Discrete-Time Quadratic Hedging of Barrier Options in Exponential Lévy Model

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Abstract We examine optimal quadratic hedging of barrier options in a discretely sampled exponential Lévy model that has been realistically calibrated to reflect the leptokurtic nature of equity returns. Our main finding is that the impact of hedging errors on prices is several times higher than the impact of other pricing biases studied in the literature.

1 Introduction

We study quadratic hedging and pricing of European barrier options with a particular focus on the magnitude of risk of optimal hedging strategies. In a discretely sampled exponential Lévy model, calibrated to reflect the leptokurtic nature of equity returns, we compute the hedging error of the optimal strategy and evaluate prices that yield reasonable risk-adjusted performance for the hedger. We also confirm what traders already know empirically, namely that the hedging risk of barrier options substantially outstrips that of plain vanilla options.

European barrier options are derivative contracts based on standard European calls or puts with the exception that the option becomes active (or inactive) when the stock price hits a prespecified barrier before the maturity of the option. Options activated in this way are called knock-ins; those deactivated are called knock-outs.

Under the assumptions of the Black–Scholes model barrier options have been valued first by [32] and in more detail by [33]. Early literature on numerical evaluation of barrier option prices concentrates on slow convergence of binomial method, which is due to the difference between the *nominal barrier* specified in the option contract and the *effective barrier* implied by the position of nodes in the stock price lattice. This discrepancy, if not properly controlled, may lead to sizeable mispricing,

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especially for options whose barrier is close to the initial stock price. [4] and [34] suggest better positioning of nodes in binomial and trinomial lattices to minimize the discrepancy between nominal and effective barrier, whereas [14] propose interpolation between two adjacent values of the effective barrier. [21] devise an adaptive mesh allowing for more nodes (and shorter time steps) around the barrier.

The papers above are concerned with continuously-monitored barriers in the Black-Scholes model. Discrete monitoring, too, can have significant impact on option valuation, and, unlike the continuous monitoring case, does not allow for a simple closed form pricing formula, cf. [22]. A simple asymptotic correction, which works well for barriers not too close to initial stock price, was developed by [5]. For barriers in close proximity of the stock price the Markov chain representation of stock prices developed by [15] is more appropriate. Other papers dealing with discrete monitoring in the log-normal framework include [23], [25], [28], [29] and [38]. [1] describe a systematic way of handling discretization errors by means of quadrature. There is also an extensive literature on barrier option pricing by Monte Carlo simulation which we will not touch upon in this paper.

The models discussed above are complete in the sense that one can devise a self-financing trading strategy that perfectly replicates the barrier option. In practice, however, one encounters considerable difficulties in maintaining a delta-neutral position when close to the barrier. This has motivated study of static replication of barrier options with plain vanilla options. [8] use the reflection principle known from barrier option pricing combined with so-called put–call symmetry to write a down-and-out call as a sum of a long call and a short put. Their methodology is to some extent model-free but it only works if the market is complete and if the aforementioned symmetry holds, requiring that risk-free rate be equal to the dividend rate in addition to a certain symmetry of local volatilities. [6] analyze static super- and sub-replication. The latter results are completely model-free at the cost of generating price bounds that are potentially very wide. Other papers on static replication include [2], [7], [12] and [13].

Several studies allow for parametric departures from the Black–Scholes model. [16] and [26] use Bates' stochastic volatility jump-diffusion model while [30] allows for IID jumps. Several numerical approaches now exist for dealing with a wide class of (possibly infinite variation) Lévy models, see [20], [19].

The paper is organized as follows. In Section 2 we specify the theoretical model, describe its calibration and computation of optimal strategies. Section 3 provides economic analysis of the numerical results and Section 4 explains the relationship between barrier option prices and hedger's risk-adjusted performance.

2 The model

We have at our disposal nominal log returns on FT100 equity index in the period January 1st, 1993 to December 31st, 2002, sampled at a 1 minute interval. Eventually we wish to say something about optimal hedging of barrier options in a model with rebalancing frequency $\Delta \in [5 \text{ minutes}, 1 \text{ day}]$ and a daily monitoring of the barrier. We will assume independence and time homogeneity of underlying asset returns at any given rebalancing frequency. This is not to say that stochastic volatility is unimportant in practice, instead we may think of the IID assumption as a useful limiting case when the (unobserved) volatility state changes either very slowly or very quickly. In this view of the world the leptokurtic nature of returns is a source of risk that does not vanish even after stochastic volatility has been factored in appropriately.

The analysis is performed under two self-imposed constraints. The first is to use the available data in a non-parametric way and the second is to perform all numerical analysis in a multinomial lattice.

In these circumstances there are essentially two strategies for calibrating the stock price process. One option is to simply take the data series sampled at time interval Δ, generate a discretized distribution of returns and construct a multinomial lattice using this distribution. An alternative is to consider an underlying continuous-time model from which the daily or hourly returns are extracted. [17] argue that equity return data display sufficient amount of time consistency for such an approach to make sense. The underlying model is then necessarily a geometric Lévy model, cf. Lemma 4.1 in [9]. Such approach also offers an alternative avenue to obtaining asymptotics as Δ tends to zero by studying quadratic hedging for barrier options directly in the underlying Lévy model – a task which at present is still outstanding and well beyond the scope of this paper.

2.1 Calibration

We take the original log return data sampled at $\Delta_0 = 1$ minute intervals and construct an equidistantly spaced sequence $m_0 < m_1 < \ldots < m_{N+1}$ with spacing δ, such that m_N is the highest and m_1 the lowest log return in the sample. We set $N = 1000$. We then identify the frequency of log returns in each interval of length δ centred on m_j, $j = 0, \ldots, N+1$ and store this information in the vector $\{f_j\}_{j=0}^{N+1}$. We construct an empirical Lévy measure F_{raw} as an absolutely continuous measure with respect to the Lebesgue measure on $\mathbb{R}$

$$F_{\text{raw}}(dx) = \frac{\hat{f}(x)}{\Delta_0} dx,$$

where $\hat{f} = f$ at the points m_j, $\hat{f} = 0$ outside (m_0, m_{N+1}) and elsewhere $\hat{f}$ is obtained as a linear interpolation of f. This construction is motivated by an asymptotic result that links transition probability measure of a Lévy process to its Lévy measure over short time horizons, see [35], Corollary 8.9.

In the next step we normalize the empirical characteristic function of log returns to achieve a pre-specified annualized mean μ and volatility σ. Since the raw empirical Lévy process is square-integrable and therefore a special semimartingale we will use the (otherwise forbidden) truncation function $h(x)=x$. We will construct the log return process by setting

$$\begin{aligned} \ln S &= \ln S_0 + \mu t + \frac{\sigma x}{\sigma_{\text{raw}}} * (J^{L_{\text{raw}}} - \nu^{L_{\text{raw}}}), \\ \sigma^2_{\text{raw}} &= \int_{\mathbb{R}} x^2 F_{\text{raw}}(dx), \end{aligned}$$

where $J^{L_{\text{raw}}}$ is the jump measure of a Lévy process with Lévy measure F_{raw}, $\nu^{L_{\text{raw}}}$ is its predictable compensator and $*$ denotes a certain stochastic integral as defined in [27]. II.1.27. This yields

$$\kappa(u) := \mu u + \int_{\mathbb{R}} (\mathrm{e}^{ux} - 1 - ux) F(\mathrm{d}x), \tag{1}$$

$$F(G) := \int_{\mathbb{R}} 1_G\left(\frac{\sigma x}{\sigma_{\text{raw}}}\right) F_{\text{raw}}(dx). \tag{2}$$

We fix the annualized volatility of log returns at $\sigma = 0.2$, but to check the robustness of our results we allow the mean log return to take 2 different values $\mu \in \{-0.1, 0.1\}$, the first representing a bear market and the second representing a bull market.

Instead of the non-parametric calibration procedure above one could instead estimate a model from a convenient parametric family, such as the generalized hyperbolic family, as outlined in [18]. The parametric route offers in some special cases an explicit expression for the log return density at all time horizons which avoids the need for numerical inversion of the characteristic function employed below.

2.2 Multinomial lattice

If Z denotes the log return on time horizon Δ its characteristic function is of the form

$$\mathrm{E}[\exp(\mathrm{i}\nu Z)] = \mathrm{e}^{\kappa(\mathrm{i}\nu)\Delta},$$

where the cumulant generating function κ is given by equations (1) and (2). Provided that Z has no atom at z the cumulative distribution is given by the inverse Fourier formula, see [35], 2.5xi,

$$P(Z \le z) = \mathscr{H}(c) - \frac{1}{2\pi} \lim_{l\to\infty} \int_{-l}^{l} \frac{\mathrm{e}^{\kappa(\mathrm{i}\lambda - c)\Delta - z(\mathrm{i}\lambda - c)}}{\mathrm{i}\lambda - c} d\lambda,$$

where c is an arbitrarily chosen real number[1] and $\mathscr{H}$ is a step function,

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[1] In numerical calculations with a fixed value of z we choose c so as to minimize the value of the integrand at $\lambda = 0$, see [36], equation (3).

$$\mathscr{H}(x) = \begin{cases} 0 & \text{for } x > 0, \\ \frac{1}{2} & \text{for } x = 0, \\ 1 & \text{for } x < 0, \end{cases} .$$

We now define a discretized distribution of log returns to populate our lattice. The discretized random variable $\hat{Z}$ will take values

$$\hat{z}_j = j\eta \text{ with } j \in [-n_{\text{down}}, n_{\text{up}}] \cap \mathbb{Z},$$

where $n_{\text{down}}, n_{\text{up}}$ are the smallest numbers in $\mathbb{N}$ such that $P(Z \leq -n_{\text{down}}\eta) \leq \alpha$ and $F_Z(Z \leq n_{\text{up}}\eta) \geq 1 - \alpha$, respectively. We use the values $\eta = 0.0005$ and $\alpha = 10^{-5}$. For comparison, the corresponding value of η in [15] with 1001 price nodes is 0.0089. Table 5 shows the number of standard deviations.

The transition probabilities corresponding to different values of log return are defined by

$$\begin{aligned} \hat{p}_j &:= P(Z \leq (j+1/2)\,\eta) - P(Z \leq (j-1/2)\,\eta) \text{ for } j \in \left(-n_{\text{down}}, n_{\text{up}}\right) \cap \mathbb{Z}, \\ \hat{p}_j &:= P(Z \leq (j+1/2)\,\eta) \text{ for } j = -n_{\text{down}}, \\ \hat{p}_j &:= 1 - P(Z \leq (j-1/2)\,\eta) \text{ for } j = n_{\text{up}}. \end{aligned}$$

To limit the effect of the discretization errors arising from an arbitrary position of the barrier we limit computations to barrier levels that satisfy $\ln B - \ln S \in (\mathbb{Z} + 1/2)\eta$ and use interpolation otherwise.

2.3 Optimal hedging

In the multinomial lattice constructed above we compute the optimal hedging strategy and the minimal hedging error according to the following theorem.

Theorem 1. *Suppose that there is an $\mathscr{F}_n$-measurable contingent claim H such that $\mathrm{E}[H^2] < \infty$. In the absence of transaction costs the dynamically optimal hedging strategy φ solving*

$$\inf_{\vartheta} \mathrm{E}\left[\left(G_n^{x,\vartheta} - H\right)^2\right],$$

subject to ϑ_i being $\mathscr{F}_i$-measurable with G being the value of a self-financing portfolio,

$$\begin{aligned} G_i^{x,\vartheta} &= RG_{i-1}^{x,\vartheta} + \vartheta_{i-1}(S_i - RS_{i-1}), \\ G_0^{x,\vartheta} &= x, \end{aligned}$$

is given by

$$
\begin{aligned}
\varphi_i &= \xi_i + aR\frac{V_i - G_i^{x,\varphi}}{S_i},\\
V_i &:= \mathrm{E}_i[(1-aX_{i+1})V_{i+1}]/(bR), \qquad (3)\\
V_n &:= H \qquad (4)\\
\xi_i &:= \mathrm{Cov}_i\left(V_{i+1},S_{i+1}\right)/\mathrm{Var}_i\left(S_{i+1}\right)\\
&= \mathrm{E}_i\left[\left(V_{i+1}-RV_i\right)X_{i+1}\right]/\left(S_i\mathrm{E}_i\left[X_{i+1}^2\right]\right),\\
X_i &:= \exp(Z_i) - R,\\
a &:= \mathrm{E}_i\left[X_{i+1}\right]/\mathrm{E}_i\left[X_{i+1}^2\right],\\
b &:= 1-\left(\mathrm{E}_i\left[X_{i+1}\right]\right)^2/\mathrm{E}_i\left[X_{i+1}^2\right].
\end{aligned}
$$

The hedging performance of the dynamically optimal strategy φ *and of the locally optimal strategy* ξ *is given by*

$$
\begin{aligned}
\mathrm{E}\left[\left(G_n^{x,\varphi}-H\right)^2\right] &= \left(R^2b\right)^n\left(x-V_0\right)^2+\varepsilon_0^2(\varphi),\\
\mathrm{E}\left[\left(G_n^{x,\xi}-H\right)^2\right] &= \left(R^2\right)^{n-j}\left(x-V_0\right)^2+\varepsilon_0^2(\xi),\\
\varepsilon_0^2(\varphi) &= \sum_{j=0}^{n-1}\left(R^2b\right)^{n-j-1}\mathrm{E}\left[\psi_j\right], \qquad (5)\\
\varepsilon_0^2(\xi) &= \sum_{j=0}^{n-1}R^{2(n-j-1)}\mathrm{E}\left[\psi_j\right],\\
\psi_j &:= \mathrm{E}_j\left[\left(RV_j+\xi_jS_jX_{j+1}-V_{j+1}\right)^2\right]\\
&= \mathrm{Var}_j\left(V_{j+1}^2\right)-\frac{\left(\mathrm{Cov}_j\left(S_{j+1},V_{j+1}\right)\right)^2}{\mathrm{Var}_j\left(S_{j+1}\right)}.
\end{aligned}
$$

Proof. See [9], Theorem 3.3. □

3 Numerical results

We first fix the rebalancing period to $\Delta = 1$ day and examine the behaviour of hedging errors across maturities, strikes and barrier levels.We then analyze the asymptotics of the hedging error as the rebalancing interval Δ approaches 0, keeping the monitoring frequency of the barrier constant. We do so initially for a range of strikes and barrier levels with rebalancing interval $\Delta = 1$ hour and then with fixed strike and barrier level we examine asymptotics going down to $\Delta = 5$ minutes.

3.1 Effect of barrier position, maturity and drift

We consider an up-and-out European call and two maturity dates: 1 and 6 months. During a detailed preliminary analysis we have found that changes in risk-free rate have a very small impact upon hedging errors and therefore we fix the risk-free rate in all computations to $r = 0$. Volatility is normalized to $\sigma = 0.2$ as explained in Section 2.1, while the drift takes two values $\mu \in \{-0.1, 0.1\}$. The time units reflect trading time; specifically we assume there are 8 hours in a day and 250 days in a year. To be able to compare the size of hedging error across maturities we measure the position of the barrier and of the striking price relative to the initial stock price in terms of their Black–Scholes delta.

For each set of parameters we report five quantities: i) the Black–Scholes price of a continuously monitored option C, ii) the Black–Scholes price of a daily monitored option[2] $\hat{V}$, iii) the standard deviation of the hedging error in a discretely rebalanced Black-Scholes model $\hat{\varepsilon}_0$ obtained from (5) using multinomial approximation of Black-Scholes normal transition probabilities[3] with daily monitoring and daily or hourly rebalancing; iv) the mean value process V obtained from (3) and (4) using multinomial approximation of Lévy transition probabilities; and v) the standard deviation of the unconditional expected squared hedging error ε_0 obtained from (5) using multinomial Lévy transition probabilities. The barrier of an up-and-out call has to be above the stock price for the option to be still alive, we therefore parametrize the delta of the barrier by values starting at[4] 10^{-100} and going up to 0.49. The deltas of the striking price range between 0.01 and 0.99. Numerical results for different values of Δ, T and μ are shown in Tables 1-4.

We commence with the base case parameters $\Delta = 1$ day, $\mu = 0.1$ in Tables 1 and 2. The mean value process V coincides to a large extent with the Black-Scholes value of a discretely monitored option. This is a striking result, since the model in which V is computed is substantially incomplete, whereas the reasoning behind $\hat{V}$ relies on continuous rebalancing and perfect replication. For $T = 1$ month (Table 1) the difference between V and $\hat{V}$ is always less than 6.4 cents in absolute value, and in relative terms it is less than 3.6% across all strikes and barrier levels.

The difference between V and $\hat{V}$ tends to diminish with increasing maturity. For $T = 6$ months (Table 2) the difference between V and $\hat{V}$ is less than 6.1 cents in absolute value, and less than 2.7% in relative terms. The signs of $V - \hat{V}$ follow a pattern across strikes and barrier levels whereby the difference tends to be negative for very high barrier levels in combination with high strike prices, and to be positive elsewhere.

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[2] Computation of the discretely monitored option price in Black–Scholes model follows the methodology of [15]. Effectively, the calculation is the same as for V in the empirical model, but the multinomial transition probabilities approximate the Black-Scholes risk-neutral distribution $N\left(\left(r-\sigma^2/2\right)\Delta, \sigma^2\Delta\right)$.

[3] Objective probability distribution of log returns in the Black-Scholes model is $N\left(\left(\mu-\sigma^2/2\right)\Delta, \sigma^2\Delta\right)$.

[4] The barrier with delta of 10^{-100} is so high that the corresponding results are, for all intents and purposes, indistinguishable from a plain vanilla option.

strike		**barrier (delta/level)**					
		1E-100	0.01	0.10	0.30	0.45	0.49
		343.8	114.6	107.9	103.3	100.9	100.3
delta	level						
		0.019					
		0.019					
0.01	114.6	0.104					
		0.020					
		0.122					
		0.268	0.151				
		0.268	0.172				
0.1	107.9	0.286	0.417				
		0.267	**0.170**				
		0.326	**0.443**				
		1.071	0.874	0.182			
		1.071	0.916	0.255			
0.3	103.3	0.408	0.734	0.521			
		1.066	**0.910**	**0.257**			
		0.469	**0.809**	**0.545**			
		1.900	1.663	0.608	0.023		
		1.900	1.716	0.752	0.052		
0.45	100.9	0.430	0.889	0.839	0.212		
		1.894	**1.709**	**0.759**	**0.053**		
		0.491	**0.960**	**0.912**	**0.223**		
		2.162	1.915	0.767	0.044	0.000	
		2.162	1.971	0.930	0.089	0.001	
0.49	100.3	0.427	0.913	0.931	0.290	0.020	
		2.156	**1.964**	**0.938**	**0.092**	**0.001**	
		0.491	**1.007**	**0.991**	**0.300**	**0.020**	
		4.563	4.247	2.447	0.515	0.054	0.013
		4.563	4.321	2.750	0.754	0.141	0.071
0.75	96.3	0.348	1.135	1.471	0.836	0.357	0.252
		4.560	**4.317**	**2.770**	**0.773**	**0.146**	**0.073**
		0.397	**1.222**	**1.610**	**0.915**	**0.379**	**0.265**
		12.488	12.020	8.764	3.512	0.808	0.262
		12.488	12.134	9.385	4.427	1.598	1.037
0.99	87.5	0.065	1.607	2.671	2.138	1.318	1.092
		12.489	**12.134**	**9.429**	**4.491**	**1.632**	**1.050**
		0.077	**1.761**	**2.874**	**2.316**	**1.452**	**1.193**

Table 1 Mean value and hedging error for a daily monitored up-and-out call option. $T = 1$ month, $\Delta = 1$ day, $\mu = 0.1, r = 0$. For each strike and barrier level we report 5 values: i) Black–Scholes value of continuously monitored option, ii) mean value for normally distributed log returns and discretely (daily) monitored option, iii) hedging error corresponding to ii); iv) the mean value process V_0 for the empirical distribution of log returns(discrete monitoring); v) standard deviation of the unconditional hedging error corresponding to iv). Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

		barrier (delta/level)					
		1E-100	0.01	0.10	0.30	0.45	0.49
		2071.0	140.5	121.2	108.8	102.8	101.4
strike delta	strike level						
		0.046					
		0.046					
0.01	140.5	0.152					
		0.046					
		0.176					
		0.635	0.364				
		0.635	0.387				
0.1	121.2	0.329	0.674				
		0.631	**0.386**				
		0.381	**0.740**				
		2.514	2.073	0.447			
		2.514	2.118	0.522			
0.3	108.8	0.437	1.238	0.711			
		2.506	**2.115**	**0.526**			
		0.506	**1.364**	**0.778**			
		4.434	3.910	1.475	0.059		
		4.435	3.966	1.622	0.087		
0.45	102.8	0.425	1.366	1.214	0.293		
		4.426	**3.963**	**1.632**	**0.088**		
		0.493	**1.505**	**1.335**	**0.316**		
		5.038	4.493	1.854	0.114	0.000	
		5.039	4.552	2.020	0.157	0.001	
0.49	101.4	0.428	1.526	1.226	0.350	0.029	
		5.030	**4.549**	**2.032**	**0.160**	**0.001**	
		0.496	**1.679**	**1.351**	**0.380**	**0.030**	
		10.490	9.812	5.793	1.296	0.163	0.055
		10.491	9.888	6.094	1.525	0.240	0.102
0.75	91.8	0.314	1.746	2.040	1.013	0.361	0.234
		10.485	**9.890**	**6.120**	**1.543**	**0.246**	**0.105**
		0.364	**1.921**	**2.251**	**1.118**	**0.398**	**0.258**
		27.452	26.507	19.689	8.316	2.280	1.036
		27.452	26.618	20.272	9.155	2.972	1.659
0.99	72.6	0.048	2.605	3.244	2.244	1.307	0.989
		27.452	**26.628**	**20.326**	**9.217**	**3.012**	**1.689**
		0.056	**2.858**	**3.589**	**2.494**	**1.455**	**1.104**

Table 2 Mean value and hedging error for a daily monitored up-and-out call option. $T = 6$ month, $\Delta = 1$ day, $\mu = 0.1, r = 0$. For each strike and barrier level we report 5 values: i) Black–Scholes value of continuously monitored option, ii) mean value for normally distributed log returns and discretely (daily) monitored option, iii) hedging error corresponding to ii); iv) the mean value process V_0 for the empirical distribution of log returns(discrete monitoring); v) standard deviation of the unconditional hedging error corresponding to iv). Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

		barrier (delta/level)					
		1E-100	0.01	0.10	0.30	0.45	0.49
		2070.99	140.55	121.17	108.82	102.83	101.37
strike							
delta	level						
		0.046					
		0.046					
0.01	140.55	0.071					
		0.047					
		0.084					
		0.635	0.364				
		0.635	0.387				
0.1	121.17	0.236	0.351				
		0.637	**0.387**				
		0.270	**0.377**				
		2.514	2.073	0.447			
		2.514	2.118	0.522			
0.3	108.82	0.374	0.612	0.515			
		2.514	**2.116**	**0.524**			
		0.438	**0.705**	**0.546**			
		4.434	3.910	1.475	0.059		
		4.435	3.966	1.622	0.087		
0.45	102.83	0.419	0.755	0.798	0.239		
		4.434	**3.962**	**1.626**	**0.088**		
		0.481	**0.822**	**0.913**	**0.268**		
		5.038	4.493	1.854	0.114	0.000	
		5.039	4.552	2.020	0.157	0.001	
0.49	101.37	0.419	0.761	0.901	0.326	0.026	
		5.037	**4.548**	**2.024**	**0.159**	**0.001**	
		0.490	**0.876**	**0.968**	**0.342**	**0.028**	
		10.490	9.812	5.793	1.296	0.163	0.055
		10.491	9.888	6.094	1.525	0.240	0.102
0.75	91.79	0.380	0.939	1.367	0.866	0.391	0.262
		10.489	**9.883**	**6.104**	**1.538**	**0.245**	**0.105**
		0.435	**1.019**	**1.564**	**0.992**	**0.419**	**0.281**
		27.452	26.507	19.689	8.316	2.280	1.036
		27.452	26.618	20.272	9.155	2.972	1.659
0.99	72.60	0.094	1.199	2.382	2.109	1.300	1.019
		27.452	**26.613**	**20.294**	**9.205**	**3.014**	**1.694**
		0.111	**1.371**	**2.591**	**2.306**	**1.487**	**1.165**

Table 3 Mean value and hedging error for a daily monitored up-and-out call option. $T = 6$ months, $\Delta = 1$ day, $\mu = -0.1, r = 0$. For each strike and barrier level we report 5 values: i) Black–Scholes value of continuously monitored option, ii) mean value for normally distributed log returns and discretely (daily) monitored option, iii) hedging error corresponding to ii); iv) the mean value process V_0 for the empirical distribution of log returns(discrete monitoring); v) standard deviation of the unconditional hedging error corresponding to iv). Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

		barrier (delta/level)					
		1E-100	0.01	0.10	0.30	0.45	0.49
strike delta	strike level	342.09	114.57	107.86	103.25	100.90	100.31
		0.019					
		0.019					
0.01	114.63	0.039					
		0.020					
		0.074					
		0.268	0.151				
		0.268	0.172				
0.1	107.89	0.104	0.214				
		0.268	**0.171**				
		0.197	**0.303**				
		1.071	0.874	0.182			
		1.071	0.916	0.255			
0.3	103.26	0.149	0.380	0.279			
		1.068	**0.912**	**0.257**			
		0.282	**0.547**	**0.381**			
		1.900	1.663	0.608	0.023		
		1.900	1.716	0.752	0.052		
0.45	100.90	0.155	0.453	0.451	0.129		
		1.897	**1.710**	**0.758**	**0.053**		
		0.295	**0.651**	**0.628**	**0.165**		
		2.162	1.915	0.767	0.044	0.000	
		2.162	1.971	0.930	0.089	0.001	
0.49	100.31	0.155	0.478	0.487	0.164	0.017	
		2.159	**1.965**	**0.937**	**0.092**	**0.001**	
		0.295	**0.684**	**0.681**	**0.216**	**0.018**	
		4.563	4.247	2.447	0.515	0.054	0.013
		4.563	4.321	2.750	0.754	0.141	0.071
0.75	96.33	0.126	0.595	0.784	0.451	0.193	0.137
		4.562	**4.317**	**2.767**	**0.771**	**0.145**	**0.073**
		0.240	**0.839**	**1.103**	**0.631**	**0.265**	**0.186**
		12.488	12.020	8.764	3.512	0.808	0.262
		12.488	12.134	9.385	4.427	1.598	1.037
0.99	87.53	0.024	0.883	1.409	1.144	0.734	0.606
		12.489	**12.132**	**9.420**	**4.484**	**1.630**	**1.049**
		0.047	**1.224**	**1.975**	**1.604**	**1.020**	**0.837**

Table 4 Mean value and hedging error for a daily monitored up-and-out call option. $T = 1$ month, $\Delta = 1$ hour, $\mu = 0.1, r = 0$. For each strike and barrier level we report 5 values: i) Black–Scholes value of continuously monitored option, ii) mean value for normally distributed log returns and discretely (daily) monitored option, iii) hedging error corresponding to ii); iv) the mean value process V_0 for the empirical distribution of log returns(discrete monitoring); v) standard deviation of the unconditional hedging error corresponding to iv). Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

Let us now turn to the hedging errors. Hedging errors of barrier options (columns 4-8) behave differently to those of plain vanilla options (column 3). The hedging error of plain vanilla options are the largest at the money and become smaller for deep-in and deep-out-of-the-money options. In contrast, the hedging error of an up-and-out barrier option increases with decreasing strike price. This happens because for vanilla options the only source of the hedging error is the non-linearity of option pay-off around the strike price, whereas for barriers the main source of the hedging errors is the barrier itself. The lower the strike the higher the pay-off near the barrier and the higher the hedging errors.

Consider an (at-the-money) plain vanilla option with $T = 1$ month to maturity and strike at 100.3 (see Table 1, column 3). The Black-Scholes value of this option is 2.162, and the standard deviation of the unconditional hedging error is 0.427, due to daily rebalancing. If we consider the empirical distribution of log returns, which exhibits excess kurtosis, the hedging error increases to 0.491. Take now a barrier option with the same strike, and barrier at 107.9. The Black–Scholes price of the barrier option is less than a half at 0.930 but the standard deviation of the hedging error is more than double at 0.931. Thus if selling a plain vanilla option at the Black–Scholes price based on historical volatility is not a profitable enterprise, doing the same for barrier options is positively counterproductive. This conclusion is more pronounced for longer maturities and lower strikes, see Table 2 ($T = 6$ months).

Next we examine the effect of the change in the market direction, by contrasting Table 2 ($\mu = 0.1$) with Table 3 ($\mu = -0.1$). The difference between the Black–Scholes no-arbitrage price of a daily monitored barrier option $\hat{V}$ and the mean value process V remains small. The mean value is higher in the bear market for plain vanilla options (column 3) but it is generally marginally lower for barrier options, with the exception of very low strikes in combination with very low barrier levels. The difference in absolute value is less than two cents and less than 1% in relative terms (with the exception of the two vanilla option with highest strikes). We conclude that V is largely insensitive to the changes in μ and that the Black–Scholes price $\hat{V}$ is a very good proxy for V.

The change in the market direction has a more dramatic effect on the size of unconditional hedging errors. Recall that the standard deviation of the unconditional error is given as a weighted average of one-period hedging errors,

$$\varepsilon_0^2(\varphi) = \sum_{j=0}^{n-1} \left(R^2 b\right)^{n-j-1} \mathrm{E}\left[\psi_j\right],$$

$$\psi_j = \mathrm{Var}_j\left(V_{j+1}^2\right) - \frac{\left(\mathrm{Cov}_j\left(S_{j+1}, V_{j+1}\right)\right)^2}{\mathrm{Var}_j\left(S_{j+1}\right)},$$

where R and b are close to 1. Since V is largely insensitive to the value of μ the values of ψ (as a function of time, stock price and option status) will very much coincide between the bull and the bear market. What will be different is the *expectation* of ψ.

The instantaneous hedging error ψ arises from two non-linearities in the option pay-off – one around the strike price and one along the barrier. The hedging error along the barrier tends to be more significant unless the barrier is either very far away from the stock price or the option is just about to be knocked out. In a bull market prices rise on average and the barrier, being above the initial stock price, contributes more significantly to $\mathrm{E}[\psi_j]$. $\mathrm{E}[\psi_j]$ will also contain more significant contribution from the strike region if the option is initially out of the money. In contrast, in a bear market price falls on average and $\mathrm{E}[\psi_j]$ will put less weight on the barrier region. It will contain a more significant contribution from the strike price, if the option is in the money to begin with. For barrier deltas equal to 10^{-100} and 0.49 we expect the strike region to dominate and therefore the hedging errors in the bear market to be larger for in-the-money options. This intuition is borne out by the numerical results shown in Tables 2 and 3.

3.2 Asymptotics

Let us now examine examine the effect of more frequent rebalancing by considering $\Delta = 1$ hour (Table 4). Although hedging now occurs *hourly* we maintain the *daily monitoring* frequency of the barrier to make the results comparable with those in Table 1.

In the Black–Scholes model the *standard deviation* of the hedging error for *plain vanilla* options decreases with the square root of rebalancing interval, see [3] and [37]. With hourly rebalancing this implies standard deviation equal to $\sqrt{1/8} \approx 35\%$ of the daily figure (with 8-hour trading day). The theoretical prediction turns out to be accurate, as can be seen by comparing entries marked ii) in each row of column 3 of Tables 1 and 4 which yields the range 36% to 38% across all strikes.

In the empirical Lévy model the standard deviation of the unconditional hedging error of plain vanilla options is seen to decay more slowly, see entries marked v) in each row of column 3 of Tables 1 and 4. With hourly rebalancing it is in the range 60%-62% of the daily rebalancing figures across all strikes. In this instance the higher frequency of hedging is (partially) offset by higher kurtosis of hourly returns. [10], Section 13.7, derives an approximation of the hedging error for leptokurtic returns and shows that rebalancing interval must be multiplied by kurtosis minus one to obtain the correct scaling of hedging errors. In our case Table 5 shows the kurtosis of daily returns is 3.72 and the kurtosis of hourly returns is 8.73, thus we should expect hourly errors to equal $\sqrt{1/8 \times 7.73/2.72} \approx 60\%$ of the daily errors which matches the actual range of 60% to 62% mentioned earlier.

Table 5 compares the kurtosis of returns and log returns in the calibrated Lévy model with the kurtosis achieved in its multinomial lattice approximation. The last two columns show the number of standard deviations of one-period log return (rounded up to the nearest quarter) corresponding to the 10^{-5} and $1-10^{-5}$ quantiles of the one-period log return distribution. This is the range represented by the lattice approximation of the Lévy process. As an aside, we observe that the lattice begins

	kurtosis lattice		kurtosis Lévy			
Δ	log	level	log	level	$\frac{n_{\text{down}}\eta}{\sigma}$	$\frac{n_{\text{up}}\eta}{\sigma}$
5 min	**69.11**	**69.05**	**72.54**	**72.51**	**27**	**25**
15 min	25.77	25.76	26.18	26.17	17	16
30 min	14.42	14.42	14.59	14.59	12.75	12
1 hr	**8.73**	**8.73**	**8.79**	**8.80**	**10**	**9.5**
2 hr	5.86	5.87	5.90	5.90	8	7.75
4 hr	4.44	4.44	4.45	4.45	6.75	6.75
1 day (8 hr)	**3.72**	**3.72**	**3.72**	**3.73**	**5.75**	**5.75**

Table 5 Kurtosis as a function of rebalancing interval

to struggle to approximate the kurtosis of the Lévy process well at the 5-minute rebalancing interval.

For *barrier options* (columns 4-8 of Tables 1 and 4) the Black-Scholes situation is more complicated because part of the error is caused by the barrier itself and this part has different Δ-asymptotics. Conjecturing that the barrier contributes an error whose *variance* is proportional to the square root of rebalancing interval, see [24], and assuming that fraction α of the error is generated by the strike region and the rest by the barrier, the approximate expression for the hourly total error as a fraction of daily error would read

$$\sqrt{0.35\alpha + \sqrt{0.35}(1-\alpha)}. \tag{6}$$

For barrier options in columns 4 and 5 of Tables 1 and 4 the percentage reduction in hedging error in the Black-Scholes model stands between 51% and 54% which implies α values in formula (6) between 0.25 and 0.4. Variability of α is to be expected since the relative importance of the two types of errors will depend on barrier and strike levels.

One can conjecture that for barrier options in the presence of excess kurtosis the formula (6) will remain the same, only the time scaling factor will be adjusted for excess kurtosis from 0.35 to 0.6 as in the case of plain vanilla options. We thus expect the ratio of hourly to daily errors in the Lévy model to be

$$\sqrt{0.6\alpha + \sqrt{0.6}(1-\alpha)}. \tag{7}$$

With α in the range 0.25 to 0.4 heuristic (7) predicts error reduction in the range 71%-74% while the actual figures from columns 4 and 5 of Tables 1 and 4 yield the range 68%-70%, which for practical purposes is a perfectly adequate approximation.

Table 6 provides 5-minute error data for one specific strike/barrier combination corresponding to $\alpha = 0.25$. It reports the hedging error ε_0 obtained from (5) and the mean value V_0 obtained from (3) and (4) using the multinomial approximation of the empirical Lévy process and analogous quantitities $\hat{\varepsilon}_0$ and $\hat{V}_0$ obtained from a multinomial approximation of the Black-Scholes model.

The Black-Scholes 5-minute time scaling factor is $1/(8 \times 12) = 1/96 = 0.0104$ and the heuristic (6) yields error reduction ratio of

	Black-Scholes		empirical Lévy	
$\triangle$	$\hat{V}_0$	$\hat{\varepsilon}_0$	V_0	ε_0
5 min	0.2525	0.142	0.2548	0.330
15 min	0.2535	0.191	0.2556	0.345
30 min	0.2535	0.231	0.2558	0.360
1 hr	0.2536	0.282	0.2559	0.380
2 hr	0.2537	0.345	0.2560	0.421
4 hr	0.2537	0.424	0.2560	0.470
8 hr	0.2538	0.519	0.2561	0.548

Table 6 Mean value V_0 and unconditional standard deviation of the hedging error ε_0 for parameter values $T = 1, S_0 = 100, B = 107.9, K = 103.3$

$$\sqrt{0.0104 \times 0.25 + \sqrt{0.0104}(1-0.25)} \approx 28\%,$$

while in Table 6 we find this ratio to be $0.142/0.519 \approx 27\%$. The leptokurtic empirical 5-minute distribution leads to the time scaling factor of $68.05/2.72/96 \approx 0.26$ hence the 5-minute empirical error is predicted to be

$$\sqrt{0.26 \times 0.25 + \sqrt{0.26}(1-0.25)} \approx 67\%$$

of the daily error. The actual figure in Table 6 is $0.33/0.548 \approx 60\%$. For practical purposes this is again an acceptable approximation.

Our exploratory analysis above points to two open questions in this area of research: 1) calculation of explicit asymptotic expression for hedging error of barrier options in discretely rebalanced Black-Scholes model analogous to the formula of [24] for path-independent options; 2) asymptotic formula for hedging error of barrier options in a continuously rebalanced Lévy model with small jumps. There is a good reason to believe that 1) and 2) are closely linked because similar link has already been established for plain vanilla options, see [11].

4 Sharpe ratio price bounds

In this model, as in reality, the sale of an option and subsequent hedging is a risky activity. If one sells an option at its Black-Scholes value corresponding to historical volatility one effectively enters into an investment with zero mean and non-zero variance. In addition this investment is by construction uncorrelated with the stock returns. To make option trading profitable the trader must aim for a certain level of risk-adjusted returns, which implies selling derivatives above their Black–Scholes value. The question then arises as to what is a sensible measure of risk-adjusted returns and what is a sensible level of compensation for the residual risk.

[10] proposes to measure profitability of investment by its certainty equivalent growth rate adjusted for investor's risk aversion. When this measure is applied to mean-variance preferences, it yields a one-to-one relationship with the ex-ante

Sharpe ratio of the investment strategy. Thus, in the present context, the unconditional Sharpe ratio appears as a natural measure of risk-adjusted returns.

It is well known that the square of maximal Sharpe ratio available by trading in two uncorrelated assets equals the sum of squared Sharpe ratios of the individual assets. Since the hedged option position is uncorrelated with the stock we can regard the Sharpe ratio of the hedged position as a meaningful measure of *incremental* performance (i.e. performance over and above optimal investment in the stock).

Suppose that the trader targets a certain level of annualized incremental Sharpe ratio h (say $h = 0.5$). Assuming that he or she can sell the option at price $\tilde{C}$ above the mean value V_0 the resulting Sharpe ratio of the hedged option position equals

$$\frac{e^{rT}(\tilde{C}-V_0)}{\varepsilon_0}.$$

If T is maturity in years the trader should look for a price $\tilde{C}$ such that

$$\frac{e^{rT}(\tilde{C}-V_0)}{\varepsilon_0} = h\sqrt{T},$$

which yields

$$\tilde{C} = V_0 + e^{-rT}h\sqrt{T}\varepsilon_0. \tag{8}$$

For plain vanilla options the price adjustment corresponding to annualized incremental Sharpe ratio of 1 gives rise to a gap between implied volatility and historical volatility of about 150 basis points, robustly across maturities and strikes. If the same price adjustment is performed for barrier options its magnitude is as important as, and often several times dominates, the price adjustment due to discrete monitoring. The fraction $\frac{\sqrt{T}\varepsilon_0}{V_0}$ is reported in Tables 7 and 8.

		barrier (delta/level)					
strike		1E-100	0.01	0.10	0.30	0.45	0.49
delta	level	343.8	114.6	107.9	103.3	100.9	100.3
0.01	114.6	**177%**					
0.10	107.9	**35%**	**75%**				
0.30	103.3	**13%**	**26%**	**61%**			
0.45	100.9	**7%**	**16%**	**35%**	**121%**		
0.49	100.3	**7%**	**15%**	**30%**	**94%**	**491%**	
0.75	96.3	**3%**	**8%**	**17%**	**34%**	**75%**	**104%**
0.99	87.5	**0.2%**	**4%**	**9%**	**15%**	**26%**	**33%**

Table 7 Risk premium as a percentage of mean value for up-and-out call. $T = 1$ month, $\Delta = 1$ day, $\mu = 0.1, r = 0$. Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

One obvious conclusion to draw from formula (8) is that prices in an incomplete market are likely to contain both a linear (V_0) and a non-linear (ε_0) component. The

strike delta	strike level	barrier (delta/level) 1E-100 / 2071.0	0.01 / 140.5	0.10 / 121.2	0.30 / 108.8	0.45 / 102.8	0.49 / 101.4
0.01	114.6	**111%**					
0.10	107.9	**17%**	**55%**				
0.30	103.3	**6%**	**19%**	**43%**			
0.45	100.9	**3%**	**11%**	**24%**	**103%**		
0.49	100.3	**3%**	**11%**	**19%**	**69%**	**665%**	
0.75	96.3	**1%**	**6%**	**11%**	**21%**	**47%**	**71%**
0.99	87.5	**0%**	**3%**	**5%**	**8%**	**14%**	**19%**

Table 8 Risk premium as a percentage of mean value for up-and-out call. $T = 6$ months, $\Delta = 1$ day, $\mu = 0.1, r = 0$. Strike and barrier levels are parametrized by the Black–Scholes delta of their position.

prevailing market practice is to use just the linear part V_0 for calibration which often requires distorting the historical distribution of returns to match observed market prices across strikes and maturities. For example, in their calibration of plain vanilla option prices [31] report historical annualized excess kurtosis at 0.002 but risk-neutral excess kurtosis at 0.18 which is a level that the variance-optimal martingale measure that generates V_0 simply cannot reach. This phenomenon gets worse in the presence of exotic options. Formula (8) offers a flexible alternative that may offer better fit of model dynamics to historical return distributions and at the same time provide closer calibration to market prices thanks to the non-linear term ε_0 which has very different characteristics for different types of exotic options, as we have seen in the previous section.

5 Conclusions

In place of conclusions a personal confession. At around 2002 I was performing numerical experiments somewhat similar to the ones presented here, but without the underlying Lévy structure, just purely driven by empirical data and with plain vanilla options. In the process of doing so I convinced myself that Lévy models, which emerge as continuous-time limits of multinomial lattices, are the key mathematical tool to describe market incompleteness. I wrote to Ernst Eberlein, out of the blue, to ask whether I might join his group for a few months to learn properly about this exciting and for me completely new and difficult theory. The result were two stimulating months in Freiburg in early 2004 and a lifetime of mathematical inspiration. Thank you Ernst!

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