INFINITE PRIMITIVE DIRECTED GRAPHS

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ABSTRACT. A group G of permutations of a set Ω is *primitive* if it acts transitively on Ω, and the only G-invariant equivalence relations on Ω are the trivial and universal relations.

A digraph Γ is *primitive* if its automorphism group acts primitively on its vertex set, and is *infinite* if its vertex set is infinite. It has *connectivity one* if it is connected and there exists a vertex α of Γ, such that the induced digraph $\Gamma \setminus \{\alpha\}$ is not connected. If Γ has connectivity one, a *lobe* of Γ is a connected subgraph that is maximal subject to the condition that it does not have connectivity one. Primitive graphs (and thus digraphs) with connectivity one are necessarily infinite.

The primitive graphs with connectivity one have been fully classified by Jung and Watkins: the lobes of such graphs are primitive, pairwise-isomorphic and have at least three vertices. When one considers the general case of a primitive digraph with connectivity one, however, this result no longer holds. In this paper we investigate the structure of these digraphs, and obtain a complete characterisation.

1. PRELIMINARIES

Throughout this note, a *graph* Γ will be a pair $(V\Gamma, E\Gamma)$, where $V\Gamma$ is the set of *vertices* of Γ, and $E\Gamma$ the set of *edges*. The set $E\Gamma$ consists of unordered pairs of distinct elements of $V\Gamma$.

A *digraph* Γ is a pair $(V\Gamma, A\Gamma)$, where $A\Gamma$ is the set of *arcs* of Γ. Each arc is an ordered pair of distinct elements of $V\Gamma$. All paths in a digraph will be undirected, unless otherwise stated. A *directed cycle* in Γ is a path $\alpha_0 \alpha_1 \ldots \alpha_n$ such that $(\alpha_n, \alpha_0) \in A\Gamma$ and $(\alpha_i, \alpha_{i+1}) \in A\Gamma$ for all integers i satisfying $0 \leq i < n$.

All graphs and digraphs will be free of loops and multiple edges. They are said to be *infinite* if their vertex sets are infinite.

The *distance* between two connected vertices α and β in a graph or digraph Γ will be denoted by $d_\Gamma(\alpha, \beta)$.

Groups, and in particular groups of automorphisms, will play a leading role in many of the arguments presented herein. Throughout this work, G will be a group of permutations of a set Ω, where Ω will usually be the vertex set of some infinite digraph.

If $\alpha \in \Omega$ and $g \in G$, we denote the image of α under g by α^g. Following this notation, all permutations will act on the right. The orbit of α under the action of G will be denoted by α^G.

If $\alpha \in \Omega$, we denote the stabiliser of α in G by G_α, and if $\Sigma \subseteq \Omega$ we denote the setwise and pointwise stabilisers of Σ in G by $G_{\{\Sigma\}}$ and $G_{(\Sigma)}$ respectively.

A transitive group G is *primitive* on Ω if the only G-congruences admitted by Ω are the trivial and universal equivalence relations; otherwise G is said to be *imprimitive*. It is said to act *regularly* on Ω if $G_\alpha = 1$ for each $\alpha \in \Omega$.

A subset Δ of Ω is called a *block* if for all $g \in G$ we have either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. A block is called *trivial* if $|\Delta| = 1$, and proper if $\Delta \neq \Omega$. Since the existence of a non-trivial proper block permits the construction of a non-trivial and non-universal G-congruence on Ω, the group G is primitive if and only if Ω does not contain a non-trivial proper block.

The following is well known, and is often a very useful test for primitivity.

Theorem 1.1. ([1, Theorem 4.7]) *If G is a transitive group of permutations on Ω, and $|\Omega| > 1$, then G is primitive on Ω if and only if, for every $\alpha \in \Omega$, the stabiliser G_α is a maximal subgroup of G.* $\qquad\square$

A graph or digraph Γ is *primitive* if its automorphism group $\operatorname{Aut} \Gamma$ acts primitively on the set $V\Gamma$, and is *automorphism-regular* if $\operatorname{Aut} \Gamma$ acts regularly on $V\Gamma$.

A primitive graph or digraph Γ with at least one edge or arc is always connected. Indeed, the connected components of Γ form a set of $\operatorname{Aut} \Gamma$-congruence classes.

The *connectivity* of an infinite connected graph or digraph Γ is the smallest possible size of a subset W of $V\Gamma$ for which the induced graph $\Gamma \setminus W$ is disconnected. A *lobe* of Γ is a connected subgraph that is maximal subject to the condition it has connectivity strictly greater than one. If Γ has connectivity one, then the vertices α for which $\Gamma \setminus \{\alpha\}$ is disconnected are called the *cut vertices* of Γ.

2. Local structure

Consider the following construction. Let V_1 be the set of cut vertices of a connected graph Γ, and let V_2 be a set in bijective correspondence

with the set of lobes of Γ. We let T be a bipartite graph whose parts are V_1 and V_2. Two vertices $\alpha \in V_1$ and $x \in V_2$ are adjacent in T if and only if α is contained in the lobe of Γ corresponding to x. In fact, this construction yields a tree, which is called the *block-cut-vertex tree* of Γ. Note that if Γ has connectivity one and block-cut-vertex tree T, then any group G acting on Γ has a natural action on T.

It is perhaps helpful to the reader at this point to describe a graph that is typical of those in which we are interested. Let P_5 denote the Petersen Graph. To each vertex α in P_5 we adjoin another two copies of P_5 in such a way that α is contained in three distinct copies of P_5 that intersect only in α. We continue this process for each additional vertex whenever a new P_5 is adjoined. In this way we obtain an infinite graph with connectivity one, whose lobes are isomorphic to P_5. The block-cut-vertex tree of this graph is a biregular tree, in which one set of the natural bipartition has valency 3, and the other valency 10. As we shall see, this graph is primitive.

Let Γ be a primitive digraph with connectivity one whose lobes have at least three vertices, and suppose G is a vertex- and arc-transitive group of automorphisms of Γ. Since Γ is vertex-transitive with connectivity one, every vertex is a cut vertex. Fix some lobe Λ of Γ, and let H be the subgroup of the automorphism group $\mathrm{Aut}\,\Lambda$ induced by the setwise stabiliser $G_{\{\Lambda\}}$ of $V\Lambda$ in G. Let T be the block-cut-vertex tree of Γ, and let x be the vertex of T that corresponds to the lobe Λ. Our aim in this section is to show H is primitive but not regular.

If x_1 and x_2 are distinct vertices of the tree T, we use $C(T \setminus \{x_1\}, x_2)$ to denote the connected component of $T \setminus \{x_1\}$ that contains the vertex x_2.

Lemma 2.1. *If G acts primitively on the vertices of Γ, then H acts primitively on the vertices of Λ.*

Proof. If H acts transitively but not primitively on $V\Lambda$, then there exists a non-trivial proper block $\Delta \subseteq V\Lambda$. For any two distinct vertices $\alpha, \beta \in \Delta$, the digraph $(V\Lambda, (\alpha, \beta)^H)$ is not connected, since it does not contain a path from α to any vertex in $V\Lambda \setminus \Delta$.

If H does not act transitively on $V\Lambda$, then one may choose distinct vertices $\alpha, \beta, \gamma \in V\Lambda$ such that $\beta \in \alpha^H$ but $\gamma \notin \alpha^H$. Again the digraph $(V\Lambda, (\alpha, \beta)^H)$ is not connected, as there is no path from α to γ. Thus, if H is not transitive on $V\Lambda$, or if H is transitive but not primitive on $V\Lambda$, then there exist distinct vertices $\alpha, \beta \in V\Lambda$ such that the digraph $\Lambda' := (V\Lambda, (\alpha, \beta)^H)$ is not connected.

Suppose this is the case, and choose distinct vertices $\alpha, \beta \in V\Lambda$ such that Λ' is not connected. We will show this assumption implies the digraph $\Gamma' := (V\Gamma, (\alpha, \beta)^G)$ cannot be connected, and is therefore not primitive; whence G cannot be primitive.

Recall that T is the block-cut-vertex tree of Γ and x is the vertex of T corresponding to the lobe Λ. Let $\{\Delta_i\}_{i \in I}$ be the set of connected components of Λ' and let

$$\mathcal{C}_i := \bigcup_{\delta \in \Delta_i} C(T \setminus \{x\}, \delta) \cap V\Gamma.$$

Suppose $\delta_i \in \mathcal{C}_i$ and $\delta_j \in \mathcal{C}_j$, with $i \neq j$. We claim δ_i and δ_j are not adjacent in Γ'. Indeed, since the distance $d_T(\alpha, \beta)$ between α and β in T is equal to 2, if δ_i and δ_j are to be adjacent in the arc-transitive digraph Γ', it must be the case that $d_T(\delta_i, \delta_j) = 2$. If either δ_i or δ_j is not adjacent to x in T then $d_T(\delta_i, \delta_j) > 2$, so they cannot be adjacent in Γ'. On the other hand, if δ_i and δ_j are adjacent to x in T, then they both lie in $V\Lambda = V\Lambda'$, and therefore $\delta_i \in \Delta_i$ and $\delta_j \in \Delta_j$. In this case, if they are adjacent in Γ' then there exists $g \in G$ such that either (δ_i, δ_j) or (δ_j, δ_i) is equal to $(\alpha, \beta)^g$. Such an automorphism must fix $V\Lambda$ setwise, and therefore lies in $G_{\{\Lambda\}}$. Thus, there exists an element $h \in H$ such that either (δ_i, δ_j) or (δ_j, δ_i) is equal to $(\alpha, \beta)^h$, meaning that δ_i and δ_j are adjacent in Λ'; however, this contradicts the fact that δ_i and δ_j are in distinct components of Λ'. Hence, δ_i and δ_j are not adjacent in Γ'.

Hence, there can be no path in Γ' between a vertex in $\mathcal{C}_i$ and a vertex in $\mathcal{C}_j$ whenever $i \neq j$, and so the digraph Γ' is not connected. Whence, Γ' cannot be primitive, and G cannot act primitively on $V\Gamma$. $\qquad\square$

Fix distinct vertices $\alpha, \beta \in V\Gamma$ and recall that α and β are also vertices of the block-cut-vertex tree T.

A *geodesic* between two vertices is a shortest path between them. In a tree, there is a unique geodesic between any two vertices. Let $[\alpha, \beta]_T$ be the T-geodesic between α and β, and let $(\alpha, \beta)_T$ be the T-geodesic $[\alpha, \beta]_T$ excluding both α and β. This notation extends obviously to $[\alpha, \beta)_T$ and $(\alpha, \beta]_T$.

Since α and β are vertices of both Γ and T, the distance $d_T(\alpha, \beta)$ is even, so we may choose a vertex $y \in (\alpha, \beta)_T$ that is distinct from α and β.

Lemma 2.2. *If $g \in G_\alpha$ does not fix $y \in VT$, and $\delta \notin C(T \setminus \{y\}, \alpha)$, then $\delta^g \notin C(T \setminus \{y\}, \beta)$.*

Proof. If $\delta \notin C(T \setminus \{y\}, \alpha)$ and $\delta^g \in C(T \setminus \{y\}, \beta)$ then $\delta, \delta^g \notin C(T \setminus \{y\}, \alpha)$, so we must have $g \in G_{\alpha, y}$. $\qquad\square$

Lemma 2.3. *If $g \in G_\alpha$ does not fix the vertex y and $\delta \notin C(T \setminus \{y\}, \alpha)$ then $d_T(y, \delta^g) > d_T(y, \delta)$.*

Proof. If $\delta \notin C(T \setminus \{y\}, \alpha)$ then $y \in [\alpha, \delta]_T$. Thus $d_T(\delta, \delta^g) = d_T(\delta, y) + d_T(y, y^g) + d_T(y^g, \delta^g)$ and $d_T(\delta, \delta^g) = d_T(\delta, y) + d_T(y, \delta^g)$. Therefore $d_T(y, \delta^g) = d_T(y^g, \delta^g) + d_T(y, y^g)$. Now $d_T(y^g, \delta^g) = d_T(y, \delta)$, and $d_T(y, y^g) \geq 1$. Whence $d_T(y, \delta^g) > d_T(y, \delta)$. $\qquad\square$

Henceforth, if H is a subgroup of G, then we will write $H \leq G$; if we wish to exclude the possibility of $H = G$ we will instead write $H < G$.

Lemma 2.4. *Let $g_1, \ldots, g_n \in G_\alpha$ and $h_1, \ldots, h_n \in G_\beta$, and suppose $G_{\alpha,y} = G_{\beta,y}$. If there exists $\gamma \in VT$ such that $G_{\alpha,y} \leq G_\gamma$ then, for some $m \leq n$, there exist $g_2', \ldots, g_m' \in G_\alpha \setminus G_y$ and $g_1' \in G_\alpha \setminus G_y \cup \{1\}$ together with $h_1', \ldots, h_{m-1}' \in G_\beta \setminus G_y$ and $h_m' \in G_\beta \setminus G_y \cup \{1\}$ such that*

$$\gamma^{g_1' h_1' \ldots g_m' h_m'} = \gamma^{g_1 h_1 \ldots g_n h_n}.$$

Proof. The proof of this lemma will be an inductive argument. Suppose there exists $\gamma \in VT$ such that $G_{\alpha,y} \leq G_\gamma$.

Let $n = 1$. When considering $h_1 \in G_\beta$ we have two cases: either $h_1 \in G_y$ or $h_1 \in G_\beta \setminus G_y$. If $h_1 \in G_y$ then $h_1 \in G_{\beta,y} = G_{\alpha,y}$, so $g_1 h_1 \in G_\alpha$. In this case, redefine $g_1 := g_1 h_1$ and set $h_1' := 1$. Alternatively, if $h_1 \in G_\beta \setminus G_y$ then set $h_1' := h_1$. Having found a suitable h_1', we will now construct g_1' from the (possibly redefined) element $g_1 \in G_\alpha$. We again have two cases: either $g_1 \in G_y$ or $g_1 \in G_\alpha \setminus G_y$. If $g_1 \in G_y$ then $g_1 \in G_{\alpha,y}$ and so $g_1 \in G_\gamma$. In this case we can choose $g_1' := 1$. Otherwise, if $g_1 \in G_\alpha \setminus G_y$, then choose $g_1' := g_1$. In choosing g_1' and h_1' in this way we ensure that

$$\gamma^{g_1 h_1} = \gamma^{g_1' h_1'},$$

so the hypothesis holds when $n = 1$.

Let k be a positive integer, and suppose the hypothesis is true for all integers $n \leq k$. Fix $g_1, \ldots, g_{k+1} \in G_\alpha$ and $h_1, \ldots, h_{k+1} \in G_\beta$, and set

$$\gamma' := \gamma^{g_1 h_1 \ldots g_{k+1} h_{k+1}}.$$

We will use induction to construct elements $g_2', \ldots, g_m' \in G_\alpha \setminus G_y$ and $g_1' \in G_\alpha \setminus G_y \cup \{1\}$ together with $h_1', \ldots, h_{m-1}' \in G_\beta \setminus G_y$ and $h_m' \in G_\beta \setminus G_y \cup \{1\}$ such that

$$\gamma^{g_1' h_1' \ldots g_m' h_m'} = \gamma',$$

where m is some integer less than or equal to $k + 1$.

We begin by considering $h_{k+1} \in G_\beta$. There are two cases: either $h_{k+1} \in G_y$ or $h_{k+1} \in G_\beta \setminus G_y$. If $h_{k+1} \in G_y$ then $h_{k+1} \in G_{\beta,y} = G_{\alpha,y}$, so $g_{k+1} h_{k+1} \in G_\alpha$. In this case, redefine $g_{k+1} := g_{k+1} h_{k+1}$ and set $h' := 1$. If, on the other hand, $h_{k+1} \in G_\beta \setminus G_y$, then set $h' := h_{k+1}$.

If we now consider the (possibly redefined) element $g_{k+1} \in G_\alpha$, there are again two cases: either $g_{k+1} \in G_y$, or $g_{k+1} \in G_\alpha \setminus G_y$. If $g_{k+1} \in G_y$ then $g_{k+1} \in G_{\alpha,y} = G_{\beta,y}$, so $h_k g_{k+1} h' \in G_\beta$. In this case, let $h'' := h_k g_{k+1} h'$; then

$$\gamma' = \gamma^{g_1 h_1 \ldots g_k h''},$$

so we can apply the induction hypothesis to $\gamma^{g_1 h_1 \ldots g_k h''}$ and we are done. If, on the other hand, $g_{k+1} \in G_\alpha \setminus G_y$, then set $g' := g_{k+1}$, and observe

$$\gamma' = \gamma^{g_1 h_1 \ldots g_k h_k g' h'}.$$

By the induction hypothesis, for some $l \leq k$ there exist $g_2', \ldots, g_l' \in G_\alpha \setminus G_y$ and $g_1' \in G_\alpha \setminus G_y \cup \{1\}$ together with $h_1', \ldots, h_{l-1}' \in G_\beta \setminus G_y$ and $h_l' \in G_\beta \setminus G_y \cup \{1\}$ such that

$$\gamma^{g_1 h_1 \ldots g_k h_k} = \gamma^{g_1' h_1' \ldots g_l' h_l'}.$$

At this final stage in the proof, we again face two possibilities: either $h_l' = 1$ or $h_l' \in G_\beta \setminus G_y$. In the first instance define $g'' := g_l' h_l' g'$, so $\gamma' = \gamma^{g_1' h_1' \ldots g_{l-1}' h_{l-1}' g'' h'}$. Since $g'' = g_l' g' \in G_\alpha$ and $h' \in G_\beta$ and $l \leq k$, we may apply the induction hypothesis.

On the other hand, if $h_l' \in G_\beta \setminus G_y$, then set $g_{l+1}' := g' \in G_\alpha \setminus G_y$ and $h_{l+1}' := h' \in G_\beta \setminus G_y \cup \{1\}$, and observe $\gamma' = \gamma^{g_1' h_1' \ldots g_{l+1}' h_{l+1}'}$. Now $l \leq k$, so defining m to be $l+1$ we have $m \leq k+1$. Thus in both cases the hypothesis holds. It is therefore true for $n = k+1$. $\qquad\square$

We are now in a position to present the main result of this section which describes necessary conditions for a vertex-transitive subgroup of the automorphism group of an infinite primitive digraph with connectivity one to be imprimitive.

Theorem 2.5. *Let G be a vertex-transitive group of automorphisms of a connectivity-one digraph Γ whose lobes have at least three vertices, and let T be the block-cut-vertex tree of Γ. If there exist distinct vertices $\alpha, \beta \in V\Gamma$ such that, for some vertex $x \in (\alpha, \beta)_T$,*

$$G_{\alpha,x} = G_{\beta,x},$$

then G does not act primitively on $V\Gamma$.

Proof. Suppose G acts primitively on $V\Gamma$ and there exist distinct vertices $\alpha, \beta \in V\Gamma$ and $x \in (\alpha, \beta)_T$ such that $G_{\alpha,x} = G_{\beta,x}$. We will begin by showing the group $\langle G_\alpha, G_\beta \rangle$ generated by G_α and G_β is not equal to G; then we shall show it is not equal to G_α. Whence, $G_\alpha < \langle G_\alpha, G_\beta \rangle < G$ which, by applying Theorem 1.1, will contradict the assumption that G is primitive.

Without loss of generality, suppose $d_T(x, \alpha) \leq d_T(x, \beta)$. If the orbit $\beta^{\langle G_\alpha, G_\beta \rangle}$ contains α, then there exist elements $g_1, \ldots, g_n \in G_\alpha$ and $h_1, \ldots, h_n \in G_\beta$ such that $\alpha = \beta^{g_1 h_1 \ldots g_n h_n}$. By Lemma 2.4, we can find $m \leq n$ and $g_2', \ldots, g_m' \in G_\alpha \setminus G_x$ and $g_1' \in G_\alpha \setminus G_x \cup \{1\}$ together with $h_1', \ldots, h_{m-1}' \in G_\beta \setminus G_x$ and $h_m' \in G_\beta \setminus G_x \cup \{1\}$ such that

$$\alpha = \beta^{g_1' h_1' \ldots g_m' h_m'}.$$

Suppose these automorphisms are chosen so that m is minimal.

Now either $g_1' \in G_\alpha \setminus G_x$ or $g_1' = 1$. If $g_1' = 1$ then $\beta^{g_1'} = \beta$ and therefore $\beta^{g_1' h_1'} = \beta$. Thus $\beta^{g_2' h_2' \ldots g_m' h_m'} = \alpha$, contradicting the minimality of m. So we must have $g_1' \in G_\alpha \setminus G_x$. Since $\beta \notin C(T \setminus \{x\}, \alpha)$, we may apply Lemma 2.2 and Lemma 2.3 to obtain $d_T(x, \beta^{g_1'}) > d_T(x, \beta)$ and $\beta^{g_1'} \notin C(T \setminus \{x\}, \beta)$.

We now observe $h_1' \neq 1$. Indeed, if $h_1' = 1$ then $m = 1$ and $\alpha = \beta^{g_1'}$; since $g_1' \in G_\alpha$ this is clearly not possible.

Thus, $h_1' \in G_\beta \setminus G_x$ and $\beta^{g_1'} \notin C(T \setminus \{x\}, \beta)$, and we can again deduce from Lemma 2.2 and Lemma 2.3 that $d_T(x, \beta^{g_1' h_1'}) > d_T(x, \beta^{g_1'}) > d_T(x, \beta)$, and $\beta^{g_1' h_1'} \notin C(T \setminus \{x\}, \alpha)$.

We may continue to apply Lemmas 2.2 and 2.3 to obtain $\beta^{g_1' h_1' \cdots g_m'} \notin C(T \setminus \{x\}, \beta)$ and $d_T(x, \beta^{g_1' h_1' \cdots g_m'}) > d_T(x, \beta)$. Now either $h_m' \in G_\beta \setminus G_x$ or $h_m' = 1$. If $h_m' = 1$ then $\alpha = \beta^{g_1' h_1' \cdots g_m'}$, and so $d_T(x, \alpha) = d_T(x, \beta^{g_1' h_1' \cdots g_m'}) > d_T(x, \beta)$. If $h_m' \in G_\beta \setminus G_x$ then, by Lemma 2.3, $d_T(x, \beta^{g_1' h_1' \cdots g_m' h_m'}) > d_T(x, \beta)$; that is, $d_T(x, \alpha) > d_T(x, \beta)$. Thus, in both cases $d_T(x, \alpha) > d_T(x, \beta)$. This contradicts our assumption that $d_T(x, \alpha) \leq d_T(x, \beta)$. Hence $\alpha \notin \beta^{\langle G_\alpha, G_\beta \rangle}$, and so $\langle G_\alpha, G_\beta \rangle$ cannot act transitively on the set $V\Gamma$. This ensures that $\langle G_\alpha, G_\beta \rangle \neq G$.

By Theorem 1.1, we must therefore have $\langle G_\alpha, G_\beta \rangle = G_\alpha$. Thus, the set of vertices $\mathrm{Fix}\,(G_\alpha)$ fixed by G_α contains both α and β. This set is a block of imprimitivity. Thus, every vertex in $V\Gamma$ must be fixed by G_α, and so $G_\alpha = \langle 1 \rangle$. However, G_α is a maximal subgroup of G, so $G_\alpha = \langle 1 \rangle$ implies that G is a finite cyclic group of prime order. This, however, is impossible, as G acts transitively on the infinite set $V\Gamma$.

Hence $\langle G_\alpha, G_\beta \rangle \neq G_\alpha$, and $\langle G_\alpha, G_\beta \rangle \neq G$, which contradicts our assumption that G is primitive. $\square$

Theorem 2.6. *Let G be a vertex-transitive group of automorphisms of a connectivity-one digraph Γ whose lobes have at least three vertices. If G acts primitively on $V\Gamma$ and Λ is some lobe of Γ then $G_{\{\Lambda\}}$ is primitive and not regular on $V\Lambda$.*

Proof. Suppose G acts primitively on $V\Gamma$, and Λ is a lobe of Γ. By Lemma 2.1, $G_{\{\Lambda\}}$ acts primitively on $V\Lambda$. Suppose this action is regular. If T is the block-cut-vertex tree of Γ then there exists a vertex $x \in VT$ corresponding to the lobe Λ. Choose distinct vertices α and β in $V\Lambda$, and observe $G_{\alpha,x} = G_{\alpha,\{\Lambda\}} \leq G_\beta$ and $G_{\beta,x} = G_{\beta,\{\Lambda\}} \leq G_\alpha$; furthermore, $x \in (\alpha, \beta)_T$. This, however, is impossible, as it implies G is imprimitive by Theorem 2.5. $\square$

3. Global structure

In this section we shall employ Theorem 2.6 to give a complete characterisation of the primitive connectivity-one digraphs.

Lemma 3.1. *Suppose Γ is a vertex-transitive digraph with connectivity one, whose lobes are vertex-transitive, have at least three vertices and are pairwise isomorphic.*

If γ is a vertex in some lobe Λ of Γ and $\alpha \in V\Gamma$ is γ or lies in a component of $\Gamma \setminus \{\gamma\}$ distinct from the component containing $V\Lambda \setminus \{\gamma\}$, then the subgroup of $\mathrm{Aut}\,\Lambda$ induced by the action of $(\mathrm{Aut}\,\Gamma)_{\alpha,\{\Lambda\}}$ on Λ

is $(\operatorname{Aut} \Lambda)_\gamma$*, and the group induced by the action of* $(\operatorname{Aut} \Gamma)_{\{\Lambda\}}$ *on* Λ *is* $\operatorname{Aut} \Lambda$*.*

Proof. Let T denote the block-cut-vertex tree of Γ, and let Λ be a lobe of Γ. Choose $\gamma \in V\Lambda$ and let C' be a component of $\Gamma \setminus \{\gamma\}$ distinct from that which contains $V\Lambda \setminus \{\gamma\}$. Let C be the subgraph of Γ induced by $C' \cup \{\gamma\}$, and suppose α is any vertex in C.

We begin by asserting that if Λ_1 and Λ_2 are lobes of Γ, and α_1 and α_2 are vertices in Λ_1 and Λ_2 respectively, then there exists an isomorphism $\rho : \Lambda_1 \to \Lambda_2$ such that $\alpha_1^\rho = \alpha_2$. Indeed, by assumption there exists an isomorphism $\rho' : \Lambda_1 \to \Lambda_2$. Define $\alpha_1' := \alpha_1^{\rho'}$. Since the lobe Λ_2 is vertex-transitive, there exists an automorphism τ of Λ_2 such that $\alpha_1'^\tau = \alpha_2$. Let $\rho := \rho'\tau$. Then $\rho : \Lambda_1 \to \Lambda_2$ is an isomorphism, with $\alpha_1^\rho = \alpha_1^{\rho'\tau} = \alpha_1'^\tau = \alpha_2$.

Let x be the vertex of T that corresponds to Λ. For $k \geq 0$, define Γ_k to be the subgraph of Γ induced by the set $\{\xi \in V\Gamma \mid d_T(x, \xi) \leq 2k+1\}$, and $C_k := C \cap \Gamma_k$. Note that $\Gamma_0 = \Lambda$. We will show any automorphism $\sigma_k : \Gamma_k \to \Gamma_k$ which fixes C_k admits an extension $\sigma_{k+1} : \Gamma_{k+1} \to \Gamma_{k+1}$ which fixes C_{k+1}.

Fix $k \geq 0$ and let $\sigma_k : \Gamma_k \to \Gamma_k$ be an automorphism that fixes C_k pointwise; in particular, $\sigma_0 \in (\operatorname{Aut} \Lambda)_\gamma$. Let $\{\alpha_i\}_{i \in I}$ be the set of vertices in $V\Gamma_k \setminus V\Gamma_{k-1}$ (where $V\Gamma_{-1} := \emptyset$). Each vertex α_i belongs to a unique lobe Λ_i of Γ_k, and, if $k \geq 1$, the lobe Λ_i possesses precisely one vertex in Γ_{k-1}. Since Γ is vertex transitive, any two vertices lie in the same number of lobes of Γ, so let $\{\Lambda_{i,j}\}_{j \in J}$ be the set of lobes of Γ that contain α_i and are distinct from Λ_i. Each lobe $\Lambda_{i,j}$ is wholly contained in Γ_{k+1} and has exactly one vertex in Γ_k, namely α_i. If $i \in I$, set $\alpha_i' := \alpha_i^{\sigma_k}$ and $\Lambda_i' := \Lambda_i^{\sigma_k}$. Then $\Lambda_i' = \Lambda_{i'}$ for some $i' \in I$. For all $j \in J$ there exists an isomorphism $\rho_{i,j} : \Lambda_{i,j} \to \Lambda_{i',j}$ such that $\alpha_i^{\rho_{i,j}} = \alpha_i'$. Thus, we may define a mapping $\sigma_{k+1} : \Gamma_{k+1} \to \Gamma_{k+1}$ with

$$\beta^{\sigma_{k+1}} := \begin{cases} \beta^{\sigma_k} & \text{if } \beta \in V\Gamma_k; \\ \beta & \text{if } \beta \in C; \\ \beta^{\rho_{i,j}} & \text{if } \beta \in V\Lambda_{i,j} \setminus C. \end{cases}$$

This is clearly a well-defined automorphism of Γ_{k+1}.

Hence if $\sigma_0 \in (\operatorname{Aut} \Lambda)_\gamma$, then we may extend it to an automorphism σ of Γ that fixes C pointwise, and therefore fixes α. Since each automorphism in $(\operatorname{Aut} \Lambda)_\gamma$ may be extended in this way, the subgroup of $\operatorname{Aut} \Lambda$ induced by $(\operatorname{Aut} \Gamma)_{\alpha, \{\Lambda\}}$ must contain $(\operatorname{Aut} \Lambda)_\gamma$. Clearly no automorphism of Γ may fix α and Λ setwise whilst not also fixing γ, so these two groups must in fact be equal.

We now adjust the above argument to show that the group induced by the action of $(\operatorname{Aut} \Gamma)_{\{\Lambda\}}$ on Λ is $\operatorname{Aut} \Lambda$. Fix $k \geq 0$ and consider an automorphism $\sigma_k : \Gamma_k \to \Gamma_k$ that fixes $V\Lambda$ setwise; in particular, $\sigma_0 \in (\operatorname{Aut} \Lambda)$.

Using the above notation, we may define a map $\sigma_{k+1} : \Gamma_{k+1} \to \Gamma_{k+1}$ with

$$\beta^{\sigma_{k+1}} := \begin{cases} \beta^{\sigma_k} & \text{if } \beta \in V\Gamma_k; \\ \beta^{\rho_{i,j}} & \text{if } \beta \in V\Lambda_{i,j}. \end{cases}$$

This is a well-defined automorphism of Γ_{k+1}. Thus we may extend any automorphism $\sigma_0 \in (\text{Aut } \Lambda)$ to an automorphism σ of Γ that fixes Λ setwise. Whence the group induced by the action of $(\text{Aut } \Gamma)_{\{\Lambda\}}$ on Λ is $\text{Aut } \Lambda$. $\square$

The primitive graphs with connectivity one have the following complete characterisation.

Theorem 3.2. ([3, Theorem 4.2]) *If Γ is a vertex-transitive graph with connectivity one, then it is primitive if and only if the lobes of Γ are primitive, pairwise isomorphic and each has at least three vertices.* $\square$

Jung and Watkins' result, while impressive, cannot be applied to primitive digraphs without some modification. Indeed, consider the following counterexample. Let Γ be the connectivity-one primitive graph whose lobes are undirected 3-cycles, in with each vertex lies in precisely two lobes. It is of course possible to verify this graph is primitive using Theorem 3.2.

We assign to each vertex in Γ the label 1, 2 or 3 in such a way that no two vertices in a common lobe of Γ share the same label. Whence, each lobe of Γ has a vertex labelled 1, a vertex labelled 2 and a vertex labelled 3.

For each lobe in Γ we replace its edge set with a set of three arcs, from the vertex labelled i to the vertex labelled $(i+1) \mod 3$, for $i = 1, 2, 3$. In this way we obtain a vertex-transitive connected digraph Γ' whose vertex set is $V\Gamma$. Furthermore, the lobes of this digraph are primitive, pairwise isomorphic, and have at least three vertices. However, the set of vertices labelled 1 is a non-trivial proper block of the group $\text{Aut } \Gamma'$, so this group cannot act primitively on the digraph Γ'.

The approach taken by Jung and Watkins in their proof of Theorem 3.2 is broadly similar to the argument presented thus far. They first prove that any automorphism of a lobe Λ of a vertex-transitive graph Γ with connectivity one may be extended to an automorphism of Γ. It is then shown that if the lobes of Γ are vertex primitive, have at least three vertices, and are pairwise isomorphic, then Γ is primitive. It is here that their proof fails to apply to digraphs; by citing Theorem 1 of [2] they claim the lobes of Γ cannot be automorphism-regular. While this is indeed true of graphs, it is not true of digraphs, as our previous example illustrates.

Imrich's result states that the automorphism group of a graph with more than two vertices cannot be regular and primitive. It relies on two results, Lemmas 2 and 3 of [2]. Lemma 2 is the well-known result that a

regular primitive group of permutations must be cyclic; Lemma 3 states that any transitive abelian automorphism group of a non-trivial graph is the direct product of two cyclic groups of order 2. Any primitive and regular automorphism group of a graph must therefore equal this direct product; Imrich shows that no such graph exists, and correctly deduces that automorphism-regular primitive graphs are not possible. It is in the proof of the latter lemma that Imrich's result ceases to be applicable to digraphs: his argument requires the existence of a specific graph automorphism ψ. On inspection it transpires that ψ is not a digraph automorphism, since it reverses the direction of edges. Thus Theorem 1 of [2] is not applicable to digraphs, which in turn causes Jung and Watkins' result to fail.

Although their result does not extend immediately to digraphs, a complete characterisation is still possible.

Theorem 3.3. *If Γ is a vertex-transitive digraph with connectivity one, then it is primitive if and only if the lobes of Γ are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices.*

Proof. Let Γ be a vertex-transitive digraph with connectivity one. Suppose the lobes of Γ are primitive but not automorphism-regular, pairwise isomorphic and each has at least three vertices. Let $\approx$ be an Aut Γ-congruence on $V\Gamma$ such that there exist distinct vertices $\alpha, \beta \in V\Gamma$ with $\alpha \approx \beta$. We will show this relation must be universal, and thus that Γ is a primitive digraph.

Let T be the block-cut-vertex tree of Γ, let $\gamma \in V\Gamma$ be the vertex in the geodesic $[\alpha, \beta]_T$ such that $d_T(\beta, \gamma) = 2$, and let Λ be the lobe of Γ containing β and γ.

By Lemma 3.1 the group $(\text{Aut } \Gamma)_{\{\Lambda\}}$ acts primitively and not regularly on the lobe Λ. Thus there exits an automorphism $h \in (\text{Aut } \Gamma)_{\gamma, \{\Lambda\}}$ which does not fix β. By restricting the action of h to the vertices of Λ, we see that there must be an element in Aut Λ_γ which does not fix β. By Lemma 3.1, the subgroup of Aut Λ induced by $(\text{Aut } \Gamma)_{\alpha, \{\Lambda\}}$ is equal to $(\text{Aut } \Lambda)_\gamma$, so there must therefore exist an element $g \in (\text{Aut } \Gamma)_{\alpha, \gamma, \{\Lambda\}}$ that does not fix β.

Thus, β and β^g are distinct vertices in Λ. Now $\alpha \approx \beta$, so $\alpha \approx \beta^g$, and therefore $\beta \approx \beta^g$. Since $(\text{Aut } \Gamma)_{\{\Lambda\}}$ is primitive on $V\Lambda$ and $\approx$ induces a non-trivial $(\text{Aut } \Gamma)_{\{\Lambda\}}$-congruence on $V\Lambda$, this relation must be universal in Λ. By assumption, Aut Γ acts transitively on the lobes of Γ, so if two vertices lie in the same lobe then they must lie in the same congruence class. Thus, if γ is any vertex of Γ, and $\alpha x_1 \alpha_1 x_2 \ldots x_n \gamma$ is the geodesic in T between α and γ, then α and α_1 lie in a common lobe, so $\alpha \approx \alpha_1$. Similarly, $\alpha_1 \approx \alpha_2$ and $\alpha_2 \approx \alpha_3$, so $\alpha \approx \alpha_2$ and $\alpha \approx \alpha_3$. Continuing in this way we eventually obtain $\alpha \approx \gamma$. Hence, this congruence relation is universal on $V\Gamma$.

Conversely, suppose the group Aut Γ acts primitively on $V\Gamma$. Since Γ is a primitive digraph with connectivity one, we can obtain an graph Γ' with vertex set $V\Gamma$ and edge set $\{\{\alpha, \beta\} \mid (\alpha, \beta) \in A\Gamma\}$. Two vertices are adjacent in Γ if and only if they are adjacent in Γ'. As Aut Γ is primitive on $V\Gamma$ and Aut $\Gamma \leq$ Aut Γ', it follows that Aut Γ' must be primitive on $V\Gamma$, and hence Γ' is a primitive graph. Since Γ has connectivity one, the same is true of Γ', so we may apply Theorem 3.2 to deduce the lobes of Γ' are primitive, pairwise isomorphic and each has at least three vertices. Now, given a lobe Λ of Γ, there is a lobe Λ' of Γ' such that $V\Lambda = V\Lambda'$. Therefore, the lobes of Γ have at least three vertices, and are primitive but not automorphism-regular by Theorem 2.6.

It remains to show they are pairwise isomorphic. Fix some lobe Λ of Γ and an arc $(\alpha, \beta) \in A\Lambda$. Let Γ_1 be the digraph $(V\Gamma, (\alpha, \beta)^{\text{Aut } \Gamma})$. As Aut Γ is primitive, this digraph is a connected subgraph of Γ. Thus, every lobe of Γ must contain an arc in $A\Gamma_1$. Furthermore, if Δ is a lobe of Γ, then any automorphism of Γ mapping the arc (α, β) to an arc in Δ must map Λ to Δ. Since Γ_1 is arc-transitive, the lobes of Γ must be pairwise isomorphic. $\qquad\square$

It is now a simple exercise to classify those vertex-transitive digraphs with connectivity one which are counterexamples to the application of Jung and Watkins' result to digraphs without modification. Any such counterexample must be one of two types: digraphs that satisfy the conditions of Jung and Watkins' theorem, but are nevertheless imprimitive; and digraphs that are primitive, but fail to satisfy Jung and Watkins' characterisation.

Since the conditions given in Theorem 3.3 under which one may be certain a vertex-transitive digraph with connectivity one is primitive include the corresponding conditions given in Jung and Watkins' theorem, no primitive digraph with connectivity one is of the latter type. Thus, any counterexample must be imprimitive, yet still satisfy the conditions of Jung and Watkins result. We begin with a lemma.

Lemma 3.4. *For each prime p, a digraph Λ on p vertices is a directed cycle if and only if its automorphism group is the cyclic group C_p of order p.*

Proof. Suppose p is prime and Λ is a digraph on p vertices. If Λ is a directed cycle then clearly its automorphism group is C_p.

Conversely, suppose Λ has automorphism group C_p. Without loss of generality, we may label the vertices of Λ as integers $0, 1, \ldots, p-1$ such that $(0, 1)$ is an arc in Λ and C_p is generated by the permutation ρ with

$$i^\rho = i + 1 \mod p.$$

Since C_p is transitive on $V\Lambda$, for some set $J \subseteq \{1, \ldots, p-1\}$ we may write

$$A\Lambda = \bigcup_{j \in J} (0, j)^{C_p}.$$

Note that $1 \in J$. Let σ be a permutation of $V\Lambda$ fixing the vertex 0 and fixing J setwise. Choose $(a, b) \in A\Lambda$ and observe $(a, b) = (0, j)^{\rho^k}$ for some $j \in J$ and some integer k. Now $(a, b)^{\rho^{-k}\sigma} = (0, j)^{\sigma} = (0, j')$ for some $j' \in J$, and $(0, j') \in A\Lambda$. Thus σ lies in $\text{Aut } \Lambda$ and, if J contains at least two elements, is not in C_p. Whence, $J = \{1\}$ and Λ is a directed cycle. $\square$

Recall that any counterexample to the unmodified extension of Jung and Watkins' result to digraphs must be imprimitive, yet still satisfy the conditions of their theorem. By Theorem 3.3, all such digraphs must have automorphism-regular primitive lobes which are pairwise isomorphic, with each possessing at least three vertices. Let Γ be such a digraph.

If Λ is a lobe of Γ, then Theorem 1.1 tells us that $\text{Aut } \Lambda$ is cyclic of prime order p. Since this group is transitive, it implies that Λ must have precisely p vertices. Thus Λ is a p-vertex digraph whose automorphism group is the cyclic group C_p of order p, and is necessarily a directed cycle by Lemma 3.4.

Conversely, we note that for any odd prime p, a vertex-transitive digraph with connectivity one whose lobes have p vertices and automorphism group C_p, satisfies the primitivity conditions given by Jung and Watkins for graphs.

Thus the counterexamples to the unmodified extension of Jung and Watkins' result are precisely those digraphs whose lobes have an odd prime p number of vertices, and are directed cycles. This is summarised in our concluding theorem. Here the undirected graph *associated* with the digraph Γ is the graph with vertex set $V\Gamma$ and edge set $\{\{\alpha, \beta\} \mid (\alpha, \beta) \in A\Gamma \text{ or } (\beta, \alpha) \in A\Gamma\}$.

Theorem 3.5. *If Γ is a vertex-transitive imprimitive digraph with connectivity one, then its associated (undirected) graph is primitive if and only if the lobes of Γ are pairwise isomorphic directed p-cycles, for some odd prime p.* $\square$

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