Open Research Online

The Open University's repository of research publications
and other research outputs

Hyperbolic geometry and the Hillam-Thron theorem

Journal Item

For guidance on citations see FAQs.

Version: Accepted Manuscript

oro.open.ac.uk

HYPERBOLIC GEOMETRY AND THE HILLAM-THRON THEOREM

IAN SHORT

ABSTRACT. Every open ball within $\mathbb{R}_\infty^N$ has an associated hyperbolic metric and Möbius transformations act as hyperbolic isometries from one ball to another. The Hillam-Thron Theorem is concerned with images of balls under Möbius transformation, yet existing proofs of the theorem do not make use of hyperbolic geometry. We exploit hyperbolic geometry in proving a generalisation of the Hillam-Thron Theorem and examine the precise configurations of points and balls that arise in that theorem.

1. INTRODUCTION

An *(infinite complex) continued fraction* is a formal expression

$$(1.1) \qquad \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}},$$

where the a_i and b_j are complex numbers and no a_i is equal to 0. This continued fraction will be denoted by $\mathbf{K}(a_n | b_n)$. We define Möbius transformations $t_n(z) = a_n/(b_n + z)$, for $n = 1, 2, \ldots$, and let $T_n = t_1 \circ \cdots \circ t_n$. The continued fraction is said to *converge classically* if the sequence $T_1(0), T_2(0), \ldots$ converges. Observe that $t_n(\infty) = 0$ for every $n \in \mathbb{N}$, which is equivalent to $T_n(\infty) = T_{n-1}(0)$ for each $n \in \mathbb{N}$, with the convention that T_0 is the identity map.

The Hillam-Thron Theorem is stated in [5, Theorem 4.37] as follows.

Theorem A. *Let D be the circular region defined by*

$$(1.2) \qquad D = \{w : |w - c| < r\}, \ \text{where } |c| < r.$$

Let the continued fraction $\mathbf{K}(a_n | b_n)$ be such that

$$(1.3) \qquad t_n(D) \subseteq D, \quad n = 1, 2, \ldots,$$

where $t_n(z) = a_n/(b_n + z)$. Then the continued fraction converges to a value $f \in \overline{D}$.

It is our intention to establish a generalisation of Theorem A in a geometric context that is independent of dimension and is conjugation invariant. To this end, we first interpret Theorem A within this framework. In geometric terms, equation (1.2) says that D is an open Euclidean disc that contains the origin within its interior. The transformations t_n map ∞, which is strictly exterior to $\overline{D}$, to 0, which is strictly interior to D. This is of particular significance to the Hillam-Thron Theorem; other results of continued fraction theory such as Van Vleck's Theorem and the Parabola Theorem also involve transformations t_n mapping certain sets D within themselves, but with these results the points ∞ and 0 do not lie on opposite sides of the boundary ∂D. We can generalise this strong condition associated with the Hillam-Thron Theorem as follows. Let D be any open ball in $\mathbb{R}_\infty^N = \mathbb{R}^N \cup \{\infty\}$, $N \geqslant 1$, defined with the chordal metric (the chordal metric is described in §2). Choose a point $a \in D$ and another point $b \in \mathbb{R}_\infty^N \setminus \overline{D}$. An open ball in the chordal metric is either a Euclidean half-space, one of the components of the complement of an $(N-1)$-dimensional Euclidean hypersphere, or $\mathbb{R}_\infty^N$ itself (the final possibility is implicitly excluded from proceedings through the assumption that $b \in \mathbb{R}_\infty^N \setminus \overline{D}$). We work with sequences t_n of N-dimensional Möbius transformations that satisfy $t_n(b) = a$ and $t_n(D) \subseteq D$, for all $n \in \mathbb{N}$. The hypotheses of Theorem A may be recovered upon choosing $a = 0$, $b = \infty$ and $N = 2$, and declaring D to be a Euclidean disc.

We have now described the assumptions of Theorem A in geometric terms that make sense in all dimensions. In this paper we amend the assumption that $t_n(D) \subseteq D$ in the following manner. Equation (1.4) ensures that

$$(1.4) \qquad \overline{D} \supseteq T_1(\overline{D}) \supseteq T_2(\overline{D}) \supseteq \cdots .$$

It is intuitively clear, and will subsequently be proved, that the intersection of this nested sequence of closed discs is itself a closed disc. The discs $T_n(\overline{D})$ converge, in a sense that will later be made precise, to this intersection of closed discs. It is our contention that it is the convergence of $T_n(\overline{D})$ that is significant in the Hillam-Thron Theorem, not the nested requirement. Moreover, all the geometry should be set within $\mathbb{C}_\infty$ (or $\mathbb{R}_\infty^N$ in higher dimensions), rather than $\mathbb{C}$, since Möbius transformations are conformal bijections of the former space and not the latter. We state our generalisation of Theorem A bearing in mind all the above modifications.

Theorem 1.1. *Let D be an open ball in $\mathbb{R}_\infty^N$, and choose two points $a \in D$ and $b \in \mathbb{R}_\infty^N \setminus \overline{D}$. Let $T_1, T_2, \ldots$ be N-dimensional Möbius transformations that satisfy*

(i) $T_n(b) = T_{n-1}(a)$, *for* $n = 1, 2, \ldots$;
(ii) $T_n(\overline{D})$ *converges to a compact set* $X \neq \mathbb{R}^N_\infty$.

Then T_n *converges locally uniformly within* D *to a point.*

If X consists of a single point x, then in fact T_n converges uniformly within $\overline{D}$ to x, and given a, b, D and X, the set $\overline{D} \cup \{b\}$ is the largest set on which pointwise convergence of T_n to x is assured. If X is a closed ball of positive chordal radius then T_n converges locally uniformly to a point x on the complement of ∂D, and given a, b, D and X, the complement of ∂D is the largest set on which pointwise convergence of T_n to x is assured. Theorem 1.1 and these stronger deductions are proved in §4. The best possibility of the theorem is discussed in §5.

Condition (ii) of Theorem 1.1 is that the sequence $T_n(\overline{D})$ converges to X with the chordal Hausdorff metric, which may be defined on the set of compact subsets of $\mathbb{R}^N_\infty$. This metric is defined in §2 and there we outline the basic theory of convergence of closed balls within $\mathbb{R}^N_\infty$. The limit of this sequence $T_n(\overline{D})$ must be another closed ball, although we have excluded the possibility that $X = \mathbb{R}^N_\infty$ in Theorem 1.1 as the result fails in that case. To see this, choose $T_{2n-1}(z) = 1/(nz)$ and $T_{2n}(z) = nz$, with D the unit disc in $\mathbb{C}$, $a = 0$ and $b = \infty$. Then it is easily proven (after the chordal Hausdorff metric has been defined in §2) that $T_n(\overline{D})$ converges to $\mathbb{C}_\infty$, whilst T_n diverges at every point of $\mathbb{C}_\infty$. It can never happen that X is equal to $\mathbb{C}_\infty$ in the classic Hillam-Thron Theorem as the nested condition (1.4) ensures that $X \subseteq D$.

Theorem A may be recovered from Theorem 1.1 upon restricting D to be a Euclidean disc, choosing $N = 2$, $a = 0$, $b = \infty$, and assuming that the balls $T_n(D)$ are nested. Unlike Theorem A, our result is conjugation invariant, in the sense that the sequence $g \circ T_n \circ g^{-1}$, where g is an N-dimensional Möbius transformation, also satisfies the hypotheses and conclusion of Theorem 1.1, but with alternative associated ball and points $g(D)$, $g(a)$ and $g(b)$.

Theorem 1.1 is a result about Möbius transformations mapping balls to other balls. Each open chordal ball $D \subsetneq \mathbb{R}^N_\infty$ admits a hyperbolic metric that we denote by ρ_D. Any Möbius transformation f that maps such an open ball D to another open ball E is an isometry from the metric space (D, ρ_D) to the metric space (E, ρ_E). Thus the geometry of the hypotheses of Theorem A and Theorem 1.1 is hyperbolic, although the deduction of both theorems is Euclidean convergence. (More precisely, it is convergence in the chordal metric, but the chordal metric is locally equivalent to the Euclidean metric in $\mathbb{R}^N$.) This interaction between hyperbolic and Euclidean metrics is reflected in our proof of Theorem 1.1. In contrast, the standard proofs of the Hillam-Thron Theorem have little geometric insight and generally consist of opaque

algebraic manipulations. Throughout this article we assume that the reader has basic knowledge of standard properties of Möbius transformations and the hyperbolic metric that can be found in [1] and [6].

To appreciate the geometric simplicity of the principle behind Theorem 1.1, we encourage the reader to first comprehend Theorem 1.2, which is an extension of [2, Theorem 1.1]. In Theorem 1.2 we use hyperbolic geometry to calculate the precise locations of the balls $T_n(D)$ and points $T_n(a)$ that arise in Theorem 1.1. With this precision attained, the proof of Theorem 1.1 is then a careful exercise in converting the exact hyperbolic distance measurements to Euclidean distance estimates that are necessary to establish Euclidean convergence.

For the purposes of concise exposition, we encapsulate the hypotheses of Theorem 1.1 in a single definition. Let a *Hillam-Thron sequence* (T_n, a, b, D) be a sequence of Möbius transformations T_n, an open ball $D \subsetneq \mathbb{R}_\infty^N$ and points $a \in D$ and $b \in \mathbb{R}_\infty^N \setminus \overline{D}$, such that $T_n(\overline{D})$ converges in the chordal Hausdorff metric to a limit that is not $\mathbb{R}_\infty^N$, and such that $T_n(b) = T_{n-1}(a)$, for $n = 1, 2, \ldots$. We use the notation ι_D for inversion in the boundary ∂D of D (ι_D is a Euclidean reflection if D is a half-space). If f is a Möbius transformation that maps one open ball D to another open ball E, then for any $z \in D$

$$f(\iota_D(z)) = \iota_E(f(z))$$

(see [1, Theorem 3.2.5]).

Theorem 1.2. *Let $D_n \subsetneq \mathbb{R}_\infty^N$, $n = 1, 2, \ldots$, be a sequence of open balls such that $\overline{D_n}$ converges to a compact set $X \neq \mathbb{R}_\infty^N$. Choose a constant $k \geqslant 0$. Let $z_1, z_2, \ldots$ be a sequence of points in $\mathbb{R}_\infty^N$ such that for each $n \geqslant 2$,*

    (i) $z_{n-1} \in D_{n-1} \setminus \overline{D_n}$;
    (ii) $\rho_{D_n}(\iota_{D_n}(z_{n-1}), z_n) = k$.

Then there is a Hillam-Thron sequence (T_n, a, b, D) with $T_n(D) = D_n$ and $T_n(a) = z_n$, for $n = 1, 2, \ldots$. Conversely, the sequence of balls $D_n = T_n(D)$ and points $z_n = T_n(a)$ associated with any Hillam-Thron sequence (T_n, a, b, D) satisfy the above conditions with $k = \rho_D(a, \iota_D(b))$.

Theorem 1.2 extends [2, Theorem 1.1], which is a result of a similar nature that applies to *Pringsheim's Theorem*. Pringsheim's Theorem is commonly recognised as the special case of the Hillam-Thron Theorem when D is the unit disc (and $a = 0$ and $b = \infty$). From a geometric perspective, Pringsheim's Theorem should be considered as the special case of the Hillam-Thron Theorem when a and b are inverse points with respect to ∂D. In Theorem 1.2 and the proof of Theorem 1.1, this

amounts to choosing $k = 0$, which negates the need to introduce hyperbolic geometry, making Pringsheim's Theorem a significantly simpler special case of the Hillam-Thron Theorem.

All of our results are true for $N = 1, 2, \ldots$. When we speak of the hyperbolic metric of an open ball D in $\mathbb{R}_\infty$, we refer to the restriction to D of the two-dimensional hyperbolic metric of the open chordal ball E in $\mathbb{C}_\infty$, where E is the unique open chordal ball in $\mathbb{C}_\infty$ such that $E \cap \mathbb{R}_\infty = D$ and such that ∂E cuts $\mathbb{R}_\infty$ orthogonally.

2. Convergence of sequences of balls in $\mathbb{R}_\infty^N$

The purpose of this section is to formalise the notion of convergence of a sequence of closed balls in $\mathbb{R}_\infty^N$. The results are all intuitive and the proofs are deliberately terse as they are straightforward.

It is simplest to first consider what it means for a sequence of balls in $\mathbb{R}^N$ to converge. To this end, we define the *Euclidean Hausdorff metric* α by the equation

$$\alpha(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b| + \sup_{b \in B} \inf_{a \in A} |a - b|,$$

for compact subsets A and B of $\mathbb{R}^N$. The metric α is complete on the set of compact subsets of $\mathbb{R}^N$ (see [4, Theorem 2.4.4] for proof).

Lemma 2.1. *If A is the closed Euclidean ball with centre c and radius $r \geqslant 0$ and B is the closed Euclidean ball with centre d and radius $s \geqslant 0$, then*

$$\alpha(A, B) = |c - d| + \max\{|c - d|, |r - s|\}.$$

Proof. This follows from adding the equation

$$\sup_{a \in A} \inf_{b \in B} |a - b| = \max\{|c - d| + r - s, 0\}$$

to a similar equation for $\sup_{b \in B} \inf_{a \in A} |a - b|$. $\square$

Corollary 2.2. *Let $B_1, B_2, \ldots$ and B be closed Euclidean balls with centres $c_1, c_2, \ldots$ and c, and non-negative radii $r_1, r_2, \ldots$ and r. Then B_n converges to B in the Euclidean Hausdorff metric if and only if $c_n \to c$ and $r_n \to r$ as $n \to \infty$.*

Corollary 2.3. *Let $B_1, B_2, \ldots$ be closed Euclidean balls that converge in the Euclidean Hausdorff metric to a compact set B. Then B is also a closed Euclidean ball.*

Proof. Define c_n and r_n to be the centre and radius of B_n, for $n = 1, 2, \ldots$. Lemma 2.1 may be applied to show that c_n and r_n are Cauchy sequences. Completeness of $\mathbb{R}^N$ ensures that c_n and r_n both converge, then our result may be deduced from Corollary 2.2. $\square$

We now switch to working with closed balls in $\mathbb{R}_\infty^N$. The *chordal metric* σ on $\mathbb{R}_\infty^N$ is defined by identifying $\mathbb{R}_\infty^N$ with the N-dimensional unit sphere $\mathbb{S}^N$ through stereographic projection, then transferring the Euclidean metric on $\mathbb{S}^N$ over to $\mathbb{R}_\infty^N$ via this bijection. See [1] for details. The metric σ is complete, since the Euclidean metric is complete when restricted to closed sets. Formulae for the chordal metric follow:

$$\sigma(x,y) = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad \sigma(x,\infty) = \frac{2}{\sqrt{1+|x|^2}},$$

where x and y are distinct points in $\mathbb{R}^N$. The chordal and Euclidean metrics are locally equivalent within $\mathbb{R}^N$. We define the *chordal Hausdorff metric* β by the equation

$$\beta(A,B) = \sup_{a\in A}\inf_{b\in B}\sigma(a,b) + \sup_{b\in B}\inf_{a\in A}\sigma(a,b),$$

for compact subsets A and B of $\mathbb{R}_\infty^N$. The metric β is complete on the set of compact subsets of $\mathbb{R}_\infty^N$ (see [4, Theorem 2.4.4] for proof).

Lemma 2.4. *If f is a Möbius map of $\mathbb{R}_\infty^N$ and $K_1, K_2, \ldots$ is a sequence of compact sets in $\mathbb{R}_\infty^N$ that converges to another compact set K in the chordal Hausdorff metric, then $f(K_n)$ converges to $f(K)$ in the chordal Hausdorff metric.*

Proof. This is true as f satisfies a Lipschitz condition

$$\frac{1}{k_f}\sigma(x,y) \leqslant \sigma(f(x),f(y)) \leqslant k_f\sigma(x,y),$$

for some $k_f > 0$ and all $x,y \in \mathbb{R}_\infty^N$. We refer the reader to [1, Theorem 3.6.1] for information about this Lipschitz condition for Möbius maps. $\square$

Lemma 2.5. *Let $B_1, B_2, \ldots$ be closed Euclidean balls and let B be a compact subset of $\mathbb{R}^N$. Then $B_n \to B$ as $n \to \infty$ in the Euclidean Hausdorff metric if and only if $B_n \to B$ as $n \to \infty$ in the chordal Hausdorff metric.*

Proof. This follows quickly from local equivalence of the Euclidean and chordal metrics within $\mathbb{R}^N$. $\square$

Lemma 2.6. *The limit B of a convergent sequence of closed chordal balls $B_1, B_2, \ldots$ in the chordal Hausdorff metric is itself a closed chordal ball (this includes the possibilities that B is a single point or the whole of $\mathbb{R}_\infty^N$).*

Proof. If $B \neq \mathbb{R}_\infty^N$, there is a point $w \notin B$. Choose a Möbius map f such that $f(w) = \infty$. Then $f(B_n)$ converges to $f(B)$ by Lemma 2.4 and for large enough n, the $f(B_n)$ are Euclidean balls. Lemma 2.5 shows that $f(B_n)$ converges to $f(B)$ in the Euclidean Hausdorff metric, therefore

Corollary 2.3 shows that $f(B)$, and hence B, are closed chordal balls.
$\square$

It remains to show that a nested sequence of closed chordal balls converges, so that Theorem 1.1 includes Theorem A.

Lemma 2.7. *Let* $B_1 \supseteq B_2 \supseteq \cdots$ *be closed chordal balls. Then* B_n *converges to* $\bigcap_{n=1}^{\infty} B_n$ *in the chordal Hausdorff metric.*

Proof. Through application of Lemma 2.4 and Lemma 2.5, it suffices to prove Lemma 2.7 for sequences of closed Euclidean balls B_n. Let B_n have centre c_n and radius r_n, for $n = 1, 2, \ldots$. The inclusion $B_n \supseteq B_{n+1}$ is equivalent to

$$r_{n+1} \leqslant r_n, \quad |c_n - c_{n+1}| \leqslant r_n - r_{n+1},$$

from which we can deduce that both c_n and r_n converge. Thus B_n converges to a limit ball B by Corollary 2.2, which is easily seen to be equal to $\bigcap_{n=1}^{\infty} B_n$. $\square$

3. Proof of Theorem 1.2

The proof of Theorem 1.2 precedes the proof of Theorem 1.1 as the geometry of the former theorem provides motivation for the proof of the latter. Figure 1 shows the first few points z_n and balls D_n associated with a Hillam-Thron sequence. We have used the notation z_{n-1}^* for the point $\iota_{D_n}(z_{n-1})$. The dashed circles about the points z_n^* represent hyperbolic spheres in D_n of radius k centred on z_n. The convergence of $z_n = T_n(a)$ in Theorem 1.1 is suggested by continuation of this diagram for higher integers n when the balls D_n are close to the limit ball X. (It is then a short step in hyperbolic geometry from convergence at the point $a \in D$ to locally uniform convergence within D.)

The next lemma in hyperbolic geometry is pivotal in the proof of Theorem 1.2 and it is also used in the examples of §5. We give only a sketch proof as it is an elementary exercise in hyperbolic geometry.

Lemma 3.1. *Choose two open chordal balls* $A, B \subsetneq \mathbb{R}_\infty^N$, *and two points* $a_1 \in A$ *and* $b_1 \in B$. *Choose two further points* a_2 *and* b_2 *such that either* $a_2 \in \partial A$ *and* $b_2 \in \partial B$, *or* $a_2 \in A$ *and* $b_2 \in B$ *with* $\rho_A(a_1, a_2) = \rho_B(b_1, b_2)$. *Then there exists an* N-*dimensional Möbius map* f *with* $f(A) = B$, $f(a_1) = b_1$ *and* $f(a_2) = b_2$.

Sketch proof. It suffices to prove the result when B is the unit ball $\mathbb{B}^N$ and $b_1 = 0$. Choose any Möbius map g with $g(A) = \mathbb{B}^N$ and $g(a_1) = 0$, then choose an orthogonal map h with $h(g(a_2)) = b_2$. Such an orthogonal map clearly exists when $a_2 \in \partial A$, and it exists when $a_2 \in A$ since, by preservation of hyperbolic distance

$$\rho_{g(A)}(0, g(a_2)) = \rho_A(a_1, a_2) = \rho_{\mathbb{B}^N}(0, b_2),$$
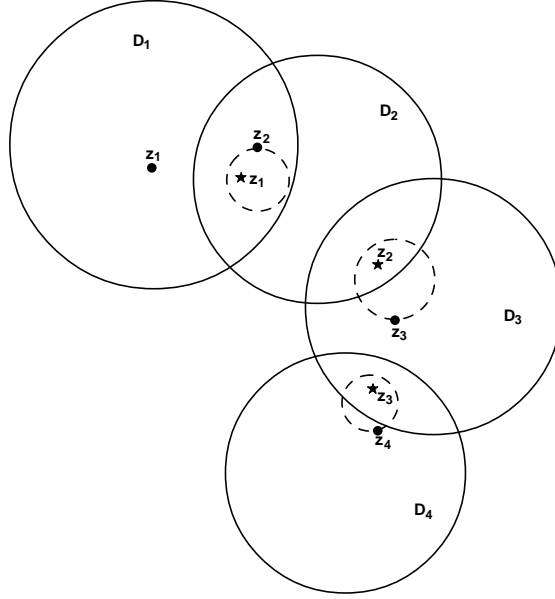
FIGURE 1. Typical geometry of Theorem 1.2.

so that $|g(a_2)| = |b_2|$. The map $f = h \circ g$ has the required properties.
$\square$

Proof of Theorem 1.2. Suppose that we are given sequences D_n and z_n and a constant $k \geqslant 0$ as described in Theorem 1.2. We define a Hillam-Thron sequence (T_n, a, b, D) with $T_n(D) = D_n$ and $T_n(a) = z_n$, for $n = 1, 2, \dots$. Choose a point $a \notin \overline{D_1}$, with $\rho_{D_1}(\iota_{D_1}(a), z_1) = k$, then choose any open ball D containing a and any point $b \notin \overline{D}$ with $\rho_D(a, \iota_D(b)) = k$.

Define $z_0 = a$, then for each $n \in \mathbb{N}$: $z_n \in D_n$, $\iota_{D_n}(z_{n-1}) \in D_n$ and $\rho_{D_n}(\iota_{D_n}(z_{n-1}), z_n) = \rho_D(a, \iota_D(b))$, therefore Lemma 3.1 may be applied to deduce the existence of a Möbius map T_n with $T_n(D) = D_n$, $T_n(a) = z_n$ and $T_n(\iota_D(b)) = \iota_{D_n}(z_{n-1})$. Preservation of inverse points ensures that $T_n(b) = z_{n-1}$ so that (T_n, a, b, D) is the required Hillam-Thron sequence.

It remains to check that a given Hillam-Thron sequence (T_n, a, b, D) satisfies the conditions of Theorem 1.2. Let $D_n = T_n(D)$ and $z_n = T_n(a)$, then since (T_n, a, b, D) is a Hillam-Thron sequence, $\overline{D_n}$ converges to a closed ball which is not $\mathbb{R}^N_\infty$. As $a \in D$ whilst $b \notin \overline{D}$, we have that $z_{n-1} = T_{n-1}(a) = T_n(b) \in D_{n-1} \setminus \overline{D_n}$. This is condition (i). Condition (ii) is also quickly verified with $k = \rho_D(\iota_D(b), a)$, since the Möbius map T_n is a hyperbolic isometry from D to D_n and it preserves inverse points between these two balls, so that

$$\rho_{D_n}(\iota_{D_n}(z_{n-1}), z_n) = \rho_{D_n}(T_n(\iota_D(b)), T_n(a)) = \rho_D(\iota_D(b), a),$$

as required.

□

The beauty of Theorem 1.2 is that it allows us to understand the Hillam-Thron Theorem in terms of sequences of points and balls. One can construct analogous results for other continued fraction theorems such as Van Vleck's Theorem and the Parabola Theorem, but the absence of a nesting condition (1.4) in Theorem 1.2 distinguishes it from the geometry that is associated with these classic theorems. Whilst Theorem 1.2 allows one to construct all possible Hillam-Thron sequences in a geometric manner whether the nesting condition is satisfied or not, we have yet to discount the unlikely possibility that every given T_n that does not satisfy (1.4), does in fact satisfy (1.4) for a different choice of a, b and D. Examples that remove this possibility are plentiful. For instance, let $t_1, t_2, \ldots$ be Euclidean rotations that map -1 to 0, and choose any open Euclidean disc D with $0 \in D$ and $-1 \notin \overline{D}$. (The maps t_n are of the form $t_n(z) = \alpha_n(z + 1)$, for $\alpha_n \in \mathbb{C}$ such that $|\alpha_n| = 1$.) Define $T_n = t_1 \circ \cdots \circ t_n$, then $(T_n, 0, -1, D)$ is a Hillam-Thron sequence provided that the α_n are chosen suitably such that $T_n(\overline{D})$ converges. On the other hand, T_n cannot satisfy (1.4) for a different choice of D since the maps t_n are elliptic and thus cannot map a disc strictly inside itself.

It will usually be difficult to apply Theorem 1.1 to a particular given continued fraction because one must find a disc D for which $T_n(\overline{D})$ converges. That may very well be a more troublesome task than determining that $T_n(z)$ converges through some other means. Of course, convergence of $T_n(\overline{D})$ is guaranteed by the condition $t_n(D) \subseteq D$ of the classic Hillam-Thron Theorem (see Lemma 2.7).

4. Proof of Theorem 1.1

In Proposition 4.2 we supply a set of conditions that ensure that a sequence x_n in $\mathbb{R}^N$ converges, then we show that the sequence $T_n(b)$ of Theorem 1.1 satisfies these conditions. Lemma 4.1 contains the key geometric step in Proposition 4.2.

Lemma 4.1. *Let a, b and c be three distinct points in $\mathbb{R}^N$ such that the angle θ between the segments $[a, b]$ and $[a, c]$ lies in the interval $[0, \pi/2)$ and such that $|a - c| \leqslant |a - b| \cos \theta$. Then*

$$|a - b| - |b - c| \geqslant \tfrac{1}{3} \cos \theta |a - c|.$$

Proof. If $\theta = 0$ then $c \in (a, b)$ and the result is clearly true. If $\theta \in (0, \pi/2)$ we may apply the Cosine Rule to the triangle with vertices a, b and c to yield,

$$|a - b|^2 - |b - c|^2 = 2|a - b||a - c| \cos \theta - |a - c|^2.$$

Using this equation and the inequality $|a - c| \leqslant |a - b| \cos \theta$ we obtain

$$
\begin{aligned}
|a - b| - |b - c| &= \left(\frac{2|a - b| \cos \theta - |a - c|}{|a - b| + |b - c|} \right) |a - c| \\
&\geqslant \left(\frac{2|a - b| \cos \theta - |a - b| \cos \theta}{|a - b| + |a - b| + |a - b| \cos \theta} \right) |a - c|,
\end{aligned}
$$

from which the result follows. $\qquad\square$

Proposition 4.2. *Choose a constant $\theta \in [0, \pi/2)$ and let $c_1, c_2, \ldots$ and $x_1, x_2, \ldots$ be sequences in $\mathbb{R}^N$ with $c_n \to 0$ and $|x_n| \to 1$ as $n \to \infty$. Suppose that there is a natural number N such that for every $n \geqslant N$,*

  (i) *the angle θ_n between the Euclidean line segments $[x_n, x_{n+1}]$ and $[x_n, c_n]$ satisfies $0 \leqslant \theta_n \leqslant \theta$;*
  (ii) *$|x_n - x_{n+1}| \leqslant |x_n - c_n| \cos \theta_n$.*

Then $x_1, x_2, \ldots$ converges.

Proof. Since $|x_n| - |c_n| \leqslant |x_n - c_n| \leqslant |x_n| + |c_n|$, it is true that $|x_n - c_n| \to 1$, and similarly $|x_{n+1} - c_n| \to 1$, as $n \to \infty$. Choose $M \geqslant N$ such that whenever $n \geqslant M$ the next set of four inequalities hold,

$$
|x_n| \leqslant 2, \quad |x_n - c_n| \geqslant 1/2, \quad |x_{n+1} - c_n| \geqslant 1/2, \quad |c_n| \leqslant \tfrac{1}{12} \cos \theta.
$$

From Lemma 4.1 we know that

$$
\tfrac{1}{3} \cos \theta |x_{n+1} - x_n| \leqslant |x_n - c_n| - |x_{n+1} - c_n|,
$$

therefore for $n \geqslant M$,

$$
\begin{aligned}
\tfrac{1}{3} \cos \theta |x_{n+1} - x_n| &\leqslant |x_n - c_n|^2 - |x_{n+1} - c_n|^2 \\
&= |x_n|^2 - |x_{n+1}|^2 + 2(x_{n+1} - x_n) \cdot c_n \\
&\leqslant |x_n|^2 - |x_{n+1}|^2 + 2|x_{n+1} - x_n||c_n| \\
&\leqslant 4(|x_n| - |x_{n+1}|) + \tfrac{1}{6} \cos \theta |x_{n+1} - x_n|,
\end{aligned}
$$

therefore

$$
|x_{n+1} - x_n| \leqslant \tfrac{24}{\cos \theta}(|x_n| - |x_{n+1}|).
$$

This shows that

$$
\sum_{n=M}^{L} |x_{n+1} - x_n| \leqslant \tfrac{24}{\cos \theta} \sum_{n=M}^{L} (|x_n| - |x_{n+1}|) = \tfrac{24}{\cos \theta}(|x_M| - |x_{L+1}|).
$$

The right hand side of this inequality converges, therefore the left hand side converges also. Hence the sum $\sum_{n=M}^{L}(x_{n+1} - x_n) = x_{L+1} - x_M$ converges, hence $x_1, x_2, \ldots$ converges. $\qquad\square$

In the next lemma, the precision of hyperbolic geometry is employed in obtaining Euclidean distance estimates that are necessary in reconciling Theorem 1.1 with Proposition 4.2. We assume in the proofs of Lemma 4.3 and Theorem 1.1 that it is known that the Euclidean centre and hyperbolic centre of a ball within the Poincaré ball model of hyperbolic space both lie on the same radius of the Poincaré ball. This follows from the symmetry of hyperbolic and Euclidean metrics and is proven in most introductory texts in hyperbolic geometry. We also use standard formulae for the hyperbolic metric in a disc (and a half-plane in §5), such as can be found in [1, Chapter 7].

Lemma 4.3. *Let Δ be an open Euclidean ball in $\mathbb{R}^N$ with centre c and radius $r > 0$, and let $x \in \mathbb{R}^N \setminus \overline{\Delta}$ and $y \in \Delta$ be inverse points with respect to $\partial\Delta$. Let B be the closed ball in Δ with hyperbolic centre y and hyperbolic radius $k \geqslant 0$. Then the Euclidean radius s of B and the Euclidean centre b of B satisfy $(1 + e^{-k})s \leqslant |x - b|$.*

Proof. Let the Euclidean line through c, b and y intersect ∂B at points u and v, where the label v is chosen for the intersection point such that c, b, v and x occur in that order along the line. Then

$$(4.1) \qquad \rho_\Delta(c, u) = |\rho_\Delta(c, y) - k|, \quad \rho_\Delta(c, v) = \rho_\Delta(c, y) + k.$$

Using the well known formula (see [1, §7.2])

$$\rho_\Delta(c, z) = \log\left(\frac{r + |c - z|}{r - |c - z|}\right), \quad z \in \Delta,$$

with equation (4.1), one obtains the formulae

$$|c - v| = r\frac{\gamma e^k - 1}{\gamma e^k + 1}, \quad |c - u| = r\frac{|\gamma e^{-k} - 1|}{\gamma e^{-k} + 1}, \quad \gamma = e^{\rho_\Delta(c,y)} = \frac{r + |c - y|}{r - |c - y|}.$$

If u lies between c and b then $2s = |c - v| - |c - u|$ (this corresponds to $\gamma \geqslant e^k$). On the other hand, if c lies between u and b then $2s = |c - v| + |c - u|$ (this corresponds to $\gamma \leqslant e^k$). In either case, it follows that

$$s = r\gamma\frac{e^k - e^{-k}}{(\gamma e^k + 1)(\gamma e^{-k} + 1)}$$

and this may be simplified to yield

$$s = \frac{\delta r(r^2 - |y - c|^2)}{r^2 - \delta^2|y - c|^2}, \quad \delta = \frac{e^k - 1}{e^k + 1}.$$

Since $2\delta/(1 - \delta^2) = \sinh k \leqslant e^k$ and $|y - c| \leqslant r$, we have that

$$(4.2) \qquad s \leqslant \frac{2\delta r^2(r - |y - c|)}{r^2(1 - \delta^2)} \leqslant e^k(r - |y - c|).$$

Now

$$(4.3) \qquad\qquad |x - b| = |x - v| + s$$

and

$$(4.4) \qquad |x - v| \geqslant |x - c| - r \geqslant \tfrac{r}{|x-c|}(|x - c| - r) = r - |y - c|.$$

Equations (4.2), (4.3) and (4.4) may be combined to give the result.   □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. If $T_n(\overline{D})$ converges to a single point x then T_n converges to x uniformly on $\overline{D}$. Thus we assume that $T_n(\overline{D})$ converges to a closed ball X of positive radius. Choose a Möbius transformation f that maps X to the closed unit ball $\overline{\mathbb{B}^N}$ in $\mathbb{R}^N$. Lemma 2.4 shows that $f \circ T_n \circ f^{-1}(f(\overline{D})) = f \circ T_n(\overline{D})$ converges to $\overline{\mathbb{B}^N}$. Since the hypotheses and conclusion of Theorem 1.1 are preserved under conjugation, it suffices to assume that $X = \overline{\mathbb{B}^N}$. Corollary 2.2 and Lemma 2.5 show that we may henceforth restrict to large enough n for which $T_n(D)$ is a Euclidean ball with centre c_n and radius r_n, where $c_n \to 0$ and $r_n \to 1$ as $n \to \infty$.

Define b^* to be the inverse point of b in ∂D. Let A be the closed ball with hyperbolic centre b^* and hyperbolic radius k, where $k \geqslant \rho_D(a, b^*)$, so that $a \in A$. We define $x_n = T_n(b)$ and then match the hypotheses of Proposition 4.2, which then shows that x_n must converge. That $c_n \to 0$ as $n \to \infty$ has been assured. To see that $|x_n| \to 1$ as $n \to \infty$, observe that $x_n = T_n(b) \notin T_n(D)$, so that $|x_n - c_n| \geqslant r_n$, and $x_n = T_{n-1}(a) \subseteq T_{n-1}(D)$, so that $|x_n - c_{n-1}| \leqslant r_{n-1}$. Therefore

$$r_n \leqslant |x_n - c_n| \leqslant |x_n - c_{n-1}| + |c_{n-1} - c_n| \leqslant r_{n-1} + |c_{n-1} - c_n|.$$

Since $c_n \to 0$ and $r_n \to 1$ as $n \to \infty$, these inequalities show that both sequences with nth terms $|x_n - c_n|$ and $|x_n|$ converge to 1.

To see that x_n converges we have only to verify properties (i) and (ii) from Proposition 4.2. Möbius maps preserve inverse points, therefore $y_n = T_n(b^*)$ and $x_n = T_n(b)$ are inverse points with respect to $\partial T_n(D)$, hence

$$(4.5) \qquad |x_n - y_n| = |x_n - c_n| - \frac{r_n^2}{|x_n - c_n|} \to 0 \text{ as } n \to \infty.$$

The map T_n preserves hyperbolic distance from D to $T_n(D)$, therefore the closed ball $T_n(A)$ has hyperbolic centre y_n and hyperbolic radius k in $T_n(D)$. Let $T_n(A)$ have Euclidean centre b_n and Euclidean radius s_n. As $x_{n+1} = T_n(a) \in T_n(A)$, we have the inequality

$$(4.6) \qquad |x_n - b_n| - s_n \leqslant |x_n - x_{n+1}| \leqslant |x_n - b_n| + s_n.$$

We apply Lemma 4.3 with $\Delta = T_n(D)$ and $B = T_n(A)$ to deduce that

$$(4.7) \qquad (1 + e^{-k})s_n \leqslant |x_n - b_n|.$$

Since b_n lies on the Euclidean line segment $[x_n, c_n]$, the angle $\theta_n \in [0, \pi)$ between the Euclidean line segments $[x_n, c_n]$ and $[x_n, x_{n+1}]$ is equal to

the angle between the Euclidean line segments $[x_n, b_n]$ and $[x_n, x_{n+1}]$. If $\theta_n > 0$, we may apply the Cosine Rule to the triangle with vertices x_n, b_n and x_{n+1} which, along with equations (4.6) and (4.7), yields

$$
\begin{aligned}
\cos\theta_n &= \frac{|x_n - b_n|^2 + |x_n - x_{n+1}|^2 - |b_n - x_{n+1}|^2}{2|x_n - b_n||x_n - x_{n+1}|} \\
&\geqslant \frac{|x_n - b_n|^2 + (|x_n - b_n| - s_n)^2 - s_n^2}{2|x_n - b_n|(|x_n - b_n| + s_n)} \\
&= \frac{|x_n - b_n| - s_n}{|x_n - b_n| + s_n} \\
&\geqslant \frac{1}{2e^k + 1}.
\end{aligned}
$$

Hence all θ_n lie in the interval $[0, \theta)$, where θ is the unique solution in $[0, \pi/2)$ of $\cos\theta = 1/(2e^k + 1)$. Thus property (i) is true. Property (ii) is also true since $|x_n - c_n|\cos\theta_n$ is greater than $\frac{1}{2}\cos\theta$ for large n, whilst we now show that $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. Using (4.6) and (4.7) we have that

$$
\begin{aligned}
(4.8) \qquad |x_n - x_{n+1}| &\leqslant |x_n - b_n| + s_n \\
&\leqslant \left(1 + \frac{1}{1+e^{-k}}\right)|x_n - b_n| \\
&= (2e^k + 1)\left(1 - \frac{1}{1+e^{-k}}\right)|x_n - b_n| \\
&\leqslant (2e^k + 1)(|x_n - b_n| - s_n) \\
(4.9) \qquad &\leqslant (2e^k + 1)|x_n - y_n|,
\end{aligned}
$$

and $|x_n - y_n| \to 0$ as $n \to \infty$ by (4.5), hence property (ii) is verified. All the hypotheses of Proposition 4.2 have now been satisfied and that proposition demonstrates convergence of x_n to a limit x.

It remains to show that T_n converges locally uniformly within D to x. Since $|x_n - y_n| \to 0$ as $n \to \infty$, certainly $T_n(b^*) \to x$ as $n \to \infty$. Now

$$
\sup_{w \in A} |y_n - T_n(w)| \leqslant 2s_n,
$$

and that $s_n \to 0$ as $n \to \infty$ can be seen from (4.8) and (4.9). Therefore T_n converges to x uniformly within A. The result follows as A may have been chosen to be arbitrarily large within D. $\qquad\square$

We remark that it has just been shown that $T_n(w) \to x$ as $n \to \infty$ for points $w \in D$, where $|x| = 1$. If $w \in \mathbb{R}_\infty^N \setminus \overline{D}$ and $w^* = \iota_D(w)$ then $T_n(w^*) \to x$ as $n \to \infty$ and

$$
|T_n(w) - T_n(w^*)| = |\iota_{D_n}(T_n(w^*)) - T_n(w^*)| = \frac{r_n^2 - |T_n(w^*) - c_n|^2}{|T_n(w^*) - c_n|}.
$$

This latter expression converges to 0 as $n \to \infty$, therefore also $T_n(w) \to x$ as $n \to \infty$. That T_n converges to x on $\mathbb{R}_\infty^N \setminus \overline{D}$ when X is a ball of positive radius is not included in the statement of Theorem 1.1 since it is not true when X is a single point. An example verifying this assertion is provided in §5.

5. Best possibility of Theorem 1.1

In this section we provide two examples to demonstrate the strength and necessity of certain aspects of Theorem 1.1. Such examples are notably missing from existing accounts of the Hillam-Thron Theorem for two reasons. Firstly, those accounts are predominantly algebraic and lack the geometric machinery we employ in constructing our examples. Secondly, those accounts tend to focus on convergence of the sequence T_n only at the point 0, as this is the classical definition of continued fraction convergence. In contrast, we are interested in the convergence or divergence of T_n at every point in $\mathbb{R}_\infty^N$, since the point 0 has no particular geometric significance for general Möbius transformations.

In Example 1 we examine Theorem 1.1 when the limit ball X is chosen to be a single point and in Example 2 we examine Theorem 1.1 when X is a closed ball with positive chordal radius. We conclude that given a, b and D in Theorem 1.1, the set $D \cup \{b\}$ is the largest set on which we can be certain of convergence of T_n to the limit value.

Let us first assume that X is a single point x. Evidently T_n converges uniformly on $\overline{D}$ to x, and since $T_n(b) \in T_{n-1}(D)$ for every n, also $T_n(b) \to x$ as $n \to \infty$. We give an example of this limit point circumstance for which T_n diverges on a chosen dense subset of the complement of $\overline{D} \cup \{b\}$, thereby proving that the conclusion of Theorem 1.1 cannot be strengthened to include convergence on a larger set than $\overline{D} \cup \{b\}$.

Example 1. Let $D = \{z \in \mathbb{C} : \mathrm{R}e[z] > 0\}$, $a = 1$ and $b = -1$. Choose a countable set of points S that is dense in $\mathbb{C} \setminus (\overline{D} \cup \{-1\})$ and a sequence $\zeta_1, \zeta_2, \ldots$ in S such that if $s \in S$ then $s = \zeta_n$ for infinitely many n. Let $\rho_n = \rho_D(\iota_D(\zeta_n), 1)$, for $n \in \mathbb{N}$.

We define two sequences u_n and v_n by the formulae

$$u_1 = 1, \quad v_n = u_n/(1 - e^{-\rho_n}), \quad u_n = 2v_{n-1} - u_{n-1}.$$

Let $D_n = \{z : \mathrm{R}e[z] > v_n\}$, then it can be verified that

$$u_1 < v_1 < u_2 < v_2 < \cdots, \quad \iota_{D_n}(u_{n+1}) = u_n, \quad \rho_{D_n}(u_{n+1}, \iota_{D_n}(0)) = \rho_n.$$

Furthermore, one may show that

$$v_n > \left(\Pi_{i=1}^n (1 - e^{-\rho_i})\right)^{-1},$$

from which we deduce that the sequence v_n is unbounded. Lemma 2.7 shows that $\overline{D_n}$ converges in the chordal metric to $\{\infty\}$. By Lemma 3.1 we may choose Möbius maps $T_1, T_2, \ldots$ with $T_n(D) = D_n$, $T_n(1) = u_{n+1}$ and $T_n(\iota_D(\zeta_n)) = \iota_{D_n}(0)$. Preservation of inverse points from D to D_n ensures that $T_n(-1) = u_n$ and $T_n(\zeta_n) = 0$. Thus we conclude that $(T_n, 1, -1, D)$ is a Hillam-Thron sequence such that T_n converges uniformly on $\overline{D}$ to ∞, whilst $T_n(\zeta_n) = 0$ for every n. Hence T_n does not converge to the limit point $x = \infty$ on S. It is sufficient for our purposes to have shown that T_n does not converge to x on S, although one may verify the stronger assertion that T_n diverges on S using the equality $\rho_{\iota_{D_n}(D_n)}(T_n(\zeta_m), T_n(\zeta_n)) = \rho_{\iota_D(D)}(\zeta_m, \zeta_n)$ and comparing hyperbolic and Euclidean distances. $\square$

When X is a closed ball with radius between 0 and 2, that is, when X is neither a single point nor the whole of $\mathbb{R}^N_\infty$, it was shown in the proof of Theorem 1.1 in §4 and the comment following that proof that T_n converges to x on $\mathbb{R}^N_\infty \setminus \partial D$. We supply an example of a Hillam-Thron sequence (T_n, a, b, D) such that T_n diverges on a given dense set of points in ∂D, thereby completing our argument to show that the conclusion of Theorem 1.1 cannot be strengthened to include convergence on a larger set than $D \cup \{b\}$.

Example 2. Let

$$D = \{z : \mathrm{Re}[z] < 3/2\}, \quad a = 1, \quad b = 2 \quad \text{and} \quad U_n(z) = z/2^n,$$

for $n = 1, 2, \ldots$. Then

$$U_1(2) = 1, \quad U_n(2) = U_{n-1}(1) = 1/2^{n-1} \quad \text{and} \quad U_n(D) = D_n,$$

for $n = 1, 2, \ldots$, where $D_n = \{z : \mathrm{Re}[z] < 3/2^{n+1}\}$.

Choose a countable dense subset S of ∂D and a sequence $\zeta_1, \zeta_2, \ldots$ of elements of S such that every $s \in S$ occurs in this sequence infinitely many times. By Lemma 3.1, we may define V_n to be an automorphism of D_n that fixes the interior point $U_n(1)$ and maps the boundary point $U_n(\zeta_n)$ to ∞. Let $T_n = V_n \circ U_n$. Then $T_n(1) = U_n(1)$ and $T_n(D) = D_n$. As Möbius maps preserve inverse points we see that

$$T_n(2) = T_n(\iota_D(1)) = \iota_{D_n}(T_n(1)) = 1/2^{n-1} = T_{n-1}(1),$$

so that $(T_n, 1, 2, D)$ is a Hillam-Thron sequence such that $T_n(z) \to 0$ as $n \to \infty$ for $z \in D$, and such that $T_n(\overline{D})$ converges to the closed left half-plane. If $s \in S$ then $T_n(s) = \infty$ for infinitely many n so that T_n does not converge to 0 on S (in fact, T_n diverges on S), as required. $\square$

6. Concluding remarks

A. F. Beardon has proven with Euclidean geometry a version of the Hillam-Thron Theorem that is valid in all dimensions (see [3]). He replaces the condition $T_n(b) = T_{n-1}(a)$ with the more general condition $T_n(b) \in T_{n-1}(A)$, where A is a compact subset of D. It is not difficult to adjust Theorem 1.1 and its proof to accommodate this generalisation. The author has extended this assumption further still in [7] whilst proving a several dimensional version of the Parabola Theorem.

Most convergence theorems in the analytic theory of continued fractions involve nested sequences of discs $D \supseteq T_1(D) \supseteq \cdots$, and it seems probable that in results other than the Hillam-Thron Theorem one may replace this nested condition with a suitable notion of convergence of discs. The author has not looked into this possibility.

Finally, we remark that it may be possible to extend Theorem 1.1 to include more general domains D than discs and more general conformal (or possibly quasiconformal) maps T_n than Möbius maps, but we have discovered only counterexamples and not positive results in this direction.

References

[1] A. F. Beardon. *The Geometry of Discrete Groups*. Springer-Verlag, 1983.
[2] A. F. Beardon. The geometry of Pringsheim's continued fractions. *Geom. Dedicata*, 84:125–134, 2001.
[3] A. F. Beardon. The Hillam-Thron Theorem in higher dimensions. *Geom. Dedicata*, 96:205–209, 2003.
[4] G. A. Edgar. *Measure, topology and fractal geometry*. Undergraduate Texts in Mathematics. Springer-Verlag, 1990.
[5] W. B. Jones and W. J. Thron. *Continued Fractions: Analytic Theory and Applications, Encyclopedia of Mathematics and its Applications*. Addison-Wesley, 1980.
[6] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*. Number 149 in Graduate Texts. Springer-Verlag, 1994.
[7] I. Short. The Paraboloid Theorem. *Paper under preparation*, 2006.

E-mail address: Ian.Short@nuim.ie

Mathematics Department, NUI Maynooth, Ireland