An unconditional proof of the André-Oort conjecture for Hilbert modular surfaces.

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1 Introduction.

The purpose of this paper is to prove the following special case of the André-Oort conjecture

Theorem 1.1 *Let S be a Hilbert modular surface and $C \subset S$ an irreducible algebraic curve containing an infinite set of special points. Then C is a special subvariety of S.*

We refer to [3] for the notions of special points and special subvarieties. For background on on Hilbert modular surfaces we refer to [3] and [16]. For generalities, history and results obtained so far on the André-Oort conjecture, we refer to [14], [15] and [8].

Theorem 1.1 was proved by Edixhoven in [3] under the assumption of the Generalised Riemann Hypothesis. Recently Klingler, Ullmo and the second author proved the André-Oort conjecture in full generality under the assumption of the GRH (see [13] and [6]). In this paper we give an unconditional proof of the André-Oort conjecture for Hilbert modular surfaces using the ideas of Pila (see [10] and [11]) and results of Peterzil-Starchenko, Pila-Wilkie, Edixhoven, Ullmo and the second author.

Let us briefly outline the strategy. Let F be a real quadratic field, O_F its ring of integers and $\Gamma := \mathrm{SL}_2(O_F)$. By Hilbert modular surface we

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mean $S = \Gamma\backslash\mathbb{H}^2$. This is a connected component of the Shimura variety $Sh_K(G,X)$ defined by the Shimura datum $(G,X) = (\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}, \mathbb{H}^{\pm 2})$ and $K = \mathrm{GL}_2(\widehat{O_F})$. One can also consider quotients of $\mathbb{H}\times\mathbb{H}$ by other congruence subgroups of $\mathrm{SL}_2(O_F)$. The André-Oort conjecture for such quotients is equivalent to the one for S (see Proposition 2.1 of [4]). Furthermore, as a subvariety is special if and only if irreducible components of its images by Hecke correspondences are special, the statement holds for a curve contained in *any* component of $Sh_K(G,X)$.

Let us outline our strategy. The Shimura variety $Sh_K(G,X)$ is a coarse moduli space for pairs (A,i) where A is an abelian surface and $i\colon O_F \longrightarrow \mathrm{End}(A)$ is a morphism. It admits a canonical model over $\mathbb{Q}$ and S is defined over a certain explicit abelian extension. Let $\pi\colon \mathbb{H}^2 \longrightarrow S$ be the uniformisation map. We choose a certain fundamental set $\mathcal{F}\subset\mathbb{H}^2$ (actually a certain part thereof) for the action of Γ. Let C be a curve in S containing an infinite set Σ of special points (in particular C is defined over a number field). We let $\mathcal{Z} := \pi^{-1}C\cap\mathcal{F}$ and $\mathcal{Z}^{alg}$ the algebraic part of $\mathcal{Z}$ i.e. the union of all real semi-algebraic subsets contained in $\mathcal{Z}$ when $\mathbb{H}\times\mathbb{H}$ is viewed as a subset of $\mathbb{R}^4$. Suppose that C is not special. A result of Ullmo and the second author then implies that $\mathcal{Z}^{alg}$ contains no special points (see section 3). A theorem of Peterzil and Starchenko shows that $\mathcal{Z}$ is definable in a certain o-minimal structure on $\mathbb{R}^4$ (see section 4). Then, by a theorem of Pila and Wilkie, the number of algebraic points on $\mathcal{Z}$ of degree at most four and up to a height T is $\ll_\epsilon T^\epsilon$ for any $\epsilon > 0$. For a special point x of S, we let (A_x, i_x) be the corresponding pair as above. The ring $\mathrm{End}_{O_F}(A_x)$ of endomorphisms commuting with the action of O_F is an order in a totally imaginary quadratic extension of F. We let $d_x := |\mathrm{Disc}(\mathrm{End}_{O_F}(A_x))|$. In section 2 we show that the height of a special point in $\mathcal{F}$ is bounded by a power of its discriminant. Hence Pila-Wilkie's result shows that the size of the Galois orbit of the special point x is $\ll_\epsilon d_x^\epsilon$ where d_x is the discriminant of x and ϵ can be chosen arbitrary small. This contradicts the result of Edixhoven who showed that the size of the Galois orbit is $\gg d_x^{1/8}$. It seems very likely that the methods of this paper generalise to the mixed case i.e. the analog of the André-Oort conjecture for the universal abelian scheme over a Hilbert modular surface. To generalise the result to the case of Hilbert modular varieties of higher dimension, one needs unconditional lower bounds for the Galois orbits of special points in terms of a positive power of the discriminant. Obtaining such bounds seems to be a very hard problem.

We would like to express our thanks to Jonathan Pila for explaining his method to us and for interesting discussions on the subject of this paper. We are extremely grateful fo Emmanuel Ullmo for several very useful discussions and remarks.

2 Bounds on the heights of special points.

In this section we give upper bounds on the height of coordinates of special points contained in a certain fundamental set in terms of 'their discriminant'. For an element α of F, we denote by α' the image of α by the non-trivial automorphism of F. A point $z = (z_1, z_2)$ of $\mathbb{H}^2$ is called special if $\pi(z)$ is a special point of S. Let $z = (z_1, z_2)$ be a special point in $\mathbb{H}^2$. Then z is fixed by a certain semisimple element of $\mathrm{SL}_2(F)$. From this it immediately follows that z_1 satisfies an equation $az_1^2 + bz_1 + c = 0$ with $a, b, c \in O_F$ and z_2 satisfies $a'z_2^2 + b'z_2 + c' = 0$. The field $K = F(z_1)$ is an imaginary quadratic extension of F.

We follow [17], section 1.1. Let $z = (z_1, z_2)$ be a point of $\mathbb{H}^2$ and consider the embedding $\mathcal{L}_z \colon F \times F \longrightarrow \mathbb{C}^2$ sending (α, β) to $(\alpha z_1 + \beta, \alpha' z_2 + \beta')$. To a point $z = (z_1, z_2)$, on associates the complex torus

$$A_z = \mathbb{C}^2 / \mathcal{L}_z(O_F \oplus \mathcal{I})$$

where $\mathcal{I}$ is an invertible rank one O_F-module contained in $O_F^{\vee}$, the $\mathbb{Z}$-dual of O_F with respect to the trace. The action of O_F on A_z is given by $m(a) \colon (\zeta_1, \zeta_2) \longrightarrow (a\zeta_1, a'\zeta_2)$. In [17], section 1.1, it is shown that A_z is a polarised abelian variety. By [16], section I.7, corollary 7.3, the abelian variety corresponding to a point x of the component S of $Sh_K(G, X)$ is $A_z = \mathbb{C}^2 / \mathcal{L}_z(O_F \oplus O_F)$ where $z \in \pi^{-1}(x)$.

Suppose that z is a special point. Denoting $\Lambda_z := \mathcal{L}_z(O_F \oplus O_F)$, we have

$$\mathrm{End}_{O_F}(A_z) = \{k \in K : k\Lambda_z \subset \Lambda_z\}$$

The ring $\mathrm{End}_{O_F}(A_z)$ is an order in K containing O_F. We first prove the following:

Lemma 2.1 *The relative discriminant ideal* $\mathrm{Disc}_{K/F}(\mathrm{End}_{O_F}(A_z))O_F$ *is generated by the* $b^2 - 4ac$ *where* $az_1^2 + bz_1 + c = 0$ *(with* $a, b, c \in O_F$*) is a quadratic equation satisfied by* z_1.

Proof. Let R be $\mathrm{End}_{O_F}(A_z)$ and let I be the ideal in O_F generated by the $b^2 - 4ac$ where $az_1^2 + bz_1 + c = 0$ (with $a, b, c \in O_F$) be any equation satisfied by z_1. For any such equation, R contains az_1 and hence the relative discriminant ideal $\mathrm{Disc}_{K/F}(R)O_F$ contains I.

To prove the other inclusion, fix a prime ideal P of O_F and let $O_{F,P}$ be the completion of O_F at P. We let M be a maximal ideal of O_K above P and K_M the completion of K with respect to the corresponding valuation. Let $az_1^2 + bz_1 + c = 0$ be an equation satisfied by z_1 with $v_P(abc)$ minimal (v_P denotes the P-adic valuation). It follows, in particular, that a, b and c are relatively prime in the ring $O_{F,P}$. Then, the proof of lemma 7.5 of [2] goes through and shows that the local order $\{k \in K_M : k(\Lambda_z \otimes O_{F,P}) \subset \Lambda_z \otimes O_{F,P}\}$ is $O_{F,P}[az_1]$. It follows that $\mathrm{Disc}_{K/F}(R)O_{F,P}$ is generated by $b^2 - 4ac$, hence is contained in $IO_{F,P}$. As this holds for all primes P, we conclude that $\mathrm{Disc}_{K/F}(R)O_F = I$ □

We write $z_i = x_i + iy_i$ and we define

$$H(z) := \max(H(x_1), H(x_2), H(y_1), H(y_2))$$

where H denotes the standard multiplicative height of an algebraic number (see [1], Chapter I). Our aim is to give an upper bound for $H(z)$ for z in a fundamental set for Γ in terms of a power of $d_z := |\mathrm{Disc}(\mathrm{End}_{O_F}(A_z))|$.

Choose an equation $az_1^2 + bz_1 + c = 0$ where a, b, c are such that the norm $|N_{F/\mathbb{Q}}(b^2 - 4ac)|$ is minimal. The above discussion shows that

$$|N_{F/\mathbb{Q}}(\mathrm{Disc}_{K/F}(\mathrm{End}_{O_F}(A_z)))| = |N_{F/\mathbb{Q}}(b^2 - 4ac)|$$

In [5], Chapter I, Proposition 2.11, it is proved that there exists a fundamental set (that is, a set containing a fundamental domain) for the action of $\Gamma = \mathrm{SL}_2(O_F)$ of the form

$$\mathcal{K} \cup V_1 \cup \ldots \cup V_h,$$

where h is the class number of F, $\mathcal{K}$ is compact and the V_i are the so-called cusp sectors.

Here V_1 is the cusp sector at infinity ∞. By definition, there is a constant $C > 0$ and $T > 0$ such that

$$V_1 = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} : y_1 y_2 > C, |x_1| \leq T, |x_2| \leq T\}$$

Noticing that on $\mathcal{K}$, y_is are bounded below and the $|x_i|$s are bounded, we may and do (after possibly altering C and T), assume that $\mathcal{K} \subset V_1$. Furthermore, for $\epsilon \in O_F^*$, the transformation $(z_1, z_2) \mapsto (\epsilon^2 z_1, \epsilon^{-2} z_2)$ is in Γ. We can therefore assume that (y_1, y_2) is in the fundamental set for the action $(y_1, y_2) \mapsto (\epsilon^2 y_1, \epsilon^{-2} y_2)$, $\epsilon \in O_F^*$. We therefore have an inequality

$$A^{-1} \leq \frac{y_i^2}{y_1 y_2} \leq A$$

where A is a constant depending on F only. For reasons explained in section five, for our purposes it is enough to consider the special points in $\mathcal{K} \cup V_1$. According to the discussion above, we consider special points in the set

$$\mathcal{F} = \{(z_1, z_2) \in \mathbb{H}^2 : y_1 y_2 > C, A^{-1} \leq \frac{y_i^2}{y_1 y_2} \leq A, |x_1| \leq T, |x_2| \leq T\}$$

Theorem 2.2 *There exists a real $c_1 > 0$ such that for any special point $z = (z_1, z_2) \in \mathcal{F}$ we have,*

$$H(z) \leq c_1 {d_z}^{1/4}$$

Remark 2.3 *The proof below generalises to the case of Hilbert modular varieties of arbitrary dimension.*

Proof. On $\mathcal{F}$, we have

$$y_1^2, y_2^2 \geq U = A^{-1} C$$

Let $a z_1^2 + b z_1 + c = 0$ with $a, b, c \in O_F$ be the equation satisfied by z_1 with $|N_{F/\mathbb{Q}}(b^2 - 4ac)|$ minimal. Then z_2 satisfies $a' z_2^2 + b' z_2 + c' = 0$ and we let $D_1 = |b^2 - 4ac|$ and $D_2 = |b'^2 - 4a'c'|$. We have $|N_{F/\mathbb{Q}}(\mathrm{Disc}_{K/F}(\mathrm{End}_{O_F}(A_z)))| = D_1 D_2$ and

$$d_z = D_1 D_2 \Delta_F^2$$

where $\Delta_F = |\mathrm{Disc}(O_F)|$.

Note that we have

$$|b| = 2|a||x_1| \leq 2T|a|, \;\; |b'| = 2|a'||x_2| \leq 2T|a'|$$

Secondly, since $D_1 = 4a^2 y_1^2$ and $D_2 = 4a'^2 y_2^2$, we have

$$|a| \leq \sqrt{\frac{D_1}{4U}}, \;\; |a'| \leq \sqrt{\frac{D_2}{4U}}$$

We have

$$H(x_1)^2 \leq |4aa'\frac{bb'}{4aa'}| = |bb'| \leq 4T^2|aa'| \leq T^2\frac{\sqrt{D_1D_2}}{U}$$

hence

$$H(x_1) \leq \frac{T}{\sqrt{U}}(D_1D_2)^{1/4} = \frac{T}{\sqrt{U\Delta_F}}d_z^{1/4}$$

Finally,

$$H(y_1)^4 \leq 16a^2a'^2 \leq \frac{D_1D_2}{U^2}.$$

Hence

$$H(y_1) \leq \frac{1}{\sqrt{U\Delta_F}}d_z^{1/4}.$$

The argument proceeds identically for x_2 and y_2. For c_1 it suffices to take $\max(\frac{T}{\sqrt{U\Delta_F}}, \frac{1}{\sqrt{U\Delta_F}})$. □

3 Characterisation of special subvarieties.

In this section we show that proper special subvarieties of S may be characterised by the property that the 'algebraic part' of their preimages in $\mathbb{H}$ contains no special points.

Let C be an irreducible algebraic curve in S and let $\mathcal{Z} := \pi^{-1}C$. Let (τ_1, τ_2) be a point of $\mathbb{H}^2$. Writing $\tau_1 = x + iy$ and $\tau_2 = u + iv$, we can view $\mathbb{H}^2$ as a subset of $\mathbb{R}^4$. A semi-algebraic subsets of $\mathbb{H}^2 \subset \mathbb{R}^4$ is by definition the intersection of a semi-algebraic subsets of $\mathbb{R}^4$ with $\mathbb{H}^2$. Following Pila, we define $\mathcal{Z}^{alg}$ to be the union of all connected *positive dimensional* semi-algebraic subsets of $\mathcal{Z}$. We also define $\mathcal{Z}^{ca}$ to be the union of positive dimensional *complex* algebraic subsets contained in $\mathcal{Z}$. The argument of the proof of Proposition 2.2 of [10] shows that

$$\mathcal{Z}^{alg} = \mathcal{Z}^{ca}$$

The characterisation we are going to use is the following.

Theorem 3.1 *If C is not special then $\mathcal{Z}^{alg}$ contains no special points.*

Proof. As remarked above $\mathcal{Z}^{ca} = \mathcal{Z}^{alg}$. Suppose that $\mathcal{Z}^{ca}$ is not empty (otherwise there is nothing to prove). Let $\mathcal{Z}'$ be an analytic component of

$\mathcal{Z}^{ca}$. As the dimension of $\mathcal{Z}$ is one, $\pi(\mathcal{Z}') = \pi(\mathcal{Z}) = C$. In particular $\pi(\mathcal{Z}')$ is an algebraic subvariety of S. By [12], $\pi(\mathcal{Z}') = C$ is a weakly special subvariety of S. By [7], theorem 4.3, a weakly special subvariety (or totally geodesic in Moonen's terminology) of a Shimura variety is special (or Hodge type) if and only if it contains a special point. Therefore, if C is not special, then C, and hence $\mathcal{Z}^{alg}$, contains no special points. □

4 Definability.

We refer to section 3 of [10] and references contained therein for notions of o-minimal structures and definability. We just mention here that an o-minimal structure on $\mathbb{R}^n$ is a collection of subsets of $\mathbb{R}^n$ which contains all semi-algebraic subsets, stable under the natural set-theoretic operations and satisfy certain geometric finiteness properties.

In what follows we consider the o-minimal structure $\mathbb{R}_{an,exp}$ which is generated by $\mathbb{R}_{an}$ and $\mathbb{R}_{exp}$. Here $\mathbb{R}_{an}$ is the structure afforded by the so-called globally subanalytic sets and $\mathbb{R}_{exp}$ is the structure consiting of the sets defined by the exponential. In what follows by 'definable' we mean definable in $\mathbb{R}_{an,exp}$. A function from $A \subset \mathbb{R}^n$ to $B \subset \mathbb{R}^m$ is said to be definable if its graph in $A \times B \subset \mathbb{R}^{n+m}$ is definable.

We will use recent results of Peterzil-Starchenko (see [9]) which we now describe. We follow section 6.3 of [9]. Let Sp_{2g} be the algebraic group (over $\mathbb{Q}$) of symplectic $2g \times 2g$ matrices with determinant one. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the Siegel upper half space $\mathbb{H}_g$. There exists a *semi-algebraic* subset $\mathcal{F}_g \subset \mathbb{H}_g$ which contains finitely many representatives for each orbit of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (hence $\mathcal{F}_g$ contains a fundamental domain). For $(a, b) \in \mathbb{R}^g$, let $\vartheta_{(a,b)}$ be the corresponding theta function on $\mathbb{H}_g$, using the notations of [9]. By definition

$$\vartheta_{(a,b)}(\tau, z) = \sum_{m \in \mathbb{Z}^g} e^{i\pi((m-a)^t \tau (m+a) + 2(m-a)^t (z+b))}$$

for $\tau \in \mathbb{H}_g$ and z in the fundamental domain of the lattice in $\mathbb{C}^g$ defined by τ.

A special case of the result of Peterzil and Starchenko (theorem 6.5 of [9]), relevant to us, is the following:

Theorem 4.1 (Peterzil-Starchenko) *For every $(a, b) \in \mathbb{R}^g$, $\vartheta_{a,b}(z, 0)$, $z \in \mathcal{F}_g$ is definable.*

For a subset $\mathcal{Z} \subset \mathbb{R}^n$, as in the previous section, the algebraic part of Z, denoted by $\mathcal{Z}^{alg}$, is the union of all connected positive-dimensional semi-algebraic subsets of $\mathcal{Z}$. Following Pila, we also denote for $T \geq e$,

$$Z(k,T) = \{z \in \mathcal{Z}(k) : [\mathbb{Q}(z_i) : \mathbb{Q}] \leq k, \max_i H(z_i) \leq T\} \text{ and } N_k(\mathcal{Z},T) := |Z(k,T)|$$

Theorem 4.2 (Pila-Wilkie) *Let $\mathcal{Z} \subset \mathbb{R}^n$ be a set definable in an o-minimal structure over $\mathbb{R}$. Let $k \geq 1$ and $\epsilon > 0$, there exists $c(\mathcal{Z}, k, \epsilon)$,*

$$N_k(\mathcal{Z} \backslash \mathcal{Z}^{alg}, T) \leq c(\mathcal{Z}, k, \epsilon) T^{\epsilon}$$

The consequence of the results of Peterzil-Starchenko and Pila-Wilkie relevant to us is the following:

Theorem 4.3 *Let C be an irreducible algebraic curve contained in S. Let $\mathcal{F}$ be the set as in section 2. Suppose that $\mathcal{Z} := \pi^{-1}C \cap \mathcal{F}$ is positive dimensional.*

Then for $k \geq 1$ and $\epsilon > 0$, there exists $c(\mathcal{Z}, k, \epsilon)$,

$$N_k(\mathcal{Z} \backslash \mathcal{Z}^{alg}, T) \leq c(\mathcal{Z}, k, \epsilon) T^{\epsilon}$$

Proof. Let

$$S = \Gamma \backslash \mathbb{H}^2 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_2$$

be the modular embedding (see [16], Chapter IX, §1). This embedding is induced by an equivariant embedding $\phi \colon \mathbb{H}^2 \longrightarrow \mathbb{H}_2$. After, if necessary replacing the set $\mathcal{F}_g$ with a finite union of its images by some $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ (this does not affect the conclusion of Peterzil-Starchenko's theorem), we assume that $\phi(\mathcal{F}) \subset \mathcal{F}_g$. The set $\mathcal{F}$ is definable (it is semi-algebraic).

The functions $\vartheta_{a,b}(\tau, 0)$ restricted to $\mathbb{H}^2$ induce a Γ-equivariant holomorphic embedding of S into some $\mathbb{P}^N(\mathbb{C})$. As C is an algebraic curve, its image in $\mathbb{P}^N(\mathbb{C})$ is given by a collection of polynomial equations in the $\vartheta_{(a,b)}(\tau, 0)$ with $\theta \in \mathcal{F}$. It follows from the Paterzil-Starchenko theorem, that the set $\mathcal{Z}$ is definable. The conclusion now follows from the Pila-Wilkie theorem 4.2. □

5 Proof of the main result.

Let C be a curve in S containing an infinite set Σ of special points. Suppose that the closure of C in $\overline{S}$ (Baily-Borel compactification of S) contains a cusp

P. After, if necessary, replacing C by a component of its image by suitable Hecke correspondence (which does not affect the property of C being special), we assume that $P = \infty$. Let $\mathcal{F}$ be the subset of $\mathbb{H} \times \mathbb{H}$ as in section 2. Our assumption that $P = \infty$ implies that after possibly replacing Σ by an infinite subset, the preimages of the points in Σ lie in $\mathcal{F}$. Suppose that C is not special. Let $\mathcal{Z}^{alg}$ be as in section 3 i.e the algebraic part of $\mathcal{Z} := \mathcal{F} \cap \pi^{-1}C$. Then $\mathcal{Z}^{alg}$ contains no special points of $\mathbb{H} \times \mathbb{H}$ by 3.1.

Let z be a point in $\mathcal{F}$ such that $x := \pi(z) \in \Sigma$. We write d_x for the discriminant d_z and A_x for the isomorphism class of the abelian variety A_z as in section 2. Notice that d_x and A_x depend on x only and not on the choice of a point in $\pi^{-1}(x)$.

As special points are $\overline{\mathbb{Q}}$-valued, C is defined over $\overline{\mathbb{Q}}$ and we can choose a number field L such that C is defined and geometrically irreducible over L Hence for all $x \in \Sigma$, $\mathrm{Gal}(\overline{\mathbb{Q}}/L) \cdot x$ is contained in C.

Let x be a point in Σ. By theorem 6.2 of [3],

$$|Gal(\overline{\mathbb{Q}}/L) \cdot x| \geq c_2 d_x^{1/8}$$

for some absolute constant $c_2 > 0$.

As the points in $Gal(\overline{\mathbb{Q}}/L) \cdot x$ have the same discriminant d_x, for any $z \in \mathcal{Z}$ such that $\pi(z) \in Gal(\overline{\mathbb{Q}}/L) \cdot x$, we have by 2.2

$$H(z) \leq c_1 d_x^{1/4}$$

On the other hand, by the theorem 4.3, we have for every $\epsilon > 0$,

$$N_4(\mathcal{Z} \backslash \mathcal{Z}^{alg}, c_1 d_x^{1/4}) = N_4(\mathcal{Z}, c_1 d_x^{1/4}) \leq c_\epsilon c_1^\epsilon d_x^{\epsilon/4}$$

for some c_ϵ depending on $\mathcal{Z}$ and ϵ only. It follows that

$$c_\epsilon c_1^\epsilon d_x^{\epsilon/4} \geq c_2 d_x^{1/8}$$

i.e

$$d_x^{\frac{1-2\epsilon}{8}} \leq \frac{c_\epsilon c_1^\epsilon}{c_2}$$

Notice that d_x tends to infinity when x ranges through Σ. Indeed there are only finitely many orders of degree two over O_F with a given discriminant and for each of these orders there are finitely many special points x with $\mathrm{End}_{O_F}(A_x)$ isomorphic to it.

Choose any $0 < \epsilon < \frac{1}{2}$. Then the left hand side of the last inequality goes to infinity as x ranges through Σ while the right hand side remains bounded. This yields a contradiction, hence C is a special subvariety.

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