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Monotone and Bounded Interval Equilibria in a Coordination Game with Information Aggregation[1]

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Abstract

We analyze how private learning in a class of games with common stochastic payoffs affects the form of equilibria, and how properties such as player welfare and the extent of strategic miscoordination relate across monotone and non-monotone equilibria. Researchers typically focus on monotone equilibria. We provide conditions under which non-monotone equilibria also exist, where players attempt to coordinate to obtain the stochastic payoff whenever signals are in a bounded interval. In bounded interval equilibria (BIE), an endogenous fear of miscoordination discourages players from coordinating to obtain the stochastic payoff when their signals suggest coordination is most beneficial. In contrast to monotone equilibria, expected payoffs from successful coordination in BIE are lower than the ex-ante expected payoff from ignoring signals and always trying to coordinate to obtain the stochastic payoff. We show that BIE only exist when, absent private information, the game would be a coordination game.

Keywords: Non-monotone Equilibria, Total Positivity, Coordination Games, Information Aggregation.

1 Introduction

In this paper, we analyze how private learning in a class of coordination games with common stochastic payoffs affects the nature and structure of equilibria, and how properties such as player welfare and the extent of strategic miscoordination are related across the monotone and non-monotone equilibria that may result. We analyze the game in Figure 1, where h, l and w are known parameters with $h > l$, but θ is an unknown common stochastic payoff drawn from a distribution with support $\mathbb{R}$, and players receive noisy private signals about θ.

		player B 0	player B 1
player A	0	h, h	w, l
	1	l, w	θ, θ

Figure 1

The game captures some key aspects of strategic interactions in a variety of economic and political settings. For example, it describes:

- A coup game between officers who must decide whether to mount a coup based on private signals about the successful coup payoff θ, where the coup only succeeds if both officers act. If one officer acts, the coup fails, and the disloyal officer is punished. The setting captures the possibilities that, when a coup fails, the loyal officer's payoff can fall ($w < h$) if the ruler increases surveillance of the military or reduces its budget to weaken it; or it can rise ($w > h$) when a loyal officer who informs the ruler is rewarded, or the ruler raises the military budget to keep officers happy.[1]

- A technology adoption game in which firms must decide whether to stick to an existing platform or to pursue a new network/platform investment (with uncertain value) that

[1]Shadmehr and Bernhardt (2011) analyze monotone equilibria in a revolution setting in which citizens receive signals about the uncertain revolution payoffs (θ), and a regime punishes a leader of a failed revolt. They impose $w = h$, and use the special properties of the additive normal signal structure to establish the existence of a threshold $\mu^* > 0$ such that two symmetric finite-cutoff equilibria exist if $\mu < \mu^*$, but no such equilibrium exists if $\mu > \mu^*$. Bueno de Mesquita (2014) also uses a normal-noise signal structure in his study of the consequences of the "location of uncertainty" and limit dominance (one-sided vs. two-sided), proving that zero or two finite-cutoff equilibria generically exist. It is not clear which properties drive these findings.

only pays off if broadly adopted.

- A "relationship game" in which individuals, who know the value of their relationships with their current partners, receive signals about their payoffs if they break those relationships to form a new one together (action 1), where $h > l$ reflects that breaking an existing relationship is costly if a potential partner does not reciprocate.

- A tariff war, in which countries choosing between high tariffs and free trade, where the countries know the payoffs from high tariffs, but receive signals about the uncertain political economy payoffs of free trade.

In these settings, the noisy signals typically leave players very uncertain about whether the actual payoffs from "successful" coordination on action 1 exceed the sure payoffs from coordinating on action 0, and the other player's equilibrium action will contain material information about θ—players face both information aggregation and miscoordination concerns.[2]

Were θ public information, then when $\theta \geq w$, equilibrium imposes no restrictions on how players coordinate: if a player believes the other will take action 1, then she will take action 1 to receive θ; but if she believes that the other will take action 0, the miscoordination cost $\mu \equiv h - l > 0$ causes her to take action 0, expecting only to receive h. Concerns about strategic miscoordination can induce players to take action 0, no matter how high θ is. However, in most strategic settings, rather than θ being public information, players instead receive private noisy signals, leaving substantial uncertainty about its value. In such settings, we ask: what forms can equilibria take, and how do the properties of these equilibria relate to each other?

Researchers typically focus on equilibria in finite-cutoff strategies in which a player takes action 0 when her signal about θ is below some critical cutoff, but takes action 1 when the signal is above. We show that such equilibria exist if and only if μ is sufficiently small.[3] We then provide conditions under which non-monotone equilibria also exist in which each player takes action 1 if and only if her signal is in a bounded interval. In these bounded interval equilibria (BIE), an endogenous fear of miscoordination discourages players from coordinating on action 1 to obtain θ exactly when their signals suggest that coordination is most beneficial. We

[2] Chassang and Padró i Miquel (2010) analyze a related setting in which signal noise is vanishingly small, so that only miscoordination concerns are present, and a unique monotone equilibrium obtains.

[3] A coordination failure equilibrium in which both players always take action 0 always exists.

prove that bounded interval equilibria exist only *when* finite-cutoff equilibria exist, and that if, following a signal, a player takes action 1 in a bounded interval equilibrium, then she also does so in the largest finite-cutoff equilibrium. Moreover, bounded interval equilibria only exist if $w < E[\theta]$—there must be sufficient potential surplus from coordinating on action 1. Thus, for bounded interval equilibria to exist, absent private information, the game must be a coordination game with both $h > l$ *and* $E[\theta] > w$. This means that in our private information setting, in any bounded interval equilibrium, players face an expected risk of miscoordination even when taking the "safe" action 0: unconditionally, were the other player to always take action 1, it must be better to always take action 1 than to always take action 0. We then highlight the bad welfare properties of bounded interval equilibria, showing that while in any finite-cutoff equilibrium, a player's ex-ante expected payoff from *successfully* coordinating on action 1 always exceeds $E[\theta]$, in any bounded interval equilibria, the opposite *must* hold.

We first analyze monotone equilibria. Given an affiliation assumption on signal structures plus some tail restrictions on distributions, we show that equilibria in finite-cutoff strategies exist if and only if the miscoordination cost μ is not too high. We then investigate the number of finite-cutoff equilibria that can exist. Previous results based on additive normal noise signal structures suggest that generically either zero or two finite-cutoff equilibria exist—see footnote 1. We show that this result does not generalize. For example, with an additive logistic noise signal structure, a *unique* finite-cutoff equilibrium obtains when μ is small.

We then ask whether and when more limited coordination can emerge in equilibrium. We search for equilibria in bounded interval strategies (BIS) in which players take action 1 whenever their signals about the common coordination payoff are high, but not *too* high, i.e., whenever signals are inside a bounded interval (k_L, k_R). In such bounded interval equilibria (BIE),[4] a self-fulfilling fear of miscoordination causes players not to take action 1 precisely when their signals suggest that the payoff from coordinating on action 1 is highest.

We first use Karlin's variation diminishing theorem to show that when the conditional pdf of signals given θ is totally positive of order three (TP_3), then the best response to a BIS is either a BIS or always to take action 0. We next establish necessary and sufficient conditions for an arbitrary BIS (k_L, k_R) to be supported as an equilibrium for *some* set of

[4]By a bounded interval equilibrium, we mean an equilibrium in bounded interval strategies. We use BIE as an abbreviation for both bounded interval equilibrium and bounded interval equilibria.

primitive payoff parameters, $\{w, \mu\}$: player indifference at k_L and k_R means that they must be more likely to succeed in coordinating on action 1 following signal k_L than signal k_R, since the expected payoff from successful coordination is higher given k_R.[5]

We show that *if* a bounded interval equilibrium (BIE) exists, then so does a finite-cutoff equilibrium. Moreover, any BIE must be "contained" in the largest monotone equilibrium—letting $\underline{k}$ be the cutoff for the monotone equilibrium in which players take action 1 the most, we establish that $\underline{k} < k_L$, i.e., players are strictly more likely to take action 1 in the $\underline{k}$ equilibrium. Intuitively, the finite-cutoff $\underline{k}$ equilibrium offers the promise of a better upside from coordination—the other player may have a really high signal—and miscoordination is less likely, making it worthwhile to risk trying to coordinate on action 1 following a less promising signal. We then show that if multiple finite-cutoff equilibria exist, and $\bar{k}$ is the cutoff for the smallest one, then $k_L < \bar{k}$: miscoordination is more likely following signal $\bar{k}$ in the monotone equilibrium than it is following signal k_L in the BIE; but the potentially high payoff if the other player has an especially high signal makes it worthwhile.[6]

To illustrate the extent to which the endogenously-generated fear of miscoordination must *necessarily* be severe in a BIE, we specialize to the classical additive normal noise signal structure. Normalizing $E[\theta] = 0$, we show that if (k_L, k_R) is a BIE, then $k_L < 0$ and $|k_L| > k_R$, implying that $\underline{k} < k_L < k_R < -\underline{k}$. Thus, on average, in a BIE, a player seeks to coordinate

[5]Chen and Suen (2015) investigate a game with model uncertainty where individuals update from their incomes about whether a revolution would lead to an improved distribution of incomes. Equilibria take a bounded interval form, because wealthy individuals do not want to have their income re-allocated and hence have dominant strategies not to revolt; and poor individuals are pessimistic about the prospect that an improved distribution of incomes would result (and revolution is costly). From a technical perspective, Chen and Suen prove the existence of bounded interval equilibria using Tarski's fixed point theorem. In contrast to our setting, in their setting, monotone equilibria do not exist (reflecting that the very poor and very rich never revolt), the best response to a BIS is always a BIS, the best response to a larger BIS is also a larger BIS (best responses are monotone with respect to set inclusion order), and one can construct a natural complete lattice consisting of non-degenerate intervals that the best response correspondence maps into itself. Therefore, Tarski's theorem applies. However, in our setting, the best response to a BIS can be a monotone strategy. Moreover, when BIS is sufficiently large (with lower bound above that for the largest monotone equilibria), iterating on best responses can generate consecutively larger BISs (each containing the preceding one, implying a monotone mapping), but one cannot construct a complete lattice because each interation makes the resulting BIS closer to a monotone strategy, as the upper end of the BIS grows without bound. This means that one cannot apply Tarski's theorem, necessitating a completely different approach to establishing the existence of a BIE.

[6]Because this game is not supermodular, one cannot invoke the result that the largest and smallest equilibria are monotone (see Van Zandt and Vives (2007) and their literature review). For a method of constructing non-monotone equilibria in dynamic supermodular games, see Balbus, Reffett, and Woźny (2014).

to receive θ on a very limited set of signals, and the expected fruits of such coordination are low relative to the potential benefits: conditional on coordinating on action 1 and obtaining θ, a player's expected payoff is lower than $E[\theta] = 0$ in any BIE, while it is higher than $E[\theta]$ in any finite-cutoff equilibrium. Moreover, we show that for a BIE to exist, w must be less than $E[\theta]$—BIE exist only when, absent private information, we have a coordination game.

We conclude with a discussion of non-monotone equilibria, (a) when stochastic payoffs from coordinating on action 1 are private rather than common values; and (b) in global games that feature two-sided limit dominance (see e.g., Morris and Shin (2003)).

2 The Model

Two players A and B must choose between actions 0 and 1. Payoffs are depicted in Figure 1; players receive the common stochastic payoff θ if and only if they both take action 1, and payoffs h, l and w are common knowledge, with $h > l$. Each player $i \in \{A, B\}$ receives a private signal s^i about θ. After receiving signals, players simultaneously take actions.

The signals and θ are jointly distributed according to a strictly positive, continuously differentiable density $f(\theta, s^A, s^B)$ on $\mathbb{R}^3$. Players are symmetrically situated in the sense that the signals are exchangeable, i.e., $f(\theta, s, s') = f(\theta, s', s)$, for all θ, s, s'. We assume that s^A, s^B and θ are strictly affiliated:[7] a higher signal s^i represents good news about s^j and θ. We also impose standard additional structure on the tail properties of $f(\theta, s^A, s^B)$:

Assumption A1 *For $i, j \in \{A, B\}$ with $i \neq j$:*

(**a**) $\lim_{s^i \to -\infty} Pr(s^j > k | s^i) = 0$ *and* $\lim_{s^i \to \infty} E[\theta | s^j > k, s^i] = \infty$, *for all* $k \in \mathbb{R}$.

(**b**) $\lim_{k \to -\infty} E[\theta | s^j > k, s^i = k] = -\infty$ *and* $\lim_{k \to \infty} Pr(s^j > k | s^i = k)\ E[\theta | s^j > k, s^i = k] < \infty$.

Assumption **A1** implies that $\lim_{k \to \infty} Pr(s^j > k | s^i = k) = 0$ and $\lim_{s^i \to -\infty} E[\theta | s^i] = -\infty$. In the Online Appendix, we show that **A1** holds with an additive noise signal structure, $s^i = \theta + \nu^i$, where θ and ν^is are independent normal or logistic random variables.

A pure strategy for player i is a function ρ_i mapping her signal s^i about θ into an action

[7] s^i, s^j and θ are strictly affiliated if, for all $z,\ z' \in \mathbb{R}^3$, with $z \neq z'$, $f(\min\{z, z'\}) f(\max\{z, z'\}) > f(z) f(z')$, where min and max are defined component-wise (see de Castro (2010) for a review).

choice. That is, $\rho_i : \mathbb{R} \to \{0, 1\}$, where $\rho_i(s^i) = 1$ indicates that player i takes action 1, and $\rho_i(s^i) = 0$ indicates that i takes action 0.

2.1 Monotone Equilibria

We first analyze monotone strategies, where a player's strategy is (weakly) monotone in her signal. With finite-cutoff strategies, a player j's strategy is summarized by a critical cutoff k^j:

$$\rho_j(s^j) = 1 \text{ if } s^j > k^j \quad \text{and} \quad \rho_j(s^j) = 0 \text{ if } s^j \leq k^j.$$

Let $\Delta(s^i; k^j)$ be player i's expected net payoff from taking action 1 rather than 0, given her signal s^i and the other player's cutoff k^j. Then,[8]

$$\Delta(s^i; k^j) = Pr(s^j > k^j | s^i)\ (E[\theta | s^j > k^j, s^i] - w) - Pr(s^j \leq k^j | s^i)\ \mu, \tag{1}$$

where $\mu \equiv h - l > 0$ is the net miscoordination cost of taking action 1 when the other player takes action 0. Player i takes action 1 if and only if $\Delta(s^i; k^j) > 0$.[9]

Because the payoff θ from coordinating on action 1 can be very low, the game is not supermodular. When player j reduces her cutoff k^j (takes action 1 more), player i is less likely to pay the miscoordination cost, i.e., $Pr(s^j \leq k^j | s^i)\ \mu$ falls, raising i's incentive to take action 1, creating a force for strategic complements. However, when player j reduces her cutoff k^j, player i's expected payoff from successful coordination on action 1, $E[\theta | s^j > k^j, s^i]$, *also* falls, reducing her incentive to take action 1, creating a force for strategic substitutes.

Always taking action 1 is never a best response. To see this, first note that if player j always takes action 1, then player i takes action 1 if and only if $E[\theta | s^i] > w$, and **A1** implies $\lim_{s^i \to -\infty} E[\theta | s^i] = -\infty < w$. Next, note that if, instead, player j adopts a finite-cutoff strategy with cutoff k^j, then from equation (1),

$$\lim_{s^i \to -\infty} \Delta(s^i; k^j) = \lim_{s^i \to -\infty} Pr(s^j > k^j | s^i)\ (E[\theta | s^j > k^j, s^i] - w) - Pr(s^j \leq k^j | s^i)\ \mu < 0,$$

since $\lim_{s^i \to -\infty} E[\theta | s^j > k^j, s^i] < \infty$ and $\lim_{s^i \to -\infty} Pr(s^j \leq k^j | s^i) = 1$ from **A1**. Thus, i takes action 0 if her signal is sufficiently low. Further, since $h > l$, if player j always takes

[8] Throughout the paper, we assume the existence of the expectations that we use.

[9] To ease exposition, we assume that in the non-generic event that a player is indifferent between taking actions 0 and 1, she takes action 0.

action 0, then so should player i: always taking action 0 *is always* an equilibrium. Besides this equilibrium, all other monotone equilibria are in finite-cutoff strategies. Unless stated otherwise, in what follows, when we write a cutoff strategy, we mean a finite-cutoff strategy.

Lemma 3 in the Appendix establishes that the best response to a finite-cutoff strategy is a unique finite-cutoff strategy. Thus, a pair of finite cutoffs (k^i, k^j) is an equilibrium if and only if $\Delta(k^i; k^j) = \Delta(k^j; k^i) = 0$. Throughout, we focus on symmetric equilibria where $k^j = k^i = k$. Thus, (k, k) is a (symmetric) cutoff equilibrium if and only if $\Delta_1(k) = 0$, where

$$\Delta_1(k) \equiv \Delta(k; k) = Pr(s^j > k | s^i = k) \, (E[\theta | s^j > k, s^i = k] - w + \mu) - \mu. \tag{2}$$

We say that a cutoff equilibrium is larger when its cutoff is smaller, so that players are more likely to take action 1 in that equilibrium. Proposition 1 characterizes cutoff equilibria.

Proposition 1 *There exists a threshold $\mu^* > 0$ on miscoordination costs such that a finite-cutoff equilibrium exists if $\mu < \mu^*$, but not if $\mu > \mu^*$.*[10] *Moreover, if $\Delta_1(x)$ is either single-peaked or strictly increasing, then at most one other finite-cutoff equilibrium exists.*

Proposition 1 reveals that when the miscoordination cost μ is large, the potential gains from coordinating on action 1 and obtaining θ are swamped by the risk of taking action 1 when the other player takes action 0. One might posit that as μ rises and incentives to take action 1 fall, equilibrium cutoffs could adjust, preserving finite-cutoff equilibria. However, because the expected value of coordinating on action 1 is bounded from above (by **A1 (b)**), sufficiently high miscoordination costs μ always swamp the possible coordination gains.

The properties of best response curves provide further insights. Let $k^i(k^j)$ be player i's unique best response cutoff to player j's cutoff k^j. Even were j to always take action 1, player i only takes action 1 when her signal is sufficiently high, i.e., $\lim_{k^j \to -\infty} k^i(k^j)$ is finite. Thus, *if* $k^i(k^j)$ crosses the 45° line, the first crossing must occur from above. Moreover, raising the punishment cost μ or the predatory payoff w makes a player more hesitant to take action 1, i.e., $k^i(k^j)$ shifts upward, $\frac{\partial k^i(k^j;\mu)}{\partial \mu}$, $\frac{\partial k^i(k^j;w)}{\partial w} > 0$. Once the miscoordiantion cost μ exceeds a threshold, $k^i(k^j) > k^j$ for all k^j, and hence no finite-cutoff equilibrium exists. When μ is below that threshold, at least one finite-cutoff equilibrium exists.

[10]Note that μ^* will vary with the primitives describing payoffs and signals of the model.

When $\Delta_1(x)$ is single-peaked or strictly increasing, there can be at most two finite-cutoff equilibria. In the Online Appendix, we show that when $s^i = \theta + \nu^i$, (1) if θ and ν^is are independent normal random variables, then $\Delta(x)$ is single-peaked for all $\mu > 0$ and w; and (2) if $\theta, \nu^A, \nu^B \sim iid \frac{e^{-x}}{(1+e^{-x})^2}$ (logistic), then $\Delta(x)$ is single-peaked when μ is large enough that $w < 1+\mu$, but is strictly increasing when $w \geq 1+\mu$. We also provide alternative general sufficient conditions for $\Delta_1(k)$ to be single-peaked or increasing: (1) $Pr(s^j > k | s^i = k)$ is decreasing and logconcave in k, and (2) $E[\theta | s^j > k, s^i = k] - w + \mu$ is logconcave in k when positive.

Let $\underline{k}$ be the cutoff for the largest finite-cutoff equilibrium, and $\bar{k}$ be the cutoff for the smallest finite-cutoff equilibrium when multiple equilibria exist. Because raising μ or w shifts $\Delta_1(k)$ downward, when $\Delta_1(x)$ is single-peaked or strictly increasing, increases in μ or w raise $\underline{k}$, but reduce $\bar{k}$: the largest finite-cutoff equilibrium has "natural" comparative statics, but the smallest one has the opposite comparative statics, reflecting that it is locally unstable.

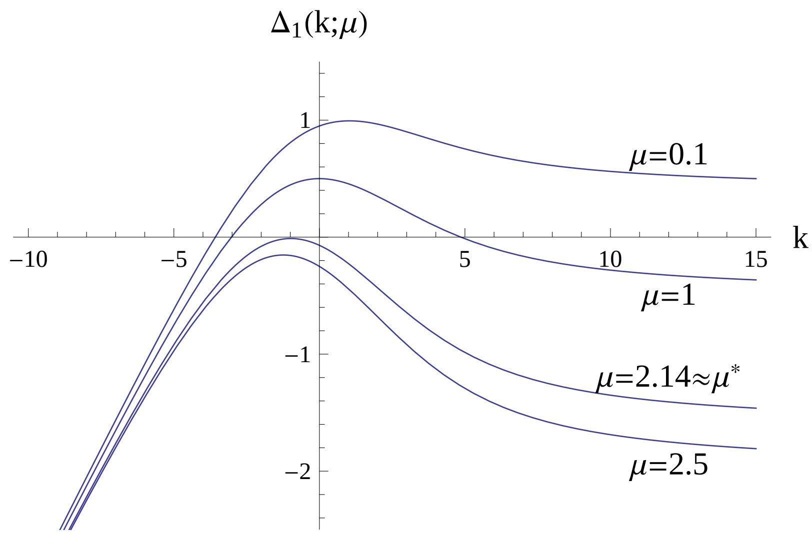

Figure 2: $\Delta_1(k;\mu)$ as a function of k for different values of miscoordination costs μ. $s^i = \theta + \nu^i$, $i \in \{A, B\}$, where θ and ν^is are iid logistic $\frac{e^{-x}}{(1+e^{-x})^2}$. When μ is small, there is a unique finite-cutoff equilibrium. Parameters: $w = -1.5$.

When $\Delta_1(x)$ is single-peaked or strictly increasing, the right-tail properties of $\Delta_1(k)$ determines the number of finite-cutoff equilibria. With an additive normal signal structure, Shadmehr and Bernhardt (2011) show that $\lim_{k\to\infty} \Delta_1(k) < 0$, so either zero or two finite-cutoff equilibria exist generically. This result does *not* generalize. For example, as Figure 2 illustrates, with an additive logistic noise signal structure, when μ is small, $\lim_{k\to\infty} \Delta_1(k)$

can be positive, implying a *unique* finite-cutoff equilibrium—see the Online Appendix.

2.2 Bounded Interval Equilibria

We next investigate when other types of equilibria can exist. In particular, we determine when an endogenously-generated fear of miscoordination can give rise to equilibria in which players sometimes seek to coordinate on action 1, but fail to do so precisely when their signals suggest that payoffs from successful coordination would be very high. We then relate the properties of monotone and these non-monotone, bounded interval equilibria (BIE), and characterize the primitive parameters for which BIE do and do not exist.

We say that player j adopts the bounded interval strategy (k_L^j, k_R^j) when

$$\rho_j(s^j) = 1 \quad \text{if and only if} \quad k_L^j < s^j < k_R^j, \quad \text{with} \;\; k_L^j, k_R^j \in \mathbb{R}. \tag{3}$$

Let $\Gamma[s^i; k_L^j, k_R^j]$ be player i's net expected payoff from taking action 1 when her signal is s^i and player j adopts interval strategy (k_L^j, k_R^j). Mirroring the derivation of equation (1),

$$\Gamma[s^i; k_L^j, k_R^j] = Pr(k_L^j < s^j < k_R^j | s^i) \; (E[\theta | k_L^j < s^j < k_R^j, s^i] - w + \mu) - \mu. \tag{4}$$

To characterize best responses to a bounded interval strategy (BIS), it helps to link $\Gamma[s^i; k_L^j, k_R^j]$ to player i's net expected payoff from taking action 1 if she knew θ. Let $\pi(\theta; k_L^j, k_R^j)$ be i's incremental return from taking action 1 given θ and (k_L^j, k_R^j):

$$\begin{aligned} \pi(\theta; k_L^j, k_R^j) &= (\theta - w) \; Pr(k_L^j < s^j < k_R^j | \theta) + (l - h) \; (1 - Pr(k_L^j < s^j < k_R^j | \theta)) \\ &= (\theta - (w - \mu)) \; Pr(k_L^j < s^j < k_R^j | \theta) - \mu. \end{aligned} \tag{5}$$

Then we can write player i's net expected payoff from taking action 1 given signal s^i as

$$\Gamma(s^i; k_L^j, k_R^j) = \int_{-\infty}^{\infty} \pi(\theta; k_L^j, k_R^j) \; f(\theta | s^i) \; d\theta, \tag{6}$$

where $f(\theta|s^i)$ is the pdf of θ given s^i. To analyze the properties of $\Gamma(s^i; k_L^j, k_R^j)$, we use Karlin's theorem on the variation diminishing property of totally positive functions (Karlin 1968, Ch. 1, Theorem 3.1). A function $K(x, y)$ is Totally Positive of order n, TP_n, whenever

$$\begin{vmatrix} K(x,y) & \frac{\partial}{\partial y} K(x,y) & \cdots & \frac{\partial^{m-1}}{\partial y^{m-1}} K(x,y) \\ \frac{\partial}{\partial x} K(x,y) & \frac{\partial^2}{\partial x \partial y} K(x,y) & \cdots & \frac{\partial^m}{\partial x \partial y^{m-1}} K(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{m-1}}{\partial x^{m-1}} K(x,y) & \frac{\partial^m}{\partial x^{m-1} \partial y} K(x,y) & \cdots & \frac{\partial^{2(m-1)}}{\partial y^{m-1} \partial x^{m-1}} K(x,y) \end{vmatrix} \geq 0, \;\; \text{for } m = 1, \cdots, n.$$

From Karlin's theorem, if $K(s^i,\theta) = f(\theta|s^i)$ is totally positive of order n and $\pi(\theta)$ has $r \leq n-1$ sign changes, then $\Gamma(s^i;k^j_L,k^j_R)$ has *at most* r sign changes. We assume that our information structure has the following properties:

Assumption A2 *For $i,j \in \{A,B\}$ with $i \neq j$, the probability density of s^i conditional on θ, $f(s^i|\theta)$, is totally positive of order three (TP_3), and $Pr(k_L < s^i < k_R|\theta)$ is strictly logconcave in θ. Moreover, $\lim_{s^i\to\infty} E[\theta|s^j,s^i]f(s^j|s^i) = \lim_{s^i\to\infty} f(s^j|s^i) = 0$ for any s^j.*

We show in an Online Appendix that **A2** holds if $s^i = \theta + \nu^i$, where θ and ν^is are independent normal or iid logistic random variables.[11] Given this structure, we use Karlin's theorem to derive the shapes of $\Gamma(s^i;k^j_L,k^j_R)$ and $\pi(\theta;k^j_L,k^j_R)$, using equations (4) to (6). This allows us to characterize the best response to a BIS:

Lemma 1 *Suppose player j adopts a bounded interval strategy. Then, either player i's best response is a bounded interval strategy, or it is to always take action 0.*

We first use logconcavity to show that (i) $\pi(\theta;k^j_L,k^j_R)$ has a unique maximum whenever it is positive, and (ii) when θ increases unboundedly, $Pr(k_L < s^i < k_R|\theta)$ approaches zero at a rate faster than exponential functions (An 1998), and hence $\lim_{\theta\to\infty} Pr(k_L < s^i < k_R|\theta)(\theta-(w-\mu)) = 0$. We use these to show that $\pi(\theta;k^j_L,k^j_R)$ has at most two sign changes. We then use the fact that because $f(s^i|\theta)$ is TP_3, Karlin's theorem implies that $\Gamma(s^i;k^j_L,k^j_R)$ has at most two sign changes. Moreover, $\lim_{s^i\to\pm\infty}\Gamma[s^i;k^j_L,k^j_R] < 0$.[12] This implies that either $\Gamma[s^i;k^j_L,k^j_R]$ has two sign changes or no sign change, in which case it must be negative. And, if $\Gamma[s^i;k^j_L,k^j_R]$ has exactly two sign changes, then as s^i traverses from $-\infty$ to ∞, $\Gamma(s^i;k^j_L,k^j_R)$ must first change sign from negative to positive and then from positive to negative; this is precisely the property of a bounded interval strategy (BIS).

Throughout, we focus on symmetric BIE. The necessary and sufficient condition for (k_L,k_R) to be a symmetric BIE is $\Gamma[s^i;k_L,k_R] > 0$ if and only if $s^i \in (k_L,k_R)$. To find a symmetric BIE, one must solve the system of equations that describe a player's indifference between taking actions 0 and 1 at the cutoffs: $\Gamma[s^i=k_L;k_L,k_R]=0$ and $\Gamma[s^i=k_R;k_L,k_R]=0$.

[11] Suppose $\nu^i \sim h$. Then, affiliation (TP_2) requires h to be log-concave: $(h')^2 \geq hh''$. TP_3 involves higher derivatives of h, in particular, $\frac{\partial^3 f(s^i|\theta)}{\partial s^i \partial^2\theta} = -\frac{\partial^3 f(s^i|\theta)}{\partial^2 s^i \partial\theta} = h^{(3)}(s^i-\theta)$ and $\frac{\partial^4 f(s^i|\theta)}{\partial^2 s^i \partial^2\theta} = h^{(4)}(s^i-\theta)$.

[12] From **A2**, (i) $\lim_{s^i\to\infty} E[\theta|s^j,s^i]f(s^j|s^i)=0$, which implies $\lim_{s^i\to\infty} Pr(k_L < s^j < k_R|s^i)\, E[\theta|k_L < s^j < k_R, s^i] = \lim_{s^i\to\infty}\int_{k_L}^{k_R} E[\theta|s^j,s^i]f(s^j|s^i) = 0$; and (ii) $\lim_{s^i\to\infty} f(s^j|s^i)=0$, which implies $\lim_{s^i\to\infty} Pr(k_L < s^j < k_R|s^i)$. Therefore, $\lim_{s^i\to\infty}\Gamma[s^i;k^j_L,k^j_R] < 0$. Moreover, **A1 (a)** implies $\lim_{s^i\to-\infty}\Gamma[s^i;k^j_L,k^j_R] < 0$.

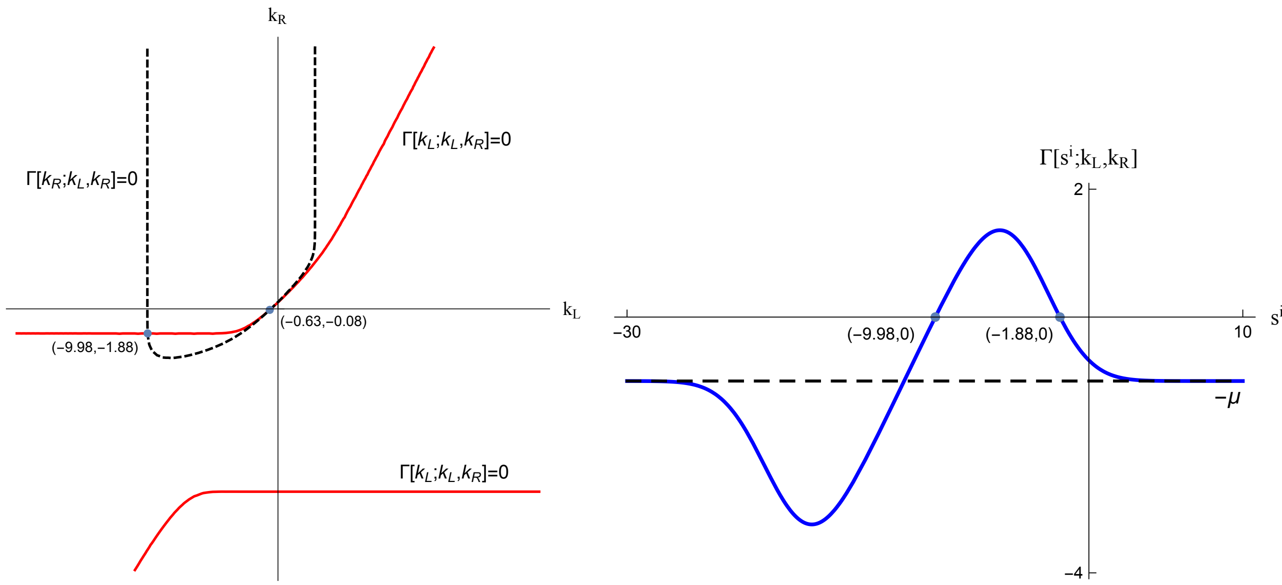

Figure 3: A bounded interval equilibrium. The left panel depicts contour curves $\Gamma[k_L; k_L, k_R] = 0$, $\Gamma[k_R; k_L, k_R] = 0$, and their intersections. The right panel depicts $\Gamma[s^i; k_L = -9.98, k_R = -1.88]$. The panels together show that $(-9.98, -1.88)$ is a bounded interval equilibrium. The numbers are two-decimal approximations. Parameters: $w = -5$, $\mu = 1$, and $s^i = \theta + \nu^i$, $i \in \{A, B\}$, with $\theta, \nu^A, \nu^B \sim iidN(0, 1)$.

These equations are only necessary conditions. To show that they are also sufficient, Lemma 2 shows that Γ changes sign at any solution of these equations. Therefore, any solution to these equations describes a BIE:

Lemma 2 *(k_L, k_R) with $k_L, k_R \in \mathbb{R}$ and $k_L < k_R$ is a bounded interval equilibrium strategy if and only if $\Gamma[k_L; k_L, k_R] = 0$ and $\Gamma[k_R; k_L, k_R] = 0$.*

Figure 3 illustrates an equilibrium in bounded interval strategies. Next, we determine for a given signal structure $f(\theta, s^A, s^B)$ that satisfies **A1** and **A2** and an *arbitrary* bounded interval strategy (k_L, k_R), whether there exist primitive payoff parameters $\{\mu, w\}$ that support (k_L, k_R) as an equilibrium. Proposition 2 details necessary and sufficient conditions for (k_L, k_R) to be an equilibrium for *some* set of primitives.

Proposition 2 *There exist payoff parameters $\{\mu, w\}$ for which (k_L, k_R) is an equilibrium BIS if and only if $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$, for $i \neq j$.*

To see the necessity, observe that the indifference conditions at the bounds of a symmetric equilibrium BIS, (k_L, k_R), are $\Gamma[s^i = k_L; k_L, k_R] = \Gamma[s^i = k_R; k_L, k_R] = 0$. Rearranging these indifference conditions yields

$$\begin{aligned}\mu &= Pr(k_L < s^j < k_R | s^i = k_L)\ (E[\theta | k_L < s^j < k_R, s^i = k_L] - w + \mu) \\ &= Pr(k_L < s^j < k_R | s^i = k_R)\ (E[\theta | k_L < s^j < k_R, s^i = k_R] - w + \mu).\end{aligned}$$

A player with high signal $s^i = k_R$ is more optimistic about the expected value of θ than if she sees signal $s^i = k_L$. To be indifferent between taking actions 0 and 1 after both signals, she must be more pessimistic about the likelihood of coordinating on θ given $s^i = k_R$. Thus,

$$Pr(s^j \in (k_L, k_R) | s^i = k_L) > Pr(s^j \in (k_L, k_R) | s^i = k_R).$$

The indifference conditions also shed light on the primitives that support (k_L, k_R) as a BIE. Since $Pr(s^j \in (k_L, k_R) | s^i = k_L) < 1$, it must be that

$$E[\theta | k_L < s^j < k_R, s^i] - w + \mu > \mu, \quad \text{for} \quad s^i \in \{k_L, k_R\}.$$

Rearranging yields a bound on the predatory payoff w:

$$E[\theta | k_L < s^j < k_R, s^i] > w, \quad \text{for} \quad s^i \in \{k_L, k_R\}. \tag{7}$$

The intuition is simple: player i is hurt when she takes action 1 rather than action 0 if j takes action 0, receiving l rather than h. Thus, to be willing to take action 1, i must expect to gain from taking action 1 rather than action 0 if j instead takes action 1. This inequality also holds for all $s^i > k_R$—were such a player to *know* that the other agent had a signal $s^j \in (k_L, k_R)$, she would also want to take action 1 for higher signals; but she assigns such a low probability to the event $s^j \in (k_L, k_R)$, and hence such a high probability to miscoordination, that she chooses to take action 0 despite her high signal.

Properties of bounded interval equilibria. To characterize bounded interval equilibria, we next relate BIE and monotone equilibria. We then specialize to the classical additive normal noise signal structure to highlight the surprisingly strong structure that Proposition 2 places on the nature and extent of coordination that can occur in any BIE. Recall that $\underline{k}$ is the cutoff for the largest finite-cutoff equilibrium, and $\bar{k}$ is the cutoff for the smallest finite-cutoff equilibrium when there are multiple equilibria.

Proposition 3 *If a BIE exists, then finite-cutoff monotone equilibria exist. If (k_L, k_R) is a bounded interval equilibrium strategy, then $\underline{k} < k_L$, i.e., players take action 1 following worse signals in the largest finite-cutoff ($\underline{k}$) equilibrium than in any BIE.*

If $\lim_{k\to\infty} \Delta_1(k) < 0$ and a BIE exists, then multiple finite-cutoff monotone equilibria exist. If (k_L, k_R) is a bounded interval equilibrium strategy, then $k_L < \bar{k}$, where $\bar{k}$ is the cutoff for the smallest finite-cutoff equilibrium.

The intuition comes from a decomposition of $\Gamma[s^i = k_L; k_L, k_R]$:

$$\Gamma[s^i = k_L; k_L, k_R] = -Pr(s^j > k_R | s^i = k_L)\ (E[\theta | s^j > k_R, s^i = k_L] - w + \mu) + \Delta_1(k_L), \quad (8)$$

where $E[\theta | s^j > k_R, s^i = k_L] - w + \mu > 0$.[13] A player must be indifferent between actions 0 and 1 when his signal just equals the lower cutoff k_L of a BIE. Thus, when a player j chooses the *cutoff strategy* k_L instead of a BIS, so that j takes action 1 after receiving all "better signals," player i has more incentive to take action 1 than in the BIE. In particular, where she was indifferent between actions (at $s^i = k_L$) in the BIE, she now strictly prefers to take action 1: $\Gamma[s^i = k_L; k_L, k_R] = 0$ implies $\Delta_1(k_L) = \Delta(s^i = k_L; k_L) > 0$. Because $\lim_{k\to-\infty} \Delta_1(k) < 0$, there exists a cutoff $\underline{k} < k_L$ at which $\Delta_1(\underline{k}) = 0$.

If, in addition, $\lim_{k\to\infty} \Delta_1(k) < 0$, then there are multiple solutions to $\Delta_1(k) = 0$, i.e., there are multiple finite-cutoff equilibria. The largest such cutoff, $\bar{k}$ exceeds k_L: this follows since $\Delta_1(k_L) > 0$ but $\Delta_1(k) < 0$ for $k > \bar{k}$. Moreover, because the gains from successful coordination following signal $\bar{k}$ in the cutoff equilibrium exceed those in the BIE given signal k_L, the probability of successful coordination must be *lower* given $\bar{k}$ than k_L in the corresponding equilibrium.

Corollary 1 *Players are more likely to take action 1 in the largest finite-cutoff equilibrium than in any BIE.*

When $w > l$, players take action 1 too little in the $\underline{k}$ equilibrium from a welfare perspective (Shadmehr and Bernhardt (2015)), implying that BIE are welfare dominated.

Additive normal noise signal structure. To illustrate the sharp bite of the necessary condition in Proposition 2 that $Pr(s^j \in (k_L, k_R) | s^i = k_L) > Pr(s^j \in (k_L, k_R) | s^i = k_R)$, we

[13]Note that $(E[\theta | s^j > k_R, s^i = k_L] - w + \mu) > E[\theta | k_L < s^j < k_R, s^i = k_L] - w > 0$, where the second inequality follows from (7).

now specialize to the classical additive normal noise signal structure. The key structure that we exploit is that the signals shift the distribution, so that single-peakedness and symmetry are preserved and the dispersions of conditional distributions do not depend on signals.

Suppose $s^i = \theta + \nu^i$, $i \in \{A, B\}$, where θ and ν^i's are independent with $\theta \sim N(0, \sigma^2)$ and $\nu^i \sim N(0, \sigma_\nu^2)$. Let $b \equiv \frac{\sigma^2}{\sigma^2+\sigma_\nu^2}$, and recall that $pdf(s^j|s^i) \sim N(bs^i, (1+b)\sigma_\nu^2)$. First, observe that when the noise in signals goes to zero, in the limit, $Pr(s^j \in (k_L, k_R)|s^i = k_L) = Pr(s^j \in (k_L, k_R)|s^i = k_R) = \frac{1}{2}$, implying that BIE do not exist.[14] Proposition 4 details the sharp restrictions imposed on possible BIE using the fact that raising s^i from k_L to k_R just shifts the conditional distribution $f(s^j|s^i)$ to the right:

Proposition 4 *If (k_L, k_R) is a bounded interval equilibrium strategy, then $k_L < E[\theta] = 0$ and $|k_L| > |k_R|$.*

Recall that if (k_L, k_R) is an equilibrium BIS, a player i must be more pessimistic about the likelihood of coordinating on θ given $s^i = k_R$: $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$. With an additive normal noise signal structure, this inequality becomes

$$|bk_L - (k_L + k_R)/2| < |bk_R - (k_L + k_R)/2|. \tag{9}$$

Inspection of (9) reveals that for it to hold, we must have $k_L < 0$ and $|k_L| > |k_R|$.

Corollary 2 *Relative to finite-cutoff equilibria, coordination in BIE on action 1 is limited and it is on less promising values: In particular, $\underline{k} < k_L < k_R < -\underline{k}$, where $\underline{k} < 0$ and $E[\theta|k_L < s^j < k_R, s^i = k_L] < 0$.*

The corollary highlights the severity of the endogenously-generated fear of miscoordination after high signals in any BIE. For a BIE to exist, we must have $\underline{k} < k_L < 0$; and $\underline{k} < 0$ implies that in the finite-cutoff equilibrium supporting the most coordination on θ, players seek to coordinate on action 1 for a majority of signals. In contrast, in a BIE, players fail to coordinate following *all* good signals $s^i > -\underline{k}$.

[14] $\lim_{\sigma_\nu \to 0} Pr(k_L < s^j < k_R|s^i = k_L) = \lim_{\sigma_\nu \to 0} \Phi\big(\frac{k_R - bk_L}{\sqrt{(1+b)\sigma_\nu^2}}\big) - \lim_{\sigma_\nu \to 0} \Phi\big(\frac{(1-b)k_L}{\sqrt{(1+b)\sigma_\nu^2}}\big) = 1 - \frac{1}{2} = \frac{1}{2}$, where we use $\lim_{\sigma_\nu \to 0} \frac{1-b}{\sqrt{(1+b)\sigma_\nu^2}} = 0$. Similarly, $\lim_{\sigma_\nu \to 0} Pr(k_L < s^j < k_R|s^i = k_R) = \frac{1}{2} - 0 = \frac{1}{2}$.

These results highlight that BIE have particularly bad welfare properties relative to finite-cutoff equilibria. In finite-cutoff equilibria, a player's expected payoff, given just her own information and conditional on successfully coordinating on action 1 to obtain θ exceeds $E[\theta]$:

$$\int_{\underline{k}}^{\infty} E[\theta|s^i]\ pdf(s^i|s^i > \underline{k})\ ds^i > E[\theta] = 0.$$

The fact that coordination succeeds only when the other player also has a good signal, $s^j > \underline{k}$ reinforces this. In sharp contrast, in BIE, $|k_L| > k_R$ and $k_L < 0$ imply that a player's expected payoff, given just her own information and conditional on *successfully* coordinating on action 1 to obtain θ, is *worse* than $E[\theta]$:

$$\int_{k_L}^{k_R} E[\theta|s^i]\ pdf(s^i|s^i \in (k_L, k_R))\ ds^i < E[\theta] = 0.$$

The fact that coordination only succeeds when the other player also has a signal $s^j \in (k_L, k_R)$ further reduces the expected payoff from 'successful' coordination in a BIE.

Because expected payoffs in BIE are lower than those for finite-cutoff equilibria, BIE cease to exist for lower miscoordination costs than for finite-cutoff equilibria:

Corollary 3 *Recall that finite-cutoff equilibria exist whenever $\mu < \mu^*$. There exists a $\mu^{**} < \mu^*$ such that if $\mu > \mu^{**}$, then no bounded interval equilibrium exists.*

Why then do BIE exist and when? Summarizing our previous results, recall that players' indifference at the bounds of an equilibrium BIS (k_L, k_R) has two implications. First, equation (7) revealed that $E[\theta|k_L < s^j < k_R, s^i = k_L] > w$. Second, Proposition 2 revealed that $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$, which Proposition 4 shows for a normal-noise signal structure implies $k_L < 0$ *and* $|k_L| > |k_R|$. These latter results imply that $E[\theta|k_L < s^j < k_R, s^i = k_L] < E[\theta] = 0$. Combining these results yields:

Corollary 4 *A necessary condition for a BIE to exist is $w < E[\theta] = 0$.*

Thus, for a BIE to exist, absent any private information, the game must be a coordination game with $h > l$ and $E[\theta] > w$. That is, a player must face a miscoordination risk when taking the 'safe' action 0: unconditionally, were the other player to always take action 1, it must be better to always take action 1 than to always take action 0. The fact that $w < E[\theta]$

is necessary for a BIE to exist highlights that there must be sufficient expected potential surplus from coordinating on action 1 to compensate for the facts that in a BIE, (i) players fail to coordinate on action 1 precisely when the payoff from doing so is expected to be high, thereby foregoing much of the benefits of coordinating on action 1; and (ii) a player is hurt when she tries to coordinate on action 1, but the other player does not.

In many of the economic settings described by the payoffs in Figure 1, the payoff h from coordinating on action 0 is less than the payoff w from taking action 0 when the other player takes action 1. For example, in an international trade game, the payoff when both countries set high tariffs is less than the payoff from setting a high tariff when the other country lowers its barriers. In such settings, the results in Corollary 4 are reinforced: BIE only exist when the status quo payoff of h (e.g., trade wars) is also low relative to the ex-ante expected benefits of coordinating on action 1 (e.g., free trade)—but the agents nonetheless get the low status quo payoff when signals about coordination are promising.

3 Discussion

A thorough analysis of non-monotone equilibria in a broader class of games is beyond the scope of our paper. However, it is straightforward to establish that BIE can exist in the private value analogue of our common value coordination setting, in which a player i's expected payoff when both players coordinate on action 1 is given by her signal s^i. Indeed, the intuitions and qualitative properties that we have emphasized carry over directly.

One may also wonder if the one-sided limit dominance property[15] of our game is essential for the existence of non-monotone equilibria. Clearly, in games featuring two-sided limit dominance, a bounded interval strategy is never a best response, and hence bounded interval equilibria cannot exist. Still, more complex non-monotone equilibria may exist. To see this, consider the classical private value investment game of Morris and Shin (2003) in Figure 4, where action 1 corresponds to invest and action 0 corresponds to not invest.

Clearly, no BIE exists because a player always takes action 0 when her signal s^i is less than h, and she always takes action 1, when her signal exceeds $h+\mu$. The question is: can slightly

[15] A player has a dominant strategy to take action 0 if her signal is low enough, but does not have a dominant strategy if her signal is high: our game features one-sided limit dominance, not two-sided limit dominance.

		player B	
		0	*1*
player A	*0*	h, h	$h, s^2 - \mu$
	1	$s^1 - \mu, h$	s^1, s^2

Figure 4: Private Value Game.

more complex non-monotone equilibria exist in which players take action 1 when their signals are either in a bounded interval or exceed a threshold, $(k_1, k_2) \cup (k_3, \infty)$ for $k_1 < k_2 < k_3 \in \mathbb{R}$?

To show that such non-monotone equilibria can exist, we construct a non-monotone equilibrium for an additive normal noise signal structure, $s^i = \theta + \nu^i$, where $\theta, \nu^A, \nu^B \sim iidN(0,1)$. In such an equilibrium, players must be indifferent at the cutoffs, k_1, k_2 and k_3. Defining $\Omega \equiv (-\infty, k_1) \cup (k_2, k_3)$, the indifference conditions are:

$$k_i - Pr(\Omega | s^j = k_i)\ \mu = h,\ \ i \in \{1, 2, 3\},\ \ j \in \{A, B\}.$$

In particular, we must have

$$\frac{k_2 - k_1}{Pr(\Omega|k_2) - Pr(\Omega|k_1)} = \frac{k_3 - k_2}{Pr(\Omega|k_3) - Pr(\Omega|k_2)}. \tag{10}$$

Then $(k_1, k_2, k_3) = (k, 0, -k)$ satisfies equation (10). For example, $(k_1, k_2, k_3) = (-5, 0, 5)$ is supported as an equilibrium by $\mu = \frac{k_2 - k_1}{Pr(\Omega|k_2) - Pr(\Omega|k_1)} \approx 10.8986$ and $h = k_i - Pr(s^2 \in (-\infty, k_1) \cup (k_2, k_3) | s^1 = k_i) \times \mu \approx -5.4493$. Figure 5 illustrates that, for these parameters, the non-monotone strategy $(k_1, k_2, k_3) = (-5, 0, 5)$ is the best response to itself, and hence constitutes an equilibrium.

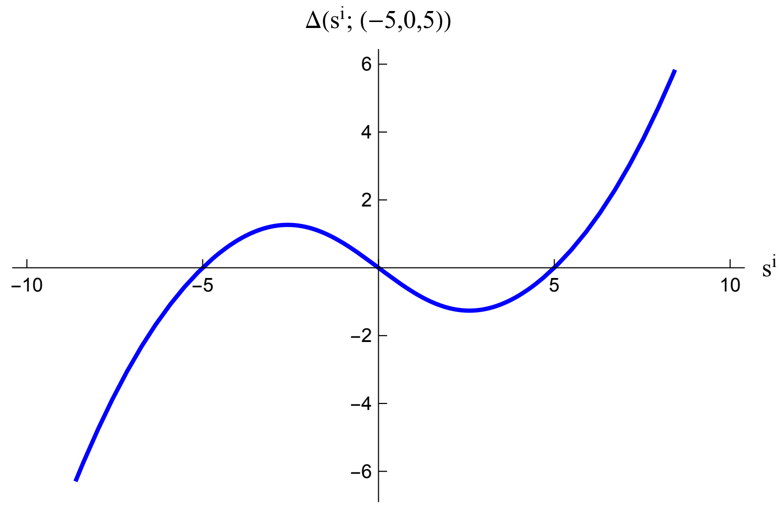

Figure 5: Net expected payoffs as a function of signal s^i given the other player's non-monotone strategy $(k_1, k_2, k_3) = (-5, 0, 5)$. Parameters: $\mu = 10.8986$ and $h = -5.4493$.

4 Appendix: Proofs

Proof of Proposition 1: First, we prove a lemma.

Lemma 3 *The best response to a finite-cutoff strategy is a unique finite-cutoff strategy.*

Proof: Fix a finite cutoff k^j. From equation (1), if $\Delta(s^i = x; k^j) = 0$, then $(E[\theta|s^j > k^j, s^i = x] - w) \geq 0$; and by affiliation, both $Pr(s^j > k^j|s^i)$ and $E[\theta|s^j > k^j, s^i]$ increase with s^i. Thus, if $\Delta(s^i = x; k^j) = 0$, then $\Delta(s^i; k^j) > 0$ for all $s^i > x$. Furthermore, from **A1 (a)**, $\lim_{s^i \to -\infty} \Delta(s^i; k^j) < 0 < \lim_{s^i \to +\infty} \Delta(s^i; k^j)$. Thus, for every k^j, there exists a unique $s^i = k^i$ such that $\Delta(k^i; k^j) = 0$. In addition, at $s^i = k^i$,

$$\left.\frac{\partial \Delta(s^i; k^j)}{\partial s^i}\right|_{s^i = k^i} > 0. \quad \square \tag{11}$$

Now, rewrite equation (2) as

$$\Delta_1(k) = Pr(s^j > k|k)\ E[\theta|s^j > k, k] - Pr(s^j > k|k)\ w - (1 - Pr(s^j > k|k))\ \mu. \tag{12}$$

For any k, $\frac{\partial \Delta_1(k;\mu)}{\partial \mu} = -(1 - Pr(s^j > k|k)) < 0$: raising μ uniformly lowers the curve $\Delta_1(k; \mu)$.

Next, we show that when μ is sufficiently small, $\Delta_1(k)$ crosses zero *at least* once; and that when μ is sufficiently large, there is no solution to $\Delta_1(k) = 0$. At $\mu = 0$, $\Delta_1(k; \mu =$

$0) = Pr(s^j > k|k)\ (E[\theta|s^j > k, k] - w)$. $E[\theta|s^j > k, s^i = k]$ increases in k by affiliation, and from **A1**, $\lim_{k\to\pm\infty} E[\theta|s^j > k, s^i = k] = \pm\infty$. Therefore, for a given w, $\Delta_1(k;\mu = 0)$ crosses zero from below—at a unique point. Thus, from the continuity of $\Delta_1(k;\mu)$ in k and μ, $\Delta_1(k;\mu) = 0$ has a solution for sufficiently small $\mu > 0$.

Lemma 4 *There exists a $\bar{\mu} > 0$ such that if $\mu > \bar{\mu}$, then $\Delta_1(k) < 0$ for all k.*

Proof: Consider $\Delta_1(k;\mu)$ from equation (12), where we have made the dependence of Δ_1 on the parameter μ explicit. First, observe that $\Delta_1(k;\mu)$ is uniformly decreasing in μ: for any k, $\Delta_1(k;\mu_1) > \Delta_1(k;\mu_2)$ for all $\mu_2 > \mu_1$. Thus, if for some $\mu = \hat{\mu}$, $\Delta_1(k;\hat{\mu}) < 0$ for all k, then $\Delta_1(k;\mu) < 0$ for all $\mu > \hat{\mu}$. Now, suppose that the statement of the lemma is false. Then, it must be the case that for all μ, there is some k such that $\Delta_1(k;\mu) \geq 0$. We seek a contradiction.

From **A1 (b)**, $\lim_{k\to-\infty} E[\theta|s^j > k, k] = -\infty$, and hence $\lim_{k\to-\infty} \Delta_1(k;\mu) < 0$ for all $\mu > 0$. Recall our premise that, for all μ, there is some k such that $\Delta_1(k;\mu) \geq 0$. Thus, because $\Delta_1(k;\mu)$ is continuous in k, there exists a smallest $k \in \mathbb{R}$ at which $\Delta_1(k;\mu) \geq 0$. Call it $\underline{k}(\mu) \in \mathbb{R}$, and observe that $\underline{k}(\mu)$ is weakly increasing in μ.

Let $M(\mu) \equiv \sup_{k\in[\underline{k}(\mu),\infty)} Pr(s^j > k|k)\ E[\theta|s^j > k, k]$ and $m(\mu) \equiv \inf_{k\in[\underline{k}(\mu),\infty)}(1-Pr(s^j > k|k))$. Clearly,

$$\Delta_1(k;\mu) \leq M(\mu) + |w| - m(\mu)\ \mu.$$

Because $\underline{k}(\mu)$ is weakly increasing in μ, $M(\mu)$ is weakly decreasing in μ and $m(\mu)$ is weakly increasing in μ. Moreover, from **A1 (b)**, $\lim_{k\to\infty} Pr(s^j > k|k)\ E[\theta|s^j > k, k] < \infty$, i.e., $Pr(s^j > k|k)\ E[\theta|s^j > k, k]$ is bounded from above, and hence $M(\mu)$ is bounded from above. Further, **A1 (b)** implies that $\lim_{k\to\infty}(1 - Pr(s^j > k|k)) = 1$. Because this limit is strictly positive and $f(\theta, s^A, s^B)$ has full support by assumption, $m(\mu) > 0$. Recalling that $M(\mu)$ is weakly decreasing in μ and $m(\mu)$ is weakly increasing in μ, we have, $M(\mu)+|w|-m(\mu)\ \mu < 0$ for sufficiently large μ. Since $\Delta_1(k;\mu) \leq M(\mu) + |w| - m(\mu)\ \mu$, then for μ sufficiently large, we have $\Delta_1(k;\mu) < 0$ for all k, a contradiction. □

Therefore, there exists a $\mu^* > 0$ such that $\Delta_1(k) = 0$ has a solution if $\mu < \mu^*$ and it does not have a solution if $\mu > \mu^*$. It is immediate that when $\Delta_1(k)$ is either single-peaked or strictly increasing, $\Delta_1(k) = 0$ has at most two solutions. □

Proof of Lemma 1. We proceed in two steps.

STEP 1: We show that $\pi(\theta; k_L^j, k_R^j)$, from equation (5), has at most two sign changes. Observe that $[\theta - (w - \mu)]\ Pr(k_L^j < s^j < k_R^j|\theta)$ has a unique root at $w - \mu$. Let $g(\theta) \equiv Pr(k_L^j < s^j < k_R^j|\theta)$. Differentiating $[\theta - (w - \mu)]\ Pr(k_L^j < s^j < k_R^j|\theta)$ with respect to θ yields $g(\theta) + [\theta - (w-\mu)]\ g'(\theta)$. Thus, $g(\theta) + [\theta-(w-\mu)]\ g'(\theta) > 0$ if and only if $g(\theta) > -[\theta-(w-\mu)]\ g'(\theta)$. If $\theta > w - \mu$, this inequality is equivalent to $-\frac{1}{\theta-(w-\mu)} < \frac{g'(\theta)}{g(\theta)}$. The left-hand side is strictly increasing in θ and the right-hand side is strictly decreasing because $g(\theta)$ is logconcave by **A2**. Thus, they can cross at most once, in which case the intersection identifies a local maximum. Next, observe that $[\theta-(w-\mu)]\ g(\theta) = 0$ at $\theta = w-\mu$, and $\lim_{\theta\to\infty}[\theta-(w-\mu)]\ g(\theta) = 0$ from log-concavity (An 1998, Corollary 1). Thus, the crossing, indeed, happens exactly once. That is, $(\theta - (w - \mu))\ Pr(k_L^j < s^j < k_R^j|\theta)$ has a unique maximum for $\theta \in (w - \mu, \infty)$, and is negative for $\theta \in (-\infty, w - \mu)$. Since $\pi(\theta; k_L^j, k_R^j)$ equals $(\theta - (w - \mu))\ Pr(k_L^j < s^j < k_R^j|\theta)$ minus $\mu > 0$, it inherits the shape, and has at most two sign changes (see Figure 3).

STEP 2: Step 1 shows that $\pi(\theta; k_L^j, k_R^j)$ has at most two sign changes. Now, consider equation (6), and recall that **A2** implies that $f(\theta|s^i)$ is TP_3. Thus, by Karlin's theorem, $\Gamma(s^i; k_L^j, k_R^j)$ has *at most* two sign changes. Footnote 12 shows that $\lim_{s^i\to\pm\infty}\Gamma[s^i; k_L^j, k_R^j] < 0$. Thus, (i) $\Gamma(s^i; k_L^j, k_R^j)$ has either no sign change or two sign changes, (ii) *if* it has no sign change, then $\Gamma(s^i; k_L^j, k_R^j) < 0$ for all s^i, i.e., player i never takes action 1, and (iii) if $\Gamma(s^i; k_L^j, k_R^j)$ has two sign changes, then as s^i traverses $\mathbb{R}$ from $-\infty$ to ∞, $\Gamma(s^i; k_L^j, k_R^j)$ is first negative, then positive, and then negative, which implies a bounded interval strategy. $\square$

Proof of Lemma 2: The "only if" part is immediate from the continuity of $\Gamma[s^i; k_L, k_R]$ in s^i for $i \in \{A, B\}$. Next, we prove the "if" part. If $\Gamma[s^i; k_L, k_R]$ changes sign at both k_L and k_R, then from Lemma 1, $\Gamma[s^i; k_L, k_R] > 0$ if and only if $s^i \in (k_L, k_R)$. Otherwise, Γ does not change sign at k_L or k_R or both. We consider two cases:

Case I: Suppose Γ changes sign at only one of k_L and k_R. WLOG, suppose Γ changes sign at k_R, but not k_L. Then k_L must be a local maximum (minimum). Adding a small positive (negative) constant ϵ to π in equation (6) adds a constant to Γ,

$$\int_{\theta=-\infty}^{\infty} (\pi(\theta; k_L^j, k_R^j) + \epsilon)\ f(\theta|s^i)\ d\theta = \Gamma(s^i; k_L^j, k_R^j) + \epsilon,$$

and hence creates k_{Ll} and k_{Lr} at which $\Gamma + \epsilon$ changes sign, with $k_{Ll} < k_L < k_{Lr} < k_R$. Thus, $\Gamma + \epsilon$ changes sign at least three times: at k_{Ll}, k_{Lr}, and $k_R + \epsilon_R$ for some small ϵ_R. But from the proof of Lemma 1, $\pi + \epsilon$ has at most two sign changes, which together with the TP_3 property of $f(\theta|s^i)$ implies that $\Gamma + \epsilon$ has at most two sign changes, which is a contradiction.

Case II: Suppose Γ does not change sign at k_L and k_R. If k_L and k_R are both local maxima or both local minima, an argument similar to Case I leads to a contradiction. If one is a local maximum and the other is a local minimum, then there exists a k_M with $k_L < k_M < k_R$ at which Γ changes sign. Now apply the argument in Case I with k_M instead of k_R. $\square$

Proof of Proposition 2: From Lemma 2, (k_L, k_R) is a symmetric bounded interval equilibrium strategy if and only if $\Gamma[k_L; k_L, k_R] = \Gamma[k_R; k_L, k_R] = 0$. Solving for $-w$ from (4) and using the notation in (3), yields

$$-w = \frac{\mu}{Pr(\rho_j = 1|s^i = k_L)} - \mu - E[\theta|k_L, \rho_j = 1] = \frac{\mu}{Pr(\rho_j = 1|s^i = k_R)} - \mu - E[\theta|k_R, \rho_j = 1], \tag{13}$$

where $E[\theta|z, \rho_j = 1]$ means $E[\theta|s^i = z, \rho_j = 1]$. Rearranging the second equality yields,

$$\mu \left\{ \frac{1}{Pr(\rho_j = 1|s^i = k_L)} - \frac{1}{Pr(\rho_j = 1|s^i = k_R)} \right\} = E[\theta|s^i = k_L, \rho_j = 1] - E[\theta|s^i = k_R, \rho_j = 1],$$

which implies

$$\frac{\mu}{Pr(\rho_j = 1|s^i = k_L)\ Pr(\rho_j = 1|s^i = k_R)} = \frac{E[\theta|s^i = k_L, \rho_j = 1] - E[\theta|s^i = k_R, \rho_j = 1]}{Pr(\rho_j = 1|s^i = k_R) - Pr(\rho_j = 1|s^i = k_L)}. \tag{14}$$

The left-hand side is positive. Therefore, a necessary condition for the equilibrium to exist is that the right-hand side be positive. Since $E[\theta|s^i = k_R, \rho_j = 1] > E[\theta|s^i = k_L, \rho_j = 1]$, existence requires that $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$. This proves the "only if" part. To prove the "if" part, note that the right-hand side of equation (14) is positive whenever $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$, and it does not depend on μ. Thus, there exists a $\mu > 0$ such that this equation holds. Finally, substitute that μ into equation (13). There exists a w that satisfies this equation. $\square$

Proof of Proposition 3: Suppose (k_L, k_R) is a symmetric bounded interval equilibrium strategy. From Lemma 2, $\Gamma[k_L; k_L, k_R] = 0$. From equation (4),

$$\begin{aligned}
\Gamma[s^i = k_L; k_L, k_R] &= Pr(k_L < s^j < k_R|s^i = k_L)\ (E[\theta|k_L < s^j < k_R, s^i = k_L] - w + \mu) - \mu \\
&= Pr(k_L < s^j|s^i = k_L)\ (E[\theta|k_L < s^j, s^i = k_L] - w + \mu) - \mu \\
&\quad -Pr(k_R < s^j|s^i = k_L)\ (E[\theta|k_R < s^j, s^i = k_L] - w + \mu) \\
&= \Delta_1(k_L) - Pr(s^j > k_R|s^i = k_L)\ (E[\theta|s^j > k_R, s^i = k_L] - w + \mu),
\end{aligned} \tag{15}$$

where the last equality follows from equation (2). Footnote 13 shows that $(E[\theta|s^j > k_R, s^i = k_L] - w + \mu) > 0$. Hence, $\Gamma[k_L; k_L, k_R] = 0 < \Delta_1(k_L)$. This together with $\lim_{k\to-\infty} \Delta_1(k) < 0$ implies that $\Delta_1(k)$ has at least one solution to the left of k_L, and hence finite-cutoff equilibria exist. In addition, if $\lim_{k\to\infty} \Delta_1(k) < 0$, then $\Delta_1(k)$ must cross the horizontal axis from above at least once, i.e., another finite-cutoff equilibrium exists. Let $\bar{k}$ be the cutoff for the smallest finite-cutoff equilibrium. Because $\Delta_1(k_L) > 0$ and $\Delta_1(k) < 0$ for $k > \bar{k}$, we must have $k_L < \bar{k}$. $\square$

Proof of Proposition 4: Suppose $s^i = \theta + \nu^i$, $i \in \{A, B\}$, where θ and ν^i's are independent with $\theta \sim N(0, \sigma^2)$ and $\nu^i \sim N(0, \sigma_\nu^2)$. Let $b \equiv \frac{\sigma^2}{\sigma^2+\sigma_\nu^2}$, and note that $pdf(s^j|s^i) \sim N(bs^i, (1+b)\sigma_\nu^2)$. From Proposition 2, $Pr(s^j \in (k_L, k_R)|s^i = k_L) > Pr(s^j \in (k_L, k_R)|s^i = k_R)$, which holds if and only if

$$|bk_L - (k_L + k_R)/2| < |bk_R - (k_L + k_R)/2|. \tag{16}$$

Because $k_L < k_R$ and $b \in (0, 1)$, (16) holds only if both $k_L < 0$ and $|k_L| > |k_R|$. $\square$

Proof of Corollary 3: Suppose (k_L, k_R) is an equilibrium BIS. By Lemma 2, $\Gamma[s^i = k_L; k_L, k_R, \mu] = 0$, where we have made the dependence of Γ on μ explicit. Below, we show that when $\mu < \mu^*$ is sufficiently close to μ^*, $\Gamma[s^i = k_L; k_L, k_R, \mu] < 0$, a contradiction. The result then follows because, from equation (4), $\Gamma[s^i = k_L; k_L, k_R, \mu]$ is decreasing in μ.

Recall equation (15) from the proof of Proposition 3,

$$\begin{aligned} \Gamma[s^i = k_L; k_L, k_R, \mu] &= -Pr(s^j > k_R|s^i = k_L)\ (E[\theta|s^j > k_R, s^i = k_L] - w + \mu) + \Delta_1(k_L; \mu) \\ &< -Pr(s^j > k_R|s^i = k_L)\ \mu + \Delta_1(k_L; \mu), \end{aligned} \tag{17}$$

where the inequality follows from (7) and affiliation. From the proof of Proposition 1, $\Delta_1(k; \mu)$ is continuous and decreasing in μ with $\Delta_1(k) < 0$ for $\mu > \mu^*$. Thus, $\lim_{\mu\to\mu^{*-}} \Delta_1(k_L) \leq 0$, where $\mu \to \mu^{*-}$ means that μ approaches μ^* from below.

Now, fix $\epsilon > 0$. For $\mu \in (\mu^* - \epsilon, \mu^*)$: (i) $\underline{k}(\mu)$ is increasing in μ (Shadmehr and Bernhardt 2011, p. 836), and hence bounded from below, and (ii) from Corollary 2, $\underline{k} < k_L < 0$, and $k_R < -\underline{k}$, and hence k_L and k_R are bounded. Thus, $Pr(s^j > k_R|s^i = k_L)$ is bounded away from 0. Combining these results, from (17), $\lim_{\mu\to\mu^{*-}} \Gamma[s^i = k_L; k_L, k_R, \mu] < 0$. $\square$

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Online Appendix: Examples

First, we present examples of signal structures that satisfy our assumptions. Then, we provide sufficient conditions (for signal structures) under which $\Delta_1(k)$ is single-peaked.

Additive Normal Distribution Signal Structure. Suppose that $s^i = \theta + \nu^i$, where θ and ν^i are independent normal random variables, with $\theta \sim N(0, \sigma^2)$ and $\nu^i \sim N(0, \sigma_\nu^2)$. The analysis of Shadmehr and Bernhardt (2011) shows with signal structure **A1** is satisfied and $\Delta_1(k)$ is single-peaked. Here, we show that this signal structure also satisfies **A2**.

Assumption A2. Conditional normal distributions are TP_n, $n \in \mathbb{N}$ (Karlin 1968), and hence are TP_3. Moreover, $\frac{dLn[Pr(k_L<s^i<k_R|\theta)]}{d\theta} = -\frac{1}{\sigma_\nu} \frac{\phi(\frac{k_R-\theta}{\sigma_\nu})-\phi(\frac{k_L-\theta}{\sigma_\nu})}{\Phi(\frac{k_R-\theta}{\sigma_\nu})-\Phi(\frac{k_L-\theta}{\sigma_\nu})}$, which is decreasing in θ, implying the logconcavity of $Pr(k_L < s^i < k_R|\theta)$ in θ. Finally, it is easy to see that, with the additive normal signal structure, $\lim_{s^i\to\infty} E[\theta|s^j, s^i] f(s^j|s^i) = \lim_{s^i\to\infty} f(s^j|s^i) = 0$ for a given s^j.

Additive Logistic Distribution Signal Structure. Suppose that $s^i = \theta + \nu^i$, where θ, ν^i and ν^j are iid according to the logistic distribution $\frac{e^{-x}}{(1+e^{-x})^2}$, where we have normalized the mean to zero without loss of generality and set the scale parameter to 1 to ease exposition. Because the logistic distribution is logconcave, θ, s^1 and s^2 are affiliated (de Castro 2010).

Assumption A1 (a). Direct calculations show that $Pr(s^j > k|s^i)$ and $E[\theta|s^j > k, s^i]$ both have closed-form expressions, and that this assumption holds.

Assumption A1 (b), and the Shape of $\Delta_1(k)$. We show that, $\Delta_1(k)$ is single-peaked when $w < 1+\mu$, and is strictly increasing when $w > 1+\mu$. To calculate $\Delta_1(k)$, we condition the terms on θ, and integrate over θ.

$$\Delta_1(k) = \int_{-\infty}^{\infty} (\theta - w + \mu)\ Pr(s^j > k|\theta)\ f(\theta|s^i = k) d\theta - \mu. \tag{18}$$

We have $Pr(s^j > k|\theta) = \int_k^\infty f(s^j|\theta) ds^j = \frac{e^\theta}{e^k + e^\theta}$. Using Bayes rule, $f(\theta|s^i = k) = \frac{f(s^i=k|\theta)\ f(\theta)}{\int_{-\infty}^{\infty} f(s^i=k|\theta) f(\theta) d\theta}$. Substituting these into equation (18), and integrating yields

$$\Delta_1(k) = \frac{1}{4}\left(k^2 \frac{k + 2(\mu - w)}{4 - e^k(2-k)^2 - k^2} + 3\frac{k + 2(\mu - w)}{1 - e^k} + 2\frac{1 - k - (\mu - w)}{2 - k}\right) - \mu. \tag{19}$$

The limiting properties of $\Delta_1(k)$ as $k \to \pm\infty$ are revealing about its shape. We have

$$\lim_{k\to-\infty} \Delta_1(k) = \lim_{k\to-\infty} \frac{1}{4}(-k + 3k + 2) - \mu = -\infty,$$

and as k increases unboundedly, the first two terms go to zero, and

$$\lim_{k\to\infty}\Delta_1(k)=\lim_{k\to\infty}\frac{1}{4}\left(2\frac{1-k-(\mu-w)}{2-k}\right)-\mu=\begin{cases}(\frac{1}{2}-\mu)^- & ;w\geq 1+\mu\\ (\frac{1}{2}-\mu)^+ & ;w<1+\mu,\end{cases}$$

where $(\frac{1}{2}-\mu)^-$ means that the function approaches its limit from below (it is increasing), and $(\frac{1}{2}-\mu)^+$ means that the function approaches its limit from above (it is decreasing). To see this, suppose k is large enough, and differentiate $\frac{a-k}{b-k}$ with respect to k to get $\frac{a-b}{(b-k)^2}$, which is negative if and only if $a<b$.[16] Thus, $\Delta_1(k)$ approaches $1/2-\mu$ from above if and only if $1-\mu+w<2$, i.e., $w<1+\mu$. Because $\lim_{k\to\infty}\Delta_1(k)=\frac{1}{2}-\mu$, we have $\lim_{k\to\infty}Pr(s^j>k|s^i=k)\ E[\theta|s^j>k,s^i=k]=\frac{1}{2}$. Direct calculations show

$$E[\theta|s^j>k,s^i=k]=-\frac{(1-k)e^{2k}+2(k^2-k-1)e^k+(k^2+3k+1)}{e^{2k}+4(1-k)e^k-(2k+5)},$$

and hence $\lim_{k\to-\infty}E[\theta|s^j>k,s^i=k]=-\infty$. Therefore, **A1 (b)**, holds.

The asymptotic behavior of $\Delta_1(k)$ means that when $w\geq 1+\mu$, $\Delta_1(k)$ cannot be single-peaked—it must either be monotone increasing, or it has at least one maximum and one minimum. From equation (19), note that $\Delta_1(k)$ and $4(\Delta_1(k)+\mu)$ have the same shape. Let $\eta\equiv\mu-w$, so that $w\geq 1+\mu$ corresponds to $\eta\leq -1$, and $w<1+\mu$ corresponds to $\eta>-1$. From equation (19),

$$\Omega(k,\eta)\equiv 4(\Delta_1(k;\eta)+\mu)=k^2\frac{k+2\eta}{4-e^k(2-k)^2-k^2}+3\frac{k+2\eta}{1-e^k}+2\frac{1-k-\eta}{2-k},\qquad(20)$$

where we make the dependency of $\Delta_1(k)$ on η explicit. Plotting $\Omega(k,\eta)$ in (k,η) space reveals that $\Delta_1(k)$ is monotone increasing when $\eta\leq -1$ (i.e., when $w\geq 1+\mu$), and that it is single-peaked when $\eta>-1$ (i.e., when $w<1+\mu$).

Assumption A2. A function $f(s^i|\theta)$ is TP_3 when s^i and θ are affiliated (TP_2), and the associated determinant is non-negative for $m=n=3$. With an additive logistic signal structure, direct calculations show that this determinant is $24\frac{e^{6(s+\theta)}}{(e^s+e^\theta)^{12}}>0$. Moreover, $\frac{dLn[Pr(k_L<s^i<k_R|\theta)]}{d\theta}=\frac{e^{(k_R+k_L)}-e^{2\theta}}{(e^{k_L}+e^\theta)(e^{k_R}+e^\theta)}$ decreases in θ, so $Pr(k_L<s^i<k_R|\theta)$ is logconcave in θ. Direct calculations show that $\lim_{s^i\to\infty}E[\theta|s^j,s^i]f(s^j|s^i)=\lim_{s^i\to\infty}f(s^j|s^i)=0$ for a given s^j.

Sufficient Conditions for $\Delta_1(k)$ to be Either Single-peaked or Strictly Increasing. Suppose $Pr(s^j>k|s^i=k)$ is decreasing in k, $Pr(s^j>k|s^i=k)$ is logconcave in k, and

[16] One can also show that $\lim_{k\to\infty}\Delta_1(k)=(\frac{1}{2}-\mu)^-$ when $w=1+\mu$.

$E[\theta|s^j > k, s^i = k] - w + \mu$ is logconcave in k whenever it is positive. Then, $\Delta_1(k)$ is either single-peaked or strictly increasing in k.

Proof: Let $h(k) = Pr(s^j > k|s^i = k)$ and $g(k) = E[\theta|s^j > k, s^i = k]$, where $h'(k) < 0 < g'(k)$ by assumption. Rewrite $\Delta_1(k)$ as $\Delta_1(k) = h(k)[g(k) - w + \mu] - \mu$. Then $\Delta_1'(k) = h'(k)[g(k) - w + \mu] + h(k)g'(k)$. If $g(k) - w + \mu \leq 0$, then $\Delta_1(k) < 0 < \Delta_1'(k)$. If, instead, $g(k) - w + \mu > 0$, then $\Delta_1'(k) = h'(k)[g(k) - w + \mu] + h(k)g'(k) = 0$ if and only if $\frac{h'(k)}{h(k)} = -\frac{g'(k)}{g(k)-w+\mu}$, which has at most one solution if $h(k)$ and $g(k) - w + \mu$ are logconcave. Therefore, if $Pr(s^j > k|s^i = k)$ is logconcave, and $E[\theta|s^j > k, s^i = k] - w + \mu$ is logconcave whenever it is positive, then $\Delta_1'(k) = 0$ has at most one solution, which must be a maximum. Thus, $\Delta_1(k)$ is either strictly increasing or single-peaked. $\square$