Transfer Operators for Deterministic and Stochastic Coupled Map Lattices

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Declaration

This thesis is composed of two chapters. Each chapter is written as an independent paper with its own introduction and notation. Chapter Two uses some results from Chapter One and refers to it by citation of the corresponding paper. The numbers in these citations refer to the numbering in Chapter One.
The paper constituting Chapter One has been accepted for publication in *Ergodic Theory & Dynamical Systems* and is based on joint work with Hans Henrik Rugh. It is not possible to distinguish exactly the individual contributions. However, each of us contributed a genuine part of proving the results.
Chapter Two has not been submitted for publication yet.

Summary

In Chapter One we consider analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with exponentially decaying interaction. We introduce Banach spaces for the infinite-dimensional system that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. Using residue calculus and 'cluster expansion'-like techniques we define transfer operators on these Banach spaces. We get a unique (in the considered Banach spaces) probability measure that exhibits exponential decay of correlations.
In Chapter Two we consider on $M = (S^1)^{\mathbb{Z}^d}$ a family of continuous local updatings, of finite range type or Lipschitz-continuous in all coordinates with summable Lipschitz-constants. We show that the infinite-dimensional dynamical system with independent identically Poisson-distributed times for the individual updatings is well-defined. In the setting of analytically coupled uniformly expanding, analytic circle maps with weak, exponentially decaying interaction, we define transfer operators for the infinite-dimensional system, acting on Banch-spaces that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. We prove existence and uniqueness (in the considered Banach space) of a probability measure and its exponential decay of correlations.

Chapter 1

Transfer Operators for Coupled Analytic Maps

1.0 Introduction

Coupled map lattices were introduced by K. Kaneko (cf. [20] for a review) as systems that are mixing wrt. spatio-temporal shifts. L.A. Bunimovich and Ya.G. Sinai proved in [7] (cf. also the remarks on that in [4]) the existence of an invariant measure and its exponential decay of correlations for a one-dimensional lattice of weakly coupled maps by constructing a Markov partition and relating the system to a two-dimensional spin system.
JḂricmont and A. Kupiainen extend this result in [3] and [4, 5] to coupled circle maps over the $\mathbb{Z}^d$-lattice with analytic and Hölder-continuous weak interaction, respectively. They use a 'polymer' or 'cluster'-expansion for the Perron-Frobenius operator for the finite-dimensional subsystems over $\Lambda \subset \mathbb{Z}^d$ and write the nth iterate of this operator applied to the constant function 1 in terms of potentials for a $d+1$-dimensional spin system. Taking the limit as $n \to \infty$ and $\Lambda \to \mathbb{Z}^d$ they get existence and uniqueness (among measures with certain properties) of the invariant probability measure and exponential decay of correlations.
V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää and A. Kupiainen define in [1], for infinite-dimensional systems over the $\mathbb{Z}^d$ lattice, transfer operators on a Frechet space, and, for $d = 1$, on a Banach space; they study the spectral properties of these operators, viewing the coupled operator as a perturbation of the uncoupled one in the Banach case.
In [21] G. Keller and M. Künzle consider periodic or infinite one-dimensional lattices of weakly coupled maps of the unit interval. In particular they define transfer operators on the space BV of measures whose finite-dimensional marginals have densities of bounded variation and prove the existence of an invariant probability measure. For the infinite-dimensional system they further show that for a small perturbation of the uncoupled map any invariant measure in BV is close (in a specified sense) to the one they found.

Coupled map lattices with multi-dimensional local systems of hyperbolic type have been studied by Ya.B. Pesin and Ya.G. Sinai [27], M. Jiang [16, 17], M. Jiang and A. Mazel [18], M. Jiang and Ya.B. Pesin [19] and D.L. Volevich [31, 32].
Detailed surveys on coupled map lattices can be found in [6], [19] and [4].

In the above papers (except [1], [21]) the analysis has been done only for Banach spaces defined for finite subsets Λ of the lattice, and the (weak) limit of the invariant measure for $\Lambda \to \mathbb{Z}^d$ was taken afterwards.
Here we present a new point of view in which a natural Banach space and transfer operators are defined for the infinite lattice of weakly coupled analytic maps (Section 1.1). The space contains consistent families of analytic densities over finite subsets of $\mathbb{Z}^d$. We take a weighted sup-norm so that the sup-norms of the densities for the sub-systems over finitely many (say N) lattice points is bounded exponentially in N (Section 1.2). We identify an ample subset of this space with a set of *rca* measures (Section 1.4) that contains the unique invariant probability density (Section 1.2).
We derive exponential decay of correlations for this measure and a certain class of observables from (the proof of) the spectral properties of our transfer operators. (Sections 1.2, 1.7). The operator for the coupled system and also the invariant measure are (for a small interaction) in fact perturbations of their counterparts in the uncoupled case. So the mixing properties are inherited from the single site systems. Section 1.8 contains the proofs.
Our approach provides a natural setting for an analysis of the full $\mathbb{Z}^d$ Perron-Frobenius operator in terms of cluster expansions over finite subsets of the lattice. Using residue calculus we introduce an integral representation for the Perron-Frobenius operator for finite-dimensional sub-systems (Section 1.3) which yields a uniform control over the perturbation and also gives rise to an easy approach to stochastic perturbation (cf. [26]) which however we do not consider here.
Our 'cluster expansion' combinatorics (Section 1.5) uses ideas from the work of C. Maes A. Van Moffaert [26] who have introduced a simplified (compared to the one in [3]) polymer expansion. Apart from the analysis of the one-dimensional operator, which is fairly standard and for which we refer to e.g. [3], the paper should be self-contained.

1.1 General Setting

We consider coupled map lattices in the following setting: The state space is $M = (S^1)^{\mathbb{Z}^d}$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the complex plane and $d \in \mathbb{N}$.
The map $S : M \to M$ is the composition $S = F \circ T^\epsilon$ of a coupling map T^ϵ depending on a (small) non-negative parameter ϵ and another parameter for the decay of interaction (cf. (1.1)) with an (uncoupled) map F that acts on each component of M separately. We make the following assumptions:

Assumption I $F(\mathbf{z}) = (f_p(z_p))_{p\in\mathbb{Z}^d}$ where $f_p : S^1 \to S^1$ are real analytic and expanding (i.e. $f_p' \geq \lambda_0 > 1$) maps that extend for some δ_1 holomorphically to the interior of an annulus $A_{\delta_1} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid -\delta_1 \leq \ln|z| \leq \delta_1\}$ and the family of Perron-Frobenius operators $\mathcal{L}_{f_p}$ for the single site systems satisfies uniformly a condition specified in Section (1.5.1) below (1.31). (We need some more definitions to specify these conditions. But note that they are in particular satisfied if all f_p are the same.)

We write $T^\epsilon : M \to M$ as $T^\epsilon(\mathbf{z}) = (T_p^\epsilon(\mathbf{z}))_{p\in\mathbb{Z}^d}$ and $T_p^\epsilon(\mathbf{z}) = z_p \exp[2\pi i \epsilon g_p(\mathbf{z})]$ with $g_p(\mathbf{z}) = \sum_{k=1}^{\infty} g_{p,k}(\mathbf{z})$. The function $g_{p,k}$ is real valued on $(S^1)^{\mathbb{Z}^d}$ and depends only on those z_q with $\|p-q\| \leq k$ (neighbours of distance at most k) where $\|p\| \stackrel{\text{def}}{=} \sum_{i=1}^d |p_i|$. We write $B_k(p) = \{q \in \mathbb{Z}^d \mid \|p-q\| \leq k\}$ and also denote by $g_{p,k}$ the function from the finite-dimensional torus $(S^1)^{B_k(p)}$ to $\mathbb{R}$.
We assume the following for the functions $g_{p,k}$:

Assumption II For all $p \in \mathbb{Z}^d$ and $k \geq 1$ each map $g_{p,k}$ extends to a holomorphic map $g_{p,k} : A_{\delta_1}^{B_k(p)} \to \mathbb{C}$ and its sup-norm (of modulus) is exponentially bounded by

$$\| g_{p,k} \|_{A_{\delta_1}^{B_k(p)}} \leq c_1 \exp\left(-c_2 k^d\right) \tag{1.1}$$

with $c_1 > 0$ and c_2 bigger than a certain constant specified in (1.100).

The parameter c_1 is actually redundant as it is multiplied by ϵ in the definition of T_p^ϵ. We also have $\exp(-c_2 k^d) \leq \exp(-\xi)\exp(-c_2^* k^d)$ for $c_2^* = c_2 - \xi$, $\xi > 0$, i.e. for any ϵ we can make the interaction small only by taking c_2 large. But once we have chosen c_2 large enough to guarantee the convergence of the infinite sums in our analysis we can consider perturbations of the uncoupled map depending on the parameter ϵ only.
With the metric

$$d_\gamma(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \sup_{p\in\mathbb{Z}^d} \gamma^{\|p\|} \|x_p - y_p\| \tag{1.2}$$

for $0 < \gamma < 1$ (M, d_γ) is a compact metric space. Its topology is the product topology on $(S^1)^{\mathbb{Z}^d}$. The Borel σ-algebra $\mathcal{B}$ on M is the same as the product σ-algebra. F and T^ϵ are continuous and measurable. Let $\mathcal{C}(M)$ denote the space of real-valued continuous functions on (M, d_γ) with the sup-norm and μ the Lebesgue (product) measure on M.
For $\Lambda_1 \subseteq \Lambda_2 \subseteq \mathbb{Z}^d$, with Λ_1 finite and an integrable function ψ on M depending only on the Λ_2-coordinates, we define the projection

$$(\pi_{\Lambda_1}\psi)(\mathbf{z}_{\Lambda_1}) \stackrel{\text{def}}{=} \int_{(S^1)^{\Lambda_2\setminus\Lambda_1}} d\mu^{\Lambda_2\setminus\Lambda_1}(\mathbf{z}_{\Lambda_2\setminus\Lambda_1})\psi(\mathbf{z}_{\Lambda_1} \vee \mathbf{z}_{\Lambda_2\setminus\Lambda_1}). \tag{1.3}$$

1.2 Main Results

For finite $\Lambda \subset \mathbb{Z}^d$ let $H(A_\delta^\Lambda)$ be the space of continuous functions on the closed polyannulus A_δ^Λ that are holomorphic on its interior and write $\|\cdot\|_\Lambda$ for the sup-norm (of modulus) on $H(A_\delta^\Lambda)$. Let $\mathcal{F}$ be the set of all finite subsets (including $\emptyset$) of $\mathbb{Z}^d$. We denote by $\mathcal{H}$ the vectorspace of all consistent families $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ of functions $\phi_\Lambda \in H(A_\delta^\Lambda)$. Consistency means $\pi_{\Lambda_1}\phi_{\Lambda_2} = \phi_{\Lambda_1}$ for $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$. We write $\mu(\phi) \stackrel{\text{def}}{=} \phi_\emptyset$.

We want to define a norm on a (sufficiently large) subspace of $\mathcal{H}$ that should at least contain 'product densities' like $h = (h_\Lambda)_{\Lambda\in\mathcal{F}}$ with $h_\Lambda(\mathbf{z}) = \prod_{p\in\Lambda} h_p(z_p)$, where $h_p \in H(A_\delta^{\{p\}})$ is the invariant probability density for the single system over $\{p\}$ (cf. Section 1.5.1).

Because of (1.32) the sup-norm $\|h_{\Lambda_1}\|_{\Lambda_1}$ does not grow faster than exponentially in $|\Lambda_1|$. Therefore we take a weighted sup-norm. For $0 < \vartheta < 1$ we define

$$\|\phi\|_\vartheta \stackrel{\text{def}}{=} \sup_{\Lambda\in\mathcal{F}} \vartheta^{|\Lambda|}\|\phi_\Lambda\|_\Lambda \tag{1.4}$$

and set $\mathcal{H}_\vartheta \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} \mid \|\phi\|_\vartheta < \infty\}$. Then $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ is a Banach space. In fact, if $(\phi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ then for each $\Lambda \in \mathcal{F}$ the sequence $(\phi^n_\Lambda)_{n\in\mathbb{N}}$ is Cauchy in the Banach space $\left(H(A_\delta^\Lambda), \|\cdot\|_{A_\delta^\Lambda}\right)$ and so converges to ϕ_Λ. Consistency of $(\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ follows from taking the limit (as $n \to \infty$) of $\pi_{\Lambda_1}\phi^n_{\Lambda_2} = \phi^n_{\Lambda_1}$ using the continuity of π_{Λ_1} for any $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$. Analogously we define for $\Lambda \in \mathcal{F}$ the weighted norm on spaces $\mathcal{H}_{\Lambda,\vartheta}$ of consistent sub-families $(\phi_{\Lambda_1})_{\Lambda_1\subseteq\Lambda}$:

$$\|\phi\|_{\Lambda,\vartheta} \stackrel{\text{def}}{=} \sup_{\Lambda_1\subseteq\Lambda} \vartheta^{|\Lambda_1|}\|\phi_{\Lambda_1}\|_{\Lambda_1}. \tag{1.5}$$

We get the same (topological) vector space as $\left(H(A_\delta^\Lambda), \|\cdot\|_\Lambda\right)$, but the constants for the estimates of the norms are unbounded as $|\Lambda|$ increases.

For given $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ and $N \in \mathbb{N}$ we have a map,

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} : (\mathcal{H}_\vartheta, \|\cdot\|_\vartheta) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}), \tag{1.6}$$

where $\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$ is the Perron-Frobenius operator for the finite-dimensional system over Λ_2 (cf. Section 1.3) with fixed boundary conditions (not included in the notation). The following definition of transfer operators for the infinite system does not depend on the choice of the boundary conditions.

Theorem 1.2.1 *For ϑ, ϵ sufficiently small, c_2, N_0 sufficiently big and any $\Lambda_1 \in \mathcal{F}$:*

1. *The limit*

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} \stackrel{\text{def}}{=} \lim_{\Lambda_2\to\mathbb{Z}^d} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} \tag{1.7}$$

$\in L\left((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\Lambda_1,\vartheta_N}, \|\cdot\|_{\Lambda_1,\vartheta_N})\right)$ *exists for suitably chosen* $0 < \vartheta_1 \leq \cdots \leq \vartheta_{N_0} = \vartheta_{N_0+1} = \cdots = \vartheta$ *and the family of these operators is uniformly (in* Λ_1*) bounded. This defines operators* $\mathcal{L}^N_{F\circ T^\epsilon}$

$\in L\left((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\vartheta_N}, \|\cdot\|_{\vartheta_N})\right)$ *by* $\left(\mathcal{L}^N_{F\circ T^\epsilon}\phi\right)_{\Lambda_1} \stackrel{\text{def}}{=} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}\phi$.

In particular for $N \geq N_0$ *we have* $\mathcal{L}^N_{F\circ T^\epsilon} \in L\left(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta\right)$.

In the case of finite-range interaction we can define a linear map $\mathcal{L}_{F\circ T^\epsilon}$ *on* $\mathcal{H}$ *in the same way, i.e. if* r *is the range of interaction we set for any* $\Lambda_1 \in \mathcal{F}$

$$\pi_{\Lambda_1} \circ \mathcal{L}_{F\circ T^\epsilon} \stackrel{\text{def}}{=} \pi_{\Lambda_1} \circ \mathcal{L}_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} \tag{1.8}$$

where $\Lambda_2 = B_r(\Lambda_1)$.

2. *There is an* $F \circ T^\epsilon$*-invariant, non-negative probability measure* ν^*. *It is unique in the set of non-negative probability measures whose marginal densities can be identified with a* $\nu = (\nu_{\Lambda_1})_{\Lambda_1 \in \mathcal{F}} \in \mathcal{H}_\vartheta$.

   In $L\left(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta\right)$ *the sequence* $\left(\mathcal{L}^N_{F\circ T^\epsilon}\right)_{N \geq N_0}$ *converges exponentially fast:*

$$\left\|\mathcal{L}^N_{F\circ T^\epsilon} - \mu(\cdot)\nu^*\right\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta))} \leq c_3 \tilde{\eta}^N \tag{1.9}$$

for some $c_3 > 0$ *and* $0 < \tilde{\eta} < 1$.

Remark 1) The relation between measures and elements of $\mathcal{H}$ is explained in Section 1.4, in particular in (1.23).
2) A formula for ν is given in (1.59).

For the invariant measure ν we have exponential decay of correlations for spatio-temporal shifts on the system:
Let $(e_1, \ldots, e_d)$ be a linearly-independent system of unit vectors in $\mathbb{Z}^d$. We define translations $\tau_{e_i}(p) \stackrel{\text{def}}{=} p + e_i$ for $p \in \mathbb{Z}^d$ and $(\tau_{e_i}(z))_p \stackrel{\text{def}}{=} z_{\tau_{e_i}(p)}$ for $z \in M$.
In the following theorem we denote by τ (acting on M from the right) compositions $\tau = \tau_1 \circ \ldots \circ \tau_{m(\tau)}$ and by σ a composition of spatio-temporal shifts (on M): $\sigma = \sigma_1 \circ \ldots \circ \sigma_{m(\sigma)+n(\sigma)}$ with $\sigma_i \in \{S, \tau_{e_1}, \ldots, \tau_{e_d}\}$. We denote by $n(\sigma)$ the number of factors S and by $m(\sigma)$ the number of spatial translations in this product. For a translation-invariant system, i.e. $f_p = f$ and $g_p(\mathbf{z}) = g_{\tau_{e_i}^{-1}(p)}(\tau_{e_i}(z))$ for all $p \in \mathbb{Z}^d$ and $i = 1, \ldots, d$, the time-shift S commutes with the translations.

Theorem 1.2.2 *For* ϑ,ϵ *as in Theorem 1.2.1 and* c_2 *sufficiently large there is a* $\kappa \in (0,1)$ *such that for all nonempty* $\Lambda_1, \Lambda_2 \in \mathcal{F}$ *the following holds with the constant* $c(\Lambda_1, \Lambda_2, \kappa) \stackrel{\text{def}}{=} \kappa^{-\max\{\|p-q\|: p\in\Lambda_1, q\in\Lambda_2\}}$:

1. *If* $g \in C((S^1)^{\Lambda_1})$ *and* $f \in C((S^1)^{\Lambda_2})$ *then*
$\left|\int_M d\nu^* gf - \left(\int_M d\nu^* g\right)\left(\int_M d\nu^* f\right)\right| \leq c_4 \vartheta^{-|\Lambda_1|-|\Lambda_2|} \|g\|_\infty \|f\|_\infty \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$,
where $\mathrm{dist}(\Lambda_1,\Lambda_2) \stackrel{\mathrm{def}}{=} \min\{\|p-q\| : p \in \Lambda_1, q \in \Lambda_2\}$.

2. *If* $g \in C((S^1)^{\Lambda_1})$ *and* $f \in \mathcal{H} \cap C((S^1)^{\Lambda_2})$ *then*

$$\begin{aligned} &\left|\int_M d\nu^*\, g \circ \tau \circ S^n f - \left(\int_M d\nu^*\, g \circ \tau\right)\left(\int_M d\nu^* f\right)\right| \\ \leq\ & c(\Lambda_1,\Lambda_2,\kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\tau)} \tilde{\eta}^n \end{aligned} \tag{1.10}$$

with suitable c_5 *and* $\tilde{\eta}$ *as in Theorem 1.2.1.*

3. *If the system is translation-invariant and* g, f *are as in (2.), then*

$$\begin{aligned} &\left|\int_M d\nu^*\, g \circ \sigma f - \left(\int_M d\nu^* g\right)\left(\int_M d\nu^* f\right)\right| \\ \leq\ & c(\Lambda_1,\Lambda_2,\kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\sigma)} \tilde{\eta}^{n(\sigma)}. \end{aligned} \tag{1.11}$$

4. *If* $g, f \in C(M)$ *then*

$$\lim_{\max\{m(\tau),n\}\to\infty} \left|\int_M d\nu^*\, g \circ \tau \circ S^n\, f - \left(\int_M d\nu^*\, g \circ \tau\right)\left(\int_M d\nu^* f\right)\right| = 0. \tag{1.12}$$

5. *If the system is translation-invariant and* $g, f \in C(M)$ *then*

$$\lim_{\max\{m(\sigma),n(\sigma)\}\to\infty} \int_M d\nu^*\, g \circ \sigma f = \left(\int_M d\nu^* g\right)\left(\int_M d\nu^* f\right). \tag{1.13}$$

Remarks: 1) Statement (5.) means that for a translation-invariant system ν is mixing wrt. spatio-temporal shifts. According to (3.), the decay of correlations for observables g and h as specified in (2.) is exponentially fast.
2) The proof of Theorem 1.2.2 shows that the statements hold for any $\kappa \in (0,1)$ if ϵ is sufficiently small and c_2 sufficiently large (both depending on κ). So a small interaction leads to small spatial correlations.

1.3 Finite-Dimensional Systems

We first consider 'finite-dimensional versions' of the maps F, T^ϵ etc. Let $\xi = (\xi_p)_{p\in\mathbb{Z}^d} \in M$ be a fixed configuration. For a finite subset $\Lambda \subset \mathbb{Z}^d$ we define $T^{\Lambda,\epsilon} : A_\delta^\Lambda \to \mathbb{C}^\Lambda$ by

$$(T^{\Lambda,\epsilon}(\mathbf{z}_\Lambda))_p \stackrel{\text{def}}{=} z_p \exp(2\pi i \epsilon g_p(\mathbf{z}_\Lambda \vee \xi_{\Lambda^C})), \tag{1.14}$$

where $\mathbf{z}_\Lambda \vee \xi_{\Lambda^C} \in M$ agrees with $\mathbf{z}_\Lambda$ on its Λ-sites and with ξ_{Λ^C} on its Λ^C-sites.
We do not specify ξ_{Λ^C} in the notation of $T^{\Lambda,\epsilon}$. The restriction of F to A_δ^Λ is denoted by F^Λ.
With the following two propositions we ensure that for sufficiently small δ and ϵ (independent of Λ and $\mathbf{z}_{\Lambda^C}$), the image of A_δ^Λ wrt. $F^\Lambda \circ T^{\Lambda,\epsilon}$ contains a bigger polyannulus (cf. [3]) and the image of the boundary, $F^\Lambda \circ T^{\Lambda,\epsilon}\left(\partial A_\delta^\Lambda\right)$, has positive distance from A_δ^Λ.
For $\Lambda \subset \mathbb{Z}^d$ we have the metric d_Λ on $(S^1)^\Lambda$ defined by

$$d_\Lambda(\mathbf{z}, \mathbf{w}) \stackrel{\text{def}}{=} \sup\{|z_p - w_p| \mid p \in \Lambda\}. \tag{1.15}$$

Proposition 1.3.1 *For all $c_7 \in (0,1)$, sufficiently small δ and ϵ (depending on c_7), and arbitrary $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$, $T^{\Lambda,\epsilon}$ maps A_δ^Λ biholomorphically onto its image and $T^{\Lambda,\epsilon}\left(A_\delta^\Lambda\right) \supset A_{c_7\delta}^\Lambda$, i.e. the image contains a sufficiently thick polyannulus. Also $T^{\Lambda,\epsilon}\left(\partial A_\delta^\Lambda\right) \cap A_{c_7\delta}^\Lambda = \emptyset$, i.e. the image of the boundary (the same as the boundary of the image) does not intersect the smaller polyannulus.*

Proposition 1.3.2 *Let the expanding maps $f_p : S^1 \to S^1$ satisfy Assumption **I** for some δ_1 and an expansion constant λ_0 and let $1 < \lambda < \lambda_0$. Then for all sufficiently small δ ($0 < \delta < \delta_0$) and all finite $\Lambda \subset \mathbb{Z}^d$ the map $F^\Lambda : A_\delta^\Lambda \to \mathbb{C}^\Lambda$ is locally biholomorphic, $A_{\lambda\delta}^\Lambda \subset F^\Lambda\left(A_\delta^\Lambda\right)$, i.e. the image contains a thicker polyannulus, and furthermore all $\mathbf{z} \in A_{\lambda\delta}^\Lambda$ have the same number of preimages. We also have $A_{\lambda\delta}^\Lambda \cap F^\Lambda\left(\partial A_\delta^\Lambda\right) = \emptyset$.*

Combining Propositions 1.3.1 and 1.3.2 we have for fixed c_7 (from Proposition 1.3.1) and (small) δ

$$F^\Lambda \circ T^{\Lambda,\epsilon}\left(A_\delta^\Lambda\right) \supset A_{c_7\lambda\delta}^\Lambda \tag{1.16}$$

and

$$F^\Lambda \circ T^{\Lambda,\epsilon}\left(\partial A_\delta^\Lambda\right) \cap A_{c_7\lambda\delta}^\Lambda = \emptyset. \tag{1.17}$$

In particular, if we choose $c_7 > \frac{1}{\lambda}$ there is a disc of radius $(c_7\lambda - 1)\delta > 0$ around each point in A_δ^Λ that is entirely contained in $F^\Lambda \circ T^{\Lambda,\epsilon}\left(A_\delta^\Lambda\right)$. We will need this for Cauchy estimates. From now on we keep δ fixed.
In the next proposition we establish a special representation of the Perron-Frobenius operator for our finite system with $(S^1)^N = (S^1)^\Lambda$, $S^\epsilon = F^\Lambda \circ T^{\Lambda,\epsilon}$, ψ continuous (the proposition holds also for $\psi \in L^\infty(M)$) and ϕ continuous on the closed polyannulus $A_{\delta_1}^\Lambda$ and analytic in its interior.
First we give the definition of the Perron-Frobenius operator (cf. for example [23]).

Definition 1.3.1 Let λ be a measure on a metric space M (with the Borel σ-algebra) and let $S : M \to M$ be a measurable map which is non-singular wrt. λ (i.e. for all measurable $A \in M$, $\lambda(A) = 0$ implies $\lambda(S^{-1}(A)) = 0$). The Perron-Frobenius operator $\mathcal{L}_S$, acting on $L^1(M)$, is defined via the equation

$$\int_M d\lambda\, \psi \circ S\, \phi = \int_M d\lambda\, \psi\, \mathcal{L}_S \phi \tag{1.18}$$

that, for given $\phi \in L^1(M)$, must hold for all $\psi \in L^\infty(M)$. The existence and uniqueness of $\mathcal{L}_S\phi \in L^1(M)$ is equivalent by the Radon-Nikodym Theorem to the absolute continuity (wrt. λ) of the measure associated to the functional $\psi \mapsto \int_M d\lambda\, \psi \circ S\, \phi$ (the functional here is restricted to continuous functions ψ), and this follows from the nonsingularity of S.

Remark Setting $\psi \equiv 1$ in (1.18) we get that $\mathcal{L}_S$ preserves the integral:

$$\int_M d\lambda\, \mathcal{L}_S \phi = \int_M d\lambda\, \phi. \tag{1.19}$$

The normalized Lebesgue measure μ on S^1 is given by $d\mu(z) = \frac{dz}{2\pi i}\frac{1}{z}$ (this lifts wrt. the map $t \to e^{it}$ to the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $[0, 2\pi)$) and the product measure μ^Λ on $(S^1)^\Lambda$ is given by

$$d\mu^\Lambda(\mathbf{z}) = \frac{d\mathbf{z}}{(2\pi i)^{|\Lambda|}}\frac{1}{\mathbf{z}} \stackrel{\text{def}}{=} \prod_{p\in\Lambda} \frac{dz_p}{2\pi i}\frac{1}{z_p}. \tag{1.20}$$

We also use $d\mu^\Lambda(\mathbf{z})$ as a shorthand notation for the right-hand side of (1.20) for $\mathbf{z} \in A_\delta^\Lambda$.

The following representation of the Perron-Frobenius operator for finite-dimensional subsystems of our coupled map lattice by means of Cauchy kernels is essential for our analysis. Similar Cauchy kernels were used in [28].

Proposition 1.3.3 *With F^Λ and $T^{\Lambda,\epsilon}$ defined as above set $S^\epsilon = F^\Lambda \circ T^{\Lambda,\epsilon}$ and let S_p^ϵ be the projection onto its p-th component. Then the Perron-Frobenius-Operator (for S^ϵ), acting on $\phi \in \mathcal{H}_\Lambda$, can be written in the following way:*

$$\mathcal{L}_{S^\epsilon}\phi(w) = \int_{\Gamma^\Lambda} d\mu^\Lambda(\mathbf{z})\phi(z) \prod_{p\in\Lambda} \left(\frac{1}{S_p^\epsilon(\mathbf{z}) - w_p} S_p^\epsilon(\mathbf{z}) \right) \tag{1.21}$$

where $\Gamma = \Gamma_+ \cup \Gamma_-$ is the positively-oriented boundary of A_δ.

1.4 Further Remarks on the Infinite-Dimensional System

The subspace of complex-valued functions that depend only on finitely many variables is dense in $(\mathcal{C}(M), \|\cdot\|_\infty)$, and each such function (say depending on $\mathbf{z}_\Lambda$ only) can be uniformly approximated by (the restriction of) functions in $\mathcal{H}(A_\delta^\Lambda)$. The dual space of $\mathcal{C}(M)$ is $rca(M)$ (see e.g. [11]), the space of bounded, regular, countably additive, complex-valued set functions on $(M, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra. The norm on $rca(M)$ is the total variation. For given ϑ, Λ we consider rca measures whose marginals have densities $\phi_{\Lambda|(S^1)^\Lambda}$ over $(S^1)^\Lambda$ (restriction of ϕ_Λ to $(S^1)^\Lambda$) s.t. $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}} \in \mathcal{H}_\vartheta$. We remark that not every $\phi \in \mathcal{H}_\vartheta$ with real-valued $\phi_{\Lambda|(S^1)^\Lambda}$ corresponds to an element in $rca(M)$ because its variation might not be bounded as $\int_\Lambda d\mu^\Lambda|\phi_\Lambda|$ might be unbounded with Λ. So we define for $\phi \in \mathcal{H}$

$$\|\phi\|_{var} \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda |\phi_\Lambda|. \tag{1.22}$$

We set $\mathcal{H}^{bv} \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} : \|\phi\|_{var} < \infty\}$ and $\mathcal{H}^{bv}_\vartheta \stackrel{\text{def}}{=} \mathcal{H}^{bv} \cap \mathcal{H}_\vartheta$. In particular all real-analytic and non-negative $\phi \in \mathcal{H}$, i.e. $\phi_{\Lambda|(S^1)^\Lambda} \geq 0$ for all $\Lambda \in \mathcal{F}$, belong to this space.

We can view every $\phi \in \mathcal{H}^{bv}$ as an element of $rca(M)$: For $g \in \mathcal{C}(M)$ the net $(g_\Lambda)_{\Lambda\in\mathcal{F}}$ given by $g_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g)$ converges uniformly to g. We set

$$\phi(g) \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda g_\Lambda \phi_\Lambda. \tag{1.23}$$

The limit exists because for $\Lambda_1 \subset \Lambda_2$

$$\begin{aligned} & \left| \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g_{\Lambda_1}\phi_{\Lambda_1} - \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} g_{\Lambda_2}\phi_{\Lambda_2} \right| \\ = & \left| \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} (g_{\Lambda_1} - g_{\Lambda_2})\phi_{\Lambda_2} \right| \\ \leq & \|g_{\Lambda_1} - g_{\Lambda_2}\|_{(S^1)^{\Lambda_2}} \|\phi\|_{var} \end{aligned} \tag{1.24}$$

gets arbitrarily small as $\Lambda_1 \to \mathbb{Z}^d$, i.e. the net has the Cauchy property.
We further see

$$\begin{aligned} \|\phi\|_{var} &= \sup_{\Lambda\in\mathcal{F}} \int_{(S^1)^\Lambda} d\mu^\Lambda |\phi_\Lambda| \\ &= \sup_{\Lambda\in\mathcal{F}} \sup_{\substack{g\in\mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty\leq 1}} \int_{(S^1)^\Lambda} d\mu^\Lambda g\,\phi_\Lambda \\ &= \sup_{\substack{g\in\mathcal{C}(M) \\ \|g\|_\infty\leq 1}} |\phi(g)|, \end{aligned} \tag{1.25}$$

so $\|\phi\|_{var}$ is in fact the total variation (the operator-norm, cf. [11]) of the corresponding linear functional on $\mathcal{C}(M)$.
Let $\mathcal{H}(\mathcal{F}) \stackrel{\text{def}}{=} \bigcup_{\Lambda\in\mathcal{F}} H(A_\delta^\Lambda)$ be the subspace of functions depending on only finitely many variables. We define the product $g^1\phi \in \mathcal{H}_\vartheta$ of $g^1 \in \mathcal{H}(A_\delta^{\Lambda_1})$ and $\phi \in \mathcal{H}_\vartheta$ by

$$(g^1\phi)_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g^1\phi_{\Lambda_1\cup\Lambda}). \tag{1.26}$$

Lemma 1.4.1 *If $g^1 \in H(A_\delta^{\Lambda_1})$, $g^2 \in H(A_\delta^{\Lambda_2})$, $g \in \mathcal{C}(M)$ and $\phi \in \mathcal{H}_\vartheta$ the following holds*

1. *The product in (1.26) is well-defined and $\|g^1\phi\|_\vartheta \leq \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta$.*
2. *$(g^1g^2)\phi = g^1(g^2\phi)$.*
3. *g^2 can be considered as an element of $\mathcal{H}_\vartheta$ and the product g^1g^2 as defined in (1.26) is the same as the usual product between functions on M.*
4. *$(g^1\phi)(g) = \phi(g^1g)$ where $(g^1\phi)$ and ϕ act as functionals in the sense of (1.23).*
5. *$\mathcal{H}_\vartheta^{bv}$ is also a module over the ring $\mathcal{H}(\mathcal{F})$, i.e. in particular $\|g^1\phi\|_{var} \leq \|g^1\|_{\Lambda_1}\|\phi\|_{var}$.*

1.5 Expansion of the Perron-Frobenius Operator

We split the integral kernel of the Perron-Frobenius operator for a finite-dimensional system. Recall that $T_p^\epsilon(\mathbf{z}) = z_p \exp\left(2\pi i\epsilon\sum_{k=1}^\infty g_{p,k}(\mathbf{z})\right)$
$= z_p\prod_{k+1}^\infty \exp(2\pi i\epsilon g_{p,k}(\mathbf{z}))$ and that $S_p(\mathbf{z}) = f_p \circ T_p^\epsilon(\mathbf{z})$.
If we consider only finite range interaction, say up to distance l, we have

$$T_{p,l}^\epsilon(\mathbf{z}) \stackrel{\text{def}}{=} z_p \exp(2\pi i\epsilon\sum_{k=1}^{l} g_{p,k}(\mathbf{z})). \tag{1.27}$$

For a finite-dimensional system (say on $(S^1)^{\Lambda_2}$) with fixed boundary conditions we have a special representation of $\mathcal{L}_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$ in terms of the integral kernel (Proposition 1.3.3).

Lemma 1.5.1 *For the factors in the integral kernel in (1.21) we have the following splitting :*

$$\begin{aligned}\frac{1}{f_p\circ T_p^\epsilon(\mathbf{z}) - w_p} f_p\circ T_p^\epsilon(\mathbf{z}) &= \frac{1}{f_p(z_p) - w_p} f_p(z_p) \\ &+ w_p\sum_{k=1}^\infty \frac{f_p\circ T_{p,k-1}^\epsilon(\mathbf{z}) - f_p\circ T_{p,k}^\epsilon(\mathbf{z})}{\left(f_p\circ T_{p,k-1}^\epsilon(\mathbf{z}) - w_p\right)\left(f_p\circ T_{p,k}^\epsilon(\mathbf{z}) - w_p\right)}.\end{aligned} \tag{1.28}$$

The sum in the right hand side converges uniformly in $\mathbf{z} \in \Gamma^\Lambda$ *and* $w_p \in A_\delta$.

1.5.1 The Unperturbed Operator

The first summand in (1.28) is just the one which appears in the uncoupled system (i.e. $T^{\epsilon=0} = \mathrm{id}$) and in this case each lattice site can be considered separately. We denote by $\mathcal{L}_{f_p}$ the restriction of the Perron-Frobenius operator to the Banach space of functions on S^1 that extend continuously on the closed annulus A_δ and holomorphically on the interior A_δ. $\|\cdot\|_{A_\delta}$ denotes the uniform norm over A_δ. The operator

$$\mathcal{L}_{f_p} : (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta}) \to (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta})$$

has 1 as simple eigenvalue and the rest of its spectrum is contained in a disc around 0 of radius strictly smaller than 1. It splits into

$$\mathcal{L}_{f_p} = Q_p + R_p \tag{1.29}$$

with

$$R_p Q_p = Q_p R_p = 0 \tag{1.30}$$

and

$$\|R_p^n\|_{L(\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta})} \leq c_r \eta^n \tag{1.31}$$

with $c_r > 0$, $0 < \eta < 1$. For proofs of these statements see e.g. [3].
Q_p is the projection onto the one-dimensional eigenspace spanned by $h_p \in \mathcal{H}(A_\delta)$, whose restriction to S^1 is positive and has integral $\int_{S^1} d\mu\, h_p = 1$.
We assume in Assumption **I** regarding the family $(f_p)_{p \in \mathbf{Z}^d}$ that

$$\|h_p\|_{A_\delta} \leq c_h \tag{1.32}$$

and the exponential bound in (1.31) both hold uniformly in p. This is the case for example if the f_p are uniformly close to each other as is shown using analytic perturbation theory.
$\mathcal{L}_{f_p}$ preserves the integral (cf. (1.19)) and so does Q_p, as follows e.g. from (1.29)-(1.31). Since Γ_+ is homologous to S^1 we can write Q_p as

$$\begin{aligned} Q_p g(w) &= h_p(w) \int_{S^1} d\mu\, g & (1.33) \\ &= h_p(w) \int_{\Gamma_+} \frac{dz}{2\pi i} \frac{1}{z} g(z) \\ &= \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z} h_p(w, z) g(z) & (1.34) \end{aligned}$$

where we have used that g is holomorphic in A_δ and defined:

$$h_p(w_p, z_p) \stackrel{\text{def}}{=} \begin{cases} h_p(w_p) & \text{for} \quad z_p \in \Gamma_+ \\ 0 & \text{for} \quad z_p \in \Gamma_- \end{cases} \tag{1.35}$$

The idempotency $Q_p^2 = Q_p$ results in the integral representation

$$\int_\Gamma \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_\Gamma \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} h_p(w_p, z_p^2) h_p(z_p^2, z_p^1) g(z_p^1) = \int_\Gamma \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} h_p(w_p, z_p^1) g(z_p^1). \tag{1.36}$$

Here and throughout the section the upper indices in z_p^1, z_p^2 etc. refer to the temporal and the lower ones to the spatial coordinate in the space-time lattice $\mathbb{Z} \times \mathbb{Z}^d$. According to Proposition 1.3.3 the operator R_p can be written

$$R_p g(w_p) = \int_\Gamma \frac{dz}{2\pi i} \frac{1}{z} r_p(w_p, z_p) g(z_p) \tag{1.37}$$

with

$$r_p(w_p, z_p) = \frac{1}{f_p(z) - w_p} f_p(z_p) - h_p(w_p, z_p). \tag{1.38}$$

Then equation (1.30) results in the integral representation

$$\int_\Gamma \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_{S^1} \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} r_p(w_p, z_p^2) h_p(z_p^2, z_p^1) g(z_p^1) \;=\; 0, \tag{1.39}$$

$$\int_{S^1} \frac{dz_p^2}{2\pi i} \frac{1}{z_p^2} \int_\Gamma \frac{dz_p^1}{2\pi i} \frac{1}{z_p^1} r_p(z_p^2, z_p^1) g(z_p^1) \;=\; 0. \tag{1.40}$$

1.5.2 The Perturbed Operator

In view of (1.28) we set

$$\beta_{p,k}(w_p, \mathbf{z}) \stackrel{\text{def}}{=} w_p \frac{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p)} \quad . \tag{1.41}$$

This corresponds to the difference between the operators for systems with interaction of finite-range of order k and $k-1$, respectively. Using (1.1) we have the estimate

$$\begin{aligned} &|\beta_{p,k}(w_p, \mathbf{z})| \\ &\leq\; |w_p| \, |f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p|^{-1} |f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p|^{-1} \\ &\qquad \times |f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})| \\ &\leq\; (1+\delta)|c_7\lambda - 1|^{-1}|c_7\lambda - 1|^{-1} \|f_p'\|_{\{p\}} c_1 \epsilon \exp(-c_2 k^d) \\ &\leq\; \tilde{c}_8 \epsilon \exp(-c_2 k^d) \end{aligned} \tag{1.42}$$

uniformly in $p \in \mathbb{Z}^d$, $w_p \in A_\delta$, $\mathbf{z} \in \Gamma^\Lambda$.

1.5.3 Time N Step

Now we want to estimate the norm of (1.6) or equivalently that of

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} : (\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}) \tag{1.43}$$

$$\begin{aligned} \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}\phi(\mathbf{z}^0) &= \int_{\Gamma^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{z}^{-1}) \cdots \int_{\Gamma^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{z}^{-N}) \prod_{t=-N}^{-1} \prod_{p\in\Lambda_2} \\ &\times \left(h_p(z_p^{t+1}, z_p^t) + r_p(z_p^{t+1}, z_p^t) + \sum_{k=1}^{\infty} \beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)\right) \phi(\mathbf{z}^{-N}) \end{aligned} \tag{1.44}$$

(cf. also the beginning of Section 1.3.)
Distributing the product we get infinitely many summands. In each factor there is for each $-N \le m \le -1$, $p \in \Lambda_2$ a choice between h_p, r_p and $\beta_{p,k}$ $(1 \le k < \infty)$ and we can interpret such a choice graphically as a *configuration* (similar objects were introduced in [26] where they were named polymers):
On $\Lambda_2 \times \{-N, \dots, 0\}$ we represent

- $h_p\left(z_p^{t+1}, z_p^t\right)$ by an *h-line* from (p,t) to $(p,t+1)$
- $r_p\left(z_p^{t+1}, z_p^t\right)$ by an *r-line* from (p,t) to $(p,t+1)$

Figure 1.1: h-line and r-line

- $\beta_{p,k}\left(z_p^{t+1}, \mathbf{z}^t\right)$ by a *k-triangle* (actually rather a cone or pyramid but in our pictures for $d = 1$ it is a triangle) with apex $(p, t+1)$ and base points (q,t) with $\|p - q\| \le k$. (So some of the base points might not lie in $\Lambda_2 \times \{-N, \dots, -1\}$ but all the apices lie in $\Lambda_2 \times \{-N+1, \dots, 0\}$.)

Note that if

$$v(k) \stackrel{\text{def}}{=} |B_k(0)| \tag{1.45}$$

denotes the number of base points of a k-triangle, we have the estimate $v(k) \le (3k)^d$. Each summand, that we get by distributing the product in (1.44), corresponds to a configuration and for each configuration C we have an operator $\mathcal{L}_C$. So we can write

$$\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} = \sum_C \mathcal{L}_C. \tag{1.46}$$

Some of these summands are zero namely if

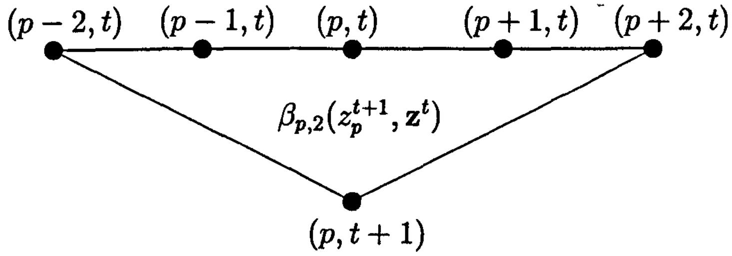


Figure 1.2: 2-triangle

- a factor $h_p\left(z_p^{t+2}, z_p^{t+1}\right) r_p\left(z_p^{t+1}, z_p^t\right)$ or $r_p\left(z_p^{t+2}, z_p^{t+1}\right) h_p\left(z_p^{t+1}, z_p^t\right)$ appears, but no factor $\beta_{q,k}\left(z_q^{t+2}, \mathbf{z}^{t+1}\right)$ with $\|p-q\| \leq k$ (i.e. an h-line follows or is followed by an r-line and at their common endpoint no triangle is attached with any of its basepoints. cf. Figure 1.3.) This follows since, by Fubini's Theorem, one can first perform the $dz_p^{t+1}dz_p^t$-integration and get zero by (1.39) or (1.40). (Note that the other factors in the integrand do not depend on z_p^{t+1}. So they can be considered as the function $g(z_p^1)$ in (1.39) or (1.40).)

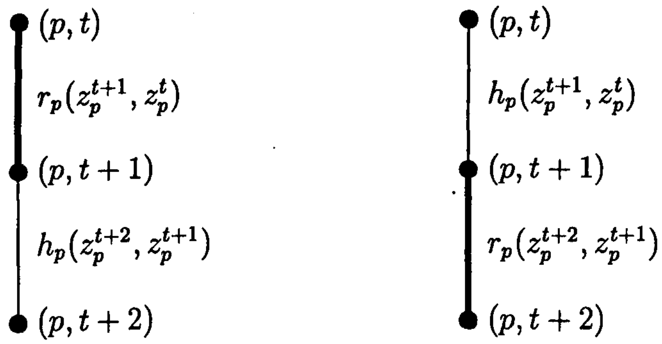


Figure 1.3: Consecutive r-line and h-line

- if a term $h_p\left(z_p^{t+2}, z_p^{t+1}\right) \beta_{p,k}\left(z_p^{t+1}, \mathbf{z}^t\right)$ appears but no $\beta_{q,l}\left(z_q^{t+2}, \mathbf{z}^{t+1}\right)$ with $\|p-q\| \leq l$ (i.e. a triangle is followed by an h-line and at their common endpoint (the apex of the triangle) no other triangle is attached with any of its basepoints. Cf. Figure 1.4.) Indeed:

$$\beta_{p,k}\left(w_p, \mathbf{z}\right) = w_p \left[\frac{1}{f_p \circ T_{p,k}^{\epsilon}(\mathbf{z}) - w_p} - \frac{1}{f_p \circ T_{p,k-1}^{\epsilon}(\mathbf{z}) - w_p}\right] \tag{1.47}$$

By the Residue Theorem:

$$\int_{S^1} \frac{dw_p}{2\pi i} \frac{1}{w_p} \beta_{p,k}\left(w_p, \mathbf{z}\right) = 0 \tag{1.48}$$

because the poles at $w_p = f_p \circ T^\epsilon_{p,k}(\mathbf{z})$ and $w_p = f_p \circ T^\epsilon_{p,k-1}(\mathbf{z})$ (with $\mathbf{z} \in \Gamma^N$, in particular $z_p \in \Gamma_+$ or Γ_-) both lie either outside Γ_+ or inside Γ_- as f_p is expanding, $T^\epsilon_{p,k}$ is close to $T^\epsilon_{p,k-1}$ and the two summands have residue -1 and 1, respectively.

Identity (1.48) is a consequence of the fact that $\beta_{p,k}$ is the kernel of a difference between two transfer operators (for the systems with interaction of range k and $k-1$) both preserving the Lebesgue integral in the sense of (1.19). So the range of this operator difference consists of functions with integral zero and these are annihilated by the operator corresponding to h_p (cf. (1.33) and (1.34).)

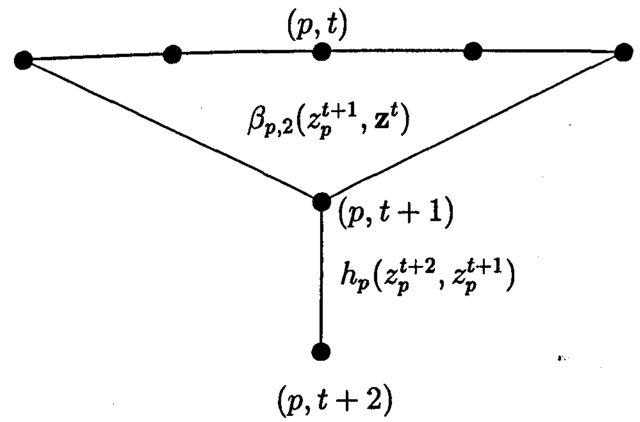


Figure 1.4: Combination 2-triangle and h-line

Furthermore we note that in

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} = \sum_C \pi_{\Lambda_1} \circ \mathcal{L}_C \tag{1.49}$$

we get $\pi_{\Lambda_1} \circ \mathcal{L}_C = 0$ unless C ends with h-lines in all points of $(\Lambda_2 \setminus \Lambda_1) \times \{0\}$ because of (1.40), (1.48) and the fact that π_{Λ_1} means integration over $(S^1)^{\Lambda_2 \setminus \Lambda_1}$.

Definition 1.5.1 We call a configuration $\mathcal{L}_C$ in the expansion (1.49) a *zero configuration* if it does not end with h-lines in all points of $(\Lambda_2 \setminus \Lambda_1) \times \{0\}$ or contains one of the constellations (consecutive r-line and h-line or k-triangle and h-line) mentioned above. Otherwise we call it a *non-zero configuration.*

Remark For a zero configuration C we have just shown that its corresponding summand in (1.49) is 0. So we just have to sum over non-zero configurations. We note that the notion non-zero configuration does not exclude that $\mathcal{L}_C = 0$.

We have to find an upper bound for the norm of each $\mathcal{L}_C$. We do so by collecting r- and h-lines into chains and estimating the contributions of integrating the factors corresponding to these parts of the configuration.

Definition 1.5.2 • Let C be a non-zero configuration with exactly $n_{\beta,k}$ k-triangles for $1 \leq k < \infty$. We define

$$n_\beta \stackrel{\text{def}}{=} (n_{\beta,1}, n_{\beta,2}, \ldots) \tag{1.50}$$

and

$$|n_\beta| \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} n_{\beta,k} < \infty. \tag{1.51}$$

- A sequence of h-lines from (p,t) to $(p,t+1)$, ..., $(p,t+k-1)$ to $(p,t+k)$ with $p \in \Lambda_2$ and $-N \leq t \leq t+k \leq 0$ such that to the points $(p,t+1)\ldots(p,t+k-1)$ no triangles are attached is called an *h-chain of length k.*
- If such an h-chain is not contained in a longer one it is called a *maximal h-chain.* Then (p,t) and $(p,t+k)$ are denoted its *endpoints.*
- The definitions for a *maximal r-chain* and its *endpoints* are analogous.
- $\tilde{\Lambda}_C$ denotes the set of points $p \in \Lambda_2$ that appear as the $\mathbb{Z}^d$-coordinate of a base point (p,t) of a triangle in C and Λ_C the set of those points $p \in \mathbb{Z}^d$ that appear as the $\mathbb{Z}^d$-coordinate of an apex (p,t) that does not lie above (i.e. having the same spatial coordinate) any other triangle.
- Λ_r is the set of $p \in \mathbb{Z}^d \setminus \tilde{\Lambda}_C$ that appear as the $\mathbb{Z}^d$ coordinate of an r-line (this implies that there is an r-chain from $(p,-N)$ to $(p,0)$ for otherwise an r-line would have a common endpoint (p,t) with an h-line and C would be a zero configuration.)
- We write $\Lambda(C) \stackrel{\text{def}}{=} \tilde{\Lambda}_C \cup \Lambda_r$.

In Figure 1.5 there are for example maximal r-chains from $(1,-3)$ to $(1,0)$ or from $(2,-3)$ to $(2,-2)$. $\Lambda_2 = \{1,\ldots,8\}$, $\tilde{\Lambda}_C = \{2,\ldots,7\}$, $\Lambda_C = \{4\}$ and $\Lambda_r = \{1\}$.
As each k-triangle has $v(k) \leq (3k)^d$ base points we have

$$|\tilde{\Lambda}_C| \leq \sum_{k=1}^{\infty} (3k)^d n_{\beta,k} \tag{1.52}$$

To get the estimate for the norm of (1.43) we proceed in the following order:

1. We integrate in $\left|\pi_{\Lambda_1} \circ \mathcal{L}_C\phi\left(\mathbf{z}^0_{\Lambda_1}\right)\right|$ over all dz^t_p for which a factor $r_p(z^{t+1}_p, z^t_p)$ appears. For each maximal r-chain of length l we get according to (1.31) a factor not greater than $c_r\eta^l$.

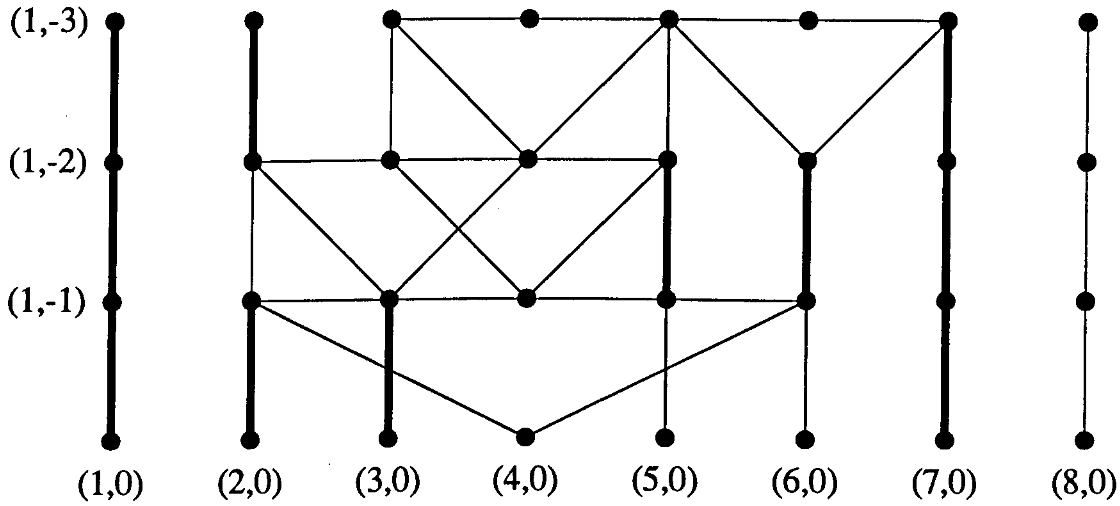


Figure 1.5: Example for a configuration

2. For each maximal h-chain starting at (p,t) and ending at $(p,t+l)$ we perform the integration

$$\int_\Gamma d\mu(z_p^{t+l-1}) \cdots \int_\Gamma d\mu(z_p^{t+1})\, h_p(z_p^{t+l}, z_p^{t+l-1}) \cdots h_p(z_p^{t+1}, z_p^t) = h_p(z_p^{t+l}). \quad (1.53)$$

3. We perform the integration corresponding to π_{Λ_1}

$$\prod_{p\in\Lambda_2\setminus\Lambda_1} \int_{S^1} d\mu(z_p^0)\, h_p(z_p^0) = 1 \quad (1.54)$$

4. We estimate the contribution of each (from step 2 and 3 remaining) factor $h_p(z_p^t)$ by $\|h_p\|_{A_\delta} \le c_h$ and, using (1.42), the contribution of each factor $\beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)$ via

$$\begin{aligned} &\left| \int_\Gamma \frac{dz_p^t}{2\pi i} \frac{1}{z_p^t} \beta_{p,k}(z_p^{t+1}, \mathbf{z}^t) \psi(z_p^t) \right| \quad (1.55) \\ \le\; & \frac{|\Gamma|}{2\pi} \frac{1}{1-\delta} \tilde{c_8} \epsilon \exp(-c_2 k^d)\, \|\psi\| \\ \le\; & c_8 \epsilon \exp(-c_2 k^d)\, \|\psi\|\,. \end{aligned}$$

Here $|\Gamma|$ denotes the euclidean length of Γ and ψ the remaining factors, containing other integrals. Finally the contribution of the factors $|\phi(\mathbf{z}^{-N})|$ is estimated by $\|\phi_{\tilde{\Lambda}_C\cup\Lambda_r}\|_{\tilde{\Lambda}_C\cup\Lambda_r}$ (cf. remark below).

Remark For all points $q \notin \tilde{\Lambda}_C \cup \Lambda_r$ we must have h-chains in C from $(q, -N)$ to $(q, 0)$. Therefore we have

$$\pi_{\Lambda_1} \circ \mathcal{L}_C \phi_{\Lambda_2}(\mathbf{z}^0_{\Lambda_1}) = \pi_{\Lambda_1} \circ \mathcal{L}_C \phi_{\tilde{\Lambda}_C \cup \Lambda_r}(\mathbf{z}^0_{\Lambda_1}) \tag{1.56}$$

where on the right-hand side we use the same notation '$\mathcal{L}_C$' for the operator on $H_{\tilde{\Lambda}_C \cup \Lambda_r, \vartheta}$.

So if n_r denotes the number of r-lines, $\tilde{n}_r$ the number of maximal r-chains and $\tilde{n}_h$ the number of maximal h-chains having spatial coordinates in $\tilde{\Lambda}_C \cup \Lambda_1$ (for otherwise they are 'integrated away' giving a factor of 1) we get, using (1.31) and (1.55),

$$\begin{aligned} &\|\pi_{\Lambda_1} \circ \mathcal{L}_C \phi\|_{\Lambda_1} \\ \leq\ & (c_8 \epsilon)^{|n_\beta|} \exp\left(-c_2 \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r} \eta^{n_r} \|\phi_{\tilde{\Lambda}_C \cup \Lambda_r}\|_{\tilde{\Lambda}_C \cup \Lambda_r} \end{aligned} \tag{1.57}$$

and, using (1.52),

$$\begin{aligned} \|\phi_{\tilde{\Lambda}_C \cup \Lambda_r}\|_{\tilde{\Lambda}_C \cup \Lambda_r} &\leq \vartheta^{-|\Lambda_r| - \sum_{k=1}^{\infty} (3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2, \vartheta} \\ &\leq \vartheta^{-|\Lambda_r|} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2, \vartheta} \end{aligned} \tag{1.58}$$

for all $\Lambda_2 \in \mathcal{F}$ and with $\|\cdot\|_{\Lambda_2, \vartheta}$ defined in (1.5).
(1.57) and (1.58) are the basic estimates for a single configuration. We use refined versions of them throughout the paper.
In particular the idea of taking the norm of $\phi_{\tilde{\Lambda}_C \cup \Lambda_r}$ rather than that of ϕ_{Λ_2} which grows with the size of Λ_2, is the key point in our analysis.

1.6 Operators for the Infinite-Dimensional System

Estimates (1.57) and (1.58) bound the particular summands in an expansion like (1.49). We see that triangles and maximal r-chains in a configuration C lead to small factors on the right-hand side of (1.57). (A maximal r-chain consisting of n r-lines contributes a factor $c_r \eta^n$. The factor c_r is greater than 1 in general. But either it will be compensated for by a small factor due to a triangle e.g. as in (1.99) or n will be large, cf. e.g. (1.103)). This motivates the following definition of the length of a configuration. The length gives rise to a lower bound for the number of triangles or r-lines, i.e. a long configuration will lead to a small contribution in the total sum in (1.49).

Definition 1.6.1 • The *length*, length($\mathcal{C}$), of a non-zero configuration $\mathcal{C}$ (that we got in an expansion like (1.46)) is the maximal difference $0 - t$ such that there are points (p, t) and $(q, 0)$ being end-points of r-lines or base points or apices of triangles. (Note that if there are any triangles or r-lines, there is also a triangle or an r-line ending at $\Lambda \times \{0\}$.) If there are no triangles or r-lines in $\mathcal{C}$ its length is zero.

- We *identify* two non-zero configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ if they agree in their triangles, r-lines and their number of max h-chains that go upwards from base-points of triangles (but might be defined on space-time boxes $\Lambda_2 \times \{-t_0, \ldots, 0\}$ of different sizes, i.e. with different Λ_2 and t_0). We still speak of configurations rather than equivalence classes. For a configuration $\mathcal{C}$ length($\mathcal{C}$), $\tilde{\Lambda}_\mathcal{C}$, $\Lambda(\mathcal{C})$ (as in the Definition 1.5.2) and the operator

  $\pi_\Lambda \circ \mathcal{L}_\mathcal{C} \in L((\mathcal{H}(A_\delta^{\Lambda(\mathcal{C})}), \|\cdot\|_{\Lambda(\mathcal{C})}), (\mathcal{H}(A_\delta^\Lambda), \|\cdot\|_\Lambda))$ is well-defined.

- For $\Lambda_1 \in \mathcal{F}$ we define $E(\Lambda_1)$ as the set of all non-zero configurations $\mathcal{C}$ in some $\Lambda_2 \times \{-t_0, \ldots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$, $t_0 \in N$ and $t_0 >$ length($\mathcal{C}$), and that do not end in $\Lambda_1 \times \{0\}$ with triangles or r-lines.

- $E_N(\Lambda_1)$ is the set of non-zero configurations $\mathcal{C}$ in $\Lambda_2 \times \{-N, \ldots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$ and $\Lambda(\mathcal{C}) \subseteq \Lambda_2$.

We define

$$\nu_\Lambda \stackrel{\text{def}}{=} \sum_{\mathcal{C} \in E(\Lambda)} \pi_\Lambda \circ \mathcal{L}_\mathcal{C} h_{\Lambda(\mathcal{C})}. \tag{1.59}$$

The convergence of this infinite sum and other properties of ν are stated in the following proposition additional to Theorem 1.2.1.

Proposition 1.6.1 *Let ϑ, the sequence of ϑ_i, ϵ, c_2, N_0 and Λ_1 be as in Theorem 1.2.1 and $N \geq N_0$.*

1.

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F \circ T^\epsilon} = \sum_{\mathcal{C} \in E_N(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_\mathcal{C}. \tag{1.60}$$

2.

$$\|\mathcal{L}^N_{F \circ T^\epsilon} - \mathcal{L}^{N+1}_{F \circ T^\epsilon}\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta))} \leq c_9 \tilde{\eta}^N. \tag{1.61}$$

3. For $N_1, N_2 \in \mathbb{N}$ the operator $\mathcal{L}^{N_2}_{F \circ T^\epsilon}$ is defined on $\mathcal{L}^{N_1}_{F \circ T^\epsilon}(\mathcal{H}_\vartheta) \subset \mathcal{H}_{\vartheta_{N_1}}$. It maps this space to $\mathcal{H}_{\vartheta_{N_1+N_2}}$ and

$$\mathcal{L}^{N_2}_{F \circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F \circ T^\epsilon} = \mathcal{L}^{N_1+N_2}_{F \circ T^\epsilon}. \tag{1.62}$$

4. *For $\phi \in \mathcal{H}^{bv}_\vartheta$ we have the estimate*

$$\|\mathcal{L}_{F\circ T^\epsilon}\phi\|_{var} \leq \|\phi\|_{var}. \tag{1.63}$$

For $g \in \mathcal{C}(M)$ and $\phi \in \mathcal{H}^{bv}_\vartheta$ we have the identity

$$\int_M d\mu\, g \circ S\, \phi = \int_M d\mu\, g \mathcal{L}_{F\circ T^\epsilon}\phi \tag{1.64}$$

and in particular

$$\mu(\phi) = \mu(\mathcal{L}_{F\circ T^\epsilon}\phi). \tag{1.65}$$

For finite-range interaction the inequality and both equations also hold for $\phi \in \mathcal{H}^{bv}$.

5. *$\mathcal{L}_{F\circ T^\epsilon}$ is non-negative, i.e. $\phi \geq 0$ implies $\mathcal{L}_{F\circ T^\epsilon}\phi \geq 0$. ($\phi \geq 0$ means $\phi_{\Lambda|(S^1)^\Lambda} \geq 0$ for all $\Lambda \in \mathcal{F}$.)*

1.7 Decay of Correlations

We have found the unique invariant $\nu \in \mathcal{H}_\vartheta$ with $\mu(\nu) = 1$. This corresponds to a non-negative measure on $(M, \mathcal{B})$ whose marginal on $(S^1)^\Lambda$ has density $\nu^\Lambda_{|(S^1)^\Lambda}$ wrt. μ^Λ. In the next theorem we state the decay of correlation for ν in terms of the weighted norms. We will use these results in the proof of Theorem 1.2.2.

Theorem 1.7.1 *For sufficiently small ϑ and ϵ, big c_2, finite disjoint Λ_1, Λ_2 and $f \in H(A^{\Lambda_2}_\delta)$ there are a $\kappa \in (0,1)$ and a $\tilde{\vartheta} \in (0,1)$ such that*

1. $\|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2,\vartheta} \leq c_{10}\kappa^{dist(\Lambda_1,\Lambda_2)}$,
2. $\|\pi_{\Lambda_1}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\vartheta} \leq c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{dist(\Lambda_1,\Lambda_2)}$,
3. $\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\tilde{\vartheta}} \leq c_{12}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{dist(\Lambda_1,\Lambda_2)}\tilde{\eta}^N$
   for every $N \geq 0$.

Remark As in Theorem 1.2.2 we can choose the rate of decay κ first and then the other parameters.

1.8 Proofs

In the proof of Proposition 1.3.1 we use the following lemma which is rather standard in real analysis. Here we formulate it in the setting of holomorphic functions.

Lemma 1.8.1 *If $T : U \to \mathbb{C}^n$ is a holomorphic map on a convex set $U \subset \mathbb{C}^n$ and satisfies the estimate $\|DT(z) - id\| \le c_{18} < 1$ then T is biholomorphic onto its image (in this lemma the chosen norm on $\mathbb{C}^n$ and the corresponding operator norm are both denoted by $\|\cdot\|$).*

Proof T is locally biholomorphic by the Inverse Function Theorem. So we only have to show injectivity. Let $z^0, z^1 \in U$ with $T(z^0) = T(z^1)$ and $\gamma : [0,1] \to U$, $\gamma(t) = z^0 + t(z^1 - z^0)$. Then

$$\begin{aligned} \|z^1 - z^0\| &= \|T(z^1) - z^1 - T(z^0) + z^0\| \\ &= \|T \circ \gamma(1) - \gamma(1) - T \circ \gamma(0) + \gamma(0)\| \\ &= \left\| \int_0^1 (DT(\gamma(t)) - \mathrm{id}) (z^1 - z^0)\, dt \right\| \\ &\le \|z^1 - z^0\| \int_0^1 \|DT(\gamma(t)) - \mathrm{id}\|\, dt \\ &\le \|z^1 - z^0\|\, c_{18} \end{aligned} \tag{1.66}$$

which implies $z^1 = z^0$.

□

Proof of Proposition 1.3.1 We have a Cauchy estimate for the partial derivatives of the functions $g_{p,k} : A_\delta^{B_k(p)} \to \mathbb{C}$ on a smaller polyannulus. Let $q \in B_k(p)$, Then

$$\left\| \frac{\partial}{\partial z_q} g_{p,k} \right\|_{A_\delta^{B_k(p)}} \le \frac{1}{|e^\delta - e^{\delta_1}|} c_1 \exp(-c_2 k^d) \tag{1.67}$$

$$= c_{13} \exp(-c_2 k^d). \tag{1.68}$$

Also note that $\frac{\partial}{\partial z_q} g_{p,k} = 0$ for $q \notin B_k(p)$. Therefore

$$\begin{aligned} \left\| \frac{\partial}{\partial z_q} g_p \right\|_{A_\delta^{\mathbb{Z}^d}} &= \left\| \frac{\partial}{\partial z_q} \sum_{k=\|p-q\|}^{\infty} g_{p,k} \right\|_{A_{\delta_1}^{\mathbb{Z}^d}} \\ &\le c_{13} \sum_{k=\|p-q\|}^{\infty} \exp(-c_2 k^d) \\ &\le c_{13} \frac{1}{1 - \exp(-c_2)} \exp\left(-c_2 \|p-q\|^d\right) \\ &= c_{14} \exp\left(-c_2 \|p-q\|^d\right). \end{aligned} \tag{1.69}$$

Now we consider the lift given by $pr : \mathbb{C}_\delta^\Lambda \to A_\delta^\Lambda$, $(\tilde{z}_p)_{p\in\Lambda} \mapsto \left(e^{i\tilde{z}_p}\right)_{p\in\Lambda}$, where $\mathbb{C}_\delta \stackrel{\text{def}}{=} \{w \in \mathbb{C} \mid \text{Im} w \in [-\delta, \delta]\}$.

Then we have for the lifted functions $\left(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}})\right)_p = \tilde{z}_p + 2\pi\epsilon\tilde{g_p}(\tilde{\mathbf{z}})$. The function $\tilde{g_p}(\tilde{\mathbf{z}}) = g_p(pr(\tilde{\mathbf{z}}))$ satisfies the same estimate (1.1) with a different constant $\tilde{c_1}$ for $\delta < \delta_1$ sufficiently small since pr and its partial derivatives are uniformly bounded on $\mathbb{C}_\delta^\Lambda$.

Then we have

$$\left| D\left(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}})\right)_{p,q} - \delta_{p,q} \right| \leq 2\pi\epsilon\tilde{c_1}\exp\left(-c_{13}\|p-q\|^d\right).$$

In particular the row sum norm (the operator-norm induced by the l^∞-norm on $\mathbb{C}^\Lambda$) of $\left(D\widetilde{T^{\Lambda,\epsilon}} - \text{id}\right)$ is smaller than 1 for ϵ small enough, independent of Λ. According to Lemma 1.8.1 (note that $\mathbb{C}_\delta$ is convex), $\widetilde{T^{\Lambda,\epsilon}}$ is a biholomorphic map onto its image and so is $T^{\Lambda,\epsilon}$.

Now fix $\delta < \delta_1$ according to the first part of the proof. If $\mathbf{z} \in \partial A_\delta^\Lambda$ we have $z_p \in \partial A_\delta$ for at least one $p \in \Lambda$. From the formula $z_p' \stackrel{\text{def}}{=} T_p^{\Lambda,\epsilon}(\mathbf{z}) = z_p \exp(2\pi i\epsilon g_p(\mathbf{z}))$ and the assumption that g_p is uniformly bounded on A_{δ_1} we see that

$$\left|\ln|z_p'|\right| \geq \delta - c_{16}\delta\epsilon > c_7\delta \tag{1.70}$$

for sufficiently small ϵ.

Now assume $\emptyset \neq A_{c_7\delta} \setminus T^{\Lambda,\epsilon}(A_\delta) \ni \mathbf{z}$. Let s be the line-segment between $\mathbf{z}$ and its nearest point $\mathbf{w}$ on $(S^1)^\Lambda$ (wrt. the metric d_Λ). For each point $\mathbf{y}$ on s the inequality $\ln d_\Lambda(\mathbf{w},\mathbf{y}) \leq \ln d_\Lambda(\mathbf{w},\mathbf{z}) \leq c_{17}\delta$ holds.

In particular there is a $\mathbf{y} \in T^{\Lambda,\epsilon}\left(\partial A_\delta^\Lambda\right)$ on s with $|y_p| \leq c_7\delta$ for all $p \in \Lambda$, but this contradicts the estimate (1.70) above.

□

Proof of Proposition 1.3.2 As F acts on each coordinate separately by an f_p we have (in view on the chosen metric (1.15)) to show the statement just for the map f (we drop the index p), i.e. the case when Λ contains just one element.

Consider the lift $\mathbb{R}_\delta \times \mathbb{R} \ni (r,\phi) \mapsto re^{i\phi}$ where $\mathbb{R}_\delta \stackrel{\text{def}}{=} [1-\ln\delta, 1+\ln\delta]$. This defines (modulo $(0,2\pi)$) a $(0,2\pi)$-periodic map $\tilde{f} = \left(\tilde{f}_r, \tilde{f}_\phi\right)$ via $f\left(re^{i\phi}\right) = \tilde{f}_r(r,\phi)e^{i\tilde{f}_\phi(r,\phi)}$. On $\{1\}\times\mathbb{R}$ one has $\frac{\partial}{\partial r}\tilde{f}_r \geq \lambda_0$ and so because of periodicity and a compactness argument, $\frac{\partial}{\partial r}\tilde{f}_r \geq \lambda$ on a thin ($0 < \delta < \delta_0$ small) strip $\mathbb{R}_\delta \times \mathbb{R}$. It follows similarly, as in the proof of Proposition 1.3.1, that $\tilde{f}(\mathbb{R}_\delta \times \mathbb{R}) \supset \mathbb{R}_{\lambda\delta} \times \mathbb{R}$, $\tilde{f}$ is diffeomorphic onto its image and each point in $\mathbb{R}_\delta \times \mathbb{R}$ has the same number of preimages (which is equal to $\left(\tilde{f}(1,2\pi) - \tilde{f}(1,0)\right)/2\pi$). From this the claim about f follows.

□

Proof of Proposition 1.3.3 We substitute the expression (1.21) into the right-hand side of equation (1.18) and get

$$\int_{(S^1)^\Lambda} \frac{d\mathbf{w}}{(2\pi i)^{|\Lambda|}} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \int_{\Gamma^\Lambda} \frac{d\mathbf{z}}{(2\pi i)^{|\Lambda|}} \phi(\mathbf{z}) \prod_{p\in\Lambda} \left(\frac{1}{S_p^\epsilon(\mathbf{z}) - w_p} \frac{S_p^\epsilon(\mathbf{z})}{z_p} \right). \tag{1.71}$$

To simplify notation we assume that $\Lambda = \{1, \ldots, N\}$. As (1.18) is linear in ψ we can assume (by using a continuous partition of unity) that ψ vanishes outside a small set $K \subset (S^1)^N$ having distinct preimages under S^t (for all $0 \le t \le \epsilon$) contained in $K_\alpha = K_{\alpha_1} \times \cdots \times K_{\alpha_N}$ such that each K_α is contained in a polydisc $D_\alpha = D_{\alpha_1} \times \cdots \times D_{\alpha_N}$. These are mutually disjoint and $S_\alpha^t \stackrel{\text{def}}{=} S^t_{|D_\alpha}$ is biholomorphic onto its image (for all $0 \le t \le \epsilon$). (To make this more precise we note that for $t = 0$ the map S^0 is the product of maps f_i $(1 \le i \le N)$ and each f_i gives rise to an M_i-fold covering map of A_δ. So locally we can index the disjoint preimages of K under S^0 by $\alpha = (\alpha_1, \ldots, \alpha_N)$ where $1 \le \alpha_i \le M_i$. If we take the set K small enough this is still true under small $(0 \le t \le \epsilon)$ perturbations.)
For given $\mathbf{w} \in K$, index α as above, $k \in \{1, \ldots, N\}$ and fixed $z_l \in A_{\delta_1}$ $(l \neq k)$ the function $z_k \mapsto (S_k^\epsilon(z_1, \cdots, z_k, \cdots, z_N) - w_k)^{-1}$ has exactly one simple pole in each D_{α_k} and is holomorphic in $A_{\delta_1}^\Lambda$ away from these poles. Therefore we get the same if we just integrate around these poles.

$$= \int_K \frac{d\mathbf{w}}{(2\pi i)^N} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \sum_\alpha \left(\prod_{k=1}^N \int_{\partial D_{\alpha_k}} \frac{dz_k}{2\pi i} \right) \phi(\mathbf{z}) \prod_{k=1}^N \frac{S_{\alpha,k}^\epsilon(\mathbf{z})}{z_k} \prod_{k=1}^N \frac{1}{S_{\alpha,k}^\epsilon(\mathbf{z}) - w_k}. \tag{1.72}$$

For each α we can write each of the inner integrals as an integral of a differential form over the distinguished boundary $b_0 D_\alpha \stackrel{\text{def}}{=} \partial D_{\alpha_1} \times \ldots \times \partial D_{\alpha_N}$, parameterized by $[0,1)^N \ni t \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_N})$, whence

$$\int_{b_0 D_\alpha} \phi(\mathbf{z}) \prod_{k=1}^N \frac{S_{\alpha,k}^\epsilon(\mathbf{z})}{z_k} \prod_{k=1}^N \frac{1}{S_{\alpha,k}^\epsilon(\mathbf{z}) - w_k} dz_1 \wedge \ldots \wedge dz_N. \tag{1.73}$$

We want to split the singular factor into a product of single poles in each variable. So we apply the transformation $\mathbf{u} = S_\epsilon(\mathbf{z}) \stackrel{\text{def}}{=} S_\alpha^\epsilon(\mathbf{z})$ to get:

$$\int_{S_\epsilon(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^N \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^N \frac{1}{u_k - w_k} du_1 \wedge \ldots \wedge du_N \tag{1.74}$$

where $(S_\epsilon^{-1})'$ is the complex derivative and so is holomorphic in $\mathbf{u}$. To apply Cauchy's formula we have to integrate over a product of cycles (each lying in $\mathbb{C}$). For example $b_0 D$ or $S_0(b_0 D)$ are such products of cycles, but $S_\epsilon(b_0 D)$ in general is not. So first we have to deform $S_\epsilon(b_0 D)$ into $S_0(b_0 D)$. The map $t \mapsto S_t \stackrel{\text{def}}{=} S_\alpha^t$ is a smooth homotopy between S_ϵ and the product map S_0 and avoids singularities of the integrand in (1.74) since for ϵ small enough the set $\{S_t(b_0 D_\alpha) \mid 0 \le t \le \epsilon\}$ has positive distance (uniformly in Λ) from the set of singularities $\bigcup_{k=1}^N \{u \in D_\alpha : u_k = w_k\}$. $S_0(b_0 D_\alpha) = S_{0,1}(\partial D_{\alpha_1}) \times \ldots \times S_{0,N}(\partial D_{\alpha_N})$ is a product of cycles and hence a cycle. The differential n-form in (1.74) is a cocycle because its coefficient is holomorphic. So we get by Stokes' theorem

$$= \int_{S_0(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^N \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^N \frac{1}{u_k - w_k} du_1 \wedge \ldots \wedge du_N \quad (1.75)$$

and by Cauchy's formula

$$= \phi \circ S_\epsilon^{-1}(\mathbf{w}) \prod_{k=1}^N \frac{w_k}{(S_\epsilon^{-1}(\mathbf{w}))_k} \frac{1}{\det(S_\epsilon'(S_\epsilon^{-1}(\mathbf{w})))}. \quad (1.76)$$

So (1.72) is equal to

$$\sum_\alpha \int_K \frac{d\mathbf{w}}{(2\pi i)^N} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \phi \circ (S_\alpha^\epsilon)^{-1}(\mathbf{w}) \frac{1}{\det(S^\epsilon)'((S_\alpha^\epsilon)^{-1}(\mathbf{w}))} \prod_{k=1}^N \frac{w_k}{((S_\alpha^\epsilon)^{-1}(\mathbf{w}))_k}. \quad (1.77)$$

For each index α, the coordinate transformation $\mathbf{u} = (S_\alpha^\epsilon)^{-1}(\mathbf{w})$ yields

$$= \sum_\alpha \int_{K_\alpha} \frac{d\mathbf{u}}{(2\pi i)^N} \frac{1}{\mathbf{u}} \psi \circ S_\alpha^\epsilon(\mathbf{u}) \phi(\mathbf{u}). \quad (1.78)$$

As $\psi \circ F = 0$ outside $\bigcup_\alpha K_\alpha$ and the K_α are mutually disjoint this equals

$$\begin{aligned} &= \int_{(S^1)^N} \frac{d\mathbf{u}}{(2\pi i)^N} \frac{1}{\mathbf{u}} \psi \circ S(\mathbf{u}) \phi(\mathbf{u}) && (1.79) \\ &= \int_{(S^1)^N} d\mu^N \psi \circ S \, \phi && (1.80) \end{aligned}$$

as was to be shown.

□

Proof of Lemma 1.4.1 Consistency follows from

$$\begin{aligned} \pi_{\Lambda_3}(g^1\phi)_{\Lambda_4} &= \pi_{\Lambda_3} \circ \pi_{\Lambda_4}(g^1\phi_{\Lambda_1 \cup \Lambda_4}) && (1.81) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda_1 \cup \Lambda_4}) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda_1 \cup \Lambda_3}) \\ &= (g^1\phi)_{\Lambda_3} \end{aligned}$$

for all $\Lambda_3 \subset \Lambda_4 \in \mathcal{F}$.
As g^1 depends only on the Λ_1-coordinates we have

$$\begin{aligned} \|(g^1\phi)_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} &= \|g^1\phi_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} \\ &\leq \|g^1\|_{\Lambda_1}\|\phi_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} \\ &\leq \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|-|\Lambda|}\|\phi\|_\vartheta \end{aligned} \tag{1.82}$$

and so

$$\vartheta^{|\Lambda|}\|(g^1\phi)_\Lambda\|_\Lambda \leq \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta \tag{1.83}$$

and

$$\|g\phi\|_\vartheta \leq \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta. \tag{1.84}$$

For Λ_1 fixed the product is continuous in both factors.
(2.) follows from

$$\begin{aligned} ((g^1g^2)\phi)_\Lambda &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2}) \\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2})) \\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2\phi)) \\ &= (g^1(g^2\phi))_\Lambda. \end{aligned} \tag{1.85}$$

To see (3.) we note that the projection of the product of g^1 and g^2 is

$$\pi_\Lambda(g^1g^2) = \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}) \tag{1.86}$$

and the product in the sense of (1.26) projects to

$$\begin{aligned} \pi_\Lambda(g^1g^2) &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda\cup\Lambda_2}) \\ &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}) \end{aligned} \tag{1.87}$$

as g^2 does not depend on $\Lambda \setminus \Lambda_2$-coordinates.
If $\Lambda_1 \subseteq \Lambda_2$ then

$$\begin{aligned} g_{\Lambda_2}(g^1\phi)_{\Lambda_2} &= g_{\Lambda_2}g^1\phi_{\Lambda_2} \\ &= (g^1g)_{\Lambda_2}\phi_{\Lambda_2} \end{aligned} \tag{1.88}$$

and so (4.) follows from

$$
\begin{aligned}
(g^1\phi)(g) &= \lim_{\Lambda_2\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} g_{\Lambda_2}(g^1\phi)_{\Lambda_2} \\
&= \lim_{\Lambda_2\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2}(g^1 g)_{\Lambda_2}\phi_{\Lambda_2} \\
&= \phi(g^1 g)
\end{aligned}
\tag{1.89}
$$

(5.) follows from

$$
\begin{aligned}
\|g^1\phi\|_{var} &= \lim_{\Lambda\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda}} d\mu^{\Lambda}|(g^1\phi)_{\Lambda}| \\
&= \lim_{\substack{\Lambda\to\mathbf{Z}^d\\ \Lambda\supset\Lambda_1}}\int_{(S^1)^{\Lambda}} d\mu^{\Lambda}|g^1||\phi_{\Lambda}| \\
&\leq \|g^1\|_{\Lambda_1}\|\phi\|_{var}
\end{aligned}
\tag{1.90}
$$

□

Proof of Lemma 1.5.1 We get recursively

$$
\begin{aligned}
&\frac{1}{f_p\circ T^{\epsilon}_{p,l}(\mathbf{z})-w_p}f_p\circ T^{\epsilon}_{p,l}(\mathbf{z}) \\
&= \frac{1}{f_p\circ T^{\epsilon}_{p,l-1}(\mathbf{z})-w_p}f_p\circ T^{\epsilon}_{p,l-1}(\mathbf{z}) \\
&\quad +w_p\frac{f_p\circ T^{\epsilon}_{p,l-1}(\mathbf{z})-f_p\circ T^{\epsilon}_{p,l}(\mathbf{z})}{(f_p\circ T^{\epsilon}_{p,l-1}(\mathbf{z})-w_p)(f_p\circ T^{\epsilon}_{p,l}(\mathbf{z})-w_p)} \\
&= \frac{1}{f_p(z_p)-w_p}f_p(\mathbf{z})+w_p\sum_{k=1}^{l}\frac{f_p\circ T^{\epsilon}_{p,k-1}(\mathbf{z})-f_p\circ T^{\epsilon}_{p,k}(\mathbf{z})}{(f_p\circ T^{\epsilon}_{p,k-1}(\mathbf{z})-w_p)(f_p\circ T^{\epsilon}_{p,k}(\mathbf{z})-w_p)}
\end{aligned}
\tag{1.91}
$$

The estimate (1.42) yields uniform convergence of this sum as $l\to\infty$. So we get (1.28).

□

In (1.57) we estimate the norm of the operator corresponding to one particular configuration in terms of the lines and triangles it contains. Now we have to bound sums over all such configurations as they arise in expansions for the full operators. For this we use our analysis and some combinatorics at the same time. The idea is that a configuration of a given length must have at least a certain number of triangles and r-chains that lead to small factors in the estimates. In fact, certain r-chains could not be replaced by h-chains in the configuration as we would get a zero configuration.

Definition 1.8.1 • A maximal r-chain going from an apex of a triangle downwards to the next basepoint of a triangle or to a bottom point is called an *a-r-chain*. (If the apex coincides with a base or bottom point the a-r-chain has length zero.)

- The *a-r-length* of a configuration $\mathcal{C}$ is the sum of the lengths of all its a-r-chains plus the number of its triangles, i.e. if $\mathcal{C}$ has $|n_\beta|$ triangles with corresponding a-r-chains of length $l_1, \ldots, l_{|n_\beta|}$ then

$$\begin{aligned} \text{a-r-length}(\mathcal{C}) &\stackrel{\text{def}}{=} l_1 + \cdots + l_{|n_\beta|} + |n_\beta| \\ &= (l_1 + 1) + \cdots + (l_{|n_\beta|} + 1) \end{aligned} \tag{1.92}$$

(In particular a-r-length$(\mathcal{C}) \geq |n_\beta|$.)

- We call a maximal r-chain going from a base point (p, t) of a triangle to $(p, -N)$ (such that $(p, -N)$ is not a base point of another triangle) a *u-r-chain* (upwards going r-chain), a maximal r-chain going downwards from a basepoint a *d-r-chain* (*d-h-chains* are defined analogously).

- A maximal r-chain going from a bottom point $(p, 0)$ to $(p, -N)$ is called an *l-r-chain* (long r-chain). We denote the number of l-r-chains of $\mathcal{C}$ by $l(\mathcal{C})$.

The configuration in Figure 1.5 has length 3, a-r-length 6, only one a-r-chain of positive length from $(6, -2)$ to $(6, -1)$, only one u-r-chain of positive length from $(2, -3)$ to $(2, -2)$, and only one l-r-chain from $(1, -3)$ to $(1, 0)$.

We prepare the proofs of Theorem 1.2.1 and Proposition 1.6.1 in the following technical proposition that provides the key bounds and basic analysis and combinatorics for the other proofs.

Proposition 1.8.1 *For sufficiently small ϑ, ϵ and big c_2 and N we have for all $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ the following bound for the terms in the expansion of (1.49) for $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$ with constants c_{19}, c_{20}:*

1.

$$\sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_\mathcal{C}\|_{L((\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}))} \leq c_{19} \tilde{\eta}^N \tag{1.93}$$

with $\tilde{\eta} \stackrel{\text{def}}{=} \sqrt{\eta} < 1$

2.

$$\left\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}\right\|_{L((\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}))} \leq c_{20} \tag{1.94}$$

For the proof of Proposition 1.8.1 we need a graph-theoretical lemma. We consider labelled tree graphs that are constructed in the following way (cf. Figure 1.6): We start with a star graph with a *root*-vertex, labelled (0), to which K edges are attached, each connecting to one *leaf*. The leaves are labelled by $(0,1),\dots,(0,K)$. Then we add successively star graphs (Each of these has a certain finite number $v(k)$ of leaves. These numbers are defined in (1.45).) to the already built up tree: We identify one of the leaves of the tree, say labelled by $s=(s_1,\dots,s_n)$, with the root of the added star and label the new leaves by $(s_1,\dots,s_n,1),\dots,(s_1,\dots,s_n,v(k))$. In total we use besides the star graph with K leaves exactly $n_{\beta,k}$ star graphs with exactly $v(k)$ leaves. We say *the tree has parameters* K and $n_\beta=(n_{\beta,1},n_{\beta,2},\dots)$
We also introduce a linear order on the set of tuples (and so on the set of vertices of the labelled graph):
We say $s=(s_1,\dots,s_n)\prec t=(t_1,\dots,t_m)$ if $n<m$ and $s_i=t_i$ for $1\le i\le n$ or if $s_i=t_i$ $(1\le i<k)$ and $s_k<t_k$ for some k.

Lemma 1.8.2 *1. The number of labelled tree graphs with exactly n edges is not greater than* 2^{2n}

2. Given $K, n_{\beta,1}, n_{\beta,2},\dots$ *with* $K+\sum_{k=1}^{\infty}n_{\beta,k}<\infty$. *The number of labelled tree graphs with parameters* K *and* n_β *is bounded from above by*
$4^K\prod_{k=1}^{\infty}c_{21}^{k^d n_{\beta,k}}$ *with* $c_{21}=4^{3^d}$.

Proof of Lemma 1.8.2 We first prove (1.) For every labelled tree graph in question we can define a path starting and ending at the root point (0) and running through each edge exactly twice in the following way. From a (labelled) vertex $t=(t_1,\dots,t_k)$ we go to the next greater (wrt. $\prec$) vertex where we haven't yet been (*going up*), or if this is not possible (i.e. t is a leaf or we have already been at all vertices $(t_1,\dots,t_{k+1})$) back to $(t_1,\dots,t_{k-1})$ (*going down*). So we return to (0) after $2n$ steps. We encode the path in a word $(a_1,\dots,a_{2n})$ with $a_i=1$ if we go up in the ith step and $a_i=0$ otherwise. Obviously the labelled graph is uniquely determined by its word. (Note that not every word of length $2n$ with symbols '0' and '1' corresponds to such a labelled graph. But the map between these two data is injective.) As there are 2^{2n} words of length $2n$ with at most two different symbols this is also an upper bound for the number of graphs in question, so (1.) is proved.
To see (2.) we note, using the estimate for $v(k)$ that we got after (1.45), that the number of edges in such a tree graph is not greater than $K+\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}$.

□

Proof of Proposition 1.8.1 We estimate the norm of each $\mathcal{L}_C$ in (1.93) in terms of the number of particular triangles, r-chains etc. of C as we do in (1.57). We also have to bound the number of configurations in (1.93) that have the same set of triangles. We do so by assigning in (i) to (iv) to each configuration a labelled tree graph and estimating the number of graphs that have certain properties.

(i) We fix $0 \leq K \leq |\Lambda_1|$ and $\Lambda_3 \subseteq \Lambda_1$ with $|\Lambda_3| = K$ (so there are $\binom{|\Lambda_1|}{K}$ possible choices for Λ_3) and want to estimate the number of configurations $\mathcal{C}$ such that $\Lambda_{\mathcal{C}} = \Lambda_3$. So let us consider such a configuration. We call the triangles whose apex lies at, or whose a-r-chain ends in, $\Lambda_3 \times \{0\}$, *root triangles.* We can assign to $\mathcal{C}$ a graph of the type we consider in Lemma 1.8.2 as follows: We start with a star graph with a star point labelled (0) and K leaves, labelled $(0, 1), \dots, (0, K)$. These leaves are in bijection with $\Lambda_3 \times \{0\}$. Now we add successively for each l-triangle (cf. def. on page 13) in $\mathcal{C}$ a star graph with one star point and $v(l)$ leaves (cf. def. of $v(l)$ in (1.45)) to the graph and label the new vertices: If an l-triangle lies with its apex or ends with its a-r-chain on a basepoint of another triangle (for which we have already assigned a small tree) or on a point in $\Lambda_3 \times \{0\}$ (this point is labelled say $s = (s_1, \dots, s_n)$) we add a small l-tree to the graph by identifying its star point with s and label the $v(l)$ new leaves in the graph $(s_1, \dots, s_n, 1), \dots, (s_1, \dots, s_n, v(l))$. Since, for example, an apex could coincide with more than one other triangle's basepoint we use the linear order $\prec$ (defined on page 28) to define an order in our successive assignment of triangles to star graphs. We always choose the next triangle such that the corresponding star graph is attached to the smallest (wrt. $\prec$) labelled leaf in the graph. This also defines a unique choice of the triangle and the leaf where we attach the star graph. So the position of triangles and the a-r-chains of $\mathcal{C}$ are completely determined by the datum consisting of the corresponding labelled graph *and* the lengths of its a-r-chains. Note that it is not the case that for every graph together with a choice of lengths for the particular a-r-chains there was a corresponding configuration.
For the configuration in Figure 1.5, for example, we get the labelled graph in Figure 1.6.

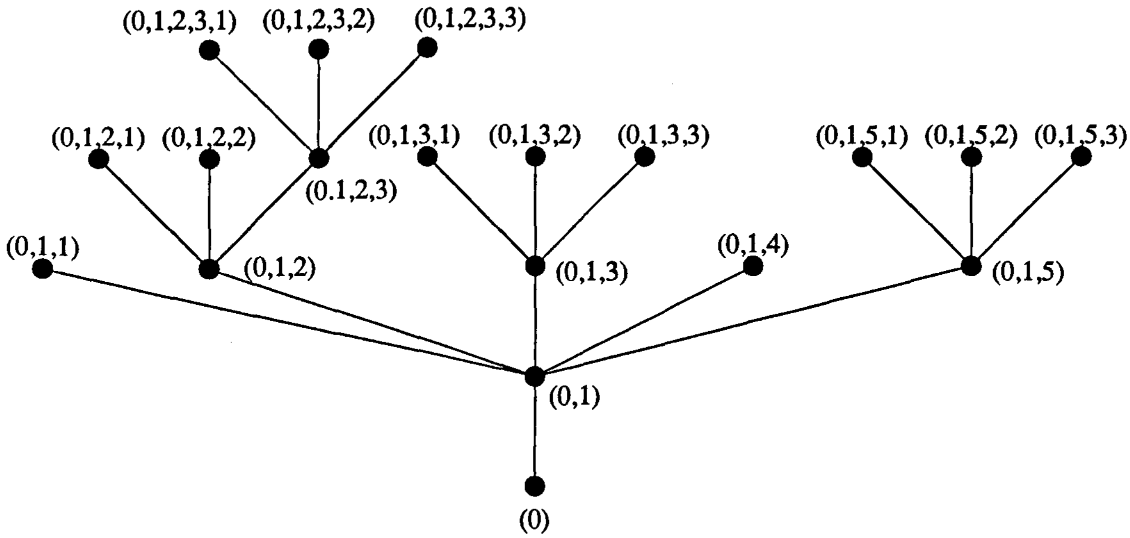


Figure 1.6: The labelled graph for the configuration in Figure 1.5

Let $n_{\beta,k}$ be, as in Definition 1.5.2, the total number of k-triangles. The number of graphs with parameters K and n_β is bounded by $4^K \prod_{k=1}^{\infty} c_{21}^{k^d n_{\beta,k}}$ (by Lemma 1.8.2). As mentioned above we have for each of the $|n_\beta|$ a-r-chains a length $0 \leq l_i < \infty$. The a-r-length is

$$L = (l_1 + 1) + \cdots + (l_{|n_\beta|} + 1). \tag{1.95}$$

So $L \geq |n_\beta|$. For a given n_β with $|n_\beta| \geq 1$ and $L \geq 1$ there are $\binom{L-1}{|n_\beta|-1}$ different choices of $(l_1, \ldots, l_{|n_\beta|})$ that satisfy (1.95). For $|n_\beta| = 0$ we have $L = 0$ and the (unique) configuration without triangles or r-lines. So in any case the number of choices is bounded from above by $\binom{L}{|n_\beta|}$. The integration over these $|n_\beta|$ a-r-chains leads to a factor $c_r^{|n_\beta|}\eta^L$ in our estimates (cf. (1.57)) and each k-triangle contributes by (1.55) a factor $c_8\epsilon \exp(-c_2 k^d)$.
(ii) There are choices between d-r-chains and d-h-chains in the configuration.
There are not more than $\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}$ base points for which we can choose between a d-h-chain (giving factor c_h in our estimates) and a d-r-chain (giving factor at most $c_r\eta$). So the total sum over these combinations is bounded from above by

$$(c_h + c_r\eta)^{\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}} \leq \prod_{k=1}^{\infty} \left(\exp(c_{22}k^d)\right)^{n_{\beta,k}}.$$

(iii) There are choices between u-r-chains and u-h-chains in the configuration.
There are not more than $\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}$ basepoints. To each of them we can attach either a u-h-chain, giving a factor c_h, or a u-r-chain, giving a factor $c_r\eta^{\max\{0,N-L\}}$, because if $N - L > 0$, such a u-r-chain cannot have length smaller than $N - L$, for otherwise it would not end in $\Lambda_2 \times \{-N\}$. We get in total a factor not greater than

$$(c_h + c_r)^{\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}} = \prod_{k=1}^{\infty} \left(\exp(c_{23}k^d)\right)^{n_{\beta,k}}. \tag{1.96}$$

(iv) There are choices left between l-h-chains and l-r-chains in $(\Lambda_1 \backslash \tilde{\Lambda}_C) \times \{-N, \ldots, 0\}$, giving factor c_h or $c_r\eta^N$ respectively. Let l $(0 \leq l \leq |\Lambda_1 \setminus \tilde{\Lambda}_C| \leq |\Lambda_1| - K)$ denote the number of l-r-chains in such a choice. For given l there are $\binom{|\Lambda_1 \setminus \tilde{\Lambda}_C|}{l} \leq \binom{|\Lambda_1| - K}{l}$ different subsets Λ_r of $\Lambda_1 \setminus \tilde{\Lambda}_C$ of cardinality l (that corresponds to a particular choice of exactly l l-r-chains.) The configuration $\mathcal{C}$ is determined by all the choices mentioned up to now.
Consider now a $\mathcal{C}$ with length$(\mathcal{C}) = N$. If $N - L > 0$ then there must be at least one u-r-chain giving rise to an extra factor $\eta^{\max\{0,N-L\}}$ or an l-r-chain giving rise to a factor η^n. To get (1.98) we split

$$\begin{aligned} \eta^{\max\{0,N-L\}} &= \tilde{\eta}^{\max\{0,N-L\}}\tilde{\tilde{\eta}}^{\max\{0,N-L\}} \\ \text{or} \qquad \eta^N &= \tilde{\eta}^N\tilde{\tilde{\eta}}^N \end{aligned}$$

with $\tilde{\eta} \stackrel{\text{def}}{=} \sqrt{\eta}$. Therefore we get the factor $\tilde{\eta}^{\max\{0,N-L\}}$.

In the configuration $\mathcal{C}$ there are h-chains at points with $\mathbb{Z}^d$-coordinate in $\Lambda_1 \setminus (\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r)$. The operator $\mathcal{L}_{\mathcal{C}}$ acts on ϕ_{Λ_2} by integration over these coordinates. So for the uniform estimate of $\mathcal{L}_{\mathcal{C}}\phi_{\tilde{\Lambda}(\mathcal{C})}$ we use (1.58).
First we estimate in (1.97)-(1.104) the sum over $\mathcal{C}$ with length($\mathcal{C}$) $= N$ and then in (1.105)-(1.107) the sum over $\mathcal{C}$ with length($\mathcal{C}$) $= m < N$. We do that separately because in the second case $\mathcal{C}$ has no l-r-chains while in the first case every l-r-chain leads to a small factor $c_r\eta^N$. The idea of making this distinction is similar to the idea of 'vacuum polymers' in other papers (cf. [3, 26, 1]).

$$\vartheta^{|\Lambda_1|} \sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1} \tag{1.97}$$

$$\leq \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq|n_\beta|<\infty}} 4^K \prod_{k=1}^{\infty} \left(\exp(c_{21}k^d)\right)^{n_{\beta,k}} (c_1\epsilon)^{|n_\beta|} \tag{1.98}$$

$$\times \prod_{k=1}^{\infty} \left(\exp(-c_2k^d)\right)^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|} \tilde{\eta}^L \prod_{k=1}^{\infty} \left(\exp(c_{22}k^d)\right)^{n_{\beta,k}}$$

$$\times \prod_{k=1}^{\infty} \left(\exp(c_{23}k^d)\right)^{n_{\beta,k}} \tilde{\eta}^{\max\{0,N-L\}} \sum_{l=0}^{|\Lambda_1|-K} \binom{|\Lambda_1|-K}{l} (c_r\tilde{\eta}^N)^l$$

$$\times c_h^{|\Lambda_1|-K-l} \vartheta^{-l} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta}$$

$$= \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq|n_\beta|<\infty}} 4^K (c_1\epsilon c_r)^{|n_\beta|} \tag{1.99}$$

$$\times \prod_{k=1}^{\infty} \exp((c_{21} - c_2 + c_{22} + c_{23} - 3^d \ln\vartheta)k^d)^{n_{\beta,k}}$$

$$\times \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^{\max\{N,L\}} (\vartheta^{-1}c_r\tilde{\eta}^N + c_h)^{|\Lambda_1|-K} \|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N.$$

We assume $\epsilon < 1$. We set $\epsilon_1 \stackrel{\text{def}}{=} 4\epsilon c_1 c_r$ and $\epsilon_2 \stackrel{\text{def}}{=} \sqrt{\epsilon_1}$. Then we have $\epsilon_1^{|n_\beta|} \leq \epsilon_2^K \epsilon_2^{|n_\beta|}$. We set $\tilde{c_2} \stackrel{\text{def}}{=} c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln\vartheta$. Then $\tilde{c_2} > 0$ if

$$c_2 > c_{21} + c_{22} + c_{23} - 3^d \ln\vartheta. \tag{1.100}$$

(We assume this condition on the decay of the coupling. Note that we first have to choose ϑ below, after (1.104), depending on the other parameters of the system (but not on c_2) and then condition (1.100) is well-defined.)

Then (1.99) can be bounded as follows:

$$\leq \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_r\tilde{\eta}^N + \vartheta c_h)^{|\Lambda_1|-K} \epsilon_2^K \sum_{\substack{n_\beta \\ K\leq|n_\beta|<\infty}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \epsilon_2^{|n_\beta|} \tag{1.101}$$

$$\times \prod_{k=1}^{\infty} \left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N$$

$$\leq (c_r\tilde{\eta}^N + \vartheta c_h + \epsilon_2)^{|\Lambda_1|} \sum_{L=0}^{\infty}\sum_{n=0}^{L} \binom{L}{n} \tilde{\eta}^L \epsilon_2^n \sum_{\substack{n_\beta \\ |n_\beta|=n}} \prod_{k=1}^{\infty} \left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}}$$

$$\times \|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N.$$

We have

$$\sum_{\substack{n_\beta \\ |n_\beta|=n}} \prod_{k=1}^{\infty} \left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}} \leq \prod_{k=1}^{\infty} \sum_{n_{\beta,k}=0}^{\infty} \left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}} \tag{1.102}$$

and the last infinite product converges (to c_{24} say) since for k sufficiently large $\exp(-\tilde{c_2}k^d) < \frac{1}{2}$ and $\sum_{n_{\beta,k}=0}^{\infty} \left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}} \leq 1 + 2\exp(-\tilde{c_2}k^d)$ and $\sum_{k=0}^{\infty} \exp(-\tilde{c_2}k^d) < \infty$. (Recall $\prod_{k=1}^{\infty}(1+u_k)$ *convergent* $\Leftarrow \sum_{k=1}^{\infty}|u_k| < \infty$.)

$$\leq (\epsilon_2 + c_r\tilde{\eta}^N + c_h\vartheta)^{|\Lambda_1|} c_{24} \sum_{L=0}^{\infty} (\epsilon_2 + \tilde{\eta})^L \|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N$$

$$= (\epsilon_2 + c_r\tilde{\eta}^N + c_h\vartheta)^{|\Lambda_1|} \frac{1}{1-\epsilon_2-\tilde{\eta}} c_{24}\|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N \tag{1.103}$$

$$\leq c_{19}\tilde{\eta}^N \|\phi\|_{\Lambda_2,\vartheta} \tag{1.104}$$

for ϑ and ϵ sufficiently small and N sufficiently large. This also holds for $\Lambda \subset \Lambda_1$. So (1.) is proved.
To show (2.) we have to estimate in addition to (1.93) the contribution of non-zero configurations $\mathcal{C}$ of length $0 \leq m < N$ in the expansion of $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$. These have no l-r-chains. So this time we have $l(\mathcal{C}) = 0$. Using the splitting $\eta^L \leq \tilde{\eta}^L\tilde{\eta}^m$ we get in a similar way

$$
\begin{aligned}
&\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\text{length}(\mathcal{C})=m,\\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} && (1.105)\\
\leq\ & \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq |n_\beta| < \infty}} 4^K \prod_{k=1}^{\infty} \left(\exp(c_{21}k^d)\right)^{n_{\beta,k}} \\
&\times (c_1\epsilon)^{|n_\beta|} \prod_{k=1}^{\infty} \left(\exp(-c_2 k^d)\right)^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|} \tilde{\eta}^L \prod_{k=1}^{\infty} \left(\exp(c_{22}k^d)\right)^{n_{\beta,k}} \\
&\times \prod_{k=1}^{\infty} \left(\exp(c_{23}k^d)\right)^{n_{\beta,k}} c_h^{|\Lambda_1|-K} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^m \\
\leq\ & \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h\vartheta)^{|\Lambda_1|-K} \sum_{\substack{n_\beta \\ K\leq |n_\beta| < \infty}} (c_1\epsilon 4 c_r)^{|n_\beta|} \prod_{k=1}^{\infty} \left(\exp(-\tilde{c}_2 k^d)\right)^{n_{\beta,k}} \\
&\times \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta} \\
\leq\ & \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h\vartheta)^{|\Lambda_1|-K} \epsilon_2^K \sum_{\substack{n_\beta \\ K\leq |n_\beta| < \infty}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \epsilon_2^{|n_\beta|} \\
&\times \prod_{k=1}^{\infty} \left(\exp(-\tilde{c}_2 k^d)\right)^{n_{\beta,k}} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta} \\
\leq\ & (\epsilon_2 + c_h\vartheta)^{|\Lambda_1|} \frac{1}{1-\epsilon_2-\tilde{\eta}} c_{25} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta} \\
\leq\ & c_{26} \tilde{\eta}^m \|\phi\|_{\Lambda_2,\vartheta}. && (1.106)
\end{aligned}
$$

Again this also holds for $\Lambda \subset \Lambda_1$ and so

$$
\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\text{length}(\mathcal{C})=m,\\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1,\vartheta} \leq c_{26} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^m. \qquad (1.107)
$$

Therefore

$$
\begin{aligned}
\left\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}\right\|_{L((\mathcal{H}_{\Lambda_2,\vartheta},\|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta}\|\cdot\|_{\Lambda_1,\vartheta}))} &\leq \sum_{m=0}^{N} c_{26}\tilde{\eta}^m && (1.108)\\
&\leq \sum_{m=0}^{\infty} c_{26}\tilde{\eta}^m \\
&\leq c_{20}
\end{aligned}
$$

which was to be shown.

□

Proof of Theorem 1.2.1 First we consider the case $N \geq N_0$. The difference between $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2}$ and $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_3} \circ T^{\Lambda_3,\epsilon}} \circ \pi_{\Lambda_3}$ for $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \in \mathcal{F}$ is due to the summands involving configurations that do not lie completely (with all their triangles) in $\Lambda_2 \times \{0, -1, \ldots\}$. For those summands we have the lower bound for the spatial extension of the set of triangles:

$$\begin{aligned} b(\mathcal{C}) &\stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k n_{\beta,k} \\ &\geq \operatorname{dist}(\Lambda_1, \Lambda_2^C) \end{aligned} \tag{1.109}$$

As the analysis in the proof of Proposition 1.8.1 shows we have in the estimate for each such configuration a factor

$$\begin{aligned} &\prod_{k=1}^{\infty} \left(\exp(-\tilde{c_2} k^d)\right)^{n_{\beta,k}} \\ &\leq \prod_{k=1}^{\infty} \left[\exp(-(\tilde{c_2} - \xi) k^d)\right]^{n_{\beta,k}} \prod_{k=1}^{\infty} \left(\exp(-\xi k n_{\beta,k})\right) \\ &\leq \prod_{k=1}^{\infty} \left[\exp(-(\tilde{c_2} - \xi) k^d)\right]^{n_{\beta,k}} \exp\left(-\xi \operatorname{dist}(\Lambda_1, \Lambda_2^C)\right). \end{aligned} \tag{1.110}$$

If we take $\xi > 0$ small enough we can take out a factor $\exp\left(-\xi \operatorname{dist}(\Lambda_1, \Lambda_2^C)\right)$ and do the analysis with the remaining factor as before since $\tilde{c_2} - \xi > 0$. So we get

$$\begin{aligned} &\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} - \pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_3} \circ T^{\Lambda_3,\epsilon}} \circ \pi_{\Lambda_3}\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}))} \\ &\qquad \leq c_{27} \exp\left(-\xi \operatorname{dist}(\Lambda_1, \Lambda_2^C)\right) \end{aligned} \tag{1.111}$$

with some constant c_{27} and the limit in (1.7) exists for $N \geq N_0$. The proof for the case $N < N_0$ is similar. We use the modified estimates that we get by replacing in (1.97) and (1.105) ϑ by a sufficiently small $\tilde{\vartheta}$. For example, (1.97) and (1.103) become

$$\begin{aligned} &\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}:\text{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} \\ &\leq c_{28}(\epsilon_2 + c_\tau \tilde{\eta}^N \frac{\tilde{\vartheta}}{\vartheta} + c_h \tilde{\vartheta})^{|\Lambda_1|} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \end{aligned} \tag{1.112}$$

and the term in parentheses is smaller than 1 if $\tilde{\vartheta}$ and $\frac{\tilde{\vartheta}}{\vartheta}$ are small enough. The statement for systems with finite-range interaction follows from the fact that in that case all limits are already attained for some sufficiently large $\Lambda_2 \in \mathcal{F}$ and that all considered sums are finite.
For the proof of (2.) we use results from Proposition 1.6.1 that we prove below.
By (1.7) the operators $\mathcal{L}^N_{F \circ T^\epsilon} \in L(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ are well defined for $N \geq N_0$ and, by part (2.) of Proposition 1.6.1, give rise to a Cauchy sequence. With the same argument we see that the infinite sum in the definition of ν_Λ (cf. (1.59)) converges and $\nu \in \mathcal{H}_\vartheta$. $\nu \geq 0$ and so $\nu \in \mathcal{H}^{bv}$ follow from (6.) of Proposition 1.6.1.
The difference in (1.9) is only due to configurations of length $\geq N$ and we estimate it, using part (2.) of Proposition 1.6.1, by $c_3\tilde{\eta}^N$. So $\nu = \lim_{N\to\infty} \mathcal{L}^N_{F\circ T^\epsilon} h$ and by (3.) and (4.) of Proposition 1.6.1, $\mathcal{L}_{F\circ T^\epsilon}\nu = \nu$ and $\mu(\nu) = 1$, respectively. For any $\phi \in \mathcal{H}_\vartheta$ with $\mathcal{L}_{F\circ T^\epsilon}\phi = \phi$ and $\mu(\phi) = 1$ we have by (1.9)

$$\phi = \lim_{N\to\infty} \mathcal{L}^N_{F\circ T^\epsilon}\phi = \mu(\phi)\nu = \nu. \tag{1.113}$$

That shows uniqueness of ν and so of ν^* and the proof of (2.) is complete.

□

Proof of Proposition 1.6.1 Using the same argument as in the proof of (1.) in Theorem 1.2.1, we see that the right-hand side term in (1.60) differs from the operator in (1.49) only in summands for $\mathcal{C}$ with $b(\mathcal{C}) \geq \text{dist}(\Lambda_1, \Lambda_2^C)$. So the difference is bounded by $c_{29} \exp\left(-\xi \text{dist}(\Lambda_1, \Lambda_2^C)\right)$ for some $c_{29} > 0$ and (1.60) follows from taking the limit $\Lambda_2 \to \mathbb{Z}^d$.
In order to prove (2.) we first observe that configurations $\mathcal{C} \in E_N(\Lambda_1)$ of length $\leq N-1$ extend canonically to $\mathcal{C}' \in E_{N+1}(\Lambda_1)$ with $\mathcal{L}_\mathcal{C} = \mathcal{L}_{\mathcal{C}'}$ because there are only h-lines in the step from time $-N$ to $-N+1$. So we can extend $\mathcal{C}$ to $\mathcal{C}'$ on $\Lambda_2 \times \{-N-1, \ldots, 0\}$ (where Λ_2 is so big that $\Lambda_2 \times \{-N-1, \ldots, 0\}$ contains all triangles of $\mathcal{C}$) by adding h-lines from $(p, -N-1)$ to $(p, -N)$ for all $p \in \Lambda_2$ and obviously $\mathcal{L}_\mathcal{C} = \mathcal{L}_{\mathcal{C}'}$.
Note that a configuration $\mathcal{C}'$ in $\Lambda_2 \times \{-N-1, \ldots, 0\}$ of length $\leq N-1$ is the extension in the above sense of a (uniquely defined) $\mathcal{C}$.
So in the difference (1.61), all terms $\mathcal{L}_\mathcal{C}$ with length$(\mathcal{C}) \leq N-1$ are cancelled. Using (1.) of Proposition 1.8.1, (1.107) and (1.) of this proposition we get for all $\Lambda_1 \in \mathcal{F}$

$$\begin{aligned} \left\|\left(\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} - \pi_{\Lambda_1} \circ \mathcal{L}^{N+1}_{F\circ T^\epsilon}\right)\phi\right\|_{\Lambda_1,\vartheta} &\leq \left(c_{19}\tilde{\eta}^N + c_{20}\tilde{\eta}^N + c_{19}\tilde{\eta}^{N+1}\right)\|\phi\|_\vartheta \\ &\leq c_{30}\tilde{\eta}^N\|\phi\|_\vartheta \end{aligned} \tag{1.114}$$

with c_{30} independent of Λ_1. This proves (2.) Next we prove (3.)

For $\Lambda_1 \in \mathcal{F}$,

$$\begin{aligned}
&\pi_{\Lambda_1} \circ \mathcal{L}^{N_2}_{F\circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F\circ T^\epsilon}\phi \qquad (1.115)\\
&= \sum_{C_2 \in E_{N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{C_2}\left(\mathcal{L}^{N_1}_{F\circ T^\epsilon}\phi\right)\\
&= \sum_{C_2 \in E_{N_2}(\Lambda_1)} \left(\pi_{\Lambda_1} \circ \mathcal{L}_{C_2} \circ \sum_{C_1 \in E_{N_2}(\Lambda(C_2))} \pi_{\Lambda(C_2)} \circ \mathcal{L}_{C_1}\phi_{\Lambda(C_1)}\right)\\
&= \sum_{\substack{C_2 \in E_{N_2}(\Lambda_1)\\ C_1 \in E_{N_2}(\Lambda(C_2))}} \pi_{\Lambda_1} \circ \mathcal{L}_{C_2 \circ C_1}\phi_{\Lambda(C_1)}\\
&= \sum_{C_3 \in E_{N_1+N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{C_3}\phi_{\Lambda(C_3)}\\
&= \pi_{\Lambda_1} \circ \mathcal{L}^{N_1+N_2}_{F\circ T^\epsilon}\phi.
\end{aligned}$$

Note that we sum over infinitely many C_1, C_2. A priori, the distribution is only valid for finite partial sums. In terms of configurations we 'put C_1 on C_2' to get $C_3 = C_2 \circ C_1$ (which might be a zero configuration) and in fact such a splitting exists and is unique for every non-zero C_3. So the net of finite partial sums over C_3 converges to the infinite expansion (1.60) of the right-hand side of (1.62) and (3.) is proved.
To prove (1.64), we consider first the special case $g \in \mathcal{C}((S^1)^\Lambda)$.

$$\begin{aligned}
\int_M d\mu\, g \circ S\,\phi &= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_M d\mu\, g \circ S_{\Lambda_1}\phi \qquad (1.116)\\
&= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S_{\Lambda_1}\phi_{\Lambda_1}\\
&= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g\, \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1}\\
&= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda}} d\mu^{\Lambda} g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}} \circ \pi_{\Lambda_1}\phi\\
&= \int_M d\mu\, g\mathcal{L}_{F\circ T^\epsilon}\phi.
\end{aligned}$$

So (1.64) is true for $g \in \mathcal{C}((S^1)^\Lambda)$. Taking $g \equiv 1$, we get (1.65).
Now we show (1.63), using the special case of (1.64) for the second equality.

$$\begin{aligned}
\|\mathcal{L}_{F\circ T^\epsilon}\phi\|_{var} &= \sup_{\Lambda\in\mathcal{F}} \sup_{\substack{g\in\mathcal{C}((S^1)^\Lambda)\\ \|g\|_\infty\le 1}} \int_M d\mu\, g\, \mathcal{L}_{F\circ T^\epsilon}\phi \\
&= \sup_{\Lambda\in\mathcal{F}} \sup_{\substack{g\in\mathcal{C}((S^1)^\Lambda)\\ \|g\|_\infty\le 1}} \int_M d\mu\, g\circ S\,\phi \\
&\le \sup_{\Lambda\in\mathcal{F}} \sup_{\substack{g\in\mathcal{C}((S^1)^\Lambda)\\ \|g\|_\infty\le 1}} \|g\|_\infty \|\phi\|_{var} \\
&= \|\phi\|_{var}.
\end{aligned} \tag{1.117}$$

We can conclude (1.64) for any $g \in \mathcal{C}(M)$. By assumption ϕ and then by (1.63) $\mathcal{L}_{F\circ T^\epsilon}\phi$ are in $\mathcal{H}^{bv}$, i.e. the integrals in (1.64) correspond to continuous linear functionals on $\mathcal{C}(M)$. The net $(g_\Lambda)_{\Lambda\in\mathcal{F}}$ converges uniformly to g as $\Lambda \to \mathbb{Z}^d$, as does $(g_\Lambda \circ S)_{\Lambda\in\mathcal{F}}$ to $g \circ S$, so (1.64) follows by uniform approximation of g by functions g_Λ and (4.) is proved.
We show (5.) by indirect proof. We have, by definition, $(\mathcal{L}_{F\circ T^\epsilon}\phi)_\Lambda \stackrel{\text{def}}{=} \lim_{\Lambda_1\to\mathbb{Z}^d} \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1}$. If that was negative somewhere there would be a $\Lambda_1 \in \mathcal{F}$ with $\pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1}$ having negative values and we could find a non-negative $g \in \mathcal{C}((S^1)^\Lambda)$ such that

$$\int_{(S^1)^\Lambda} d\mu^\Lambda g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}}\phi_{\Lambda_1} < 0 \tag{1.118}$$

But by (4.) the integral equals

$$\int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S\, \phi_{\Lambda_1} \ge 0. \tag{1.119}$$

So $\mathcal{L}_{F\circ T^\epsilon}$ is non-negative.

□

Proof of Theorem 1.7.1

$$\begin{aligned}
\nu_{\Lambda_1\cup\Lambda_2} &= \sum_{\mathcal{C}\in E(\Lambda_1\cup\Lambda_2)} \pi_{\Lambda_1\cup\Lambda_2} \circ \mathcal{L}_\mathcal{C} h \\
&= \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\\ b(\mathcal{C})\le\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_1} h)(\pi_{\Lambda_2} \circ \mathcal{L}_{\mathcal{C}_2} h) \\
&\quad + \sum_{\substack{\mathcal{C}\\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2} \circ \mathcal{L}_\mathcal{C} h
\end{aligned} \tag{1.120}$$

In estimating the second summand we note that if we sum in formula (1.97) and (1.105) just over $\mathcal{C}$ for which $b(\mathcal{C}) \ge \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)$ ($b(\mathcal{C})$ was defined in (1.109)), we can take out from $\prod_{k=1}^{\infty}\left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}}$ a factor $\exp(-\xi\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2))$(like in the proof of Proposition 1.6.1): We do so by choosing a $\kappa \in (0,1)$ so that

$$\tilde{c_2} + \ln\kappa = c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln\vartheta + \ln\kappa > 0 \tag{1.121}$$

and by defining ξ by $\exp(-\xi\frac{1}{2}) = \kappa$. Note that such a choice exists as $\tilde{c_2} > 0$ by (1.100).

The rest of the analysis is as in the proof of Proposition 1.8.1. We get

$$\|\sum_{\substack{\mathcal{C} \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2}\circ\mathcal{L}_{\mathcal{C}}h\|_{\Lambda_1\cup\Lambda_2} \tag{1.122}$$

$$\begin{aligned} &\le \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} c_{31}\|h\|_{\vartheta}\vartheta^{-|\Lambda_1|-|\Lambda_2|} \\ &\le c_{32}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \end{aligned} \tag{1.123}$$

We write for the first summand in (1.120)

$$\begin{aligned} &\sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})\le\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \\ &= \nu_{\Lambda_1}\nu_{\Lambda_2} - \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \end{aligned} \tag{1.124}$$

and estimate in a similar way

$$\|\sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h)\|_{\Lambda_1\cup\Lambda_2} \le c_{33}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{1.125}$$

(1.124) and (1.125) also hold for all $\Lambda_1' \subseteq \Lambda_1$, $\Lambda_2' \subseteq \Lambda_2$ and (1.) follows.

$$\begin{aligned} \pi_{\Lambda_1}(f\nu) &= \pi_{\Lambda_1}(f\nu_{\Lambda_1\cup\Lambda_2}) \\ &= \pi_{\Lambda_1}(f\nu_{\Lambda_1}\nu_{\Lambda_2} - f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2})) \\ &= \nu(f)\nu_{\Lambda_1} - \pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2})) \end{aligned} \tag{1.126}$$

and, using $\|\pi_{\Lambda_1}\|_\infty = 1$, we get

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \le \|f\|_{\Lambda_2}\|\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}\|_{\Lambda_1\cup\Lambda_2} \tag{1.127}$$

and so by (1.)

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \le c_{16}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{1.128}$$

This holds for all $\Lambda_1' \subset \Lambda_1$, so (2.) is proved.
We set $\phi = f\nu - \nu(f)\nu$. So $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\Lambda_1} = \pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}\phi$. We estimate the $\|\cdot\|_{\Lambda_1,\tilde\vartheta}$-norm of the last term as in the proof of Proposition 1.8.1, but this time using the finer estimates from (2.)

$$\begin{aligned}\|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} &\leq \vartheta^{-|\Lambda(\mathcal{C})|} c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda(\mathcal{C}),\Lambda_2)} \\ &\leq c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\vartheta^{-|\Lambda_r(\mathcal{C})|-\sum_{k=1}^\infty (3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty k n_{\beta,k}}\end{aligned} \tag{1.129}$$

where as before $\Lambda(\mathcal{C}) \stackrel{\text{def}}{=} \tilde\Lambda_{\mathcal{C}} \cup \Lambda_r$. So we get analogously to formulae (1.97) and (1.98):

$$\begin{aligned}&\tilde\vartheta^{|\Lambda_1|} \sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1} \\
&\leq \tilde\vartheta^{|\Lambda_1|}\sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq|n_\beta|<\infty}} 4^K \prod_{k=1}^\infty \left(\exp(c_{21}k^d)\right)^{n_{\beta,k}} (c_1\epsilon)^{|n_\beta|} \\
&\quad\times \prod_{k=1}^\infty \left(\exp(-c_2k^d)\right)^{n_{\beta,k}} \sum_{L=|n_\beta|}^\infty \binom{L}{|n_\beta|} c_r^{|n_\beta|}\eta^L \prod_{k=1}^\infty \left(\exp(c_{22}k^d)\right)^{n_{\beta,k}} \\
&\quad\times \prod_{k=1}^\infty \left(\exp(c_{23}k^d)\right)^{n_{\beta,k}} \eta^{\max\{0,N-L\}} \sum_{l=0}^{|\Lambda_1|-K}\binom{|\Lambda_1|-K}{l}(c_r\eta^N)^l c_h^{|\Lambda_1|-K-l} \\
&\quad\times c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\vartheta^{-l-\sum_{k=1}^\infty (3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty k n_{\beta,k}} \\
&\leq c_{11}\tilde\vartheta^{|\Lambda_1|}\sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq|n_\beta|<\infty}} 4^K (c_1\epsilon c_r)^{|n_\beta|} \\
&\quad\times \prod_{k=1}^\infty \left(\exp((c_{21}-c_2+c_{22}+c_{23}-3^d\ln\vartheta-\ln\kappa)k^d)\right)^{n_{\beta,k}} \\
&\quad\times \sum_{L=|n_\beta|}^\infty \binom{|\Lambda_1|}{K}\eta^{\max\{L,N\}}(\vartheta^{-1}c_r\eta^N + c_h)^{|\Lambda_1|-K}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}.\end{aligned} \tag{1.130}$$

Using (1.121) we get with the same analysis as from (1.98) to (1.103):

$$\leq c_{34}(\epsilon_2 + c_r\tilde\eta^N\frac{\tilde\vartheta}{\vartheta} + c_h\tilde\vartheta)^{|\Lambda_1|}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde\eta^N. \tag{1.131}$$

For sufficiently small ϵ_2 and $\tilde\vartheta$ the term in brackets is smaller than one. Note that there is no condition on N. So we get the same estimates for all $n \geq 0$ and these also hold for $\Lambda \subset \Lambda_1$. So in analogy with (1.61) we get

$$\|\mathcal{L}^N_{F\circ T^\epsilon}\phi - \mathcal{L}^{N+1}_{F\circ T^\epsilon}\phi\|_{\Lambda_1} \le c_{35}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde{\eta}^N \tag{1.132}$$

and as $\mu(\phi) = 0$ we conclude (3.)

□

Proof of Theorem 1.2.2 Applying (1.) of Theorem 1.7.1 we get

$$\begin{aligned}
&\left|\int_M \nu d\mu\, gf - \left(\int_M \nu d\mu\, g\right)\left(\int_M \nu d\mu\, f\right)\right| \\
\le\ & \left|\int_{(S^1)^{\Lambda_1\cup\Lambda_2}} d\mu^{\Lambda_1\cup\Lambda_2}(\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2})gf\right| \\
\le\ & \|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2}\|g\|_\infty\|f\|_\infty \\
\le\ & c_{10}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|g\|_\infty\|f\|_\infty\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)},
\end{aligned} \tag{1.133}$$

so (1.) is proved.

$$\begin{aligned}
&\left|\int_M \nu d\mu\, g\circ\tau\circ S^n f - \left(\int_M \nu d\mu\, g\circ\tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\
=\ & \left|\int_M d\mu\, g\circ\tau\left(\pi_{\tau^{-1}(\Lambda_1)}\circ\mathcal{L}^n_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\tau^{-1}(\Lambda_1)}\right)\right| \\
\le\ & c_{12}c_5^{|\Lambda_1|+|\Lambda_2|}\|f\|_{\Lambda_2}\|g\|_\infty\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)}\tilde{\eta}^n.
\end{aligned} \tag{1.134}$$

Here we have used (3.) of Theorem 1.7.1 and set $c_5 \stackrel{\mathrm{def}}{=} \tilde{\vartheta}^{-1}$. From

$$\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2) \ge m(\tau) - \max\{\|p-q\| : p\in\Lambda_1, q\in\Lambda_2\} \tag{1.135}$$

follows

$$\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)} \le c(\Lambda_1,\Lambda_2,\kappa)\kappa^{m(\tau)} \tag{1.136}$$

where $c(\Lambda_1,\Lambda_2,\kappa)$ is as defined in Theorem 1.2.2. If τ and S commute, (3.) follows from (2.).
We prove (4.) by approximating g and f by functions for which we can apply estimate (2.). For any $\gamma > 0$ we can choose $\Lambda_1 \in \mathcal{F}$ so large that $\|g - g_{\Lambda_1}\|_\infty \le \gamma$. Further there exists an $\tilde{f}_{\Lambda_2} \in \mathcal{H}(A_\delta^{\Lambda_2})$ with $\|f - \tilde{f}_{\Lambda_2}\|_\infty \le \gamma$ (sup-norm on $(S^1)^{\mathbb{Z}^d}$).

So

$$
\begin{aligned}
&\left|\int_M \nu d\mu\, g \circ \tau \circ S^n f - \left(\int_M \nu d\mu\, g \circ \tau\right)\left(\int_M \nu d\mu\, f\right)\right| \qquad (1.137)\\
\leq\ & \left|\int_M \nu d\mu\,(g - g_{\Lambda_1}) \circ \tau \circ S^n f\right| + \left|\int_M \nu d\mu\, g_{\Lambda_1} \circ \tau \circ S^n (\tilde{f}_{\Lambda_2} - f)\right| \\
&+ \left|\int_M \nu d\mu\, g_{\Lambda_1} \circ \tau \circ S^n \tilde{f}_{\Lambda_2} - \left(\int_M \nu d\mu\, g_{\Lambda_1} \circ \tau\right)\left(\int_M \nu d\mu\, \tilde{f}_{\Lambda_2}\right)\right| \\
&+ \left|\left(\int_M \nu d\mu\, g_{\Lambda_1} \circ \tau\right)\left(\int_M \nu d\mu\,(f - \tilde{f}_{\Lambda_2})\right)\right| \\
&+ \left|\left(\int_M \nu d\mu\,(g - g_{\Lambda_1}) \circ \tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\
\leq\ & \|g - g_{\Lambda_1}\|_\infty \|f\|_\infty + \|g_{\Lambda_1}\|_\infty \|f - \tilde{f}_{\Lambda_2}\|_\infty \\
&+ c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} \|g_{\Lambda_1}\|_\infty \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)} \\
&+ \|g_{\Lambda_1}\|_\infty \|f - f_{\Lambda_2}\|_\infty + \|g - g_{\Lambda_1}\|_\infty \|f_{\Lambda_2}\|_\infty \\
\leq\ & (2\|f\|_\infty + 2\|g\|_\infty + 3\gamma)\,\gamma \\
&+ c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} (\|g\|_\infty + \gamma) \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)}
\end{aligned}
$$

and this gets arbitrarily small as we can first choose γ, and then (depending on γ) Λ_1, Λ_2 and $\tilde{f}_{\Lambda_2}$ and finally $\max\{m(\sigma), n(\sigma)\}$.
(5.) follows from (4.) and the commutation of the τ_{e_i} with S.

□

Chapter 2

Weakly Coupled Circle Maps with Asynchronous Updatings

2.0 Introduction

In this paper we study coupled map lattices with indepentent identically (i.i.) Poisson-distributed updatings at the individual sites.
A deterministic coupled map lattice (CML) is given by a $\mathbb{Z}^d$-lattice with a copy of the same Riemannian manifold at each lattice point (i.e. the state space is the product of these manifolds with index set $\mathbb{Z}^d$) and a map on the infinite space that can be decomposed into an uncoupled map that acts individually on each component and an 'interaction step' where the change of each coordinate depends also on the other sites.
L.A. Bunimovich and Y.G. Sinai prove in [7] (cf. also the remarks on this in [4]) the existence and uniqueness of an invariant measure and its exponential decay of correlations for a one-dimensional lattice of interval maps with weak coupling. By constructing a Markov partition they relate the system to a two-dimensional spin system whose Gibbs measure corresponds to the invariant measure of the CML.
G. Keller and M. Künzle prove in [21] the existence and uniqueness of an invariant measure for periodic or infinite one-dimensional lattices of weakly coupled interval maps by studying the transfer operators on the space of measures whose finite-dimensional marginals have densities of bounded variation. For small perturbation of the uncoupled map any invariant measure is 'close' to the one they found.
J. Bricmont and A. Kupiainen extend in [3] and [4, 5] the result of Bunimovich and Sinai [7] to $\mathbb{Z}^d$-lattices of weakly coupled circle maps with analytic and Hölder-continuous interaction, respectively.
They represent the iterates of the Perron-Frobenius operator for finite-dimensional subsystems (over $\Lambda \subset \mathbb{Z}^d$) by a 'polymer'- or 'cluster'-expansion that gives rise to a representation of the corresponding invariant measure in terms of a $(d+1)$-dimensional spin system. The weak limit (as $\Lambda \to \mathbb{Z}^d$) of these measures is the unique (in a certain class) invariant probability measure and it is exponentially

mixing with respect to (wrt) spatio-temporal shifts.
C. Maes A. Van Moffaert introduce in [26] for a similar setting as in [3] a simplified 'cluster'-expansion for the truncated Perron-Frobenius operator and show stochastic stability of the CML under stochastic perturbation.
In [1] V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää and A. Kupiainen define Frechet spaces, and, for $d = 1$, a Banach space and transfer operators for the infinite-dimensional systems, considered by Bricmont and Kupiainen in [3], and study the spectral properties of these operators.
In [12] we consider analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with exponentially decaying interaction and introduce Banach spaces for the infinite-dimensional system that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. We define transfer operators on these spaces, get a unique (in the considered Banach spaces) probability measure and prove its exponential decay of correlations.
CMLs with multi-dimensional local systems of hyperbolic type have been studied by Ya.B. Pesin and Ya.G. Sinai [27], M. Jiang [16, 17], M. Jiang and A. Mazel [18], M. Jiang and Ya.B. Pesin [19] and D.L. Volevich [31, 32].
For detailed reviews on mathematical results on CMLs we refer to [1], [4], [6] and [19].

An interacting particle system (IPS) is given by an infinite lattice with a copy of the same state space (that is usually a finite or countable set but can also be a Riemannian manifold) at each site. The updating at an individual site is a deterministic or stochastic map (e.g. in the case of finite local state spaces it is given by a stochastic matrix with transition probabilities as its coefficients) that is applied with 'exponential waiting times', i.e. like the waiting times for jumps in a Poisson process. The jump rates for the updating depends also on the other sites. R.J. Glauber introduces in [13] (a special case of) the stochastic Ising model as a model for magnetism. The total state space $\{-1,+1\}^{\mathbb{Z}^d}$ represents the spins of the atoms at all sites. The rate for a flip of an individual spin depends on the states of the neighbour sites. F. Spitzer [29, 30] and R.L. Dobrushin [8, 9] study more general systems where the individual jump rates do not only depend on the nearest neighbours.
A basic problem is to establish the existence of infinite systems with asynchronous updatings. The infinitely many jumps in a finite time-interval cannot be 'listed', i.e. there is no order preserving bijection between the time-ordered set of jumps and $\mathbb{N}$.
R.L. Dobrushin obtains in [8] the infinite system as the limit of subsystems over finitely many sites.
By using a percolation argument T.E. Harris proves in [14] that for systems of finite range interaction and a sufficiently small time interval the history of an individual particle depends on only finitely many sites, and so he provides a natural definition of the infinite system. With probability 1 the set $\mathbb{Z}^d$ splits into finite clusters such that each site is affected at most by sites in the same cluster.
R. Holley shows in [15] for generators, corresponding to one-dimensional models,

and T.M. Liggett in [24] for the d-dimensioanal case, that these operators generate, in fact, a semigroup, acting on continuous functions.
Here we have only mentioned different methods to establish the existence of the infinite systems. For detailed reviews on IPSs and results on invariant measures, mixing properties, phase transitions and applications to physics and other sciences we refer to [10] and [25].

In this paper we consider the infinite topological product $M = (S^1)^{\mathbb{Z}^d}$ and continuous updating maps for the individual coordinates that are of finite range or Lipschitz-continuous wrt all coordinates with a summable family of Lipschitz constants (cf. Section 2.2.2 for the definition). The times for the updatings at the individual sites are independently Poisson-distributed with the same constant rate $\lambda > 0$. For the finite range case we show that with probability 1 the updatings at any finite set of sites and any finite time-interval depend on only finitely many sites. Our proof uses time- and space-oriented percolation and is different from the one in [14]. This result provides a natural definition of the infinite dynamical system.
For the systems with infinite range interaction we show that with probability 1 it is well-defined as the net-limit of its finite-dimensional subsystems with arbitrary boundary conditions. We combine standard estimates for error growth with ideas from percolation theory. The limit of the corresponding Markov kernels, acting on continuous functions, exists and provides a definition of the infinite process, different from the widely used generator approach.
Our proofs still work if we replace S^1 by any compact Riemannian manifold or stochastic systems with finite state spaces. The assumption of having the same constant jump rate at all sites is by no means essential and can be weakened to the case of upper bounded individual jump rates that depend on other states as well. However we do not consider these generalizations in this paper.
In a setting similar to that of [12], i.e. for analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with weak, exponentially decaying interaction but with asynchronous updatings as described above, we define transfer operators for the Markov kernels of the infinite system. The operators act on the Banach space $\mathcal{H}_\vartheta$ (introduced in [12]) that includes measures whose finite-dimensional marginals have analytic, exponentially bounded densities. Using 'cluster-expansion'-like techniques, we represent these integral operators in terms of configurations and prove the existence and uniqueness (in $\mathcal{H}_\vartheta$) of an invariant probability measure and its exponential decay of correlations.
The paper is organized as follows. Section 2.1 provides definitions, notation and some propositions about stochastic processes and metric spaces. In Section 2.2 we define the infinite-dimensional systems for finite range (Section 2.2.1) and infinite range interaction (Section 2.2.2) and the corresponding Markov kernels (Section 2.2.3). In Section 2.3 we study the transfer operators for a specific class of interactions. In Section 2.4 we prove the mixing properties of the invariant measure (found in Section 2.3) wrt spatio-temporal shifts.

2.1 Basic Definitions and Examples

In this section we present definitions from probability theory and topology and also introduce most of the notation used in this paper. We have taken most definitions and statements on probability theory from [2].

Definition 2.1.1 $\mathbb{N}$ denotes the set of natural numbers including zero. Let $(E, \mathcal{A}_2)$ be a measurable space, $(\Omega, \mathcal{A}_1, P)$ a probability space and $(X_t)_{t\in I}$ a family (with index set $I \neq \emptyset$) of random variables on $(\Omega, \mathcal{A}_1, P)$ with values in E.

- Then $(\Omega, \mathcal{A}_1, P, (X_t)_{t\in I})$ is called a **stochastic process with values in** $(\mathbf{E}, \mathcal{A}_2)$.
- If $I = \mathbb{N}$ or $I = \{0, 1, \ldots, N\}$ the process is called a **discrete time stochastic process.** If $I = \mathbb{R}^{\geq 0}$, $[0, T]$ or $[0, T)$ for some $T > 0$ the process is called a **continuous time stochastic process.**
- For fixed $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is called the **trajectory** of ω. It is also denoted by $X_\bullet(\omega)$.
- We consider the set $\mathbb{N}$ as measurable space with the discrete σ-algebra. For any set Λ we denote by $\mathbb{N}^\Lambda$ the product space, established with the product σ-algebra.

  A discrete or continuous time stochastic process with values in $\mathbb{N}^\Lambda$ and with index set I and P-a.a. of whose trajectories are non-decreasing (i.e. the functions $t \mapsto \pi_q \circ X_t(\omega)$ are non-decreasing for all $q \in \Lambda$ and P-a.a. $\omega \in \Omega$. 'π_q' denotes the projection on the qth coordinate.), is called **a counting process** with values in $\mathbb{N}^\Lambda$ We say that such a process is **of finite expectation** if for all $t \in I$ the random variable $\omega \mapsto \sum_{q\in\Lambda} \pi_q \circ X_t(\omega)$ has finite expectation.

Remark 2.1.1
1. We will also use the short-hand-notation $X_\bullet$ for a stochastic process if Ω, $\mathcal{A}_1$ and P are obvious from the context.
2. The term *path* seems to be more common than *trajectory* but we will denote something else later on by *path.*
3. Finite expectation means that with probability 1 there are only finitely many jumps (cf. Definition 2.1.2 below) in every finite time-interval.

Definition 2.1.2 (cf. [2]) Let $(\Omega, \mathcal{A}_1, P, (X_t)_{t\in I})$ be a discrete time counting process with values in $\mathbb{N}$ as in Definition 2.1.1 and $\omega \in \Omega$. We say that $X_\bullet(\omega)$ **jumps,** or has a **jump,** at time $t \in I \setminus \{0\}$ if $X_{t-1}(\omega) < X_t(\omega)$. Then $X_t(\omega) - X_{t-1}(\omega)$ is called the **size of the jump.**
Now let $(\Omega, \mathcal{A}_1, P, (X_t)_{t\in I})$ be a continuous time counting process with values in $\mathbb{N}$ as in Definition 2.1.1 and $\omega \in \Omega$. We define

$$X_t^+(\omega) \stackrel{\text{def}}{=} \lim_{s \searrow t} X_s(\omega) \tag{2.1}$$

$$X_t^-(\omega) \stackrel{\text{def}}{=} \begin{cases} \lim_{s \nearrow t} X_s(\omega) & \text{if } \; t > 0 \\ X_0(\omega) & \text{if } \; t = 0 \end{cases} \tag{2.2}$$

We say that $X_.(\omega)$ **jumps** at time $t \geq 0$ if $X_t^-(\omega) < X_t^+(\omega)$. Then $X_t^+(\omega) - X_t^-(\omega)$ is called the **size of the jump.**
Let $X_.(\omega)$ be a (discrete or continuous time) counting process with values in $\mathbb{N}^\Lambda$ and $\omega \in \Omega$. We say that $X_.(\omega)$ jumps at time t and site $q \in \Lambda$ if $\pi_q \circ X_.(\omega)$ jumps at t. Then we also say that ω jumps at (q, t).
We define the **jump set $\Lambda(\omega, t)$ of ω at time t** as the set of all $q \in \Lambda$ such that ω jumps at (q, t).

Definition 2.1.3 (cf. [2]) Let $I = \mathbb{R}^{\geq 0}$ or $I = [0, T)$ for some $T > 0$. A stochastic process $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ with values in $\mathbb{N}$ is called **(normalized) Poisson process with parameter $\lambda > 0$** if the following holds:

1. The process has stationary and independent increments which for all $s < t \in I$ satisfy

$$P(\{\omega : X_t(\omega) - X_s(\omega) = n\}) = p_\lambda(t - s, n) \tag{2.3}$$

with

$$p_\lambda(t, n) \stackrel{\text{def}}{=} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{2.4}$$

2. P-almost every trajectory $X_.(\omega)$ is a right-continuous, increasing function having at most jumps of size 1.

3. At time 0 P-a.a. trajectories have value 0:

$$P(\omega : X_0(\omega) = 0) = 1 \tag{2.5}$$

Theorem 2.1.1 *(cf. [2], Satz 41.2) For any $\lambda > 0$ and I as in Definition 2.1.3 there exists a (normalized) Poisson process with parameter λ. Any two such processes are equivalent (i.e. if $X^1_.$ and $X^2_.$ are two such processes then for any finite sequence $t_1 < \ldots < t_n$ in I the projections $(X^1_{t_1}, \ldots, X^1_{t_n})$ and $(X^2_{t_1}, \ldots, X^2_{t_n})$ have the same distribution.)*

□

Definition 2.1.4 Let Λ be a nonempty set and $(\Omega_q, \mathcal{A}_q, P_q, (X_t^q)_{t\in I})_{q\in\Lambda}$ be a family of stochastic processes with values in $(E_q, \mathcal{A}^q)$, respectively. We set

$$\Omega \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} \Omega_q, \tag{2.6}$$

$$\tilde{\mathcal{A}} \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} \mathcal{A}_q, \tag{2.7}$$

$$\tilde{P} \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} P_q, \tag{2.8}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \text{completion of } \tilde{\mathcal{A}} \text{ wrt } \tilde{P}, \tag{2.9}$$

$$P \stackrel{\text{def}}{=} \text{extension of } \tilde{P} \text{ to } \mathcal{A} \tag{2.10}$$

$$\text{and} \quad X_t \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} X_t^q. \tag{2.11}$$

Then the process $(\Omega, \mathcal{A}, P, (X_t)_{t\in I}$ with values in $(\bigotimes_{q\in\Lambda} E_q, \bigotimes_{q\in\Lambda} \mathcal{A}^q)$ is called the **product of the family of processes.**

Remark 2.1.2

1. Products of stochastic processes as in Definition 2.1.4 exist. For example the existence of the non-completed product measure follows from standard measure theory (cf. [2].)
2. For non-empty, at most countable Λ and a family (indexed by Λ) of Poisson processes two such products $X_\cdot^1$ and $X_\cdot^2$ are equivalent because for all $q \in \Lambda$ the Poisson processes $\pi_q \circ X_\cdot^1$ and $\pi_q \circ X_\cdot^2$ are equivalent (cf. Theorem 2.1.1). It follows from the definition of the product σ-algebra $\bigotimes_{q\in\Lambda} \mathcal{A}^q$ that $X_\cdot^1$ and $X_\cdot^2$ are equivalent.

Definition 2.1.5 Let $\lambda > 0$ and Λ a nonempty, at most countable set. A **Poisson process on Λ with parameter λ** is the product of a family, indexed by Λ, of Poisson processes with parameter λ.

Remark 2.1.3

1. For $\lambda > 0$ the Poisson process on $\mathbb{Z}^d$ with parameter λ is clearly not of finite expectation. In fact for any $t > 0$ there are P-almost surely infinitely many jumps in $[0, t]$, i.e.

$$P(\{\omega : \sum_{q\in\mathbb{Z}^d} \pi_q \circ X_t(\omega) = \infty\}) = 1. \tag{2.12}$$

2. But if $\Lambda_1 \subset \mathbb{Z}^d$ is finite then $\pi_{\Lambda_1} \circ X_\cdot(\omega)$ has finitely many jumps in $[0, t]$ for P-a.a. $\omega \in \Omega$ and $t > 0$.

3. There are P-almost surely no simultaneous jumps at two different sites:

$$P(\{\omega : \exists q_1 \neq q_2 \in \mathbb{Z}^d, t \geq 0 \text{ such that } \omega \text{ jumps at } (q_1, t) \text{ and } (q_2, t)\}) = 0. \tag{2.13}$$

4. For $0 \leq t_0 < t$

$$P(\{\omega : \omega \text{ jumps at } t_0\}) = 0. \tag{2.14}$$

Proof of Remark 2.1.3 We only show (2.13). The proofs of the other statements are similar. We set

$$A(q_1, q_2, T) \stackrel{\text{def}}{=} \{\omega : \exists t \in [0, T) \text{ such that } \omega \text{ jumps at } (q_1, t) \text{ and } (q_2, t)\}. \tag{2.15}$$

We have to prove that the set

$$\bigcup_{T \in \mathbb{N}} \bigcup_{q_1, q_2 \in \mathbb{Z}^d} A(q_1, q_2, T) \tag{2.16}$$

has P-measure zero and it is sufficient to show that

$$P(A(q_1, q_2, T)) = 0 \tag{2.17}$$

for fixed $q_1 \neq q_2 \in \mathbb{Z}^d$ and $T > 0$. For this we set

$$I_{N,k} \stackrel{\text{def}}{=} \left[(k-1)\frac{T}{N}, k\frac{T}{N}\right) \tag{2.18}$$

for $N \in \mathbb{N} \setminus \{0\}$ and $1 \leq k \leq N$. We have for $i = 1, 2$:

$$P\left(\{\omega : \text{ jumps at } (q_i, t) \text{ for some } t \in I_{N,k}\}\right) = 1 - e^{-\lambda \frac{T}{N}} \tag{2.19}$$

and so, using the estimate $e^x \geq 1 + x$:

$$\begin{aligned} & P\left(\{\omega : \exists k; t_1, t_2 \in I_{N,k} \text{ such that } \omega \text{ jumps at } (q_1, t_1) \text{ and } (q_1, t_2)\}\right) \\ \leq\ & N\left(1 - e^{-\lambda \frac{T}{N}}\right)^2 \\ \leq\ & \lambda^2 T^2 \frac{1}{N} \end{aligned} \tag{2.20}$$

which converges to 0 as $N \to \infty$.

□

In Sections 2.2.3 and 2.3 we will use discrete time processes to *approximate* Poisson processes. (The convergence in distribution will be made precise in Lemma 2.1.2.)

Definition 2.1.6 For $d \geq 1$ we denote by $\mathcal{F}$ the set of finite subsets of $\mathbb{Z}^d$. Let $\lambda, T > 0$, $N > \lambda T$, $I = \{1, \ldots, N\}$, $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$,

$$\Omega_{\Lambda,N} \stackrel{\text{def}}{=} \{0,1\}^{\Lambda \times I} \tag{2.21}$$

and $\mathcal{A}_{\Lambda,N}$ the discrete σ-algebra on $\Omega_{\Lambda,N}$. Elements of Ω are denoted by $\omega = (\omega(q,n))_{(q,n)\in\Lambda\times I}$. We set:

$$p \stackrel{\text{def}}{=} \frac{\lambda T}{N} \tag{2.22}$$

$$|\omega| \stackrel{\text{def}}{=} \sum_{(q,n)\in\Lambda\times I} \omega(q,n) \tag{2.23}$$

$$P_{\Lambda,N}(\{\omega\}) \stackrel{\text{def}}{=} p^{|\omega|}(1-p)^{|\Lambda|N-|\omega|} \tag{2.24}$$

$$x_{q,n}(\omega) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \omega(q,k) \tag{2.25}$$

$$X_n(\omega) \stackrel{\text{def}}{=} (x_{q,n})_{q\in\Lambda} \tag{2.26}$$

The discrete time counting process $(\Omega_{\Lambda,N}, \mathcal{A}_{\Lambda,N}, P_{\Lambda,N}, (X_t)_{t\in\{1,\ldots,N\}})$ with values in $\mathbb{N}^\Lambda$ is called **Bernoulli process with parameters λ, T, N and values in $\mathbb{N}^\Lambda$.**

The following two definitions prepare Definition 2.1.9 that we will need in Section 2.2.

Definition 2.1.7 In view of Definition 2.1.3 and Remark 2.1.3 we define (for a given Poisson process like in that remark) the set $\mathcal{N}_1$ of P-measure zero:

$$\begin{aligned}\mathcal{N}_1 \stackrel{\text{def}}{=} \{\omega : X_\cdot(\omega) &\text{ is not non-decreasing, has jumps at 0,} \\ &\text{simultaneous jumps or jumps of size greater than 1}\}.\end{aligned} \tag{2.27}$$

Definition 2.1.8 For $q = (q_1, \ldots, q_n) \in \mathbb{Z}^d$ we define

$$||q|| \stackrel{\text{def}}{=} |q_1| + \ldots + |q_n|. \tag{2.28}$$

For $R \geq 0$

$$B_R(q) \stackrel{\text{def}}{=} \{\tilde{q} \in \mathbb{Z}^d : ||q - \tilde{q}|| \leq R\} \tag{2.29}$$

is the set of points that have distance at most R from q.

Definition 2.1.9 Let $a, b \in \mathbb{Z}^d$ and $n \geq 0$. A **path from a to b** is a finite sequence $Q = (q_0 = a, q_1, \ldots, q_n = b)$ of points $q_i \in \mathbb{Z}^d$. We call $\max_{0\leq i\leq n-1} ||q_{i+1} - q_i||$ the **step size** of Q. Note the special case of a path $Q = (q_0)$. It is called the **empty path at site q_0** and we define its step size to be 0.

Definition 2.1.10 Let $(\Omega, \mathcal{A}, P, (X_t)_{t\geq 0})$ be a Poisson process with parameter $\lambda > 0$ and with values in $\mathbb{N}^{\mathbb{Z}^d}$. Let $T > 0$, $\omega \in \Omega$ and $Q = (q_0 = a, q_1, \ldots, q_n = b)$ a path. We extend Q to the infinite sequence $\widetilde{Q} = (q_0, q_1, \ldots, q_n, q_{n+1} = q_n, \ldots)$ in which q_n is repeated.
We define a process $(\Omega, \mathcal{A}, P, (Z_t)_{t\in[0,T]})$ with values in $\mathbb{N}$ as follows.

$$\begin{aligned} Z : [0,t] \times \Omega &\to \mathbb{N} \\ (t,\omega) &\mapsto Z_t(\omega) \end{aligned} \tag{2.30}$$

If $\omega \in \mathcal{N}_1$ or it does not jump at (q_0, t) for any $t \in (0,T)$ we set $Z_{\bullet}(\omega) = 0$ on $[0,T]$. Otherwise there is a maximal sequence $T > t_0 > t_1 > \ldots > t_{m(\omega)}$ such that with $t_{-1} \stackrel{\text{def}}{=} T$:

$$t_i \stackrel{\text{def}}{=} \max\{t \in (0, t_{i-1}) : \omega \text{ jumps at } (q_i, t)\} \quad \text{for} \quad 0 \leq i \leq m(\omega). \tag{2.31}$$

'Maximal' means that ω does not jump at $q_{m(\omega)+1}$ in the time interval $(0, t_{m(\omega)})$ and the sequence cannot be extended. (Intuitively one can think that one sits at time T at site q_0 and, going *backwards* in time, waits for the next jump of ω at q_0 (which happens at time t_0), then jumps (instantly) to q_1 and waits (*backwards* in time) for the next jump of ω at q_1, then jumps to q_2 etc. After n jumps (should this occur) one does not change the sites any more, but possibly jumps from q_n to q_n. $m(\omega)$ is the total number of jumps. It is P-a.s. finite because P-a.a. ω have only finitely many jumps at q_n.)
We set for $t \in [0,T]$:

$$\widetilde{Z}_t(\omega) \stackrel{\text{def}}{=} \begin{cases} i & \text{for} \quad t \in [t_i, t_{i-1}) \\ m(\omega) & \text{for} \quad t \in [0, t_{m(\omega)}] \end{cases} \tag{2.32}$$

And $Z_{\bullet}(\omega)$ is the (uniquely defined) right-continuous function, such that $Z_{\bullet}(\omega) = \widetilde{Z}_{T-\bullet}(\omega)$ everywhere, except possibly where these functions jump. Then $(\Omega, \mathcal{A}, P, (Z_t)_{t\in[0,T]})$ is a Poisson process with parameter λ. (A precise proof of this uses that the constructed process is 'made of' independent Poisson processes and that these have independent increments.) We call it the **Poisson process induced by the path Q**.

Definition 2.1.11 In the setting of Definition 2.1.10 we call Q a **causal path wrt** $(\mathbf{t}, \omega)$ if $Z_T(\omega) \geq n$ and a **maximal causal path wrt** $(\mathbf{t}, \omega)$ if $Z_T(\omega) = n$. (The latter means that $Q = (q_0, \ldots, q_n)$ cannot be extended to any causal path $(q_0, \ldots, q_n, q_{n+1})$.)
We define:

- $\mathbf{Path(q, n, R)}$ to be the set of paths that start at q, have exactly n steps and are of step size at most R.

- **Path(q → Λ)** for any $\emptyset \neq \Lambda \in \mathbb{Z}^d$ to be the set of paths starting at q and ending in Λ.

- **Path$_{\mathbf{C}}$(t, ω, q, Λ)** for $q \in \Lambda$ to be the set of causal wrt (t, ω) paths $Q = (q_0 = q, \ldots, q_n)$ such that

  1. Q is maximal causal and $q_0, \ldots, q_n \in \Lambda$

  or 2. $q_0, \ldots, q_{n-1} \in \Lambda$ and $q_n \in \Lambda^C$.

- **Path$_{\mathbf{C}}$(t, ω, q → Λ$^{\mathbf{C}}$)** for $q \in \Lambda$ to be the set of causal paths $(q_0 = q, \ldots, q_n)$ such that $q_0, \ldots, q_{n-1} \in \Lambda$ and $q_n \in \Lambda^C$. (So this is the subset of elements in **Path$_{\mathbf{C}}$**(t, ω, q, Λ) for which case 2. applies.)

Remark 2.1.4 1. We have defined the property of being causal for general paths and not related this definition to any kind of interaction. When we study finite range interaction, of range R say, we will consider only causal paths of step size at most R.

2. A term like *inverse causal path* from a to b instead of *causal path* would actually be more appropiate as it corresponds to b affecting a (cf. Definition 2.2.1) but not necessarily the other way around. However, we prefer the shorter notion.

Definition 2.1.12 (cf. [2]) Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. A map $K : \Omega_1 \times \mathcal{A}_2 \to [0, 1]$ is called a **Markov kernel from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$** if the two following conditions are satisfied:

MK1 $\omega_1 \mapsto K(\omega_1, A_2)$ is $\mathcal{A}_1$-measurable for all $A_2 \in \mathcal{A}_2$.
MK2 $A_2 \mapsto K(\omega_1, A_2)$ is a probability measure on $\mathcal{A}_2$ for all $\omega_1 \in \mathcal{A}_1$.

If $(\Omega_1, \mathcal{A}_1) = (\Omega_2, \mathcal{A}_2)$ then K is called a **Markov kernel on $(\Omega_1, \mathcal{A}_1)$**.

Example 2.1.1 Let (Y, ϱ_Y) be a metric space and $\mathcal{B}_Y$ its Borel σ-algebra. $C^0(Y, Y)$ is the space of continuous maps from Y to Y. It has a uniform metric, defined by $\varrho_{C^0(Y,Y)}(g_1, g_2) = \sup_{\mathbf{y} \in \mathbf{Y}} \varrho_Y(g_1(\mathbf{y}), g_2(\mathbf{y}))$ and the Borel σ-algebra $\mathcal{B}_{C^0(Y,Y)}$ wrt this metric. Further let $(\Omega, \mathcal{A}, P)$ be a probability space and

$$\begin{aligned} S : \Omega &\to C^0(Y, Y) \\ \omega &\mapsto S_\omega \end{aligned} \tag{2.33}$$

a measurable (wrt the σ-algebras $\mathcal{A}$ and $\mathcal{B}_{C^0(Y,Y)}$) map.
Then

$$K_S(y, Y_1) \stackrel{\text{def}}{=} P(\{\omega : S_\omega(y) \in Y_1\}) \tag{2.34}$$

for all $y \in Y, Y_1 \in \mathcal{B}_Y$, defines a Markov kernel on $(Y, \mathcal{B}_Y)$.

Proof To verify **MK1** we fix an $Y_1 \in \mathcal{B}_Y$ and show that the map $y \mapsto K_S(y, Y_1)$ is measurable. First we note that S can be seen as a measurable map from $\Omega \times Y$ to Y. We write it as the composite of measurable maps $S \times \mathrm{id}_Y$ and the 'evaluation map':

$$(\omega, y) \mapsto (S_\omega, y) \mapsto S_\omega(y). \tag{2.35}$$

The map $S \times \mathrm{id}_Y$ is measurable by assumption and the definition of the product σ-algebra of $\mathcal{C}^0(Y, Y) \times Y$. The evaluation map is continuous (wrt to the product topology), hence measurable wrt the Borel σ-algebras. So the composite in (2.35) is measurable in $\Omega \times M$. It follows that the map $y \mapsto P(\{\omega : S_\omega(y) \in Y_1\})$ is measurable (cf. Lemma 8.1 on p. 159 in [22]) and so **MK1** holds.
Next we show **MK2**. Consider for fixed $y \in Y$ the composite of measurable maps

$$\omega \mapsto (\omega, y) \mapsto S_\omega(y) \tag{2.36}$$

that maps Ω to Y. We see that $K(y, \cdot)$ is the image of P wrt this map and so a probability measure which was to be shown.

□

Definition 2.1.13 (cf. [2]) Let K be a Markov kernel from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$ and $E^*(\mathcal{A}_i)$ $(i = 1, 2)$ the set of $\mathcal{A}_i$-measurable functions with values in $[0, \infty]$. Then K defines a map from $E^*(\mathcal{A}_2)$ to $E^*(\mathcal{A}_1)$ as follows:

$$(Kf)(\omega_1) \stackrel{\text{def}}{=} \int_{\Omega_2} K(\omega_1, d\omega_2) f(\omega_2) \tag{2.37}$$

for any $f \in E^*(\mathcal{A}_2)$.

Example 2.1.2 (cf. [2]) For the characteristic function χ_{A_2} of an $\mathcal{A}_2$-measurable set A_2 we get

$$K\chi_{A_2}(\omega_1) = K(\omega_1, A_2). \tag{2.38}$$

Now we consider a special case of Example 2.1.1.

Example 2.1.3 Let $S : Y \to Y$ be a continuous map map on (Y, ϱ_Y) and let $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ be a counting process with values in $\mathbb{N}$ and $t \in I$.
The map

$$\begin{aligned} S_\omega^t : Y &\to Y \\ y &\mapsto S^{X_t(\omega)}(y), \end{aligned} \tag{2.39}$$

where $S^{X_t(\omega)}$ denotes the $X_t(\omega)$-th iterate of S, is well-defined for all $\omega \in \Omega$. Further $S_\omega(y)$ is measurable wrt (ω, y). In fact, S_ω depends just on $X_t(\omega)$ and so we get a countable, measurable partition of Ω:

$$\Omega = \bigcup_{n\in\mathbf{N}} U(n) \tag{2.40}$$

$$\text{with} \quad U(n) \stackrel{\text{def}}{=} \{\omega \in \Omega : X_t(\omega) = n\} \tag{2.41}$$

We define a Markov kernel by

$$\begin{aligned} K_S^t(y, Y_1) &\stackrel{\text{def}}{=} P(\{\omega : S_\omega^t(y) \in Y_1\} \\ &= \sum_{n:S^n(y)\in Y_1} P(U(n)) \\ &= \int_\Omega dP(\omega)\, \chi_{Y_1} \circ S_\omega^t(y) \end{aligned} \tag{2.42}$$

for $y \in Y$ and $Y_1 \in \mathcal{B}_Y$.

We prepare a generalization of Example 2.1.3 with a definition and a technical lemma.

Definition 2.1.14 Let $\Lambda \in \mathcal{F}\setminus\{\emptyset\}$ be fixed. We define $\mathcal{J}$ to be the union of a one-point set $\{j_\infty\}$ and the set of finite sequences $(\Lambda_1, \ldots, \Lambda_n)$ of subsets of Λ. Then $\mathcal{J}$ is countable and we consider it as a measurable space, established with the discrete σ-algebra.
Let $(\Omega, \mathcal{A}, P, (X_t)_{t\in I})$ be a discrete or continuous time counting process with values in $\mathbb{N}^\Lambda$ and index-set $I = \{1, \ldots, N\}$ or $[0, T]$, respectively. We define a map

$$\begin{aligned} \mathbf{j} : \Omega &\to \mathcal{J} \\ \omega &\mapsto \mathbf{j}(\omega) \end{aligned} \tag{2.43}$$

If $X_{\bullet}(\omega)$ is non-decreasing, has only finitely many jumps and at most jumps of size 1 then we define $\mathbf{j}(\omega)$ to be the (time-ordered) sequence of jump sets of ω. Otherwise we set $\mathbf{j}(\omega) = j_\infty$. We define for $j \in \mathcal{J}$:

$$U(j) \stackrel{\text{def}}{=} \{\omega : \mathbf{j}(\omega) = j\} \tag{2.44}$$

Lemma 2.1.1 *Let $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ be fixed and $(\Omega, \mathcal{A}, P, (X_t)_{t\in I})$ a discrete or continuous time counting process with bounded index-set I and values in $\mathbb{N}^\Lambda$ such that for P-a.a. ω the trajectory $X_{\bullet}(\omega)$ is non-decreasing, has only finitely many jumps and at most jumps of size 1. Then the map $\mathbf{j}$, as defined in Definition 2.1.14, is measurable.*

Proof For the discrete time case this is obvious because then all subsets of Ω are measurable. Now we consider the continuous time case with $I = [0,T]$. By assumption $\mathcal{N} = U(j_\infty)$ is measurable and has measure zero. We have to show that $U(j)$ is measurable for any $j = (\Lambda_1, \ldots, \Lambda_n)$. For any $q_1, q_2 \in \Lambda$, $n_1, n_2 \in \mathbb{N} \setminus \{0\}$ and $t \in [0,T]$ we define $A_1(q_1, n_1, q_2, n_2)$ to be the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at q_1 and at least n_2 jumps at q_2 and the n_1th jump at q_1 happens at the same time as the n_2th jump at q_2. Similarly $A_2(q_1, n_1, q_2, n_2)$ is the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at site q_1 and the n_1th jump at q_1 occurs before the n_2th jump (if any) at q_2. We only show the measurability of the sets $A_2(\cdot)$. The proof of the measurability of the sets $A_1(\cdot)$ uses similar arguments.

$$A^{\geq}(q_1, n_1, t) \overset{\text{def}}{=} \{\omega \in \Omega \setminus \mathcal{N} : \pi_{q_1} \circ X_t(\omega) \geq n_1\} \tag{2.45}$$

is the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at site q_1 and the n_1th of these jumps happens at the latest at time t. We define sets $A^{>}(q_1, n_1, t)$ etc. analogously. $A^{\geq}(q_1, n_1, t)$ and $A^{<}(q_1, n_1, t)$ are measurable, and so is $A_2(q_1, n_1, q_3, n_3)$ since

$$A_2(q_1, n_1, q_3, n_3) = \bigcup_{t \in [0,T) \cap \mathbb{Q}} \left(A^{\geq}(q_1, n_1, t) \cap A^{<}(q_3, n_3, t)\right). \tag{2.46}$$

Now ω belongs to $U(j)$ if and only if, for all $1 \leq k \leq n$ and $q_1, q_2 \in \Lambda_k$ and $q_3 \in \Lambda \setminus \Lambda_k$ the following holds:

- If for exactly n_1 indices $1 \leq i \leq k$ the point q_1 belongs to Λ_i and for exactly n_2 indices $1 \leq j \leq k$ the point q_2 belongs to Λ_j then $\omega \in A_1(q_1, n_1, q_2, n_2)$.
- If for exactly n_1 indices $1 \leq i \leq k$ the point q_1 belongs to Λ_i and for exactly $n_3 - 1$ indices $1 \leq j < k$ the point q_3 belongs to Λ_j then $\omega \in A_2(q_1, n_1, q_3, n_3)$.
- If for exactly $l \geq 0$ indices $1 \leq k_1 < k_2 < \ldots < k_l \leq n$ a point $q \in \Lambda$ belongs to A_{k_i} then $\omega \in \{\tilde{\omega} \in \Omega \setminus \mathcal{N} : \pi_q \circ X_t(\tilde{\omega}) = l\}$.

We see that $U(j)$ is the intersection of finitely many measurable sets and hence measurable.

□

Example 2.1.4 Let us consider a generalization of Example 2.1.3. Let (Y, ϱ_Y) be a measurable space, Λ a non-empty finite set and $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ a counting process with values in $\mathbb{N}^\Lambda$ that has finite expectation and with P-almost surely only jumps of size at most 1. Let $(S_{\Lambda_1})_{\Lambda_1 \subset \Lambda}$ be a family of continuous maps on Y^Λ, such that S_{Λ_1} changes at most the Λ_1-coordinates, i.e. if $\mathbf{y}_\Lambda \in Y^\Lambda$ and $q \in \Lambda \setminus \Lambda_1$ we have for the qth coordinate $\pi_q \circ S_{\Lambda_1}(\mathbf{y}_\Lambda) = y_q$.
For $t \in I$ and P-a.a. $\omega \in \Omega$ with $X_t(\omega) \in \mathbb{N}^\Lambda$ we have a finite sequence of jump-sets $\mathbf{j}(\omega) = (\Lambda_1, \ldots, \Lambda_n)$, as defined in Definition 2.1.14, and it depends measurably on ω, as was shown in Lemma 2.1.1. We define

$$S(\omega): \quad Y^\Lambda \to Y^\Lambda \tag{2.47}$$
$$\mathbf{y}_\Lambda \mapsto S_{\mathbf{j}(\omega)} \stackrel{\text{def}}{=} S_{\Lambda_n} \circ \ldots \circ S_{\Lambda_1}(\mathbf{y}_\Lambda) \tag{2.48}$$

We get a representation of $K_S^t(y, Y_1)$, similar to the one in (2.42):

$$\begin{aligned} K_S^t(y, Y_1) &= P(\{\omega : S_\omega(\mathbf{x}) \in Y_1\}) \\ &= \int_\Omega dP(\omega)\, \chi_{Y_1} \circ S_\omega(y) \\ &= \sum_{j \in \mathcal{J}: S_j(y) \in Y_1} P(U(j)) \end{aligned} \tag{2.49}$$

for $y \in Y$ and $Y_1 \in \mathcal{Y}$.

We have seen in Example 2.1.4 that $S(\omega)$ depends on $\mathbf{j}(\omega)$ only. In Section 2.2.3 we will approximate the Markov kernels for Poisson processes by kernels for Bernoulli processes, and in Section 2.3 do an analogous approximation for transfer operators. We prepare this in the following lemma.

Lemma 2.1.2 *Let $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$, $T > 0$, $\lambda > 0$, $(\Omega, \mathcal{A}, P, (X_t)_{t \in [0,T)})$ a Poisson process with parameter λ and with values in $\mathbb{N}^\Lambda$ and for each integer $N > \lambda T$, $(\Omega_{\Lambda,N}, \mathcal{A}_{\Lambda,N}, P_{\Lambda,N}, (X_t)_{t \in \{1,\ldots,N\}})$ the Bernoulli process with parameters λ, T, N and values in N^Λ. Let*

$$\mathbf{j}: \quad \Omega \to \mathcal{J} \tag{2.50}$$
$$\mathbf{j}_N: \quad \Omega_{\Lambda,N} \to \mathcal{J}, \tag{2.51}$$

be as defined in Definition 2.1.14, for the Poisson process and the Bernoulli processes, respectively. We consider $\mathbf{j}_N$ and $\mathbf{j}$ as random variables with different probability spaces but the same range.
Then the sequence $(\mathbf{j}_N)_{N > \lambda T}$ converges to $\mathbf{j}$ in distributuion.

Proof We define for all $J \subset \mathcal{J}$ the sets $U(J) \stackrel{\text{def}}{=} \mathbf{j}^{-1}(J)$ and $U_N(J) \stackrel{\text{def}}{=} \mathbf{j}_N^{-1}(J)$.
We have to show that for all $J \subset \mathcal{J}$

$$\lim_{N \to \infty} P_{\Lambda,N}(U_N(J)) = P(U(J)). \tag{2.52}$$

Because of

$$\sum_{j \in \mathcal{J}} P_{\Lambda,N}(U_N(j)) = \sum_{j \in \mathcal{J}} P(U(j)) = 1 \tag{2.53}$$

and Lemma 2.1.3 (see below) we have to show (2.52) only for the special case $J = \{j\}$.
For $J = \{j_\infty\}$ equation (2.52) holds because $P(U(j_\infty)) = 0$ (by definition of the Poisson process) and $U_N(j_\infty) = \emptyset$.
If j contains a set with more than one element then $P(U(j)) = 0$ by Remark 2.1.3.2, and for the Bernoulli process with parameters λ, T, N the probability of simultaneous jumps is

$$\begin{aligned} & P_{\Lambda,N}(\{\omega : \omega \text{ has simultaneous jumps}\}) \\ \leq\ & N \sum_{n=2}^{|\Lambda|} \binom{|\Lambda|}{n} \left(\frac{\lambda T}{N}\right)^n \left(1 - \frac{\lambda T}{N}\right)^{|\Lambda|-n}, \end{aligned} \tag{2.54}$$

and this tends to 0 (as $N \to \infty$) which was to be shown.
Finally we consider $j = (\{q_1\}, \dots, \{q_n\})$. We have

$$\begin{aligned} & P_{\Lambda,N}(U_N(j)) \\ = & \binom{N}{n} \left[\frac{\lambda T}{N}\left(1 - \frac{\lambda T}{N}\right)^{|\Lambda|-1}\right]^n \left[\left(1 - \frac{\lambda T}{N}\right)^{|\Lambda|}\right]^{N-n} \\ = & \frac{N(N-1)\cdot \ldots \cdot (N-n+1)}{n!} \frac{(\lambda T)^n}{N^n} \left(1 - \frac{\lambda T}{N}\right)^{(|\Lambda|-1)n} \left(1 - \frac{\lambda T}{N}\right)^{(N-n)|\Lambda|} \\ = & \frac{N(N-1)\cdot \ldots \cdot (N-n+1)}{N^n} \left(1 - \frac{\lambda T}{N}\right)^{-n} \left(1 - \frac{\lambda T}{N}\right)^{N|\Lambda|} \frac{(\lambda T)^n}{n!} \end{aligned} \tag{2.55}$$

and so

$$\lim_{N \to \infty} P_{\Lambda,N}(U(j)) = e^{-|\Lambda|\lambda T} \frac{(\lambda T)^n}{n!}. \tag{2.56}$$

This is equal to $P(U(j))$ because $\sum_{q \in \Lambda} \pi_q \circ X_\cdot$ is a Poisson process with parameter $|\Lambda|\lambda$ and for any $n \in \mathbb{N}$ the $|\Lambda|^n$ (ordered) sequences of jump-sites have all the same probability.

□

In the proof of Lemma 2.1.2 we have used the following lemma.

Lemma 2.1.3 *Let* $(a_n)_{n\in\mathbb{N}}$, $\left(a_n^{(k)}\right)_{n\in\mathbb{N}}$ *(with* $k \in \mathbb{N}$*) be sequences of non-negative real numbers such that*

$$\sum_{n\in\mathbb{N}} a_n = 1, \tag{2.57}$$

$$\sum_{n\in\mathbb{N}} a_n^{(k)} = 1 \quad \textit{for all } k \tag{2.58}$$

$$\textit{and} \quad \lim_{k\to\infty} a_n^{(k)} = a_n \quad \textit{for all } n. \tag{2.59}$$

Then for any $\tilde{N} \subset \mathbb{N}$:

$$\lim_{k\to\infty} \sum_{n\in\tilde{N}} a_n^{(k)} = \sum_{n\in\tilde{N}} a_n. \tag{2.60}$$

Proof Let $\epsilon > 0$. Choose n_0 and k_0 such that for all $k \geq k_0$:

$$\sum_{n=0}^{n_0} a_n > 1-\epsilon \tag{2.61}$$

$$\text{and} \quad \sum_{n=0}^{n_0} |a_n - a_n^{(k)}| < \epsilon. \tag{2.62}$$

Then we also have for $k \geq k_0$

$$\begin{aligned} \sum_{n=0}^{n_0} a_n^{(k)} &\geq \sum_{n=0}^{n_0} a_n - \sum_{n=0}^{n_0} |a_n - a_n^{(k)}| \\ &> 1-2\epsilon \end{aligned} \tag{2.63}$$

$$\begin{aligned} \text{and} \quad \sum_{n=n_0+1}^{\infty} |a_n - a_n^{(k)}| &\leq \sum_{n=n_0+1}^{\infty} a_n^{(k)} + \sum_{n=n_0+1}^{\infty} a_n \\ &\leq 3\epsilon \end{aligned} \tag{2.64}$$

We conclude from (2.62) and (2.64) that

$$\begin{aligned} \left| \sum_{n\in\tilde{N}} a_n^{(k)} - \sum_{n\in\tilde{N}} a_n \right| &\leq \sum_{n=0}^{\infty} |a_n - a_n^{(k)}| \\ &< 4\epsilon. \end{aligned} \tag{2.65}$$

□

As we are interested in spatially extended systems we need some definitions and facts about infinite-dimensional systems.

Definition 2.1.15 S^1 is the one-dimensional sphere. We define it to be isometric as Riemannian manifold to $\mathbb{R}/2\pi\mathbb{Z}$. This defines in particular a metric ϱ_{S^1} on S^1 and also the normalized Lebesgue measure on the (completed) Borel σ-algebra. The diameter of S^1 is

$$c_s \stackrel{\text{def}}{=} \operatorname{diam}_{\varrho_{S^1}}(S^1) = \pi. \tag{2.66}$$

(It seems a bit redundant to introduce the constant c_S instead of using π in the following. But we indicate that the proofs in Section 2.2 work if S^1 is replaced by any compact Riemannian manifold or more general by a bounded metric space with a Borel probability measure. Further we use the letter 'π' as notation for projections.)
We set

$$M \stackrel{\text{def}}{=} (S^1)^{\mathbb{Z}^d} \tag{2.67}$$

and give it the product topology and product Lebesgue measure on the (completed) Borel σ-algebra.

For $\Lambda \subset \mathbb{Z}^d$ we denote by π_Λ the projection on the Λ-coordinates.
Note that the product of the Borel σ-algebras is the same as the Borel σ-algebra for the product space. M is compact and metrizable in the following way:

Definition 2.1.16 Let $(b(q))_{q\in\mathbb{Z}^d}$ be a family of positive numbers such that

$$\lim_{R\to\infty} \sup_{\|q\|\geq R} b(q) = 0. \tag{2.68}$$

Then the metric ϱ_M on M, associated to $(b(q))_{q\in\mathbb{Z}^d}$, is defined by

$$\varrho_M(\mathbf{x},\mathbf{y}) \stackrel{\text{def}}{=} \sup_{q\in\mathbb{Z}^d} b(q)\, \varrho_{S^1}(x_q, y_q) \tag{2.69}$$

for $\mathbf{x},\mathbf{y} \in M$.

Remark 2.1.5 1. One can easily show that ϱ_M, as defined in Definition 2.1.16, is in fact a metric and also compatible with the product topology.

2. A sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ in M converges wrt the product topology iff it converges wrt to each coordinate, i.e. $(x_q^{(n)})_{n\in\mathbb{N}}$ converges for every $q \in \mathbb{Z}^d$. The same holds also for nets $(\mathbf{x}^\Lambda)_{\Lambda\in\mathcal{F}}$.

3. The product topology does not distinguish any particular sites despite the the fact that the weights $b(q)$ depend on q. Spatial shifts, like $\mathbf{x} \mapsto \tilde{\mathbf{x}}$ with $\tilde{x}_q = x_{q-r}$ for some $r \in \mathbb{Z}^d$, are homeomorphisms.

4. The space $C^0(M, M)$ of continuous maps on (M, ϱ_M) is complete wrt the metric defined by

$$\varrho_{C^0(M,M)}(f,g) \stackrel{\text{def}}{=} \sup_{\mathbf{x}\in M} \varrho_M(f(\mathbf{x}), g(\mathbf{x})). \tag{2.70}$$

We denote by $\mathcal{B}_{C^0(M,M)}$ the Borel σ-algebra wrt this metric.

Lemma 2.1.4 *Let $(\Omega, \mathcal{A})$ be a measurable space and $(f^{\Lambda})_{\Lambda \in \mathcal{F}\setminus\{\emptyset\}}$ be a net of measurable maps*

$$\begin{aligned} f^{\Lambda} : \Omega &\to C^0(M, M) \\ \omega &\mapsto f^{\Lambda}_{\omega} \end{aligned} \tag{2.71}$$

such that for all $\Lambda_1 \in \mathcal{F}\setminus\{\emptyset\}$ and $\omega \in \Omega$ the net $(\pi_{\Lambda_1} \circ f^{\Lambda}_{\omega})_{\Lambda_1 \subset \Lambda \in \mathcal{F}\setminus\{\emptyset\}}$ converges (as $\Lambda \to \mathbb{Z}^d$) in $C^0(M, (S^1)^{\Lambda_1})$, say to $\pi_{\Lambda_1} \circ f_{\omega}$.
Then

$$f_{\omega,q}(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{\Lambda \to \mathbb{Z}^d} \pi_q \circ f^{\Lambda}_{\omega}(\mathbf{x}) \tag{2.72}$$

defines a measurable map

$$\begin{aligned} f : \Omega &\to C^0(M, M) \\ \omega &\mapsto f_{\omega} \end{aligned} \tag{2.73}$$

whose qth coordinate function is given by (2.72).

Proof Fix $\omega \in \Omega$, $\mathbf{x} \in M$ and a metric ϱ_M like in Definition 2.1.16 . We show that f_{ω} is continuous in $\mathbf{x}$. For that let $\epsilon > 0$ and choose $R_0 \in \mathbb{N}$ such that

$$c_s\, b(q) < \epsilon \tag{2.74}$$

for all q with $\|q\| > R_0$. We note that the qth coordinate function of $\pi_{\Lambda_1} \circ f_{\omega} \in C^0(M, (S^1)^{\Lambda_1})$ is the same as the qth coordinate function $f_{\omega,q}$ of f_{ω}.
By continuity of $\pi_{B_{R_0}} \circ f_{\omega}$ we can choose a $\delta > 0$ such that for all $\mathbf{y} \in B_{\delta}(\mathbf{x})$ and all q with $\|q\| \leq R_0$:

$$c_s\, b(q)\, \varrho_{S^1}(f_{\omega,q}(\mathbf{x}), f_{\omega,q}(\mathbf{y})) < \epsilon. \tag{2.75}$$

From (2.74) and (2.75) we conclude that for all $\mathbf{y} \in B_{\delta}(\mathbf{x})$

$$\varrho_M(f_{\omega}(\mathbf{x}), f_{\omega}(\mathbf{y})) < \epsilon \tag{2.76}$$

which was to be shown. Finally f depends measurably on ω because it is pointwise limit of measurable functions with values in a metric space (cf. [22], p 117, for example).

□

Remark 2.1.6 1. Lemma 2.1.4 is in particular based on the compactness on M wrt the product toplology.

M is not compact wrt the different metric, defined by

$$\tilde{\varrho}_M(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \sup_{q \in \Lambda} \varrho_{S^1}(x_q, y_q).$$

In this case the conclusion from 'local' to 'global' does not hold.

2. As f in (2.73) is $(\mathcal{A}, \mathcal{B}_{C^0(M,M)})$-measurable, the map $(\omega, \mathbf{x}) \mapsto f_\omega(\mathbf{x})$ is $(\mathcal{A} \times \mathcal{B}_M, \mathcal{B}_M)$-measurable. We have proved this fact in Example 2.1.1.

2.2 Infinite-Dimensional Systems

In Example 2.1.4 we used a counting process with values in $\mathbb{N}^\Lambda$ (for finite Λ) and a family of updating-maps on Y^Λ to define Markov kernels on the product Y^Λ. These kernels act on the product space $C^0(Y^\Lambda)$ of continuous functions (cf. Definition 2.1.13 and Proposition 2.2.4). In view of spatially extended systems like coupled map lattices or interacting particle systems we would like to define analogous operators for infinite-dimensional systems ($\Lambda = \mathbb{Z}^d$). As counting process we take the Poisson process $(\Omega, \mathcal{A}, P, (X_t)_{t \geq 0})$ with parameter $\lambda > 0$ and values in $\mathbb{N}^{\mathbb{Z}^d}$.
Recall that the set $\mathcal{N}_1$, defined in Definition 2.1.7, of all $\omega \in \Omega$ such that $X_\cdot(\omega)$ is not nondecreasing, jumps at time 0, has simultaneous jumps or jumps of size greater than one, has P-measure zero. So we have to consider updatings only at single sites. They are given by a family of continuous maps $(S_q)_{q \in \mathbb{Z}^d}$ such that $S_q : M \to M$ changes only the qth coordinate (cf. Example 2.1.4 for a definition.) We remark that such a family naturally gives rise to updatings at more than one point at the same time. We will use this when we approximate continuous time processes by discrete time processes.)
A problem is obviously that the Poisson process, restricted to any finite interval $[0, t]$ of length $t > 0$ is not of finite expectation (cf. Definition 2.1.1 and Remark 2.1.3. 1). P-a.s. there are infinitely many jumps and it is even impossible to define an order preserving bijection between them and $\mathbb{N}$. However in Subsection 2.2.1 we will for systems with finite range interaction show that for P-a.a. $\omega \in \Omega$, any $q \in \mathbb{Z}^d$ and $t > 0$ the site q is affected in $[0, t]$ (cf. Definition 2.2.1) by only finitely many sites, so that maps '$\pi_q \circ S^t(\omega)$' from M to $(S^1)^{\{q\}}$ and then also '$S^t(\omega)$' from M to M can be defined in a natural way. The proof is based on a percolation argument. Percolation techniques, but different from the ones presented here, were already used by Harris in [14] for proving the existence of certain interacting particle systems of finite range. It follows in particular that $\pi_\Lambda \circ S^t(\omega) : M \to (S^1)^\Lambda$ for finite $\Lambda \neq \emptyset$ is the limit (as $\tilde{\Lambda} \to \mathbb{Z}^d$) of maps that are constructed by using the 'cut offs' $\pi_\Lambda \circ S^t_{\tilde{\Lambda},\xi}(\omega)$, corresponding to a finite $\tilde{\Lambda} \supset \Lambda$ and boundary conditions ξ. In fact this limit also exists and is independent of the boundary conditions for a huge class of infinite range interactions as we will show in Subsection 2.2.2. It gives rise to a natural definition of the system. But we also note that for infinite range interaction

each site depends with positive probability on infinitely many other sites. So we cannot use the same definition as for finite range interaction.
In Section 2.2.3 we define Markov kernels K^t_S for the infinite system S^t and $K^t_{S,\tilde{\Lambda}}$ for the system $S^t_{\tilde{\Lambda}}$ that fixes the $\tilde{\Lambda}^C$-coordinates for a finite $\tilde{\Lambda}$. We show that K^t_S is the weak limit of $K^t_{S,\tilde{\Lambda}}$ (as $\tilde{\Lambda} \to \mathbb{Z}^d$), i.e. the corresponding operators on continuous functions converge weakly.

2.2.1 Finite Range Interaction

Now we are considering an interaction of range $R \in \mathbb{N} \setminus \{0\}$, i.e. $\pi_q \circ S_q(\mathbf{x})$ depends only on $\mathbf{x}_{B_R(q)}$. (Recall that $B_R(q)$ was defined in (2.29).)

Definition 2.2.1 Given R as above, $q, \tilde{q} \in \mathbb{Z}^d$, $T > 0$, $\omega \in \Omega$. We say that $\tilde{\mathbf{q}}$ **affects q wrt** $(\mathbf{R}, \mathbf{t}, \omega)$ if there is a causal path from q to $\tilde{q}$ of step size at most R. (Recall that we defined *path* etc. in Definitions 2.1.9 to 2.1.11). If $\Lambda \subset \mathbb{Z}^d$ we say that $\tilde{\mathbf{q}}$ **affects** $\boldsymbol{\Lambda}$ **wrt** $(\mathbf{R}, \mathbf{t}, \omega)$ if $\tilde{q}$ affects any point in Λ wrt (R, t, ω).
We set

$$\mathrm{Aff}_{(R,t,\omega)}(\Lambda) \stackrel{\text{def}}{=} \{\tilde{q} \in \mathbb{Z}^d : \tilde{q} \text{ affects } \Lambda \text{ wrt } (R, t, \omega)\} \tag{2.77}$$

$$\text{and} \quad \Omega_R \stackrel{\text{def}}{=} \{\omega : \exists t > 0, q \in \mathbb{Z}^d \text{ such that } |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\} \tag{2.78}$$

where $|\cdot|$ denotes the cardinality.

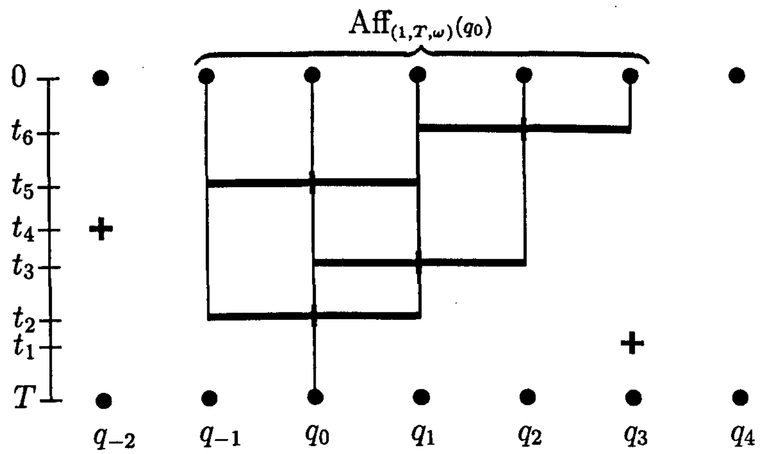


Figure 2.1: The history of q_0

Figure 2.1 is a picture of $\mathrm{Aff}_{(1,T,\omega)}(q_0)$. We consider the finite time-interval $(0, T]$ and nearest neighbour interaction and a particular ω. For each jump we draw a cross at the particular point (q, t). There are jumps at (q_2, t_6), (q_0, t_5), (q_{-2}, t_4), (q_1, t_3), (q_0, t_2) and (q_3, t_1) The last jump at q_0 is at time t_2. We draw a thick horizontal line between (q_0, t_2) and (q, t_2) for all nearest neighbours q of q_0 because the updating of q_2 depends also on these sites. So we have to consider the 'histories' of q_0 *and* its nearest neighbours *before* time t_2. Note that $q_3 \in \mathrm{Aff}_{(1,T,\omega)}(q_0)$ and it is updated at time t_1 (and so affected by q_4 for example) but that updating has no influence on q_0 (at time T). We also note that, for example, q_{-1} affects q_0 (wrt $(1, T, \omega)$) but not the other way around. So we have to consider only the time- and space-ordered percolation.

Proposition 2.2.1 *Ω_R has P-measure zero:*

$$P(\Omega_R) = 0. \tag{2.79}$$

Proof $\mathrm{Aff}_{(R,t,\omega)}(q)$ is increasing in t and so

$$\Omega_R = \bigcup_{t \in \mathbf{N}} \bigcup_{q \in \mathbf{Z}^d} \{\omega : |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\}. \tag{2.80}$$

So it is sufficient to show that for fixed $q \in \Lambda$ and $t > 0$ the set $\{\omega : |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\}$ has P-measure zero. If we set

$$A_N \stackrel{\text{def}}{=} \{\omega : \mathrm{Aff}_{(R,t,\omega)}(q) \not\subset B_N(q)\} \tag{2.81}$$

it is sufficient to show that

$$\lim_{N \to \infty} P(A_N) = 0. \tag{2.82}$$

If q is affected by some $\tilde{q} \notin B_N(q)$ wrt (R, t, ω) then there is a maximal causal path of step size at most R from q to $\tilde{q}$ with at least N_0 steps, where N_0 is the smallest integer greater than $\frac{N}{R}$.
Consider any maximal causal path $Q = (q_0 = q, \ldots, q_n)$ of step size at most R and with $n \geq N_0$. Q is a maximal causal path wrt (t, ω) iff the trajectory of ω wrt the Poisson process induced by Q has exactly n jumps. The probability of this is $p_\lambda(n, t)$ (which was defined in (2.4).)
We set

$$c_{d,R} \stackrel{\text{def}}{=} |B_R(q)|. \tag{2.83}$$

(Recall that $B_R(q)$ was defined in (2.29) and $|\cdot|$ denotes the cardinality.)
Then

$$|\mathrm{Path}(q, n, R)| = c_{d,R}^n \tag{2.84}$$

because at each step in the path one can choose between $c_{d,R}$ lattice-points.
So we have

$$A_N \subset \bigcup_{n > N_0} \bigcup_{Q \in \mathrm{Path}(q,n,R)} \{\omega : Q \text{ is maximal causal wrt}(R, n, \omega)\} \tag{2.85}$$

and so

$$P(A_N) \leq \sum_{n\geq N_0} c_{d,R}^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{2.86}$$

$$\leq e^{(c_{d,R}-1)\lambda t}(c_{d,R}\lambda t)^{N_0}\frac{1}{N_0!} \tag{2.87}$$

which converges to 0 as $N_0 \to \infty$ which was to show. For the last inequality we have used the estimate for the Lagrange remainder in Taylor's formula .

□

Definition 2.2.2 Let a finite range interaction (i.e. a family of updatings) be given by $(S_q)_{q\in\mathbb{Z}^d}$. Fix $\omega \in \Omega \setminus (\Omega_R \cup \mathcal{N}_1)$, $\emptyset \neq \Lambda \subset \tilde{\Lambda} \in \mathcal{F}$, $\xi \in M$ and $t > 0$. Then ω has only finitely many jumps in $\tilde{\Lambda} \times (0,t)$, say at $(q_1,t_1),\ldots,(q_n,t_n)$ with $0 < t_1 < \ldots < t_n < t$.
We denote by $\mathbf{x}_{\tilde{\Lambda}} \vee \xi_{\tilde{\Lambda}^C}$ the point in M that has the same $\tilde{\Lambda}$-coordinates as $\mathbf{x}$ and the same $\tilde{\Lambda}^C$-coordinates as ξ.
We define

$$\begin{aligned} S_{q,\tilde{\Lambda},\xi} : (S^1)^{\tilde{\Lambda}} &\to (S^1)^{\tilde{\Lambda}} \\ S_{q,\tilde{\Lambda},\xi}(\mathbf{x}_{\tilde{\Lambda}}) &\stackrel{\text{def}}{=} \pi_{\tilde{\Lambda}} \circ S_q(\mathbf{x}_{\tilde{\Lambda}} \vee \xi_{\tilde{\Lambda}^C}) \end{aligned} \tag{2.88}$$

and

$$\begin{aligned} \omega \in \Omega \setminus (\mathcal{N}_1 \cup \Omega_R) &\mapsto S^t_{\tilde{\Lambda},\xi}(\omega) \in C^0\left((S^1)^{\tilde{\Lambda}},(S^1)^{\tilde{\Lambda}}\right) \\ S^t_{\tilde{\Lambda},\xi}(\omega)(\mathbf{x}) &\stackrel{\text{def}}{=} S_{q_n,\tilde{\Lambda},\xi} \circ \ldots \circ S_{q_1,\tilde{\Lambda},\xi}(\mathbf{x}_{\tilde{\Lambda}}). \end{aligned} \tag{2.89}$$

The maps $S^t_{\tilde{\Lambda},\xi}(\omega)$ are continuous as composites of continuous maps. Furthermore $S^t_{\tilde{\Lambda},\xi}(\omega)$ depends only on $\omega_{\tilde{\Lambda}}$ (i.e. on $\pi_{\tilde{\Lambda}} \circ X_{\cdot}(\omega)$) and (2.89) gives rise to a countable, measurable partition of $\Omega \setminus (\mathcal{N}_1 \cup \Omega_R)$: ω and $\tilde{\omega}$ belong to the same set of this partition if they have the same list of jump sites $(q_1,\ldots,q_n)$ (ordered wrt the jump times).
Now let $\tilde{\Lambda} \supset \mathrm{Aff}_{(R,t,\omega)}(\Lambda)$ and $\xi \in M$ and define

$$\pi_\Lambda \circ S^t : \Omega \setminus (\mathcal{N}_1 \cup \Omega_R) \to C^0(M,(S^1)^\Lambda), \tag{2.90}$$

$$\pi_\Lambda \circ S^t(\omega)(\mathbf{x}) \stackrel{\text{def}}{=} \pi_\Lambda \circ S^t_{\tilde{\Lambda},\xi}(\omega)(\mathbf{x}_{\tilde{\Lambda}}). \tag{2.91}$$

The definition does not depend on the choice of $\tilde{\Lambda}$ or ξ because the right-hand side (rhs) of (2.91) depends, by definition, on the $\mathrm{Aff}_{(R,t,\omega)}(\Lambda)$-coordinates of $\mathbf{x}$ only.

Further the family $(\pi_\Lambda \circ S^t(\omega)(x))_{\Lambda \in \mathcal{F}\setminus\emptyset}$ is consistent in the sense that for any $\Lambda_1 \subset \Lambda_2 \in \mathcal{F} \setminus \emptyset$:

$$\pi_{\Lambda_1} \left(\pi_{\Lambda_2} \circ S^t(\omega)(x) \right) = \pi_{\Lambda_1} \circ S^t(\omega)(x), \tag{2.92}$$

and so defines a map

$$S^t(\omega) : M \to M. \tag{2.93}$$

Proposition 2.2.2 *The map $S^t(\omega)$, defined in (2.93) and (2.91) is continuous and depends measurably on ω.*

Proof We want to apply Lemma 2.1.4. For that we define for $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \Omega_R)$ maps

$$\begin{aligned} \tilde{S}^t_{\tilde{\Lambda},\xi}(\omega) : M &\to M \\ \mathbf{x} &\mapsto S^t_{\tilde{\Lambda},\xi}(\omega)(\mathbf{x}_{\tilde{\Lambda}}) \vee \xi_{\tilde{\Lambda}^c} \end{aligned} \tag{2.94}$$

The net $(\tilde{S}^t_{\tilde{\Lambda},\xi})_{\tilde{\Lambda} \in \mathcal{F}\setminus\{\emptyset\}}$ satisfies the assumptions in Lemma 2.1.4 and so all statements of Proposition 2.2.2 follow.

□

2.2.2 Infinite Range Interaction

We extend our notion of '$S^t(\omega)$' to interactions that are not necessarily of finite range.

Consider a family $(S_q)_{q \in \mathbb{Z}^d}$ of maps $S_q : M \to M$ such that S_q does not change the $\mathbb{Z}^d \setminus \{q\}$-coordinates and $\pi_q \circ S_q : M \to S^1$ is *Lipschitz-continuous wrt all coordinates* and the Lipschitz constants depend only on the relative positions of the sites, i.e. there are constants $w(r)$ for all $r \in \mathbb{Z}^d$ such that for all $q, \tilde{q} \in \mathbb{Z}^d$ and $\mathbf{x}, \mathbf{y} \in M$ with $\mathbf{x}_{\mathbb{Z}^d \setminus \{\tilde{q}\}} = \mathbf{y}_{\mathbb{Z}^d \setminus \{\tilde{q}\}}$ (i.e. $\mathbf{x}$ and $\mathbf{y}$ differ at most in their $\tilde{q}$-coordinates.)

$$\varrho_{S^1}(\pi_q \circ S_q(\mathbf{x}), \pi_q \circ S_q(\mathbf{y})) \le w(\tilde{q} - q)\, \varrho_{S^1}(y_{\tilde{q}}, z_{\tilde{q}}). \tag{2.95}$$

We further assume summability of the Lipschitz-constants, i.e.

$$\sum_{q \in \mathbb{Z}^d} w(q) = c_1 \tag{2.96}$$

with a positive constant c_1.

We need the following technical lemma.

Lemma 2.2.1 *If* $(w(q))_{q\in\mathbb{Z}^d}$ *is a family of non-negative real numbers satisfying (2.96) then there are families* $(w_1(q))_{q\in\mathbb{Z}^d}$ *and* $(w_2(q))_{q\in\mathbb{Z}^d}$ *of non-negative numbers such that*

$$w(q) = w_1(q)\, w_2(q) \quad \text{for all } q \in \mathbb{Z}^d, \tag{2.97}$$

$$\sum_{q\in\mathbb{Z}^d} w_1(q) \leq 2c_1 + 1 \tag{2.98}$$

$$\text{and} \quad \lim_{R\to\infty} a(R) = 0, \tag{2.99}$$

where $a(R)$ *is defined by*

$$a(R) \stackrel{\text{def}}{=} \sup_{\|r_1\|+\ldots+\|r_n\|=R} w_2(r_1)\cdot\ldots\cdot w_2(r_n) \tag{2.100}$$

(The empty product is defined to be equal to 1.)

Proof We can choose $r_0 = 0 < r_1 < \ldots \in \mathbb{N}$ such that

$$\sum_{\|q\|<r_i} w(q) \geq c_1 - 4^{-(i+1)} \quad \text{for } i \geq 1. \tag{2.101}$$

Then we have

$$\sum_{\|q\|<r_1} w(q) \leq c_1 \tag{2.102}$$

$$\text{and} \quad \sum_{r_i\leq\|q\|<r_{i+1}} w(q) \leq 4^{-(i+1)} \quad \text{for } i \geq 1.$$

We set for $i \geq 1$ and $r_{i-1} \leq \|q\| < r_i$:

$$w_2(q) \stackrel{\text{def}}{=} 2^{-i} \tag{2.103}$$

$$w_1(q) \stackrel{\text{def}}{=} 2^i\, w(q). \tag{2.104}$$

Then (2.97) is obviously satisfied. To prove (2.98) we use (2.102) and (2.104):

$$\begin{aligned}\sum_{q\in\mathbb{Z}^d} w_1(q) &= \sum_{i=0}^{\infty} \sum_{r_i\leq\|q\|<r_{i+1}} w_1(q) \qquad (2.105)\\ &\leq 2c_1 + \sum_{i=1}^{\infty} 2^{-i}\\ &= 2c_1 + 1.\end{aligned}$$

Now we prove (2.99). We show by induction (wrt i) that for every $i \geq 1$ there is an n_i such that

$$a(R) < 2^{-i} \quad \text{for all } R \geq n_i. \tag{2.106}$$

For $i = 1$ the statement is true with $n_1 = 1$ because $a(R) \leq \frac{1}{2}$ for every $R \geq 1$ as there is at least one factor on the right-hand-side in (2.100) and each such factor is at most $\frac{1}{2}$.
Now we assume now that that the statement holds for i and n_i. We set

$$n_{i+1} \stackrel{\text{def}}{=} r_i + 2n_i. \tag{2.107}$$

Then every path $(q_0, \ldots, q_n)$ of length $R \geq n_{i+1}$ has at least one step of size at least r_i (i.e. there is an $1 \leq l \leq n$ such that $\|q_l - q_{l-1}\| \geq r_i$) or it can be divided into two paths both of length at least n_i (i.e. there is an $1 \leq l \leq m-1$ such that $\|q_0 - q_1\| + \ldots \|q_{l-1} + q_l\| \geq n_i$ and $\|q_{l+1} - q_l\| + \ldots \|q_n + q_{n-1}\| \geq n_i$). So each product on the right-hand side of (2.100) has at least one factor less than or equal to $2^{-(i+1)}$ or two factors less than or equal to 2^{-i}. As the other factors are smaller than 1 the product is bounded by $2^{-(i+1)}$ as was to be shown.

□

Now we fix (like in Lemma 2.2.1) a choice of $(w_1(q))_{q\in\mathbb{Z}^d}$ and $(w_2(q))_{q\in\mathbb{Z}^d}$ and so the function a.

Definition 2.2.3 We fix the metric ϱ_M on M by

$$\varrho_M(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \sup_{r\in\mathbb{Z}^d} a(\|r\|)\, \varrho_{S^1}(x_r, y_r). \tag{2.108}$$

Remark 2.2.1 It follows from Remark 2.1.5.1 and (2.99) that ϱ_M is a metric and compatible with the product topology.

Lemma 2.2.2 *The maps $S_q : M \to M$ are continuous (wrt the product topology on M).*

Proof According to Remark 2.1.5.2 and the uniform choice of the Lipschitz-constants (cf. (2.95)) we only have to show that the maps $\pi_q \circ S_0 : M \to S^1$ are continuous.
If $q \neq 0$ then the qth coordinate is not changed by S_0 and

$$\begin{aligned} a(\|q\|)\, \varrho_{S^1}(\pi_q \circ S_0(\mathbf{x}), \pi_q \circ S_0(\mathbf{y})) &= a(\|q\|)\, \varrho_{S^1}(x_q, y_q) \\ &\leq a(\|q\|) \frac{1}{a(\|q\|)} \varrho_M(\mathbf{x}, \mathbf{y}) \\ &\leq \varrho_M(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{2.109}$$

If $q = 0$ we estimate

$$\begin{aligned} a(0)\, \varrho_{S^1}(\pi_0 \circ S_0(\mathbf{x}), \pi_0 \circ S_0(\mathbf{y})) \\ \leq\ a(0) \sum_{r\in\mathbb{Z}^d} w(r)\, \varrho_{S^1}(x_r, y_r) \\ \leq\ a(0) \sum_{r\in\mathbb{Z}^d} w(r) \frac{1}{a(\|r\|)}\, \varrho_M(\mathbf{x}, \mathbf{y}) \\ \leq\ a(0)\,(2c_1 + 1)\, \varrho_M(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{2.110}$$

where we have used (2.95) for the first, the definition of ϱ_M for the second and (2.97) for the third inequality. So $\pi_q \circ S_0$ is continuous for all $q \in \mathbb{Z}^d$.

□

In the following we estimate the distance (wrt the uniform norm) of $\pi_0 \circ S^t_{\Lambda,\xi}(\omega)$ and $\pi_0 \circ S^t_{\Lambda,\tilde{\xi}}(\omega)$ for different boundary conditions ξ_{Λ^C} and $\tilde{\xi}_{\Lambda^C}$ (that might even depend on the time) at the Λ^C-sites. Conditions (2.95) and (2.96) allow us to apply standard estimates for the 'error-growth' for composites of maps. Using the linear nature of the 'Lipschitz-condition' (2.95), we write the products of sums (over all coordinates, like in (2.95)) as sums (over paths) of products (corresponding to the particular paths).

We fix $t > 0$, $\Lambda \in \mathcal{F}$ and $\omega \in \Omega \setminus \mathcal{N}_1$. By definition of $\mathcal{N}_1$ (cf. (2.27)) ω has no jumps at 0, no simultaneous jumps and only finitely many jumps in $\Lambda \times (0, t)$, say at $(q_1, t_1), \ldots, (q_N, t_N)$ with $0 < t_1 < \ldots < t_N < t$. We set $t_0 \overset{\text{def}}{=} 0$ and fix arbitrary $\xi = (\xi(t_0), \ldots, \xi(t_N))$, $\tilde{\xi} = (\tilde{\xi}(t_0), \ldots, \tilde{\xi}(t_N)) \in M^{N+1}$ and $\mathbf{x}, \mathbf{y} \in M$.

We set $\mathbf{x}(0) \overset{\text{def}}{=} \mathbf{x}_\Lambda \vee \xi_{\Lambda^C}(0)$, $\mathbf{y}(0) \overset{\text{def}}{=} \mathbf{y}_\Lambda \vee \tilde{\xi}_{\Lambda^C}(0)$ and define for $1 \leq i \leq N$ recursively:

$$\mathbf{x}_q(t_i) \overset{\text{def}}{=} \begin{cases} \pi_q \circ S_q(\mathbf{x}(t_{i-1})) & \text{for} \quad q = q_i \\ x_q(t_{i-1}) & \text{for} \quad q \in \Lambda \setminus \{q_i\} \\ \xi_q(t_i) & \text{for} \quad q \in \Lambda^C \end{cases}. \tag{2.111}$$

We define $\mathbf{y}(t_i)$ analogously, using $\mathbf{y}$ and $\tilde{\xi}$ instead of $\mathbf{x}$ and ξ, respectively.

Two points in S^1 can have distance at most $c_s = \operatorname{diam}_{\varrho_{S^1}}(S^1)$. For estimating the distance between $x_q(t_i)$ and $y_q(t_i)$ we define

$$\Delta_q(0) \overset{\text{def}}{=} \tilde{\Delta}_q(0) \overset{\text{def}}{=} \begin{cases} \varrho_{S^1}(x_q(0), y_q(0)) & \text{for} \quad q \in \Lambda \\ c_s & \text{for} \quad q \in \Lambda^C \end{cases}, \tag{2.112}$$

and for $1 \leq i \leq N$

$$\Delta_q(i) \overset{\text{def}}{=} \begin{cases} \sum_{r\in\mathbb{Z}^d} w(r-q)\, \Delta_r(i-1) & \text{for} \quad q = q_i \\ \Delta_q(i-1) & \text{for} \quad q \in \Lambda \setminus \{q_i\} \\ c_s & \text{for} \quad q \in \Lambda^C \end{cases} \tag{2.113}$$

$$\tilde{\Delta}_q(i) \stackrel{\text{def}}{=} \begin{cases} \max\{c_s, \sum_{r\in\mathbb{Z}^d} w(r-q)\,\tilde{\Delta}_r(i-1)\} & \text{for } q = q_i \\ \tilde{\Delta}_q(i-1) & \text{for } q \in \Lambda \setminus \{q_i\} \\ c_s & \text{for } q \in \Lambda^C \end{cases}$$

The functions Δ_q and $\tilde{\Delta}_q$ depend on $\mathbf{x}$, $\mathbf{y}$ and Λ but we do not refer to this in our notation. We have introduced them for estimating the difference between $x_q(t_i)$ and $y_q(t_i)$ (cf. 2.114)) and so the difference between $x_q(t)$ and $y_q(t)$. This difference depends also on ω and so do the corresponding estimates for Δ_q and $\tilde{\Delta}_q$. In Definition 2.2.4 we will relate them to families of random variables $(Y_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ and $(\tilde{Y}_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$, respectively. For Δ_q we find a particularly nice expansion (cf. (2.115)). From this follows the convergence of Y_Λ to zero in expectation (as $\Lambda \to \mathbb{Z}^d$). We will show that $\tilde{Y}_\Lambda$ is bounded by Y_Λ and decreasing and so converges P-almost surely to zero by the Monotone Convergence Theorem (cf. Theorem 2.2.1).

Proposition 2.2.3 *The following holds for* $0 \le i \le N$*:*

1.

$$\varrho_{S^1}(x_q(t_i), y_q(t_i)) \le \tilde{\Delta}_q(i) \le \Delta_q(i) \tag{2.114}$$

2.

$$\Delta_q(i) = \sum_{\substack{(r_0=q_i,r_1,\ldots,r_n) \\ \in\mathrm{Path}_C(t_i,\omega,q,\Lambda)}} w(r_1-r_0)\cdot\ldots\cdot w(r_n-r_{n-1})\,\Delta_{r_n}(0) \tag{2.115}$$

3. If in particular $\mathbf{x}_\Lambda = \mathbf{y}_\Lambda$ *then*

$$\begin{aligned} &\Delta_0(N(\omega)) \\ &\quad\le\ c_S\, a(\mathrm{dist}_{\mathbb{Z}^d}(q,\Lambda^C)) \\ &\qquad \sum_{\substack{(r_0=q,r_1,\ldots,r_n) \\ \in\mathrm{Path}_C(t,\omega,q\to\Lambda^C)}} w_1(r_1-r_0)\cdot\ldots\cdot w_1(r_n-r_{n-1}) \end{aligned} \tag{2.116}$$

where $N(\omega)$ *is the number of jumps of* ω *in* $\Lambda\times(0,t)$.

Proof We prove (2.114) and (2.115) by induction wrt i.
$\underline{i=0}$: (2.114) holds by definition of $\Delta_q(0)$ and $\tilde{\Delta}_q(0)$ (cf. (2.112)). At time 0 no jump has happened and the only summand on the right-hand-side in (2.115) corresponds to the empty path at site q and so the equality in (2.115) holds.
$\underline{i-1\to i}$: (2.115) holds obviously for i and $q \ne q_i$ as there is no updating at site q and

$$\mathrm{Path}_C(t_i,\omega,q,\Lambda) = \mathrm{Path}_C(t_{i-1},\omega,q,\Lambda) \tag{2.117}$$

At site q_i there is a jump at time t_i and so we have

$$\Delta_{q_i}(i) = \sum_{r\in\mathbb{Z}^d} w(r-q_i)\,\Delta_r(i-1) \tag{2.118}$$

Using the representation (2.115) for $\Delta_{r-q}(i-1)$ and the fact that every $(q_i, r_1, \dots, r_n) \in \mathrm{Path}_{\mathrm{C}}(t_i, \omega, q_i, \Lambda)$ can be (uniquely) split into (q_i, r_1) and $(r_1, \dots, r_n) \in \mathrm{Path}_{\mathrm{C}}(t_{i-1}, \omega, r_1, \Lambda)$, we see that (2.115) holds for i.
Next we show the first inequality in (2.114) for i. For $q \in \Lambda^C$ the distances between $x_q(t_i) = \xi_q(t_i)$ and $y_q(t_i) = \tilde{\xi}_q(t_i)$ is bounded by c_s and for $q \in \Lambda \setminus \{q_i\}$ we have $x_q(t_i) = x_q(t_{i-1})$ and $y_q(t_i) = y_q(t_{i-1})$. So in both cases the first inequality in (2.114) holds.
Now we consider the site q_i where a jump happens at time t_i. Using (2.95), assumption (2.114), for $i-1$, and (2.118), we get

$$\begin{aligned} \varrho_{S^1}(x_q(t_i), y_q(t_i)) &\leq \sum_{r \in \mathbf{Z}^d} w(r - q_i)\, \varrho_{S^1}(x_q(t_{i-1}), y_q(t_{i-1})) \\ &\leq \sum_{r \in \mathbf{Z}^d} w(r - q_i)\, \Delta_r(i-1) \\ &\leq \Delta_{q_i}(i) \end{aligned} \tag{2.119}$$

So the first inequality in (2.114) is proved for i. The second follows immediately from (2.113). So statements 1 and 2 are proved.
Finally (2.116) follows from (2.115)): $\Delta_q(0) = 0$ for $q \in \Lambda$. So we only have to sum over paths $(r_0 = 0, \dots, r_n)$ that end in $r_n \in \Lambda^C$.
In particular, if we set $R \stackrel{\mathrm{def}}{=} \|r_n\|$, then

$$\tilde{\Delta}_{r_n}(0) = c_s, \tag{2.120}$$

$$\mathrm{dist}_{\mathbf{Z}^d}(q, \Lambda^C) \leq R, \tag{2.121}$$

$$R \leq \|r_n - r_{n-1}\| + \dots + \|r_1 - r_0\|, \tag{2.122}$$

and so by the choice of w_1, w_2 and a, made before Definition 2.2.3, we get

$$\begin{aligned} & w(r_1 - r_0) \cdot \ldots \cdot w(r_n - r_{n-1}) \\ \leq\ & w_1(r_1 - r_0) \cdot \ldots \cdot w_1(r_n - r_{n-1})\, a(R) \\ \leq\ & w_1(r_1 - r_0) \cdot \ldots \cdot w_1(r_n - r_{n-1})\, a(\mathrm{dist}_{\mathbf{Z}^d}(q, \Lambda^C)). \end{aligned} \tag{2.123}$$

Using (2.115), (2.120) and (2.123), we get (2.116) .

□

Remark 2.2.2 The summing over causal paths in Proposition 2.2.3 reflects that the result of an updating depends only on what has happened before.

Definition 2.2.4 We define two families $(Y_\Lambda)_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$ and $(\tilde{Y}_\Lambda)_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$ of random variables on $\Omega \setminus \mathcal{N}_1$. Let $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\omega \in \Omega \setminus \mathcal{N}_1$, say with exactly $N(\omega)$ jumps in $\Lambda \times [0,t]$. If we choose $\mathbf{x}, \mathbf{y} \in M$ with $\mathbf{x}_\Lambda = \mathbf{y}_\Lambda$ the value of $\Delta_0(N(\omega))$ (as defined by (2.112) and 2.113) does not depend on $\mathbf{x}$ or $\mathbf{y}$. We define $Y_\Lambda(\omega)$ to be equal to this value:

$$Y_\Lambda(\omega) \stackrel{\text{def}}{=} \Delta_0(N(\omega)) \tag{2.124}$$

$\tilde{Y}_\Lambda$ is defined analogously, using $\tilde{\Delta}_0(N(\omega))$ instead of $\Delta_0(N(\omega))$.

Remark 2.2.3 1. We remark that Y_Λ depends measurably on ω. In fact there is a countable, measurable partition of $\Omega \setminus \mathcal{N}_1$ such that ω and $\tilde{\omega}$ belong to the same set (of that partition) if the sums for $\Delta_0(t_{N(\omega)})$ and $\Delta_0(t_{N(\tilde{\omega})})$ (cf. (2.115)) are over the same paths (This gives rise to a measurable partition of Ω, as considered in Lemma 2.1.2).

2. From (2.114) we see that

$$\tilde{Y}_\Lambda \le Y_\Lambda. \tag{2.125}$$

Now we fix $\xi \in M$, $\mathbf{x} \in S^1$ and define the map $S^t_{\Lambda,\xi}(\omega)$ like in (2.89).

Theorem 2.2.1 *1. There is a set $\mathcal{N}_2$ of P-measure zero such that*

$$\lim_{\Lambda \to Z^d} \tilde{Y}_\Lambda = 0 \qquad \text{for } \omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2). \tag{2.126}$$

2. The limit

$$\pi_0 \circ S^t(\omega) \stackrel{\text{def}}{=} \lim_{\Lambda \to \mathbb{Z}^d} \pi_0 \circ S^t_{\Lambda,\xi}(\omega) \tag{2.127}$$

exists in $C^0(M, S^1)$ for all $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$. It is measurable in ω and does not depend on ξ.

3. There is a set $\mathcal{N} \subset \Omega$ of P-measure zero such that we can define maps

$$\pi_q \circ S^t(\omega) \stackrel{\text{def}}{=} \lim_{q \in \Lambda \to \mathbb{Z}^d} \pi_q \circ S^t_{\Lambda,\xi}(\omega) \tag{2.128}$$

for all $q \in \mathbb{Z}^d$ and $\omega \in \Omega \setminus \mathcal{N}$.
Further we can define a map $S^t(\omega) \in C^0(M, M)$ by

$$\left(S^t(\omega)(\mathbf{x})\right)_q \stackrel{\text{def}}{=} \pi_q \circ S^t(\omega). \tag{2.129}$$

$S^t(\omega)$ depends measurably on ω.

Proof First we show that

$$\lim_{\Lambda \to \mathbb{Z}^d} E(Y_\Lambda) = 0 \tag{2.130}$$

We set $R \stackrel{\text{def}}{=} \text{dist}_{\mathbb{Z}^d}(q, \Lambda^C)$. Using (2.116) we get

$$\begin{aligned} & E(Y_\Lambda) && (2.131)\\ \leq & \int_\Omega dP(\omega)\, c_S\, a(R) \\ & \times \sum_{\substack{(r_0=q, r_1, \ldots, r_n) \\ \in \text{Path}_C(t,\omega,q\to\Lambda^C)}} w_1(r_1 - r_0) \cdot \ldots \cdot w_1(r_n - r_{n-1}) \\ = & \; c_s\, a(R) \sum_{Q \in \text{Path}(0\to\Lambda^C)} w_1(r_1 - r_0) \cdot \ldots \cdot w_1(r_n - r_{n-1}) \\ & \times P(\{\omega : Q \in \text{Path}_C(t,\omega,0\to\Lambda^C)\}) \end{aligned}$$

A path $Q = (q_0, q_1, \ldots, q_n)$ with $q_n \in \Lambda^C$ is causal wrt (t,ω) (i.e. $Q \in \text{Path}_C(t,\omega,0,\Lambda)$) iff the Poisson process induced by Q has at least n jumps. So we can estimate the probability

$$P(\{\omega : Q \in \text{Path}_C(t,\omega,0,\Lambda)\}) = \sum_{m\geq n} e^{-\lambda t} \frac{(\lambda t)^m}{m!} \tag{2.132}$$

$$\leq \frac{(\lambda t)^n}{n!} \tag{2.133}$$

For the last line we have used Taylor's formula, as we did in (2.87). So we get, using (2.98),

$$E(Y_\Lambda) \leq c_s\, a(R) \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \left(\sum_{r\in\mathbb{Z}^d} w_1(r) \right)^n \tag{2.134}$$

$$\leq c_2\, a(R) \tag{2.135}$$

whith $c_2 = c_s\, e^{\lambda t(2c_1+1)}$. (Recall that we consider a fixed t at the moment, so c_2 is a constant.) By (2.99) we get

$$\lim_{\Lambda \to Z^d} E(Y_\Lambda) = 0 \tag{2.136}$$

and, using (2.125),

$$\lim_{\Lambda \to Z^d} E(\tilde{Y}_\Lambda) = 0. \tag{2.137}$$

$\tilde{Y}_\Lambda(\omega)$ is decreasing for all $\omega \in \Omega \setminus \mathcal{N}_1$: For a fixed ω and $0 \in \Lambda_1 \subset \Lambda_2 \in \mathcal{F}$ we consider the (time-ordered) sequence of jumps $(q_1, t_1), \ldots, (q_n, t_n)$ of ω at sites $q_1, \ldots, q_2 \in \Lambda_1$. It is a subsequence of the sequence of jumps $(\overline{q}_1, \overline{t}_1), \ldots, (\overline{q}_m, \overline{t}_m)$ of ω at sites $\overline{q}_1, \ldots, \overline{q}_m \in \Lambda_2$. The jumps (q_i, t_i) in the first sequence correspond to jumps $(\overline{q}_{j(i)}, \overline{t}_{j(i)})$ in the second one. Then $q_i = \overline{q}_{j(i)}$ and $t_i = \overline{t}_{j(i)}$ but the indices are different in general.
We define $\tilde{\Delta}^1_q(i)$ and $\tilde{\Delta}^2_q(j)$ as in (2.112) and (2.113) for the sets Λ_1 and Λ_2, respectively. We show that

$$\tilde{\Delta}^1_q(i) \geq \tilde{\Delta}^2_q(j(i)) \tag{2.138}$$

If $q \in {\Lambda_1}^C$ then (2.138) obviously holds because $\tilde{\Delta}^1_q(i) = c_S$ is an upper bound for $\tilde{\Delta}^2_q(j)$. For $q \in \Lambda_1$ we show (2.138) by induction wrt i.
If $i = 0$ then (2.138) is true by (2.112). Now assume that (2.138) holds for all q and a particular $i < n$. For $q \in \Lambda_1 \setminus \{q_{i+1}\}$ we have

$$\tilde{\Delta}^1_q(i+1) = \tilde{\Delta}^1_q(i) \geq \tilde{\Delta}^2_q(j(i)) = \tilde{\Delta}^2_q(j(i+1)) \tag{2.139}$$

where the inequality holds by assumption and the equalities by (2.113). For the site $q = q_{i+1}$ we have by (2.113)

$$\begin{aligned}
\tilde{\Delta}^1_q(i+1) &= \max\{c_S, \sum_{r \in \mathbb{Z}^d} w(r-q)\tilde{\Delta}^1_r(i)\} \qquad (2.140)\\
&\geq \max\{c_S, \sum_{r \in \mathbb{Z}^d} w(r-q)\tilde{\Delta}^2_r(j(i+1)-1)\}\\
&= \tilde{\Delta}^2_q(j(i+1))
\end{aligned}$$

which was to be shown. Here we have used that $\tilde{\Delta}^1_r(i) \geq \tilde{\Delta}^2_r(j(i+1)-1)$. This follows for $r \in \Lambda_1$ from the definition of $\tilde{\Delta}^1_r$ and $\tilde{\Delta}^2_r$ and for $r \in \Lambda_1$ from assumption (2.138) and the fact that $\tilde{\Delta}^2_r(j(i+1)-1) = \tilde{\Delta}^2_r(j(i))$.
Using the definition of $\tilde{Y}_{\Lambda_1}(\omega)$ and $\tilde{Y}_{\Lambda_2}(\omega)$ (cf. Definition 2.2.4) we conclude

$$\tilde{Y}_{\Lambda_1}(\omega) \geq \tilde{Y}_{\Lambda_2}(\omega) \tag{2.141}$$

which was to be shown.
We have proved (2.137) and that $(\tilde{Y}_\Lambda)_{\Lambda \in \mathcal{F}\setminus\{\emptyset\}}$ is decreasing. So we conclude (2.126), by using the Monotone Convergence Theorem.
Now we prove the second statement in Theorem 2.2.1, using the first one. First we note that for $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ the map $S^t_{\Lambda,\xi}(\omega)$ is continuous since it is the composite of finitely many continuous (cf. Lemma 2.2.2) updating maps.
For $\Lambda \subset \tilde{\Lambda}$ we have

$$\varrho_{C^0(M,S^1)}(\pi_0 \circ S^t_{\Lambda,\xi}(\omega), \pi_0 \circ S^t_{\tilde{\Lambda},\xi}(\omega)) \leq \tilde{Y}_\Lambda(\omega). \tag{2.142}$$

So by (2.126) the net $(\pi_0 \circ S^t_{\Lambda,\xi}(\omega))_{\Lambda \in \mathcal{F} \setminus \emptyset}$ is a Cauchy net with values in $C^0(M, S^1)$ for $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ and so converges. Furthermore it is a pointwise limit, i.e. for each particular ω, and so $\pi_0 \circ S^t(\cdot)$ is measurable in ω. (The last conclusion uses the theorem that the pointwise limit of measurable functions with values in a metric space is measurable. (cf. for example [22], p. 117)).
As mentioned in Remark 2.1.5.3 there is no distinction of the point 0 by the product topology. So for all $q \in \mathbb{Z}^d$ we can define $\pi_q \circ S^t(\omega)$ for all $\omega \in \Omega \setminus \mathcal{N}^q$ where $P(\mathcal{N}^q) = 0$. In the same way we can define for each $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\omega \in \Omega \setminus \mathcal{N}^\Lambda$ (with $P(\mathcal{N}^\Lambda) = 0$) maps $\pi_\Lambda \circ S^t(\omega) \in C^0(M, (S^1)^\Lambda)$ that depend measurably on ω, and such that $S^t(\omega)(\mathbf{x})$ depends measurably on $(\omega, \mathbf{x})$.
The set

$$\mathcal{N} \stackrel{\text{def}}{=} \bigcup_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}} \mathcal{N}^\Lambda \tag{2.143}$$

has P-measure zero. So by Lemma 2.1.4 the map $S^t(\omega)$ is well-defined for $\omega \in \Omega \setminus \mathcal{N}$ and the statements in 3. hold.

□

2.2.3 Markov Kernels

In Section 2.1 we defined the Poisson process $(\Omega, \mathcal{A}, P, (X_t)_{t \in [0,T]})$ with parameter λ and values in $\mathbb{N}^\Lambda$, the measure space $(M, \mathcal{B}_M, \mu)$ and the measurable space $(C^0(M, M), \mathcal{B}_{C^0(M,M)})$.
We have nets $(S^T_\Lambda)_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$ of maps $S^T_\Lambda : \Omega \setminus \mathcal{N} \to C^0(M, M)$ with limit $S^T \in C^0(M, M)$, and the following statements hold:

1. S^T_Λ and S^T are $(\mathcal{A}, \mathcal{B}_{C^0(M,M)})$-measurable.

2. S^T is the pointwise limit of the net $(S^T_\Lambda)_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$.

3. For fixed $\mathbf{x} \in M$ the map $S^T_\Lambda(\cdot)(\mathbf{x}) : \Omega \to M$ is $(\mathcal{A}, \mathcal{B}_M)$-measurable.

More precisely, for finite range interaction (cf. Section 2.2.1) S^T was defined in (2.93) and the approximate S^T_Λ in (2.94) (Now we drop the fixed boundary condition ξ and the '$\tilde{\ }$' in the notation for convenience.) For infinite range interaction (cf. Section 2.2.2) S^T_Λ is defined in the same way as for finite range interaction (cf. (2.128)) and the existence of the limit S^T is established in (2.129). Note that these maps are a priori not defined on a set of P-measure zero. For these exceptional $\omega \in \Omega$ we define $S^T(\omega)$ and $S^T_\Lambda(\omega)$ to be equal to the identity on M.
Statement 3. follows from measurability wrt $(\omega, \mathbf{x})$ of $S^T_\Lambda(\omega, \mathbf{x})$ (Proposition 2.2.2 and Remark 2.1.6.2 for finite range interaction and statement 3. of Theorem 2.2.1 and Remark 2.1.6.2 for infinite range interaction), the fact that one-point-sets in M are measurable, and Fubini's Theorem.

Like in Example 2.1.1 we set

$$\begin{aligned} K_S^T : M \times \mathcal{B}_M &\to [0,1] \\ K_S^T(\mathbf{x}, A) &\stackrel{\text{def}}{=} P(\{\omega : S^T(\omega)(\mathbf{x}) \in A\}). \end{aligned} \tag{2.144}$$

The corresponding operator, applied to an $f \in \mathcal{C}^0(M)$, is

$$(K_S^T f)(\mathbf{x}) = \int_M K_S^T(\mathbf{x}, d\mathbf{y})\, f(\mathbf{y}) \tag{2.145}$$

$$= \int_\Omega dP(\omega)\, f \circ S^T(\omega)(\mathbf{x}). \tag{2.146}$$

(2.145) is the definition (cf. (2.37)), and (2.146) is a consequence of (2.144).
We define analogously the Markov kernels $K_{S,\Lambda}^T$ and corresponding operators for the Poisson process with values in $\mathbb{N}^\Lambda$ and $K_{S,\Lambda,N}^T$ for the Bernoulli process with parameters λ, T, N and values in $\mathbb{N}^\Lambda$ (cf. Definition 2.1.6).

Proposition 2.2.4 *K_S^T and $K_{S,\Lambda}^T$ are bounded linear operators on $\mathcal{C}^0(M)$.*

Proof We give the proof for K_S^T. The one for $K_{S,\Lambda}^T$ is analogous. Let $\omega \in \Omega \setminus \mathcal{N}$, $f \in \mathcal{C}^0(M)$ and $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ a sequence in M with limit $\mathbf{x}$. Then

$$\lim_{n\to\infty} S^T(\omega)(\mathbf{x}^{(n)}) = S^T(\omega)(\mathbf{x}) \tag{2.147}$$

$$\text{and so} \quad \lim_{n\to\infty} f \circ S^T(\omega)(\mathbf{x}^{(n)}) = f \circ S^T(\omega)(\mathbf{x}). \tag{2.148}$$

Further

$$\|f \circ S^T(\omega)\|_\infty \le \|f\|_\infty \tag{2.149}$$

Using the Dominated Convergence Theorem, we conclude

$$\begin{aligned} \lim_{n\to\infty} (K_S^T f)(\mathbf{x}^{(n)}) &= \lim_{n\to\infty} \int_\Omega dP(\omega)\, f \circ S^T(\omega)(\mathbf{x}^{(n)}) \\ &= \int_\Omega dP(\omega)\, f \circ S^T(\omega)(\mathbf{x}) \\ &= (K_S^T f)(\mathbf{x}). \end{aligned} \tag{2.150}$$

So $K_S^T f$ is continuous. Continuity of the operator follows from (2.146) and (2.149).
□

Proposition 2.2.5 *The net* $(K^T_{S,\Lambda})_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ *converges weakly to* K^T_S *(as* $\Lambda\to\mathbb{Z}^d$*), i.e. for all* $f\in C^0(M)$*:*

$$\lim_{\Lambda\to\mathbb{Z}^d} K^T_{S,\Lambda} f = K^T_S f. \tag{2.151}$$

Proof We have

$$\|K^T_S f - K^T_{S,\Lambda} f\|_\infty \;\le\; \int_\Omega dP(\omega)\,\|f\circ S^T(\omega) - f\circ S^T_\Lambda(\omega)\|_\infty. \tag{2.152}$$

Because of condition 2 on page 73 and (2.149) the rhs converges to 0 (as $\Lambda\to\mathbb{Z}^d$). □

2.3 Transfer Operators

We recall some definitions and notations from [12].
For $\delta>0$ we denote by A_δ the annulus

$$A_\delta \overset{\text{def}}{=} \{z\in\mathbb{C}\mid -\delta\le \ln|z|\le\delta\} \tag{2.153}$$

and by Γ its positively oriented boundary.
For $\emptyset\ne\Lambda\subset\mathbb{Z}^d$ the normalized Lebesgue-measure on $(S^1)^\Lambda$ is denoted by μ^Λ. For finite Λ it is given by

$$d\mu^\Lambda(\mathbf{z}) = \frac{d\mathbf{z}}{(2\pi i)^{|\Lambda|}}\frac{1}{\mathbf{z}} \overset{\text{def}}{=} \prod_{p\in\Lambda}\frac{dz_p}{2\pi i}\frac{1}{z_p}. \tag{2.154}$$

We also use $d\mu^\Lambda(\mathbf{z})$ as a shorthand notation for the right-hand side of (2.154) for $\mathbf{z}\in A_\delta^\Lambda$.
In **Assumption I** (see below) we will fix a $\delta>0$. For $\Lambda\in\mathcal{F}$ we denote by $\mathcal{H}_\Lambda$ the space of continuous functions on the polyannulus A_δ^Λ that are holomorphic on its interior and write $\|\cdot\|_\Lambda$ for the uniform norm on $\mathcal{H}_\Lambda$. As a function on A_δ^Λ is also a function on $A_\delta^{\mathbb{Z}^d}$ we can drop the index Λ and mean the uniform norm on the infinite-dimensional polyannulus. $\mathcal{H}$ is the vectorspace of all consistent families $\phi=(\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ of functions $\phi_\Lambda\in\mathcal{H}_\Lambda$ where consistency means

$$\begin{aligned}(\pi_{\Lambda_1}\phi_{\Lambda_2})(\mathbf{z}_{\Lambda_1}) &\overset{\text{def}}{=} \int_{(S^1)^{\Lambda_2\setminus\Lambda_1}} d\mu^{\Lambda_2\setminus\Lambda_1}(\mathbf{z}_{\Lambda_2\setminus\Lambda_1})\,\phi(\mathbf{z}_{\Lambda_1}\vee\mathbf{z}_{\Lambda_2\setminus\Lambda_1}) \\ &= \phi_{\Lambda_1}\end{aligned} \tag{2.155}$$

for all $\Lambda_1\subseteq\Lambda_2\in\mathcal{F}$ and $\mathbf{z}_{\Lambda_1}\in A_\delta^{\Lambda_1}$. (Note that we use the same symbol 'π_Λ' for projections of functions and projections of coordinates, for example from M to $(S^1)^\Lambda$.)

For $0 < \vartheta < 1$ and $\phi \in \mathcal{H}$ we define

$$\|\phi\|_\vartheta = \sup_{\Lambda\in\mathcal{F}} \vartheta^{|\Lambda|}\|\phi_\Lambda\|_\Lambda \tag{2.156}$$

$$\|\phi\|_{var} \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda\, |\phi_\Lambda|. \tag{2.157}$$

We set

$$\mathcal{H}_\vartheta \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} : \|\phi\|_\vartheta < \infty\} \tag{2.158}$$

$$\mathcal{H}^{bv} \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} : \|\phi\|_{var} < \infty\} \tag{2.159}$$

$$\mathcal{H}^{bv}_\vartheta \stackrel{\text{def}}{=} \mathcal{H}^{bv} \cap \mathcal{H}_\vartheta. \tag{2.160}$$

Then $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ is a Banach space. For $\phi \in \mathcal{H}^{bv}$ and $\psi \in C^0(M)$ we define

$$\psi_\Lambda(\mathbf{z}_\Lambda) \stackrel{\text{def}}{=} \int_{(S^1)^{\Lambda^C}} d\mu^{\Lambda^C}(\mathbf{z}_{\Lambda^C})\, \psi(\mathbf{z}_\Lambda \vee \mathbf{z}_{\Lambda^C}) \tag{2.161}$$

$$\int_M d\mu\, \psi\, \phi \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbf{Z}^d} \int_M d\mu_\Lambda\, \psi_\Lambda\, \phi_\Lambda. \tag{2.162}$$

Finally we recall the definition of a transfer operator: Let $\tilde{\mu}$ be a measure on the (completed) Borel σ-algebra of a metric space $\widetilde{M}$ and $\widetilde{S} : \widetilde{M} \to \widetilde{M}$ be a non-singular measurable map. The **Perron-Frobenius operator** (or **transfer operator**) $\mathcal{L}_{\widetilde{S}}$, acting on $L^1(\widetilde{M})$, is defined via the equation

$$\int_{\widetilde{M}} d\tilde{\mu}\, \psi \circ \widetilde{S}\, \phi = \int_{\widetilde{M}} d\tilde{\mu}\, \psi\, \mathcal{L}_{\widetilde{S}}\phi \tag{2.163}$$

that must hold for all $\psi \in L^\infty(\widetilde{M})$ and $\phi \in L^1(\widetilde{M})$.

The Markov kernels for our stochastic systems are analogous to the composition operator '$\circ S$' (with deterministic S), acting on functions. We define transfer operators for this case analogously. These operators act on elements of $\mathcal{H}_\vartheta$ that do not in general correspond to elements of $L^1(M)$. Recall (see [12]) that $\mathcal{H}^{bv}_\vartheta$ can be identified with a subset of $rca(M)$ (or, in other words, a subset of Borel measures). So for example in Theorem 2.3.1 we will show that the equation analogous to (2.163) holds for $\psi \in C^0(M)$ (rather than $L^\infty(M)$) and $\phi \in \mathcal{H}^{bv}_\vartheta$.

Now we consider a special class of interactions (cf. [12]), namely a family $(S_\Lambda)_{\Lambda\in\mathcal{F}}$ of maps on M that can be written as

$$\begin{aligned} S_\Lambda : M &\to M \\ S_\Lambda(\mathbf{z}) &= F_\Lambda \circ T_\Lambda(\mathbf{z}) \vee \mathbf{z}_{\Lambda^C} \end{aligned} \tag{2.164}$$

where

$$\begin{aligned} F_\Lambda : (S^1)^\Lambda &\to (S^1)^\Lambda \\ \mathbf{z}_\Lambda = (z_q)_{q\in\Lambda} &\mapsto (f_q(z_q))_{q\in\Lambda} \end{aligned} \tag{2.165}$$

and

$$T_\Lambda : M \to (S^1)^\Lambda \tag{2.166}$$

$$(T_\Lambda(\mathbf{z}))_q \stackrel{\text{def}}{=} z_q \exp\left(2\pi i\epsilon \sum_{k=1}^{\infty} g_{q,k}(\mathbf{z})\right) \quad \text{for } q \in \Lambda \tag{2.167}$$

and f_q and $g_{q,k}$ satisfy the following assumptions:

Assumption I $F(\mathbf{z}) = (f_p(z_q))_{q\in\mathbf{Z}^d}$ where $f_q : S^1 \to S^1$ are real analytic and expanding (i.e. $f_q' \geq \lambda_0 > 1$) maps that extend for some δ_1 holomorphically to the interior of an annulus A_{δ_1}. In Proposition 1.3.1 and 1.3.2 of [12] we have shown that the holomorphic extension to a sufficiently thin annulus A_δ is expanding in the sense that the preimage of A_δ wrt f_q lies in the interior of A_δ. We fix such a δ_1. Then for every $q \in \mathbb{Z}^d$ the Perron-Frobenius operator $\mathcal{L}_{f_q}$, acting on $\mathcal{H}_{\{q\}}$, has a simple largest eigenvalue 1 with eigenvector h_q , such that $\pi_\emptyset(h_q) = 1$ and the restriction of h_q to S^1 is positive' and it splits into

$$\mathcal{L}_{f_q} = \mathcal{Q}_q + \mathcal{R}_q, \tag{2.168}$$

where $\mathcal{Q}_q$ is a projection onto span(h_q). We assume that there are positive constants c_h and c_r such that the following two estimates hold for all $q \in \mathbb{Z}^d$:

$$\|h_q\|_{\{q\}} \leq c_h \tag{2.169}$$

$$\|\mathcal{R}_q^n\|_{\{q\}} \leq c_r\eta^n \tag{2.170}$$

where $\|\cdot\|_{\{q\}}$ denotes the uniform norm on $\mathcal{H}_{\{q\}}$ (for this we might have to take δ_1 even smaller) and the induced operator-norm, respectively. We note that this holds in particular if f_q does not depend on q.

Assumption II For all $q \in \mathbb{Z}^d$ and $k \geq 1$ each map $g_{q,k}$ extends to a holomorphic map $g_{q,k} : A_{\delta_1}^{B_k(q)} \to \mathbb{C}$ and its sup-norm (of modulus) is exponentially bounded by

$$\| g_{q,k} \|_{A_{\delta_1}^{B_k(q)}} \leq c_3 \exp\left(-c_g k^d\right) \tag{2.171}$$

with $c_3 > 0$ and 'large' $c_g > 0$. (In several statements in Section 2.3 and 2.4 a lower bound for c_g will come out of our computations. The idea is always that our estimates work, provided c_g is bigger than a certain constant.)

For $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ we denote by h_Λ the function

$$h_\Lambda(\mathbf{z}_\Lambda) \stackrel{\text{def}}{=} \prod_{q \in \Lambda} h_q(z_q), \tag{2.172}$$

where h_q is as in Assumption I. We set $h_\emptyset = 1$ and

$$h_{\mathbf{Z}^d} \stackrel{\text{def}}{=} (h_\Lambda)_{\Lambda \in \mathcal{F}} \in \mathcal{H}. \tag{2.173}$$

We further define for a fixed $\xi \in M$ and $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\Lambda_1 \subseteq \Lambda$ the *updating at the Λ_1-sites with fixed boundary conditions ξ_{Λ^c} outside Λ* (or *cut-off* of S_{Λ_1}):

$$\begin{aligned} S_{\Lambda_1,\Lambda} : (S^1)^\Lambda &\to (S^1)^\Lambda \\ \mathbf{z}_\Lambda &\mapsto \pi_\Lambda \circ S_{\Lambda_1}(\mathbf{z}_\Lambda \vee \xi_{\Lambda^c}). \end{aligned} \tag{2.174}$$

And for $\mathbf{z}_{\Lambda \setminus \Lambda_1} \in (S^1)^{\Lambda \setminus \Lambda_1}$ we define

$$\begin{aligned} \pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\cdot \vee \mathbf{z}_{\Lambda \setminus \Lambda_1}) : (S^1)^{\Lambda_1} &\to (S^1)^{\Lambda_1} \\ \mathbf{z}_{\Lambda_1} &\mapsto \pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{z}_{\Lambda \setminus \Lambda_1}). \end{aligned} \tag{2.175}$$

Remark 2.3.1 1. The map defined in (2.175) is the cut-off of S wrt Λ_1 and boundary conditions $\mathbf{z}_{\Lambda \setminus \Lambda_1} \vee \xi_\Lambda$. So we can use the special representation in terms of integral kernels for its transfer operator, restricted to $\mathcal{H}_{\Lambda_1}$, for the proposition below.

2. The family $(S_q)_{q \in \mathbf{Z}^d}$, defined by (2.164), satisfies conditions (2.95) and (2.96) as one can see from [12]: The partial derivatives are estimated in the proof of Proposition 1.3.1 there.

Lemma 2.3.1 *Let $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ be the disjoint union of Λ_1 and Λ_2. The transfer operator, restricted to $\mathcal{H}_{\Lambda_1}$, of the map $S_{\Lambda_1,\Lambda} : (S^1)^\Lambda \to (S^1)^\Lambda$, defined in (2.174) has the following representation in terms of integral kernels:*

$$\begin{aligned} &\left(\mathcal{L}_{S_{\Lambda_1,\Lambda}} \phi\right)(\mathbf{w}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}) \\ &= \int_{\Gamma^{\Lambda_1}} d\mu^{\Lambda_1}(\mathbf{z}_{\Lambda_1}) \phi(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}) \prod_{q \in \Lambda_1} \frac{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q}{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q - w_q} \end{aligned} \tag{2.176}$$

for $\phi \in \mathcal{H}_\Lambda$.

Proof Let $\psi \in \mathcal{C}^0((S^1)^\Lambda)$. We use the notation $\phi_{\mathbf{w}_{\Lambda_2}}$ for the function $\mathbf{w}_{\Lambda_1} \mapsto \phi(\mathbf{w}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2})$.

$$
\begin{aligned}
&\int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{w}_\Lambda)\,\psi\circ S_{\Lambda_1,\Lambda}(\mathbf{w}_\Lambda)\,\phi(\mathbf{w}_\Lambda) && (2.177)\\
&= \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{w}_{\Lambda_2}) \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1}(\mathbf{w}_{\Lambda_1})\,\psi(\cdot\vee\mathbf{w}_{\Lambda_2})\circ\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\mathbf{w}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2})\\
&\qquad\times\phi(\mathbf{w}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2})\\
&= \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2}(\mathbf{w}_{\Lambda_2}) \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1}(\mathbf{w}_{\Lambda_1})\,\psi(\mathbf{w}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2})\\
&\qquad\times(\mathcal{L}_{\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\cdot\vee\mathbf{w}_{\Lambda_2})}\phi_{\mathbf{w}_{\Lambda_2}})(\mathbf{w}_{\Lambda_1})\\
&= \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{w}_\Lambda)\,\psi(\mathbf{w}_\Lambda)\,(\mathcal{L}_{\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\cdot\vee\mathbf{w}_{\Lambda_2})}\phi_{\mathbf{w}_{\Lambda_2}})(\mathbf{w}_{\Lambda_1})
\end{aligned}
$$

Using the representation of the transfer operator for $\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\cdot\vee\mathbf{w}_{\Lambda_2})$ that we established in Proposition 1.3.3 of [12], we obtain the rhs of (2.176).

□

Remark 2.3.2 We see in particular that $\mathcal{L}_{S_{\Lambda_1,\Lambda}}$ 'acts on the Λ_1-coordinates' only. There is no integration wrt the Λ_2-coordinates.

For $q \in \Lambda_1$ we can split the factor

$$
\frac{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2}))_q}{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2}))_q - w_q} = h_q(w_q,z_q) + r_q(w_q,z_q) + \sum_{k=1}^{\infty}\beta_{q,k}(w_q,\mathbf{z}_{\Lambda_1}\vee\mathbf{w}_{\Lambda_2}\vee\xi_{\Lambda^c}) \tag{2.178}
$$

as in [12] and we can represent the particular summands graphically as h-line, r-line or k-triangles. For $q \in \Lambda_2$ there is no integration and we draw an *identity-line* in the configuration.

Definition 2.3.1 We define for fixed $\Lambda \in \mathcal{F}\setminus\{\emptyset\}$, $\xi \in M$ and a finite sequence $j = (\Lambda_1,\ldots,\Lambda_n) \in J$ of subsets of Λ the map

$$
\begin{aligned}
S_{j,\Lambda} : (S^1)^\Lambda &\to (S^1)^\Lambda && (2.179)\\
S_{j,\Lambda} &\stackrel{\text{def}}{=} S_{\Lambda_n,\Lambda}\circ\ldots\circ S_{\Lambda_1,\Lambda}.
\end{aligned}
$$

Recall that in Lemma 2.1.2 we defined the maps $\mathbf{j}$ and $\mathbf{j}_N$ for the Poisson and Bernoulli process, respectively. For almost all $\omega \in \Omega$ there is a finite sequence $\mathbf{j}(\omega) = (\Lambda_1,\ldots,\Lambda_n)$ and so

$$
\mathcal{L}_{S_{\mathbf{j}(\omega),\Lambda}} \stackrel{\text{def}}{=} \mathcal{L}_{S_{\Lambda_n,\Lambda}}\circ\ldots\circ\mathcal{L}_{S_{\Lambda_1,\Lambda}} \tag{2.180}
$$

is well-defined.

We set

$$\mathcal{L}^T_{S,\Lambda,N} \stackrel{\text{def}}{=} \sum_{\omega\in\Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\mathcal{L}_{S_{\mathbf{j}(\omega),\Lambda}} \tag{2.181}$$

where $P_{\Lambda,N}$ is the probability measure for the Bernoulli process with parameters λ, T, N and values in $\mathbb{N}^\Lambda$, as defined in Definition 2.1.6.
Formula (2.181) defines the transfer operator for $K^T_{S,\Lambda,N}$ (cf. (2.144) - (2.146)) as we will show in the following proposition. The limit (as $N \to \infty$) is the transfer operator for $K^T_{S,\Lambda}$. Our proof of the latter statement is quite long and technical and will be completed in Proposition 2.3.6.

Proposition 2.3.1 *$\mathcal{L}^T_{S,\Lambda,N}$ is the transfer operator for $K^T_{S,\Lambda,N}$, i.e.*

$$\begin{aligned}\int_{(S^1)^\Lambda} & d\mu^\Lambda(\mathbf{z}_\Lambda)\ \left(K^T_{S,\Lambda,N}\psi\right)(\mathbf{z}_\Lambda)\,\phi(\mathbf{z}_\Lambda) \\ &= \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_\Lambda)\left(\mathcal{L}^T_{S,\Lambda,N}\phi\right)(\mathbf{z}_\Lambda)\end{aligned} \tag{2.182}$$

for all $\psi \in L^\infty((S^1)^\Lambda)$ and $\phi \in L^1((S^1)^\Lambda)$.

Proof

$$\begin{aligned}\int_{(S^1)^\Lambda} & d\mu^\Lambda(\mathbf{z}_\Lambda)\ (K^T_{S,\Lambda,N}\psi)(\mathbf{z}_\Lambda)\,\phi(\mathbf{z}_\Lambda) \\ &= \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda) \sum_{\omega\in\Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\,\psi\circ S_{\mathbf{j}(\omega),\Lambda}(\mathbf{z}_\Lambda)\,\phi(\mathbf{z}_\Lambda) \\ &= \sum_{\omega\in\Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi\circ S_{\mathbf{j}(\omega),\Lambda}(\mathbf{z}_\Lambda)\,\phi(\mathbf{z}_\Lambda) \\ &= \sum_{\omega\in\Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_\Lambda)\left(\mathcal{L}_{S_{\mathbf{j}(\omega),\Lambda}}\phi\right)(\mathbf{z}_\Lambda) \\ &= \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_\Lambda) \sum_{\omega\in\Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\left(\mathcal{L}_{S_{\mathbf{j}(\omega),\Lambda}}\phi\right)(\mathbf{z}_\Lambda) \\ &= \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_\Lambda)\,(\mathcal{L}^T_{S,\Lambda,N}\phi)(\mathbf{z}_\Lambda)\end{aligned} \tag{2.183}$$

□

Now we are studying the rhs of (2.181), for the restriction of the operators to $\mathcal{H}_\Lambda$, in more detail for. The sum is over all $\omega \in \Omega_{\Lambda,N}$. If $\omega(q,i) = 1$ then the qth site is updated at time i. The updating $z_q(i) = S_q(\mathbf{z}_\Lambda(i-1) \vee \xi_{\Lambda^c})$ depends in general on

all other sites at time $i-1$. In the representation (2.176) for the transfer operator a factor like (2.178) and an integration wrt the corresponding coordinate z_q occurs. We represent the particular summands on the rhs of (2.178) graphically as h-line, r-line or k-triangle (cf. [12] and also Figure 2.2).
If $\omega(q,i)=0$ then the site q is not changed at time i. We have $z_q(i)=z_q(i-1)$ and represent this by an identity-line from $(q,i-1)$ to (q,i).
By Definition 2.1.6 of the Bernoulli process we have for each (q,i):

$$\omega(q,i)=\begin{cases} 1 & \text{with probability } \; p \\ 0 & \text{with probability } \; 1-p \end{cases}, \tag{2.184}$$

where $p=\frac{\lambda T}{N}$ as in (2.22). The family $(\omega(q,i))_{(q,i)\in\Lambda\times I}$ is independent. For a particular $\omega\in\Omega_{\Lambda,N}$ we get a *reduced configuration* $\mathcal{C}_r$ by choosing h-lines, r-lines or k-triangles at all (q,i) for which $\omega(q,i)=1$. At the other points (q,i) (with $\omega(q,i)=0$) there are identity-lines. This reduced configuration corresponds to an operator $\mathcal{L}_{\mathcal{C}_r}$.

Definition 2.3.2 Let $\Lambda_1\subseteq\Lambda\in\mathcal{F}\setminus\{\emptyset\}$ and $I=\{1,\ldots,N\}$.

1. A **full configuration $\mathcal{C}$ on $\Lambda\times$ I** is an assignment of each point in $\Lambda\times I$ to either an h-line, r-line or k-triangle. We denote the set of all full configurations on $\Lambda\times I$ by **Conf(Λ, N)**. (Figure 2.3 shows a full configuration. The r-lines are drawn thick and the h-lines thin.)

2. A **reduced configuration $\mathcal{C}_r$ on $\Lambda\times$ I** is an assignment of each point in $\Lambda\times I$ to either an h-line, r-line, k-triangle or an identity-line. We denote the set of all reduced configurations on $\Lambda\times I$ by **Conf$_r$(Λ, N)**. (Figure 2.2 shows a reduced configuration. The h-lines and r-lines are drawn as in Figure 2.3 and the identity-lines are dotted.)

3. We call the h-lines, r-lines, k-triangles or an identity-lines the **items of the configuration**.

4. If $(q,i)\in\Lambda\times I$ is assigned to an h-line then we also say that there is an **h-line from $(\mathbf{q},\mathbf{i-1})$ to $(\mathbf{q},\mathbf{i})$**. In this case we also say that there is an **h-line at $(\mathbf{q},\mathbf{i})$**. Our terminology for the other items is analogous.

5. The **basepoints of a k-triangle** at (q,i) are the points $(\tilde{q},i-1)$ with $\tilde{q}\in B_k(q)$.

$$\omega(\mathcal{C}_r)(q,i)\stackrel{\text{def}}{=}\begin{cases} 0 & \text{if } (q,i) \text{ is assigned to an identity-line} \\ 1 & \text{otherwise} \end{cases}. \tag{2.185}$$

We set

$$|\omega(\mathcal{C}_r)| \stackrel{\text{def}}{=} \sum_{(q,i)\in\Lambda\times I} \omega(\mathcal{C}_r)(q,i). \quad (2.186)$$

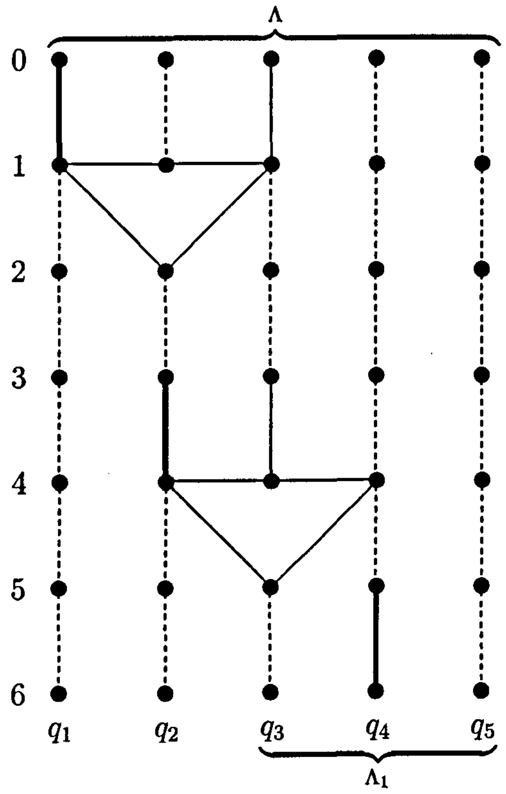


Figure 2.2: A reduced configuration

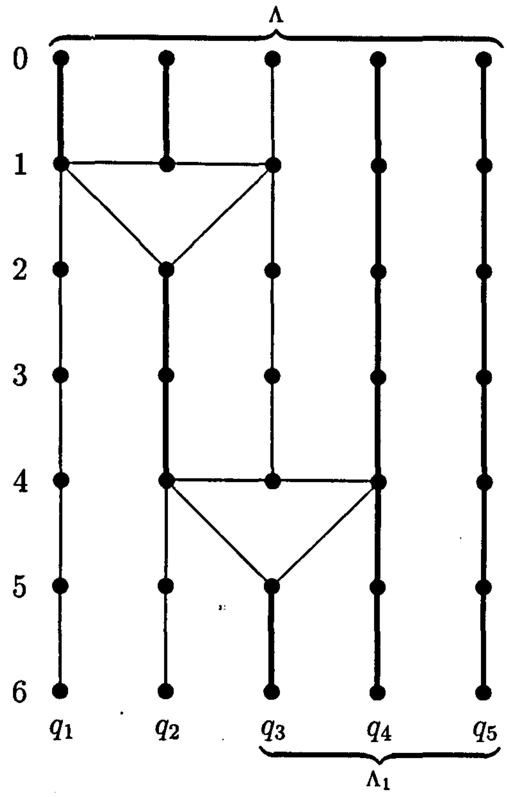


Figure 2.3: A full configuration

Remark 2.3.3 A full configuration is, of course, the same as a configuration defined in [12]. In contrast the reduced configurations can have identity-lines as items. Recall the definitions of *maximal* chains (see Definition 1.5.2 of [12]) for full configurations. We define these for reduced configurations analogously.

We remark that, as in [12], certain combinations of lines and triangles lead to the operator $\mathcal{L}_{\mathcal{C}_r}$ being equal to zero, namely if

1. an h-line follows (wrt the time-order) or is followed by an r-line and their common endpoint is not a basepoint of any triangle (cf. Figure 2.4(a).)
2. a triangle is followed by an h-line and their common point is not a basepoint of any triangle (cf. Figure 2.4(b).)

3. an h-line is followed by an identity-chain and then an r-line and no endpoint of the identity-chain is a basepoint of a triangle (cf. Figure 2.4(c).)
4. an r-line is followed by an identity-chain and an h-line (in this order) and no endpoint of the identity-chain is a basepoint of a triangle (cf. Figure 2.4(c).)
5. (the apex of) a triangle is followed by an identity-chain and an h-line (in this order) and no endpoint of the identity-chain is a basepoint of a triangle (cf. Figure 2.4(d).)

The proofs for the first two cases are given in [12]. The proofs of the other statements are (modulo notation) the same.
We further note that for $\Lambda_1 \subseteq \Lambda$ and the expansion

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda,N} = \sum_{\omega \in \Omega_{\Lambda,N}} P_{\Lambda,N}(\omega)\, \pi_{\Lambda_1} \circ \mathcal{L}_{S_{j(\omega)}} \tag{2.187}$$

$$= \sum_{\mathcal{C}_r \in \mathrm{Conf}_\Gamma(\Lambda,N)} P_{\Lambda,N}(\omega(\mathcal{C}_r))\, \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_r}, \tag{2.188}$$

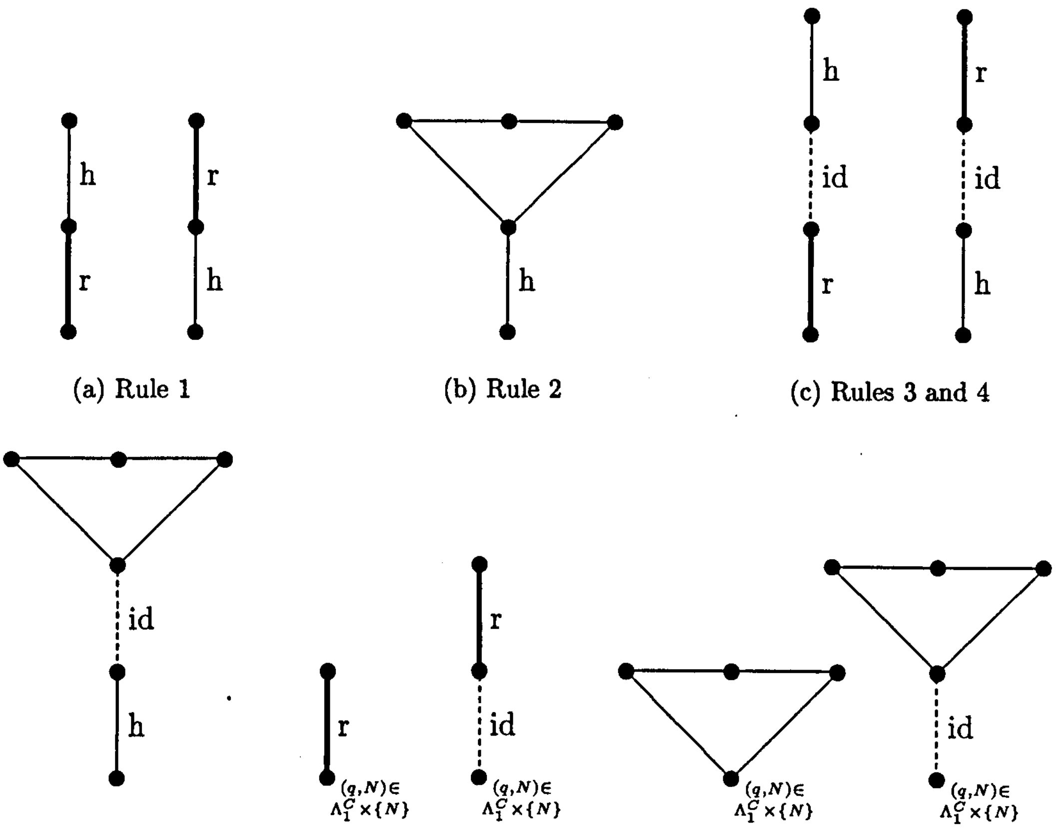
where the second sum is over all reduced configurations, we get $\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_r} = 0$ if

6. $\mathcal{C}_r$ ends with an r-line or a triangle in $(\Lambda \setminus \Lambda_1) \times \{N\}$ (cf. Figure 2.4(e) and Figure 2.4(f).)
7. $\mathcal{C}_r$ ends with a maximal identity-chain in $(\Lambda \setminus \Lambda_1) \times \{N\}$, say from $(q, i_l - 1)$ to (q, N), such that $(q, i_l - 1)$ is the endpoint of an r-line or the apex of a triangle but not a basepoint of any triangle (cf. Figure 2.4(e) and Figure 2.4(f).)

In view of this we make the following definitions.

Definition 2.3.3 Let $\Lambda_1 \subset \Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and N be fixed.

1. We call a reduced configuration, as considered in the expansion (2.188), a **non-zero reduced configuration on $\Lambda \times$ I that ends in Λ_1** if none of the cases 1 - 7 occurs. (Figure 2.2 shows a non-zero reduced configuration on $\Lambda = \{q_1, \ldots, q_5\}$ ending in $\Lambda_1 = \{q_3, q_4, q_5\}$. Note that such a configuration could also end with an h-line in $\Lambda_1 \times \{N\}$.)
2. We define **$\mathrm{Conf}_\Gamma(\Lambda, \mathrm{N}, \Lambda_1)$** to be the set of all non-zero reduced configurations on $\Lambda \times I$ that end in Λ_1.
3. We call a full configuration on $\Lambda \times I$ a **non-zero full configuration that ends in Λ_1** if none of the cases 1,2 or 6 occurs. (Figure 2.3 shows a non-zero full configuration on $\Lambda = \{q_1, \ldots, q_5\}$ ending in $\Lambda_1 = \{q_3, q_4, q_5\}$.)



(a) Rule 1 (b) Rule 2 (c) Rules 3 and 4

(d) Rule 5 (e) Rules 6 and 7 (f) Rules 6 and 7

Figure 2.4: Patterns in reduced configurations that give rise to the zero-operator

4. We denote by $\mathbf{Conf}(\mathbf{\Lambda}, \mathbf{N}, \mathbf{\Lambda_1})$ the set of all non-zero full configurations on $\Lambda \times I$ that end in Λ_1.

We would like to find for our stochastic system similar estimates as for the deterministic coupled map lattices (cf. [12]). For this we *extend* reduced to a full configurations and estimate the sums, weighted by probability factors, by introducing an *effective decay rate* and an *effective coupling parameter.*

Definition 2.3.4 We define the **extension map**

$$\mathrm{Ext}: \ \mathrm{Conf}_r(\Lambda, N, \Lambda_1) \to \mathrm{Conf}(\Lambda, N, \Lambda_1) \times \Omega_{\Lambda,N} \tag{2.189}$$

$$\mathcal{C}_r \mapsto \mathrm{ext}(\mathcal{C}_r) \times \omega(\mathcal{C}_r). \tag{2.190}$$

$\omega(\mathcal{C}_r)$ was defined in (2.185) and $\mathrm{ext}(\mathcal{C}_r)$ is defined as follows:
If $\mathcal{C}_r$ has an h-line, r-line or k-triangle at (q, i) then so has $\mathrm{ext}(\mathcal{C}_r)$. Suppose $\mathcal{C}_r$ has a maximal identity-chain, say from $(q, i_l - 1)$ to (q, i_r). Then $\mathrm{ext}(\mathcal{C}_r)$ has an h-chain from $(q, i_l - 1)$ to (q, i_r) in any of the following three cases:

1. $(q, i_l - 1)$ is the endpoint of an h-line, but not a basepoint of any triangle.

2. (q, i_r) endpoint of an h-line, but not a basepoint of any triangle.

3. $(q, i_r) \in (\Lambda \setminus \Lambda_1) \times \{N\}$.

Otherwise $\operatorname{ext}(\mathcal{C}_r)$ has a maximal r-chain from $(q, i_l - 1)$ to (q, i_r).

We remark that $\operatorname{ext}(\mathcal{C}_r)$ has the same triangles as $\mathcal{C}_r$.
The full configuration $\mathcal{C} \in \operatorname{Conf}(\Lambda, 6, \Lambda_1)$ in Figure 2.3 is the extension $\operatorname{ext}(\mathcal{C}_g)$ of the reduced configuration $\mathcal{C}_g \in \operatorname{Confg}(\Lambda, 6, \Lambda_1)$. $\mathcal{C}$ has for example a maximal h-chain from $(q_3, 1)$ to $(q_3, 4)$ because $\mathcal{C}_g$ has a maximal identity-chain from $(q_3, 1)$ to $(q_3, 3)$ and $(q_3, 3)$ is the endpoint of an h-line but not the basepoint of a triangle, so case 2 applies.
The maximal identity-chain of $\mathcal{C}_g$ from $(q_2, 4)$ to $(q_2, 6)$ corresponds to a maximal h-chain of $\mathcal{C}$ because it ends in $(q_2, 6) \in (\Lambda \setminus \Lambda_1) \times \{N\}$ and case 3 applies.
As $q_5 \in \Lambda_1$ the identity-chain of $\mathcal{C}_g$ from $(q_5, 0)$ to $(q_5, 6)$ gives rise to a maximal r-chain of $\mathcal{C}$.

The map Ext is a bijection onto its image. So we can rewrite the representations in (2.187) and (2.188):

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda,N} = \sum_{\mathcal{C} \in \operatorname{Conf}(\Lambda,N,\Lambda_1)} \sum_{\substack{(\mathcal{C}_r,\omega) \in \operatorname{Confr}(\Lambda,N,\Lambda_1) \times \Omega_{\Lambda,N}: \\ \operatorname{Ext}(\mathcal{C}_r) = (\mathcal{C},\omega)}} P_{\Lambda,N}(\omega)\, \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_r}. \tag{2.191}$$

In general $\mathcal{C}_r$ is not uniquely determined by the condition $\mathcal{C} = \operatorname{ext}(\mathcal{C}_r)$. If $\mathcal{C}$ has for example a maximal r-chain, from $(q, i_l - 1)$ to (q, i_r) say, then $\mathcal{C}_r$ can have any sequence of r-lines and identity-lines from $(q, i_l - 1)$ to (q, i_r), corresponding to the different values of $(\omega(q, i_l), \dots, \omega(q, i_r)) \in \{0, 1\}^{i_r - i_l}$. For any chosen $\omega = \omega(\mathcal{C}_r)$ the reduced configuration $\mathcal{C}_r$ has exactly

$$k = \operatorname{card}\{(q, i) : i_l \le i \le i_r \text{ and } w(q, i) = 1\} \tag{2.192}$$

r-lines between $(q, i_l - 1)$ and (q, i_r) and the sequence of r-lines and identity-lines corresponds to an operator $\mathcal{R}^k_q$.
The event that $\omega(q, i) = 1$ for exactly k values $i_l \le i \le i_r$ has probability $\binom{n}{k} p^k (1 - p)^{n-k}$ where $n = i_r - i_l + 1$. So if $\mathcal{C}$ (in (2.191)) has an r-chain of length n the sum over all possible corresponding sequences of r-lines and identity-lines in $\mathcal{C}_r$, weighted with the corresponding probabilities, gives rise to an operator

$$\begin{aligned} \mathcal{R}_{N,q}(n) &\stackrel{\text{def}}{=} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \mathcal{R}^k_q \\ &= (p\mathcal{R}_q + (1-p)\operatorname{id}_q)^n \\ &= (\operatorname{id}_q - p\,(\operatorname{id}_q - \mathcal{R}_q))^n \end{aligned} \tag{2.193}$$

with $p = \frac{\lambda T}{N}$, as in (2.22). We note that

$$\mathcal{R}_{N,q}(n_1)\, \mathcal{R}_{N,q}(n_2) = \mathcal{R}_{N,q}(n_1 + n_2). \tag{2.194}$$

Using (2.170), we estimate the norm of (2.193) by

$$\begin{aligned} R_N(n) &\stackrel{\text{def}}{=} \|\mathcal{R}_{N,q}(n)\|_{L(\mathcal{H}_{\{q\}})} \\ &\leq \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} c_r \eta^k \\ &= c_r(1-(1-\eta)p)^n. \end{aligned} \tag{2.195}$$

We therefore make the following definition.

Definition 2.3.5 The **effective decay rate** is defined as:

$$\eta_e \stackrel{\text{def}}{=} 1 - (1-\eta)\, p. \tag{2.196}$$

Then (2.195) reads

$$R_N(n) \leq c_r \eta_e^n. \tag{2.197}$$

$\eta_e = (1-p) \cdot 1 + p \cdot \eta$ is the convex combination of 1 and the original decay rate η. For $p \to 0$ (equivalently $N \to \infty$) η_e tends to 1.

Now we consider a maximal h-chain in $\text{ext}(\mathcal{C}_r)$, say from $(q, i_l - 1)$ to (q, i_r). If $(q, i_r) \in (\Lambda \setminus \Lambda_1) \times \{N\}$ then $\mathcal{C}_r$ can have any sequence of h-lines and identity-lines from $(q, i_l - 1)$ to (q, N) (in particular a maximal identity-chain). This sequence corresponds to the composite of operators $\mathcal{Q}_q$ and id_q. Summing over all possible sequences, we see that the corresponding operators, weighted with their particular probabilities, give rise to an operator:

$$(1-p)^n \,\text{id}_q + \sum_{k=1}^{n} \binom{n}{k} (1-p)^k p^{n-k} \mathcal{Q}_q^{n-k} = (1-p)^n \,\text{id}_q + (1-(1-p)^n)\, \mathcal{Q}_q \tag{2.198}$$

The projection π_{Λ_1} in (2.191) is an integration wrt all $(\Lambda \setminus \Lambda_1)$-coordinates, in particular also wrt the q-coordinate. As $\pi_q = \pi_q \circ \text{id}_q = \pi_q \circ \mathcal{Q}_q$ we can replace the full operator in (2.198) simply by $\mathcal{Q}_q$.
If the maximal h-chain does not end in $(\Lambda \setminus \Lambda_1) \times \{N\}$ then $\mathcal{C}_r$ has a sequence of h-lines or identity-lines from $(q, i_l - 1)$ to (q, i_r) in which at least one h-line occurs. (For otherwise, if no h-line occured, $\text{ext}(\mathcal{C}_r)$ would have an r-chain from $(q, i_l - 1)$ to (q, i_r).) It corresponds to an operator

$$\mathcal{Q}_{N,q}(n) = \sum_{k=1}^{n} \binom{n}{k} (1-p)^k p^{n-k} \mathcal{Q}_q^k \quad (2.199)$$

$$= (1-(1-p)^n)\,\mathcal{Q}_q \quad (2.200)$$

So we see that each maximal h-chain in a full configuration $\mathcal{C}$ in (2.191) gives rise to an operator which is a scalar multiple of $\mathcal{Q}_q$.
For h-chains ending in $(\Lambda \setminus \Lambda_1) \times \{N\}$ the scalar factor is equal to 1 and for a maximal h-chain of length n and not ending in $(\Lambda \setminus \Lambda_1) \times \{N\}$ it is equal to $1-(1-p)^n$. The product of all these factors is

$$\tilde{c}_N(\mathcal{C}) \stackrel{\text{def}}{=} \prod_H (1-(1-p)^{\text{length}(H)}) \quad (2.201)$$

where the product is over all maximal h-chains H not ending in $(\Lambda \setminus \Lambda_1) \times \{N\}$.

As mentioned before, $\mathcal{C}_r$ has exactly the same triangles as $\text{ext}(\mathcal{C}_r)$, so if there is a triangle from $(q, i-1)$ to (q, i) then $\omega(q,i) = 1$ which happens with probability p. (Note that if $\omega(q,i) = 0$ then $\text{Ext}(\mathcal{C}_g, \omega) \neq \mathcal{C}$ for any $\mathcal{C}_r \in \text{Conf}_{\text{r}}(\Lambda, N, \Lambda_1)$.) So in (2.191) we just sum over such $\mathcal{C}_r$ with $w(\mathcal{C}_r)(q,i) = 1$ which leads to a factor p. In our estimates for the deterministic coupled map lattices we have seen, that each triangle contributes (among other factors) a factor ϵ in the estimates. In case of the system we are considering it also contributes an additional factor p. This motivates the following definition.

Definition 2.3.6 We define the **effective coupling parameter**:

$$\epsilon_e \stackrel{\text{def}}{=} \epsilon p. \quad (2.202)$$

Recall from [12] that for $\mathcal{C} \in \text{Conf}(\Lambda, N, \Lambda_1)$ we have the representation

$$\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} = \pi_{\Lambda_1} \circ \text{Op}(N, \mathcal{C}) \circ \cdots \circ \text{Op}(1, \mathcal{C}) \quad (2.203)$$

$$\text{with} \quad \text{Op}(i, \mathcal{C}) \stackrel{\text{def}}{=} \bigotimes_{q \in \Lambda_{\mathcal{Q}}(i,\mathcal{C})} \mathcal{Q}_q \bigotimes_{q \in \Lambda_{\mathcal{R}}(i,\mathcal{C})} \mathcal{R}_q \bigotimes_{\substack{k \geq 1 \\ q \in \Lambda_{\mathcal{B},k}(i,\mathcal{C})}} \mathcal{B}_{k,q}$$

where $\Lambda_{\mathcal{Q}}(i, \mathcal{C})$ is the set of $q \in \Lambda$ such that there is an h-line from $(q, i-1)$ to (q, i), $\Lambda_{\mathcal{R}}(i, \mathcal{C})$ is the set of $q \in \Lambda$ such that there is an r-line from $(q, i-1)$ to (q, i) and $\Lambda_{\mathcal{B},k}(i, \mathcal{C})$ is the set of $q \in \Lambda$ such that there is a k-triangle from $(q, i-1)$ to (q, i) in $\mathcal{C}$. The operators $\mathcal{Q}_q$, $\mathcal{R}_q$ and $\mathcal{B}_{k,q}$ are integral-operators with kernels h_q, r_q and $\beta_{k,q}$, respectively. (In [12] we only used the representation by integal kernels. The notation '$\otimes$' here means that the integral kernel for the operator $\mathcal{L}_{\mathcal{C}}$ in (2.203) is the product of the particular integral kernels. It should not be mixed up with the notation of tensor products. The representation in (2.203) is more convenient for the following considerations.)

Definition 2.3.7 For fixed $T > 0$ we define the **operator with effective parameters** corresponding to $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$:

$$\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^{T} \stackrel{\text{def}}{=} \tilde{c}_N(\mathcal{C})\, \pi_{\Lambda_1} \circ \widetilde{\mathrm{Op}}(N, \mathcal{C}) \circ \cdots \circ \widetilde{\mathrm{Op}}(1, \mathcal{C}) \tag{2.204}$$
$$\text{with } \widetilde{\mathrm{Op}}(i, \mathcal{C}) \stackrel{\text{def}}{=} \bigotimes_{q \in \Lambda_{\mathcal{Q}}(i,\mathcal{C})} \mathcal{Q}_q \bigotimes_{q \in \Lambda_{\mathcal{R}}(i,\mathcal{C})} \mathcal{R}_{N,q} \bigotimes_{\substack{k \geq 1 \\ q \in \Lambda_{\mathcal{B},k}(i,\mathcal{C})}} p\mathcal{B}_{k,q}.$$

Remark 2.3.4 We point out that the term *effective coupling parameter* is rather heuristic. It does *not* mean that we took a different coupling strength for our original system. But, as we see from (2.203),we use an operator $p\mathcal{B}_{q,k}$ rather than just $\mathcal{B}_{q,k}$ (like in (2.202)) for each triangle and so in our estimates we use ϵ_e rather than the original coupling parameter ϵ.

In the next proposition we summarize our study of the relation between full and reduced configurations. First we recall some definitions.

Definition 2.3.8 (cf. Definition 1.5.2 in [12])

- For $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ we denote by $\tilde{\Lambda}_{\mathcal{C}}$ the set of points $q \in \Lambda$ that appear as the $\mathbb{Z}^d$-coordinate of a base point (q, t) of a triangle in $\mathcal{C}$.
- $\Lambda_{\mathcal{C}}$ is the set of those points $q \in \Lambda$ that appear as the $\mathbb{Z}^d$-coordinate of an apex (q, t) that does not lie above any other triangle.
- Λ_r is the set of $q \in \Lambda \setminus \tilde{\Lambda}_{\mathcal{C}}$ that appear as the $\mathbb{Z}^d$ coordinate of an r-line. (So $\Lambda_r \subseteq \Lambda_1$.)
- We write $\Lambda(\mathcal{C}) \stackrel{\text{def}}{=} \tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r$.

Proposition 2.3.2 *1. We can write the transfer operators in terms of full configurations and reduced parameters:*

$$\pi_{\Lambda_1} \circ \mathcal{L}_{S,\Lambda,N}^{T} = \sum_{\mathcal{C} \in Conf(\Lambda,N,\Lambda_1)} \pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^{T} \tag{2.205}$$

2. If $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ has exactly $n_{\beta,k}$ k-triangles, n_r r-lines, n_h h-lines, $\tilde{n}_r$ maximal r-chains, $\tilde{n}_h$ maximal h-chains, and $\Lambda(\mathcal{C})$ is as in Definition 2.3.8, then we have for all $\phi \in \mathcal{H}_\vartheta$ the estimate:

$$\begin{aligned} &\|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^{T} \phi_\Lambda\|_{\Lambda_1} \\ \leq\ &(c_3\epsilon_e)^{|n_\beta|} \exp\left(-c_g \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r} \eta_e^{n_r} \|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} \end{aligned} \tag{2.206}$$

and

$$\|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} \le \vartheta^{-|\Lambda_r|-\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}}\|\phi\|_\vartheta \qquad (2.207)$$
$$\le \vartheta^{-|\Lambda_r|}\prod_{k=1}^{\infty}\vartheta^{-(3k)^d n_{\beta,k}}\|\phi\|_\vartheta .$$

Proof Representation (2.205) results from our considerations beginning with (2.191). We have seen that we can replace the operator sum over reduced configurations, weighted with probability factors, by an operator with effective parameters and corresponding to a full configuration.
Note that for each triangle in $\mathcal{C}$ we get a probability factor $p = \frac{\lambda T}{N}$. The proof of (2.206) is analogous to the one of (1.57) in [12]. We only have to use the effective parameters η_e and ϵ_e instead of η and ϵ, respectively. Note that the factor $\tilde{c}_N$, defined in (2.201), is bounded by 1. (2.207) is formula (1.58) in [12].

□

Using (2.206) and (2.207) we will estimate (2.205) analogously to the estimate of (2) in [12]. Again we sum over all $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ but this time we have to use the effective parameters ϵ_e and η_e. A problem is that η_e is not uniformly (in N) bounded away from 1 (cf. Definition 2.3.5 and the remark thereafter). However in the following proposition we establish a bound for (2.205) that holds uniformly in N.

Proposition 2.3.3 *For sufficiently small ϵ and large decay parameter c_g of the interaction we can choose $\vartheta > 0$, $\tilde{\vartheta} > 0$ and $c > 0$ such that for all $T > 0$ and $N > \lambda T$ the following estimate holds for all $\phi \in \mathcal{H}_\vartheta$ and $\Lambda_1 \subseteq \Lambda \in \mathcal{F}$:*

$$\sum_{\mathcal{C}\in\mathrm{Conf}(\Lambda,N,\Lambda_1)} \tilde{\vartheta}^{\Lambda_1}\left\|\pi_{\Lambda_1}\circ\widetilde{\mathcal{L}}_{\mathcal{C}}^T\phi_\Lambda\right\| \le c_4\,\|\phi\|_\vartheta \qquad (2.208)$$

Further for sufficiently large T we can choose $\tilde{\vartheta} = \vartheta$.

Proof The estimates in this proof hold, provided c_g is sufficiently large and ϵ small. We can choose the bounds for these parameters independently of Λ, Λ_1, T and N. We pointed out in [12] in detail how to get such bounds and do not repeat the arguments here again.
We fix Λ, N and Λ_1. First we estimate the sum over all $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ with length$(\mathcal{C}) = N$. Let $\mathcal{C}$ have exactly $n_{\beta,k}$ k-triangles. We can assign to it a set $\Lambda_{\mathcal{C}}$ (cf. Definition 2.3.8) and a labelled tree-graph with parameter $|\Lambda_{\mathcal{C}}|$ like in [12]. (Recall that, if we consider a particular $\Lambda_{\mathcal{C}}$ and set $|\Lambda_{\mathcal{C}}| = K$, there are not more than $4^K\prod_{k=1}^{\infty} c_d^{k^d n_{\beta,k}}$ such graphs.) Each triangle has an a-r-chain of length between 0 and N. The labelled tree-graph and the length of all a-r-chains (cf. the proof of

Proposition 1.8.1 in [12] for a definition) determine the positions of all triangles and a-r-chains in a configuration $\mathcal{C}$. At each k-triangle there can be attached upwards or downwards going r-chains or h-chains. Summing over their particular choices gives rise to a factor

$$c_5^{(3k)^d} \leq \exp(c_6 k^d) \tag{2.209}$$

for each k-triangle. We split

$$\begin{aligned} \epsilon_e &= \tilde{\epsilon}\tilde{\epsilon}\frac{\lambda T}{N} \\ \text{with } \tilde{\epsilon} &\stackrel{\text{def}}{=} \sqrt{\epsilon} \end{aligned} \tag{2.210}$$

So the estimate for the norm of the operator corresponding to a k-triangle gives rise to a factor

$$\tilde{\epsilon}\tilde{\epsilon}\frac{\lambda T}{N}\exp(-c_g k^d) \tag{2.211}$$

For the number l of long r-chains (i.e. long r-chains from $(q,0)$ to (q,N)) in $\mathcal{C}$ we have the bound $0 \leq l \leq |\Lambda_1| - K$.
Further we can split

$$\begin{aligned} \eta_e &= \tilde{\eta}_e\tilde{\eta}_e \\ \text{with } \tilde{\eta}_e &\stackrel{\text{def}}{=} \sqrt{\eta_e}, \end{aligned} \tag{2.212}$$

and as length$(\mathcal{C}) = N$ we can extract from each summand in (2.214) a factor $\tilde{\eta}_e^N$ (as in the proof of Proposition 1.8.1 in [12]).
In the step from (2.215) to (2.216) we make use of the bound

$$\begin{aligned} \frac{\lambda T}{N}\sum_{l=0}^{L}\tilde{\eta}_e^l &\leq \frac{\lambda T}{N}\frac{1}{1-\tilde{\eta}_e} \\ &\leq \frac{\lambda T}{N}\frac{1}{\frac{1}{2}(1-\eta)\frac{\lambda T}{N}} \\ &= \frac{2}{1-\eta}. \end{aligned} \tag{2.213}$$

We get, provided that ϵ is sufficiently small and c_g large:

$$\tilde{\vartheta}^{\Lambda_1} \sum_{\substack{\mathcal{C}\in\text{Conf}(\Lambda,N,\Lambda_1) \\ \text{length}(\mathcal{C})=N}} \|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^T \phi_\Lambda\|_{\Lambda_1} \tag{2.214}$$

$$\leq \tilde{\vartheta}^{\Lambda_1} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h + \vartheta^{-1} c_r \tilde{\eta}_e^N)^{|\Lambda_1|-K} 4^K \sum_{n=K}^{\infty} \tag{2.215}$$

$$\times \left(\epsilon_e \left(\sum_{k=1}^{\infty} \exp(-c_g k^d) \exp(c_6 k^d)\, c_d^{k^d} \vartheta^{-ck^d} \right) \sum_{l=0}^{N} \tilde{\eta}_e^l \right)^n \tilde{\eta}_e^N \|\phi\|_\vartheta$$

$$\leq \tilde{\vartheta}^{\Lambda_1} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h + \vartheta^{-1} c_r \tilde{\eta}_e^N)^{|\Lambda_1|-K}\, \tilde{\epsilon}^K \tag{2.216}$$

$$\times \sum_{n=K}^{\infty} (c_7\, \tilde{\epsilon} \frac{2}{1-\eta})^n\, \tilde{\eta}_e^N \|\phi\|_\vartheta$$

$$\leq c_8\, (\tilde{\vartheta} c_h + \frac{\tilde{\vartheta}}{\vartheta} c_r \tilde{\eta}_e^N + \tilde{\vartheta}\, \tilde{\epsilon})^{|\Lambda_1|}\, \tilde{\eta}_e^N \|\phi\|_\vartheta. \tag{2.217}$$

We remark that (2.215) also includes the estimate for the special case $K = 0$. Then the configurations have no triangle and $n_\beta = 0$. The sum '$\sum_{n=0}^{\infty}$' should then be replaced by a factor 1 (to avoid confusion). However, this sum is at least 1 and so the estimate is correct.

Now we consider the case length$(\mathcal{C}) = L$ for fixed $1 \leq L \leq N-1$. Let $\mathcal{C}$ have n triangles. At least one of them has to be assigned to a point in $\Lambda \times \{N - L + 1\}$ because $\mathcal{C}$ has length L. ($\mathcal{C}$ has to have a triangle at that level and not an r-chain because otherwise $\mathcal{C}$ would be a zero-configuration.) So for each $\mathcal{C}$ we have a labelled tree-graph and can distinguish one triangle. The number of all possible choices of a distinguished triangle (for a given tree-graph) is bounded by n and so by 2^n. With the choice of the lengths of the a-r-chains of the other $n-1$ (non-distinguished) triangles the positions of *all* n triangles and the lengths of *all* a-r-chains are determined. This time we get a factor $(\sum_{l=0}^{N-1} \eta_e^l)^{n-1}$ (with exponent $n-1$ rather than n.) We estimate

$$\tilde{\vartheta}^{\Lambda_1} \sum_{\substack{\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda_1) \\ \mathrm{length}(\mathcal{C}) = L}} \|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^T \phi_\Lambda \|_{\Lambda_1} \tag{2.218}$$

$$\leq \tilde{\vartheta}^{\Lambda_1} \sum_{K=1}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} c_h^{|\Lambda_1|-K} 4^K \tilde{\epsilon}^K \sum_{n=K}^{\infty} \left(\frac{\lambda T}{N} \sum_{l=0}^{N-1} \tilde{\eta}_e^l \right)^{n-1} \tag{2.219}$$

$$\times \left(2\, \tilde{\epsilon}\, c_9 \frac{2}{1-\eta} \sum_{k=1}^{\infty} \exp(-c_g k^d) \exp(c_6 k^d)\, c_d^{k^d} \vartheta^{-ck^d} \right)^n \frac{\lambda T}{N} \tilde{\eta}_e^L \|\phi\|_\vartheta$$

$$\leq c_{10}\, (\tilde{\vartheta} c_h + \tilde{\epsilon} \tilde{\vartheta})^{|\Lambda_1|} \frac{\lambda T}{N} \tilde{\eta}_e^L \|\phi\|_\vartheta \tag{2.220}$$

Finally we get for the special case that $\mathcal{C}$ has neither triangles nor r-lines (i.e. length$(\mathcal{C}) = 0$):

$$\tilde{\vartheta}^{\Lambda_1} \|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}_{\mathcal{C}}^T \|_{\Lambda_1} \leq (\tilde{\vartheta} c_h)^{\Lambda_1} \|\phi\|_\vartheta \tag{2.221}$$

So, using (2.217), (2.220), (2.221) and the bound $\tilde{\eta}_e < 1$, we conclude

$$\begin{aligned}\sum_{C\in\mathrm{Conf}(\Lambda,N,\Lambda_1)} \tilde{\vartheta}^{\Lambda_1}\|\pi_{\Lambda_1}\circ\widetilde{\mathcal{L}}_C^T\phi_\Lambda\|_{\Lambda_1} &\leq \left(1+c_{10}\frac{\lambda T}{N}\sum_{L=1}^{N-1}c_8\tilde{\eta}_e^L+\tilde{\eta}_e^N\right)\|\phi\|_\vartheta &(2.222)\\ &\leq \left(1+\frac{2c_{10}}{1-\eta}+c_8\tilde{\eta}_e^N\right)\|\phi\|_\vartheta &(2.223)\\ &\leq c_4\|\phi\|_\vartheta &(2.224)\end{aligned}$$

□

We want to take the limit (as $N\to\infty$) of the rhs of (2.205) for fixed Λ_1, Λ and T. Considering the sum over all full configurations, we can collect (into the same class) configurations with the same 'constellation of triangles relative to each other in time and space' but different lengths of their chains. The h- and r-chains can be thought of as being flexible (made of gum) so that we can move the triangles in time-direction. So the sum over all configurations is the double sum over all classes (outer sum) and all possible time-positions of the particular triangles (inner sum). In the limit the inner sum, say for a class with exactly n triangles, becomes an integral over a subset of $\mathbb{R}^n$. We will choose our classes (the *gum configurations*, cf. Definition 2.3.10) so that they determine a linear order on the set of *branchings* (that correspond to the triangles). So the domain of integration corresponding to a gum configuration is a simplex.
To make these ideas more precise we give some rather formal definitions. (Figure 2.5 might assist to understand these better.)

Definition 2.3.9 In a labelled tree (as defined in [12]) we call the vertices that have no maximal label (i.e. they are not leaves of the tree and there is a vertex with a greater (wrt to the partial order) label) and that are different from the root vertex, **branchings**. (Each branching corresponds to a particular *star-graph* (like we used to define the term *labelled tree* in [12].) It is called k**-branching** if it corresponds to a star-graph with exactly $v(k)$ vertices (and hence to a k-triangle). k is called the **degree of the branching**.
Recall that in [12] we have introduced a linear order $\prec$ on the set of labels of vertices. In the following we will use the same symbol for linear orders on different sets as well. It should be clear from the context to which order we refer.
We say that a linear order $\prec$ on the set of branchings is **compatible with the labelling** if the following condition is satisfied: If $\mathbf{v_1}$ and $\mathbf{v_2}$ are branchings, labelled by label($\mathbf{v_1}$) and label($\mathbf{v_2}$), respectively, then

$$\mathrm{label}(\mathbf{v_1})\succ\mathrm{label}(\mathbf{v_2}) \quad\Rightarrow\quad \mathbf{v_1}\prec\mathbf{v_2}. \qquad (2.225)$$

We introduce a linear order (also denoted by $\prec$) on $\mathbb{Z}^d$:

$$(k_1,\dots,k_d) \prec (\tilde{k}_1,\dots,\tilde{k}_d) \quad \text{if } k_i < \tilde{k}_i \text{ for the lowest index } i \text{ such that } k_i \neq \tilde{k}_i \tag{2.226}$$

Definition 2.3.10 A **gum tree** τ_g with parameters $n_\beta = (n_{\beta,1}, n_{\beta,2}, \dots)$ and $\Lambda_2 \in \mathcal{F} \setminus \{\emptyset\}$ is given by the following data:

1. A labelled tree τ with parameters n_β and $|\Lambda_2|$ (as defined before Lemma 1.8.2 of [12]).

2. A map **pin** from the set of vertices (except the root) of τ to $\mathbb{Z}^d$ that satisfies the following conditions:

   - The restriction of pin to the set of vertices, that are labelled by $(0,1),\dots,$ $(0,|\Lambda_2|)$ (We denote the restriction of pin to this set by $\text{pin}_0 \cdot$), is an order-preserving bijection onto Λ_2, i.e. for any two such vertices $\mathbf{v}$ and $\tilde{\mathbf{v}}$

$$\text{label}(\mathbf{v}) \prec \text{label}(\tilde{\mathbf{v}}) \quad \Rightarrow \quad \text{pin}(\mathbf{v}) \prec \text{pin}(\tilde{\mathbf{v}}). \tag{2.227}$$

   - If $\mathbf{v}$ with $\text{label}(\mathbf{v}) = s = (s_1,\dots,s_m)$ is a k-branching and $\text{pin}(\mathbf{v}) = q \in \mathbb{Z}^d$ then the restriction of pin to the set of vertices with labels $(s,1),\dots,(s,v(k))$ (We denote the restriction of pin to this set by $\text{pin}_{\mathbf{v}}\cdot$.) is an order-preserving bijection onto $B_k(q) \subset \mathbb{Z}^d$.

A **gum configuration** $\mathcal{C}_g$ **on** Λ **ending in** Λ_1 is given by the following data:

1. A gum tree τ_g with parameters n_β and Λ_2 such that $\Lambda_2 \subseteq \Lambda_1$. The corresponding tree has branchings $\mathbf{v}_1 \prec \dots \prec \mathbf{v}_n$, say, with branching-degrees $b_1,\dots,b_n$, respectively.

2. For each $1 \leq i \leq n$ there are maps

$$u_i \; : \; B_{b_i}(\text{pin}(\mathbf{v}_i)) \cap \Lambda \to \{0,1\} \tag{2.228}$$

$$d_i \; : \; B_{b_i}(\text{pin}(\mathbf{v}_i)) \cap \Lambda \to \{0,1\}. \tag{2.229}$$

   such that

   (a) If $q \in B_{b_i}(\text{pin}(\mathbf{v}_i)) \cap \Lambda$ and j is the smallest number greater than i such that $q \in B_{b_j}(\text{pin}(\mathbf{v}_j))$ (if such a j exists at all) then $d_i(q) = u_j(q)$.

   (b) For every $1 \leq i \leq n$

$$d_i(\text{pin}(\mathbf{v}_i)) = 1. \tag{2.230}$$

(c) If $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap (\Lambda \setminus \Lambda_1)$ and there is no $j > i$ such that $q \in B_{b_j}(q_j)$ then $d_i(q_i) = 0$.

(We will see later that the maps u_i define from a vertex upwards going *h-strips* (if $u_i = 0$) or *r-strips* (if $u_i = 1$). Similarly the maps d_i determine downwards going strips. For a strip between two vertices it should be well-defined if it is an h-strip or an r-strip. Hence we impose condition (a). Condition (b) says that a strip that goes downwards from a branching must be an r-strip.)

3. A map **long** from $\Lambda \setminus \bigcup_{i=1}^{n} B_{b_i}(\mathrm{pin}(\mathbf{v_i}))$ to $\{0, 1\}$ such that

$$\mathrm{long}(q) = 0 \quad \text{if } q \notin \Lambda_1 \tag{2.231}$$

We define in analogy to Definition 2.3.8

$$\widetilde{\Lambda}(\mathcal{C}_g) \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap \Lambda, \tag{2.232}$$

$$\Lambda_r(\mathcal{C}_g) \stackrel{\text{def}}{=} \{q \in \Lambda \setminus \widetilde{\Lambda}(\mathcal{C}_g) : \mathrm{long}(q) = 1\}, \tag{2.233}$$

$$\Lambda(\mathcal{C}_g) \stackrel{\text{def}}{=} \widetilde{\Lambda}(\mathcal{C}_g) \cup \Lambda_r(\mathcal{C}_g). \tag{2.234}$$

We introduce the following notation:

- In the situation of 2.a the point q is the image (wrt pin) of the vertices $\mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ and $\mathrm{pin}_{\mathbf{v_j}}^{-1}(q)$. We say that $\mathcal{C}_g$ has an **h-strip (r-strip)** from $\mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ to $\mathrm{pin}_{\mathbf{v_j}}^{-1}(q)$ if $d_i(q) = 0$ ($d_i(q) = 1$). (We note that we do not distinguish the order of the vertices in this notation: A strip from $\mathbf{v}$ to $\tilde{\mathbf{v}}$ is the same as a strip from $\tilde{\mathbf{v}}$ to $\mathbf{v}$.)

- If $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap \Lambda$ and $\mathbf{v} = \mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ and there is no $j > i$ such that $q \in B_{b_j}(\mathrm{pin}(\mathbf{v_j}))$ and if $d_i(q) = 0$ ($d_i(q) = 1$) we say that $\mathcal{C}_g$ has an **h-strip (r-strip) from v to the bottom.**

- If $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap \Lambda$ nd $\mathbf{v} = \mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ and there is no number $j < i$ such that $q \in B_{b_j}(\mathrm{pin}(\mathbf{v_j}))$ and if $u_i(q) = 0$ ($u_i(q) = 1$) we say that $\mathcal{C}_g$ has an **h-strip (r-strip) from v to the top.**

- In the situation of 2.b we call the corresponding r-strip an **apex-r-strip**.

- If $q \in \Lambda \setminus \widetilde{\Lambda}(\mathcal{C}_g)$ and $\mathrm{long}(q) = 0$ ($\mathrm{long}(q) = 1$) then we say that $\mathcal{C}_g$ has a **long h-strip (long r-strip)** at q. So $\Lambda_r(\mathcal{C}_g) \subset \Lambda_1$ is the set of q where $\mathcal{C}_g$ has long r-strips.

- If $\mathcal{C}_g$ has an r-strip to the top or a long r-strip we say that $\mathcal{C}_g$ **reaches the top.**

We denote by $\mathbf{Confg}(\Lambda, \Lambda_1)$ the set of all gum configurations over Λ ending in Λ_1.

Definition 2.3.11 Let $\mathcal{C}_g$ be a gum configuration over Λ ending in Λ_1 with branchings $\mathbf{v_1} \prec \ldots \prec \mathbf{v_n}$ of branching-orders $b_1, \ldots, b_n$, respectively, and let $T \in (0, \infty]$. Then we define

$$\text{Simplex}(\mathcal{C}_g, T) \stackrel{\text{def}}{=} \{(t_1, \ldots, t_n) : -T < t_1 < \ldots < t_n < 0\}. \tag{2.235}$$

Simplex$(\mathcal{C}_g, T)$ is an open subset of $\mathbb{R}^n$ and so carries the induced Lebesgue-measure. For the special case $n_\beta(\mathcal{C}_g) = 0$ we define Simplex$(\mathcal{C}_g, T)$ to be a single point having measure 1.

Definition 2.3.12 For $\mathcal{C}_g \in \text{Confg}(\Lambda, \Lambda_1)$, $T \in (0, \infty]$ and $\mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T)$ we call the triple $(\mathcal{C}_g, T, \mathbf{t})$ a **specific gum configuration**.
Specific gum configurations can be viewed graphically: The vertices are placed in $\mathbb{Z}^d \times [0, T]$ and the strips are 'spanned' between vertices, the top $(t = -T)$ and the bottom $(t = 0)$:

- We assign to each vertex $\mathbf{v}$ in $\tau(\mathcal{C}_g)$ a point in $\mathbb{Z}^d \times [-T, 0]$ in the following way. If $\mathbf{v_i}$ is a branching of degree b_i, $q \in B_{b_i}(\text{pin}(\mathbf{v_i}))$ and $\mathbf{v} = \text{pin}_{\mathbf{v_i}}^{-1}(q)$ then $\mathbf{v}$ has time-coordinate t_i. In particular $\mathbf{v_i}$ has time-coordinate t_i. As further $\text{pin}(\mathbf{v_i}) = q$ we assign $\mathbf{v}$ to (q, t_i).

  Let for the following two vertices $\mathbf{v}$ and $\tilde{\mathbf{v}}$ be assigned to (q, t) and $(q, \tilde{t})$, respectively.

- If $\mathcal{C}_g$ has an h-strip (r-strip) from $\mathbf{v}$ to $\tilde{\mathbf{v}}$ we say that $(\mathcal{C}_g, T, \mathbf{t})$ has a **maximal h-strip (maximal r-strip)** from (q, t) to $(q, \tilde{t})$. We define its **length** to be $|t - \tilde{t}|$.

- If $\mathcal{C}_g$ has an h-strip (r-strip) from $\mathbf{v}$ to the bottom (this has time-coordinate 0.) we say that $(\mathcal{C}_g, T, \mathbf{t})$ has a **maximal h-strip (maximal r-strip)** from (q, t) to $(q, 0)$. Its length is $|t|$.

- If $\mathcal{C}_g$ has an h-strip (r-strip) from $\mathbf{v}$ to the top (this has time-coordinate $-T$.) we say that $(\mathcal{C}_g, T, \mathbf{t})$ has a **maximal h-strip (maximal r-strip)** from (q, t) to $(q, -T)$. Its length is $T - |t|$. (Note that for $T = \infty$ this length is ∞.)

- If $\mathcal{C}_g$ has a long h-strip (long r-strip) at q we say that $(\mathcal{C}_g, T, \mathbf{t})$ has a **long h-strip (long r-strip)** at q. Its length is T. (Long h-strips (long r-strips) are also considered as maximal strips.)

If $(\mathcal{C}_g, T, \mathbf{t})$ has a maximal h-strip (r-strip) from $(q, \tilde{t}_1)$ to $(q, \tilde{t}_4)$ and $\tilde{t}_1 \leq \tilde{t}_2 < \tilde{t}_3 \leq \tilde{t}_4$ then we say that $(\mathcal{C}_g, T, \mathbf{t})$ has an h-strip (r-strip) from $(q, \tilde{t}_2)$ to $(q, \tilde{t}_3)$ (or from $(q, \tilde{t}_3)$ to $(q, \tilde{t}_2)$).

For a branching $\mathbf{v_i}$ and a $q \in B_{b_i}(\text{pin}(\mathbf{v_i}))$ we call the maximal h-strip (if any) from (q, t_i) to (q, t) with $t_i < t$ $(t_i > t)$ a **downwards going (upwards going) h-strip associated to the branching.** (Note that in our pictures the positively oriented time-axis goes downwards.) The notation for r-strips is analogous.
$(\mathcal{C}_g, T, \mathbf{t})$ must have a downwards going r-strip at the points $(\text{pin}(\mathbf{v_i}), t_i)$ because of condition 2.b. We call it an **apex-r-strip.**
An h-strip (r-strip) in $(\mathcal{C}_g, T, \mathbf{t})$ **goes to the bottom (to the top)** if the corresponding h-strip (r-strip) in $\mathcal{C}_g$ goes to the bottom (to the top).
In analogy to (2.201) we define

$$\tilde{c}(\mathcal{C}_g, T, \mathbf{t}) \stackrel{\text{def}}{=} \prod_H (1 - \exp(\lambda \,\text{length}(H))) \tag{2.236}$$

where the product is over all maximal h-strips H that do not end in $(\Lambda \setminus \Lambda_1) \times \{0\}$.

We draw in the specific gum configuration in Figure 2.5 thick horizontal lines for branchings and thin or thick vertical lines for h-strips or r-strips, respectively. There are two branchings of degree 1, at (q_2, t_1) and at (q_3, t_2). The specific gum configuration has, for example, a long r-strip at site q_5, an r-strip from (q_1, t_1) to the top and an h-strip from (q_1, t_1) to the bottom.
Note that the vertices in the labelled gum tree (except the root) are assigned to points in $\mathbb{Z}^d$ (in this example $d = 1$) by the map pin. For example $\text{pin}(\mathbf{v_1}) = q_2$.
Also note that the specific gum configuration in Figure 2.5 'has the same structure' as the full configuration in Figure 2.3. We will make this analogy more precise in the proof of Proposition 2.3.5 where we use the approximation of a specific gum configuration (or more precisely the approximation of the corresponding operator) by full configurations.

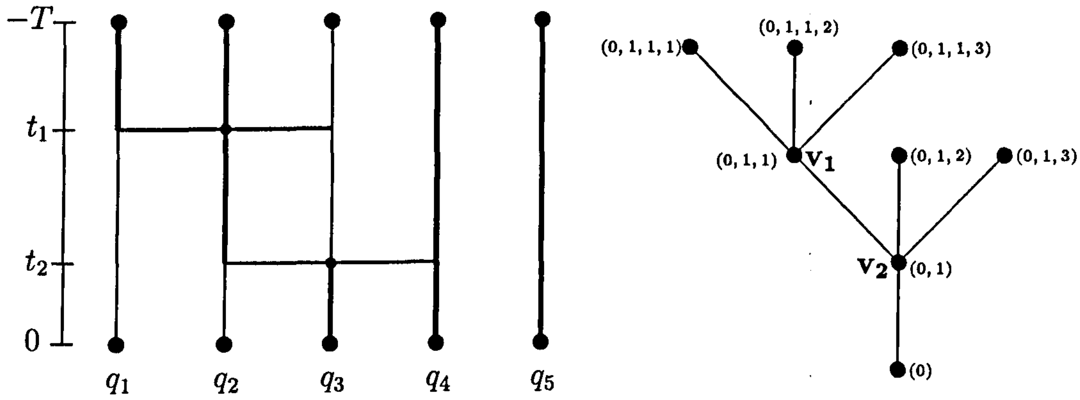


Figure 2.5: Specific gum configuration and its labelled tree

Definition 2.3.13 We define in analogy to (2.193) for $t \geq 0$:

$$\mathcal{R}_q(t) = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\mathcal{R}_q^k \tag{2.237}$$

$$= \exp(\lambda t(\mathrm{id}_q - \mathcal{R}_q)). \tag{2.238}$$

Then we have, using (2.170),

$$\mathcal{R}_q(t_1)\,\mathcal{R}_q(t_2) = \mathcal{R}_q(t_1+t_2) \tag{2.239}$$

$$\|\mathcal{R}_q(t)\| \leq c_r e^{-(1-\eta)\lambda t} \tag{2.240}$$

For $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$ with $n \geq 1$ branchings at $\mathbf{v}_1 \prec \ldots \prec \mathbf{v}_n$ of degree $b_1, \ldots, b_n$, respectively, we set $t_0 = -T$, $t_{n+1} = 0$ and define in analogy to (2.204):

$$\mathrm{Op}_1(i, \mathcal{C}_g, T, \mathbf{t}) \stackrel{\mathrm{def}}{=} \bigotimes_{q\in\Lambda_{\mathcal{Q}}(i,\mathcal{C}_g,T,\mathbf{t})} \mathcal{Q}_q \bigotimes_{q\in\Lambda_{\mathcal{R}}(i,\mathcal{C}_g,T,\mathbf{t})} \mathcal{R}_q(t_{i+1}-t_i), \tag{2.241}$$

$$\mathrm{Op}_2(i,k) \stackrel{\mathrm{def}}{=} \lambda\,\mathcal{B}_{k,\mathrm{pin}(\mathbf{v}_i)} \bigotimes_{\tilde{q}\in\Lambda\setminus\{q\}} \mathrm{id}_{\tilde{q}}, \tag{2.242}$$

$$\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \stackrel{\mathrm{def}}{=} \tilde{c}(\mathcal{C}_g,T,\mathbf{t})\,\mathrm{Op}_1(n,\mathcal{C}_g,T,\mathbf{t}) \circ \mathrm{Op}_2(n,b_n) \circ \ldots \circ \mathrm{Op}_1(1,\mathcal{C}_g,T,\mathbf{t}) \circ \mathrm{Op}_2(1,b_1) \circ \mathrm{Op}_1(0,\mathcal{C}_g,T,\mathbf{t}) \tag{2.243}$$

$$\text{and} \quad \mathcal{L}^T_{\mathcal{C}_g} \stackrel{\mathrm{def}}{=} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} d\mathbf{t}\,\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \tag{2.244}$$

where $\Lambda_{\mathcal{Q}}(i, \mathcal{C}_g, T, \mathbf{t})$ is the set of $q \in \Lambda$ such that $(\mathcal{C}_g, T, \mathbf{t})$ has an h-strip from (q, t_i) to (q, t_{i+1}) and $\Lambda_{\mathcal{R}}(i, \mathcal{C}_g, T, \mathbf{t})$ is the set of $q \in \mathbb{Z}^d$ such that $(\mathcal{C}_g, T, \mathbf{t})$ has an r-strip from (q, t_i) to (q, t_{i+1}).

If $n(\mathcal{C}_g) = 0$ we simply set

$$\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \stackrel{\mathrm{def}}{=} \bigotimes_{q:\,\mathrm{long}(q)=0} \mathcal{Q}_q \bigotimes_{q:\,\mathrm{long}(q)=1} \mathcal{R}_q \tag{2.245}$$

$$\mathcal{L}^T_{\mathcal{C}_g} \stackrel{\mathrm{def}}{=} \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}. \tag{2.246}$$

Remark 2.3.5 1. If H is an h-strip from time t_i to time t_j with $1 \leq i < j \leq n+1$ then $\mathrm{length}(H) = |t_i - t_j|$ and so the factor $1-\exp(-\lambda\,|t_i-t_j|)$ does not depend on T. However, in the case $i = 0$, i.e. $t_i = -T$, the factor $1-\exp(-\lambda\,(T-|t_j|))$ depends on T. For $T = \infty$ this is equal to 1.

2. From (2.236), (2.241), (2.242) and (2.243) we see that the map $\mathbf{t} \mapsto \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$, defined on $\mathrm{Simplex}(\mathcal{C}_g, T)$, is uniformly continuous (because all factors are uniformly continuous wrt $\mathbf{t}$), hence integrable if $T < \infty$. We will see in the next proposition that the integral also exists in the case $T = \infty$. So (2.244) is well-defined.

3. We see that if $(\mathcal{C}_g, \infty, \mathrm{t})$ has an r-strip going to the top then $\mathcal{L}^{\infty}_{\mathcal{C}_g,\mathrm{t}} = 0$.

Definition 2.3.14 Let $c, T > 0$. We set $t_0 = -T$ and $t_{n+1} = 0$ and define

$$D(n,T,c) \stackrel{\text{def}}{=} \{\mathrm{t} = (t_1, \dots t_n) \in \mathbb{R}^n : \inf_{0 \le i < j \le n+1} |t_i - t_j| \le c\} \quad \text{for } n \ge 1 \tag{2.247}$$

$$D(0,T,c) \stackrel{\text{def}}{=} \begin{cases} \{\mathrm{pt}\} & \text{if } T \le c \\ \emptyset & \text{if } T > c \end{cases}. \tag{2.248}$$

For the special case $n = 0$ we have to define the notation for some sets:

- $\{pt\}$ denotes a one-point set.
- The set $\mathrm{Simplex}(\mathcal{C}_g, T) \setminus D(0, T, c)$ for $n(\mathcal{C}_g) = 0$ is defined to be equal to $\mathrm{Simplex}(\mathcal{C}_g, T)$ if $T > c$ and to $\emptyset$ if $T \le c$.
- Similarly $\mathrm{Simplex}(\mathcal{C}_g, T) \cap D(0, T, c)$ is equal to $\emptyset$ if $T > c$ and to $\mathrm{Simplex}(\mathcal{C}_g, T)$ if $T \le c$.
- We define $\mathrm{Simplex}(\mathcal{C}_g, T) \setminus \mathrm{Simplex}(\mathcal{C}_g, \frac{T}{2}) \stackrel{\text{def}}{=} \mathrm{Simplex}(\mathcal{C}_g, T)$ for $\mathcal{C}_g$ with $n(\mathcal{C}_g) = 0$.

Proposition 2.3.4 1. For sufficiently small $\epsilon > 0$ and large c_g there is a constant $c_{12} > 0$ such that for all $T > 0$, $\Lambda_1 \subset \Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\phi \in \mathcal{H}_\vartheta$

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)} \int_{\mathrm{Simplex}(\mathcal{C}_g, T)} d\mathrm{t} \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \phi_\Lambda\| \le c_{12} \, \|\phi\|_\vartheta \tag{2.249}$$

For sufficiently large T this also holds for suitably chosen $\tilde{\vartheta} = \vartheta$.

2.

$$\lim_{c \to 0} \tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g, T) \\ \cap D(|n_\beta(\mathcal{C}_g)|, T, c)}} d\mathrm{t} \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \phi_\Lambda\| = 0 \tag{2.250}$$

Proof For each $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$ and $\mathrm{t} \in \mathrm{Simplex}(\mathcal{C}_g, T)$ we get an estimate analogous to (2.206) (or (1.57) in [12]):

$$\begin{aligned} &\|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g, \mathrm{t}} \phi_\Lambda\|_{\Lambda_1} \\ &\le (c_3 \epsilon)^{|n_\beta|} \exp\left(-c_g \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r} \\ &\quad \times \prod_R \exp(-(1-\eta)\,\lambda\, \mathrm{length}(R))\, \tilde{c}(\mathcal{C}_g, T, \mathrm{t}) \, \|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}, \end{aligned} \tag{2.251}$$

where the product is over all maximal r-strips R of $(\mathcal{C}_g, T, \mathbf{t})$.
Consider a labelled tree τ with parameters $n_{\beta,k}$ and K, a set $\Lambda_2 \subset \Lambda_1$ with $|\Lambda_2| = K$ and the set $A(\tau, \Lambda_2)$ of all $\mathcal{C}_g \in \mathrm{Conf}_g(\Lambda, \Lambda_1)$ whose labelled tree is τ and whose gum tree has parameter Λ_2. Note that there can be different linear orders on the branchings of τ. We want to estimate

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in A(\tau,\Lambda_2)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda\| \tag{2.252}$$

and consider this expression as integral over the union of all sets $\mathrm{Simplex}(\mathcal{C}_g, T)$. We change the variables of integration: Let $\tilde{t}_1, \ldots, \tilde{t}_{|n_\beta|}$ denote the lengths of the a-r-strips (where $\tilde{t}_i$ corresponds to the branching $\mathbf{v}_i$). They are bounded by T. For each $\mathbf{t} = (t_1, \ldots, t_{|n_\beta|}) \in \bigcup_{\mathcal{C}_g \in A(\tau)} \mathrm{Simplex}(\mathcal{C}_g, T)$ there is a unique $\tilde{\mathbf{t}} = (\tilde{t}_1, \ldots, \tilde{t}_{|n_\beta|})$. Note that the union is of disjoint sets and its image is a subset of $[0, T]^n$. Further the change of variables from $\mathbf{t}$ to $\tilde{\mathbf{t}}$ is linear and has a determinant of modulus 1. We see that by doing the transformation successively: $\tilde{t}_1$ is given by a linear equation

$$\tilde{t}_1 = \mathrm{Lin}_1(t_2, \ldots, t_n) - t_1 \tag{2.253}$$

and $\tilde{t}_2$ by

$$\tilde{t}_2 = \mathrm{Lin}_2(\tilde{t}_1, t_3, \ldots, t_n) - t_2 \tag{2.254}$$

etc. and the statement about the determinant follows. So we can estimate in (2.252) '$\sum_{\mathcal{C}_g \in A(\tau,A_2)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} dt$' by '$\int_{[0,T]^{|n_\beta(\tau)|}} d\tilde{\mathbf{t}}$' and so in the estimate of (2.249) we replace '$\sum_{\mathcal{C}_g} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} dt$' by '$\sum_{\Lambda_2,\tau} \int_{[0,T]^{|n_\beta(\tau)|}} d\tilde{\mathbf{t}}$' where the sum is over all $\Lambda_2 \subseteq \Lambda_1$ and labelled trees τ with parameter $|\Lambda_2|$.
We are using the 'usual estimates' (cf. Proposition 2.3.3 and also [12]). For fixed $\Lambda_2 \subseteq \Lambda_1$ and n_β with $|n_\beta| \geq |\Lambda_2|$ the number of labelled trees with parameter $|\Lambda_2|$ and n_β is bounded by $4^{|\Lambda_2|} \prod_{k=1}^\infty (\exp(\tilde{c}_d k^d))^{n_{\beta,k}}$. For each k-branching we get a factor $\lambda \epsilon \exp(-c_g k^d)$ from the uniform estimates for the corresponding operator. Summing over all possible choices of upwards or downwards going h-strips or r-strips associated with the branching, we get a factor $\exp(c_{12} k^d)$. There are not more than $|\Lambda_1| - |\Lambda_2|$ sites for which we can choose between long h-strips and long r-strips. A long r-strip gives rise to a factor $c_r \exp(-(1-\eta)\lambda T)$, and a long h-strip to a factor at most c_h. The norm $\|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}$ is estimated by (2.207). Gum configurations $\mathcal{C}_g$ without branchings (i.e. $n_\beta(\mathcal{C}_g) = 0$) can only have long r-chains (that must end in Λ_1) or long h-chains. This case corresponds to the summand for $K = 0$ in (2.256). The sum '$\sum_{n=0}^\infty$' could be replaced by the factor 1 (cf. the remark on the analogous situation after (2.217).)

We estimate the left-hand side (lhs) of (2.249):

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} d\mathbf{t}\, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda\| \tag{2.255}$$

$$\leq \tilde{\vartheta}^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_h + \vartheta^{-1} c_r \exp(-(1-\eta)\lambda T))^{|\Lambda_1|-K} 4^K \tag{2.256}$$

$$\times \sum_{n=K}^{\infty} \left(\epsilon \sum_{k=1}^{\infty} \exp(-c_g k^d) \exp((\tilde{c}_d + c_{12}) k^d) c^{k^d} \vartheta^{-ck^d} \times \int_0^T dt\, c_r \exp(-(1-\eta)\lambda T) \right)^n \|\phi\|_\vartheta$$

$$\leq c_{13} \left(\tilde{\vartheta} c_h + \frac{\tilde{\vartheta}}{\vartheta} c_r \exp(-(1-\eta)\lambda T) + \tilde{\vartheta}\epsilon_1 \right)^{|\Lambda_1|} \|\phi\|_\vartheta$$

with $\lim_{\epsilon \to 0} \epsilon_1 = 0$. So there are $\vartheta > 0$, $\epsilon > 0$ and $\tilde{\vartheta} > 0$ such that

$$\tilde{\vartheta} c_h + \frac{\tilde{\vartheta}}{\vartheta} c_r \exp(-(1-\eta)T) + \tilde{\vartheta}\epsilon_1 < 1 \tag{2.257}$$

and so (2.249) holds uniformly in Λ_1 and Λ. For sufficiently large T we can choose $\tilde{\vartheta} = \vartheta$ such that (2.257) holds. So statement 1 is proved.
(2.250) follows immediately from (2.249) and the fact that for all $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$ and $T > 0$ the family $(A_c)_{c>0}$ of sets

$$A_c = \mathrm{Simplex}(\mathcal{C}_g, T) \cap D(|n_\beta(\mathcal{C}_g)|, T, c), \tag{2.258}$$

is increasing, i.e.

$$A_{c_1} \subseteq A_{c_2} \quad \text{for} \quad c_1 \leq c_2, \tag{2.259}$$

and from

$$\emptyset = \bigcap_{c>0} A_c. \tag{2.260}$$

□

We prepare the proof of Proposition 2.3.5 which relates our representations of the transfer operators for discrete and continuous time.

Definition 2.3.15 For fixed $T > 0$, $N > T$, $n \geq 1$ and $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ we define $[\mathbf{t}]_{\mathbf{T},\mathbf{N}}$ to be the n-tuple $\mathbf{k} = (k_1, \ldots, k_n)$ such that k_i is the smallest integer greater than or equal to $\frac{t_i N}{T} + N$ (for all $1 \leq i \leq n$). Note that we add N so that $[\cdot]_{T,N}$ induces a map from $(-T, 0)$ to $\{1, \ldots, N\}$ for every single coordinate. We define $\widetilde{\mathbf{D}}(\mathbf{n}, \mathbf{T}, \mathbf{N})$ to be the set of $\mathbf{t}$ such that $\mathbf{k} = [\mathbf{t}]_{T,N}$ satisfies (at least) one of the following two conditions:

$$k_i = k_j \quad \text{for some } i \neq j \tag{2.261}$$
$$k_i \in \{1, N\} \quad \text{for some } i. \tag{2.262}$$

Consider $\mathcal{C}_g \in \text{Confg}(\Lambda, \Lambda_1)$ with branchings $\mathbf{v_1} \prec \ldots \prec \mathbf{v_n}$ of degrees $b_1, \ldots, b_n$, respectively, and $\mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T) \cap \widetilde{D}(n, T, N)$. We define $\mathbf{appr_N}(\mathcal{C}_\mathbf{g}, \mathbf{T}, \mathbf{t}) \in \text{Conf}(\Lambda, N, \Lambda_1)$ as follows:
$\text{appr}_N(\mathcal{C}_g, T, \mathbf{t})$ has a b_i-triangle from $(\text{pin}(\mathbf{v_i}), k_i - 1)$ to $(\text{pin}(\mathbf{v_i}), k_i)$. For $q \in B_{b_i}(\text{pin}(\mathbf{v_i}))$ there is an h-chain (r-chain) going upwards from $(q, k_i - 1)$ if $u_i(q) = 0$ ($u_i(q) = 1$), and, if $q \neq \text{pin}(\mathbf{v_i})$ and $d_i(q) = 0$ ($d_i(q) = 1$) an h-chain (r-chain) going downwards from $(q, k_i - 1)$. There is an a-r-chain (possibly of length 0) going downwards from $(\text{pin}(\mathbf{v_i}), k_i)$.
For $q \in \Lambda \setminus \widetilde{\Lambda}(\mathcal{C}_g)$ the configuration $\text{appr}_N(\mathcal{C}_g, T, \mathbf{t})$ has a long h-chain (long r-chain) at site q if $\text{long}(q) = 0$ ($\text{long}(q) = 1$).
We define for $\mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T)$:

$$\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N} \stackrel{\text{def}}{=} \begin{cases} \left(\frac{N}{T}\right)^{|n_\beta(\mathcal{C}_g)|} \widetilde{\mathcal{L}}^T_{\text{appr}_N(\mathcal{C}_g,T,\mathbf{t})} & \text{if} \quad \mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T) \setminus \widetilde{D}(n, T, N) \\ 0 & \text{if} \quad \mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T) \cap \widetilde{D}(n, T, N) \end{cases}. \tag{2.263}$$

where we define for $\mathcal{C}_g$ without triangles

$$\text{Simplex}(\mathcal{C}_g, T) \setminus \widetilde{D}(0, T, N) \stackrel{\text{def}}{=} \text{Simplex}(\mathcal{C}_g, T) \tag{2.264}$$
$$\text{and} \quad \text{Simplex}(\mathcal{C}_g, T) \cap \widetilde{D}(0, T, N) \stackrel{\text{def}}{=} \emptyset. \tag{2.265}$$

Recall that '$\widetilde{\mathcal{L}}$' denotes an operator with effective parameters (cf. Definition 2.204).

Remark 2.3.6 1. As $\mathbf{t} \notin \widetilde{D}(n, T, N)$, the time-coordinates of two different triangles in $\text{appr}_N(\mathcal{C}_g, T, \mathbf{t})$ are not the same and no time-coordinate of a triangle is equal to 1 or N, i.e. $k_i \neq k_j$ for $i \neq j$ and $k_i \notin \{1, N\}$ for all i.

2. We have the relation

$$\mathbf{t} \in D(n, T, \frac{T}{N}) \quad \Rightarrow \quad [\mathbf{t}]_{T,N} \in \widetilde{D}(n, T, N). \tag{2.266}$$

Definition 2.3.16 For $c, T > 0$, $N > \max\{2, \lambda T\}$ and $\emptyset \neq \Lambda_1 \subset \Lambda \subset \mathcal{F}$ we denote by $\mathbf{Conf_0}(\mathbf{\Lambda}, \mathbf{N}, \mathbf{\Lambda_1}, \mathbf{T}, \mathbf{c})$ the set of all $\mathcal{C} \in \text{Conf}(\Lambda, N, \Lambda_1)$ with the following property: If $\mathcal{C}$ has triangles at $(q_1, k_1), \ldots, (q_n, k_n)$ with $k_1 \leq \ldots \leq k_n$ (This is meant to include the case $n = 0$.) then

$$\inf_{0 \leq i < j \leq n+1} |k_i - k_j| \leq \frac{cN}{T} + 1, \tag{2.267}$$

where we have set $k_0 = 0$ and $k_{n+1} = N$.

We remark that $\mathrm{Conf}_0(\Lambda, N, \Lambda_1, T, c)$ has no configurations without triangles if $c < T$.

Lemma 2.3.2 *For all* T, Λ_1, Λ *we have*

$$\lim_{\substack{c\to 0\\ N\to\infty}} \sum_{C\in \mathrm{Conf}_0(\Lambda,N,\Lambda_1,T,c)} \|\pi_{\Lambda_1}\circ \widetilde{\mathcal{L}}_C^T\|_{L(\mathcal{H}_\emptyset,\mathcal{H}_{\Lambda_1})} = 0. \tag{2.268}$$

Proof The proof is fairly similar to that of Proposition 2.3.3 and will therefore only be sketched. We fix an N and $0 \le N_0 \le \frac{cN}{T} + 1$. First we consider a $C \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ with triangles at $(k_1, q_1), \ldots, (k_n, q_n)$ with $k_1 \le \ldots \le k_n$ and such that there are indices $1 \le i < j \le n$ with $k_j - k_i = N_0$. Let the corresponding triangles be called a and b. We note that there are $n(n-1) \le 2^n$ pairs of triangles. (The factor 2^n will be compensated for by ϵ^n in the usual way, provided ϵ is sufficiently small.) The labelled tree-graph of C together with the lengths of the a-r-chains corresponding to the $n-1$ triangles different from b determine the positions of *all* triangles and the lengths of *all* a-r-chains. So if we do the estimates like in the proof of Proposition 2.3.3 we get in the formulae analogous to (2.215) and (2.219) one factor $\sum_l \tilde{\eta}_e^l$ less (because there is no summation over possible lengths l of the a-r-chain corresponding to triangle b as there is a unique choice for that.) That gives rise to an additional (i.e. not cancelled) factor $\frac{\lambda T}{N}$ that we can extract from the sum. So the restriction of the sum on the lhs of (2.268) to such configurations that have a pair of triangles whose time-coordinates have difference N_0 is bounded from above by $c_{14}\frac{\lambda T}{N}$. And so the sum, restricted to configurations that have a pair of triangles whose time-coordinates have difference at most $\frac{cN}{T}+1$, is bounded from above by $(\frac{cN}{T}+2)\, c_{14}\frac{\lambda T}{N}$ and this tends to zero as $c \to 0$ and $N \to \infty$. We can similarly estimate the sum, restricted to configurations that have a triangle with time-coordinate in $\{1, \ldots, [\frac{cN}{T}+1]\} \cup \{N - [\frac{cN}{T}+1] \ldots, N\}$, and so (2.268) follows. (Here we have used the notation $[cN]$ for the biggest number not greater than cN.)

□

Proposition 2.3.5 *For* $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$ *the map*

$$\begin{aligned} \mathrm{Simplex}(\mathcal{C}_g, T) &\to L(\mathcal{H}_\Lambda, \mathcal{H}_{\Lambda_1}) \\ \mathrm{t} &\mapsto \pi_{\Lambda_1}\circ \mathcal{L}^T_{\mathcal{C}_g,\mathrm{t}} \end{aligned} \tag{2.269}$$

is the pointwise limit of the step-functions $\mathrm{t} \mapsto \pi_{\Lambda_1}\circ \mathcal{L}^T_{\mathcal{C}_g,\mathrm{t},N}$ *(as* $N \to \infty$*). We further have*

$$\sum_{\mathcal{C}_g\in \mathrm{Conf}(\Lambda,\Lambda_1)} \pi_{\Lambda_1}\circ \mathcal{L}^T_{\mathcal{C}_g} \tag{2.270}$$

$$= \lim_{N\to\infty} \sum_{C_g\in\mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\mathrm{Simplex}(C_g,T)} dt\, \pi_{\Lambda_1} \circ \mathcal{L}^T_{C_g,\mathbf{t},N}. \tag{2.271}$$

$$= \lim_{N\to\infty} \sum_{C\in\mathrm{Conf}(\Lambda,N,\Lambda_1)} \pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_C \tag{2.272}$$

For the proof of Proposition 2.3.5 we need the following Lemma.

Lemma 2.3.3 *Let $A_1, \ldots, A_n, \tilde{A}_1, \ldots, \tilde{A}_n$ be operators on the same Banach space, $0 < \delta < 1$ and $a_1, \ldots, a_n$ positive numbers such that:*

$$\|A_i\| \leq a_i \quad \text{for all } 1 \leq i \leq n \tag{2.273}$$

$$\text{and} \quad \|A_i - \tilde{A}_i\| \leq \delta^2 a_i. \tag{2.274}$$

Then

$$\|A_1 \circ \ldots \circ A_n - \tilde{A}_1 \circ \ldots \circ \tilde{A}_n\| \leq \delta\, (1+\delta)^n\, a_1 \cdot \ldots \cdot a_n. \tag{2.275}$$

Proof From (2.274) we get

$$\|\tilde{A}_i\| \leq (1+\delta^2)\, a_i. \tag{2.276}$$

So we get via 'telescope expansion':

$$\begin{aligned}
&\|A_1 \circ \ldots \circ A_n - \tilde{A}_1 \circ \ldots \circ \tilde{A}_n\| \\
\leq\; & \|A_1 \circ \ldots \circ A_n - \tilde{A}_1 \circ A_2 \circ \ldots \circ A_n\| + \ldots \\
& + \|\tilde{A}_1 \circ \ldots \circ \tilde{A}_{n-1} \circ A_n - \tilde{A}_1 \circ \ldots \circ \tilde{A}_n\| \\
\leq\; & \delta^2 \left(1 + (1+\delta^2) + \ldots + (1+\delta^2)^{n-1}\right) a_1 \cdot \ldots \cdot a_n \\
=\; & \left((1+\delta^2)^n - 1\right) a_1 \cdot \ldots \cdot a_n \\
=\; & \sum_{k=1}^{n} \binom{n}{k} \delta^{2k}\, a_1 \cdot \ldots \cdot a_n \\
\leq\; & \delta \sum_{k=1}^{n} \binom{n}{k} \delta^{k}\, a_1 \cdot \ldots \cdot a_n \\
\leq\; & \delta (1+\delta)^n\, a_1 \cdot \ldots \cdot a_n
\end{aligned} \tag{2.277}$$

and the lemma is proved.

□

Proof of Proposition 2.3.5 We first show that

$$\sum_{C_g\in\mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(C_g,T)\\ \cap \tilde{D}(n,T,N)}} dt\, \pi_{\Lambda_1} \circ \mathcal{L}^T_{C_g,\mathbf{t},N} = \sum_{\substack{C\in\mathrm{Conf}(\Lambda,N,\Lambda)\\ \setminus \mathrm{Conf}_0(\Lambda,N,\Lambda_1,T,0)}} \pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_C. \tag{2.278}$$

For $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$, say with exactly n triangles, we have a map

$$\mathbf{t} = (t_1, \ldots, t_n) \mapsto \mathbf{k} = (k_1, \ldots, k_n) = [\mathbf{t}]_{T,N} \tag{2.279}$$

where the k_i are as defined in Definition 2.3.16. This map is constant with value $(l_1, \ldots, l_n)$ on the cube

$$\begin{aligned} \mathrm{Cube}(T, N, l_1, \ldots, l_n) &= \left(\frac{T}{N}(-N + l_1 - 1), \frac{T}{N}(-N + l_1)\right] \times \ldots \\ &\quad \times \left(\frac{T}{N}(-N + l_n - 1), \frac{T}{N}(-N + l_n)\right], \end{aligned} \tag{2.280}$$

i.e. the configuration $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t}) \in \mathrm{Conf}(\Lambda, N, \Lambda_1)$ is the same for all $\mathbf{t} \in \mathrm{Cube}(T, N, l_1, \ldots, l_n)$. If further $\mathbf{t} \notin \widetilde{D}(n, T, N)$ then $1 \le k_1 < k_2 < \ldots < k_n \le N-1$ and $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t}) \in \mathrm{Conf}(\Lambda, N, \Lambda) \setminus \mathrm{Conf}_0(\Lambda, N, \Lambda_1, T, 0)$. We also see that each $\mathcal{C} \in \mathrm{Conf}(\Lambda, N, \Lambda) \setminus \mathrm{Conf}_0(\Lambda, N, \Lambda_1, T, 0)$ determines uniquely a $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, N, \Lambda_1)$ and a $\mathrm{Cube}(T, N, k_1, \ldots, k_n)$ such that $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t}) = \mathcal{C}$ for all $\mathbf{t}$ in this cube. And as

$$\int_{\mathrm{Cube}(T,N,l_1,\ldots,l_n)} d\mathbf{t}\, \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N} = \pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_{\mathrm{appr}_N(\mathcal{C}_g,T,\mathbf{t})}, \tag{2.281}$$

we conclude (2.278). Using (2.266), we get

$$\begin{aligned} &\Big\| \sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} d\mathbf{t}\, \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N} - \sum_{\mathcal{C} \in \mathrm{Conf}(\Lambda,N,\Lambda_1)} \pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_{\mathcal{C}} \Big\|_{L(\mathcal{H}_\Lambda, \mathcal{H}_{\Lambda_1})} \\ \le &\sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T) \\ \cap D(|n_\beta(\mathcal{C}_g)|, T, \frac{T}{N})}} d\mathbf{t}\, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\|_{L(\mathcal{H}_\Lambda, \mathcal{H}_{\Lambda_1})} \\ &+ \sum_{\mathcal{C} \in \mathrm{Conf}_0(\Lambda,N,\Lambda_1,T,1)} \|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_{\mathcal{C}}\|_{L(\mathcal{H}_\Lambda, \mathcal{H}_{\Lambda_1})}. \end{aligned} \tag{2.282}$$

Because of (2.250) and (2.268) the rhs of (2.282) tends to zero (as $N \to \infty$). So the equality between (2.271) and (2.272) is proved.

Now we fix a $c > 0$ and consider a labelled tree τ on Λ ending in Λ_1. For any $\mathcal{C}_g \in \mathrm{Conf}(\Lambda, \Lambda_1)$ whose labelled tree is equal to τ we have the estimate (2.251). Recall that in the proof of Proposition 2.3.4 we concluded formula (2.249) from this estimate (for every τ). For $\frac{N}{T} > c^{-1}$ and $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T) \setminus D(|n_\beta(\mathcal{C}_g)|, T, c)$ we compare $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ with $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}$, using representations (2.204) and (2.243). We see that these operators *have the same structure* because each k-branching (maximal h-strip, maximal r-strip) in $(\mathcal{C}_g, \mathbf{t})$ corresponds to a k-triangle (maximal h-chain, maximal r-chain) in $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t})$. More precisely:

1. Each operator $\lambda\mathcal{B}_{q,k}$ in (2.243) corresponds to an operator $p\mathcal{B}_{q,k}$ in (2.204), and so to an operator $\lambda\mathcal{B}_{q,k}$ in the representation of $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}$ (cf. (2.263) and note the cancellation $\frac{N}{T}p = \lambda$.)

2. A maximal h-strip H in $(\mathcal{C}, T, \mathbf{t})$, say from (q, t_i) to (q, t_j), corresponds to a maximal h-chain H_N in $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t})$ from (q, k_i) to (q, k_j) where k_i and k_j are as defined in Definition 2.3.15. This implies that

$$\frac{|t_i - t_j|}{T}N - 1 \leq |k_i - k_j| \leq \frac{|t_i - t_j|}{T}N + 1 \tag{2.283}$$

If H does not end in$(\Lambda\backslash\Lambda_1)\times\{T\}$ then it gives rise to a factor $(1-\exp(-\lambda|t_i - t_j|)$ in (2.243) and H_N gives rise to a factor $(1-(1-\frac{\lambda T}{N})^{|k_i-k_j|})$ in (2.204). We compare these factors. Clearly we have

$$\lim_{N\to\infty}\left(1 - \frac{\lambda T}{N}\right)^{|k_i-k_j|} = \exp(-\lambda|t_i - t_j|). \tag{2.284}$$

Further, as $|t_i - t_j| > c$, we have that

$$1 - \exp(-\lambda|t_i - t_j|) > 1 - \exp(-\lambda c), \tag{2.285}$$

and so for

$$c_{H,N} \stackrel{\text{def}}{=} \sup_{\substack{\mathbf{t}\in\mathrm{Simplex}(\mathcal{C}_g,T)\\ \backslash D(\mathcal{C}_g,T,c)}} \sup_{i\neq j}\left|\frac{1-\exp(-\lambda\,|t_i - t_j|) - (1-(1-\frac{\lambda T}{N})^{|k_i-k_j|})}{1-\exp(-\lambda\,|t_j - t_i|)}\right| \tag{2.286}$$

we have

$$\lim_{N\to\infty} c_{H,N} = 0. \tag{2.287}$$

3. A maximal r-strip in $(\mathcal{C}_g, T, \mathbf{t})$, say from (q, t_i) to (q, t_j) with $|t_i - t_j| > c$ gives rise to an operator $\mathcal{R}_q(|t_i - t_j|) = \exp(-\lambda|t_i - t_j|(\mathrm{id}_q - \mathcal{R}_q))$. There is a corresponding maximal r-chain in $\mathrm{appr}_N(\mathcal{C}_g, T, \mathbf{t})$ from (q, k_i) to (q, k_j), where k_i and k_j are as in Definition 2.3.15, and satisfy the esimates (2.283). It corresponds to an operator

$$\mathcal{R}_{N,q}(|k_i - k_j|) = \left(\mathrm{id}_q - \frac{\lambda T}{N}(\mathrm{id}_q - \mathcal{R}_q)\right)^{|k_i-k_j|} \tag{2.288}$$

We have, analogously to (2.284),

$$\lim_{N\to\infty} \mathcal{R}_{N,q}(|k_i - k_j|) = \mathcal{R}_q(|t_i - t_j|) \tag{2.289}$$

As $|t_i - t_j| < T$, we further have for our estimate (2.240) of $\|\mathcal{R}_q(|t_i - t_j|)\|$:

$$\exp(-(1-\eta)\lambda|t_i - t_j|) > \exp(-(1-\eta)\lambda T), \tag{2.290}$$

and for

$$c_{R,N} \stackrel{\text{def}}{=} \sup_{\substack{\mathbf{t}\in\text{Simplex}(\mathcal{C}_g,T)\\ \setminus D(\mathcal{C}_g,T,c)}} \sup_{i\neq j} \frac{\|\mathcal{R}_{N,q}(|k_i - k_j|) - \mathcal{R}_q(|t_i - t_j|)\|}{\exp(-(1-\eta)\lambda|t_i - t_j|)} \tag{2.291}$$

we have

$$\lim_{N\to\infty} c_{R,N} = 0. \tag{2.292}$$

We have seen that certain maximal h-strips and r-strips in $(\mathcal{C}_g T, \mathbf{t})$ give rise to operators that only differ from the ones that arise from the corresponding maximal h-lines and r-lines in $\text{appr}_N(\mathcal{C}_g, T, \mathbf{t})$. We bound the number $\tilde{n}(\mathcal{C}_g)$ of such strips. They are attached to a branching or they are long h- or r-strips in Λ_1. (Recall that every long h-strip ending at site $q \in \Lambda \setminus \Lambda_1$ and its corresponding long h-chain both give rise to the same operator $\mathcal{Q}_q$. The scalar factors are both 1 in that case. So

$$\tilde{n}(\mathcal{C}_g) \leq 2\sum_{k=1}^{\infty} n_{\beta,k}(3k)^d + |\Lambda_1| - K \tag{2.293}$$

where $n_{\beta,k}$ and K are defined by the condition that $\mathcal{C}_g$ has a labelled tree with these parameters.
We can think of $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ abstractly as

$$\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} = A_1 \circ \ldots \circ A_m. \tag{2.294}$$

(For example if A_i corresponds to a k-branching at q then
$A_i = \lambda \mathcal{B}_{q,k} \bigotimes_{\tilde{q}\in\Lambda\setminus\{q\}} \text{id}_{\tilde{q}}$. If A_i corresponds to a maximal h-chain at site q and of length $|t_j - t_l|$ then $A_i = (1 - \exp(-\lambda|t_j - t_l|))\, \mathcal{Q}_q \bigotimes_{\tilde{q}\in\Lambda\setminus\{q\}} \text{id}_{\tilde{q}}$.)
As we have seen $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}$ has the same structure as $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ and we can write

$$\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N} = \tilde{A}_1 \circ \ldots \circ \tilde{A}_m. \tag{2.295}$$

Because of (2.286) and (2.292) we can apply Lemma 2.3.3 with an arbitrarily small δ in (2.274), provided N is sufficiently large. This implies in particular that the maps $\mathbf{t} \mapsto \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}$, restricted to $\text{Simplex}(\mathcal{C}_g, T) \setminus D(n, T, c)$ converge uniformly to $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$.

We show for fixed $c > 0$ that

$$\lim_{N\to\infty} \sum_{\mathcal{C}_g\in\mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T)\\ \setminus D(|n_\beta(\mathcal{C}_g)|,T,c)}} dt\, \|\pi_{\Lambda_1}\circ\left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right)\|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})} = 0. \quad (2.296)$$

Using (2.251), Lemma 2.3.3 and (2.293), we get for every $\mathcal{C}_g \in \mathrm{Conf}(\Lambda, \Lambda_1)$

$$\begin{aligned} &\tilde{\vartheta}^{|\Lambda_1|}\|\pi_{\Lambda_1}\circ\left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right)\phi\| && (2.297)\\ &\le\ \delta\,(1+\delta)^{\tilde{n}(\mathcal{C}_g)}(c_3\epsilon)^{|n_\beta|}\exp\left(-c_g\sum_{k=1}^{\infty}k^d n_{\beta,k}\right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r}\\ &\quad\times\prod_R \exp(-(1-\eta)\,\lambda\,\mathrm{length}(R))\,\tilde{c}(\mathcal{C}_g,T,\mathbf{t})\,\|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}. \end{aligned}$$

The factor $\prod_{k=1}^{\infty}(1+\delta)^{3^d n_{\beta,k}k^d}$ will be compensated for by $\exp\left(-c_g\sum_{k=1}^{\infty}k^d n_{\beta,k}\right)$, provided that c_g is sufficiently large. So with the same argument that leads from (2.251) to (2.255) we can estimate (2.297) by

$$\begin{aligned} &\tilde{\vartheta}^{|\Lambda_1|}\|\sum_{\mathcal{C}_g\in\mathrm{Confg}(\Lambda,\Lambda_1)}\pi_{\Lambda_1}\circ\left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right)\phi_\Lambda\| && (2.298)\\ &\le\ \delta\, c\left(\tilde{\vartheta}c_h(1+\delta) + \frac{\tilde{\vartheta}}{\vartheta}c_r\exp(-(1-\eta)\lambda T) + \tilde{\vartheta}\epsilon_2\right)^{|\Lambda_1|}\|\phi\|_\vartheta, \end{aligned}$$

where $\lim_{\epsilon\to 0}\epsilon_2 = 0$.
As we take the limit (for $N\to\infty$) for fixed Λ_1 we do not have to estimate the term in brackets (which could be easily done in the usual way). However we remark that to get (2.298) we have assumed that ϵ is sufficiently small (depending on ϑ) and c_g large. We conclude

$$\sum_{\mathcal{C}_g\in\mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T)\\ \setminus D(|n_\beta(\mathcal{C}_g)|,T,c)}} dt\, \|\pi_{\Lambda_1}\circ\left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right)\|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})} \le c_{15}\,\delta \quad (2.299)$$

for every c and with $\lim_{N\to\infty}\delta = 0$.

Using (2.266), we estimate

$$\sum_{\mathcal{C}_g \in \text{Confg}(\Lambda,\Lambda_1)} \int_{\text{Simplex}(\mathcal{C}_g,T)} dt \, \|\pi_{\Lambda_1} \circ \left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right) \|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})} \tag{2.300}$$

$$\leq \sum_{\mathcal{C}_g \in \text{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\text{Simplex}(\mathcal{C}_g,T) \\ \setminus D(|n_\beta(\mathcal{C}_g)|,T,c)}} dt \, \|\pi_{\Lambda_1} \circ \left(\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} - \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\right) \|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})} \tag{2.301}$$

$$+ \sum_{\mathcal{C}_g \in \text{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\text{Simplex}(\mathcal{C}_g,T) \\ \cap D(|n_\beta(\mathcal{C}_g)|,T,c)}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})} \tag{2.302}$$

$$+ \sum_{\mathcal{C}_g \in \text{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\text{Simplex}(\mathcal{C}_g,T) \\ \cap D(|n_\beta(\mathcal{C}_g)|,T,c)}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t},N}\|_{L(\mathcal{H}_\Lambda,\mathcal{H}_{\Lambda_1})}. \tag{2.303}$$

The first and second summand on the rhs tend to zero (as $c \to 0$ and $N \to \infty$) because of (2.299) and (2.250), respectively. The third summand is bounded from above by $\sum_{\mathcal{C} \in \text{Conf}_0(\Lambda,N,\Lambda_1,T,c)} \|\pi_{\Lambda_1} \circ \widetilde{\mathcal{L}}^T_{\mathcal{C}}\|$ which also tends to zero because of (2.268).
□

The following proposition is a corollary of Proposition 2.3.5.

Proposition 2.3.6 $\mathcal{L}^T_{S,\Lambda} \stackrel{\text{def}}{=} \pi_\Lambda \circ \mathcal{L}^T_{S,\Lambda}$ *is the transfer operator, restricted to* $\mathcal{H}_\Lambda$*, for* $K^T_{S,\Lambda}$*, i.e.*

$$\int_M d\mu \, (K^T_{S,\Lambda}\psi_\Lambda) \, \phi_\Lambda = \int_M d\mu \, \psi_\Lambda \, \mathcal{L}^T_{S,\Lambda}\phi_\Lambda \tag{2.304}$$

For all $\psi \in C^0((S^1)^\Lambda)$ *and* $\phi_\Lambda \in \mathcal{H}_\Lambda$.

Proof We know from Proposition 2.3.1 that $\mathcal{L}^T_{S,\Lambda,N}$ is the transfer operator for $K^T_{S,\Lambda,N}$. Taking the limit (as $N \to \infty$) in (2.182) and using the equality of (2.270) and (2.272), we conclude (2.304).

□

For the representation of the transfer operator for the infinite dimensional system we need the following definition.

Definition 2.3.17 Let $\Lambda_1, \Lambda_2 \subseteq \Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\mathcal{C}_g \in \text{Confg}(\Lambda, \Lambda_1)$. We say that $\mathcal{C}_g$ **lies in** $\mathbf{\Lambda_2}$ if $\Lambda(\mathcal{C}_g) \cup \Lambda_1 \subseteq \Lambda_2$.
Let both $\mathcal{C}_g \in \text{Confg}(\Lambda, \Lambda_1)$ and $\widetilde{\mathcal{C}}_g \in \text{Confg}(\widetilde{\Lambda}, \Lambda_1)$ lie in $\Lambda \cap \widetilde{\Lambda}$. If further $\mathcal{C}_g$ and $\widetilde{\mathcal{C}}_g$ have the same gum tree with the same linear order and if they have the same r-strips then we say that $\mathcal{C}_g$ is **equivalent** to $\widetilde{\mathcal{C}}_g$. Then we have defined an equivalence relation and further, for $\mathcal{C}_g$ equivalent to $\widetilde{\mathcal{C}}_g$, we have:

$$\begin{aligned}
\text{Simplex}(\mathcal{C}_g, T) &= \text{Simplex}(\widetilde{\mathcal{C}}_g, T) \quad \text{for all } T \in (0, \infty], && (2.305)\\
\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \circ \pi_\Lambda &= \pi_{\Lambda_1} \circ \mathcal{L}^T_{\widetilde{\mathcal{C}}_g,\mathbf{t}} \circ \pi_{\widetilde{\Lambda}} \quad \text{for all } \mathbf{t} \in \text{Simplex}(\mathcal{C}_g, T) && (2.306)\\
\text{and} \quad \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \circ \pi_\Lambda &= \pi_{\Lambda_1} \circ \mathcal{L}^T_{\widetilde{\mathcal{C}}_g} \circ \pi_{\widetilde{\Lambda}}. && (2.307)
\end{aligned}$$

(2.306) and (2.307) say that the operators in $L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})$ are the same. We define by $\mathbf{Conf}(\mathbb{Z}^d, \Lambda_1)$ the set of equivalence classes. Because of (2.305) and (2.306) the simplices and operators for each equivalent class can be defined as being equal to the corresponding object for any representative.
We will write $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ instead of $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \circ \pi_\Lambda$ and $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}$ instead of $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \circ \pi_\Lambda$ for the operators from $\mathcal{H}_\vartheta$ to $\mathcal{H}_{\Lambda_1}$.

Theorem 2.3.1 *1. For sufficiently small ϵ, large c_g and every $T \in (0, \infty]$ we can define an operator $\mathcal{L}^T_S$ from $\mathcal{H}_\vartheta$ to $\mathcal{H}_{\tilde{\vartheta}}$ by*

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_S \phi = \sum_{\mathcal{C}_g \in \text{Confg}(\mathbb{Z}^d, \Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \phi. \tag{2.308}$$

There is a $T_0 > 0$ such that for $T \geq T_0$ the operator $\mathcal{L}^T_S$ maps $\mathcal{H}_\vartheta$ into $\mathcal{H}_\vartheta$. $\mathcal{L}^T_S$ is the transfer operator, restricted to $\mathcal{H}^{bv}_\vartheta$, for the kernel K^T_S, i.e.

$$\int_M d\mu \, (K^T_S \psi) \, \phi = \int_M d\mu \, \psi \, \mathcal{L}^T_S \phi \tag{2.309}$$

for all $\psi \in C^0(M)$ and $\phi \in \mathcal{H}^{bv}_\vartheta$.

2. The family $(\mathcal{L}^T_S)_{T \geq 0}$ in $L(\mathcal{H}_\vartheta)$ converges exponentially fast to $\mathcal{L}^\infty_S$:

$$\|\mathcal{L}^\infty_S - \mathcal{L}^T_S\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\tilde{\vartheta}})} \leq c_{16} e^{-c_{17} T} \tag{2.310}$$

for some positive constants c_{16}, c_{17}. For sufficiently large T (2.310) holds also in the norm of $L(\mathcal{H}_\vartheta)$. So among the probability measures corresponding to elements in $\mathcal{H}_\vartheta$ there is a unique K^T_S-invariant probability measure ν^* on M, say corresponding to $\nu \in \mathcal{H}_\vartheta$. The operator $\mathcal{L}^\infty_S$ is a projection onto span ν*:*

$$\mathcal{L}^\infty_S \phi = \mu(\phi) \, \nu. \tag{2.311}$$

Proof The infinite sum on the rhs of (2.308) converges as the prove of estimate (2.249) applies literally to the case $\Lambda = \mathbb{Z}^d$. Next we want to show that $\pi_{\Lambda_1} \circ \mathcal{L}^T_S$ is the limit of $\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda}$ (as $\Lambda \to \mathbb{Z}^d$). The difference between these two operators is due to configurations $\mathcal{C}_g$ in $\text{Confg}(\Lambda, \Lambda_1)$ or in $\text{Confg}(\mathbb{Z}^d, \Lambda_1)$ with $\Lambda(\mathcal{C}_g) \not\subset \Lambda$.

For these we can split in estimate (2.251) the factor that arises from the decay of interaction in the following way (which is the same as the splitting (1.110) in [12]).

$$\exp\left(-c_g \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) \leq \exp\left(-\tilde{c}_g \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) \exp(-\xi \operatorname{dist}(\Lambda_1, \Lambda^C)) \qquad (2.312)$$

with a $\xi > 0$ such that $\tilde{c}_g = c_g - \xi > 0$. (Note that we can choose ξ so small that the estimates, formerly done with c_g work with $\tilde{c}_g$ instead as well.) So we can estimate

$$\begin{aligned} & \tilde{\vartheta}^{|\Lambda_1|} \|\pi_{\Lambda_1} \circ \mathcal{L}_{S,\Lambda}^T - \pi_{\Lambda_1} \circ \mathcal{L}_S^T\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})} \qquad (2.313) \\ \leq\ & 2 \sum_{\substack{C_g \in \mathrm{Conf}(\mathbf{Z}^d, \Lambda_1), \\ \Lambda(C_g) \not\subset \Lambda}} \tilde{\vartheta}^{|\Lambda_1|} \|\pi_{\Lambda_1} \circ \mathcal{L}_{C_g}^T\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})} \\ \leq\ & c_{18} \exp(-\xi \operatorname{dist}(\Lambda_1, \Lambda^C)). \end{aligned}$$

Next we show (2.309) for the special case that ψ depends only on the Λ_1-coordinates, using (2.304):

$$\begin{aligned} & \int_M d\mu(\mathbf{z})\, (K_S^T \psi)(\mathbf{z})\, \phi(\mathbf{z}) \qquad (2.314) \\ =\ & \lim_{\Lambda \to \mathbf{Z}^d} \int_M d\mu^\Lambda(\mathbf{z}_\Lambda) (K_{S,\Lambda}^T \psi)(\mathbf{z}_\Lambda) \phi_\Lambda(\mathbf{z}_\Lambda) \\ =\ & \lim_{\Lambda \to \mathbf{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda(\mathbf{z}_\Lambda)\, \psi(\mathbf{z}_\Lambda)\, (\mathcal{L}_{S,\Lambda}^T \phi_\Lambda)(\mathbf{z}_\Lambda) \\ =\ & \lim_{\Lambda \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1}(\mathbf{z}_\Lambda)\, \psi(\mathbf{z}_{\Lambda_1})\, (\pi_{\Lambda_1} \circ \mathcal{L}_s^T \phi_\Lambda)(\mathbf{z}_{\Lambda_1}) \\ =\ & \int_M d\mu(\mathbf{z})\, \psi(\mathbf{z})\, (\mathcal{L}_S^T \phi)(\mathbf{z}). \end{aligned}$$

We conclude (2.309) for general $\psi \in C^0(M)$ by approximating it by ψ_{Λ_1} (cf. (2.161)), depending only on the Λ_1-coordinates and using continuity wrt ψ of both sides of (2.309). So 1. is proved.

Next we show (2.310). We note that for $\Lambda_1 = \emptyset$ the lhs (2.315) in the following estimate is equal to zero as both transfer operators preserve the Lebesgue integral (μ is a 'left eigenvector' with eigenvalue 1.) So we only have to consider the case $|\Lambda_1| \geq 1$.

$$\begin{aligned} & \tilde{\vartheta}^{|\Lambda_1|} \|\pi_{\Lambda_1} \circ \mathcal{L}_S^\infty - \pi_{\Lambda_1} \circ \mathcal{L}_S^T\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})} \qquad (2.315) \\ \leq\ & \tilde{\vartheta}^{|\Lambda_1|} \sum_{C_g \in \mathrm{Conf}(\mathbf{Z}^d, \Lambda_1)} \|\pi_{\Lambda_1} \circ \mathcal{L}_{C_g}^\infty - \pi_{\Lambda_1} \circ \mathcal{L}_{C_g}^T\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})} \end{aligned}$$

$$\leq \tilde{\vartheta}^{|\Lambda_1|} \|\mathcal{Q}_{\Lambda_1} \circ \pi_{\Lambda_1} - (1-e^{-\lambda T})^{|\Lambda_1|}\mathcal{Q}_{\Lambda_1} \circ \pi_{\Lambda_1}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \tag{2.316}$$

$$+\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1), \\ \mathcal{C}_g \text{reaches the top}}} \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \tag{2.317}$$

$$+\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1), \\ \mathcal{C}_g \text{does not} \\ \text{reach the top,} \\ |n_\beta(\mathcal{C}_g)| \geq 1}} \int_{\mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g,\mathbf{t}} - \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \tag{2.318}$$

$$+\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d,\Lambda_1), \\ \mathcal{C}_g \text{does not reach the top,} \\ |n_\beta(\mathcal{C}_g)| \geq 1}} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,\infty) \\ \setminus \mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \tag{2.319}$$

$$+\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1), \\ \mathcal{C}_g \text{does not reach the top,} \\ |n_\beta(\mathcal{C}_g)| \geq 1}} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T) \\ \setminus \mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \tag{2.320}$$

We have distinguished between the following classes of gum configurations. The first summand (2.316) corresponds to the operator $\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g} - \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}$ where $\mathcal{C}_g$ is the gum configuration that has on ly long h-strips (no branchings or r-strips). The second summand (2.317) takes all $\mathcal{C}_g$ into account that reach the top. So all specified configurations $(\mathcal{C}_g, T, \mathbf{t})$ have an r-strip ending at time $-T$. All $(\mathcal{C}_g, \infty, \mathbf{t})$ have an infinitely long r-strip and so the corresponding operator is zero (cf. Remark 2.3.5.3) and does not appear in (2.317). The last three summands, (2.318), (2.319) and (2.320), correspond to $\mathcal{C}_g$ that do not reach the top and do not consist only of h-strips. That implies that it has at least one branching and the corresponding domains of integration, Simplex$(\mathcal{C}_g, \infty)$ and Simplex$(\mathcal{C}_g, T)$, are not degenerated to a point. We divide them into Simplex$(\mathcal{C}_g, \frac{T}{2})$ and the particular complements. (The reason for this will become clear when we do the estimates.) In (2.318) we integrate the norm of the operator difference $\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g} - \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}$ over Simplex$(\mathcal{C}_g, \frac{T}{2})$ and in (2.319) and (2.320) we integrate the norms of the two operators separately over the particular complement sets.

Now we estimate each summand: The first summand (2.316) is estimated by

$$\begin{aligned} \tilde{\vartheta}^{|\Lambda_1|} c_h^{|\Lambda_1|}(1-(1-e^{-\lambda T})^{|\Lambda_1|}) &\leq (\tilde{\vartheta}\, c_h)^{|\Lambda_1|} \sum_{k=1}^{|\Lambda_1|} \binom{|\Lambda_1|}{k} (e^{-\lambda T})^k \\ &\leq (\tilde{\vartheta}\, c_h\, (1+e^{-\frac{1}{2}\lambda T}))^{|\Lambda_1|} e^{-\frac{1}{2}\lambda T} \\ &\leq e^{-\frac{1}{2}\lambda T} \end{aligned} \tag{2.321}$$

where the last inequality holds if $\tilde{\vartheta}$ is chosen sufficiently small.

For estimating the last summand (2.320) we note that for $\mathbf{t} \in$ Simplex$(\mathcal{C}_g, T) \setminus$ Simplex$(\mathcal{C}_g, \frac{T}{2})$ the sum of the lengths of all r-strips of $(\mathcal{C}_g, T, \mathbf{t})$ is at least $\frac{T}{2}$. (This is because $(\mathcal{C}_g, T, \mathbf{t})$ has a branching, say at time t_i with $|t_i| \geq \frac{T}{2}$ and there must be a sequence of apex-r-strips whose lengths add up to at least $\frac{T}{2}$.) Using the remark at the end of Definition 2.3.14, we see that

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T) \\ \setminus \mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_\Lambda)} \leq c_{19} \exp\left(-\lambda \frac{1-\eta}{2}\frac{T}{2}\right). \tag{2.322}$$

Similarly we can estimate the second (2.317) and the fourth (2.319) summand:

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\Lambda,\Lambda_1), \\ \mathcal{C}_g \text{reaches the top}}} \|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \leq c_{20} \exp\left(-\frac{1-\eta}{2}\lambda T\right). \tag{2.323}$$

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d,\Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,\infty) \\ \setminus \mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}} dt \, \|\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g,\mathbf{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} \leq c_{21} \exp\left(-\frac{1-\eta}{2}\lambda \frac{T}{2}\right). \tag{2.324}$$

For estimating the third summand (2.318) we use a similar idea as for the proof of Proposition 2.3.5. For $\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1)$ and $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, \frac{T}{2})$ the difference between the operators $\pi_{\Lambda_1} \circ \mathcal{L}^\infty_{\mathcal{C}_g,\mathbf{t}}$ and $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ is only due to h-strips going to the top or long h-strips in Λ_1 as we can see from representation (2.243) for $\mathcal{L}^\infty_{\mathcal{C}_g,\mathbf{t}}$ (and $\mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ and also from Remark 2.3.5.1.) So they differ only in the constants $\tilde{c}(\mathcal{C}_g, \infty, \mathbf{t})$ and $\tilde{c}(\mathcal{C}_g, T, \mathbf{t})$. More precisely, an h-strip in $\mathcal{C}_g$ that goes to the top and therefore corresponds to an h-strip in $(\mathcal{C}_g, T, \mathbf{t})$, say from (q, t_i) to $(q, -T)$, and so gives rise to a factor $1 - \exp(-\lambda(T - |t_i|))$ (note that $|t_i| < \frac{T}{2}$) whilst the corresponding h-strip in $(\mathcal{C}_g, \infty, \mathbf{t})$ ends at time $-\infty$ and gives rise to a factor 1. Similarly a long h-strip of $\mathcal{C}_g$ in Λ_1 gives rise to factors $1 - \exp(-\lambda T)$ and 1, respectively. In both cases the difference between the scalar factors (for each h-strip to the top) is bounded by

$$\delta^2 = \exp\left(-\frac{\lambda}{2}T\right). \tag{2.325}$$

The number of h-strips to the top is bounded by $\sum_{k=1}^\infty 3^d n_{\beta,k} k^d$ and the number of long h-strips at sites in Λ_1 by $|\Lambda_1| - K$ (where $n_{\beta,k}$ and K are the parameters of the labelled tree of $\mathcal{C}_g$.)

So we estimate

$$\begin{aligned}
&\|\pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g,\mathrm{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} &(2.326)\\
&\leq \delta \prod_{k=1}^{\infty}(1+\delta)^{3^d n_{\beta,k}k^d}(1+\delta)^{|\Lambda_1|-K}(c_3\epsilon_e)^{|n_\beta|}\exp\left(-c_g\sum_{k=1}^{\infty}k^d n_{\beta,k}\right)c_h^{\tilde{n}_h}c_r^{\tilde{n}_r}\\
&\quad\times\prod_R \exp(-(1-\eta)\lambda\,\mathrm{length}(R))\,\tilde{c}(\mathcal{C}_g,T,\mathrm{t})\,\|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}.
\end{aligned}$$

The factors $(1+\delta)^{3^d n_{\beta,k}k^d}$ are compensated for by $\exp(-c_g k^d n_{\beta,k})$ 'in the usual way'. If ϵ is sufficiently small and c_g large we can estimate

$$\begin{aligned}
&\tilde{\vartheta}^{|\Lambda_1|}\sum_{\substack{\mathcal{C}_g\in\mathrm{Conf}_g(\Lambda,\Lambda_1),\\ \mathcal{C}_g\text{does not reach the top},\\ |n_\beta(\mathcal{C}_g)|\geq 1}}\int_{\mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})} d\mathrm{t}\,\|\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g,\mathrm{t}}-\pi_{\Lambda_1}\circ\mathcal{L}^{T}_{\mathcal{C}_g,\mathrm{t}}\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})} &(2.327)\\
&\qquad\leq c_{22}\,e^{-\frac{1}{4}\lambda T}.
\end{aligned}$$

From (2.321), (2.323), (2.327), (2.324), and (2.322) we conclude (2.310) with $c_{17} = \frac{1-\eta}{4}\lambda$ and c_{16} sufficiently large.
For any $\phi \in \mathcal{H}_\vartheta$ and any $\Lambda \in \mathcal{F}$ we have

$$\begin{aligned}
\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{S}\phi &= \sum_{\substack{\mathcal{C}_g\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{does not reach the top}}}\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}\phi &(2.328)\\
&= \sum_{\substack{\mathcal{C}_g\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{does not reach the top}}}\left(\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}h_{\mathbb{Z}^d}\right)\mu(\phi)
\end{aligned}$$

The sum in (2.328) is a priori over all $\mathcal{C}_g \in \mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1)$ but, as we have seen before, if $\mathcal{C}_g$ reaches the top the corresponding operator $\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}$ is zero. If $\mathcal{C}_g$ does not reach the top there are only h-strips going to the top $(-\infty)$ and $\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}$ is a projection onto $\mathrm{span}(h_{\mathbb{Z}^d})$.
We set $\nu_\Lambda \stackrel{\mathrm{def}}{=} \pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}h_{\mathbb{Z}^d}$ and this defines $\nu = (\nu_\Lambda)_{\Lambda\in\mathcal{F}}$. Note that the transfer operator $\mathcal{L}^{\infty}_{S}$ preserves the Lebesgue-integral and so $\nu_\emptyset = 1$, i.e. ν corresponds to a probability measure.

□

2.4 Decay of Correlations

In the following theorem which is completely analogous to Theorem 1.7.1 in [12], we state the mixing properties for the invariant probability measure ν^* in terms of the weighted norms.

Theorem 2.4.1 *For sufficiently small ϑ, $\tilde{\vartheta}$, ϵ and big c_2 there is a $\kappa \in (0,1)$ and positive constants c_{22}, c_{23}, c_{24} and c_{25} such that for all finite disjoint $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and $\psi \in \mathcal{H}_{\Lambda_2}$ the following holds:*

$$\begin{aligned} \|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2} &\leq c_{22}\vartheta^{|\Lambda_1\cup\Lambda_2|}\kappa^{\text{dist}(\Lambda_1,\Lambda_2)}, & (2.329)\\ \|\pi_{\Lambda_1}(\psi\nu) - \nu(\psi)\nu_{\Lambda_1}\|_{\Lambda_1} &\leq c_{23}\vartheta^{-|\Lambda_1\cup\Lambda_2|}\|\psi\|_{\Lambda_2}\kappa^{\text{dist}(\Lambda_1,\Lambda_2)}, & (2.330)\\ \|\pi_{\Lambda_1}\circ\mathcal{L}_S^T(\psi\nu) - \nu(\psi)\nu_{\Lambda_1}\|_{\Lambda_1} &\leq c_{24}\vartheta^{-|\Lambda_2|}\tilde{\vartheta}^{-|\Lambda_1|}\|\psi\|_{\Lambda_2}\kappa^{\text{dist}(\Lambda_1,\Lambda_2)} & (2.331)\\ &\quad \times\exp(-c_{25}T) \end{aligned}$$

for every $T > 0$.

Proof For a gum configuration $\mathcal{C}_g$ we define in analogy to (1.109) in [12]

$$b(\mathcal{C}_g) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k\, n_{\beta,k}(\mathcal{C}_g). \tag{2.332}$$

In the following we split gum configurations $\mathcal{C}_g \in \text{Conf}(\mathbb{Z}^d, \Lambda_1 \cup \Lambda_2)$ with $b(\mathcal{C}_g) \leq \frac{1}{2}\text{dist}(\Lambda_1,\Lambda_2)$ into $\mathcal{C}_g = \mathcal{C}_g^1 \cup \mathcal{C}_g^2$ with $\mathcal{C}_g^1 \in \text{Conf}(\mathbb{Z}^d, \Lambda_1)$, $\mathcal{C}_g^2 \in \text{Conf}(\mathbb{Z}^d, \Lambda_2)$ and $\Lambda(\mathcal{C}_g^1) \cap \Lambda(\mathcal{C}_g^2) = \emptyset$.
We write, using (2.311) and the notation of (2.173):

$$\begin{aligned} \nu_{\Lambda_1\cup\Lambda_2} &= \sum_{\substack{\mathcal{C}_g=\mathcal{C}_g^1\cup\mathcal{C}_g^2\in\text{Confg}(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)\leq\frac{1}{2}\text{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g^1}^T h_{\mathbb{Z}^d})(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_g^2}^T h_{\mathbb{Z}^d}) \\ &+ \sum_{\substack{\mathcal{C}_g\in\text{Confg}(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)>\frac{1}{2}\text{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_g}^T h_{\mathbb{Z}^d} \end{aligned}$$

In estimating the norm of the second summand in (2.333) we can take out from the estimate for $\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^\infty h_{\mathbb{Z}^d}\|$ a factor

$$\exp\left(-\xi\,\frac{1}{2}\,\text{dist}(\Lambda_1,\Lambda_2)\right) = \kappa^{\text{dist}(\Lambda_1,\Lambda_2)} \tag{2.333}$$

like in (2.312) such that we get

$$\|\sum_{\substack{\mathcal{C}_g\in\text{Confg}(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)>\frac{1}{2}\text{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_g}^T h_{\mathbb{Z}^d}\| \leq c_{26}\,\vartheta^{|\Lambda_1\cup\Lambda_2|}\kappa^{\text{dist}(\Lambda_1,\Lambda_2)}. \tag{2.334}$$

We write the first summand in (2.333) as

$$\sum_{\substack{\mathcal{C}_g=\mathcal{C}_g^1\cup\mathcal{C}_g^2\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)\le\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}^T_{\mathcal{C}_g^1}h_{\mathbb{Z}^d})(\pi_{\Lambda_2}\circ\mathcal{L}^T_{\mathcal{C}_g^2}h_{\mathbb{Z}^d}) \tag{2.335}$$
$$= \nu_{\Lambda_1}\nu_{\Lambda_2} - \sum_{\substack{\mathcal{C}_g=\mathcal{C}_g^1\cup\mathcal{C}_g^2\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}^T_{\mathcal{C}_g^1}h_{\mathbb{Z}^d})(\pi_{\Lambda_2}\circ\mathcal{L}^T_{\mathcal{C}_g^2}h_{\mathbb{Z}^d})$$

and estimate

$$\|\sum_{\substack{\mathcal{C}_g=\mathcal{C}_g^1\cup\mathcal{C}_g^2\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}^T_{\mathcal{C}_g^1}h_{\mathbb{Z}^d})(\pi_{\Lambda_2}\circ\mathcal{L}^T_{\mathcal{C}_g^2}h_{\mathbb{Z}^d})\| \le c_{27}\,\vartheta^{|\Lambda_1\cup\Lambda_2|}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}. \tag{2.336}$$

From (2.334), (2.335) and (2.336) we conclude (2.329). The proof of (2.330), using (2.329), is the same as in [12].
To prove (2.331) we set $\phi = \psi\nu - \nu(\psi)\nu$. So

$$\pi_{\Lambda_1}\circ\mathcal{L}_S^T\phi = \pi_{\Lambda_1}\circ\mathcal{L}_S^T(\psi\nu) - \nu(\psi)\nu_{\Lambda_1} \tag{2.337}$$

and in particular

$$\mathcal{L}_S^\infty\phi = 0. \tag{2.338}$$

We estimate (2.337), analogously to (1.129) in [12], using the finer estimate

$$\|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} \le c_{23}\vartheta^{-|\Lambda_2|}\|\psi\|_{\Lambda_2}\vartheta^{-|\Lambda_r(\mathcal{C})|-\sum_{k=1}^\infty(3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty k n_{\beta,k}} \tag{2.339}$$

that we get from (2.330). For each $\mathcal{C}_g$ we get a 'good' factor $\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$ that we can take out of the sum (over gum configurations), and a 'bad' factor $\kappa^{-\sum_{k=1}^\infty k n_{\beta,k}}$. The latter is compensated for in the usual way by the factor $\exp(-c_g\sum_{k=1}^\infty k^d n_{\beta,k})$, provided that c_g is sufficiently large.
Using (2.338) and (2.339), we get with the same argument as for the proof of (2.310):

$$\tilde{\vartheta}^{|\Lambda_1|}\|\pi_{\Lambda_1}\circ\mathcal{L}_S^T\phi\| \le c_{28}\vartheta^{-|\Lambda_2|}\|\psi\|\,\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\exp(-c_{25}T) \tag{2.340}$$

and (2.331) follows.

□

We can state the mixing properties of ν^* wrt spatio-temporal shifts in terms of correlation functions for observables $\psi_1,\psi_2\in C^0(M)$ like in Theorem 1.2.2 of [12]. The proof of the following theorem, using Theorem 2.4.1, is analogous to the one of Theorem 1.2.2 in [12]. For the definition of the spatial shift τ and its size $m(\tau)$ (or number of its single steps) we refer to the definition before Theorem 1.2.2 in [12].

Theorem 2.4.2 *For sufficiently small ϑ, ϵ and large c_g there is a $\kappa \in (0,1)$ such that for all nonempty $\Lambda_1, \Lambda_2 \in \mathcal{F}$ the following holds with the constant $c(\Lambda_1, \Lambda_2, \kappa) \stackrel{\text{def}}{=} \kappa^{-\max\{\|p-q\|: p\in\Lambda_1, q\in\Lambda_2\})}$ and some positive constants c_{29}, c_{30}:*

1. *If $\psi_1 \in \mathcal{C}((S^1)^{\Lambda_1})$ and $\psi_2 \in \mathcal{C}((S^1)^{\Lambda_2})$ then*

$$\left|\int_M d\nu^*\, \psi_1\, \psi_2 - \left(\int_M d\nu^*\, \psi_1\right)\left(\int_M d\nu^*\, \psi_2\right)\right| \qquad (2.341)$$
$$\leq\ c\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|\psi_1\|_\infty\|\psi_2\|_\infty\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}.$$

2. *If $\psi_1 \in \mathcal{C}((S^1)^{\Lambda_1})$ and $\psi_2 \in \mathcal{H} \cap \mathcal{C}((S^1)^{\Lambda_2})$ then*

$$\left|\int_M d\nu^*\, K_S^T(\psi_1 \circ \tau)\, \psi_2 - \left(\int_M d\nu^*\, \psi_1 \circ \tau\right)\left(\int_M d\nu^*\, \psi_2\right)\right| \qquad (2.342)$$
$$\leq\ c(\Lambda_1, \Lambda_2, \kappa) c^{|\Lambda_1|+|\Lambda_2|}\|\psi_1\|_\infty\|\psi_2\|_{\Lambda_2}\, \kappa^{m(\tau)}\exp(-cT).$$

3. *If $g, f \in \mathcal{C}(M)$ then*

$$\lim_{\max\{m(\tau),T\}\to\infty}\int_M d\nu^*\, K_S^T(\psi_1 \circ \tau)\, \psi_2 = \left(\int_M d\nu^*\, \psi_1\right)\left(\int_M d\nu^*\, \psi_2\right). \qquad (2.343)$$

□

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