill}, \mathfrak{D}^{kill}_{[a,b]})$ *denotes the the generator of* $X^{[a,b]}$*, then for any* $f \in \mathfrak{D}^{stop}_{[a,b]*}$ *such that* $f(x) = 0$ *for* $x \in \{a,b\}$*, it holds that* $f \in \mathfrak{D}^{kill}_{[a,b]}$ *and* $-L_{kill}f = -L_{[a,b]}f$*.*

Proof. See proof in Section 5.6. ∎

To guarantee the regularity of the boundary points $\{a, b\}$ for the cases when $-A$ vanishes or $-A = \gamma(\cdot)\frac{d}{dx}$, the following assumption will be needed:

(H1') There exists a constant $C > 0$ and $q \in (0, 1)$ such that

$$\int_{-\infty}^{0} \min(|y|, \epsilon)\nu(a, y)dy > C\epsilon^{q}$$
$$\int_{0}^{\infty} \min(y, \epsilon)\nu(b, y)dy > C\epsilon^{q}.$$

Theorem 5.2.7. *Suppose that ν satisfies assumption (H0).*

(i) If $-A$ vanishes and condition (H1') also holds, then the statements (i)-(ii) of Theorem 5.2.6 hold for the operators $-L^{(\nu)}_{[a,b]}$ and $-L^{(\nu)}_{[a,b]*}$ (see Remark 5.2.1 for notation).

(ii) If $-A = \gamma(\cdot)\frac{d}{dx}$ with $\gamma \in C^1_0[a, b]$ and additionally assumption (H1') holds, then statements (i)-(ii) of Theorem 5.2.6 hold for the operators $-L^{(\nu,\gamma)}_{[a,b]}$ and $-L^{(\nu,\gamma)}_{[a,b]*}$.

Proof. The regularity in expectation of $\{a, b\}$ is obtained via the Lyapunov method (see proof of Proposition 5.2.5). Then, the proof follows a similar reasoning to the one used in the proof of Theorem 5.2.6. ■

Remark 5.2.8. *The operator $-L_{[a,b]*}$ can be obtained from the generator $(L, \mathfrak{D}_L)$ of a Feller process X_x given by*

$$(Lf)(x) = \int_{-\infty}^{\infty} (\, f(x + y) - f(x)\,)\nu(x, y)dy + \gamma(x)f'(x) + \alpha(x)f''(x), \qquad (5.2.5)$$

by modifying it in such a way that it forces the jumps aimed to be out of the interval (a, b) to land at the nearest (boundary) point (see also [55]). If, instead, the process is killed upon leaving (a, b), then the corresponding process has the operator $-L_{[a,b]}$ as generator (with a suitable domain). Hence, when starting at the same state $x \in (a, b)$, it holds that

1. *the first exit times from the interval* (a,b) *of the processes* X_x, $\hat{X}_x$, $X_x^{[a,b]*}$ *and* $X_x^{[a,b]}$ *all have the same distribution. Thus, in all cases, the first exit time will always be denoted by* $\tau_{(a,b)}(x)$ *indistinctly.*

2. *The paths of the processes* $X_x^{[a,b]*}$ *and* $X_x^{[a,b]}$ *coincide with the paths of the process* X_x *until the first exit time* $\tau_{(a,b)}(x)$.

We refer to X_x, $\hat{X}_x$, $X_x^{[a,b]*}$ *and* $X_x^{[a,b]}$ *as the underlying process, the interrupted process, the stopped process and the killed process, respectively.*

5.3 Two-sided equations involving RL type operators

We are now able to study the two-sided linear equation of RL type given in (5.1.1). This equation will also be referred to as the equation $\left(-L_{[a,b]}, \lambda, g, u_a, u_b\right)$ where u_a and u_b stand for the boundary conditions. In the case of RL type operators (and due to their relationship with generators of killed processes) we will always assume $u_a = 0 = u_b$.

The standard theory of Feller processes ensures that, for any function $g \in C_0[a,b]$, a solution to the resolvent equation $-L_{[a,b]*}u = \lambda u - g$ belonging to $\hat{\mathfrak{D}}_*$ is given by the resolvent operator $u = \hat{R}_\lambda g$ corresponding to the process $\hat{X}$ (see, e.g. Theorem A.1.1 and Theorem A.1.2 in Appendix). Moreover, by Theorem 5.2.6, if $-L_{[a,b]*}u(x) = 0$ on the boundary points $\{a,b\}$ and $u \in C_0[a,b]$, then $-L_{[a,b]}u = -L_{[a,b]*}u$. Using this, we shall introduce two notions of solutions for RL type equations. These definitions are similar to those used in the case of the one-sided operators $-D_{a+}^{(\nu)}$ (see Chapter 3).

Definition 5.3.1. ***(Solutions to RL type equations)*** *Let* $g \in B[a,b]$ *and* $\lambda \geq 0$. *A function* $u \in C_0[a,b]$ *is said to solve the two-sided linear equation of RL type* $(-L_{[a,b]}, \lambda, g, 0, 0)$ *as*

(i) *a solution in the domain of the generator if* u *satisfies (5.1.1) and* u *belongs to* $\mathfrak{D}_{[a,b]}^{kill}$;

(ii) *a generalized solution if for all sequence of functions* $g_n \in C_0[a,b]$ *such that* $\sup_n \|g_n\| < \infty$ *uniformly on* n *and* $\lim_{n\to\infty} g_n \to g$ *a.e., it holds that* $u(x) = \lim_{n\to\infty} w_n(x)$ *for all* $x \in [a,b]$*, where* w_n *is the unique solution (in the domain of the generator) to the two-sided RL type equation* $(-L_{[a,b]}, \lambda, g_n, 0, 0)$.

Definition 5.3.2. *We will say that the equation* $(-L_{[a,b]}, \lambda, g, 0, 0)$ *is well-posed in the generalized sense if it has a unique generalized solution for* $g \in B[a,b]$ *and* $\lambda \geq 0$.

5.3.1 Well-posedness results

Theorem 5.3.1. *Let* $\lambda \geq 0$, $\gamma \in C_0^3[a,b]$ *and* $\alpha \in C^3[a,b]$ *with derivative* $\alpha' \in C_0[a,b]$*. Suppose that* α *is positive function. Assume that the function* ν *associated with* ν_+ *and* ν_- *(defined via the equalities in (5.2.1)) satisfies assumption (H0). Let* $\hat{R}_\lambda$ *denote the resolvent operator (or the potential operator if* $\lambda = 0$*) of the process* $\hat{X}_x$.

(i) *If* $g \in C_0[a,b]$ *and* $\hat{R}_\lambda g(x) = 0$ *for* $x \in \{a,b\}$*, then there exists a unique solution* $u \in C_0[a,b]$ *in the domain of the generator to the two-sided RL type equation*

$$(-D_{a+}^{(\nu_+)} - D_{b-}^{(\nu_-)} + \gamma(\cdot)d/dx + \alpha(\cdot)d^2/dx^2, \lambda, g, 0, 0). \quad (5.3.1)$$

The solution is given by $u(x) = R_\lambda^{[a,b]} g(x)$*, where* $R_\lambda^{[a,b]}$ *denotes the resolvent operator (or the potential operator if* $\lambda = 0$*) of the process* $X_x^{[a,b]}$*. Furthermore,* u *takes the stochastic representation given in (5.3.2) below.*

(ii) *For any* $g \in B[a,b]$*, the equation (5.3.1) has a unique generalized solution* $u \in C_0[a,b]$ *given by*

$$u(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X_x(t)\right) dt\right], \quad (5.3.2)$$

where $\tau_{(a,b)}(x)$ *denotes the first exit time from the interval* (a,b) *of the underlying process* X_x *generated by the operator (5.2.5).*

Proof. *(i)* Theorem 5.2.3 implies that the operator $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ on (a,b), with semigroup strongly continuous on $C[a,b]$. Then, by Theorem A.1.1, the resolvent equation $-L_{[a,b]*}u = \lambda u - g$ has a unique solution u in the domain of the generator given by $u = \hat{R}_\lambda g$, the resolvent operator of the process $\hat{X}$, whenever $\lambda > 0$ and $g \in C[a,b]$. In particular, the latter statement holds for $g \in C_0[a,b]$ satisfying $\hat{R}_\lambda g(x) = 0$ for $x \in \{a,b\}$. Since the resolvent operator satisfies

$$\lambda \hat{R}_\lambda g - g = -L_{[a,b]*}u,$$

the assumptions on g ensure that $-L_{[a,b]*}u(x) = 0$ for $x \in \{a,b\}$. Hence, Theorem 5.2.6 implies that $u \in \mathfrak{D}^{kill}_{[a,b]}$ and $-L_{[a,b]*}u = -L_{[a,b]}u$, which in turn implies that the boundary problem $(-L_{[a,b]}, g, \lambda, 0, 0)$ is equivalent to the resolvent equation $-L_{[a,b]*}u = \lambda u - g$. Therefore, by Theorem A.1.1 in Appendix, the unique solution $u \in \mathfrak{D}^{kill}_{[a,b]}$ is given by the resolvent operator $u = \hat{R}_\lambda g$ which also coincides with $R^{[a,b]}_\lambda g$, where $R^{[a,b]}_\lambda$ stands for the resolvent operator of the process $X^{[a,b]}$ whenever $\lambda > 0$.

Note that $\tau_{(a,b)}(x) := \inf\{t \geq 0 : X^{[a,b]}_x(t) \notin (a,b)\}$ is the lifetime of the process $X^{[a,b]}_x$. Since Proposition 5.2.5 ensures that the boundary points $\{a,b\}$ are regular in expectation, the definition of the resolvent operator and Fubini's theorem imply

$$R^{[a,b]}_\lambda g(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X^{[a,b]}_x(t)\right) dt\right], \tag{5.3.3}$$

yielding (5.3.2) as the paths of the processes $X^{[a,b]}_x$ and X_x coincide before the time $\tau_{(a,b)}(x)$. If $\lambda = 0$, then observe that setting $\lambda = 0$ in (5.3.3) implies (as $\tau_{(a,b)}(x)$ has a finite expectation) that

$$||R^{[a,b]}_0 g|| \leq \sup_{x \in [a,b]} \mathbf{E}\left[\tau_{(a,b)}(x)\right] < +\infty.$$

Therefore, the potential operator $R^{[a,b]}_0 g$ provides the unique solution belonging to the domain $\mathfrak{D}^{kill}_{[a,b]}$ (by Theorem A.1.2), as required.

(ii) Take $g \in B[a,b]$ and any sequence $\{g_n\}$ satisfying the conditions from Definition 5.3.1. Fubini's theorem and the dominated convergence theorem (DCT) applied to (5.3.3) imply the convergence of $\lim_{n\to\infty} R_\lambda^{[a,b]} g_n(x) =: u(x)$, $x \in [a,b]$, which in turn implies that u is the unique generalized solution to (5.3.1), as required.

■

Cases $\gamma(\cdot) = 0$ **and** $\alpha(\cdot) = 0$

Theorem 5.3.2. *Let $\lambda \geq 0$. Assume that the function ν associated with ν_+ and ν_- (defined via the equalities in (5.2.1)) satisfies assumptions (H0) and (H1'). Then, the statements (i)-(ii) of Theorem 5.3.1 hold with $\alpha \equiv 0$ and with either $\gamma \equiv 0$ or $\gamma \in C_0^1[a,b]$.*

Proof. Follows similar arguments to those used above but using Theorem 5.2.7.

■

Corollary 5.3.3. *Let $\lambda \geq 0$ and $g_k \in B[a,b]$, for $k \in \{1,2\}$. If u_k is the generalized solution to the two-sided RL type equation $(-L_{[a,b]}, \lambda, g_k, 0, 0)$, then*

$$\|u_1 - u_2\| \leq \|g_1 - g_2\| \frac{1}{\lambda}, \qquad \text{if} \quad \lambda > 0 \tag{5.3.4}$$

and

$$\|u_1 - u_2\| \leq \|g_1 - g_2\| \sup_{x \in [a,b]} \mathbf{E}\left[\, \tau_{(a,b)}(x)\right], \qquad \text{if} \quad \lambda = 0. \tag{5.3.5}$$

In particular, the solution u to the two-sided generalized equation of RL type given by $(-L_{[a,b]}, \lambda, g, 0, 0)$ depends continuously on the function g.

Proof. Follows from the bounds of the resolvent $R_\lambda^{[a,b]} g$ and the potential operator $R_0^{[a,b]} g$, respectively. ■

5.4 Two-sided equations involving Caputo type operators

We now turn our attention to the well-posedness for the Caputo type equation given in (5.1.2). This equation will also be referred to as the equation $(-L_{[a,b]*}, \lambda, g, u_a, u_b)$. To introduce the notion of a solution in this case, we first observe that equation (5.1.2) can be rewritten in terms of the RL type operator $-L_{[a,b]}$ due to the following relation (see equalities (2.2.5) and (2.2.6))

$$-L_{[a,b]*}h(x) = -L_{[a,b]}h(x) + h(a)\int_{x-a}^{\infty} \nu(x,y)dy + h(b)\int_{b-x}^{\infty} \nu(x,y)dy.$$

Consequently, both operators coincide on functions h vanishing at the boundary points $\{a, b\}$. With this in mind, assume now that u solves (5.1.2). Take any function $\phi \in \mathfrak{D}^{stop}_{[a,b]*}$ satisfying $\phi(a) = u_a$ and $\phi(b) = u_b$. By Theorem 5.2.3 and Theorem 5.2.6, we can take for example, $\phi \in C^2[a,b]$ such that $\phi' \in C_0[a,b]$, $\phi(a) = u_a$, $\phi(b) = u_b$ and ϕ satisfying $\left(-L_{[a,b]*}\phi\right)(x) = 0$ for $x \in \{a, b\}$.

Define $w(x) := u(x) - \phi(x)$ for all $x \in [a,b]$ and observe that w vanishes at the boundary, then

$$-L_{[a,b]}w(x) = -L_{[a,b]*}w(x) = -L_{[a,b]*}u(x) + L_{[a,b]*}\phi(x).$$

Thus

$$\begin{aligned} -L_{[a,b]}w(x) &= \lambda u(x) - g(x) + L_{[a,b]*}\phi(x), \\ &= \lambda w(x) + \lambda\phi(x) - g(x) + L_{[a,b]*}\phi(x), \end{aligned} \tag{5.4.1}$$

which yields the RL type equation $(-L_{[a,b]}, \lambda, g - L_{[a,b]*}\phi - \lambda\phi, 0, 0)$ for the function w. Therefore, if w is the (possibly generalized) solution to equation (5.4.1), then the function $u = w + \phi$ can be considered as a generalized solution to the original Caputo type equation $(-L_{[a,b]*}, \lambda, g, u_a, u_b)$.

The previous discussion motivates the next definition.

Definition 5.4.1. ***(Solutions to Caputo type equations)*** *Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $u \in C[a,b]$ is said to solve the linear equation (5.1.2) as*

(i) a solution in the domain of the generator if u satisfies (5.1.2) and u belongs to $\mathfrak{D}^{stop}_{[a,b]*}$;

(ii) a generalized solution if u can be written as $u = \phi + w$, where w is the (possibly generalized) solution to the RL type problem

$$(-L_{[a,b]},\, \lambda,\, g - L_{[a,b]*}\phi - \lambda\phi,\, 0,\, 0)$$

with $\phi \in C^2[a,b]$ satisfying that $\phi' \in C_0[a,b]$, $-L_{[a,b]*}\phi(x) = 0$ for $x \in \{a,b\}$, $\phi(a) = u_a$ and $\phi(b) = u_b$.

Definition 5.4.2. *We say that the two-sided linear equation (5.1.2) is well-posed in the generalized sense if it has a unique generalized solution for $g \in B[a,b]$.*

Next result guarantees the uniqueness of generalized solutions.

Theorem 5.4.1. *If a generalized solution $u = w + \phi$ exists for the Caputo type linear equation (5.1.2), then this is unique and thus independent of ϕ.*

Proof. Suppose that the equation (5.1.2) has two different solutions u_j for $j \in \{1,2\}$. Then, $u_j = w_j + \phi_j$, where w_j is the unique solution (possibly generalized) to the RL type equation $(-L_{[a,b]}, \lambda, g - L_{[a,b]*}\phi_j - \lambda\phi_j, 0, 0)$ for some ϕ_j satisfying the conditions of Definition 5.4.1. Define $u(x) := u_1(x) - u_2(x)$ for $x \in [a,b]$, then

$$\begin{aligned}-L_{[a,b]}u(x) = -L_{[a,b]*}u(x) &= -L_{[a,b]*}u_1(x) + L_{[a,b]*}u_2(x)\\ &= \big(\lambda u_1(x) - g(x)\big) - \big(\lambda u_2(x) - g(x)\big)\\ &= \lambda u(x).\end{aligned}$$

Therefore, u solves the RL type equation $(-L_{[a,b]}, \lambda, g = 0, 0, 0)$ whose unique solution (by Theorem 5.3.1) is $u \equiv 0$, which implies the uniqueness and thus the independence of ϕ. ∎

5.4.1 Well-posedness results

Theorem 5.4.2. *Let $\lambda \geq 0$. Suppose that the assumptions of Theorem 5.3.1 hold.*

(i) For any $g \in B[a,b]$, the two-sided equation

$$(-D_{a+*}^{(\nu_+)} - D_{b-*}^{(\nu_-)} + \gamma(\cdot)d/dx + \alpha(\cdot)d^2/dx^2, \lambda, g, u_a, u_b). \tag{5.4.2}$$

is well-posed in the generalized sense. The solution admits the stochastic representation

$$\begin{aligned} u(x) = u_a \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)} \mathbf{1}_{\{X_x(\tau_{(a,b)}(x)) \leq a\}}\right] + u_b \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)} \mathbf{1}_{\{X_x(\tau_{(a,b)}(x)) \geq b\}}\right] \\ + \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X_x(t)\right) dt\right], \end{aligned} \tag{5.4.3}$$

where $\tau_{(a,b)}(x)$ denotes the first exit time from the interval (a,b) of the underlying process X_x generated by the operator in (5.2.5).

(ii) If $g \in C[a,b]$ satisfying $\lambda \hat{R}_\lambda g(x) = g(x)$ for $x \in \{a,b\}$, $g(a) = \lambda u_a$ and $g(b) = \lambda u_b$, then (5.4.3) is the unique solution to (5.4.2) in the domain of the generator.

Proof. (i) By Theorem 5.2.3 the operator $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}$ on $[a,b]$, whereas Proposition 5.2.5 ensures that $\tau_{(a,b)}(x)$ has a finite expectation. Let us take any function $\phi \in C^2[a,b]$ satisfying the conditions stated in Definition 5.4.1. Then (by Theorem 5.3.1) the generalized solution w to the RL type equation $(-L_{[a,b]}, g - \lambda\phi - L_{[a,b]*}\phi, \lambda, 0, 0)$ is given by $w(x) = I - II$, where

$$I := \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t} g\left(X_x^{[a,b]}(t)\right) dt\right]$$

$$II := \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}(\lambda + L_{[a,b]*})\phi\left(X_x^{[a,b]}(t)\right) dt\right].$$

Thus, $u = w + \phi$ is (by definition) the generalized solution to (5.4.2). Using that for all $\lambda > 0$ and for all ϕ belonging to the domain of the generator $L_{[a,b]*}$ it follows (by Theorem A.1.4 in Appendix) that the process Y defined by

$$Y(r) := e^{-\lambda r}\phi\left(X_x^{[a,b]*}(r)\right) + \int_0^r e^{-\lambda s}(\lambda + L_{[a,b]*})\phi\left(X_x^{[a,b]*}(s)\right) ds, \qquad (5.4.4)$$

is a martingale. Furthermore, since the stopping time $\tau_{(a,b)}(x)$ has finite expectation, Doob's stopping theorem ([53, Theorem 3.10.1, p. 142]) applied to the martingale (5.4.4) implies that

$$\begin{aligned}\phi(x) =& \mathbf{E}\left[\ e^{-\lambda\tau_{(a,b)}(x)}\phi\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right] \quad + \\ &+ \quad \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda s}(\lambda + L_{[a,b]*})\phi\left(X_x^{[a,b]*}(s)\right) ds\right],\end{aligned}$$

yielding

$$II = \phi(x) - \mathbf{E}\left[\ e^{-\lambda\tau_{(a,b)}(x)}\phi\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right]$$

which in turn implies

$$u(x) = \mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right] + \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda t}g\left(X_x^{[a,b]*}(t)\right) dt\right], \qquad (5.4.5)$$

as $\phi\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right) = u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)$ by assumption.

Since at the random time $\tau_{(a,b)}(x)$ the process $X_x^{[a,b]*}$ takes either the value a or the value b, the term $u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)$ appearing in (5.4.5) is completely determined by the boundary conditions prescribed. Hence, the first term in the r.h.s of (5.4.5) can be written in terms of the underlying process X_x (generated by the operator in

(5.2.5)) as

$$\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}u\left(X_x^{[a,b]*}\left(\tau_{(a,b)}(x)\right)\right)\right] =$$
$$= u_a\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\le a\}}\right] + u_b\mathbf{E}\left[e^{-\lambda\tau_{(a,b)}(x)}\mathbf{1}_{\{X_x(\tau_{(a,b)}(x))\ge b\}}\right],$$

which implies (5.4.3).

(ii) Assume now that $g \in C[a,b]$ satisfying $\lambda\hat{R}_\lambda g(x) = g(x)$ for $x \in \{a,b\}$, then item (i) above ensures that the solution to (5.4.2) is given by $u = w + \phi$, where w is a RL type solution and ϕ a function satisfying the conditions in stated in Definition 5.4.1. Hence, by Theorem 5.3.1, w belongs to $\mathfrak{D}^{kill}_{[a,b]} \subset \mathfrak{D}^{stop}_{[a,b]}$ whenever

$$g(a) = \lambda u_a + (-L_{[a,b]*}\phi)(a) \quad \text{and} \quad g(b) = \lambda u_b + (-L_{[a,b]*}\phi)(b).$$

Since $(-L_{[a,b]*}\phi)(a) = (-L_{[a,b]*}\phi)(b) = 0$ because $\phi \in \mathfrak{D}^{stop}_{[a,b]*}$ by Theorem 5.2.6. Further, assumption $\lambda\hat{R}_\lambda g(x) = g(x)$ for $x \in \{a,b\}$ implies $-L_{[a,b]*}u(x) = 0$ for $x \in \{a,b\}$, which in turn implies $-L_{[a,b]*}u = -L_{stop}u$. Hence, Theorem 5.2.6 guarantees that $u \in \mathfrak{D}^{stop}_{[a,b]*}$ whenever $g(a) = \lambda u_a$ and $g(b) = \lambda u_b$, as required.

■

Cases $\alpha \equiv 0$ and $\gamma \equiv 0$

Theorem 5.4.3. *Let $\lambda \ge 0$. Suppose that the function ν associated with ν_+ and ν_- (defined via the equalities in (5.2.1)) satisfies assumptions (H0) and (H1'). Then, the statements (i)-(ii) of Theorem 5.4.2 hold with $\alpha \equiv 0$ and with either $\gamma \equiv 0$ or $\gamma \in C^1_0[a,b]$.*

Proof. Follows similar arguments to those used previously but using Theorem 5.2.7 and Theorem 5.3.2. We omit the details. ■

Corollary 5.4.4. *Let $\lambda \ge 0$. Suppose that $g_k \in B[a,b]$ and $u_a^k, u_b^k \in \mathbb{R}$, for $k \in \{1,2\}$. If u_k is the generalized solution to the two-sided Caputo type equation*

$(-L_{[a,b]*},\, \lambda,\, g_k,\, u_a^k,\, u_b^k)$ *for* $k \in \{1,2\}$*, then*

$$||u_1 - u_2|| \le |u_a^1 - u_a^2| + |u_b^1 - u_b^2| + ||g_1 - g_2||\frac{1}{\lambda}, \quad \textit{for} \quad \lambda > 0,$$

and

$$||u_1 - u_2|| \le |u_a^1 - u_a^2| + |u_b^1 - u_b^2| + ||g_1 - g_2|| \sup_{x\in[a,b]} \mathbf{E}\left[\ \tau_{(a,b)}(x)\right], \quad \textit{for } \lambda = 0.$$

In particular, the solution u *to the two-sided equation of Caputo type given by* $(-L_{[a,b]*}, \lambda, g, u_a, u_a)$ *depends continuously on the function* g *and on the boundary conditions* $\{u_a, u_b\}$*.*

Remark 5.4.5. *In all the equations above* λ *was considered as a constant parameter. When* λ *is replaced by a positive function* $\lambda(\cdot)$*, the equations (5.1.1)-(5.1.2) become linear equations with nonconstant coefficients. From a probabilistic point of view, the term* $\lambda(\cdot)$ *is added to the corresponding generator and this term is then interpreted as the instantaneous killing rate. The solution in this case admits a Feynman-Kac type stochastic representation (see, e.g., the left-sided case* $-D^{(\nu)}_{a+*} - \lambda(\cdot)$ *studied in Chapter 4).*

5.5 Applications

Let us consider the following results related to the exit time of Feller processes from bounded intervals and generalized fractional equations of Caputo type and RL type.

5.5.1 Two-sided exit problems.

Let X_x be the underlying process generated by the operator L in (5.2.5). Define the events $\Pi_a(x)$ and $\Pi_b(x)$ by

$$\Pi_a(x) := \left\{X_x\left(\tau_{(a,b)}(x)\right) \le a\right\} \quad \text{and} \quad \Pi_b(x) := \left\{X_x\left(\tau_{(a,b)}(x)\right) \ge b\right\}. \tag{5.5.1}$$

Then, $\Pi_a(x)$ and $\Pi_b(x)$ denote the events that the process X_x leaves the interval (a,b) through the lower boundary a, and through the upper boundary b, respectively. Using the stochastic representation (5.4.3), the solution to $(-L_{[a,b]*},\, \lambda = 0,\, g,\, u_a,\, u_b)$ can be rewritten

$$u(x) = u_a \mathbf{P}\left[\Pi_a(x)\right] + u_b \mathbf{P}\left[\Pi_b(x)\right] + \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} g\left(X_x(t)\right) dt\right]. \tag{5.5.2}$$

Let $H^D(x,\cdot)$ be the *potential measure* for the process X_x (see, e.g. [9]) defined by

$$H^D(x,dy) := \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{X_x(t)\in dy\}} \mathbf{1}_{\{\forall s\le t, X_x(s)\in D\}} dt\right].$$

Corollary 5.5.1. *Suppose that the assumptions of Theorems 5.4.2 or 5.4.3 (depending on the operator $-A^{(\gamma,\alpha)}$) hold. Then, the generalized solution to the Caputo type equation $(-L_{[a,b]*},\, \lambda = 0,\, g,\, u_a,\, u_b)$ can be rewritten*

$$u(x) = u_a \mathbf{P}\left[\Pi_a(x)\right] + u_b \mathbf{P}\left[\Pi_b(x)\right] + \int_a^b g(y) H^{(a,b)}(x,dy). \tag{5.5.3}$$

Corollary 5.5.2. *Under the assumptions of Theorems 5.3.1 or 5.3.2 (depending on the operator $-A^{(\gamma,\alpha)}$), the function $u(x) = \mathbf{E}\left[\tau_{(a,b)}(x)\right]$ (the mean exit time from the interval (a,b) of the underlying process X_x) is the generalized solution to the two-sided RL type equation*

$$-L_{[a,b]}u(x) = -1, \quad x \in (a,b), \qquad u(a) = u(b) = 0.$$

Moreover, under the assumptions of Theorems 5.4.2 or 5.4.3 (depending on the operator $-A^{(\gamma,\alpha)}$), the probability of exit through the point $x = a$, $\mathbf{P}\left[\Pi_a(x)\right]$, is the generalized solution to the two-sided Caputo type equation

$$-L_{[a,b]*}u(x) = 0, \quad x \in (a,b), \qquad u(a) = 1,\ u(b) = 0.$$

Analogously, the probability of exit through $x = b$, $\mathbf{P}\left[\Pi_b(x)\right]$, is the generalized solu-

tion to the two-sided Caputo type equation

$$-L_{[a,b]*}u(x) = 0, \quad x \in (a,b), \qquad u(a) = 0,\ u(b) = 1.$$

5.5.2 Examples: classical fractional setting

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Example 1. Consider the two-sided fractional differential equation

$$\begin{aligned} D^{\beta}_{-1+}w(x) + D^{\beta}_{+1-}w(x) &= -\lambda w(x) + g(x), \quad x \in (-1,1) \\ w(-1) = 0 &= w(1). \end{aligned} \tag{5.5.4}$$

By Theorem 5.3.2 this equation is well-posed in the generalized sense. In this case the process $X_x^{[-1,1]}$ is obtained from a symmetric stable process X_x^{β} with exponent $\beta \in (0,1)$ by killing it upon leaving the interval $(-1,1)$.

1. If $g \in B[-1,1]$, then the unique generalized solution can be rewritten

   $$w(x) = \mathbf{E}\left[\int_0^{\tau_{(-1,1)}(x)} e^{-\lambda t} g\left(X_x^{\beta}(t)\right) dt\right],$$

   where

   $$\tau_{(-1,1)}(x) := \inf\left\{t \geq 0 \,:\, X_x^{\beta}(t) \notin (-1,1)\right\}.$$

2. If $g = 1$ and $\lambda = 0$, then the *mean exit time* $\mathbf{E}\left[\tau_{(-1,1)}(x)\right]$ is the unique generalized solution to the two-sided equation (5.5.4). Moreover, by Theorem 2.1 in [85], we obtain the explicit solution

   $$w(x) = \frac{(1-x^2)^{\beta/2}}{\Gamma(\beta+1)}.$$

Example 2. Consider now the two-sided Caputo fractional equation:

$$D^{\beta}_{-1+*}h(x) + D^{\beta}_{+1-*}h(x) = 0, \quad x \in (-1,1) \quad \beta \in (0,1),$$

$$h(-1) = 0, \quad h(1) = 1. \tag{5.5.5}$$

Corollary 5.5.2 gives the unique generalized solution

$$h(x) = \mathbf{P}\left[X_x^{[-1,1]}(\tau_{(-1,1)}(x)) = 1\right] = \mathbf{P}\left[X_x^{\beta}(\tau_{(-1,1)}(x)) \in [1,\infty)\right].$$

Using [85, Formula 3.2] one obtains the explicit solution

$$h(x) = 2^{1-\beta}\frac{\Gamma(\beta)}{\Gamma(\beta/2)^2}\int_{-1}^{x}(1-y^2)^{\frac{\beta}{2}-1}dy. \tag{5.5.6}$$

Furthermore, again by Corollary 5.5.2, the equation

$$D^{\beta}_{-1+*}v(x) + D^{\beta}_{+1-*}v(x) = 0, \quad x \in (-1,1), \quad \beta \in (0,1),$$
$$v(-1) = 1, \quad v(1) = 0. \tag{5.5.7}$$

has solution

$$v(x) = 1 - h(x).$$

Example 3. The two-sided Caputo fractional equation

$$D^{\beta}_{-1+*}u(x) + D^{\beta}_{1-*}u(x) = g(x) \quad x \in (-1,1), \quad \beta \in (0,1),$$
$$u(-1) = u_{-1}, \quad u(1) = u_1, \tag{5.5.8}$$

has the unique generalized solution given by (5.5.3), which rewrites

$$u(x) = (u_1 - u_{-1})h(x) + u_{-1} + \int_{-1}^{1} g(y)H_\beta^{(-1,1)}(x,y)dy,$$

where $h(x)$ is the function given in (5.5.6), and $H_\beta^{(-1,1)}(x,y)$ (the density of the potential measure of the process X_x^β) is given by [85]

$$H_\beta^{(-1,1)}(x,y) = 2^{-\beta}\pi_{-1/2}\frac{\Gamma(1/2)}{(\Gamma(\beta/2))^2}\int_0^z (r+1)^{-\frac{1}{2}}r^{\frac{\beta}{2}-1}|x-y|^{\beta-1}dr,$$

with

$$z = (1 - x^2)(1 - y^2)/(x - y)^2.$$

Remark 5.5.3. *Observe that all the explicit solutions w, v, h and u above are smooth solutions since they belong to $C[-1,1] \cap C^1(-1,1)$.*

5.6 Proofs

Let us first recall the notation

$$-L^{(\nu)}_{[a,b]} := -D^{(\nu_+)}_{a+} - D^{(\nu_-)}_{b-} \quad \text{and} \quad -L^{(\nu)}_{[a,b]*} := -D^{(\nu_+)}_{a+*} - D^{(\nu_-)}_{b-*}.$$

The definitions of left- and right-sided generalized derivatives given in (2.2.1)-(2.2.4) yield

$$-L^{(\nu)}_{[a,b]} f(x) = \int_{x-a}^{b-x} (f(x+y) - f(x))\nu(x,y)dy - f(x) \int_{\mathbb{R}\setminus(a-x,b-x)} \nu(x,y)dy,$$

and

$$\begin{aligned} -L^{(\nu)}_{[a,b]*} f(x) = &\int_{x-a}^{b-x} (f(x+y) - f(x))\nu(x,y)dy + \\ &+ (f(b) - f(x)) \int_{b-x}^{\infty} \nu(x,y)dy + (f(a) - f(x)) \int_{-\infty}^{a-x} \nu(x,y)dy. \end{aligned} \tag{5.6.1}$$

Let us now define the bounded operators $M^{(\nu)}$ and $M^{(\nu)}_*$ (acting on functions from $C[a,b]$) by

$$-M^{(\nu)} g(x) = \int_{a-x}^{b-x} \int_{x}^{x+y} g(z)dz\nu(x,y)dy, \tag{5.6.2}$$

and

$$\begin{aligned} M^{(\nu)}_* g(x) := &\int_{a-x}^{b-x} \int_{x}^{x+y} g(z)dz\nu(x,y)dy \\ &+ \int_{x}^{b} g(z)dz \int_{b-x}^{\infty} \nu(x,y)dy + \int_{x}^{a} g(z)dz \int_{-\infty}^{a-x} \nu(x,y)dy, \end{aligned} \tag{5.6.3}$$

respectively. Then $-L^{(\nu)}_{[a,b]*}f(x)$ coincides with $M^{(\nu)}_{*}f'(x)$ for functions $f \in C^1[a,b]$, whereas

$$-L^{(\nu)}_{[a,b]}f(x) = -M^{(\nu)}f'(x) - f(x)\int_{\mathbb{R}\setminus(a-x,b-x)}\nu(x,y)dy,$$

whenever $f \in C^1[a,b]$.

5.6.1 Proof of Theorem 5.2.3

Proof. We follow a similar strategy to the one used in [55] for the operator without diffusion term (therein denoted by $A_{[a,b]*}$). Namely, we approximate $-L_{[a,b]*}$ by a family of operators $(-L_{h*})_{h\in(0,1]}$ defined by

$$-L_{h*} := -L^{(\nu_h)}_{[a,b]*} - A^{(\gamma,\alpha)}, \tag{5.6.4}$$

where

- for each $h \in (0,1]$, the function ν_h is defined by $\nu_h(x,y) := \Phi_h(x,y)\nu(x,y)$. Here $\Phi_h(x,y)$ is a smooth function on $[a,b]\times\mathbb{R}$ such that $\Phi_h(x,y)$ is equal 1 on the set $\{|y| > h,\ x \in [a+h, b-h]\}$ and vanishes near the boundary.
- the operator $(-A^{(\gamma,\alpha)}, \mathfrak{D}_A)$ is a diffusion on $[a,b]$ with reflecting boundaries {a,b} (see, e.g. [6, Chapter V, Section 6]) given by

$$\left(-A^{(\gamma,\alpha)}f\right)(x) = \gamma(x)f'(x) + \alpha(x)f''(x), \tag{5.6.5}$$

  with drift and diffusion terms $\gamma \in C^3_0[a,b]$ and $\alpha \in C^3[a,b]$ (with derivative $\alpha' \in C_0[a,b]$), respectively, and with a domain

$$\mathfrak{D}_A := \left\{f \in C[a,b] : -A^{(\gamma,\alpha)}f \in C[a,b],\ f'(a) = 0,\ f'(b) = 0\right\}.$$

In a similar way we shall define the operator

$$-L_h := -L_{[a,b]}^{(\nu_h)} - A^{(\gamma,\alpha)}. \tag{5.6.6}$$

Notice that for each $h \in (0,1]$ the operator $-L_{h*}$ decomposes as a diffusion on $[a,b]$ perturbed by the operator $-L_{[a,b]*}^{(\nu_h)}$ which is bounded on $C[a,b]$, so that by perturbation theory (see, e.g., [53, Theorem 1.9.2]) for each h the operator $(-L_{h*}, \mathfrak{D}_A)$ generates a Feller semigroup T_t^h on $C[a,b]$.
This semigroup is the unique (bounded) solution to the evolution equation

$$\frac{d}{dt} f_t(x) = -L_{h*} f_t(x), \quad f_0 = f \in \mathfrak{D}_A. \tag{5.6.7}$$

Moreover, due to the smoothness assumptions on the functions γ, α and ν, the spaces $\{f \in C^j[a,b] : f' \in C_0[a,b]\}$ for $j \in \{2,3\}$ are invariant cores for the operator $-L_{h*}$ [53, Theorem 1.9.2,(iii)].
Uniform boundedness of the derivatives of T_t^h. To prove that the semigroup T_t^h converges to a Feller semigroup on $C[a,b]$ as $h \to 0$, we will use that the first derivative with respect to x of the semigroup $\left(T_t^h f\right)(x)$ remains uniformly bounded in h and t for $t \leq t_0$ and $t_0 \in \mathbb{R}$.
To prove the boundedness of the derivative we proceed as follows. Since the space $H := \{f \in C^3[a,b] : f' \in C_0[a,b]\}$ is an invariant space for the semigroup T_t^h, it follows that $T_t^h f \in H$ whenever $f \in H$. Furthermore, note that $-L_{h*} f \in C^1[a,b]$ whenever $f \in H$, and hence $-L_{h*} T_t^h f \in C^1[a,b]$ for any $f \in H$. Thus, differentiating the evolution equation (5.6.7) with respect to the space variable x (we use prime notation for this derivative) one obtains

$$\frac{d}{dt} f_t'(x) = \frac{d}{dx} \left(-L_{h*} f_t(x)\right) = -L_h f_t'(x) - L_{h*}' f_t(x) \tag{5.6.8}$$

where $-L_h$ is the operator given in (5.6.6) and

$$\begin{aligned}-L'_{h*}f_t(x) :&= -L^{(\partial_x\nu_h)}_{[a,b]*}f_t(x) - A^{(\gamma',\alpha')}f_t(x)\\ &= -M^{(\partial_x\nu_h)}_{*}f'_t(x) - A^{(\gamma',\alpha')}f_t(x).\end{aligned}$$

The second equality above is obtained by the definition of the operator $-M^{(\cdot)}_*$ given in (5.6.3) with $\partial_x\nu$ instead of ν (notation ∂_x means the partial derivative with respect to x). Hence

$$\frac{d}{dx}\left(-L_h f_t(x)\right) = -\boldsymbol{L}^{h,(1)}f'_t(x), \tag{5.6.9}$$

where

$$-\boldsymbol{L}^{h,(1)}g(x) := -A^{(\gamma+\alpha',\alpha)}g(x) + \left[-L^{(\nu_h)}_{[a,b]} - M^{(\partial_x\nu_h)}_* + \gamma'(x)\right]g(x). \tag{5.6.10}$$

Using that (by assumption) α' also vanishes in the boundary points $\{a,b\}$, it follows that for each h the operator $-\boldsymbol{L}^{h,(1)}$ decomposes as a diffusion $-A^{(\gamma+\alpha',\alpha)}$ on $[a,b]$ (with reflecting boundaries) perturbed by the operator K_h given by

$$K_h := -L^{(\nu_h)}_{[a,b]} - M^{(\partial_x\nu_h)}_* + \gamma'(\cdot),$$

which is bounded on $C[a,b]$ (due to assumption (H0)). Hence, $(-\boldsymbol{L}^{h,(1)}, \mathfrak{D}'_A)$ generates a strongly continuous semigroup on $C[a,b]$, denoted by $T^{h,(1)}_t$, with the domain given by

$$\mathfrak{D}'_A := \left\{g \in C[a,b] \,:\, -A^{(\gamma+\alpha',\alpha)}g \in C[a,b],\, g'(a) = 0,\, g'(b) = 0\right\}$$

Setting $g_t(x) = f'_t(x)$ yields the evolution equation

$$\frac{d}{dt}g_t(x) = -\boldsymbol{L}^{h,(1)}g_t(x), \quad g_0 = g \in \mathfrak{D}'_A. \tag{5.6.11}$$

Due to the invariance of the space H, it follows that $\frac{d}{dx}(T_t^h f)(x) = \left(T_t^{h,(1)} f'\right)(x)$ for each $f \in H$, i.e., the derivative with respect to x of the semigroup $T_t^h f$ coincides with the semigroup (applied to f') generated by $-\boldsymbol{L}^{h,(1)}$ whenever $f \in H$.

Now, using the perturbation series representation for the semigroup $T_t^{h,(1)}$ [53, Equality 1.78, p. 52]) and the fact that the semigroup generated by the diffusion term in (5.6.10) is a contraction semigroup, one obtains

$$\|T_t^{h,(1)} f'\| \le \|f'\| + \sum_{m=1}^{\infty} \frac{(t\|K_h\|)^m}{m!} \|f'\|. \tag{5.6.12}$$

Therefore, as K_h is uniformly bounded in h due to the bounds from assumption (H0), the derivative $\frac{d}{dx}\left(T_t^h f\right)(x)$ is uniformly bounded in h and $t \le t_0$ whenever $f \in H$.

Let us now write (see [42, Lemma 19.26, p. 385])

$$(T_t^{h_1} - T_t^{h_2}) f = \int_0^t T_{t-s}^{h_2} (-L_{h_1 *} + L_{h_2 *}) T_s^{h_1} f \, ds,$$

for $0 < h_2 \le h_1 < 1$ and $f \in H$. Since $T_t^{h_1} f$ is differentiable (with derivative uniformly bounded in h given by $T_t^{h_1,(1)} f'$), we can estimate (by mean value theorem)

$$\begin{aligned}
\left|(-L_{h_1 *} + L_{h_2 *}) T_s^{h_1} f(x)\right| &\le \int_{h_2 \le |y| \le h_1} \left|T_s^{h_1} f(x+y) - T_s^{h_1} f(x)\right| \nu(x,y) dy \\
&\le \int_{h_2 \le |y| \le h_1} \|T_s^{h_1,(1)} f'\| |y| \nu(x,y) dy \\
&= o(1) \|T_s^{h_1,(1)} f'\| = o(1) \|f\|_{C^1}, \quad h_1 \to 0.
\end{aligned}$$

The last equality holds due to the assumption (H0) (i.e, the uniform bound of the first moment of ν and its tightness property). Therefore,

$$\|\left(T_t^{h_1} - T_t^{h_2}\right) f\| = o(1) t \|f\|_{C^1}. \tag{5.6.13}$$

Thus, for each $f \in C^3[a,b]$ satisfying $f' \in C_0[a,b]$, the family $\{T_t^h f\}$ converges to a limiting family $\{T_t f\}$ as $h \to 0$. It follows then that the limiting family forms a semigroup of contractions on $C[a,b]$ (by standard approximation arguments) yielding the strongly continuity in $C[a,b]$. Now write

$$\frac{T_t f - f}{t} = \frac{T_t f - T_t^h f}{t} + \frac{T_t^h f - f}{t}.$$

Using the estimate (5.6.13), we conclude that $\{f \in C^3[a,b] : f' \in C_0[a,b]\}$ belongs to the domain of the generator and further the generator is given by $-L_{[a,b]*}$ as

$$\lim_{t\downarrow 0} \frac{T_t f - f}{t} = \lim_{h\downarrow 0} \lim_{t\downarrow 0} \frac{T_t f - T_t^h f}{t} + \frac{T_t^h f - f}{t} = -L_{[a,b]*} f.$$

Now, take $f \in C^2[a,b]$ and $\{f_n\} \subset H$ such that $f_n \to f$ uniformly as $n \to \infty$. Since the operator $-L_{[a,b]*}$ is closed [20, Corollary 1.6] and $-L_{[a,b]*} f_n \to g$ as $n \to \infty$, it follows that $g = -L_{[a,b]*} f$ and $f \in \hat{\mathfrak{D}}_*$. Therefore, the space $\{f \in C^2[a,b] : f' \in C_0[a,b]\}$ also belongs to the domain of the generator, as required.

■

5.6.2 Proof of Proposition 5.2.5

Proof. Using the method of Lyapunov functions (see, e.g., [53, Proposition 6.3.2]), we take the continuous function $f_w(x) = (x-a)^w$ for some sufficiently small $w \in (0,1)$. This function satisfies that $f_w(a) = 0$, $f_w(x) > 0$ for $x > a$, and f_w is differentiable in (a,b). To prove that a is regular in expectation we need to see that $\left(-L_{[a,b]*} f_w\right)(x) < -K$ for all $x \in (a,c)$ and for some $c \in (a,b)$ and a positive constant K. Since

$$\left(-L_{[a,b]*} f_w\right)(x) = \left(-D_{a+*}^{(\nu_+)} - D_{b-*}^{(\nu_-)}\right) f_w(x) + w\gamma(x)(x-a)^{w-1} + w(w-1)\alpha(x)(x-a)^{w-2},$$

we obtain that $\left(-L_{[a,b]*} f_w\right)(x) < -K$ for some positive constant K when $\gamma(a) = 0$ and $\alpha(a) > 0$ due to the fact that the first two terms in the r.h.s of the previous equality are dominated by the last term which tends to $-\infty$ as $x \to a$. The regularity in expectation for $x = b$ is proved analogously but with the Lyapunov function $f_w(x) = (b-x)^w$.

The finite expectation of the first exit time of the process $\hat{X}_x$ from the interval (a,b), denoted by $\hat{\tau}_{(a,b)}(x)$, follows from the regularity of $\{a,b\}$ [53, Proposition 6.3.1]. ■

5.6.3 Proof of Theorem 5.2.6

Proof. (i) Theorem 5.2.3 ensures that $(-L_{[a,b]*}, \hat{\mathfrak{D}}_*)$ generates a Feller process $\hat{X}_x$ on the closed interval $[a,b]$. Further, Proposition 5.2.5 guarantees that $\{a,b\}$ are regular in expectation. Hence, the stopped process $X_x^{[a,b]*} := \{\hat{X}_x(s \wedge \tau_{(a,b)}(x))\}_{s\geq 0}$ is also a Feller process on $[a,b]$ [53, Theorem 6.2.1, Chapter 6]. Let us denote by $(-L_{stop}, \mathfrak{D}^{stop}_{[a,b]*})$ the generator of the stopped process where $\mathfrak{D}^{stop}_{[a,b]*}$ stands for the domain of the operator $-L_{stop}$. By definition of the process $X_x^{[a,b]*}$, the states $\{a,b\}$ are absorbing states, which implies that for any $f \in D^{stop}_{[a,b]*}$ the equality $(-L_{[a,b]*}f)(x) = 0$ for $x \in \{a,b\}$ holds, as required.
For the second part, take $f \in \hat{\mathfrak{D}}_*$ such that $-L_{[a,b]*}f(x) = 0$ in $\{a,b\}$. Since the domain of the generator is given by the image of the corresponding resolvent operator (say $\hat{R}_\lambda$), given $f \in \hat{\mathfrak{D}}_*$ there exists $g \in C[a,b]$ such that $f = \hat{R}_\lambda g$.
Using that f solves the resolvent equation

$$\lambda \hat{R}_\lambda g + L_{[a,b]*} f = g,$$

and that (by assumption) $-L_{[a,b]*}f(x) = 0$ for $x \in \{a,b\}$, we get

$$f(a) = \hat{R}_\lambda g(a) = g(a)/\lambda \quad \text{and} \quad f(b) = \hat{R}_\lambda g(b) = g(b)/\lambda. \tag{5.6.14}$$

Moreover, Dynkin's formula implies

$$f(x) = \hat{R}_\lambda g(x) = \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda s} g\left(\hat{X}_x(s)\right) ds\right] + \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)} f\left(\hat{X}_x(\tau_{(a,b)}(x))\right)\right],$$

for each $x \in (a,b)$. Using that the paths of the processes $\hat{X}_x$ and $X_x^{[a,b]*}$ coincide before the first exit time $\tau_{(a,b)}(x)$, the previous expression becomes

$$\begin{aligned}\hat{R}_\lambda g(x) &= \mathbf{E}\left[\int_0^{\tau_{(a,b)}(x)} e^{-\lambda s} g\left(X_x^{[a,b]*}(s)\right) ds\right] + \\ &+ \mathbf{E}\left[e^{-\lambda \tau_{(a,b)}(x)}\left(f(a)\mathbf{1}_{\{\tau_a<\tau_b\}} + f(b)\mathbf{1}_{\{\tau_b<\tau_a\}}\right)\right],\end{aligned}$$

where τ_a and τ_b denote the first exit time through the boundary point a and b, respectively. Finally, plugging the equalities (5.6.14) into the second term of the r.h.s of the last formula we get that $f = \hat{R}_\lambda g = R_\lambda^{[a,b]*} g$, where $R_\lambda^{[a,b]*}$ denotes the resolvent operator of the stopped process $X^{[a,b]*}$. Therefore, for any $f \in \hat{\mathfrak{D}}_*$ such that $\left(-L_{[a,b]*} f\right)(x) = 0$ on $\{a,b\}$, there exist $g \in C[a,b]$ such that $f = R_\lambda^{[a,b]*} g$ implying that $f \in \mathfrak{D}_{[a,b]*}^{stop}$ and $-L_{stop} f = -L_{[a,b]*} f$.

(ii) Follows the same reasoning as before, so that we omit the details. ■

Chapter 6

Generalized fractional evolution equations of Caputo type

This chapter is devoted to the study of generalized fractional evolution equations involving Caputo type derivatives. Using analytical methods and probabilistic arguments we obtain well-posedness (in the generalized sense) and integral (stochastic) representations for the solutions. These results encompass known equations from classical FPDE's such as the time-fractional diffusion equation and the time-space-fractional diffusion equation, as well as their far reaching extensions.

6.1 Introduction

The *generalized fractional evolution equations of Caputo type* studied in this chapter can be thought of as classical evolution equations wherein the first-order time derivative has been replaced by the non-local operator of Caputo type $-D^{(\nu)}_{a+*}$. Based on the notion of *Green's functions* for differential operators, we shall study:

i) the nonhomogeneous generalized fractional evolution equation

$$\begin{aligned} -{}_tD^{(\nu)}_{a+*}u(t,x) &= A_x u(t,x) - g(t,x), && t \in (a,b],\ x \in \mathbb{R}^d, \\ u(a,x) &= \phi_a(x), && x \in \mathbb{R}^d \end{aligned} \tag{6.1.1}$$

for given functions g and ϕ_a defined on $[a,b]\times\mathbb{R}^d$ and $\mathbb{R}^d$, respectively;

ii) the generalized fractional *nonlinear* equation

$$\begin{aligned} -{}_tD^{(\nu)}_{a+*}u(t,x) &= A_x u(t,x) + f(t,x,u(t,x)), && t\in(a,b],\ x\in\mathbb{R}^d, \\ u(a,x) &= \phi_a(x), && x\in\mathbb{R}^d \end{aligned} \tag{6.1.2}$$

where f is a given function on $[a,b]\times\mathbb{R}^d\times\mathbb{R}$.

Notation $-{}_tD^{(\nu)}_{a+*}$ means the Caputo type operator $-D^{(\nu)}_{a+*}$ acting on the (time) variable t, whereas $-A_x$ stands for the generator of a Feller process acting on the (space) variable x.
Since Caputo derivatives are special cases of the operators $-D^{(\nu)}_{a+*}$, the generalized equations in (6.1.1)-(6.1.2) include, as particular cases, a variety of equations studied in the theory of fractional partial differential equations (FPDE's). The latter equations have been successfully used for describing diffusions in disordered media, also called *anomalous diffusions*, which include *subdiffusions* as well as *enhanced diffusions* (or *superdiffusions*). Subdiffusion phenomena are usually related to time-FPDE's, whereas superdiffusions are related to space- FPDE's.

In the classical fractional setting, *the fractional Cauchy problems* are special cases of equation (6.1.1). Fractional Cauchy problems are initial value problems involving the Caputo derivative of order $\beta\in(0,1)$:

$$\begin{aligned} {}_tD^{\beta}_{0+*}u(t,x) &= A_x u(t,x), && (t,x)\in[0,b]\times\mathbb{R}^d, \\ u(0,x) &= \phi_0(x), && x\in\mathbb{R}^d \end{aligned} \tag{6.1.3}$$

Equations of the type in (6.1.3) have been actively studied in the literature. Amongst the standard analytical approaches to solve FPDE's, *the Laplace-Fourier transform* technique plays an important role (see, e.g., [15], [18], [45], [73], [76], and references therein). From a probabilistic point of view, interesting connections have been found between the solution of (time-) FPDE's and the transition densities of time-changed Markov processes (see, e.g., [26], [52], [53], [67], [71], [78]).

A very standard example of the equation (6.1.3) is given by the (time-) *fractional diffusion equation* (or *fractional-kinetic equation*) [10], [62], [67] corresponding to the case $A_x = -\frac{1}{2}\Delta_x$, where Δ_x denotes the Laplace operator. Its fundamental solution was first studied by Schneider and Wyss [79] and Kochubei [51]. In this case the fundamental solution corresponds to the time-changed transition probability function of the Brownian motion by the hitting time of a β-stable subordinator.
Another example of equation (6.1.3) was studied in [19], wherein the authors consider the second-order differential operator given by

$$A_x = \sum_{i,j}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + c(x).$$

As for nonhomogeneous equations, the multi-time fractional differential equation:

$$\sum_{k=1}^{n} \lambda_k \, {}_tD_{0+*}^{\beta_k} u(t,x) - \Delta_x u(t,x) = g(t,x), \quad \lambda_k, t \in \mathbb{R}^n, \ x \in \mathbb{R}^d.$$

was investigated in [74].
More recently, the regularity of the nonhomogenous time-space fractional linear equation for the fractional Laplacian operator $A_x = -(-\Delta)^{\alpha/2}$:

$$\begin{aligned} {}_tD_{0+*}^{\beta} u(t,x) &= -c(-\Delta)^{\alpha/2}u(t,x) + g(t,x), && x \in \mathbb{R}^d, \ t \geq 0, \\ u(0,x) &= \phi_0(x), && x \in \mathbb{R}^d \end{aligned}$$

as well as the well-posedness for the fractional HJB type equation

$$\begin{aligned} {}_tD_{0+*}^{\beta} u(t,x) &= -c(-\Delta)^{\alpha/2}u(t,x) + H(t,x,\nabla u(t,x)), && x \in \mathbb{R}^d, \ t \geq 0, \\ u(0,x) &= \phi_0(x), && x \in \mathbb{R}^d, \end{aligned}$$

were addressed in [54], for $\beta \in (0,1)$, $\alpha \in (1,2]$, and a positive constant $c > 0$.

Evolution equations of the type (6.1.3) arise, for example, as the limiting evolution of an uncoupled and properly scaled *continuous time random walk* (CTRW) with the waiting times in the *domain of attraction of $\beta-$stable laws*. This probabilistic model and some of its extension have been widely studied (see, e.g., [67], [78], [53], and references therein). Yet another extension of equation (6.1.3) can be obtained, for instance, by considering the limiting evolution of properly scaled *coupled* CTRW (or semi-Markov processes) with power law waiting times depending on the state of the system. This procedure yields the generalized evolution equation with the Caputo type operator of variable order ${}_tD_{0+*}^{\beta(t,x)}$, i.e.,

$$ {}_tD_{0+*}^{\beta(t,x)} u(t,x) = A_x^{(t)} u(t,x), \quad (t,x) \in [0,b] \times \mathbb{R}^d, \quad \beta(\cdot) \in (0,1). \tag{6.1.4} $$

Using the results presented here, we are able to deduce some of the results known for the previous cases, as well as to extend the analysis to more general situations. Some specific equations of the type in (6.1.4) will be discussed in a forthcoming paper in preparation.

We highlight the fact that the (analytical) approach used in this chapter is different from the one used in chapters 3-5. Namely, in analogy with the standard analytical methods to solve classical evolution equations, we obtain well-posedness results for the nonhomogeneous equation (6.1.1) by transforming (6.1.1) into an *abstract* generalized fractional linear equation on a suitable Banach space. Then, we construct the solutions via the concept of Green's function. For the stochastic representation for the solutions, we use Dynkin's martingale and Doob's stopping theorem as usual.

As for the nonlinear case, we study the well-posedness for the 'ordinary' equation (6.1.2) following a similar strategy to the one used for the nonlinear equation in

(4.1.1). Namely, by means of the the integral representation (*mild form*) of the solution to the linear problem (6.1.1), we reduce the analysis of (6.1.2) to a fixed point problem for a suitable operator. Let us mention that, even though in this work we do not include the HJB type case, our results for the generalized nonlinear equation (6.1.2) can be used to extend the well-posedness for the corresponding equations of HJB type.

6.2 Motivation: weak formulation in classical differential setting

In order to prove the well-posedness for the equation (6.1.1), we shall study the generalized linear equation $-D^{(\nu)}_{a+*}u(t) = -g(t)$ using a different approach to the one used in Chapter 3. Namely, we are interested in the notion of solutions in the sense of distributions. For that purpose, in this section we shall recall some related concepts taken from the classical differential setting.

6.2.1 The fundamental solution and the Green function

Denote by $\mathcal{T} := C_c^\infty(\mathbb{R}^d)$ the space of *test functions* given by the space of smooth functions with compact support on $\mathbb{R}^d$. The space of distributions (or generalized functions) $\mathcal{D}'$ is defined to be the space of continuous linear functionals on $\mathcal{T}$ (see, e.g. [24] for a detailed treatment).

Let $L_1^{loc}(\mathbb{R}^d)$ denote the set of locally (Lebesgue) integrable functions on $\mathbb{R}^d$. If $f \in L_1^{loc}(\mathbb{R}^d)$, then f defines a (regular) distribution $f \in \mathcal{D}'$ [24, p. 4] by setting

$$(f,\phi) = \int f(x)\varphi(x)dx, \quad \varphi \in \mathcal{T}. \tag{6.2.1}$$

Hence, every locally integrable function can be seen as a distribution. Further, every distribution $f \in \mathcal{D}'$ is differentiable in the sense of distributions (hence, it has derivatives of all orders [24, p.20]). Its generalized derivative f' is given by the distribution

$$\left(f',\varphi\right)=-\left(f,\varphi'\right),\qquad \text{for all } \varphi\in\mathcal{T}.$$

Let us now recall the concept of *generalized solution*, *fundamental solution* and *Green's function* related to the equation $Lu=g$ when L is a general differential operator (see, e.g. [21]).

Definition 6.2.1. *The function $\Phi(x,y)$ is called the fundamental solution of the differential operator L if $\Phi(x,y)$ is the distributional solution to*

$$\left(L\Phi(\cdot,y)\right)(x)=\delta(x-y),\quad x\in\mathbb{R}^d, \tag{6.2.2}$$

for each $y\in\mathbb{R}^d$.

The importance of the fundamental solution relies on the fact that it determines the solution to the nonhomogeneous equation $Lu=g$ for any $g\in C_c^\infty(\mathbb{R}^d)$. Namely, if $\Phi(x,\cdot)\in L_1^{loc}(\mathbb{R}^d)$ for each $x\in\mathbb{R}^d$ and $g\in C_c^\infty(\mathbb{R}^d)$, then

$$u(x):=(\Phi(x,\cdot),g(\cdot))=\int_{\mathbb{R}^d}\Phi(x,y)g(y)dy \tag{6.2.3}$$

solves $Lu=g$ in $\mathbb{R}^d$. To verify this, note that

$$\begin{aligned}Lu(x)=\left(L\int\Phi(\cdot,y)g(y)dy\right)(x)&=\int(L\Phi(\cdot,y))(x)g(y)dy\\&=\int\delta(x-y)g(y)dy=g(x),\end{aligned}$$

where the last equality holds due to (6.2.2). The previous yields the following definition.

Definition 6.2.2. *A locally integrable function $u\in L_1^{loc}(\mathbb{R}^d)$ is said to be a generalized solution to equation $Lu=g$ for $g\in L_1^{loc}(\mathbb{R}^d)$ if u is given by*

$$u(x)=\int_{\mathbb{R}^d}\Phi(x,y)g(y)dy,\quad x\in\mathbb{R}^d, \tag{6.2.4}$$

where $\Phi(x,y)$ is the fundamental solution of the operator L.

Let $U \subset \mathbb{R}^d$ be an open bounded domain of $\mathbb{R}^d$ with a regular boundary. Consider now the nonhomogeneous Dirichlet problem for the operator L on U given by

$$\begin{aligned} Lu(x) &= g(x), \qquad x \text{ in } U \\ u(x) &= \phi(x), \qquad x \text{ on } \partial U, \end{aligned} \tag{6.2.5}$$

for prescribed functions g and ϕ. We will see that solutions to the boundary value problem (6.2.5) are related to the concept of the Green's function.

Definition 6.2.3. *A function $\mathcal{G}(x,y)$ is said to be the Green function of the differential operator L in U if $\mathcal{G}$ is the solution to the boundary value problem*

$$\begin{aligned} \left(L\mathcal{G}(\cdot,y)\right)(x) &= \delta(x-y), \qquad && x \in U, \\ \mathcal{G}(x,y) &= 0, && x \in \partial U \end{aligned} \tag{6.2.6}$$

for each $y \in \bar{U}$.

Then, the function u defined via the Green function by setting

$$u(x) = \int_U \mathcal{G}(x,y)g(y)dy. \tag{6.2.7}$$

provides a solution to the equation (6.2.5) for which the boundary condition is $\phi \equiv 0$.

Remark 6.2.1. *The main difference between the fundamental solution Ψ and the Green function $\mathcal{G}$ is that the former provides a solution to the equation in the whole space $\mathbb{R}^d$, whereas the latter takes into account zero boundary conditions.*

Due to the linearity of the operator L, the solution to equations with nonhomogeneous boundary conditions (i.e. for which $\phi \neq 0$) can be obtained by superposition of solutions. Namely, the solution to (6.2.5) is given by

$$u(x) = \int_U \mathcal{G}(x,y)g(y)dy + v(x),$$

where v is the solution to the boundary value problem

$$\begin{aligned} Lv(x) &= 0, \quad & x \in U, \\ v(x) &= \phi(x), \quad & x \in \partial U. \end{aligned}$$

Remark 6.2.2. *In classical PDE's theory, there are different methods to determine the Green function, e.g. the method of images and the eigenfunction method. In our case, the Green function is obtained via the probabilistic interpretation of the generalized operator* $-D_{a+*}^{(\nu)}$.

6.3 Preliminary results

Let us recall that the notion of *generalized solution* associated with Caputo type equations was introduced in Chapter 3 via an approximating sequence of solutions taken from the domain of the generator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$. In this section, we will see that that notion is also consistent with the notion of generalized solution defined via the concept of a Green's function.

By analogy with the theory of differential equations, we have the following definition.

Definition 6.3.1. *A function* $\Psi^{(\nu)}(x,y)$ *is called the fundamental solution for the operator* $-D_{a+*}^{(\nu)}$*, if the function* $\Psi^{(\nu)}(\cdot,y)$ *is the distributional solution to the equation*

$$\left(-D_{a+*}^{(\nu)}\Psi^{(\nu)}(\cdot,y)\right)(x) = \delta(x-y), \quad x \in \mathbb{R},$$

for all $y \in \mathbb{R}$.

Remark 6.3.1. *Since the operator* $-D_{a+*}^{(\nu)}$ *acts on functions defined on* $[a,+\infty)$*, the fundamental solution can be defined as* $\Psi^{(\nu)}(x,y) = 0$ *for all* $x, y \notin [a,+\infty)$.

Definition 6.3.2. *A function* $\pi^{(\nu)}(x,y)$ *is called the Green function for the operator*

$-D^{(\nu)}_{a+*}$ in $U \subset [a,\infty)$, if the function $\pi^{(\nu)}(\cdot,y)$ solves the equation

$$\left(-D^{(\nu)}_{a+*}\pi^{(\nu)}(\cdot,y)\right)(x) = \delta(x-y), \quad x \in U$$
$$\pi^{(\nu)}(x,y) = 0, \quad x \in \partial U, \tag{6.3.1}$$

for all $y \in \bar{U}$.

Due to the linearity of the operator $-D^{(\nu)}_{a+*}$, it follows that the integral function

$$u(x) = \int_{\mathbb{R}} \Psi^{(\nu)}(x,y)g(y)dy$$

solves the equation $-D^{(\nu)}_{a+*}u = g$ for any $g \in C^\infty_c(\mathbb{R}^d)$. Further, the function

$$u(x) = \int_U \pi^{(\nu)}(x,y)g(y)dy, \tag{6.3.2}$$

solves the equation $-D^{(\nu)}_{a+*}u = g$ on $U \subset \mathbb{R}$, with boundary condition $\phi \equiv 0$, yielding the following definition.

Definition 6.3.3. *Let $\pi^{(\nu)}(x,y)$ be the Green function of the operator $-D^{(\nu)}_{a+*}$ on $(a,b]$. A function $u \in B[a,b]$ is said to be a generalized solution to the Dirichlet problem*

$$-D^{(\nu)}_{a+*}u(x) = g(x), \quad x \in (a,b]$$
$$u(a) = 0,$$

for any $g \in B[a,b]$, if u is given by the integral equation (6.3.2).

6.3.1 The Green function for the Caputo type operators

Consider the boundary value problem

$$-D^{(\nu)}_{a+*}u(x) = g(x) \quad x \in (a,b]$$
$$u(a) = u_a, \quad u_a \in \mathbb{R}. \tag{6.3.3}$$

Theorem 6.3.2. *(Well-posedness) Let ν be a function satisfying assumption (H0)-(H1) (see Chapter 2). For any $g \in B[a,b]$ and $u_a \in \mathbb{R}$, the equation (6.3.3) has a generalized solution (according to Definition 6.3.3) given by*

$$u(x) = u_a + \int_a^x \pi^{(\nu)}(x,y)g(y)dy, \quad x \in [a,b], \tag{6.3.4}$$

where $\pi^{(\nu)}(x,y) := -\int_0^\infty p_s^{+(\nu)}(x,y)ds$ for all $x > a$, $y \geq a$ and $\pi^{(\nu)}(a,y) = 0$ for all $y \geq a$. Notation $p_s^{+(\nu)}(x,y)$ stands for the transition densities of the process generated by $(-G_+^{(\nu)}, \mathfrak{D}_G)$ (see 2.3.1).

Proof. Since the operator $-D_{a+*}^{(\nu)}$ is linear, it is enough (by Definition 6.3.3) to prove that the function $\pi^{(\nu)}(x,y)$ is the Green function of the operator $-D_{a+}^{(\nu)}$ on (a,b). Thus, we will see that, for each $y \in [a,b]$,

$$\left(-D_{a+*}^{(\nu)}\pi^{(\nu)}(\cdot,y)\right)(x) = \delta(x-y), \quad x \in (a,b].$$

Since $\pi^{(\nu)}(x,y)$ is defined in terms of transition densities of a decreasing Feller process, it follows that $\pi^{(\nu)}(x,y)$ is continuous on both variables except on the diagonal $x = y$ (wherein there is a singularity). Furthermore, $\pi^{(\nu)}(x,y)$ vanishes for all $x \leq a$. Let us extend $\pi^{(\nu)}(\cdot,y)$ by zero to the space $\{x < a\}$ for each $y \in [a,b]$ (we will use notation $\pi_a^{(\nu)}(\cdot,y)$ for the extension). Then, notice that

$$\left(-D_{a+*}^{(\nu)}\pi^{(\nu)}(\cdot,y)\right)(x) = \left(-G_+^{(\nu)}\pi_a^{(\nu)}(\cdot,y)\right)(x), \quad x \in [a,b].$$

Hence, the definition of $-G_+^{(\nu)}$ and Fubini's theorem yield

$$\begin{aligned}\left(-D_{a+*}^{(\nu)}\pi^{(\nu)}(\cdot,y)\right)(x) &= \left(-G_+^{(\nu)}\pi_a^{(\nu)}(\cdot,y)\right)(x) \\ &= \int_{-\infty}^{\infty} \pi_a^{(\nu)}(x-z,y) - \pi_a^{(\nu)}(x,y)\nu(x,z)dz \\ &= -\int_{-\infty}^{\infty}\int_0^\infty \left(p_s^{+(\nu)}(x-z,y) - p_s^{+(\nu)}(x,y)ds\right)\nu(x,z)dz \\ &= -\int_0^\infty \left(-G_+^{(\nu)}p_s^{+(\nu)}(\cdot,y)\right)(x)ds.\end{aligned}$$

Since $p_s^{+(\nu)}(x,y)$ are transition densities of the process generated by $(-G_+^{(\nu)}, \mathfrak{D}_G)$, for each $y \in \mathbb{R}$, $p_s^{+(\nu)}(x,y)$ solves the evolution equation

$$\frac{\partial}{\partial s} p_s^{+(\nu)}(x,y) = \left(-G_+^{(\nu)} p_s^{+(\nu)}(\cdot,y)\right)(x)$$
$$p_s^{+(\nu)}(x,y)\Big|_{s=0} = \delta(x-y)$$

Therefore,

$$\left(-D_{a+*}^{(\nu)} \pi^{(\nu)}(\cdot,y)\right)(x) = -\int_0^\infty \frac{\partial}{\partial s} p_s^{+(\nu)}(x,y) ds = \delta(x-y),$$

as required.

Finally, to see that that $u \in B[a,b]$ note that

$$|u(x)| \leq u_a + \|g\| \int_a^x \int_0^\infty p_s^{+(\nu)}(x,y) dy$$
$$\leq u_a + \|g\| \sup_{x \in [a,b]} \mathbf{E}\left[\tau_a(x)\right] < +\infty,$$

due to Lemma 2.5.2 which implies that, under assumption (H1), $\tau_a(x)$ (the first exit time from the interval (a,b) of the process generated by $(-G_+^{(\nu)}, \mathfrak{D}_G)$) has a finite expectation. ■

6.3.2 Generalized fractional integral $I_{a+}^{(\nu)}$

The Green function $\pi^{(\nu)}(x,y)$ allows us to define an integral operator $I_{a+}^{(\nu)}$ on $B[a,b]$ which can be thought of as a generalization of the Riemann-Liouville integral operator $-I_{a+}^{\beta}$ of order $\beta \in (0,1)$, (see Definition 2.1.1).

Definition 6.3.4. *Let ν be a function satisfying assumption (H0) and (H1). If $\pi^{(\nu)}(x,y)$ is the Green function of the operator $-D_{a+*}^{(\nu)}$ on $(a,b]$, then the operator $I_{a+}^{(\nu)} : B[a,b] \to B[a,b]$ defined by*

$$\left(I_{a+}^{(\nu)} f\right)(x) := \int_a^x \pi^{(\nu)}(x,y) f(y) dy, \tag{6.3.5}$$

will be called the generalized fractional integral associated with the function ν.

The generalized fractional integral $I_{a+}^{(\nu)}$ satisfies the following:

(i) For each $f \in B[a,b]$,

$$\left| \left(I_{a+}^{(\nu)} f \right)(x) \right| \leq \|f\| \sup_{x \in [a,b]} \mathbf{E}\left[\tau_a(x)\right]. \tag{6.3.6}$$

In particular, if $f(x) = \mathbf{1}$ (the constant function 1), then

$$-\left(I_{a+}^{(\nu)} \mathbf{1} \right)(x) = \int_a^x \int_0^\infty p_s^{+(\nu)}(x,y)dy = \mathbf{E}\left[\tau_a^{(\nu)}(x)\right]. \tag{6.3.7}$$

(ii) The operator $I_{a+}^{(\nu)}$ can be thought of as the left inverse operator of the Riemann-Liouville (RL) type operator $-D_{a+}^{(\nu)}$. Let us recall that the RL type operator coincides with the Caputo type operator $-D_{a+*}^{(\nu)}$ on functions vanishing at a.

Remark 6.3.3. *In particular, if the function $\nu(x,y)$ is given by (2.4.1), then $I_{a+}^{(\nu)}$ coincides with the Riemann-Liouville integral $-I_{a+}^{\beta}$ of order $\beta \in (0,1)$. Further, if $\tau_a^{\beta}(x)$ is the first exit time from the interval (a,b) of an inverted $\beta-$stable subordinator, then we obtain the known results*

$$\left(I_{a+}^{\beta} \mathbf{1} \right)(x) = \int_a^x \int_0^\infty p_s^{+\beta}(x,y)dy = \mathbf{E}\left[\tau_a^{\beta}(x)\right] = \frac{(x-a)^{\beta}}{\Gamma(\beta+1)}, \tag{6.3.8}$$

and

$$\left| \left(I_{a+}^{\beta} f \right)(x) \right| \leq \frac{1}{\Gamma(\beta+1)} \|f\| (b-a)^{\beta}. \tag{6.3.9}$$

The next result gives us an explicit bound for $|I_{a+}^{(\nu)} f(x)|$ under the assumption (H4) given in Chapter 4.

Proposition 6.3.4. *Let ν be a function satisfying assumptions (H0) and (H4).*

Then, for each $f \in B[a,b]$,

$$\left|\left(I_{a+}^{(\nu)} f\right)(x)\right| \leq \frac{1}{\Gamma(\beta+1)} \|f\|_x (x-a)^\beta, \tag{6.3.10}$$

where $\|f\|_y := \sup_{z \leq y} |f(z)|$.

Proof. By definition of the generalized fractional integral it follows that

$$\begin{aligned}\left|\left(I_{a+*}^{(\nu)} f\right)(x)\right| &\leq \int_0^\infty \int_a^x |f(y)|\, p_s^{+(\nu)}(x,y) dy \\ &\leq \int_0^\infty \int_a^x \sup_{z \leq y} |f(z)| p_s^{+(\nu)}(x,y) dy\end{aligned}$$

Denote by $X^{+(\nu)}$ the underlying process generated by $(-G_+^{(\nu)}, \mathfrak{D}_G)$ and by $X^{+\beta}$ the process given by an inverted β–stable subordinator. Then, under assumption (H4) the process $X_x^{+(\nu)}$ *dominates* the inverted β-stable subordinator $X_x^{+\beta}$ in the sense that

$$\mathbf{P}[X_x^{+(\nu)}(s) > y] \leq \mathbf{P}[X_x^{+\beta}(s) > y], \quad y \leq b,\ s \geq 0,$$

as the intensity of the jumps of the process $X^{+(\nu)}$ is at least equal to the intensity of the jumps of the process $X^{+\beta}$. Equivalently, $\mathbf{P}[X_x^{+\beta}(s) \leq y] \leq \mathbf{P}[X_x^{+(\nu)}(s) \leq y]$). Therefore,

$$\mathbf{E}\left[g\left(X_x^{+(\nu)}(s)\right)\right] \leq \mathbf{E}\left[g\left(X_x^{+\beta}(s)\right)\right], \tag{6.3.11}$$

for any nondecreasing function g, so that in particular for the function $g(y) = \sup_{z \leq y} |f(z)|$, implying

$$\begin{aligned}\left|\left(I_{a+*}^{(\nu)} f\right)(x)\right| &\leq \int_0^\infty \int_a^x |f(y)|\, p_s^{+(\nu)}(x,y) dy ds \\ &\leq \int_0^\infty \int_a^x \sup_{z \leq y} |f(z)| p_s^{+\beta}(x,y) dy ds \\ &\leq \|f\|_x \int_0^\infty \int_a^x p_s^{+\beta}(x,y) dy ds \leq \frac{1}{\Gamma(\beta+1)} \|f\|_x (x-a)^\beta,\end{aligned}$$

as required. ■

The following result shall be useful for the following sections.

Let $I_{a+}^{(\nu),n}$ denote the n-fold iteration of the operator $I_{a+}^{(\nu)}$ for all $n \in \mathbb{N}_0$. For convention $I_{a+}^{(\nu),0} \equiv \mathbb{I}$, where $\mathbb{I}$ stands for the identity operator.

Proposition 6.3.5. *Let ν be a function satisfying assumptions (H0) and (H4). If $f \in B[a,b]$, then*

$$\left|\left(I_{a+}^{(\nu),n} f\right)(x)\right| \leq \|f\|_x \frac{(x-a)^{n\beta}}{(\Gamma(\beta+1))^n} \prod_{k=0}^{n-1} B(k\beta+1,\, \beta), \quad n \geq 1, \tag{6.3.12}$$

where $\|f\|_x := \sup_{y \leq x} |f(y)|$ and $B(\cdot,\cdot)$ denotes the Beta function.

Proof. Proceeding by induction. Case $n = 1$ is given by Proposition 6.3.4. Let us assume that the inequality (6.3.12) holds for $n-1$. Then, equation (6.3.11) and the induction hypothesis yield

$$\begin{aligned}
\left|\left(I_{a+}^{(\nu),n} f\right)(x)\right| &= \left|I_{a+}^{(\nu)} \circ I_{a+}^{(\nu),n-1} f(x)\right| \\
&\leq \int_0^\infty \int_a^x \sup_{z \leq y} \left|I_{a+}^{(\nu),n-1} f(z)\right| p_s^{+\beta}(x,y)\,dy\,ds \\
&\leq \int_0^\infty \int_a^x \|f\|_y \frac{(y-a)^{(n-1)\beta}}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) p_s^{+\beta}(x,y)\,dy\,ds \\
&\leq \|f\|_x \frac{1}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) \int_0^\infty \int_a^x (y-a)^{(n-1)\beta} p_s^{+\beta}(x,y)\,dy\,ds \\
&\leq \|f\|_x \frac{1}{(\Gamma(\beta+1))^{n-1}} \prod_{k=0}^{n-2} B(k\beta+1,\, \beta) \int_a^x (y-a)^{(n-1)\beta} (x-y)^{\beta-1} \frac{1}{\Gamma(\beta+1)} dy
\end{aligned} \tag{6.3.13}$$

where the last equality holds due to Fubini's theorem (to interchange the order of integration) and because of the equalities in (A.2.6).

For the integral in (6.3.13), the change of variable $z = (y-a)/(x-a)$ yields

$$\begin{aligned}
\int_a^x (y-a)^{(n-1)\beta}(x-y)^{\beta-1} dy &= (x-a)^{n\beta} \int_0^1 z^{(n-1)\beta}(1-z)^{\beta-1} dz \\
&= (x-a)^{n\beta} B((n-1)\beta+1, \beta).
\end{aligned} \tag{6.3.14}$$

Plugging (6.3.14) into (6.3.13), and then rearranging terms yields (6.3.12), as required. ■

Remark 6.3.6. *In the classical fractional setting, the $n-$fold RL integral $I_{a+}^{\beta;n}$ has an explicit expression obtained from its semigroup property [15, Theorem 2.2]. Namely,*

$$\left(I_{a+}^{\beta;n}f\right)(x) = \left(I_{a+}^{n\beta}f\right)(x). \tag{6.3.15}$$

Hence, for $f(x) = \mathbf{1}$,

$$\left(I_{a+}^{\beta;n}f\right)(x) = \frac{1}{\Gamma(n\beta)}\int_a^x (x-y)^{n\beta-1}dy = \frac{(x-a)^{n\beta}}{\Gamma(n\beta+1)}. \tag{6.3.16}$$

6.3.3 Series representations of solutions

Using the generalized fractional integral $I_{a+}^{(\nu)}$ defined before, this section provides series representations for the solutions to linear equations (with constant and non-constant coefficients) involving Caputo type operators.

Eigenfunction of generalized fractional equations

Consider the equation

$$\begin{aligned} -D_{a+*}^{(\nu)}u(x) &= \lambda u(x), \quad x \in (a,b] \\ u(a) &= u_a. \end{aligned} \tag{6.3.17}$$

Theorem 6.3.7. *Let ν be a function satisfying assumption (H0). Suppose that the Green function $\pi^{(\nu)}(x,y)$ of the operator $-D_{a+*}^{(\nu)}$ is such that the series*

$$\sum_{n=0}^{\infty}\lambda^n\left(I_{a+}^{(\nu),n}\mathbf{1}\right)(x) \tag{6.3.18}$$

converges uniformly on $[a,b]$. Then, for any $\lambda \in \mathbb{R}$, there exists a unique bounded *solution u to equation (6.3.17) given by*

$$u(x) = u_a \sum_{n=0}^{\infty} \lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(x). \tag{6.3.19}$$

In particular, (6.3.18) converges uniformly on $[a,b]$ under assumption (H4).

Proof. Firstly, we shall prove that if u is a bounded solution, then u solves

$$u(x) = u_a \sum_{n=0}^{N} \lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(x) + \lambda^{N+1} \left(I_{a+}^{(\nu),N+1} u \right)(x), \quad \text{for all } N \geq 0. \tag{6.3.20}$$

Proceeding by induction. For the case $N = 0$, take $g(x) := \lambda u(x)$, which is a bounded function, then Theorem 6.3.2 implies that

$$u(x) = u_a + \left(I_{a+}^{(\nu)} \lambda u \right)(x), \tag{6.3.21}$$

is the generalized solution to (6.3.17). Rewriting again the function u in terms of the generalized fractional integral $I_{a+}^{(\nu)}$ yields

$$\begin{aligned} u(x) &= u_a + \left(I_{a+}^{(\nu)} \lambda \left[u_a + I_{a+}^{(\nu)} \lambda u(\cdot) \right] \right)(x) \\ &= u_a + u_a \lambda \left(I_{a+}^{(\nu)} \mathbf{1} \right)(x) + \lambda^2 \left(I_{a+}^{(\nu),2} u \right)(x), \end{aligned}$$

as required.

Let us now assume that the equality (6.3.20) holds for $N-1$, that is

$$u(x) = u_a \sum_{n=0}^{N-1} \lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(x) + \lambda^N \left(I_{a+}^{(\nu),N} u \right)(x). \tag{6.3.22}$$

Plugging (6.3.21) into the r.h.s of equation (6.3.22) implies

$$\begin{aligned} \lambda^N \left(I_{a+}^{(\nu),N} u \right)(x) &= \lambda^N \left(I_{a+}^{(\nu),N} \left[u_a + I_{a+}^{(\nu)} \lambda u(\cdot) \right] \right)(x) \\ &= u_a \lambda^N \left(I_{a+}^{(\nu),N} \mathbf{1} \right)(x) + \lambda^{N+1} \left(I_{a+}^{(\nu),N+1} u \right)(x). \end{aligned} \tag{6.3.23}$$

Substituting (6.3.23) into (6.3.22) yields (6.3.20).

Thus, to obtain (6.3.19), it is enough to see that the second term in the r.h.s of (6.3.20) vanishes as $N \to \infty$ for each x, but the latter follows from the fact that

$$\left|\lambda^N \left(I_{a+}^{(\nu),N} u\right)(x)\right| \le \|u\| \left|\lambda^N \left(I_{a+}^{(\nu),N} \mathbf{1}\right)(x)\right| \to 0, \qquad \text{as } N \to \infty,$$

since (by assumption) the series in (6.3.18) is uniformly convergent on $[a, b]$.

Finally, we need to prove that (H4) implies the uniform convergence of (6.3.18). To do so, notice that Proposition 6.3.5 guarantees that, for each $n \in \mathbb{N}$, it holds

$$\left|\lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1}\right)(x)\right| \le \lambda^n \frac{(b-a)^{n\beta}}{(\Gamma(\beta+1))^n} \prod_{k=0}^{n-1} B(k\beta+1,\, \beta).$$

Thus, proceeding by induction (using the identities in (A.3.3), see Appendix) yields

$$\prod_{k=0}^{n-1} B(k\beta+1, \beta) = \frac{(\,\Gamma(\beta)\,)^n}{n\beta\Gamma(n\beta)}, \qquad n \in \mathbb{N}.$$

Further, the inequality (A.3.4) in the Appendix implies

$$\frac{(\,\Gamma(\beta)\,)^n}{n\beta\Gamma(n\beta)} \le \frac{(\,\Gamma(\beta)\,)^n}{n\beta(n-1)!\beta^{2(n-1)}\,(\,\Gamma(\beta)\,)^n} \le \frac{1}{n!\beta^{2n}}.$$

Hence,

$$\left|\lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1}\right)(x)\right| \le \left(\lambda \frac{(b-a)^\beta}{\beta^2\Gamma(\beta+1)}\right)^n \frac{1}{n!} =: M_n.$$

Since $\sum_{n=0}^{\infty} M_n$ converges, Weierstrass $M-$test implies the uniform convergence of the series (6.3.18) on $[a, b]$, as required. ∎

Remark 6.3.8. *In the classical fractional setting, the series (6.3.19) provides the very well-known series representation for the solution to the Caputo equation*

$$D_{a+*}^{\beta} u(x) = -\lambda u(x), \quad x \in (a, b], \quad \beta \in (0, 1), \qquad u(a) = u_a.$$

Namely, the equality in (6.3.16) implies

$$u(x) = u_a \sum_{n=0}^{\infty} (-\lambda)^n \left(I_{a+}^{\beta,n} \mathbf{1} \right)(x) = u_a \sum_{n=0}^{\infty} \frac{(-\lambda(x-a)^{\beta})^n}{\Gamma(n\beta+1)} = E_{\beta}\left(-\lambda(x-a)^{\beta}\right), \quad (6.3.24)$$

where $E_\beta(\cdot)$ stands for the Mittag-Leffler function of order β (see Appendix).

Linear equation with non-constant coefficients

Consider now the equation

$$\begin{aligned} -D_{a+*}^{(\nu)} u(x) &= \lambda(x) u(x) + g(x), \quad x \in (a,b] \\ u(a) &= u_a. \end{aligned} \quad (6.3.25)$$

For any function $\lambda \in B[a,b]$, define the operator $L_\lambda^{(\nu)}$ by

$$\left(L_\lambda^{(\nu)} f \right)(x) := \left(I_{a+}^{(\nu)} \lambda \cdot g \right)(x), \quad g \in B[a,b]. \quad (6.3.26)$$

Notation $L_\lambda^{(\nu),n}$ will denote the n-fold iteration of the operator $L_\lambda^{(\nu)}$ for each $n \in \mathbb{N}_0$. As usual, $L_\lambda^{(\nu),0} \equiv \mathbb{I}$, where $\mathbb{I}$ stands for the identity operator.

Theorem 6.3.9. *Let ν be a function satisfying assumption (H0) and (H4). Suppose that $\lambda \in B[a,b]$. Then, there exists a unique bounded solution u to equation (6.3.25) given by the series*

$$u(x) = u_a \sum_{n=0}^{\infty} \left(L_\lambda^{(\nu),n} \mathbf{1} \right)(x) + \sum_{n=0}^{\infty} \left(L_\lambda^{(\nu),n} \circ I_{a+}^{(\nu)} g \right)(x). \quad (6.3.27)$$

In particular, for any constant $\lambda \in \mathbb{R}$, the solution takes the form

$$u(x) = u_a \sum_{n=0}^{\infty} \lambda^n \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(x) + \sum_{n=0}^{\infty} \lambda^n \left(I_{a+}^{(\nu),n+1} g \right)(x). \quad (6.3.28)$$

Proof. As in the proof of Theorem 6.3.7, we obtain (proceeding by induction) that if u is a bounded solution, then u solves

$$u(x) = u_a \sum_{n=0}^{N} \left(L_\lambda^{(\nu),n} \mathbf{1} \right)(x) + \sum_{n=0}^{N} \left(L_\lambda^{(\nu),n} \circ I_{a+}^{(\nu)} g \right)(x) + \left(L_\lambda^{(\nu),N+1} u \right)(x), \quad \text{for all } N \geq 0. \tag{6.3.29}$$

Note that

$$a_n(x) \;:=\; \left| \left(L_\lambda^{(\nu),n} \mathbf{1} \right)(x) \right| \;\leq\; \|\lambda\|^n \left| \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(x) \right| \;=:\; b_n(x).$$

Theorem 6.3.7 and assumption (H4) imply the uniform convergence of $\sum_{n=0}^{\infty} b_n(x)$ on $[a, b]$, which in turn yields the uniform convergence of $\sum_{n=0}^{\infty} a_n(x)$ on $[a, b]$. Similarly, the inequality

$$c_n(x) := \left| \left(L_\lambda^{(\nu),n} \circ I_{a+}^{(\nu)} g \right)(x) \right| \;\leq\; \|\lambda\|^n \|g\| \left| \left(I_{a+}^{(\nu),n+1} \mathbf{1} \right)(x) \right| = \frac{\|g\|}{\|\lambda\|} b_{n+1}(x),$$

implies the uniform convergence of $\sum_{n=0}^{\infty} c_n(x)$ on $[a, b]$. Moreover, since

$$|L_\lambda^{N+1} u(x)| \leq \|u\| |L_\lambda^{N+1} \mathbf{1}(x)| \to 0, \quad \text{as} \quad N \to \infty,$$

due to the uniform convergence of $\sum_{n=0}^{\infty} (L_\lambda^{(\nu),n} \mathbf{1})(x)$ and the boundedness of u, letting $N \to \infty$ in the equality (6.3.29) yields the result in (6.3.27). ∎

Remark 6.3.10. *Consider the Caputo fractional equation*

$$\begin{aligned} D_{a+*}^{\beta} u(x) &= -\lambda u(x) + g(x), \quad x \in (a, b] \\ u(a) &= u_a. \end{aligned} \tag{6.3.30}$$

According to Theorem 6.3.9, the solution to (6.3.30) is given by

$$u(x) = u_a \sum_{n=0}^{\infty} (-\lambda)^n \left(I_{a+}^{n\beta} \mathbf{1} \right)(x) + \sum_{n=0}^{\infty} (-\lambda)^n \left(I_{a+}^{\beta(n+1)} g \right)(x). \tag{6.3.31}$$

Further, as seen in Remark 6.3.8, the first term in the r.h.s. of (6.3.31) coincides *with the Mittag-Leffler function* $E_\beta\left(-\lambda(x-a)^\beta\right)$*; whereas*

$$\begin{aligned}\sum_{n=0}^{\infty}(-\lambda)^n I_{a+}^{n\beta+\beta}g(x) &= \sum_{n=0}^{\infty}(-\lambda)^n \int_a^x \frac{(x-y)^{n\beta+\beta-1}g(y)dy}{\Gamma(n\beta+\beta)}\\ &= \int_a^x \sum_{n=0}^{\infty} \frac{(-\lambda)^n(x-y)^{n\beta+\beta-1}g(y)dy}{\Gamma(n\beta+\beta)}\\ &= \int_a^x \sum_{n=0}^{\infty} \frac{(-\lambda)^n(x-y)^{\beta n}}{\Gamma(n\beta+\beta)}(x-y)^{\beta-1}g(y)dy\\ &= \int_a^x E_{\beta,\beta}(-\lambda(x-y)^\beta)(x-y)^{\beta-1}g(y)dy.\end{aligned}$$

Hence, one obtains the very well-known integral representation for the solution to (6.3.30) given by [15, p.136]

$$u(x) = u_a E_\beta\left(-\lambda(x-a)^\beta\right) + \int_a^x E_{\beta,\beta}(-\lambda(x-y)^\beta)(x-y)^{\beta-1}g(y)dy.$$

6.3.4 Stochastic representations of solutions

In the previous sections we proved the existence of generalized solutions to *ordinary* fractional differential equations of Caputo type. Further, some series representations for the solutions were obtained as well. Knowing the existence of solutions, we can now apply Dynkin's martingale theorem (see Theorem A.1.3, Appendix) to obtain also a stochastic representation for the corresponding solutions.

As usual, we assume that the stochastic processes considered here are defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 6.3.11. *Let* ν *be a function satisfying assumption (H0) and (H4). Suppose that* $\lambda \in \mathbb{R}$, $\lambda \le 0$ *and* $g \in C[a,b]$*. If* u *is a generalized solution to (6.3.25), then* u *admits the stochastic representation*

$$u(x) = \mathbf{E}\left[e^{\lambda\tau_a(x)}u_a\right] - \mathbf{E}\left[\int_0^{\tau_a(x)} e^{\lambda s} g\left(X_x^{+(\nu)}(s)\right)ds\right], \tag{6.3.32}$$

where $X_x^{(\nu)}$ *is the process (started at* $x \in [a,b]$*) generated by the operator* $(-G_+^{(\nu)}, \mathfrak{D}_G)$

and $\tau_a(x)$ is the first exit time from the interval $(a, +\infty)$ of $X_x^{+(\nu)}$.

Proof. By Theorem 4.1 in [55], assumption (H0) implies that the operator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ generates a Feller process on $[a, b]$. Further, assumption (H4) implies the regularity in expectation of the point a and the finite expectation of $\tau_a(x)$. Hence, if $u \in \mathfrak{D}_{a+*}^{(\nu)}$, then the result for the case $\lambda = 0$ follows from the application of Dynkin's martingale theorem (see Theorem A.1.3 in Appendix) to the process $M = \{M(s)\}_{s\geq 0}$

$$M(s) := u\left(X_x^{a+*(\nu)}(s)\right) - u\left(X_x^{a+*(\nu)}(0)\right) + \int_0^s D_{a+*}^{(\nu)} u\left(X_x^{a+*(\nu)}(r)\right) dr, \quad s \geq 0;$$

together with the use of Doob's optional theorem [53, Theorem 3.10.1] for the stopping time $\tau_a(x)$. On the other hand, the case $\lambda < 0$ and $u \in \mathfrak{D}_{[a,b]*}^{(\nu)}$ follows using the martingale (see Theorem A.1.4 in Appendix)

$$M_\lambda(s) := e^{\lambda s} u\left(X_x^{a+*(\nu)}(s)\right) + \int_0^s e^{\lambda r} \,(\lambda - D_{a+*}^{(\nu)}) u\left(X_x^{a+*(\nu)}(r)\right) dr, \quad s \geq 0;$$

For the general case $u \in C[a, b]$, the proof follows by standard approximation arguments. Similar arguments have been used before, so we omit the details. ∎

Remark 6.3.12. *Notice that, under the additional assumption (H3), Lemma 2.5.7 in Chapter 2 implies that the solution u can be written explicitly as*

$$u(x) = \int_0^\infty u_a e^{\lambda s} \mu_a^x(s) ds - \int_a^x g(y) \int_0^\infty e^{\lambda s} p_s^{+(\nu)}(x, y)\, ds\, dy, \qquad (6.3.33)$$

where

$$\mu_a^x(s) := \frac{\partial}{\partial s} \int_{-\infty}^a p_s^{+(\nu)}(x, y) dy, \quad x > a, \qquad (6.3.34)$$

is the transition density function of the r.v. $\tau_a(x)$. Otherwise, using integration by *parts, the function u rewrites*

$$u(x) = -\int_0^\infty \lambda u_a e^{\lambda s}\left(\int_{-\infty}^a p_s^{+(\nu)}(x,y)dy\right)ds - \int_a^x g(y)\int_0^\infty e^{\lambda s}p_s^{+(\nu)}(x,y)\,ds\,dy. \tag{6.3.35}$$

6.4 Well-posedness results: nonhomogeneous case

This section establishes the existence and uniqueness of generalized solutions to the nonhomogeneous evolution equation given in 6.1.1. The result relies on transforming the equation (6.1.1) into an *abstract* linear equation of the type

$$\begin{aligned} -D_{a+*}^{(\nu)}u(t) &= Au(t) - g, \qquad \text{in } C([a,b];\mathsf{B}), \\ u(a) &= \phi_a, \qquad \text{in } \mathsf{B} \end{aligned} \tag{6.4.1}$$

where $g \in C([a,b];\mathsf{B})$ and $\phi_a \in \mathsf{B}$ for a suitable Banach space B. Hence, we are seeking a solution u given by a B-valued function defined on $[a,b]$.
For any Banach space $(\mathsf{B}, \|\cdot\|_{\mathsf{B}})$, notation $C([a,b];\mathsf{B})$ stands for the space of functions $f:[a,b]\times\mathbb{R}^d \to \mathbb{R}$ such that

$$C([a,b];\mathsf{B}) := \left\{f \in C([a,b]\times\mathbb{R}^d) \,:\, f(t,\cdot)\in\mathsf{B},\ f(\cdot,x)\in C[a,b]\right\}.$$

This space equipped with the norm

$$\|f\|_{C_B} = \sup_{t\in[a,b]}\|f(t,\cdot)\|_{\mathsf{B}}.$$

If A is a bounded operator, then the solution takes a series representation similar to the one given in Theorem 6.3.9.

Theorem 6.4.1. *Let ν be a function satisfying assumption (H0) and (H4). Suppose that A is a bounded operator on a Banach space $(B, \|\cdot\|_B)$. Assume that $\phi_a \in B$ and $g \in C([a,b];B)$. Then, there exists a unique generalized solution $u(t,x) \in C([a,b];B)$*

to equation (6.1.1) given by

$$u(t) = \sum_{n=0}^{\infty} (A^n \phi_a) \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(t) + \sum_{n=0}^{\infty} \left(L_A^{(\nu),n} g \right)(t), \qquad (6.4.2)$$

where $L_A^{(\nu),0} \equiv \mathbb{I}$ *(the identity operator), and* $L_A^{(\nu),n}$ *stands for the* n*-fold iteration of the operator* $L_A^{(\nu),n}$ *defined by*

$$\left(L_A^{(\nu)} h \right)(t) \ := \ \left(I_{a+}^{(\nu)} \circ Ah \right)(t), \quad h \in B([a,b]; \mathsf{B}). \qquad (6.4.3)$$

Proof. We seek a solution $u(t) \in C([a,b]; \mathsf{B})$, with $u(a) = \phi_a(\cdot) \in \mathsf{B}$. Since A is a bounded linear operator acting on x, it can be considered as the parameter λ in Theorem 6.3.9. Then, we obtain (proceeding by induction) that if u is a bounded solution, then u satisfies

$$u(t) = \sum_{n=0}^{N} (A^n \phi_a) \left(I_{a+}^{(\nu),n} \mathbf{1} \right)(t) + \sum_{n=0}^{N} \left(L_A^{(\nu),n} \circ I_{a+}^{(\nu)} g \right)(t) + \left(L_A^{(\nu),N+1} u \right)(t). \qquad (6.4.4)$$

Since A is a bounded operator, the convergence of the first two series in the previous equality and the convergence of $\left(L_A^{(\nu),N+1} u \right)(t) \to 0$ as $N \to \infty$ is guaranteed, due to assumption (H4), similarly as in the case $\lambda \in \mathbb{R}$.

■

Let us now assume that the operator $-A$ is the generator of a Feller process on $\mathbb{R}^d$. Since in this case $-A$ is not necessarily bounded, the convergence of the series in the previous theorem cannot be guaranteed as was done before. Hence, for this case, we shall use the integral representation given in (6.3.33).

Theorem 6.4.2. *Let* ν *be a function satisfying assumptions (H0),(H3) and (H4). Suppose that* $(-A, \mathfrak{D}_A)$ *is the generator of a Feller process on* $\mathbb{R}^d$ *with semigroup* $S = \{S_s\}_{s \geq 0}$ *and domain* $\mathfrak{D}_A$*. Then, for any* $g \in C([a,b]; C_\infty(\mathbb{R}^d))$ *and* $\phi_a \in C_\infty(\mathbb{R}^d)$*, there exists a unique generalized solution* $u \in C([a,b]; C_\infty(\mathbb{R}^d))$ *to the equation*

(6.1.1) given by

$$u(t,x) = \int_0^\infty (S_s\phi_a)(x)\mu_a^x(s)ds + \int_a^t \int_0^\infty (S_s g(r,\cdot))(x) p_s^{+(\nu)}(t,r)\, dr\, ds, \quad (6.4.5)$$

where μ_a^x is given by (6.3.34).

Proof. Let us write $\mathsf{B} := C_\infty(\mathbb{R}^d)$. For each $\lambda > 0$ and $\phi_a \in \mathsf{B}$, consider the abstract equation

$$\begin{aligned} -D_{a+*}^{(\nu)} u_\lambda(t) &= A^\lambda u_\lambda(t) - g(t), \qquad \text{in} \quad C((a,b];\mathsf{B}) \\ u_\lambda(a) &= \phi_a, \end{aligned} \quad (6.4.6)$$

where $-A^\lambda$ is the Yosida approximation of the operator $-A$, i.e. $-A^\lambda := -\lambda A(\lambda + A)^{-1}$, (see, e.g., [20, p.12]). Hence, the equation (6.4.6) approximates the original equation (6.4.1) as $\lambda \to \infty$.

Since $-A^\lambda$ is a bounded operator on B, Theorem 6.4.1 ensures the existence of a unique generalized solution $u_\lambda \in C([a,b];\mathsf{B})$ to equation (6.4.6). Further, Theorem 6.3.11 provides a stochastic representation for the function u_λ, which can be written explicitly as in (6.3.33) due to assumption (H3). Namely,

$$u_\lambda(t) = \int_0^\infty \mu_a^t(s) e^{-A^\lambda s}\phi_a ds + \int_a^t \int_0^\infty e^{-A^\lambda s} g(r) p_s^{+(\nu)}(t,r)\, dr\, ds. \quad (6.4.7)$$

Let $S_s^\lambda := e^{-A^\lambda s}$ be the semigroup generated by $-A^\lambda$. Then, the dominated convergence theorem implies that $u_\lambda \to u$ as $\lambda \to +\infty$ since the Yosida approximation satisfies $\lim_{\lambda\to\infty} e^{-sA_\lambda} f = S_s f$ for all $f \in \mathsf{B}$ and all $t \geq 0$ uniformly on bounded intervals [20, Proposition 2.7, p. 14]. Therefore, the function u is the generalized solution to (6.1.1), as required. ∎

Remark 6.4.3. *If the representation given in (6.3.35) is used instead of (6.3.34), then one obtains a different representation for the generalized solution to equation 6.1.1. The use of (6.3.35) does not require the differentiability of the transition* *densities on the time variable, but it will impose the condition that the boundary function ϕ_a belongs to the domain of $(-A, \mathfrak{D}_A)$.*

6.5 Well-posedness results: nonlinear case

Let us now study the well-posedness for the nonlinear equation given in (6.1.2). Firstly, we shall introduce some definitions, and then we will proceed as in Chapter 4 via fixed point arguments.

Definition 6.5.1. *Let ν be a function satisfying (H0) and (H4). A function $u : [a,b]\times\mathbb{R}^d \to \mathbb{R}$ is said to be a generalized solution to the nonlinear equation (6.1.2) if u is a generalized solution to the linear equation (6.1.1) with $g(t,x) := f(t,x,u(t,x))$ for all $(t,x) \in [a,b]\times\mathbb{R}^d$.*

Lemma 6.5.1. *Let ν be a function satisfying conditions (H0), (H3) and (H4). Assume that $(-A, \mathfrak{D}_A)$ is the generator of a Feller process $S = \{S_s\}$ on $\mathbb{R}^d$. Suppose that $f : [a,b]\times\mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function and $\phi_a \in C_\infty(\mathbb{R}^d)$. Then, a function $u \in C_\infty([a,b]\times\mathbb{R}^d)$ is a generalized solution to equation (6.1.2) if, and only if, u solves the nonlinear integral equation*

$$u(t,x) = \int_0^\infty (S_s\phi_a)(x)\mu_a^t(s)\,ds + \\ + \int_a^t \int_0^\infty \left(S_s f(r,\cdot,u(r,\cdot))\right)(x) p_s^{+(\nu)}(t,r)\,ds\,dr, \qquad (6.5.1)$$

where $p_s^{+(\nu)}(t,r)$ is the transition density function of the process generated by $(-G_+^{(\nu)}, \mathfrak{D}_G)$.

Proof. By Definition 6.5.1, $u \in C_\infty([a,b]\times\mathbb{R}^d)$ is a generalized solution to (6.1.2) if, and only if, u is a generalized solution to the the linear equation (6.1.1) with $g(t,x) := f(t,x,u(t,x))$ and $\lambda = 0$. Note that if $u \in C_\infty([a,b]\times\mathbb{R}^d)$, then g is bounded measurable function on $[a,b]\times\mathbb{R}^d$. Theorem 6.4.2 implies the integral equation (6.4.5), as required. ∎

Using Weissenger's fixed point theorem we shall prove that the integral equation

(6.5.1) possesses a unique solution under the following additional assumption:

(H5'): The function $f : [a,b] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is bounded and fulfills a Lipschitz condition with respect to the third variable, i.e., for all $(t,x,y_1),(t,x,y_2) \in [a,b] \times \mathbb{R}^d \times \mathbb{R}$,

$$|f(t,x,y_1) - f(t,x,y_2)| < L_f|y_1 - y_2|, \tag{6.5.2}$$

for a constant $L_f > 0$ (independent of t and x).

Theorem 6.5.2. *Let $[a,b] \subset \mathbb{R}$ and $\phi_a \in C_\infty(\mathbb{R}^d)$. Suppose that ν is a function satisfying conditions (H0), (H3) and (H4). If f is a function satisfying condition (H5'), then the equation (6.1.2) has a unique generalized solution $u \in C([a,b];C_\infty(\mathbb{R}^d))$.*

Proof. By definition, the existence of a unique generalized solution to (6.1.2) means the existence of a unique solution to the integral equation (6.5.1). The latter equation can be rewritten as a fixed point problem $u(t,x) = (\Psi u)(t,x)$ for a suitable operator Ψ.

Step a) Definition of the operator Ψ. Given the function $\phi_a \in C_\infty(\mathbb{R}^d)$, denote by B_{ϕ_a} the closed convex subset of $C\left([a,b];C_\infty(\mathbb{R}^d)\right)$ consisting of functions h satisfying $h(a) = \phi_a$. This space endowed with the norm

$$||h||_{B_{\phi_a}} = \sup_{t\in[a,b]} ||h(t,\cdot)||.$$

Define the operator Ψ on B_{ϕ_a} by

$$\begin{aligned}(\Psi u)(t,x) := &\int_0^\infty (S_s\phi_a)(x)\mu_a^t(s)\, ds \ + \\ &+ \int_a^t \int_0^\infty (S_s f(r,\cdot,u(r,\cdot)))(x)p_s^{+(\nu)}(t,r)\, ds\ \ dr, \qquad t \in [a,b]. \end{aligned} \tag{6.5.3}$$

Note that if $u \in B_{\phi_a}$, then $(\Psi u)(\cdot,x) \in C[a,b]$ for each $x \in \mathbb{R}^d$ and $(\Psi u)(t,\cdot) \in C_\infty(\mathbb{R}^d)$ for each $t \in [a,b]$. Further, $(\Psi u)(a,x) = \phi_a(x)$ since $\mu_a^t(s) \to \delta_0(s)$ as $t \to a$. Therefore, $\Psi : B_{\phi_a} \to B_{\phi_a}$.

Step b) Let Ψ^n denote the n-fold iteration of the operator Ψ for $n \geq 0$, $n \in \mathbb{N}$, where Ψ^0 denotes the identity operator. Note that for $n = 1$, the Lipschitz condition of f and the fact that S_s is a contraction semigroup imply

$$\begin{aligned}\Big|(\Psi u)(t,x) - (\Psi v)(t,x)\Big| &\leq L_f \int_a^t \int_0^\infty \|u-v\|_t p_s^{+(\nu)}(t,r)\, ds \ \ dr, \\ &\leq L_f \|u-v\|_t \left(I_{a+}^{(\nu)} \mathbf{1}\right)(t),\end{aligned}$$

where

$$\|u-v\|_t := \sup_{z\leq t} \|u(z,\cdot) - v(z,\cdot)\|, \quad t \in [a,b],$$

and L_f is the Lipschitz constant of the function f.
Proceeding by induction we obtain that

$$|\Psi^n u(t,x) - \Psi^n v(t,x)| \leq \|u-v\|_t L_f^n \left(I_{a+}^{(\nu),n} \mathbf{1}\right)(t) \qquad n \geq 0, \tag{6.5.4}$$

where $I_{a+}^{(\nu),n}$ is the n-fold iteration of the generalized fractional operator $I_{a+}^{(\nu)}$. Moreover, by Theorem 6.3.9, we know that

$$\sum_{n=0}^{\infty} L_f^n \left(I_{a+}^{(\nu),n} \mathbf{1}\right)(t) \leq \left(\frac{L_f^n (b-a)^\beta}{\beta^2 \Gamma(\beta+1)}\right)^n \frac{1}{n!} = \alpha_n.$$

Hence,

$$\|\Psi^n u - \Psi^n v\|_{B_{\phi_a}} \leq \alpha_n \|u-v\|_{B_{\phi_a}}, \tag{6.5.5}$$

for every $n \geq 0$ and every $u, v \in B_{\phi_a}$, where $\alpha_n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ converges. Therefore, the Weissinger fixed point theorem [15, Theorem D.7, Appendix] guarantees the existence of a unique fixed point $u^* \in B_{\phi_a}$ to the integral equation (6.5.1), which in turn implies the existence of a generalized solution to (6.1.2), as required. ■

6.6 Stochastic representations of solutions

Using once more the probabilistic interpretation of the Caputo type operator $-D^{(\nu)}_{a+*}$, we can now obtain a stochastic representation (in terms of mathematical expectations) for the generalized solution to the nonhomogeneous evolution equation (6.1.1). The result relies on proving that (i) the operator $-{}_tD^{(\nu)}_{a+*} - A_x$ is the generator of a Feller process, and (ii) the boundary points $(a,\cdot) \in \{a\} \times \mathbb{R}^d$ are *regular in expectation* for this operator.

Proposition 6.6.1. *Let ν be a function satisfying assumptions (H0) and (H4). Suppose that $(-A_x, \mathfrak{D}_A)$ is the generator of an $\mathbb{R}^d$-valued Feller process with a domain $\mathfrak{D}_A$ and an invariant core $\mathcal{C}_A$. Define $L_{a+*} := -{}_tD^{(\nu)}_{a+*} - A_x$, then the operator $(-L_{a+*}, \mathfrak{D}_{L*})$ generates a Feller process $\boldsymbol{Z}^{a+*}$ on $[a,b]\times\mathbb{R}^d$ with a domain $\mathfrak{D}_{L*}$ and with an invariant core $\mathcal{C}_{L*} \subset \mathfrak{D}_{L*}$ given by*

$$\mathcal{C}_{L*} := \left\{ f \in C_\infty([a,b]\times\mathbb{R}^d) \,:\, f(\cdot,x) \in C^1[a,b],\ f(t,\cdot) \in \mathcal{C}_A \right\}.$$

Proof. Both statements are a direct consequence of the Trotter product formula [20, Corollary 6.7, p. 33] since (by assumption) $(-A_x, \mathfrak{D}_A)$ generates a process $X = \{X(s)\}_{s\geq 0}$ and (by [55, Theorem 4.1]) the operator $(-{}_tD^{(\nu)}_{a+*},\ \mathfrak{D}^{(\nu)}_{a+*})$ generates a Feller process, say $T^{a+*(\nu)} = \left\{T^{a+*(\nu)}(s)\right\}_{s\geq 0}$, both processes being independent of each other.

∎

Remark 6.6.2. *Notice that, for each $(t,x) \in [a,b]\times\mathbb{R}^d$, the operator L_{a+*} generates a two-coordinate Feller (Markov) process $\boldsymbol{Z}^{a+*}_{(t,x)} := \left(\boldsymbol{Z}^{a+*}_{(t,x)}(s)\right)_{s\geq 0}$ on $[a,b]\times\mathbb{R}^d$, with initial state (t,x), given by*

$$\boldsymbol{Z}^{a+*}_{(t,x)}(s) := \left(T^{a+*(\nu)}_t(s),\ X_x(s) \right), \quad s \geq 0.$$

Therefore, the monotonicity of the process $T^{a+*(\nu)}_t$ implies that, once the process

$T_t^{a+*(\nu)}$ *reaches* a*, the first coordinate is absorbed at* $t = a$ *whilst the second coordinate* X_x *continuous a free (independent) motion.*

It is also worth noting that the process $\boldsymbol{Z}_{(t,x)}^{a+*}$ *is related to an underlying process* $\boldsymbol{Z}_{(t,x)}^{+} := \left(T_t^{+(\nu)}, X_x\right)$*, where* $T_t^{+(\nu)}$ *denotes the process (started at* $t \in (a,b]$*) with the generator* $(-G^{+(\nu)}, \mathfrak{D}_G)$ *(see definition in (2.3.1)). Namely, define the stopped process* $\hat{\boldsymbol{Z}}_{(t,x)}$ *by*

$$\hat{\boldsymbol{Z}}_{(t,x)}(s) := \begin{cases} \left(T_t^{+(\nu)}(s),\ X_x(s)\right) & \text{if } s < \tau_a(t), \\ \\ (a,\ X_x(s)) & \text{if } s \geq \tau_a(t). \end{cases} \tag{6.6.1}$$

where

$$\tau_a(t) := \inf\left\{s \geq 0 \,:\, T_t^{+(\nu)}(s) \notin (a, +\infty)\right\}, \tag{6.6.2}$$

is the first exit time from the interval $(a,+\infty)$ *of the (underlying) process* $T_t^{+(\nu)}$*. It follows that the paths of the processes* $\boldsymbol{Z}_{(t,x)}^{a+*}$*,* $\hat{\boldsymbol{Z}}_{(t,x)}$ *and* $\boldsymbol{Z}_{(t,x)}^{+}$ *coincide until the (monotone) first coordinate leaves the interval* $(a,b]$*.*

Let us now turn our attention to the regularity of the boundary of $(a,b] \times \mathbb{R}^d$. For the stochastic representation of the solutions to (6.1.1), we are interested only in part of the boundary of $(a,b] \times \mathbb{R}^d$, hereafter denoted by ∂_a, defined as

$$\partial_a := \left\{\mathbf{z}_a \in \mathbb{R}^{d+1} \,:\, \mathbf{z}_a = (a,x),\ x \in \mathbb{R}^d\right\}.$$

Proposition 6.6.3. *Let* ν *be a function satisfying assumptions (H0) and (H4) and let* $(-A_x, \mathcal{D}_A)$ *be the infinitesimal generator of a Feller process as stated in Proposition 6.6.1. If* $\boldsymbol{z}_a \in \partial_a$*, then* $\boldsymbol{z}_a$ *is regular in expectation for the operator* $(-L_{a+*}, \mathfrak{D}_{L^*})$*. Moreover, the first exit time from* $(a,b] \times \mathbb{R}^d$ *for the corresponding process has finite expectation.*

Proof. Let $\mathbf{z} = (t,x) \in (a,b] \times \mathbb{R}^d$. Denote by $\tau_a^Z(\mathbf{z})$ the first time that the process $\mathbf{Z}_{\mathbf{z}}^{a+*}$ generated by $(-L_{a+*}, \mathfrak{D}_{L^*})$ leaves $(a,+\infty) \times \mathbb{R}^d$ when starting at $\mathbf{z} \in (a,b] \times \mathbb{R}^d$, i.e.

$$\tau_a^Z(\mathbf{z}) := \inf\left\{s \geq 0 : \mathbf{Z}_{\mathbf{z}}^{a+*}(s) \notin (a,+\infty) \times \mathbb{R}^d\right\}. \tag{6.6.3}$$

Analogously, define $\tau_a(t)$ as the first exit time from (a,∞) of the process $T_t^{a+*(\nu)}$ starting at $t \in (a,b]$. Observe that the events

$$\left\{ \tau_a^Z(\mathbf{z}) > s\right\}, \quad \left\{ \mathbf{Z}_{(t,x)}^{a+*(\nu)}(s) \in (a,b] \times \mathbb{R}^d\right\} \quad \text{ and } \left\{ (T_t^{+(\nu)}(s),\ X_x(s)) \in (a,b] \times \mathbb{R}^d\right\}$$

are all equivalent to the event $\{ \tau_a(t) > s\}$. Hence,

$$\mathbf{E}\left[\tau_a^Z(\mathbf{z})\right] = \int_0^\infty \mathbf{P}\left[\tau_a^Z(t,x) > s\right] ds = \int_0^\infty \mathbf{P}\left[\tau_a(t) > s\right] ds = \mathbf{E}\left[\tau_a(t)\right].$$

Therefore,

$$\mathbf{E}\left[\tau_a^Z(\mathbf{z})\right] \to 0, \quad \text{ as } \mathbf{z} \to \mathbf{z}_a, \quad \text{for } \mathbf{z}_a \in \partial_a,$$

holds due to the fact that the point $t = a$ is regular in expectation for the operator $(-D_{a+*}^{(\nu)}, \mathfrak{D}_{a+*}^{(\nu)})$ (see, Lemma 2.5.2). Further,

$$\mathbf{E}\left[\tau_a^Z(\mathbf{z})\right] < +\infty \text{ uniformly on } \mathbf{z} \in (a,b] \times \mathbb{R}^d$$

because $\tau_a(t)$ has finite expectation under assumptions (H0) and (H4). ∎

Remark 6.6.4. *As in the case of the first exit time $\tau_a(t)$ for the process $T_t^{a+*(\nu)}$, the distribution law of the first exit time from $(a,b] \times \mathbb{R}^d$ of both processes $\boldsymbol{Z}_{\boldsymbol{z}}^{+}$ and $\boldsymbol{Z}_{\boldsymbol{z}}^{a+*}$ coincide when they start at $\boldsymbol{z} = (t,x) \in (a,b] \times \mathbb{R}^d$, so that we will use indistinctly the same notation.*

We also have the following result related to the first exit time $\tau_a^Z(t,x)$.

Corollary 6.6.5. *Under the assumptions (H0), (H3) and (H4), it holds that*

$$\mathbf{E}\left[\tau_a^Z(t,x)\right] = \int_0^\infty \mathbf{P}\left[\tau_a(t) > s\right] ds = \int_0^\infty \int_a^t p_s^{+(\nu)}(t,r)\, dr\, ds, \quad t \in (a,b],\ x \in \mathbb{R}^d. \tag{6.6.4}$$

Furthermore, the distribution law of $\tau_a^Z(t,x)$ has the density function

$$\mu_a^{(t,x)}(s) := -\frac{\partial}{\partial s}\int_a^t p_s^{+(\nu)}(t,r)dr, \quad t \in (a,b]. \tag{6.6.5}$$

Proof. Since $\tau_a^Z(t,x)$ and $\tau_a(t)$ have the same distribution for each $t \in (a,b]$, the result follows directly from Lemma 2.5.2. ∎

To introduce the notion of a smooth solution, let us assume that the operator $-A$ is well-defined on twice differentiable functions.

Definition 6.6.1. *A function $u : [a,b] \times \mathbb{R}^d \to \mathbb{R}$ is said to be a smooth solution to equation (6.1.1) if u is a generalized solution belonging to $C_b([a,b]\times\mathbb{R}^d)\cap C^{1,2}((a,b]\times\mathbb{R}^d)$.*

Remark 6.6.6. *If $-A_x$ is also well-defined on differentiable functions, then the previous definition holds but with $u \in C_b([a,b]\times\mathbb{R}^d)\cap C^{1,1}((a,b]\times\mathbb{R}^d)$.*

Theorem 6.6.7. *Suppose that the assumptions in Proposition 6.6.1 hold. Assume also that the generator $(-A_x, \mathfrak{D}_A)$ has an invariant core $\mathcal{C}_A = C_\infty^2(\mathbb{R}^d)$. Let $\lambda = 0$, $g \in C_\infty([a,b]\times\mathbb{R}^d)$ and $\phi_a \in C_\infty(\mathbb{R}^d)$.*

(i) If u is a classical solution to the Caputo type equation (6.1.1), then u admits the the stochastic representation

$$u(t,x) = \mathbf{E}\left[\phi_a(X_x(\tau_a(t)))\right] + \mathbf{E}\left[\int_0^{\tau_a(t)} g\left(\boldsymbol{Z}^+_{(t,x)}(s)\right)ds\right], \tag{6.6.6}$$

where $\boldsymbol{Z}^+_{(t,x)} := \left(T_t^{+(\nu)}, X_x\right)$. Recall that $T_t^{+(\nu)}$ denotes the process (starting at $t \in (a,b]$) with the generator $(-G^{+(\nu)}, \mathfrak{D}_G)$ given in (2.3.1).

(ii) If, additionally, condition (H3) holds, then u takes the explicit form

$$\begin{aligned} u(t,x) = &\int_0^\infty \int_{\mathbb{R}^d} \phi_a(y)p_s(x,y)\,dy\,\mu_a^t(s)ds + \\ &+ \int_{\mathbb{R}^d}\int_0^{t-a} g(t-r,y)\int_0^\infty p_s^{+(\nu)}(t,t-r)p_s(x,y)\,ds\,dr\,dy, \end{aligned} \tag{6.6.7}$$

where $p_s(x,y)$ denotes the transition densities of the process generated by $(-A_x, \mathfrak{D}_A)$, $p_s^{+(\nu)}(t,r)$ stands for the transition densities of the process $T^{+(\nu)}$ and $\mu_a^t(s)$ is the density function of $\tau_a(t)$.

Proof. (i) Let $-L_{a+*} = -{}_tD_{a+*}^{(\nu)} - A_x$. By Proposition 6.6.1, the operator $(-L_{a+*}, \mathfrak{D}_{L*})$ generates a Feller process $\mathbf{Z}^{a+*}$ on $[a,b]\times\mathbb{R}^d$. Further, by Proposition 6.6.3 the stopping time $\tau_a^Z(t,x)$ is finite in expectation and it has the same distribution than $\tau_a(t)$. Take $u \in C([a,b]\times\mathbb{R}^d)$ such that $u(t,\cdot)\in\mathcal{C}_A$ and $u(\cdot,x)\in C^1[a,b]$, then $u\in\mathfrak{D}_{L*}$. Therefore, Dynkin's martingale theorem and Doob's stopping theorem (applied to the operator $(-L_{a+*}, \mathfrak{D}_{L*})$ and the stoping time $\tau_a^Z(t,x)$, see Appendix for the statements) imply that

$$u(t,x) = \mathbf{E}\left[u\left(\mathbf{Z}_{(t,x)}^{a+*}(\tau_a^Z(t,x))\right)\right] + \mathbf{E}\left[\int_0^{\tau_a^Z(t,x)} -L_{a+*}u\left(\mathbf{Z}_{(t,x)}^{a+*}(s)\right)ds\right] \tag{6.6.8}$$

$$= \mathbf{E}\left[\phi_a\left(X_x(\tau_a(t))\right)\right] + \mathbf{E}\left[\int_0^{\tau_a(t)} g\left(\mathbf{Z}_{(t,x)}^{+}(s)\right)ds\right], \tag{6.6.9}$$

where the last equality follows from the boundary condition and from the fact (by assumption) u is a solution to the equation (6.1.1) (so that $-L_{a+*}u = g$). We have used that the paths of the process $\mathbf{Z}_{(t,x)}^{a+*}$ coincide with the paths of the process $\mathbf{Z}_{(t,x)}^{+}$ before time $\tau_a(t)$.
The case when the solution u does not belong to the domain of the generator $(-L_{a+*},\ \mathfrak{D}_{L*})$ is obtained by standard approximation arguments.

(ii) Due to the independence between the coordinate processes $T_t^{+(\nu)}$ and X_x, the representation (6.6.7) is obtained by using the transition probabilities of X_x, and the joint distribution between the random variables $T_t^{+(\nu)}(s)$ and $\tau_a(t)$. The latter given in Proposition 2.5.5. ∎

Remark 6.6.8. *Observe now that if u is a classical solution to (6.1.1), then (by the previous theorem) this is necessarily unique.*

Remark 6.6.9. *If instead of having $\lambda = 0$ in Theorem 6.6.7 we consider $\lambda\in\mathbb{R}$, then* *we shall obtain an additional exponential term in the expressions given in (6.6.6) and (6.6.7). Even more, taking a function* $\lambda \in C([a,b]\times\mathbb{R}^d)$ *will yield to a stochastic representation in the form of a Feynman-Kac type formula (see, e.g., the linear case in Theorem 4.4.7).*

Chapter 7

Conclusions

This dissertation established the well-posedness (in the generalized sense) for equations involving *generalized fractional operators of Caputo and RL type*, denoted by $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$, respectively. In particular, it focused on the study of generalized linear equations (Chapter 3); nonlinear equations and linear equations with nonconstant coefficients (Chapter 4); two-sided equations (Chapter 5), as well as fractional evolution equations of Caputo type (Chapter 6).

The use of a probabilistic approach (based on the interpretation of the generalized operators as generators of *interrupted* Feller processes) allowed us to obtain stochastic representations for the solutions to the equations considered in this work, as well as smoothness results for specific cases. Further, for the case of generalized fractional evolution equations, the use of an analytical method provided the existence of generalized solutions defined via the concept of a Green's function. This analytical approach also allowed us to obtain some series representations for the solutions to certain equations. Moreover, since the classical Caputo and RL fractional derivatives of order $\beta \in (0,1)$ are particular cases arising by stopping and killing an inverted β-stable subordinator, respectively, the results presented here encompass and extend many very well-known results from the theory of (classical) fractional differential equations.

Appendix A

A.1 Feller processes: basic definitions

Let $\{S_s\}_{s\geq 0}$ be a strongly continuous semigroup of linear bounded operators on a Banach space $(\mathsf{B}, \|\cdot\|_{\mathsf{B}})$, i.e., $\lim_{s\to 0}\|S_s f - f\|_B = 0$ for all $f \in \mathsf{B}$. Its (infinitesimal) generator L with domain $\mathfrak{D}_L$, shortly $(L, \mathfrak{D}_L)$, is defined as the (possibly unbounded) operator $L : \mathfrak{D}_L \subset \mathsf{B} \to \mathsf{B}$ given by the strong limit

$$Lf := \lim_{s\downarrow 0} \frac{S_s f - f}{s}, \quad f \in \mathfrak{D}_L, \tag{A.1.1}$$

where the domain of the generator $\mathfrak{D}_L$ consists of those $f \in \mathsf{B}$ for which the limit in (A.1.1) exists in the norm sense. We also recall that, if L is a closed operator, then a linear subspace $\mathcal{C}_L \subset \mathfrak{D}_L$ is called a *core* for the generator L if the operator L is the closure of the restriction $L\big|_{\mathcal{C}_L}$ [20, Chapter 1, Section 3]. If additionally $S_s\mathcal{C}_L \subset \mathcal{C}_L$ for all $s \geq 0$, then $\mathcal{C}_L$ is said to be an *invariant core*.

The *resolvent operator* R_λ of the semigroup $\{S_s\}_{s\geq 0}$ is defined (for any $\lambda > 0$) as the Bochner integral (see, e.g., [17, Chapter 1], [20, Chapter 1])

$$R_\lambda g := \int_0^\infty e^{-\lambda s} S_s g \, ds, \quad g \in \mathsf{B}. \tag{A.1.2}$$

By taking $\lambda = 0$ in (A.1.2), one obtains the *potential operator* denoted by $R_0 g$ (whenever it exists).

We say that a E−valued (time-homogeneous) Markov process $X = (X(s))_{s\geq 0}$ is a

Feller process (see, e.g., [52, Section 3.6]) if its semigroup $\{S_s\}_{s\geq 0}$, defined by

$$S_s f(x) := \mathbf{E}\left[f\left(X(s)\right)|X(0)=x\right], \quad s \geq 0,\ x \in E,\ f \in B(E),$$

gives rise to a *Feller semigroup* when reduced to $C_\infty(E)$, i.e., it is a strongly continuous semigroup on $C_\infty(E)$ and it is formed by positive linear contractions ($0 \leq S_s f \leq 1$ whenever $0 \leq f \leq 1$).

For a stochastic process $X_x = (X_x(s))_{s\geq 0}$ with state space E, the subscript x in $X_x(s)$ means that the process starts at $x \in E$, so that notation $\mathbf{E}\left[f\left(X_x(s)\right)\right]$ shall be understood as $\mathbf{E}\left[f\left(X(s)\right)|X(0)=x\right]$.

Additional subscripts and superscripts will be used in the corresponding notations to differentiate amongst different stochastic processes, semigroups, generators (and their domains), resolvent and potential operators.

Some standard results from the theory of semigroups and stochastic processes which are used throughout this work are the following.

Theorem A.1.1. *(taken from [17, p. 24]) Let S_s be a strongly continuous semigroup of contractions on a Banach space B and let $(L, \mathfrak{D}_L)$ be its infinitesimal operator with domain $\mathfrak{D}_L$. Then for arbitrary $g \in B$ the equation*

$$\lambda f - Lf = g, \qquad \lambda > 0$$

has one and only one solution $f \in \mathfrak{D}_L$. This solution is given by the corresponding resolvent operator

$$f = R_\lambda g = \int_0^\infty e^{-\lambda s} S_s g ds.$$

Theorem A.1.2. *(taken from [17, p.26]) Let S_s be a strongly continuous semigroup of contractions on a Banach space B and let $(L, \mathfrak{D}_L)$ be its infinitesimal operator with domain $\mathfrak{D}_L$. If $f = R_0 g$, where R_0 is the potential operator corresponding to* *the semigroup S_s and $g \in B$, then $f \in \mathfrak{D}_L$, and*

$$-Lf = g. \tag{A.1.3}$$

If the operator R_0 is bounded, then for every $g \in B$, the equation (A.1.3) has a unique solution, which is given by $f = R_0 g$. In this case L is a one to one mapping of $\mathfrak{D}_L$ onto B and the potential R_0 gives the inverse mapping of B onto $\mathfrak{D}_L$.

For the previous two theorems, see reference [17, Theorem 1.1 and Theorem 1.1'] for the original statements.

The following standard result known as Dynkin's formula (or Dynkin's martingale) is also an important tool in this work.

Theorem A.1.3. ***(Dynkin's formula)*** *Let $X = \{X_s\}_{s\geq 0}$ be a Feller process with (infinitesimal) generator (L, D_L), where D_L is the domain of the generator. If $f \in D_L$, then the process*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \quad t \geq 0,$$

is a martingale (with respect to the same filtration for which X_t is a Markov process) under any initial distribution ν.

See, e.g., reference [53, Theorem 3.9.4, p.134] for the proof.

We shall also use the following more general form of Dynkin's martingale theorem (see, e.g., [53, Proposition 3.9.3, p. 136]).

Theorem A.1.4. *Suppose that the assumptions of Theorem A.1.3 hold. Let ϕ be a bounded continuously differentiable function. Then*

$$S_t = f(X_t)\phi(t) - \int_0^t \left[f(X_s)\frac{d}{ds}\phi(s) + \phi(s)Lf(X_s) \right] ds,$$

is a martingale for any $f \in D_L$*. In particular, choosing* $\phi(s) = e^{-\lambda s}$ *with* $\lambda > 0$ *yields the martingale*

$$f(X_t)e^{-\lambda t} - \int_0^t e^{-\lambda s}(\lambda - L)f(X_s)ds. \tag{A.1.4}$$

A.2 Stable subordinators

We always assume the existence of a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ such that all the stochastic processes of our interest are defined on it. Notation $\mathcal{F}_s^X$ means the completed natural filtration generated by a process $X = \{X(s)\}_{s\geq 0}$.

For $\beta \in (0,1)$, a β-stable subordinator, is a real-valued stable Lévy process $X^\beta = \{X^\beta(s) : s \geq 0\}$ started at 0 almost surely (a.s.) with independent increments $X^\beta(s) - X^\beta(r)$, for any $0 \leq r < s$, having the same distribution as the r.v. $W_\beta((s - r)^{1/\beta}, 1)$, i.e., a totally skewed positive β-stable r.v. with scale parameter $\sigma = (s - r)^{1/\beta}$ (see, e.g., [3], [77]).

This process has nondecreasing sample paths a.s. and is time-homogeneous with respect to its natural filtration. Further, since the β-stable processes are self-similar with index $1/\beta$, the process $\{c^{1/\beta}X^\beta(s) : s \geq 0\}$ has the same distribution as the process $\{X^\beta(cs) : s \geq 0\}$ for any positive constant c. Consequently, the transition probabilities $p_s^\beta(x, E) := \mathbf{P}[X^\beta(s) \in E | X^\beta(0) = x]$ for any $E \in \mathcal{B}(\mathbb{R})$ (the Borel sets of $\mathbb{R}$) satisfy

$$p_s^\beta(x, E) = s^{-1/\beta} \int_E w_\beta(s^{-1/\beta}(y - x); 1, 1)dy,$$

where $w_\beta(\cdot; \sigma, \gamma)$ is the density of β–stable r.v. $W_\beta(\sigma, \gamma)$ with scale parameter σ, skewness parameter γ and zero location parameter. The density $w_\beta(\cdot; 1, 1)$ corresponds to a standard β-stable r.v. and is given by

$$w_\beta(x; 1, 1) = \frac{1}{\pi}\Re \int_0^\infty \exp\left\{-iux - u^\beta \exp\left(-i\frac{\pi}{2}\beta\right)\right\} du, \tag{A.2.1}$$

where $\Re(z)$ means the real part of $z \in \mathbb{C}$ (see Theorem 2.2.1 in [88]).

The infinitesimal generator $(A^\beta, \mathfrak{D}_\beta)$ of a β-stable subordinator is the generator of a jump-type Markov process of the form

$$A^{\beta}h(x) = \int_0^{\infty}(h(x+y)-h(x))\nu_{\beta}(dy), \quad h \in \mathfrak{D}_{\beta}, \tag{A.2.2}$$

with a domain $\mathfrak{D}_{\beta}$ and with the jump intensity given by the Lévy measure ν supported in $\mathbb{R}_+$:

$$\nu_{\beta}(dy) = \frac{\beta}{\Gamma(1-\beta)y^{1+\beta}}dy = -\frac{1}{\Gamma(-\beta)y^{1+\beta}}dy. \tag{A.2.3}$$

The last equality holds due to the identity $\Gamma(x) = (x-1)\Gamma(x-1)$.

We say that the process $X^{+\beta} = \{X^{+\beta}(s) : s \geq 0\}$ is an *inverted* β*–stable subordinator* if $-X^{+\beta}$ is a β–stable subordinator with $\beta \in (0,1)$. Thus, $X^{+\beta}$ is a Markov process with non increasing sample paths a.s. and with the generator

$$A^{+\beta}h(x) = \int_0^{\infty}(h(x-y)-h(x))\,\nu_{\beta}(dy).$$

Notice that the relation

$$w_{\beta}(-x;\sigma,1) = w_{\beta}(x;\sigma,-1),$$

implies that $X^{+\beta}(s)-X^{+\beta}(r)$ has the same distribution as the r.v. $W_{\beta}\left((s-r)^{1/\beta},-1\right)$. Hence, the transition probabilities $p_s^{+\beta}(x,E) := \mathbf{P}[X^{+\beta}(s) \in E|X^{+\beta}(0) = x]$ are given by

$$p_s^{+\beta}(t,E) = s^{-1/\beta}\int_E w_{\beta}(s^{-1/\beta}(x-y);1,1)dy, \quad E \in \mathcal{B}(\mathbb{R}). \tag{A.2.4}$$

We shall use some of the following equalities:

1. If $p_s^{\beta}(x,y)$ denotes the transition densities of a β–stable subordinator, then

$$p_s^{\beta}(x,y) = s^{-1/\beta}\omega_{\beta}(s^{-1/\beta}(y-x);1,1). \tag{A.2.5}$$

2. The equalities

$$\int_0^\infty p_s^\beta(x,y)ds = \int_0^\infty s^{-1/\beta}\omega_\beta(s^{-1/\beta}(y-x);1,1)ds$$
$$= (y-x)^{\beta-1}\int_0^\infty u^{-1/\beta}\omega_\beta(u^{-1/\beta};1,1)du,$$

hold for $\beta \in (0,1)$. They are obtained via the change of variable $u = s(y-x)^{-\beta}$.

3. The relationship between $p_s^\beta(x,y)$ and $p_s^{+\beta}(x,y)$ implies that

$$\int_0^\infty p_s^{+\beta}(x,y)ds = (x-y)^{\beta-1}\int_0^\infty u^{-1/\beta}\omega_\beta(u^{-1/\beta};1,1)du$$
$$= \frac{1}{\Gamma(\beta)}(x-y)^{\beta-1}. \tag{A.2.6}$$

The last equality is obtained by means of the change of variable $z = u^{-1/\beta}$, and then using the Mellin transform of the β–stable densities $\omega_\beta(z;1,1)$ (see, e.g., [88, Theorem 2.6.3, p. 117]).

A.3 The Gamma and Beta function

The Euler's gamma function $\Gamma(\cdot)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \quad \mathcal{R}(z) > 0. \tag{A.3.1}$$

The Gamma function can be thought of as the generalization of the factorial function, since for $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$. For all $\alpha > 0$ and $\beta > 0$, the Euler's Beta function $B(\alpha,\beta)$ is defined by the two-parameter integral

$$B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du. \tag{A.3.2}$$

Some rather standard identities (see, e.g., [15, Theorem D.1, Theorem D.6]):

$$\Gamma(z+1) = z\Gamma(z), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{A.3.3}$$

We will also use the inequality

$$\Gamma(na) > (n-1)!a^{2(n-1)}\big(\Gamma(a)\big)^n, \tag{A.3.4}$$

for $n \in \mathbb{N}$ and $a > 0$.

Remark A.3.1. *The Gamma function can be extended to $z \in \mathbb{C}$ with negative real part except for negative integers. For $\beta < 0$, $\beta \notin \{0, -1, -2, \ldots\}$, $\Gamma(-\beta)$ is defined via the relation $\Gamma(\beta) = \frac{\Gamma(\beta+1)}{\beta}$.*

A.4 Mittag-Leffler function

The Mittag-Leffler function and their numerous generalizations take a relevant place in the solution of fractional differential equations.
The *Mittag-Leffler function* of order $\beta > 0$, E_β, is defined by

$$E_\beta(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\beta + 1)}, \quad z \in \mathbb{C}.$$

One of its generalizations is *the two-parameter Mittag-Leffler function* E_{β_1,β_2} given by

$$E_{\beta_1,\beta_2}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\beta_1 + \beta_2)}, \quad z \in \mathbb{C},\ \beta_1 > 0,\ \beta_2 \in \mathbb{R}.$$

In particular, these functions can be seen as the generalizations of the exponential function since $E_1(z) = E_{1,1}(z) = \exp(x)$. For a brief review of properties of these functions see, e.g., [15, Chapter 4]. More detailed treatments can be found, e.g., in [73], [76].

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