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School of Mathematical Sciences

Doctoral Dissertation

Towards a Comprehensive Framework for the Analysis of Anomalous Diffusive Systems

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The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka" (I found it!) but "That's funny...".

Isaac Asimov

I, Andrea Cairoli, confirm that the novel results presented in this thesis are part of my own research work, made at the School of Mathematical Sciences of Queen Mary University of London, or that, in the case they have been carried out in collaboration with or supported by others, this is clearly acknowledged and that the precise details of my contribution are specified. Previously published material is acknowledged below.

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Details of collaboration and publications:

1. Publications Based on the Content of the Thesis
   - Anomalous processes with general waiting times: functionals and multipoint structure, Cairoli A., & Baule A. (2015), *Phys. Rev. Lett.*, 115(11), 110601
   - Langevin formulation of a subdiffusive continuous time random walk in physical time, Cairoli A., & Baule A. (2015), *Phys. Rev. E*, 92(1), 012102.
   - Feynman-Kac equation for anomalous processes with space- and time- dependent forces, Cairoli A., & Baule A., submitted to *J. Phys. A*.
   - Weak Galilean Invariance of Anomalous Stochastic Processes, Cairoli A., Klages R., & Baule A., In preparation.
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   - Cairoli A., Baule A. and Hayakawa H., in Preparation.
   - Kanazawa K., Sano T., Cairoli A., & Baule A., in Preparation.
   - Interplay between consensus and coherence in a model of interacting opinions, Battiston F.*, Cairoli A.*, Nicosia V., Baule A., & Latora V. (2016), *Physica D*, 323-324, 12-19.
   - Forecasting transitions in systems with high dimensional stochastic complex dynamics: A Linear Stability Analysis of the Tangled Nature Model, Cairoli A.*, Piovani D.*, & Jensen H. J. (2014), *Phys. Rev. Lett.* 113(26), 264102.

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List of Abbreviations

MSD	Mean Square Displacement
CTRW	Continuous-Time Random Walk
SBM	Scaled Brownian Motion
PDF	Probability Density Function
CTRW	Continuous-Time Random Walk
FK	Feynman-Kac
FFK	Fractional Feynman-Kac
GFFK	Generalised Fractional Feynman-Kac
GI	Galilean Invariant
FP	Fokker-Planck
FFP	Fractional Fokker-Planck
GFFP	Generalised Fractional Fokker-Planck
RL	Riemann-Liouville
RV	Random Variable
KK	Klein-Kramers
OU	Ornstein-Uhlenbeck
EOM	Equation of Motion
FDRI	Fluctuation-Dissipation Relation of the first kind

Abstract

The modelling of transport processes in biological systems is one of the main theoretical challenges in physics, chemistry and biology. This is motivated by their essential role in the emergence of diseases, like tumour metastases, which originate from the spontaneous migration of cancer cells. Thus, improvements in their understanding could potentially pave the way for an outstanding innovation of present-day techniques in medicine. These processes often exhibit anomalous properties, which are qualitatively described by the power-law scaling of their mean square displacement, compared to the linear one of normal diffusion. Such behaviour has been often successfully explained by the celebrated continuous-time random walk model. However, recent experimental studies revealed the existence of both more complicated mean square displacement behaviour and anomalous features in other characteristic observables, e.g. the position-velocity statistics or the two point correlation functions of either the velocity or the position. Thus, in order to understand the anomalous diffusion recorded in these experiments and assess the microscopic processes underlying the observed macroscopic dynamics, one needs to have a complete tool-kit of techniques and models that can be readily compared with the experimental datasets. In this Thesis, we contribute to the construction of such a complete framework by fully characterising anomalous processes, which are described by means of a continuous-time random walk with general waiting time distributions and/or external forces that are exerted both during the jumps (as in the original model) and the waiting times. In the first case we derive both the joint statistics of these processes and their observables, specifically by obtaining a generalised fractional Feynman-Kac formula, and their multipoint correlation functions and employ them to fit the mean square displacement data of diffusing mitochondria. This result supports the experimental relevance of our formalism, which comprises general formulas for several quantities that can provide readily predictable tests to be checked in experiments. In the second case, we characterise the new anomalous processes by means of Langevin equations driven by a novel type of non Gaussian noise, which reproduces the typical fluctuations of a free diffusive continuous-time random walk. For a constant external force, we also obtain the fractional evolution equations of their position probability density function and show that, contrarily to continuous-time random walks, they are weak Galilean invariant, i.e., their position distribution in different Galilean frames is obtained by shifting the sample variable according to the relative motion of the frames. Thus, these processes provide a suitable frame-invariant framework, that could be employed to investigate the stochastic thermodynamics of anomalous diffusive processes.

CHAPTER 1

Introduction

Our aim in this Chapter is to explain the importance of developing a comprehensive understanding of diffusion phenomena in living systems, thus motivating the research work discussed in the rest of Thesis. We will provide an historical overview on both experiments and theoretical models of diffusion processes. Thus, we will first recall the pioneering work of L. Bachelier [1], A. Einstein [2], M. Smoluchowski [3], P. Langevin [4] (on the theoretical side) and of I. Ingenhousz [5] and J. Perrin [6] (on the experimental side) on normal diffusive processes and then present those mechanisms, for instance trapping or energy consumption, which causes anomalous diffusion to emerge, by ultimately driving the system out of equilibrium. We will then characterise anomalous diffusive processes in terms of their mean square displacement. In conclusion, we will provide an outline of the Thesis.

1.1 Motivation

Transport phenomena, specifically diffusion processes, are ubiquitous in both physical and biological systems [7]. For instance, nutrients and other biological macromolecules often need to be transported into a cell trough its membrane and/or to some specific places within the cytoplasm to ensure the correct functioning, and in some cases the survival, of the cell itself. Another example consists in cell migration, which is responsible for several essential mechanisms of living systems, like embryo-genesis, wound-healing or immune defence. However, it is also the mechanism generating tumour metastases and other diseases, thus suggesting that the improvement of our understanding of transport processes could represent an essential step towards the development of efficient ways to cure them. In addition, the study of the diffusive behaviour of tracer particles is often used to probe the rheological properties and the internal dynamics of complex media, e.g. the cytoplasm or other different active gels and granular systems, thus providing a straightforward technique to assess the validity of the theoretical models describing them.

All these applications, and several others not mentioned, demonstrate the relevance of diffusion processes and motivate the extraordinary effort that has been devoted to their analysis, either in the case of normal processes or in that of anomalous ones (their difference will be discussed later in this Chapter), both from the experimental and the theoretical side. In the former case, the aim is usually to observe the different types of dynamical behaviour that diverse systems may exhibit, compare their properties and construct an efficient classification of them. In the latter case, the focus resides on developing a complete theoretical framework for their interpretation, comprising both techniques for the statistical analysis of experimental datasets and mathematical models to reproduce them.

Considering that the rest of the Thesis will be devoted to this second aspect, it is instructive to elucidate what *modelling* means in this specific context. Let us consider a system, either physical, like charge carriers, or biological, like cells or other macromolecules within it or even macroscopic animals (birds or fish for instance). Depending on its specific properties and on the environmental conditions, the system may be observed in a variety of different states, corresponding to specific values of characteristic observables that can be measured experimentally, which all together constitute its *state space*. We define the *dynamics* of the system as the set of rules determining how it can evolve in time from one state to a different one [8]. When we perform an experiment on such system, we usually initialise it in a prescribed initial state and let it evolve freely or under the effect of external forces according to these rules. At the same time, we measure a suitable observable, usually either the position or the velocity in the case of diffusive systems, whose value is determined by the state of the system and thus changes continuously during its temporal evolution, along with the systems moving to different states. We call $Y(t)$ the measured time evolution of such observable and y_0 its value corresponding to the state in which the system was initialised. If the dynamics of the system is known, $Y(t)$ would be a deterministic function, as one could calculate exactly its value at each time t, once also y_0 is specified. However, this is never the case in realistic situations, because the dynamical rules of the system are usually unknown, meaning that $Y(t)$ cannot be determined a priori and it is instead a random function of time, i.e., a *stochastic process*. Nevertheless, the experimentally recorded $Y(t)$ provides essential information, which can be employed to infer the dynamical rules of the system, and thus to elucidate its physical or biological features, by comparing it with simplified models that can reproduce such observed function.

If Y is the position of the system at each time, a fundamental information to understand the dynamics of the system is provided by the Mean Square Displacement (MSD):

$$MSD_Y(t) = \left\langle [Y(t) - y_0]^2 \right\rangle \tag{1.1}$$

where the brackets denote an average over many independent realisations of the process Y. These are obtained by repeating several times its measurement under the same experimental conditions. According to the definition Eq. (1.1), the MSD provides an estimate of the spatial extent of the observed process, i.e., of the portion of space explored on average by the system. Despite its simplicity, the functional form of the MSD already provides a qualitative classification of the different types of diffusion behaviour that a system can exhibit and enables one to distinguish between normal and anomalous diffusive processes. In the rest of this Chapter, we will describe how this classification in terms of their MSD historically evolved through a close interplay between experimental evidences, elucidating the existence of such different types of diffusion, and theoretical advances in the definition of models that are capable of reproducing such experimentally observed MSDs [9, 10].

1.2 Normal Diffusion

The random motion of inorganic particles in a fluid, which had already been seen by, e.g., W. F. Gleichen, J. T. Needham, G. L. Leclerc, A. T. Brogniart and L. Spallanzani [11, 12], who though failed to correctly interpret their experimental observations, was first postulated by I. Ingenhousz in 1784 [5] and independently by J. Bywater in 1801 [13]. However, the first systematic analysis of such motion was obtained by the botanist R. Brown in 1827, after whom it was named *Brownian motion*, who was studying a system of particles trapped in cavities and immersed in a water solution enriched of pollen grains. While looking at this system with his microscope, he realised that the particles were moving through the solution while rapidly changing directions without any net drift [11]. Furthermore, he provided robust evidence that such motion could not be ascribed to bubbles, release of matter within the solution or interactions between the particles. This motion of the particles, that Brown investigated, belongs to the class of normal diffusive processes, which are characterised by a linear scaling of the MSD:

$$MSD_Y(t) \sim t. \tag{1.2}$$

A first insight into the properties of these processes was obtained by A. Fick [14]. During his studies on molecular transport through membranes, he phenomenologically proposed an equation describing the concentration of the diffusing species $n(x,t)$, which reads as:

$$\frac{\partial}{\partial t} n(x,t) = D \frac{\partial^2}{\partial x^2} n(x,t) \tag{1.3}$$

where D is a constant coefficient. As we will discuss in Sec. 2.1.1, this same equation holds for the position Probability Density Function (PDF) of a normal diffusive process.

However, it was only trough the seminal works of L. Bachelier [1], who first connected the random motion studied by Brown to diffusion equations like Eq. (1.3), and A. Einstein [2], which were later put on more robust mathematical grounds by M. Smoluchowski [3], that the microscopic origin of Brownian motion became clear. In the following, we review the calculation of the MSD Eq. (1.2) within the microscopic picture proposed by Einstein. Let us tag one of those particles observed by Brown. According to Einstein's argument, its motion can be constructed by assuming independent and identically distributed (i.i.d) position displacements s_i between consecutive samplings of the particle's position at discrete times, which are separated by an interval of fixed length τ. In physical terms, these position displacements are due to the collisions between such particle and the smaller ones forming the solution. As it will be clarified in Sec. 2.1.1, this is ultimately a random walk type description. Let now $Y(t)$ and $N(t)$ be respectively the position of the tagged particle at time t and the number of positions samplings up to t. Thus, $Y(t) = y_0 + \sum_{i=1}^{N(t)} s_i$, such that the MSD is given as below:

$$MSD_Y(t) = \langle Y^2(t) \rangle = \left\langle \sum_{i,j=1}^{N(t)} s_i\, s_j \right\rangle$$
$$= \sum_{i=1}^{N(t)} \langle s_i^2 \rangle + \sum_{\substack{i,j=1 \\ i \neq j}}^{N(t)} \langle s_i\, s_j \rangle = \sum_{i=1}^{N(t)} \langle s_i^2 \rangle = \frac{a^2}{\tau}\, t, \tag{1.4}$$

where the contribution from the sum over terms $\langle s_i\, s_j \rangle$ for $i \neq j$ is null because the variables $\{s_i\}$ are independent, thus also uncorrelated, $\langle s_i^2 \rangle = a^2$ and $N(t) = t/\tau$ by construction. Experimental confirmation of Einstein's molecular picture was provided by J. Perrin [6]

Three years later than Einstein's work, an alternative formulation of normal diffusive processes was proposed by P. Langevin in Ref. [4]. His approach consists in writing down a Newtonian like equation of motion for the tagged particle, which consists of a first deterministic part, accounting for the friction and any external force f being exerted on it, and a second stochastic contribution $\xi(t)$, which instead accounts for the effect of the collisions from the other particles in the solution in a coarse-grained, probabilistic way (an exact derivation based on the seminal Ref. [15] is discussed in Sec. 5.2.1). Thus, denoting $V(t)$ the time dependent velocity of the tagged particle, we obtain the equation [4]:

$$m\, \dot{V}(t) = -\gamma\, V(t) + f + \xi(t), \tag{1.5}$$

where m is the mass of the colloidal particle, γ is the friction coefficient and the stochastic force ξ has fixed statistical properties [in the case of diffusive processes these are specified in Sec. 2.2.2, Eq. (2.98)]. The linearity of the MSD can be easily shown also from Eq. (1.5). Indeed, recalling that $Y(t) = \int_0^t V(\tau)\, \mathrm{d}\tau$ and assuming $f = 0$, we obtain:

$$MSD_Y(t) = \int_0^t \int_0^t \langle V(t')V(t'') \rangle\, \mathrm{d}t'\, \mathrm{d}t''$$
$$= 2 \int_0^t \left[\int_{t'}^t \langle V(t')V(t'') \rangle\, \mathrm{d}t'' \right] \mathrm{d}t'$$
$$= 2 \int_0^t \left[\int_0^{t-t'} \langle V(t')V(t'+\tau) \rangle\, \mathrm{d}\tau \right] \mathrm{d}t' = 2 \int_0^t \left[\int_0^{t-t'} C_v(t';\tau)\, \mathrm{d}\tau \right] \mathrm{d}t' \tag{1.6}$$

where we first employed the symmetry of the integration region and of the integrand function, secondly we changed the integration variable t'' into the time lag $\tau = t'' - t'$ in the third line and lastly we introduced the velocity-velocity correlation function $C_v(t';\tau) = \langle V(t')V(t'+\tau) \rangle$. This quantity is equal to $C_v(t';\tau) = [(k_B\, T)/m] \exp[-(\gamma\, \tau)/m]$ for $t' \to \infty$ with k_B being the Boltzmann constant and T the temperature of the system [16]. Thus, the internal integral in Eq. (1.6) is equal to m/γ for $t \gg t' \to \infty$, such that we finally find: $MSD_Y(t) = 2\,(k_B\, T/\gamma)\, t$. We highlight that these two different approaches, i.e., the random walk one proposed by Einstein, on the one hand, and that based on the description of the system in terms of a Newton equation of motion with stochastic driving force proposed by Langevin, on the other hand, are both equivalent and complementary. Indeed, while the Langevin approach provides essential information on the stochastic trajectories of the observed process, the random walk framework enables one to derive straightforwardly evolution equations for its PDF (this will be discussed in details in both Chapters 2). Thus, in order to fully understand their properties, one needs to manipulate both these two techniques. As this argument also holds in the case of anomalous diffusive processes, in this Thesis we will characterise all the processes of interest by always employing both these two different techniques.

1.3 Out-of-Equilibrium Driving and Anomalous Diffusion

The study of Brownian motion by Brown [11] and its subsequent theoretical description developed by A. Einstein [2], M. Smoluchowski [3] and P. Langevin [4] represents a milestone in the theory of diffusion processes. For the first time, these results suggested the experimental relevance of diffusion processes, and specifically of the normal ones satisfying Eq. (1.2), and the need to put solid ground to their theoretical framework, thus inspiring several further investigations both in the Mathematics and in the Physics community.

Nevertheless, it was soon realised that normal diffusive processes only represents a subclass of the set of diffusion processes that can be observed experimentally. Indeed, many systems in Nature exhibit anomalous diffusive patterns, whose distinctive feature is a power-law scaling of the MSD [17, 18, 9, 19]:

$$MSD_Y(t) \sim t^{\alpha}. \tag{1.7}$$

Here, α is a positive real number, whose value enables us to provide a first qualitative classification of anomalous diffusive processes. Indeed, recalling that for $\alpha = 1$ we recover normal diffusion, the case $0 < \alpha < 1$ corresponds to processes with slower space exploration, whereas in the opposite case $\alpha > 1$ the resulting processes diffuse faster. These different processes are called *subdiffusive* or *superdiffusive* respectively. Within the class of superdiffusive processes we can identify a further distinction. Indeed, $\alpha = 2$, i.e., *ballistic motion*, is another special case, so that we call processes for which $\alpha > 2$ *superballistic*. In what follows, we will provide experimental evidence of both subdiffusive and superdiffusive processes and discuss the main mechanisms, which generate such anomalous behaviour by ultimately driving the system out of equilibrium, and their corresponding models.

1.3.1 Subdiffusive Processes

Subdiffusive processes are usually observed when the system is diffusing in a complex medium, which presents impediments of either energetic or geometrical nature to its motion. For instance, one of the first observation of subdiffusion was obtained while studying the motion of charge carriers moving in amorphous semiconductors [20, 21, 22, 23, 24, 25, 26, 27]. Such medium generates a complex landscape of energy wells where the electrons can get trapped for a time interval, whose duration depends on the height of the potential barrier that they need to overcome to get out of it. Thus, their motion consists in a sequence of trapping-untrapping events, such that the total time spent in free motion is smaller than the time spent in the energetic wells. According to the qualitative discussion of the previous paragraph, this type of dynamics leads to subdiffusion. Such system was originally modelled in terms of the Continuous-Time Random Walk (CTRW) model [28, 20], which was proved to reproduce its observed dynamical behaviour. Later, subdiffusive behaviour of different origin was also observed for particles being transported on fractal geometries and for Rouse or reptation dynamics in polymeric systems (see [9] for further examples and related references).

In addition, thanks to the recent improvements of experimental techniques in biology, joint position-velocity datasets have been obtained, revealing the existence of several examples of subdiffusion also in living systems. Specifically, biological macromolecules and/or organelles often exhibit a subdiffusive scaling of the MSD, while moving in molecular crowded environments, which can be prepared ad hoc for in vitro experiments, e.g. by using solutions of surfactant micelles [29] or polymer networks [30] to name just a few techniques, or found in vivo, e.g. the cytoplasm [31] or the cells' membrane [32], whose viscoelastic properties have recently been found to play a major role in determining the anomalous diffusion [33, 34]. Further examples can be found in Ref. [10].

The diverse features of these examples suggest that several different mechanisms can generate subdiffusive behaviour in both non-living and living systems. Detailed reviews of them, along with the models usually adopted to describe them, can be found in Refs. [35, 36, 19, 10, 8]. Here, we only mention the two most commonly observed in experiments:

- trapping. As for electrons moving in amorphous solids, the dynamics of the diffusing particle alternates periods of free motion (possibly in the presence of external forces) with periods of immobilisation, which are caused by its getting trapped in either energetic wells (due to the presence of binding sites or complex potential landscapes) or geometrical traps (like for particle transport in Purkinje cells [37, 38, 39]). Its molecular description is given by Einstein's argument [2], but the finite time step τ, which for normal diffusive dynamics is not an intrinsic property of the system but rather a time-scale induced by the data sampling procedure, corresponds to the physical sojourn time of the particle in each of the cages. This implies that its duration is generally not constant, but trap-dependent. Further considering that in experiments we only have information on the statistics of the height of these cages, τ needs to be randomly sampled from a specific distribution.

- Viscoelasticity of the medium. In this case, the diffusing particle is part of a more complex interacting system exhibiting viscoelastic properties, thus inducing strong correlations with the other parts of the system. Consequently, its motion is naturally forced to take place in a concerted way with that of the rest of the system.

Considering the large variety of different mechanisms that may generate subdiffusive behaviour, different models are naturally needed in order to correctly reproduce their features. The CTRW represents one specific example of such models, which is particularly suitable to describe trapping-untrapping dynamics, but other models have also been proposed, e.g. the generalised Langevin equation with power-law kernel [40], which is often adopted to describe diffusion in a viscoelastic environment, the fractional Brownian motion [41] or even ordinary Langevin dynamics in the presence of random or multiscale potential [42, 43]. All these different models are not equivalent, even if they possibly share the same subdiffusive MSD behaviour for specific choices of their characteristic parameters, because they usually differ with respect to other statistical properties, e.g. the free-force propagator or the two-point correlation functions. Consequently, one needs to choose the model according to the specific features of the system observed in the experimental study.

1.3.2 Superdiffusive Processes

Anomalous superdiffusive processes originate from different mechanisms than subdiffusive ones. In this case, indeed, the MSD grows faster than in the normal case, meaning that the diffusing particle needs to perform on average longer spatial excursions than Brownian motion in the same fixed time interval. Examples of such diffusive behaviour can be found in rotating flows, particle motion in turbulent fluids/plasma or in heterogeneous rocks (see [9] for references and further experimental results). Furthermore, cell migration experiments have recently revealed a characteristic superdiffusive scaling of the MSD along with many more features deviating from standard Brownian models, e.g. non Gaussian PDFs for the position and/or the velocity of the moving cell and power-law long time scaling of the velocity auto-correlation functions [44, 45, 46, 47, 48, 49]. Another relevant application consists in animal foraging and target search, where superdiffusive behaviour is often observed in specific environmental conditions [50, 51], i.e., when targets are sparsely distributed and non-destructible. Models often used to describe superdiffusion are Lévy flights and Lévy walks [52, 53, 54, 9, 55, 56], which will be mentioned in Chapter 2. We recall that a dynamical behaviour qualitatively resembling that of Lévy walks has been recently obtained for ordinary underdamped Langevin dynamics in two dimensions, i.e., with pure thermal fluctuations, with randomly generated potential landscapes [42, 43].

1.4 Thesis Outline

We conclude this Introduction by providing an outline of the Thesis.

- **Chapter 2: Mathematical Tools for Anomalous Stochastic Processes**
  The aim of this Chapter is to construct a complete framework for the theoretical analysis of diffusion processes, either normal or anomalous. To this aim, the Chapter is divided in two parts. The first one is focused on the random walk description of diffusive processes. Specifically, we will review the random walk model and derive the diffusion equation from its corresponding master equation. This same scheme will then be applied to the CTRW model. A tutorial on the use of fractional operators will also be provided. The second part deals with their description in terms of stochastic Langevin equations. Thus, we will review the stochastic analysis of Itô processes and present the theory of Lévy processes, in particular subordinators. A brief discussion of semimartingales and time-changed processes will end the Chapter.

- **Chapter 3: Anomalous Processes with General Waiting Times: Functionals and Multipoint Structure**
  In this Chapter, we formulate anomalous diffusive processes in terms of a CTRW with a more general waiting time distribution than the Lévy stable one. We fully characterise them, and their observables, in terms of (i) the stochastic description of their microscopic dynamics in terms of subordinated Langevin equations, (ii) the generalised Fractional Feynman-Kac (FFK) equation, whose derivation is presented in details, and (iii) their multipoint correlation functions. We compute the MSD and two-point correlation functions of specific toy models of biological relevance. We conclude the Chapter by applying our formalism to model MSD data of mitochon-

dria diffusing in S. Cerevisiae cells depleted of actin microfilaments of Ref. [57] and predict the functional form of the two point correlation function of the process, that can be readily tested experimentally.

- **Chapter 4: Langevin Formulation of a Subdiffusive Continuous-Time Random Walk in Physical Time**
  This Chapter is devoted to study if a Langevin formulation of a CTRW can be obtained without using subordination. To this aim, we define a novel non Gaussian noise and investigate processes whose dynamics is described by Langevin equations driven by it. While in the free diffusive case we find equivalence with the CTRW, external forces are exerted on the system differently, specifically both during the waiting times and the jumps, thus changing completely their properties. We will discuss in details their difference with CTRWs and conclude by comparing them with the Scaled Brownian Motion (SBM), which is a Gaussian model of anomalous diffusion.

- **Chapter 5: Galilean Invariance of Anomalous Stochastic Processes**
  This Chapter is focused on elucidating the role of Galilean Invariance (GI) in the study of diffusion processes. We first discuss how the violation of GI of the Langevin equation and the Fokker-Planck (FP) equation is caused by the coarse-graining procedure, which is employed to switch from the microscopic Hamiltonian description of the tracer particle dynamics to the macroscopic stochastic one. We then formulate the concept of "weak", or statistical, GI and show that both normal diffusive dynamics and the novel processes of Chapter 4 satisfy it, while CTRWs do not.

- **Chapter 6: Conclusions**
  In this Chapter we summarise the novel results discussed in this Doctoral Dissertation. Open questions and hints for future lines of research will be suggested.

CHAPTER 2

Mathematical Tools for Anomalous Stochastic Processes

Transport processes are usually described by adopting either a "physicist" approach, where the dynamics is described at a microscopic level in terms of random walks, or a "mathematician" approach, which consists instead of a coarse-grained description, where the dynamics is modelled in terms of stochastic Langevin equations. These different techniques equivalently lead to the same evolution equations for the probability density function of the process, i.e., the diffusion equation in the well-known case of a colloidal particle in a fluid. However, they also complement each other, by independently providing information on several different properties of the system, for instance of its stochastic trajectories in the case of the Langevin description. Thus, a full understanding of the dynamical properties of a process can only be attained, if both these two descriptions are available. In this Chapter, we build such a complete tool-kit of techniques to investigate anomalous stochastic processes.

2.1 Random Walks and Fractional Calculus

In this first section, we present an overview of random walk techniques for the analysis of diffusion processes (see Ref. [36] for a more detailed discussion). We will first introduce the concept of the random walk, i.e., the celebrated drunken sailor problem, which was first extensively studied in [58], and show how the PDF of the walker's position is naturally described by the diffusion equation [14]. Secondly, we will endow the walker with the ability of either resting in its position or making arbitrarily large random jumps. This generalisation of the random walk is called the continuous-time random walk (CTRW) [28]. We will show how this model can account for different types of anomalous dynamical behaviour, by suitably specifying the asymptotic scaling of the resting time and jump size distributions, and derive their corresponding diffusion equations [59, 60, 61, 9]. These will contain fractional operators, elucidating the non locality in space and time of the walker's dynamics in the CTRW. Finally, we will describe the Lévy walk model [56] of superdiffusion and provide a tutorial on fractional operators, which are an essential tool to investigate anomalous stochastic processes.

Let us define the notation for the integral transforms used throughout this thesis. The

Laplace transform of a function $f(t)$ with support on the positive half line will be denoted as $\widetilde{f}(\lambda)$ and it is defined as follows:

$$\widetilde{f}(\lambda) = \mathcal{L}\left\{f(t)\right\}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t)\,\mathrm{d}t\,. \tag{2.1}$$

The convolution of two such functions ϕ_1 and ϕ_2 is given by:

$$(\phi_1 * \phi_2)(t) = \int_0^t \phi_1(t-\tau)\phi_2(\tau)\,\mathrm{d}\tau\,, \tag{2.2}$$

whose Laplace transform, thanks to the convolution theorem, is given by the product of the corresponding Laplace transforms: $(\widetilde{\phi_1 * \phi_2})(\lambda) = \widetilde{\phi_1}(\lambda)\widetilde{\phi_2}(\lambda)$. On the other hand, let us now consider a function $g(x)$ defined on the all real line $\mathbb{R}$. The Fourier transform of g will be referred to as $\widehat{g}(k)$ and it is defined explicitly as:

$$\widehat{g}(k) = \mathcal{F}\left\{g(x)\right\}(k) = \int_{-\infty}^{+\infty} e^{i\,k\,x} g(x)\,\mathrm{d}x\,. \tag{2.3}$$

The convolution of two such functions g_1 and g_2 is given by:

$$(g_1 * g_2)(x) = \int_{-\infty}^{+\infty} g_1(x-y)g_2(y)\,\mathrm{d}y\,. \tag{2.4}$$

The convolution theorem also holds in this case, such that the Fourier transform of a convolution of two functions g_1 and g_2 is the product of the corresponding Fourier transforms: $(\widehat{g_1 * g_2})(k) = \widehat{g_1}(k)\widehat{g_2}(k)$. The corresponding definitions for the case of a convolution of multiple functions follow straightforwardly.

2.1.1 Random Walk and the Diffusion Equation

Let us consider a point particle on a one dimensional lattice of mesh Δx, which moves at each finite time step Δt by jumping into one of its adjacent positions. We denote with $\mathbb{P}_j(t)$ its probability of being in the lattice node j at time t. We assume that jumps to the right or to the left are independent and that they occur with probability 1/2. Our interest is in understanding how the quantity $\mathbb{P}_j$ evolves over time, as this would provide us information on both (i) the final point that the particle can reach and (ii) the area spanned by the particle if, after some finite time T, we restart its motion from the same initial position and repeat this procedure several times. According to the dynamical rules prescribed, the particle can reach a fixed position j only by jumping from one of its neighbouring sites, i.e., from the lattice sites $j \pm 1$, at the previous step. Thus, if a particle was in the lattice site $j \pm 1$ at some time t, it would arrive in the position j with probability $\mathbb{P}_{j\pm 1}(t)/2$. Thus, we obtain the following master equation [16]:

$$\mathbb{P}_j(t+\Delta t) = \frac{1}{2}\mathbb{P}_{j+1}(t) + \frac{1}{2}\mathbb{P}_{j-1}(t). \tag{2.5}$$

What happens if we consider the limit $\Delta t \to 0$, this being equivalent to the limit of a large number of jumps, and the continuum limit of the lattice mesh $\Delta x \to 0$. In the first case, one needs first to Taylor expand the lhs of Eq. (2.5) in Δt, which leads to:

$$\mathbb{P}_j(t) + \Delta t \frac{\partial}{\partial t}\mathbb{P}_j(t) + o\big(\Delta t^2\big) = \frac{1}{2}\mathbb{P}_{j+1}(t) + \frac{1}{2}\mathbb{P}_{j-1}(t). \tag{2.6}$$

In the second case, i.e., the continuum limit $\Delta x \to 0$, we consider density functions instead of probabilities: $\mathbb{P}_j(t) \to P(x,t)$ and $\mathbb{P}_{j\pm 1}(t) \to P(x'_\pm, t)$, where x represents the lattice node j and $x'_\pm$ the neighbouring sites $x \pm \Delta x$ (mathematically $\mathbb{P}_j(t) = P(x,t)\,\mathrm{d}x$). By Taylor expansion, we obtain: $P(x'_\pm, t) = P(x,t) \pm \Delta x\,\frac{\partial}{\partial x}P(x,t) + \frac{\Delta x^2}{2}\frac{\partial^2}{\partial x^2}P(x,t) + o(\Delta x^3)$. If we substitute it in Eq. (2.6) and keep only first and second order terms, we obtain the celebrated *diffusion equation* [14]:

$$\frac{\partial}{\partial t}P(x,t) = K_1 \frac{\partial^2}{\partial x^2}P(x,t) \tag{2.7}$$

where we define the *diffusion constant*: $K_1 = \lim_{\Delta x, \Delta t \to 0} \Delta x^2/(2\,\Delta t)$. By taking its Fourier-Laplace transform, we derive the solution of the corresponding initial value problem with initial condition $P(x,0) = \delta(x - x_0)$ as the following Gaussian distribution:

$$P(x,t) = \frac{1}{\sqrt{4\pi K_1 t}} \exp\left[-\frac{(x-x_0)^2}{4K_1 t}\right]. \tag{2.8}$$

The Gaussian distribution of the walker's position is a direct consequence of the central limit theorem [62, 63], here holding because we require the ratio $\Delta x^2/\Delta t$ to be finite in the limit $\Delta t, \Delta x \to 0$ in order to ensure the existence of K_1. In addition, we can easily show that $MSD(t) = 2\,K_1\,t$, i.e., the motion of the walker is an ordinary diffusive process.

2.1.2 CTRW Model and The Montroll-Weiss Equation

The celebrated Continuous-Time Random Walk model, first introduced in the seminal work of Montroll and Weiss [28], consists in an elegant generalisation of the random walk model, whose relevance is motivated by its ability of capturing complex physical phenomena, like trapping or target searching. The main idea of the CTRW is to allow the point particle of an ordinary random walk to rest for some time τ, which is random and sampled from a given distribution, before performing the next jump, which is itself sampled from a specified distribution. Thus, in order to fully characterise the dynamics of the point particle in a CTRW, we need to introduce a distribution for both the jump lengths and the waiting times between different jumps. Let us call $\psi(x,t)$ such joint distribution of jump lengths and waiting times. By integrating out one variable, one can obtain the marginal distributions. In details, we define: (i) the jump length distribution $\phi(x) = \int_0^{+\infty} \psi(x,t)\,\mathrm{d}t$ and (ii) the waiting time distribution $w(t) = \int_{-\infty}^{+\infty} \psi(x,t)\,\mathrm{d}x$. In general, both jump lengths and waiting times can be dependent (coupled CTRWs) [64, 65, 66, 67, 68, 69, 70], so that we need to use the joint distribution. In the opposite case instead, such distribution can be factorised: $\psi(x,t) = w(t)\,\phi(x)$ (uncoupled CTRWs).

Our first aim is to introduce the analogue of the master Eq. (2.5) for the dynamics of the CTRW. Let us denote $Y(t)$ the position of the walker at the time t. We further define its PDF P as the quantity: $\mathbb{P}(Y(t) \in \mathrm{d}x\,|Y(0) = 0) = P(y,t)\,\mathrm{d}y$. As both waiting times and jump lengths are independent and identically distributed (i.i.d.), we find:

$$\begin{aligned}\mathbb{P}(Y(t) \in \mathrm{d}y\,|Y(t') = y') &= \mathbb{P}(Y(t-t') \in \mathrm{d}y\,|Y(0) = y') \\ &= \mathbb{P}(Y(t-t') - y' \in \mathrm{d}y\,|Y(0) = 0) \\ &= P(y-y', t-t')\,\mathrm{d}y\,. \end{aligned} \tag{2.9}$$

Thanks to the simple structure of the CTRW dynamics outlined previously, the value of $Y(t)$ can be expressed by specifying the time of the first jump τ_1 with respect to t [71]:

$$\begin{aligned}\mathbb{P}(Y(t) \in \mathrm{d}y\,|Y(0) = 0) = \mathbb{P}(Y(t) \in \mathrm{d}y\,, \tau_1 \leq t|Y(0) = 0) \\ + \mathbb{P}(Y(t) \in \mathrm{d}y\,, \tau_1 > t|Y(0) = 0). \end{aligned} \tag{2.10}$$

Each of these terms can then be computed exactly. On the one hand, if the first jump occurs after t, i.e., $\tau_1 > t$, which happens with probability:

$$\Psi(t) = 1 - \int_0^t \int_{-\infty}^{+\infty} \psi(y,\tau)\,\mathrm{d}y\ \mathrm{d}\tau\ = 1 - \int_0^t w(\tau)\,\mathrm{d}\tau\,, \tag{2.11}$$

the particle should have been in the position y already at the initial time in order to be there at time t. As this happens with probability $\delta(y)\,\mathrm{d}y$, we find:

$$\mathbb{P}(Y(t) \in \mathrm{d}y\,, \tau_1 > t|Y(0) = 0) = \Psi(t)\,\delta(y)\,\mathrm{d}y\,. \tag{2.12}$$

If instead the first jump happens before t, i.e., $\tau_1 \leq t$, we need to condition on both the position of the walker before the jump and the exact jump time:

$$\begin{aligned}\mathbb{P}(Y(t) \in \mathrm{d}y\,, \tau_1 \leq t|Y(0) = 0) = \int_{-\infty}^{+\infty} \int_0^t \mathbb{P}(Y(t) \in \mathrm{d}y\,|Y(t') = y') \times \\ \times\, \mathbb{P}(Y(t') \in \mathrm{d}y'\,, \tau_1 \in \mathrm{d}t'\,|Y(0) = 0). \end{aligned} \tag{2.13}$$

However, the quantity $\mathbb{P}(Y(t') \in \mathrm{d}y'\,, \tau_1 \in \mathrm{d}t'\,|Y(0) = 0)$ simply represents the probability that the first jump has size $y' < Y(t') < y' + \mathrm{d}y'$ and occurs at the time τ_1 such that $t' < \tau_1 < t' + \mathrm{d}t'$. This is given in terms of the distribution ψ by the following [71]:

$$\mathbb{P}(Y(t') \in \mathrm{d}y'\,, \tau_1 \in \mathrm{d}t'\,|Y(0) = 0) = \psi(y', t')\ \mathrm{d}y'\ \mathrm{d}t'\,. \tag{2.14}$$

Thus, by substituting it into Eq. (2.13), we find:

$$\begin{aligned}\mathbb{P}(Y(t) \in \mathrm{d}y\,, \tau_1 \leq t\,|\,Y(0) = 0) &= \int_{-\infty}^{+\infty} \int_0^t \mathbb{P}(Y(t) \in \mathrm{d}y\,|Y(t') = y')\,\psi(y', t')\,\mathrm{d}y'\ \mathrm{d}t' \\ &= \left[\int_{-\infty}^{+\infty} \int_0^t P(y-y', t-t')\,\psi(y', t')\mathrm{d}y'\ \mathrm{d}t'\right]\mathrm{d}y\,, \end{aligned} \tag{2.15}$$

where we used Eq. (2.9) in the second line. Finally, by employing the results of Eqs. (2.12, 2.15) into Eq. (2.10), we obtain the following master equation:

$$P(y,t) = \int_0^t \int_{-\infty}^{+\infty} P(y-y', t-\tau)\psi(y',\tau)\,\mathrm{d}y'\ \mathrm{d}\tau\ + \delta(y)\,\Psi(t). \tag{2.16}$$

The result just derived is better understood in Laplace-Fourier space. Recalling that $\widetilde{\Psi}(\lambda) = [1 - \widetilde{w}(\lambda)]/\lambda$, we find the celebrated Montroll-Weiss equation [28]:

$$\widehat{\widetilde{P}}(k,\lambda) = \frac{1-\widetilde{w}(\lambda)}{\lambda}\frac{\widehat{P}_0(k)}{1-\widehat{\widetilde{\psi}}(k,\lambda)}, \tag{2.17}$$

where P_0 denotes the initial condition (if different from a delta function in the origin). This equation expresses the position PDF of a CTRW-type random walker in terms of the jump length and waiting time distribution. Once these are fixed, the statics of the walker's position is completely determined by inverse Fourier-Laplace transforming Eq. (2.17). As in the case of the ordinary random walk, we will be interested in the behaviour of P in the continuum and long-time limit, which in Fourier-Laplace variables is respectively equivalent to $(k, \lambda) \to (0, 0)$. Thus, the dynamics of the random walk will be classified by considering only the asymptotic behaviour of w and ϕ, which are related to the existence of a finite mean waiting time and/or jump length variance, respectively defined as:

$$\langle \tau \rangle = \int_0^\infty \tau\, w(\tau)\, \mathrm{d}\tau\,, \tag{2.18}$$

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{+\infty} x^2\, \phi(x)\, \mathrm{d}x\,. \tag{2.19}$$

We discuss in the following such classification, by first considering the case of uncoupled CTRWs. We define the following distributions for waiting times and jump lengths [56]:

$$w(\tau) = \frac{1}{\tau_0}\frac{\alpha}{[1+\tau/\tau_0]^{1+\alpha}} \qquad \alpha > 0 \tag{2.20}$$

$$\phi(x) = \frac{\Gamma(\beta+1/2)}{x_0\sqrt{\pi}\,\Gamma(\beta)\,[1+(x/x_0)^2]^{\beta+1/2}} \qquad \beta > 0 \tag{2.21}$$

We recall that the exact details of these distributions do not matter in determining the qualitative features of the dynamics of the walker [59, 9, 56]. This specific choice has been made, as their Laplace and Fourier transform respectively can be obtained analytically [72]. Their asymptotic behaviour in the limit $(k, \lambda) \to (0, 0)$ is given below:

$$\widetilde{w}(\lambda) = 1 - \frac{\tau_0}{\alpha-1}\lambda - \tau_0^\alpha\,\Gamma(1-\alpha)\,\lambda^\alpha + O(\lambda^{1+\alpha}) \tag{2.22}$$

$$\widehat{\phi}(k) = 1 - \frac{x_0^2}{\beta-1}\frac{k^2}{4} - \frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}|k|^{2\beta} + O(k^{2+2\beta}) \tag{2.23}$$

Clearly, different values of the exponents α, β corresponds to finite or divergent mean waiting time and jump length variance. We will classify these different ranges below.

Normal Diffusion Case

Let us first show that we recover normal diffusive behaviour, and specifically Eq. (2.7), if both $\langle \tau \rangle$ and $\left\langle x^2 \right\rangle$ exist finite. By looking at Eqs. (2.22, 2.23), this is obtained if we assume $\alpha > 1$ and $\beta > 1$ simultaneously. In this regime, indeed, the fractional powers become of sub-leading order compared to both the linear and quadratic terms and can be neglected.

Thus, Eqs. (2.18, 2.19) read as below:

$$\langle\tau\rangle = \frac{\tau_0}{\alpha-1}, \qquad \left\langle x^2\right\rangle = \frac{x_0^2}{2(\beta-1)}. \tag{2.24}$$

In addition, if we substitute Eqs. (2.22, 2.23), after neglecting the fractional order terms, in Eq. (2.17) and keep only first and second order terms we find:

$$\begin{aligned}\widehat{\widehat{P}}(k,\lambda) &= \frac{\tau_0}{\alpha-1}\frac{\widehat{P}_0(k)}{1-\left(1-\frac{\tau_0}{\alpha-1}\lambda\right)\left(1-\frac{x_0^2}{\beta-1}\frac{k^2}{4}\right)} \\ &= \frac{\widehat{P}_0(k)}{\lambda+\frac{x_0^2}{4(\beta-1)}\frac{\alpha-1}{\tau_0}k^2} = \frac{\widehat{P}_0(k)}{\lambda+K_1k^2},\end{aligned} \tag{2.25}$$

where we set $K_1 = \left\langle x^2\right\rangle/[2\langle\tau\rangle]$. If we now rearrange the terms, we obtain the equation:

$$\lambda\,\widehat{\widehat{P}}(k,\lambda) - \widehat{P}_0(k) = -K_1\,k^2\,\widehat{\widehat{P}}(k,\lambda), \tag{2.26}$$

whose Fourier-Laplace inverse transform leads to Eq. (2.7). In conclusion, normal diffusion, and the diffusion equation, can be recovered in the continuum and long-time limit of a CTRW with finite mean waiting time and jump length variance [9].

Long Rests Case: Subdiffusion

Let us now relax one of the conditions assumed before. Specifically, we allow for the mean waiting time to be infinite. This corresponds to a random walker that can get trapped in a fixed position for long times. This is obtained by assuming $0 < \alpha < 1$, which implies that the leading order term in Eq. (2.22) is the fractional one, which scales as λ^α. On the contrary, we still keep a jump length distribution with finite variance, i.e., $\beta > 1$. In this regime $\langle\tau\rangle$ is not finite, while $\left\langle x^2\right\rangle$ is the same as in Eq. (2.24). Thus, substituting the expansions $w(\lambda) \sim 1 - \tau_0^\alpha\,\Gamma(1-\alpha)\,\lambda^\alpha$ and $\phi(x) \sim 1 - \frac{x_0^2}{\beta-1}\frac{k^2}{4}$ in Eq. (2.17), we obtain:

$$\begin{aligned}\widehat{\widehat{P}}(k,\lambda) &= \tau_0^\alpha\Gamma(1-\alpha)\lambda^{\alpha-1}\frac{\widehat{P}_0(k)}{1-[1-\tau_0^\alpha\Gamma(1-\alpha)\lambda^\alpha]\left(1-\frac{x_0^2}{\beta-1}\frac{k^2}{4}\right)} \\ &= \frac{\lambda^{\alpha-1}\widehat{P}_0(k)}{\lambda^\alpha+\frac{x_0^2}{4(\beta-1)}\frac{1}{\tau_0^\alpha\Gamma(1-\alpha)}k^2} = \frac{\widehat{P}_0(k)}{\lambda+K_\alpha k^2\lambda^{1-\alpha}},\end{aligned} \tag{2.27}$$

where we introduce the anomalous diffusion coefficient: $K_\alpha = \left\langle x^2\right\rangle/[2\tau_0^\alpha\Gamma(1-\alpha)]$. In order to get the corresponding evolution equation, we rearrange the terms as follows:

$$\lambda\widehat{\widehat{P}}(k,\lambda) - \widehat{P}_0(k) = K_\alpha k^2\lambda^{1-\alpha}\widehat{\widehat{P}}(k,\lambda) \tag{2.28}$$

and take its Fourier-Laplace inverse transform. We note that the Laplace inverse transform of the fractional term can be derived by introducing a suitable fractional operator (details are given in Sec. 2.1.5). Thus, we obtain the fractional diffusion equation (FFPE) [28, 73]:

$$\frac{\partial}{\partial t}P(x,t) = K_\alpha\frac{\partial^2}{\partial x^2}\,{}_0D_t^{1-\alpha}P(x,t) \tag{2.29}$$

where ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville operator [74, 75, 76, 77]:

$$ {}_0D_t^{1-\alpha}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{P(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,, \tag{2.30}$$

or equivalently in Laplace transform:

$$ \mathcal{L}\left\{{}_0D_t^{1-\alpha}P(x,t)\right\}(x,\lambda) = \lambda^{1-\alpha}\widetilde{P}(x,\lambda). \tag{2.31}$$

It is now straightforward to see that $MSD(t) = 2K_\alpha t^\alpha/\Gamma(1+\alpha)$, i.e., the motion of the walker is of subdiffusive type. This is due to the presence of long waiting times, which are physically related to the occurrence of trapping events during the particle's dynamics [19]. We note that a closed form Laplace-Fourier inverse transform of Eq. (2.27) is derived in terms of a Fox H-function (see Appendix A.3 for its definitions and properties) [9]:

$$ P(x,t) = \frac{1}{\sqrt{4\,\pi\,K_\alpha\,t^\alpha}} H^{2,0}_{1,2}\left[\frac{x^2}{4\,K_\alpha\,t^\alpha}\left|\begin{matrix}\left(1-\frac{\alpha}{2},\alpha\right)\\ (0,1),\left(\frac{1}{2},1\right)\end{matrix}\right.\right] \tag{2.32}$$

Large Jump Lengths Case: Lévy Flights

We now consider the opposite case of finite mean waiting time [same as in Eq. (2.24)], but infinite jump length variance, i.e., $\alpha > 1$ and simultaneously $0 < \beta < 1$. These parameters lead to the asymptotic behaviour: $\widetilde{w}(\lambda) \sim 1 - \frac{\tau_0}{\alpha-1}\lambda$ and $\widehat{\phi}(k) \sim 1 - \frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}|k|^{2\beta}$. As before, we substitute these expressions into Eq. (2.17) to obtain [73, 78, 9, 56]:

$$\begin{aligned}\widehat{\widetilde{P}}(k,\lambda) &= \frac{\tau_0}{\alpha-1}\frac{\widehat{P}_0(k)}{1-\left(1-\frac{\tau_0}{\alpha-1}\lambda\right)\left(1-\frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}|k|^{2\beta}\right)}\\ &= \frac{\widehat{P}_0(k)}{\lambda+\frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{\alpha-1}{\tau_0}|k|^{2\beta}} = \frac{\widehat{P}_0(k)}{\lambda+K_\beta|k|^{2\beta}},\end{aligned} \tag{2.33}$$

where we defined the parameter: $K_\beta = [(x_0)^{2\beta}\Gamma(1-\beta)]/[2^{2\beta}\Gamma(1+\beta)\,\langle\tau\rangle]$. In order to derive an evolution equation, we need to rearrange its terms. In details, we obtain:

$$ \lambda\widehat{\widetilde{P}}(k,\lambda) - \widehat{P}_0(k) = -K_\beta|k|^{2\beta}\widehat{\widetilde{P}}(k,\lambda). \tag{2.34}$$

Its inverse Fourier transform can be derived exactly, if we introduce a suitable fractional operator in the x-variable. This is obtained by employing Eq. (2.81) in Sec. 2.1.5. Thus, we obtain the following fractional evolution equation:

$$ \frac{\partial}{\partial t}P(x,t) = K_\beta\,D_0^{2\beta}P(x,t) \tag{2.35}$$

where we introduce the Riesz-Feller fractional derivative:

$$ D_0^{2\beta}P(x,t) = \int_{-\infty}^{+\infty}\frac{P(y,t)}{|x-y|^{1+2\beta}}\,\mathrm{d}y \tag{2.36}$$

In this regime, the MSD of the walker is not finite (indeed the second order derivative in k of Eq. (2.37) is $\propto |k|^{2(\beta-1)}$ which diverges for $k \to 0$). This is due to the existence of instantaneous arbitrarily large jumps, which, however, lack of physical meaning for massive particles, as they would require them to have an infinite instantaneous velocity. Nevertheless, it is possible to compute an exact PDF [9], again as a Fox H-function:

$$P(x,t) = \frac{1}{\sqrt{2\,\beta\,|x|}} H^{1,1}_{2,2}\left[\frac{|x|}{(K_{2\beta}\,t)^{1/(2\beta)}}\,\middle|\,\begin{matrix}\left(1,\frac{1}{2\beta}\right),\left(1,\frac{1}{2}\right)\\ (1,1),\left(1,\frac{1}{2}\right)\end{matrix}\right], \tag{2.37}$$

which is a closed form expression of a symmetric Lévy stable distribution [63]. Such distribution has power-law decaying tails, specifically $P(x,t) \sim |x|^{-1-2\beta}$ for $|x| \to \infty$. Lévy stable processes will be reviewed in Sec. 2.2.3, but we anticipate that these processes are self-similar with index $H = 1/[2\,\beta]$, i.e., $P(x, a\,t) = a^{-H} P(a^{-H}\,x, t)$. Despite the fact that moments of order ≥ 2 are infinite, by employing Eqs. (2.37, A.26), one can determine the fractional moments of the distribution, i.e., the quantities $\left\langle |x|^{\delta}\right\rangle$, where $0 < \delta < 2\beta < 2$, which are found to scale as $t^{\delta/(2\beta)}$ for long times. After rescaling, we obtain: $MSD(t) \sim t^{1/\beta}$, i.e., the walker exhibits superdiffusive behaviour [73, 63, 17, 54, 18].

General Case: Lévy Flights with Long Rests

Let us finally consider the more general case, where both the mean waiting time and the jump length variance are not finite, which is obtained in the regime $0 < \alpha < 1$ and $0 < \beta < 1$. In this case, the fractional powers dominate the expansions given in Eqs. (2.22, 2.23), i.e., we have $w(\lambda) \sim 1 - \tau_0^{\alpha}\Gamma(1-\alpha)\lambda^{\alpha}$ and $\widehat{\phi}(k) \sim 1 - \frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}|k|^{2\beta}$. Substituting these asymptotic expressions into Eq. (2.17), we derive the following solution [9, 56]:

$$\begin{aligned}\widehat{\widehat{P}}(k,\lambda) &= \tau_0^{\alpha}\Gamma(1-\alpha)\lambda^{\alpha-1}\frac{\widehat{P}_0(k)}{1-[1-\tau_0^{\alpha}\Gamma(1-\alpha)\lambda^{\alpha}]\left(1-\frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}|k|^{2\beta}\right)}\\ &= \frac{\lambda^{\alpha-1}\widehat{P}_0(k)}{\lambda^{\alpha}+\frac{x_0^{2\beta}}{2^{2\beta}}\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)}\frac{1}{\tau_0^{\alpha}\Gamma(1-\alpha)}|k|^{2\beta}} = \frac{\widehat{P}_0(k)}{\lambda+K_{\beta,\alpha}|k|^{2\beta}\lambda^{1-\alpha}},\end{aligned} \tag{2.38}$$

where we define the parameter: $K_{\beta,\alpha} = [x_0^{2\beta}\Gamma(1-\beta)]/[2^{2\beta}\Gamma(1+\beta)\tau_0^{\alpha}\Gamma(1-\alpha)]$. Rearranging the terms, we find the following equation:

$$\lambda^{\alpha}\widehat{\widehat{P}}(k,\lambda) + K_{\beta,\alpha}|k|^{2\beta}\widehat{\widehat{P}}(k,\lambda) = \lambda^{\alpha-1}\widehat{P}_0(k), \tag{2.39}$$

whose inverse Fourier-Laplace transform is expressed in terms of fractional operators:

$${}_0D_t^{\alpha}P(x,t) = K_{\beta,\alpha}\,D_0^{2\beta}P(x,t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}P_0(x) \tag{2.40}$$

Analogously to the case before, we can compute: $MSD(t) \sim t^{\alpha/\beta}$. The dependence of the scaling exponent on the ratio α/β elucidates that the qualitative features of the walker's motion are determined by the interplay between both long waiting times and large jumps.

Parametrising the CTRW: The Subordinated Langevin Equations

We conclude this section on the CTRW model, by presenting the parametrisation of its stochastic trajectories in the continuum limit proposed by H. C. Fogedby [79], which for the first time provided a Langevin formulation of CTRWs.

In the original work of Montroll and Weiss [28], the CTRW is defined as a renewal process where both the jump lengths and the waiting times between successive jumps are selected as i.i.d RVs. A natural parametrisation of the CTRW is then obtain in terms of the number of jumps n. Let us introduce the two sets of RVs $\{(\xi_j, \eta_j)\}_{j=1,\dots,n}$, such that ξ_j and η_j respectively specify the amplitude of the jump occurring at the jth step and the waiting time between the $(j-1)$th and the jth jump. Waiting times and jump lengths may possibly be correlated. However, in the Thesis we will only focus on the uncorrelated case, i.e., the RVs $\{\xi_j\}$ are independent of $\{\eta_j\}$. Thus, the position of the walker Y, initially in the position y_0, after n jumps is obtained by summing all the variables $\{\xi_j\}$:

$$Y(n) = y_0 + \sum_{j=1}^{n} \xi_j. \tag{2.41}$$

Analogously, the total elapsed time T is obtained by summing all the variables τ_j:

$$T(n) = \sum_{j=1}^{n} \tau_j. \tag{2.42}$$

Instead of such parametrisation in terms of the discrete variable n, it is usually convenient to describe the position coordinate in terms of a continuous time variable t. By looking at Eq. (2.42), we note that T and n are complementary variables, i.e., we can employ such relation to define a stochastic process $N(t)$, which counts the number of jumps in a time interval $[0, t]$. Specifically, it can be defined as: $N(t) = \max\{n \geq 0 : T(n) \leq t\}$. Thus, the position variable can be written in terms of the continuous time t as below:

$$Y(t) = y_0 + \sum_{j=1}^{N(t)} \xi_j. \tag{2.43}$$

If we now consider the continuum limit in the number of steps of this random walk picture, $N(t)$ is no longer an integer, but it becomes a continuous variable $S(t)$, which still counts the number of waiting time increments up to t. In details, introducing a continuous parameter s, which stands for the steps of the original random walk picture, the total elapsed time after s steps [the continuum limit of Eq. (2.42)] is given by:

$$T(s) = \int_0^s \tau(s)\, \mathrm{d}s\,, \tag{2.44}$$

whereas the position after s steps [the continuum limit of Eq. (2.41)] is given by :

$$X(s) = \int_0^s \xi(s)\, \mathrm{d}s\,. \tag{2.45}$$

These equations are equivalent to the following Langevin equations for X and T [79]:

$$\dot{X}(s) = \xi(s), \tag{2.46a}$$

$$\dot{T}(s) = \eta(s). \tag{2.46b}$$

Thus, the analogue of the discrete RV $N(t)$ in the continuum limit is the continuous function of time $S(t)$ defined as the collection of first passage times (for a formal derivation in terms of functional limit theorems we refer to [80]):

$$S(t) = \inf_{s>0} \{s : T(s) > t\}. \tag{2.47}$$

Indeed, this definition ensures that S accounts exactly for the number of steps, such that the total elapsed time, i.e., the sum of the waiting time increments for each of those steps, is equal to t. Thus, the time-dependent position of the CTRW $Y(t)$ [the continuum limit of Eq. (2.43)] is naturally expressed as a time-changed process:

$$Y(t) = \int_0^{S(t)} \xi(\tau)\,\mathrm{d}\tau = X(S(t)) \tag{2.48}$$

The notion of time-changed process and its properties are addressed in details in Sec. 2.2.4. Therefore, Fogedby's approach [79] describes the resulting trajectory of the random walk in the continuum limit by parametrising both the path of the walker $X(\cdot)$ and the time elapsed $T(\cdot)$ with an arbitrary continuous arc-length s. The stochastic process $S(\cdot)$ is the inverse of $T(\cdot)$ and measures the arc-length as a function of the physical time. $S(t)$ thus represents the continuum limit of the RV $N(t)$ that counts the number of steps in the renewal picture. An illustrative representation of this procedure is reported in Fig. 2.1. We further remark that, within this framework, the effect of an external force $F(x)$ acting on the walker is accounted for by directly inserting it into Eq. (2.46a).

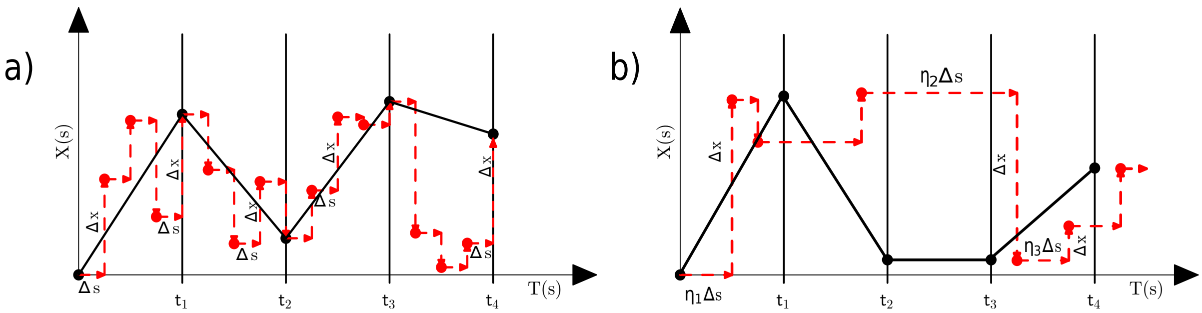

Figure 2.1: Physical interpretation of the arc-length s. We consider (i) a discretisation of the physical time t (black solid lines) of step length Δt and (ii) a discretisation of finite step length $\Delta s \ll \Delta t$ of s (not shown). We denote with Δx, ΔT the increments of the processes X and T corresponding to an increment Δs of s (red dots), which are given by Eqs. (2.46a, 2.46b). The resulting process $Y(t) = X(S(t))$, with S defined by Eq. (2.47), is plotted in black solid lines. (a) Normal Diffusion. In this case $\Delta T = \Delta s$ (T is a deterministic drift) and X is an ordinary random walk. In this setting, s coincides with the physical time t and its discretisation is a thinner time partition. (b) Anomalous Diffusion. In this case, $\Delta T = \eta_j\,\Delta s$, with η_j being a RV. In this setup, s no longer coincides with the physical time, but it provides a parametrisation of the elapsed physical time T via the definition Eq. (2.47). We note the occurrence of trapping events ($Y(t_2) = Y(t_3)$ in panel b), which are due to the variable length of ΔT.

This description of the CTRW dynamics in terms of the subordinated Langevin Eqs. (2.46a, 2.46b), together with the definition of the time-change S in Eq. (2.47), also represents a convenient method to write an algorithm for the Monte Carlo simulations of its stochastic trajectories. We present the algorithm proposed in [81], which has the advantage of not requiring an exact form of the time-change $S(t)$. An alternative algorithm can be found in [82]. In order to simulate the CTRW $Y(t)$ up to a fixed time T, we consider a partition of the interval $[0,T]$ $\pi = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ with fixed mesh Δt, i.e., $m = T/\Delta t$, and a second fixed step size Δs, such that $\Delta s \ll \Delta t$, which will serve as an update step for the arc-length parameter s. The increments of X and T corresponding to an update step in s are obtained by using an Euler scheme [83] to approximate Eqs. (2.46a, 2.46b):

$$X(s + \Delta s) = X(s) + \xi(s, \Delta s), \tag{2.49a}$$

$$T(s + \Delta s) = T(s) + \eta(s, \Delta s), \tag{2.49b}$$

where $\xi(s, \Delta s)$ and $\eta(s, \Delta s)$ for different values of s are RVs sampled respectively from the jump length distribution and the waiting time distribution of the CTRW. In the case of subdiffusive CTRWs, these are respectively a Gaussian with zero mean and variance $\propto \Delta s$ and a Lévy stable distribution of order parameter $0 < \alpha < 1$ [79]. Along with $X(s)$ and $T(s)$, also the operational time s is updated step by step: $s \to s + \Delta s$. According to Eq. (2.47), for each t_i in the partition the update is stopped whenever s reaches a value $\bar{s}$ such that: $t_i \leq T(\bar{s}) \leq t_i + \Delta t$, so that the subdiffusive process $Y(t)$ is approximated as: $Y(t_i) = X(\bar{s})$. The corresponding pseudo code is provided below.

Algorithm 1 Simulation of the Subordinated Langevin Equations

$t := 0$, $S(t) := 0$, $Y(t) := x_0$ ▷ Initialise Processes in t
$s := 0$, $X(s) := x_0$, $T(s) := 0$ ▷ Initialise Processes in s
for i=0 **to** N **do**
  $t \leftarrow i\,\Delta t$
  repeat
    if $T(s) \geq t$ **then**
      Exit For Loop
    end if
    $X(s) \leftarrow X(s) + \xi(s, \Delta s)$ ▷ Update X according to Eq. (2.49a)
    $T(s) \leftarrow T(s) + \eta(s, \Delta s)$ ▷ Update T according to Eq. (2.49b)
    $s \leftarrow s + \Delta s$
  until $T(s) < t$ ▷ Exit Update Loop when Eq. (2.47) holds.
  $S(t) \leftarrow s$
  $Y(t) \leftarrow x(s)$
end for

2.1.3 Ageing and Weak Ergodicity Breaking of Subdiffusive CTRWs

As we highlighted in the previous paragraphs, anomalous subdiffusion is obtained by the CTRW model when the mean waiting time does not exist finite. This condition is also intimately related to two other important properties of CTRWs, i.e., *ageing behaviour* and *ergodicity breaking*, that we here briefly recall. Our discussion is inspired by that of Ref. [8].

In a broad sense, a system is ageing if its statistics, measured in some time interval $[t_1, t]$, explicitly depend on the time $t_0 \neq t_1$ when the system was initially prepared. In the CTRW model such dependence is strictly related to the asymptotic properties of the waiting time distribution, which ultimately determine if the resulting process in the diffusive limit has stationary increments or not. Specifically, let us first consider a normal random walk, i.e., a normal diffusive process Y in the scaling limit, corresponding to a finite mean waiting time $\langle\tau\rangle$. In such scenario, the frequency of steps is constant and equal to $1/\langle\tau\rangle$, thus implying that Y has stationary increments. Consequently, despite the fact that the system was prepared at the earlier time t_0, no ageing effects are recorded. In particular, it is not possible to determine t_0 from the experimental measurements. For instance, its MSD, as measured in the experiment, can be computed as below [by recalling Eq. (2.25)]:

$$\left\langle [Y(t-t_0) - Y(t_1-t_0)]^2 \right\rangle = \left\langle [Y(t-t_1)]^2 \right\rangle = K_1\,(t-t_1). \tag{2.50}$$

On the contrary, let us now consider the case of a subdiffusive CTRW, which is characterised by the condition $\langle\tau\rangle \to \infty$. In this case, the frequency of steps scales as $t^{\alpha-1}/\tau_0^{\alpha}$, i.e., the dynamics slows down for long times. This effect is due to the heavy tails of the waiting time distribution. Indeed, in the long time limit the walker can sample waiting times eventually longer than the observation time, so that it can get stuck in its actual position. Consequently, Y does not have stationary increments and ageing ultimately affects its statistics. For instance, the MSD can be computed explicitly as below:

$$\begin{aligned}\left\langle [Y(t-t_0) - Y(t_1-t_0)]^2 \right\rangle &= \left\langle [Y(t-t_0) - Y(t_0)]^2 \right\rangle - \left\langle [Y(t_1-t_0) - Y(t_0)]^2 \right\rangle \\ &= K_\alpha[(t-t_0)^\alpha - (t_1-t_0)^\alpha],\end{aligned} \tag{2.51}$$

where we used the fact that the increments are still uncorrelated. Introducing the variables $t_a = t_1 - t_0$, estimating the age of the system at the beginning of the measurement, and $t_{obs} = t - t_1$, i.e., the duration of the experiment, we can rewrite it as [84, 85]:

$$MSD(t) = K_\alpha[(t_{obs} + t_a)^\alpha - t_a^\alpha], \tag{2.52}$$

which exhibit an explicit dependence on the age of the system t_a. For more details on ageing effects in CTRWs, in particular on statistical quantities, e.g., the position PDF or the two-point correlation functions, and on the fractional evolution equations, we refer to [84, 85, 86]. In [87] a more extensive study of ageing in renewal processes is discussed.

Ageing effects and non stationarity of the increments are related to the so called *weak ergodicity breaking* of subdiffusive CTRWs [88, 89, 90, 91, 92, 93]. Let us first consider a process Y with stationary increments, e.g. the scaling limit of an ordinary random walk. In this case, the ergodic hypothesis states that ensemble averages over several independent realisations of the dynamics $\langle\cdot\rangle$ and temporal averages of single trajectories of length T $\langle\cdot\rangle_T$ are the same for long measurement time T. In particular, for an ergodic process the time-averaged MSD:

$$\left\langle Y^2(t) \right\rangle_T = \frac{1}{T-t}\int_0^{T-t} \left[Y(t+t') - Y(t')\right]^2 \mathrm{d}t' \tag{2.53}$$

is the same of the ensemble averaged MSD, i.e., $\langle Y^2(t)\rangle = \lim_{T\to\infty}\left\langle Y^2(t)\right\rangle_T$. This is not the case of subdiffusive CTRWs, and more generally of random walk and trap models with power-law distributed sojourn times [94], for which $\langle Y^2(t)\rangle \neq \lim_{T\to\infty}\left\langle Y^2(t)\right\rangle_T$. Here, ergodicity breaking occurs because the time that the system waits inside a trap, before being able to escape it, becomes infinitely long in the limit $T \to \infty$. In such limit indeed the system eventually samples waiting times from the power-law tails of the waiting time distribution. Thus, despite being theoretically possible for the system to span the all phase-space, thus recovering ergodicity, this would happen in an infinite time, i.e., not in physical conditions. This mechanism, manifestly related to ageing, leads to the so called *weak* ergodicity breaking. On the contrary, other systems may exhibit *strong* ergodicity breaking, in which case the phase-space itself is divided into closed sets, which do not communicate between themselves. Consequently, after the system is initially prepared in one of this sets, it is no longer able to leave it and explore the rest of its phase-space [94].

In the specific case of subdiffusive CTRWs one can show that the time-averaged MSD scales linearly in time and that the diffusion coefficients (the slopes of the MSD) of different trajectories exhibits a broad distribution centred around $a^2\, A\, T^{\alpha-1}$. Here, a^2 is the finite variance of the jump length distribution, α is the scaling exponent of the waiting time distribution and A a numerical factor [89]. Thus, when analysing data from single-particle tracking experiments, where only a few trajectories are typically available and temporal averages are employed, one needs to take such effect into account to correctly classify the observed process.

2.1.4 Lévy Walks

As suggested in the previous section, superdiffusive behaviour in the CTRW model can be obtained by assuming a heavy-tailed jump length distribution, which, however, leads to a PDF of the walker's position, whose moments of order larger than two are infinite. Such a model can not represent the motion of massive particles, as it would require them to have instantaneous infinite velocity. An alternative model ensuring that the particle's velocity is finite at any time is the Lévy walk model [52, 73, 95, 96, 56]. In this section, we present the derivation of the position PDF of a Lévy walker, i.e., the analogue of the Montroll-Weiss Eq. (2.17). In addition to its relevance for biological systems, this derivation represents a further application of the random walk techniques and of fractional calculus.

We consider a point particle, whose dynamics consists of (i) periods of random duration during which it moves with fixed velocity, i.e., the flights, and (ii) random changes of direction of motion occurring at the end of each flight. The dynamics of the particle is completely determined by defining the characteristic speed v_0 during the flights and the flight time PDF $\psi(t)$. Similarly to the derivation of the Montroll-Weiss Eq. (2.17), our first goal is to write the corresponding master equation for the frequency of changes of direction $\nu(x,t)$. The argument at this point is the following: a change of direction occurs at the position x at time t if the particle is at that time at the end of a flight, namely if it started a flight of duration τ at an earlier time $t-\tau$ and moved straight with fixed velocity for the remaining time. This is summarised in the master equation:

$$\nu(x,t) = \int_{-\infty}^{+\infty} \int_0^t \phi(y,\tau)\, \nu(x-y,t-\tau)\, \mathrm{d}\tau\ \mathrm{d}y\ + \delta(t) f_0(x) \tag{2.54}$$

where we introduced the transition probability density $\phi(y,t)$, which couples the time duration of the flight and the distance travelled. Differently from the case of Lévy flights, this coupling, which is due to the assumption of a fixed flight velocity, naturally sets the distance that the walker can travel during a flight of finite duration t. This was indeed not the case for Lévy flights. Specifically, we assume:

$$\phi(x,t) = \frac{1}{2}\delta(|x| - v_0\, t)\psi(t), \tag{2.55}$$

meaning that the walker can only travel a distance $|v_0\, t|$ during a flight of duration t. Consequently, if x_0 is the initial position and T is the total duration of the motion, the walker's position will never exceed $x_0 \pm v_0\, T$. The source term in the rhs of Eq. (2.54), which depends on the Dirac delta function, originates from the assumption that the particle starts a new flight at the initial time. The position PDF at the point (x, t) is now obtained by accounting for all the possible flights that can get the particle at that specified position, but that do not terminate at time t. This is written as:

$$P(x,t) = \int_0^t \int_{-\infty}^{+\infty} \nu(x-y, t-\tau)\, \Phi(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau \tag{2.56}$$

with the following definition for Φ:

$$\Phi(y,\tau) = \frac{1}{2}\delta(|x| - v_0\, t)\Psi(t), \tag{2.57}$$

where $\Psi(t) = 1 - \int_0^t \psi(\tau)\, \mathrm{d}\tau$ is the probability that the flight time is longer than t. Finally, Eq. (2.56) can be solved in Fourier-Laplace space as below:

$$P(k,\lambda) = \frac{[\Phi(\lambda + i\, v_0\, k]) + \Psi(\lambda - i\, v_0\, k)]P_0(k)}{2 - [\psi(\lambda + i\, v_0\, k) + \psi(\lambda - i\, v_0\, k)]} \tag{2.58}$$

This equation is the counterpart of the Montroll-Weiss equation for CTRWs and it expresses the position PDF of the Lévy walker in terms of the flight time distribution. The next step is to investigate its asymptotic behaviour in terms of the one of ψ and relate the characteristic features of the particle motion to the exponent of the heavy-tails of the flight time distribution. We refer to the recent review [56] for this detailed discussion.

2.1.5 A Tutorial on Fractional Derivatives

As shown by the calculations of the earlier sections, fractional operators naturally appear in the coarse-grained description of anomalous processes, as they are reminiscent of the power-law tails of the distributions of waiting times and/or jump lengths, whose exponent determines the qualitative anomalous features of the dynamics [59, 9, 56]. In addition, several different complex phenomena have been successfully described within the framework of fractional calculus, which has become an essential tool to investigate systems affected by long-term memory effects, spatial heterogeneity or non stationary and/or non ergodic statistics [97]. Due to the relevance that fractional calculus has gained as a fundamental tool to describe out-of-equilibrium systems, in this section we provide the reader with a brief overview on the necessary definition and properties of fractional derivatives and integrals to understand the content of this thesis and to work with fractional equations in a broader sense. The following discussion will be based on Refs. [74, 98, 75, 9, 76, 77].

The first notion that needs to be presented is that of a fractional integral. We define the *Riemann-Liouville (RL) fractional integral* as the following convolution integral:

$$ {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau \tag{2.59} $$

for an arbitrary function f, preserving the convergence of the integral, and a general complex parameter α such that $\mathrm{Re}\,(\alpha) > 0$. These fractional integrals satisfy: (i) the semi-group property, i.e., ${}_{t_0}D_t^{-\beta}\, {}_{t_0}D_t^{-\alpha} f(t) = {}_{t_0}D_t^{-\alpha-\beta} f(t)$ and (ii) the commutative property, i.e., ${}_{t_0}D_t^{-\alpha}\, {}_{t_0}D_t^{-\beta} f(t) = {}_{t_0}D_t^{-\beta}\, {}_{t_0}D_t^{-\alpha} f(t)$. A *fractional derivative* is then defined in terms of both integer derivatives and fractional integrals. Let β be a complex parameter with $\mathrm{Re}\,(\beta) > 0$ and let n be an integer, such that $n-1 < \mathrm{Re}\,(\beta) \leq n$. We define the fractional derivative of order β as below:

$$ {}_{t_0}D_t^{\beta} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, {}_{t_0}D_t^{\beta-n} f(t). \tag{2.60} $$

An important specific case of Eqs. (2.59, 2.60) is obtained when $t_0 = 0$. In this case, we then obtain (i) the fractional integral:

$$ {}_{0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau\,, \tag{2.61} $$

which has a well defined Laplace transform:

$$ \mathcal{L}\{{}_{0}D_t^{-\alpha} f(t)\}(\lambda) = \lambda^{-\alpha}\, \widetilde{f}(\lambda) \tag{2.62} $$

and (ii) the *RL operator*:

$$ {}_{0}D_t^{\beta} f(t) = \frac{1}{\Gamma(n-p)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{1-n+\beta}}\, \mathrm{d}\tau\,, \tag{2.63} $$

again with n being an integer such that $n-1 < \mathrm{Re}\,(\beta) \leq n$. If we set $\beta = 1-\alpha$ $(0 < \alpha < 1)$, we recover Eq. (2.30). It is convenient to derive its Laplace transform. This is given by:

$$ \mathcal{L}\left\{{}_{0}D_t^{\beta} f(t)\right\}(\lambda) = \lambda^{\beta} f(\lambda) - \sum_{q=0}^{n-1} c_q \lambda^q, \qquad c_q = \lim_{t\to 0^+} {}_{0}D_t^{\beta-1-q} f(t). \tag{2.64} $$

Let us discuss two simple examples of application of the formulas provided. First, we compute the quantity ${}_{0}D_t^{p} t^{\mu}$, with $\mu \in \mathbb{R}$. Let us first discuss the case $p < 0$, which corresponds to compute the fractional integral Eq. (2.59). In details, we have:

$$ {}_{0}D_t^{p} t^{\mu} = \frac{1}{\Gamma(-p)} \int_{0}^{t} \frac{\tau^{\mu}}{(t-\tau)^{1+p}}\, \mathrm{d}\tau \; = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-p)} t^{\mu-p} \tag{2.65} $$

with the additional condition $\mu > -1$, needed to ensure the convergence of the integral. In the opposite case $p > 0$, we need to compute the Riemann-Liouville operator in Eq. (2.60).

We assume that $m-1 < \mathrm{Re}\,(p) \le m$, with $m \in \mathbb{N}_0$. Recalling Eqs. (2.63, 2.66), we obtain:

$$
\begin{aligned}
{}_0D_t^p t^\mu &= \frac{1}{\Gamma(m-p)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{\tau^\mu}{(t-\tau)^{1-m+p}}\, \mathrm{d}\tau \\
&= \frac{\mathrm{d}^m}{\mathrm{d}t^m} \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+m-p)} t^{\mu+m-p} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-p)} t^{\mu-p}.
\end{aligned} \tag{2.66}
$$

As before, we need to assume: $\mu > -1$. We also note that in the particular case $\mu = 0$, we obtain: ${}_0D_t^p 1 = t^{-p}/\Gamma(1-p)$, i.e., fractional derivatives of a constant are not zero, differently from the case of ordinary integer-order derivatives. Interestingly, we note that a direct application of Eq. (2.64) would have required a more stringent condition on the exponent μ, i.e., $\mu > m$, in order to ensure that the coefficients c_q were finite [75], and in this specific case null. As a consequence, one needs to work carefully with the quantities defined in Eqs. (2.61, 2.63), usually first solving for the fractional integral Eq. (2.61), which has a well defined Laplace transform, and then performing the integer order derivatives. As shown in the simple example above, this procedure provides results which are not affected by the explicit dependence on the boundary condition $t_0 = 0$. To clarify this issue, we discuss another general example. We consider the RL operator in Eq. (2.60) with $0 < \beta < 1$ and perform explicitly the time derivative. Naively, we would obtain:

$$
\frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, \mathrm{d}\tau = \frac{1}{\Gamma(1-\beta)} \left[\left. \frac{f(\tau)}{(t-\tau)^\beta} \right|_{\tau=t} - \beta \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\beta}}\, \mathrm{d}\tau \right] \tag{2.67}
$$

where the first term does not exist, as the integrand function has a pole in $t = \tau$. On the contrary, if we first work on the fractional integral and then take its time derivative, we obtain a well defined equation. In details, we can write the following:

$$
\begin{aligned}
\frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, \mathrm{d}\tau &= \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(\tau) \left[-\frac{1}{1-\beta} \frac{\partial}{\partial\tau} (t-\tau)^{1-\beta} \right] \mathrm{d}\tau \\
&= \frac{-1}{\Gamma(2-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \left[f(\tau)(t-\tau)^{1-\beta} \Big|_{\tau=0}^{\tau=t} - \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta-1}}\, \mathrm{d}\tau \right] \\
&= \frac{-1}{\Gamma(2-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \left[f(0) t^{1-\beta} - \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta-1}}\, \mathrm{d}\tau \right] \\
&= \frac{1}{\Gamma(2-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta-1}}\, \mathrm{d}\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0) \\
&= \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta}\, \mathrm{d}\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0),
\end{aligned} \tag{2.68}
$$

under the assumption that f is continuous in $t = 0$. As a sanity check, one can verify that the two sides of this equation are equal when we Laplace transform them. The first term in the rhs of Eq. (2.68) is a *Caputo fractional derivative of order β*. In this Thesis we will not employ such fractional derivatives. Thus, we refer the interested reader to Ref. [98] for a review on its properties compared with those of the RL fractional derivative. As a second example, we calculate the RL derivative of an exponential: ${}_0D_t^p e^{a\,t}$ ($a \in \mathbb{R}$). In details:

$$
\begin{aligned}
{}_0D_t^p e^{a\,t} &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\, {}_0D_t^p t^n \\
&= \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\Gamma(1+n)}{\Gamma(1+n-p)} t^{n-p} = \frac{t^{-p}}{\Gamma(1-p)}\, {}_1F_1(1; 1-p; a\,t),
\end{aligned}
\tag{2.69}
$$

where we used Eq. (2.66) and we introduce the confluent hypergeometric function ${}_1F_1$ (see Appendix A.1 for details).

To conclude this brief tutorial, we introduce fractional integrals, and derivatives, in space, i.e., where $t_0 \to -\infty$ in Eq. (2.59). Differently from the original case, the fractional integrals are no longer supported on a compact interval, which results in a better behaviour under transformations. In addition, they are naturally suitable to be used when fractional powers appear in the Fourier variable of the Montroll-Weiss Eq. (2.17), which is usually the characteristic feature of superdiffusive anomalous dynamics. For instance, we have shown earlier that this is the case of Lévy flights and walks. When considering fractional integrals on $\mathbb{R}$, we need to introduce both left-sided and right-sided Liouville fractional integrals. These are given respectively by the formulas below:

$$
(I_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}}\, \mathrm{d}y \tag{2.70}
$$

$$
(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(x-y)^{1-\alpha}}\, \mathrm{d}y \tag{2.71}
$$

where $x \in \mathbb{R}$, $\mathrm{Re}\,(\alpha) > 0$ and f is chosen arbitrarily, but preserving the convergence of the integral. The corresponding left-sided and right-sided fractional derivatives are defined similarly to Eq. (2.60) (with $n = 1 + [\mathrm{Re}\,(\alpha)]$, $\mathrm{Re}\,(\alpha) \geq 0$ and $x \in \mathbb{R}$):

$$
{}_{-\infty}D_x^\alpha f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} (I_+^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{\alpha-n+1}}\, \mathrm{d}y \tag{2.72}
$$

$$
{}_xD_\infty^\alpha f(x) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} (I_-^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{\alpha-n+1}}\, \mathrm{d}y \tag{2.73}
$$

For instance, when $\alpha = n$, $n \in \mathbb{N}_0$, we recover the ordinary derivatives: (i) ${}_{-\infty}D_x^0 f(x) = f(x) = {}_xD_\infty^0 f(x)$ and (ii) ${}_{-\infty}D_x^n f(x) = f^{(n)}(x) = {}_xD_\infty^n f(x)$. Another remarkable case is when $0 < \mathrm{Re}\,(\alpha) < 1$. From Eqs. (2.72, 2.73) we obtain:

$$
{}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{\alpha-[\mathrm{Re}\,\alpha]}}\, \mathrm{d}y \tag{2.74}
$$

$$
{}_xD_\infty^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{\alpha-[\mathrm{Re}\,\alpha]}}\, \mathrm{d}y \tag{2.75}
$$

As already suggested, these fractional integrals and derivatives generate fractional powers when we compute their Fourier transform. Indeed, one can show the relations [76]:

$$
\mathcal{F}\{(I_\pm^\alpha f)(x)\}(k) = (\mp i\,k)^{-\alpha} \widehat{f}(k) \tag{2.76}
$$

$$
\mathcal{F}\{{}_{-\infty}D_x^\alpha f(x)\}(k) = (-i\,k)^{\alpha} \widehat{f}(k) \tag{2.77}
$$

$$
\mathcal{F}\{{}_xD_\infty^\alpha f(x)\}(k) = (i\,k)^{\alpha} \widehat{f}(k) \tag{2.78}
$$

with the definition $(\pm i\,k)^\alpha = |k|^\alpha e^{\mp\pi i\,\mathrm{sign}\,(k)\alpha/2}$. However, in physical applications slightly different fractional powers with no imaginary part appears [look for instance Eq. (2.34)]. Thus, we would like to construct such a fractional integral and derivative from the earlier definitions. This is obtained by considering the symmetrised fractional integral:

$$(I_0^\alpha f)(x) = \frac{(I_+^\alpha f)(x) + (I_-^\alpha f)(x)}{2\cos(\alpha\pi/2)} = \frac{1}{2\cos(\alpha\pi/2)\Gamma(\alpha)} \int_{-\infty}^{+\infty} \frac{f(y)}{|x-y|^{1-\alpha}}\,\mathrm{d}y\,, \tag{2.79}$$

which is valid for $\alpha \notin \mathbb{N}$. We call this fractional integral the *Riesz potential.* From Eq. (2.76) it follows straightforwardly the following Fourier transform:

$$\mathcal{F}\{(I_0^\alpha f)(x)\}(k) = |k|^{-\alpha}\widehat{f}(k). \tag{2.80}$$

The corresponding fractional derivative, called the *Riesz fractional derivative*, is then defined as below (by analytic continuation):

$$(D_0^\alpha f)(x) = -(I_0^{-\alpha} f)(x) \qquad\qquad \mathcal{F}\{(D_0^\alpha f)(x)\}(k) = -|k|^\alpha \widehat{f}(k). \tag{2.81}$$

We further recall that Eq. (2.81) is a special case of the Riesz-Feller fractional derivative, defined in Fourier transform as $\mathcal{F}\{(D_\theta^\alpha f)(x)\}(k) = -\psi_\alpha^\theta(k)\widehat{f}(k)$, with the definition $\psi_\alpha^\theta(k) = |k|^\alpha e^{i\,\mathrm{sign}(k)\,\theta\,\alpha/2}$ for $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. As we will discuss in details later in this Chapter, $-\psi_\alpha^\theta$ is the logarithm of the characteristic function of a Lévy stable distribution with stability and skewness parameters α and θ respectively. We conclude by calculating the Riesz fractional derivative of (i) the exponential function $e^{b\,x}$, $b \in \mathbb{R}$ and (ii) the power-law function $|x|^\beta$, $\beta \neq 0$. In the former case, we find:

$$D_0^\alpha e^{b\,x} = \mathcal{F}^{-1}\{-|k|^\alpha \delta(k - i\,b)\}(x) = -|b|^\alpha \mathcal{F}^{-1}\{\delta(k - i\,b)\}(x) = -|b|^\alpha e^{b\,x}. \tag{2.82}$$

We note that if we set $b = 0$, we obtain zero. This means that the Riesz fractional derivative of a constant is null, like in the case of ordinary derivatives. In the latter case, we obtain:

$$D_0^\alpha |x|^\beta = \sqrt{\frac{2}{\pi}}\Gamma(1+\beta)\sin\left(\frac{\pi\beta}{2}\right)\mathcal{F}^{-1}\{|k|^{\alpha-\beta-1}\}(x) = -\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha-\beta)}\frac{\sin\left(\frac{\pi\beta}{2}\right)}{\sin\left(\frac{\pi(\alpha-\beta)}{2}\right)}|x|^{\alpha-\beta}. \tag{2.83}$$

2.2 Stochastic Processes and Itô Calculus

In this section, we provide an overview of the theory of stochastic processes and Itô calculus. These notions complement the random walk approach, by providing the essential methods to develop a coarse-grained description of anomalous processes in terms of subordinated Langevin dynamics, which has already been suggested in Sec. 2.1.2. Our aim will be to put robust ground to such formulation and construct a tool-kit, comprising essential notions and techniques to work with it. Purposely, we will avoid proofs of the theorems reported and refer to mathematical textbooks for them [99, 100, 101, 102, 103, 104, 105, 106, 107]. We begin with a preliminary discussion on the notation of this section and on the definitions of convergence, which will clarify the range of applicability of the results presented.

A stochastic process $Y = (Y(t), t \geq 0)$ is a collection of Random Variables (RVs) indexed by time, or equivalently a random function of time. We define its PDF $P(y,t)$ as the function such that $\forall t \geq 0$ the quantity $P(y,t)\,\mathrm{d}y$ determines the probability that $y \leq Y(t) < y + \mathrm{d}y$, with $\mathrm{d}y$ denoting an infinitesimal increment of y. We will denote with $\langle\cdot\rangle$ averages over all different realisations of Y. For a general function f of Y, this is given by the following integral expression:

$$\langle f(Y(t))\rangle = \int_{-\infty}^{+\infty} f(y)P(y,t)\,\mathrm{d}y\,. \tag{2.84}$$

We mention two important specific cases: (i) $f(y) = y^n$, $n \in \mathbb{N}$, which provides the moments of Y; (ii) $f(y) = e^{i\,k\,y}$, which leads to its characteristic function $\phi_Y(k,t)$. From the definition Eq. (2.84), it follows straightforwardly that the PDF can be written as the average: $P(y,t) = \langle\delta(y - Y(t))\rangle$. Equivalently, one can define the joint PDF of Y $P(y_1,t_1;\,\ldots;y_N,t_N)$ as that function such that $P(y_1,t_1;\,\ldots;y_N,t_N)\,\mathrm{d}y_1\,\ldots\mathrm{d}y_N$ is the probability of simultaneously satisfying the N relations: $\{y_1 \leq Y(t_1) < y_1 + \mathrm{d}y_1\,,\,\ldots,$ $y_N \leq Y(t_N) < y_N + \mathrm{d}y_N\,\}$. Multipoint averages are given by the integral expression:

$$\langle f(Y(t_1),\ldots,Y(t_N))\rangle = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(y_1,\ldots,y_N)P(y_1,t_1;\ldots;y_N,t_N)\,\mathrm{d}y_1\,\ldots\mathrm{d}y_N\,, \tag{2.85}$$

which also implies that $P(y_1,t_1;\,\ldots;y_N,t_N) = \langle\delta(y_N - Y(t_N)),\ldots\delta(y_1 - Y(t_1))\rangle$. In the specific case $f(y) = y$, we obtain the multipoint correlation functions of Y, whereas for $f(y) = e^{i\,k\,y}$ we obtain the multipoint characteristic function $\phi_Y(k_1,t_1;\ldots;k_N,t_N)$.

In many applications, it is necessary to know the PDF of Y at the time t conditioned to the knowledge of its value at an earlier time s. If we call such value x, this quantity is denoted as $P(y,t|x,s)$. Conditional averages are then defined accordingly:

$$\langle f(Y(t))|Y(s) = x\rangle = \int_{-\infty}^{+\infty} f(y)P(y,t|Y(s) = x)\,\mathrm{d}y\,. \tag{2.86}$$

The definition of conditional probability density can be extended to the case when the value of Y at multiple times $\tau_1,\ldots,\tau_M$, $M \in \mathbb{N}$ is known. The following relation holds:

$$P(y_1,t_1;\,\ldots;y_N,t_N|x_1,\tau_1;\,\ldots;x_M,\tau_M) = \frac{P(y_1,t_1;\,\ldots;y_N,t_N;x_1,\tau_1;\,\ldots;x_M,\tau_M)}{P(x_1,\tau_1;\,\ldots;x_M,\tau_M)}. \tag{2.87}$$

Finally, we review the main notions of convergence of RVs that will be needed in the following discussion. Let us consider a sequence of RVs $\{X_n\}_{n\in\mathbb{N}}$ and let Y be a different RV. We say that the RVs X_n converge to X:

- *almost surely* (a.s.) if $\lim_{n\to\infty} X_n = X$ for each realisation of the RVs;
- *in p-average* if $\lim_{n\to\infty}\langle|X_n - X|^p\rangle = 0$;
- *in probability* if $\lim_{n\to\infty}\mathbb{P}(|X_n - X| > \epsilon) = 0$ for all $\epsilon > 0$;
- *in distribution* if $\lim_{n\to\infty}\langle f(X_n)\rangle = \langle f(X)\rangle$ (denoted as $X_n \stackrel{d}{=} X$) for every continuous and bounded function f.

We note that these notions can be used to clarify in what sense any property holds for a general stochastic process Y. For instance, if a certain property is satisfied a.s., then it holds for each realisation of Y. Proving that a certain property holds a.s. is usually a challenging task, that will not be discusses in this context. Here, we will mainly justify theorems and formulas on quadratic average, i.e., in p-average with $p = 2$.

2.2.1 Finite Variation Processes

In this first section, we define stochastic processes with paths of finite variation and present the definition of the stochastic integral with respect to them. As a preliminary step, we introduce the concept of total variation of a real-valued function g, defined on an interval $[s, t]$. Intuitively, this quantity will quantify the total increment of g on such interval. In mathematical terms, we introduce a partition of the interval $\pi = \{s = t_0 < t_1 < \ldots < t_n = t\}$, whose mesh is given by the maximum of the lengths of the subintervals: $|\pi| = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$ and compute the following quantity:

$$V_t^\pi(g) = \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|, \tag{2.88}$$

whose value depends on the specific π chosen. Let us now consider the set of all possible partitions $\mathcal{P} = \{\pi_n\}$ and the corresponding variations of g with respect to them $\{V_t^{\pi_n}(g)\}$. The total variation of g on $[0, t]$ is obtained by taking the supremum of this set:

$$V_t(g) = \sup_{\pi \in \mathcal{P}} V_t^\pi(g). \tag{2.89}$$

Thus, if $V_t(g) < \infty$, then g is said to be of finite variation and $V_t(g)$ is the total variation of g on the chosen interval; otherwise, it is said to have infinite variation. If g is defined over all $\mathbb{R}$, then g has finite variation if it is of finite variation on all closed intervals of $\mathbb{R}$. In addition, if g is a non decreasing function, then it is of finite variation, as $V_t(g) = g(t_n) - g(t_0)$. Conversely, if g is of finite variation, we can always find two auxiliary non decreasing functions g_1 and g_2, such that $g = g_1 + g_2$. In a similar way, a stochastic process Y is said to be of finite variation if its stochastic trajectories $Y(t)$ have finite variation almost surely. An analogous definition holds in the opposite case of a process of infinite variation. Thus, the total variation provides fundamental information on the properties of the stochastic trajectories of Y. We further remark that ordinary integrals (in Lebesgue sense) of a continuous stochastic process are processes of finite variation.

We conclude by defining the stochastic integration with respect to a finite variation process Y. We will restrict our discussion to the subclass of processes with continuous stochastic paths. These integrals can be defined straightforwardly as Lebesgue-Stieltjes integral with the proper measure associated to Y, which exists due to the assumption of finite variation [108]. In terms of Riemann sums, if we define a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$, the stochastic integral of an arbitrary function $H(t)$ with respect to Y is defined as follows:

$$\int_0^t H(\tau)\,\mathrm{d}Y(\tau) = \lim_{\substack{n\to\infty \\ |\pi|\to 0}} \sum_{i=1}^{n} H(t_i)[Y(t_i) - Y(t_{i-1})]. \tag{2.90}$$

We note that H does not need to be continuous. The only assumption needed is that its paths are right continuous with left limits (*càdlàg*). Under these same assumptions, the following result for a general differentiable function f of Y holds:

$$f(Y(t)) = f(y_0) + \int_0^t f'(Y(\tau))\,\mathrm{d}Y(\tau)\,, \tag{2.91}$$

where we denote: $f'(x) = \frac{\partial}{\partial x} f(x)$. Considering again the partition π and employing the mean value theorem, this is proven as below:

$$\begin{aligned} f(Y(t)) - f(y_0) &= \sum_{i=1}^{N} [f(Y(t_i)) - f(Y(t_{i-1})] \\ &= \sum_{i=1}^{N} f'(Y(t_i))(t_i - t_{i-1}) = \int_0^t f'(Y(\tau))\,\mathrm{d}\tau\,. \end{aligned} \tag{2.92}$$

If the paths of Y are instead càdlàg, the Riemann sum in Eq. (2.90) still converges to the stochastic integral, which has an additional contribution coming from the jumps of Y. However, we will not discuss further details in this context.

2.2.2 Brownian Processes and the Itô Formula

Brownian Motion and Stochastic Integral

We define the Brownian Motion, also denoted as Wiener process, a stochastic process $B = (B(t), t \geq 0)$, with initial condition $B(0) = B_0$, satisfying the following properties:

A) $B_0 = 0$ a.s.;

B) non overlapping increments are independent, i.e., $\forall k \geq 2$ and for each partition $0 \leq t_0 < t_1 < \ldots < t_k$ the RVs $\{B(t_i) - B(t_{i-1})\}_{i=1,\ldots,k}$ are independent;

C) increments of the process B are stationary, i.e., for all $0 \leq s_1 < s_2 \leq t$ the increment $B(s_2) - B(s_1)$ has the same distribution of $B(s_2 - s_1)$, which is a Gaussian with null average and variance $s_2 - s_1$;

D) its stochastic trajectories are a.s. continuous.

We note that a process B, satisfying these properties, can be formally constructed (for a general derivation we refer to Ref. [109]). The definition of a d-dim Brownian motion follows straightforwardly, by simply replacing the 1-dim Gaussian distribution with a d-dim one having variance $(t - s)I_d$ (I_d is the unitary $d \times d$ matrix).

In physical applications, one usually does not introduce the process B directly, but the so called *white Gaussian noise* instead. We here clarify the relation between this new object and B. A white Gaussian noise is a continuous stochastic process $\xi = (\xi(t), t \geq 0)$ with the formal properties: $\langle \xi(t) \rangle = 0$ and $\langle \xi(t_1)\xi(t_2) \rangle = D\,\delta(t_2 - t_1)$, which is related to a finite increment of B, i.e., $B(t + \Delta t) - B(t)$ with Δt finite, trough the equation:

$$B(t + \Delta t) - B(t) = \int_t^{t+\Delta t} \xi(\tau)\,\mathrm{d}\tau\,. \tag{2.93}$$

Such relation, together with the specific two point correlation function of ξ, can be employed to show that the properties A)-D) are preserved. In particular, the independence of non overlapping increments follows straightforwardly from the calculation below:

$$\begin{aligned}\langle [B(t+\Delta t) - B(t)][B(s+\Delta t) - B(s)]\rangle &= \int_t^{t+\Delta t}\int_s^{s+\Delta t} \langle \xi(s')\xi(s'')\rangle \,\mathrm{d}s'\,\mathrm{d}s'' \\ &= \int_t^{t+\Delta t}\int_s^{s+\Delta t} \delta(s'-s'')\,\mathrm{d}s'\,\mathrm{d}s'' = 0, \end{aligned} \tag{2.94}$$

which holds for every $s < s+\Delta t < t < t+\Delta t$. In addition, we can compute the variance:

$$\begin{aligned}\langle [B(t) - B(s)]^2\rangle &= \int_t^{s}\int_t^{s} \langle \xi(s')\xi(s'')\rangle \,\mathrm{d}s'\,\mathrm{d}s'' \\ &= \int_t^{s}\int_t^{s} \delta(s'-s'')\,\mathrm{d}s'\,\mathrm{d}s'' = t-s, \end{aligned} \tag{2.95}$$

which is in agreement with property B). The remaining properties can be shown similarly. When $\Delta t \to 0$, the discrete increments become differentials and Eq. (2.93) reduces to:

$$\mathrm{d}B(t) = \xi(t)\,\mathrm{d}t\,. \tag{2.96}$$

This relation is essential to make sense of integrals over ξ, which appears in the solution of ordinary Langevin equations. These are indeed interpreted as stochastic integrals over B.

As it will be discussed extensively in Chapter 3, if one is interested in the analysis of experimental data, essential information on the nature of the observed dynamical process are provided by its higher order correlation functions, i.e., by quantities of the type of Eq. (2.85). Thus, it is important to provide methods to determine such objects analytically. In the case of Brownian motion, and more generally of Gaussian stochastic processes, these are the *Wick's theorem* [110, 111] and the *Novikov's theorem* [112]. On the one hand, the Wick's theorem directly relates the hierarchy of nth order correlation functions of Gaussian distributed RVs, and hence of Gaussian processes, to their two point one. Specifically, if we consider a random n-dimensional vector of Gaussian distributed RVs with zero mean $(Y_1, Y_2, \ldots, Y_n)$, its higher order correlation function is characterised as follows ($n, \nu \in \mathbb{N}$):

$$\langle Y_1, \ldots, Y_n\rangle = \begin{cases} 0 & n = 1+2\,\nu \\ \frac{1}{\nu!\,2^\nu} \sum_{\sigma\in S^{2\nu}} \prod_{i=1}^{\nu-1} \langle Y_{\sigma(i+1)} - Y_{\sigma(i)}\rangle & n = 2\,\nu \end{cases} \tag{2.97}$$

where $S^{2\,\nu}$ is the space of all permutations of 2ν elements. Thus, the theorem states that, while the odd correlation functions of Gaussian distributed RVs are null, the even ones can be factorised in terms of the two point one, which then provides a complete characterisation of the statistics of Y. As an example, for the white Gaussian noise $\xi(t)$ we obtain:

$$\langle \xi(t_1)\ldots\xi(t_n)\rangle = \begin{cases} 0 & n = 1+2\,\nu \\ \frac{D^\nu}{\nu!\,2^\nu} \sum_{\sigma\in S^{2\,\nu}} \prod_{i=1}^{\nu-1} \delta\big(t_{\sigma(i+1)} - t_{\sigma(i)}\big) & n = 2\,\nu \end{cases} \tag{2.98}$$

On the other hand, Novikov's theorem provides a general method to compute averages of the type $\langle Y(t)W[Y]\rangle$, where Y is a Gaussian stochastic process with general two point correlation function $\langle Y(t)Y(t')\rangle = C(t,t')$ and W is a general functional of Y, i.e., W

depends on the trajectory of Y in the interval $[0, t]$. The theorem states the following:

$$\langle Y(t)W[Y]\rangle = \int_0^t C(t,t') \left\langle \frac{\delta W[Y]}{\delta Y(t')} \right\rangle \mathrm{d}t' \,, \tag{2.99}$$

where the term inside the brackets in the rhs is a functional derivative of W. In the original formulation of [112], the Gaussian process may depend on both time and space coordinates, but Eq. (2.99) still holds possibly within a different integration region. The proof of this theorem follows straightforwardly by functional Taylor expansion of W, which then enables one to compute exactly the quantities $\langle Y(t)W[Y]\rangle$ and $\left\langle \frac{\delta W[Y]}{\delta Y(t')} \right\rangle$ by means of the Wick theorem. Eq. (2.99) then follows by combining the resulting analytical solutions. A similar argument is presented in Sec. 5.5 for a different type of non Gaussian process.

We conclude this section by introducing the concept of stochastic integral with respect to B. Let us introduce (i) a process Y a.s. continuous, (ii) a Brownian motion B on the time interval $[0, t]$ and (iii) a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ of the time interval with mesh properly defined and converging to zero. The stochastic integral of Y with respect to B is a stochastic process explicitly defined as follows:

$$\int_0^t Y(\tau)\,\mathrm{d}B(\tau) \;=\; \lim_{\substack{|\pi|\to 0 \\ n\to\infty}} \sum_{i=1}^{n} Y(t_{i-1})[B(t_i) - B(t_{i-1})]. \tag{2.100}$$

Two remarks need to be made: (1) the stochastic integral defined above can be constructed formally; (2) Eq. (2.100) holds for every sequence of partitions of the time interval with mesh converging to zero, thus implying that the convergence holds in probability. For the explicit construction and the proof of the related theorems, we refer to [104, 107, 109].

One further remark regards the choice of the specific time at which we evaluate the integrand process Y in the definition of the stochastic integral. In Eq. (2.100), this is the earlier time t_{i-1}. This specific choice is called the Itô prescription, but in general we can choose any point in the interval $[t_{i-1}, t_i]$, which however leads to integrals with completely different properties. A general definition of the stochastic integral accounting for the different prescriptions is:

$$\int_0^t Y(\tau) \star \mathrm{d}B(\tau) = \lim_{\substack{|\pi|\to 0 \\ n\to\infty}} \sum_{i=1}^{n} \left[(1-\kappa)Y(t_{i-1}) + \kappa\, Y(t_i)\right] \left[B(t_i) - B(t_{i-1})\right] \quad \kappa \in [0,1]. \tag{2.101}$$

We note that for $\kappa = 0$, we recover the Itô prescription, whereas for $\kappa = 1/2$ and $\kappa = 1$ we obtain respectively the Stratonovich and the Hänggi-Klimontovich prescriptions [113, 114]. Choosing a specific interpretation of the stochastic integral Eq. (2.101), i.e., the parameter κ, usually leads to different and possibly rather peculiar dynamics, e.g. Lévy flights [115, 116]. However, these processes all have physical meaning, so that the choice of the prescription is guided by the specific features of the system that one needs to model. In this Thesis, we will implement such different interpretations within the context of anomalous diffusive processes, by looking at subordinated Langevin equations of the type of Eqs. (2.46a, 2.46b), where we also assume a Y-dependent diffusion coefficient in Eq. (2.46a) [see Eqs. (3.33)]. After integration, this leads to the stochastic integral Eq. (2.101). As for the Itô case (see Chapter 3), we will study their Feynman-Kac equation.

The Quadratic Variation

In this section, we define the quadratic variation of a process $Y = (Y(t), t \geq 0)$ on a time interval $[s, t]$. We consider a partition of this interval $\pi = \{s = t_0 < t_1 < \ldots < t_n = t\}$ of mesh $|\pi| = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$. Associated to π, we can define the following process:

$$[Y, Y]_t^\pi = \sum_{i=1}^{n} [Y(t_i) - Y(t_{i-1})]^2, \tag{2.102}$$

which depends both on the specific realisation of Y and on the partition chosen. To avoid this latter dependence, we are interested in its properties in the limit $|\pi| \to 0$. Let us now consider sequences of partitions $\{\pi_n\}$, such that $|\pi_n| \to 0$, and compute the corresponding sequences $\{[Y, Y]_t^{\pi_n}\}$. If for every t this latter sequence converges in probability to a finite value $[Y, Y]_t$ independent on the specific choice of $\{\pi_n\}$ a.s., then $[Y, Y]_t$ is a well-defined process, which is called the quadratic variation of Y.

As a first application, we show that the quadratic variation of a process Y with paths of finite variation exists and it is null. From Eq. (2.102) and for a given π, we can write:

$$\begin{aligned}
[Y, Y]_t^\pi &\leq \left(\sum_{i=1}^{n} |Y(t_i) - Y(t_{i-1})| \right) \max_{j=1,\ldots,n} |Y(t_j) - Y(t_{j-1})| \\
&\leq V_t^\pi(Y) \max_{j=1,\ldots,n} |Y(t_j) - Y(t_{j-1})| = V_t^\pi(Y)\, |\pi|,
\end{aligned} \tag{2.103}$$

where $V_t^\pi(Y)$ is the variation of Y as defined in Eq. (2.88), which is finite by hypothesis. Thus, the rhs converges to zero a.s. in the limit $|\pi| \to 0$. In addition, as this result holds independently of the specific π, we can conclude that a.s. $[Y, Y]_t = 0$.

As a second application, we compute the quadratic variation of a Brownian motion $B(t)$. To this aim, it is convenient to rewrite Eq. (2.102) for a given π as follows:

$$[B, B]_t^\pi - (t - s) = \sum_{i=1}^{n} [B(t_i) - B(t_{i-1})]^2 - (t_i - t_{i-1}), \tag{2.104}$$

where the RVs $Y_i = [B(t_i) - B(t_{i-1})]^2 - (t_i - t_{i-1})$ are independent:

$$\begin{aligned}
\langle Y_i\, Y_j \rangle &= \left\langle [B(t_i) - B(t_{i-1})]^2\, [B(t_j) - B(t_{j-1})]^2 \right\rangle + (t_i - t_{i-1})(t_j - t_{j-1}) \\
&\quad - (t_i - t_{i-1}) \left\langle [B(t_j) - B(t_{j-1})]^2 \right\rangle - (t_j - t_{j-1}) \left\langle [B(t_i) - B(t_{i-1})]^2 \right\rangle \\
&= \left\langle [B(t_i) - B(t_{i-1})]^2 \right\rangle \left\langle [B(t_j) - B(t_{j-1})]^2 \right\rangle - (t_i - t_{i-1})(t_j - t_{j-1}) = 0.
\end{aligned} \tag{2.105}$$

In this brief calculation, we employed the independence of the increments of B and the relation $\left\langle [B(t_i) - B(t_{i-1})]^2 \right\rangle = t_i - t_{i-1}$. Thus, when we take its square and average over the realisations of B, only the squared terms remain. Thus, we obtain:

$$\begin{aligned}
\left\langle ([B, B]_t^\pi - (t - s))^2 \right\rangle &= \sum_{i=1}^{n} (t_i - t_{i-1})^2 \left\langle \left[\left(\frac{B(t_i) - B(t_{i-1})}{\sqrt{t_i - t_{i-1}}} \right) - 1 \right]^2 \right\rangle \\
&= \sum_{i=1}^{n} (t_i - t_{i-1})^2 \left\langle [Z_i - 1]^2 \right\rangle,
\end{aligned} \tag{2.106}$$

with the definition $Z_i = \frac{B(t_i) - B(t_{i+1})}{\sqrt{t_i - t_{i-1}}}$. Recalling that $B(t_i) - B(t_{i+1})$ is a Gaussian distributed RV with zero mean and variance $t_i - t_{i-1}$, Z_i is a Gaussian RV with zero mean and variance equal to one, i.e., $\left\langle [Z_i - 1]^2 \right\rangle = c$ is a constant independent on i. Thus, we can write:

$$\begin{aligned}\left\langle ([B,B]_t^\pi - (t-s))^2 \right\rangle &= c \sum_{i=1}^{n} (t_i - t_{i-1})^2 \\ &\leq c\,(t-s) \max_{i=1,\dots,n} |t_i - t_{i-1}| = c\,(t-s)\,|\pi| \end{aligned} \tag{2.107}$$

whose rhs converges to zero in the limit $|\pi| \to 0$, i.e., $[B,B]_t^\pi = t - s$. As this result holds for every sequence of such partition, we obtain the following relation:

$$[B,B]_t = (t-s). \tag{2.108}$$

The Itô Formula

We consider a process $Y = (Y(t), t \geq 0)$ described by the Langevin equation:

$$\dot{Y}(t) = F(t, Y(t)) + \sigma(t, Y(t)) \cdot \xi(t), \tag{2.109}$$

where $\xi(t)$ is a white Gaussian noise and we assume the Itô prescription. We recall that in the mathematical literature processes satisfying Eq. (2.109) are called *Itô processes*. Even though Eq. (2.109) describes the full stochastic trajectory of Y for a fixed initial condition $y_0 = Y(0)$ and for a given realisation of ξ, one often needs to investigate the dynamics of general functions of Y. For instance, the characteristic function of Y is obtained by taking the ensemble average of the function $e^{i\,k\,Y(t)}$. Such information is provided by the celebrated Itô formula. If we consider a twice differentiable function f of the process Y and we denote $f'(x) := \frac{\partial}{\partial x} f(x)$ and $f''(x) := \frac{\partial^2}{\partial x^2} f(x)$, the Itô formula is given by:

$$f(Y(t)) = f(y_0) + \int_0^t f'(Y(\tau))\,\mathrm{d}Y(\tau) + \frac{1}{2}\int_0^t f''(Y(\tau))\,\mathrm{d}[Y,Y]_\tau\,, \tag{2.110}$$

where $[Y,Y]_t$ is the quadratic variation of Y, which can be characterised, following a similar approach to what explained earlier. Let us consider a partition π and compute $[Y,Y]_t^\pi$. Recalling that the increments of Y are determined by Eq. (2.109), we can write:

$$\begin{aligned}[Y,Y]_t^\pi = \sum_{i=0}^{n-1} &[F(t_i, Y(t_i))]^2\,(t_{i+1} - t_i)^2 + \sigma^2(t_i, Y(t_i))\,[W(t_{i+1}) - W(t_i)]^2 \\ &+ 2\,F(t_i, Y(t_i))\sigma(t_i, Y(t_i))[W(t_{i+1}) - W(t_i)]. \end{aligned} \tag{2.111}$$

If we take its average over the realisations of B, the third term in its rhs cancels out, so that we are left with the following expression:

$$\begin{aligned}\langle [Y,Y]_t^\pi \rangle &= \sum_{i=0}^{n-1} \left\langle [F(t_i, Y(t_i))]^2 \right\rangle (t_{i+1} - t_i)^2 + \sum_{i=0}^{n-1} \left\langle \sigma^2(t_i, Y(t_i)) \right\rangle \left\langle [W(t_{i+1}) - W(t_i)]^2 \right\rangle \\ &= \sum_{i=0}^{n-1} \left\langle [F(t_i, Y(t_i))]^2 \right\rangle (t_{i+1} - t_i)^2 + \sum_{i=0}^{n-1} \left\langle \sigma^2(t_i, Y(t_i)) \right\rangle (t_{i+1} - t_i) \end{aligned} \tag{2.112}$$

where we factorise the average on the second term as $Y(t_i)$ is independent on the increment of B. In the limit $|\pi| \to 0$, we can further simplify this expression as below:

$$\langle [Y,Y]_t^\pi \rangle \leq |\pi| \sum_{i=0}^{n-1} \left\langle [F(t_i, Y(t_i))]^2 \right\rangle (t_{i+1} - t_i) + \sum_{i=0}^{n-1} \left\langle \sigma^2(t_i, Y(t_i)) \right\rangle (t_{i+1} - t_i) \tag{2.113}$$

such that only the second term survives in the limit $|\pi| \to 0$, which is the discretised form of an ordinary Lesbegue-Stieltjes integral over the function $\sigma^2(t, Y(t))$. Further recalling that this result holds independently on π, we conclude that:

$$[Y,Y]_t = \int_0^t \sigma^2(s, Y(s))\, \mathrm{d}s\,. \tag{2.114}$$

Eq. (2.110) is of fundamental importance to investigate the properties of Y. For instance, we show how to compute its Fokker-Planck equation (FPE), i.e., the evolution equation for its PDF $P(y,t)$. As suggested, we choose: $f(Y(t)) = e^{i\,k\,Y(t)}$ and substitute it in Eq. (2.110). In details, we find:

$$\begin{aligned} e^{i\,k\,Y(t)} &= e^{i\,k\,y_0} + i\,k \int_0^t e^{i\,k\,Y(\tau)}\, \mathrm{d}Y(\tau) \,+ \frac{k^2}{2} \int_0^t e^{i\,k\,Y(\tau)} \sigma^2(\tau, Y(\tau))\, \mathrm{d}\tau \\ &= e^{i\,k\,y_0} + i\,k \int_0^t e^{i\,k\,Y(\tau)} F(\tau, Y(\tau))\, \mathrm{d}\tau \,+ i\,k \int_0^t e^{i\,k\,Y(\tau)} \sigma(\tau, Y(\tau))\, \mathrm{d}B(\tau) \\ &\quad + \frac{k^2}{2} \int_0^t e^{i\,k\,Y(\tau)}\, \sigma^2(\tau, Y(\tau))\, \mathrm{d}\tau\,. \end{aligned} \tag{2.115}$$

Now, we need to take the ensemble average of Eq. (2.115). We note that the stochastic integral in its rhs is null on average. Indeed, if we introduce a discretisation of time of step length $\Delta\, t$ and define $n = t/\Delta t$ the number of discrete time intervals, we can write:

$$\begin{aligned} \left\langle \int_0^t e^{i\,k\,Y(\tau)}\, \sigma(\tau, Y(\tau))\, \mathrm{d}B(\tau) \right\rangle &= \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \sum_{i=0}^{n-1} \left\langle e^{i\,k\,Y(t_i)}\, \sigma(t_i, Y(t_i))\, [B(t_{i+1}) - B(t_i)] \right\rangle \\ &= \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \sum_{i=0}^{n-1} \left\langle e^{i\,k\,Y(t_i)} \sigma(t_i, Y(t_i)) \right\rangle \langle [B(t_{i+1}) - B(t_i)] \rangle = 0 \end{aligned} \tag{2.116}$$

where (i) we denote $t_i = i\,\Delta t$, (ii) the factorisation of the average in the rhs of Eq. (2.116) can be made as the increments of W are independent and (iii) it is equal to zero because $\langle B(t) \rangle = 0$ for all t. Thus, Eq. (2.115) becomes:

$$\left\langle e^{i\,k\,Y(t)} \right\rangle = e^{i\,k\,y_0} + \int_0^t \left\langle e^{i\,k\,Y(\tau)} \left[i\,k\,F(\tau, Y(\tau)) + \frac{k^2}{2}\, \sigma^2(\tau, Y(\tau)) \right] \right\rangle \mathrm{d}\tau\,. \tag{2.117}$$

Finally, if we take its time derivative and Fourier inverse transform, we obtain:

$$\frac{\partial}{\partial t} P(y,t) = \left[-\frac{\partial}{\partial y} F(\tau, y) + \frac{1}{2} \frac{\partial^2}{\partial y^2}\, \sigma^2(\tau, y) \right]\, P(y,t), \tag{2.118}$$

where we also employed the properties of the delta function. We note that in the force free case, i.e., $F(t,x) = 0$, and for $\sigma^2(t,x) = 2\,K_1$ we recover Eq. (2.7).

We conclude this section by extending Eq. (2.110) to an M-dimensional Itô process $\boldsymbol{Z}$.

In details, this indicates that we can define a vectorial process $\boldsymbol{F}(t, \boldsymbol{Z}(t))$, a matrix process $\boldsymbol{\sigma}(t, \boldsymbol{Z}(t))$ and a vectorial white Gaussian noise $\boldsymbol{\xi}(t)$ of dimensions respectively M, $M \times d$ and d $(1 \leq d \leq M)$ such that the dynamics of $\boldsymbol{Z}$ is described by the equation:

$$\dot{\boldsymbol{Z}}(t) = \boldsymbol{F}(t, \boldsymbol{Z}(t)) + \boldsymbol{\sigma}(t, Y(t)) \cdot \boldsymbol{\xi}(t) \tag{2.119}$$

where the second term is interpreted as follows:

$$(\boldsymbol{\sigma}(t, \boldsymbol{Z}(t)) \cdot \boldsymbol{\xi}(t))^{(i)} = \sum_{j=1}^{d} \sigma^{(i\,j)}(t, \boldsymbol{Z}(t)) \cdot \xi^{(j)}(t), \tag{2.120}$$

with the dot still imposing the Itô prescription. As before, if we now consider a twice differentiable function f of $\boldsymbol{Z}$ and denote $f_i'(\boldsymbol{x}) := \frac{\partial}{\partial x^i} f(\boldsymbol{x})$ and $f_{i,j}'(\boldsymbol{x}) = \frac{\partial^2}{\partial x^i \, \partial x^j} f(\boldsymbol{x})$, the multidimensional Itô formula is defined as below:

$$f(\boldsymbol{Z}(t)) = f(\boldsymbol{Z}_0) + \sum_{i=1}^{M} \int_0^t f_i'(\boldsymbol{Z}(\tau)) \, \mathrm{d}Z^{(i)}(\tau) + \frac{1}{2} \sum_{i,j=1}^{M} \int_0^t f_{i,j}''(\boldsymbol{Z}(\tau)) \, \mathrm{d}[Z^{(i)}, Z^{(j)}]_\tau \,, \tag{2.121}$$

where, in analogy with Eq. (2.114), we define the quadratic covariation $[Z^{(i)}, Z^{(j)}]_t$ as the following stochastic process (of the Itô type):

$$[Z^{(i)}, Z^{(j)}]_t = \sum_{n=1}^{d} \int_0^t \sigma^{(i\,n)}(s, \boldsymbol{Z}(s)) \sigma^{(j\,n)}(s, \boldsymbol{Z}(s)) \, \mathrm{d}s \,. \tag{2.122}$$

Relation between Generalised and Itô Prescription

As we suggested earlier, the Itô formula represents a powerful method to investigate the dynamical properties of processes described by Eq. (2.109). However, depending on the physical system we are interested in, we may need to use different prescriptions for the stochastic integrals, which would require different techniques to be studied. Here, we show that at least in the 1-dim case processes with the general prescription can be mapped into an Itô process by suitably choosing the functions F and σ in Eq. (2.109).

We consider a process $Y = (Y(t), t \geq 0)$ described by the two Langevin equations:

$$\dot{Y}(t) = F(t, Y(t)) + \sigma(t, Y(t)) \star \xi(t) \tag{2.123}$$

$$\dot{Y}(t) = a(t, Y(t)) + b(t, Y(t)) \xi(t) \tag{2.124}$$

where we use respectively the generalised prescription as in Eq. (2.101) or the Itô one. Our aim is to find suitable functions a, b, such that the resulting process is the same. Let us consider the integrated version of Eq. (2.123):

$$Y(t) - y_0 = \int_0^t F(\tau, Y(\tau)) \, \mathrm{d}\tau + \int_0^t \sigma(\tau, Y(\tau)) \star \mathrm{d}B(\tau) \,, \tag{2.125}$$

where the stochastic integral is defined as in Eq. (2.101). The first task is then to represent this term as an Itô stochastic integral. To this aim, let us consider a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$, define the auxiliary variable $Z(t_i) = Y(t_i) + \kappa \, [Y(t_{i+1}) - Y(t_i)]$, and use the explicit definition of the integral:

$$\int_0^t \sigma(\tau, Y(\tau))\,\mathrm{d}B(\tau) \;=\; \lim_{\substack{|\pi|\to 0\\ n\to\infty}} \sum_{i=0}^{n-1} \sigma(t_i, Z(t_i))[B(t_{i+1}) - B(t_i)]. \tag{2.126}$$

We note that $Z(t_i)$ depends on the increment $\Delta Y(t_i) = Y(t_{i+1}) - Y(t_i)$, which can be expressed as an Itô increment by using Eq. (2.124), i.e., we find $\Delta Y(t_i) = a(t_i, Y(t_i))\Delta t_i + b(t_i, Y(t_i))\Delta B(t_i)$, where to simplify the notation we called: $\Delta B(t_i) = [B(t_{i+1}) - B(t_i)]$ and $\Delta t_i = (t_{i+1} - t_i)$. Consequently, we can use the discretised version of the Itô formula Eq. (2.110) to express σ as follows:

$$\begin{aligned}
\sigma(t_i, Z(t_i)) &= \sigma(t_i, Y(t_i)) + \kappa\,\sigma'(t_i, Y(t_i))\Delta Y(t_i) + \frac{\kappa^2}{2}\sigma(t_i, Y(t_i))[b(t_i, Y(t_i))]^2\Delta t_i \\
&= \sigma(t_i, Y(t_i)) + \kappa\,\sigma'(t_i, Y(t_i))b(t_i, Y(t_i))\Delta B(t_i) \\
&\quad + [\kappa\, a(t_i, Y(t_i))\sigma'(t_i, Y(t_i)) + \frac{\kappa^2}{2}\sigma''(t_i, Y(t_i))(b(t_i, Y(t_i)))^2]\Delta t_i.
\end{aligned} \tag{2.127}$$

This result needs to be substituted back into Eq. (2.126). We can then further simplify such expression by recalling that $\left\langle \Delta B(t_i)^2 \right\rangle = \Delta t_i$, due to its being Gaussian distributed, and that the Δt_i dependent term cancels out in the limit of null mesh. Thus we obtain:

$$\int_0^t \sigma(\tau, Y(\tau)) \star \mathrm{d}B(\tau) \;=\; \int_0^t \sigma(\tau, Y(\tau))\,\mathrm{d}B\,(\tau) + \kappa \int_0^t \sigma'(\tau, Y(\tau))b(\tau, Y(\tau))\,\mathrm{d}\tau\,. \tag{2.128}$$

By putting everything together, we obtain the integrated equation:

$$Y(t) - y_0 = \int_0^t \big[F(\tau, Y(\tau)) + \kappa\,\sigma'(\tau, Y(\tau))b(\tau, Y(\tau))\big]\,\mathrm{d}\tau + \int_0^t \sigma(\tau, Y(\tau))\,\mathrm{d}B(\tau)\,. \tag{2.129}$$

It is now clear that the mapping between the two processes is realised if we set:

$$b(t, Y(t)) = \sigma(t, Y(t)), \tag{2.130a}$$

$$a(t, Y(t)) = F(t, Y(t)) + \kappa\,\sigma(t, Y(t))\sigma'(t, Y(t)). \tag{2.130b}$$

2.2.3 Lévy Processes, Subordinators and Time-Changed Processes

Infinitely Divisible Random Variables

Let us consider a RV Y with law P_Y and characteristic function: $\phi_X(k) = \int_{-\infty}^{+\infty} e^{i\,k\,x} P_Y(x)\,\mathrm{d}x\,.$ If $\forall n \in \mathbb{N}$ there exists i.i.d RVs $X_1^{(n)}, \ldots, X_N^{(n)}$ with law P_Y and characteristic function ϕ_Y (uniquely defined), such that:

$$Y \overset{d}{=} \sum_{i=1}^{n} X_i^{(n)} \tag{2.131}$$

then Y is said to be *infinitely divisible*. Consequently, we can show the following relation:

$$\begin{aligned}
\phi_Y(k) &= \left\langle e^{i\,k\,Y} \right\rangle = \left\langle e^{i\,k\,\sum_{i=1}^n X_i^{(n)}} \right\rangle \\
&= \prod_{i=1}^{n} \left\langle e^{i\,k\,X_i^{(n)}} \right\rangle = \left[\left\langle e^{i\,k\,X_1^{(n)}} \right\rangle\right]^n = [\phi_X(k)]^n
\end{aligned} \tag{2.132}$$

where the factorisation of the average is due to the independence of the RVs $X_i^{(n)}$. We note that Eq. (2.132) is a necessary and sufficient condition for Y to be infinitely divisible [107], i.e., we can use it as a criterion to asses if a given RV is infinitely divisible.

Let us consider two examples of infinitely divisible RVs:

- Gaussian RVs.
  Let Y be Gaussian distributed with mean m and variance σ. Thus, we can write:

$$\phi_Y(k) = \exp\left(i\,k\,m - \frac{\sigma^2}{2}\,k^2\right) = \left[\exp\left(i\,k\,\frac{m}{n} - \frac{\sigma^2}{2\,n}\,k^2\right)\right]^n = [\phi_X(k)]^n, \tag{2.133}$$

  with $\phi_X(k) = \exp\left[i\,k\,m/n - k^2\sigma^2/(2\,n)\right]$, i.e., X_i in Eq. (2.131) are Gaussian distributed with mean m/n and variance $\sigma/\sqrt{n}$.

- Poisson RVs.
  Let Y be a Poisson distributed RV with characteristic parameter λ. Thus, its law is:

$$\mathbb{P}(Y = m) = \exp(-\lambda)\frac{\lambda^m}{m!} \qquad \forall m \in \mathbb{N}_0 \tag{2.134}$$

  Its characteristic function can be computed easily as below:

$$\begin{aligned}\phi_Y(k) = \langle \exp(i\,k\,Y)\rangle &= \exp(-\lambda)\sum_{m=0}^{\infty}\exp(i\,k\,m)\frac{\lambda^m}{m!}\\ &= \exp\left[\lambda\,(e^{i\,k} - 1)\right]\end{aligned} \tag{2.135}$$

  Therefore, Eq. (2.131) is satisfied if we take Poisson RVs with parameter λ/n.

- Compound Poisson RVs.
  Let us consider Y being a compound Poisson variable. Y is a sum of M i.i.d RVs Z_i with law $P_Z(x)$ and M is itself a Poisson distributed RV of parameter λ. The characteristic function of Y is computed by conditioning on M:

$$\begin{aligned}\phi_Y(k) = \left\langle\left\langle e^{i\,k\,Y}\middle|\, M = m\right\rangle\right\rangle &= \left\langle\left[\int_{-\infty}^{+\infty} e^{i\,k\,x}P_Z(x)\,\mathrm{d}x\right]^m\right\rangle\\ &= \exp(-\lambda)\sum_{m=0}^{\infty}\left[\int_{-\infty}^{+\infty} e^{i\,k\,x}P_Z(x)\,\mathrm{d}x\right]^m\frac{\lambda^m}{m!}\\ &= \exp\left[\lambda\int_{-\infty}^{+\infty}(e^{i\,k\,x} - 1)P_Z(x)\,\mathrm{d}x\right].\end{aligned} \tag{2.136}$$

  Thus, Eq. (2.131) holds if the RVs $X_i^{(n)}$ are compound Poisson with parameter λ/n.

The relevance of infinitely divisible RVs is motivated by the fact that they can be uniquely determined in terms of their characteristic function, which is described by the *Lévy-Khintchine* formula. Before writing it down, we need to define the concept of *Lévy measure*. Let Π be a measure on $\mathbb{R}/\{0\}$. We note that so far we have always implicitly assumed that Π can be written in terms of the Lebesgue measure, i.e., $\Pi(\mathrm{d}y) = P(y)\,\mathrm{d}y$, for some density P. We call Π a Lévy measure if the following condition is satisfied:

$$\int_{\mathbb{R}/\{0\}} \mathrm{Max}(|y|^2, 1)\,\Pi(\mathrm{d}y) < \infty \tag{2.137}$$

Further introducing parameters $b \in \mathbb{R}$, $\sigma \geq 0$ the Lévy-Khintchine formula is given by the following expression:

$$\phi_Y(k) = \exp\left\{ i\, b\, k - \frac{1}{2}\sigma\, k^2 + \int_{\mathbb{R}/\{0\}} [e^{i\,k\,y} - 1 - i\,k\,y\,\mathbf{1}_{|y|<1}(y)]\, \Pi(\mathrm{d}y\,) \right\}. \tag{2.138}$$

with $\mathbf{1}_A(y) = 1$ for $y \in A$ or $\mathbf{1}_A(y) = 0$ otherwise. Thus any infinitely divisible RV Y has a characteristic function of the type of Eq. (2.138) for a specific triplet (b, σ, Π) and conversely any function of the type of Eq. (2.138) is the characteristic function of an infinitely divisible RV. For instance, Eqs. (2.133, 2.136) are obtained from Eq. (2.138) by choosing the triplets $(m, \sigma^2, 0)$ and $(0, 0, w(y)\,\mathrm{d}y\,)$ respectively. Finally, if Y is a d-dim infinitely divisible variable, Eq. (2.138) still holds by taking b a vector in $\mathbb{R}^d$, σ a positive definite $d \times d$ matrix, Π a Lévy measure on $\mathbb{R}^d/0$ and the indicator function being defined on d-dim sphere centred at the origin.

Stable Random Variables

In this section, we discuss a further example of infinitely divisible RVs, which will play a major role in the stochastic description of CTRWs. Let us consider a RV Y and n independent of its copies $\{Y_i\}_{i=1,\ldots,n}$. If there exists real-valued sequences of parameters $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$, such that:

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} c_n\, Y + d_n \tag{2.139}$$

then Y is called a *stable* RV. If $d_n = 0$, then Y is *strictly stable*. From this definition, it is straightforward to see that Y is infinitely divisible [simply set $X_i^{(n)} = [Y_i - d_n/n]/c_n$ in Eq. (2.131)] and that the existence of Y represents a generalisation of the central limit theorem. Indeed, Eq. (2.139) is equivalent to say that the sequences of partial sums $\{S_n\}_{n\in\mathbb{N}}$ with: $S_n = [Y_1 + \ldots + Y_n - d_n]/c_n$ converge in distribution to Y. With the choice $c_n = \sigma\sqrt{n}$ and $d_n = n\,m$, this is the ordinary central limit theorem and Y is a Gaussian RV with mean m and variance σ^2. However, for different choices of c_n and d_n, we obtain a generalised central limit theorem. However, the only possible choice to satisfy Eq. (2.139) is given by $c_n = \sigma\, n^{1/\alpha}$, with $0 < \alpha \leq 2$, also called *index of stability* of the stable distribution [100]. As stable distributions are infinitely divisible, their characteristic function is completely determined by Eq. (2.138). In particular, we have two possible characteristics: (i) $(b, \sigma, 0)$ for $\alpha = 2$, implying that Y is Gaussian distributed with mean b and variance σ and (ii) $(b, 0, \Pi)$ with

$$\Pi(\mathrm{d}x\,) = \begin{cases} c_1\, x^{-1-\alpha}\ \mathrm{d}x & x \in [0, \infty) \\ c_2\, |x|^{-1-\alpha}\ \mathrm{d}x & x \in (-\infty, 0) \end{cases} \tag{2.140}$$

with $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. However, it is often preferable to work with the equivalent characterisation (obtained by changing coordinates in Eq. (2.138) [103]) for $\mu \in \mathbb{R}$, $\sigma \geq 0$

and $-1 \leq \beta \leq 1$:

$$\phi_Y(k) = \exp\left(i\,\mu\,k - \frac{1}{2}\sigma^2\,k^2\right) \qquad \alpha = 2 \qquad (2.141a)$$

$$\phi_Y(k) = \exp\left(i\,\mu\,k - \sigma^\alpha\,|k|^\alpha\left[1 - i\,\beta\ \mathrm{sign}\,(k)\tan\left(\frac{\pi\,\alpha}{2}\right)\right]\right) \qquad \alpha \neq 1, 2 \qquad (2.141b)$$

$$\phi_Y(k) = \exp\left(i\,\mu\,k - \sigma\,|k|\left[1 + i\,\beta\,\frac{2}{\pi}\,\mathrm{sign}\,(k)\log\left(|k|\right)\right]\right) \qquad \alpha = 1 \qquad (2.141c)$$

We remark that Eqs. (2.141a-2.141c) uniquely define a stable RV Y and, conversely, every stable RV Y has characteristic function of the type of Eqs. (2.141a-2.141c). A symmetric stable RV Y has characteristic function ($\rho = \sigma$ for $0 < \alpha < 2$ and $\rho = \sigma/\sqrt{2}$ for $\alpha = 2$):

$$\phi_Y(k) = \exp\left(-\rho^\alpha\,|k|^\alpha\right) \qquad 0 < \alpha \leq 2 \qquad (2.142)$$

Lévy Processes

A stochastic process $Y = (Y(t), t \geq 0)$ and initial condition $Y(0) = y_0$ is a *Lévy process* if the conditions below are satisfied:

1. $Y(0) = y_0 = 0$ almost surely.
2. $Y(t)$ has independent increments, i.e., $\forall\, n \geq 2$ and for each partition $0 \leq t_0 < t_1 < \ldots < t_n \leq t$ the RVs $\{Y(t_j) - Y(t_{j-1})\}_{j=1,\ldots,n}$ are independent.
3. $Y(t)$ has stationary increments, meaning that for all $0 \leq t_1 < t_2 \leq t$ the RV $Y(t_2) - Y(t_1)$ has the same distribution as $Y(t_2 - t_1)$. Note that, if 1 is not satisfied, it would instead depend on $Y(t_2 - t_1) - y_0$.
4. The trajectories of $Y(t)$ are càdlàg, i.e., right-continuous with left limits.

If one restricts the conditions 2, 4 by assuming Gaussian distributed increments and continuous trajectories respectively, one recovers ordinary Brownian motion.

Thanks to the property 2-3, Y is infinitely divisible $\forall t \geq 0$. Indeed, we can write:

$$Y(t) = \sum_{i=1}^{n} X_i^{(n)} \qquad X_i^{(n)} = Y\left(\frac{i\,t}{n}\right) - Y\left(\frac{(i-1)\,t}{n}\right) \qquad \forall n \in \mathbb{N}, \forall t \geq 0 \qquad (2.143)$$

where the RVs $X_i^{(n)}$ are i.i.d. by assumption. This result can be employed to derive a general formula for the characteristic function of a Lévy process. Let us define the function:

$$\Psi(k,t) = \ln\langle\exp\left(i\,k\,Y(t)\right)\rangle. \qquad (2.144)$$

We further consider two integers m, n. By recalling Eq. (2.143), we can write:

$$Y(m) = Y(1) + [Y(2) - Y(1)] + \ldots + [Y(m) - Y(m-1)] \qquad (2.145a)$$

$$Y(m) = Y\Big(\frac{m}{n}\Big) + \left[Y\Big(2\frac{m}{n}\Big) - Y\Big(\frac{m}{n}\Big)\right] + \ldots + \left[Y(m) - Y\Big((n-1)\frac{m}{n}\Big)\right] \qquad (2.145b)$$

These are equivalent to: $Y(m) \stackrel{d}{=} m\,Y(1)$ and $Y(m) \stackrel{d}{=} n\,Y\left(\frac{m}{n}\right)$, as we recall that increments of Lévy processes are stationary and independent. Thus, we can compute explicitly the quantity $\Psi(k,m)$. We obtain the two equivalent expressions:

$$\Psi(k,m) = \ln\langle\exp[i\,k\,m\,Y(1)]\rangle = m\,\ln\langle\exp[i\,k\,Y(1)]\rangle \tag{2.146a}$$
$$\Psi(k,m) = \ln\left\langle\exp\left[i\,k\,n\,Y\left(\frac{m}{n}\right)\right]\right\rangle = n\,\ln\left\langle\exp\left[i\,k\,Y\left(\frac{m}{n}\right)\right]\right\rangle \tag{2.146b}$$

where the factorisation of the average is again due to the independence of the increments. Thus, we derive that simultaneously $\Psi(k,m) = m\,\Psi(k,1)$ and $\Psi(k,m) = n\,\Psi\left(k,\frac{m}{n}\right)$. Combining these two results, we obtain:

$$\Psi\left(k,\frac{m}{n}\right) = \frac{m}{n}\,\Psi(k,1). \tag{2.147}$$

This relation expresses the characteristic function of a Lévy process at some finite time m/n in terms of its value at time $t=1$. As this relation holds for every integer m,n, we can conclude that it holds for any real positive number t, i.e., $\Psi(k,t) = t\,\Psi(k,1)$. Further recalling that $\Psi(k,1)$ is given by the logarithm of Eq. (2.138), as a Lévy increment is an infinitely divisible RV, we can write the characteristic function of a Lévy process as below:

$$\phi_Y(k,t) = \exp\left\{\left[i\,b\,k - \frac{1}{2}\sigma\,k^2 + \int_{\mathbb{R}/\{0\}}[e^{i\,k\,y} - 1 - i\,k\,y\,\mathbf{1}_{|y|<1}(y)]\,\Pi(\mathrm{d}y)\right]\,t\right\}. \tag{2.148}$$

The quantity $\eta(k) = \Psi(k,1)$ is called the Lévy symbol of Y and it is given by:

$$\eta(k) = i\,b\,k - \frac{1}{2}\sigma\,k^2 + \int_{\mathbb{R}/\{0\}}[e^{i\,k\,y} - 1 - i\,k\,y\,\mathbf{1}_{|y|<1}(y)]\,\Pi(\mathrm{d}y) \tag{2.149}$$

Let us discuss discuss four special examples of Lévy processes:

- Brownian motion.
  As suggested earlier, Brownian motion is a Lévy process with Gaussian distributed increments, which are infinitely divisible RVs with characteristic triplet $(m,\sigma^2,0)$, where m and σ describes respectively their mean and variance. Its Lévy symbol is
  $$\eta(k) = i\,m\,k - \frac{1}{2}\sigma^2\,k^2. \tag{2.150}$$
- Poisson Process.
  A Poisson process of parameter λ is a Lévy process N, such that $N(t)$ is Poisson distributed with the parameter $\lambda\,t$ for every $t>0$. Recalling the characteristic function of a Poisson RV in Eq. (2.135), we deduce that its characteristic triplet is $(0,0,\Pi)$ with the Lévy measure $\Pi(\mathrm{d}y) = \lambda\,\delta(y-1)\,\mathrm{d}y$ and that its Lévy symbol is
  $$\eta(k) = \lambda\,(e^{i\,k} - 1). \tag{2.151}$$
- Compound Poisson Process.
  A Compound Poisson process $Y(t)$ on the interval $[0,t]$ is defined by sampling $N(t)$ independent RVs $\{Z_i\}_{i=1,\ldots,N(t)}$ distributed with law $P_Z(x)$, where $N(t)$ are Poisson distributed RVs $\forall\,t>0$. It can be thought as a sequence of jumps of random amplitudes occurring at exponentially distributed random times with law $P(t) = e^{-\lambda\,t}$ (here λ is the inverse of the mean time between the jumps). Such process can be shown to be

a Lévy process. Further recalling the characteristic function of a Compound Poisson RV Eq. (2.136), we deduce that the characteristic triplet of a Compound Poisson process is $(b, 0, \Pi)$ with $\Pi(\mathrm{d}y) = \lambda\, P_Z(y)\,\mathrm{d}y$ and $b = \lambda \int_{-1}^{1} y\, P_Z(y)\ \mathrm{d}y$. Finally, its Lévy symbol is given as follows:

$$\eta(k) = \lambda \int_{-\infty}^{+\infty} P_Z(x)(e^{i\,k\,x} - 1)\,\mathrm{d}x\,. \tag{2.152}$$

We remark that the Compound Poisson process can have a Lévy measure, which cannot be expressed in terms of the Lebesgue measure. In this case, if $\lambda\,\Pi(\mathrm{d}x)$ is its measure, $b = \lambda \int_{-1}^{1} y\,\Pi(\mathrm{d}y)$ and its Lévy symbol is modified to the following:

$$\eta(k) = \lambda \int_{-\infty}^{+\infty} (e^{i\,k\,x} - 1)\Pi(\mathrm{d}x). \tag{2.153}$$

- Stable Process.
  A stable process is a Lévy process with increments that are sampled from a stable distribution. Thus, $\eta(k)$ is given by Eqs. (2.141a-2.141c) or by Eq. (2.142), if we consider a process with symmetric stable distributed increments.

We conclude this section by introducing the *Lévy-Itô decomposition* of a Lévy process Y. To this aim, we rewrite Eq. (2.149) in the equivalent form: $\eta(k) = \eta_1(k) + \eta_2(k) + \eta_3(k)$, with the auxiliary definitions below:

$$\eta_1(k) = i\,b\,k - \frac{1}{2}\sigma\,k^2 \tag{2.154a}$$

$$\eta_2(k) = \int_{\mathbb{R}/\{[-1,1]\}} (e^{i\,k\,y} - 1)\,\Pi(\mathrm{d}y) \tag{2.154b}$$

$$\eta_3(k) = \int_{-1}^{1} (e^{i\,k\,y} - 1)\,\Pi(\mathrm{d}y) - i\,k\,c \tag{2.154c}$$

where we defined the constant $c = \int_{-1}^{1} y\,\Pi(\mathrm{d}y)$. Comparing these formulas with the examples discussed earlier, we recognise that (i) $\eta_1(k)$ is the Lévy exponent of a Brownian motion $B(t)$ with parameters b, σ, (ii) $\eta_2(k)$ is that of a Compound Poisson process $C(t)$ with jump lengths always greater/smaller than ± 1 respectively and (iii) $\eta_3(k)$ is that of a compensated, i.e., with null average, compound Poisson process $\widetilde{C}(t)$ with jump lengths always inside the interval $(-1, 1)$. Note that the compensation is realised by introducing a constant drift term in the Lévy exponent, which is proportional to the mean jump length c. Thus, Y has the following Lévy-Itô decomposition:

$$Y(t) = B(t) + C(t) + \widetilde{C}(t). \tag{2.155}$$

Note that for any Lévy process there exists suitable $B, C, \widetilde{C}$ satisfying Eq. (2.155).

Subordinators

We define *subordinator* a 1-dim Lévy process a.s. non-decreasing. Thus, if $T = (T(t), t \geq 0)$ is a subordinator, then the following relations hold a.s.: $T(t) \geq 0\ \forall\, t \geq 0$ and $T(t_1) \leq T(t_2)\ \forall\, t_1 \leq t_2$. Recalling that T is infinitely divisible, its characteristic function will be determined by Eq. (2.148) for a subclass of characteristic triplets (b, σ, ν) that we need to determine. First, we note that, if $X(t)$ is Gaussian with zero mean and variance $\sigma\, t$, then $P(X(t) \geq 0) = 1/2 = P(X(t) \leq 0)$, whereas we require $T \geq 0\, \forall\, t$. This means that a subordinator T cannot have any Gaussian component in its Lévy symbol, i.e., $\sigma = 0$. In addition, it also implies that no jumps of negative amplitudes nor a negative shift can be allowed in its Lévy symbols, thus implying the further conditions: $b \geq 0$ and $\Pi((-\infty, 0)) = 0$. In details, one can prove the following characterisation (in terms of the Laplace transform of T) [105]:

$$\left\langle e^{-\lambda T(t)} \right\rangle = e^{-t\,\Phi(\lambda)} \qquad \Phi(\lambda) = b\,\lambda + \int_0^\infty (1 - e^{-\lambda\, y})\Pi(\mathrm{d}y\,) \tag{2.156}$$

with the conditions explained above and the further one: $\int_0^\infty \mathrm{Max}(y, 1)\, \Pi(\mathrm{d}y\,) < \infty$. We call Φ the Laplace exponent of the subordinator. We note that only two parameters now define its form, i.e., the characteristics of T are described in terms of the couple (b, ν). Using Eq. (2.148) and Jensen's inequality, one can show that $\Phi(\lambda)$ must be a continuous, non decreasing, non negative and convex function. In addition, we note that $\Phi(0) = 0$. In general, one can prove that Φ is a Bernstein function [117, 118].

Let us consider two specific examples of subordinators that will be used in the thesis:

- Lévy stable subordinator.
  A subordinator T is Lévy stable if it has characteristic couple $(0, \Pi)$ with

$$\Pi(\mathrm{d}x\,) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\mathrm{d}x}{x^{1+\alpha}}. \tag{2.157}$$

  If we replace it inside Eq. (2.156), we obtain the following Laplace exponent:

$$\Phi(\lambda) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda\, y})\, y^{-1-\alpha}\, \mathrm{d}y\, = \frac{\lambda}{\Gamma(1-\alpha)} \int_0^\infty e^{-\lambda\, y}\, y^{-\alpha}\, \mathrm{d}y\, = \lambda^\alpha. \tag{2.158}$$

- Tempered Lévy stable subordinator.
  A subordinator T is tempered Lévy stable if it has characteristic couple $(0, \Pi)$ with Lévy measure specified by the following equation [101]:

$$\Pi(\mathrm{d}x\,) = \frac{\alpha}{\Gamma(1-\alpha)}\, e^{-\mu\, x} \frac{\mathrm{d}x}{x^{1+\alpha}} \qquad c > 0. \tag{2.159}$$

  If we substitute it inside Eq. (2.156), we obtain the following Laplace exponent:

$$\begin{aligned} \Phi(\lambda) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda\, y})\, e^{-\mu\, y}\, y^{-1-\alpha}\, \mathrm{d}y \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \left[- \int_0^\infty (1 - e^{-\mu\, y})\, y^{-\alpha}\, \mathrm{d}y\, + \int_0^\infty (1 - e^{-(\lambda+\mu) y})\, y^{-\alpha}\, \mathrm{d}y \right] \\ &= (\lambda + \mu)^\alpha - \mu^\alpha \end{aligned} \tag{2.160}$$

  where we solved the integrals in the rhs by employing the result derived in Eq. (2.158).

2.2.4 Continuous Semimartingales

We consider a process $M = (M(t), t \geq 0)$ and assume that all the information on M up to a chosen time s is known, i.e., we know $M(s) = m_s$. M is a *martingale* if (i) $M(t)$ is integrable for all t and (ii) if the following relation on its conditional average holds [83]:

$$\langle M(t)|M(s) = m_s\rangle = m_s. \tag{2.161}$$

It is instead called a sub-martingale if $\langle M(t)|M(s) = m_s\rangle \geq m_s$ or a super-martingale if $\langle M(t)|M(s) = m_s\rangle \leq m_s$. For instance, the Brownian motion $B(t)$ is a martingale, as one can easily verify by direct computation of Eq. (2.161).

Let us further consider a process $Y = (Y(t), t \geq 0)$. The process Y is a *semimartingale* if the following decomposition holds:

$$Y(t) = M(t) + A(t) \tag{2.162}$$

where M and $A = (A(t), t \geq 0)$ are respectively a martingale and a finite variation process with càdlàg paths. Semimartingales are good integrators, i.e., stochastic integration with respect to such processes, can be well defined. We refer to the monograph [102] for more details. For the sake of our discussion, we will only present their Itô formula, in the specific case of Y being a semimartingale with continuous stochastic paths. With this assumption, the Itô formula is given as follows:

$$f(Y(t)) - f(y_0) = \int_0^t f'(Y(\tau))\,\mathrm{d}Y(\tau) + \frac{1}{2}\int_0^t f''(Y(\tau))\,\mathrm{d}[Y,Y]_\tau\,, \tag{2.163}$$

where $[Y,Y]_t$ is the quadratic variation of Y, which is defined as in Sec. 2.2.2. Analogously to the case of the Itô formula Eq. (2.110), the extension of Eq. (3.15) to a M-dim semimartingale $\boldsymbol{Z}$ is given by:

$$f(\boldsymbol{Z}(t)) - f(\boldsymbol{Z}_0) = \sum_{i=1}^{M}\int_0^t f'_i(\boldsymbol{Z}(\tau))\,\mathrm{d}Z^{(i)}(\tau) + \frac{1}{2}\sum_{i,j=1}^{M}\int_0^t f''_{i,j}(\boldsymbol{Z}(\tau))\,\mathrm{d}[Z^{(i)},Z^{(j)}]_\tau\,, \tag{2.164}$$

where $[Z^{(i)},Z^{(j)}]_t$ is the joint quadratic variation of $Z^{(i)}$, $Z^{(j)}$. Both processes $[Y,Y]_t$ and $[Z^{(i)},Z^{(j)}]_t$ can be shown to be given by a continuous increasing processes. In the joint case, such process has also finite variation paths [102].

Time-Changed Processes

We conclude this Chapter by presenting the definition of time-changed process. Let us consider a Lévy process $X = (X(t), t \geq 0)$ and a subordinator $T = (T(t), t \geq 0)$. Then, we can define a new time-changed process Y trough the following relation:

$$Y(t) = X(T(t)). \tag{2.165}$$

The resulting process Y is a Lévy process [107]. This procedure can also be applied to processes X other than Lévy ones, even though the resulting time-changed process Y will no longer be a Lévy process. In the specific case of Y being a semimartingales, one can show that the resulting process is still a semimartingale [119]. This technique is particularly relevant in the theory of CTRWs, as it is employed to formulate their Langevin equations, as we suggested in Sec. 2.1.2. In this case, however, the time-change is obtained by considering an auxiliary process S called the inverse process, or hitting time, of T:

$$S(t) = \inf_{s\in\mathbb{R}} \{s > 0 : T(s) > t\}. \tag{2.166}$$

Conveniently, we can prove that S has continuous paths, if T is strictly increasing [104, 120]. We conclude by recalling the following two relations, which are proved in a more general setting in [120]. Let S be given by Eq. (2.166) for a subordinator T and Z be a continuous semimartingale. Thus, we can derive:

$$\int_0^{S(t)} H(s)\,\mathrm{d}Z(s) = \int_0^t H(S(\tau))\,\mathrm{d}Z(S(\tau)) \tag{2.167a}$$

$$[Z(S(t)), Z(S(t))]_t = [Z, Z]_{S(t)} \tag{2.167b}$$

CHAPTER 3

Anomalous Processes with General Waiting Times: Functionals and Multipoint Structure

In this Chapter we employ the mathematical tools provided in Chapter 2 to study a general class of anomalous diffusive processes that can capture more complicated MSD behaviour than a pure power-law by means of a general waiting time distribution. Specifically, we will be interested in physical systems displaying a non linear MSD, where crossovers between different scaling regimes are observed over time. Here, we will provide a complete characterisation of these processes comprising: (i) the stochastic description of their microscopic diffusive dynamics; (ii) evolution equations for the PDF of the process and its associated time-integrated observables; (iii) the multi-point correlation functions. We will show that our model includes the ordinary CTRW as a special case. We will then apply our formalism to model the MSD of mitochondria diffusing in S. Cerevisiae cells depleted of actin microfilaments [57] and predict the form of the two-point correlation function, that can be readily compared with the experimental data. These results suggest the relevance of our formalism for both theorists and experimentalists in the field of anomalous stochastic processes.

3.1 Motivation

In Chapter 1 we extensively discussed how several experiments provided evidence of anomalous diffusive behaviour in both physical and biological systems, which motivated the definition of the CTRW model [28]. In all these experiments (see [18, 9, 35, 19, 10, 56] and references therein) the MSD was observed to exhibit a pure power-law scaling behaviour, which naturally suggested the classification between subdiffusive and superdiffusive transport processes, according to its characteristic scaling exponent being respectively less or greater than one. However, despite the fact that many models have been proposed that are able to reproduce such power-law MSD scaling, the modelling of empirical data displaying anomalous diffusive behaviour still represents a sound theoretical challenge. Indeed, recent experiments of diffusion in biological systems have shown that the MSD can be a general non linear function, where different scaling regimes and crossovers regions between them can be identified over time [44, 45, 46, 48, 57, 49, 121, 122, 123, 124, 125, 57, 126, 127, 128, 129, 130]. Thus, these results suggest the need to formulate more general anomalous stochastic processes, which can account for such complicated MSD behaviour, and to characterise them in terms of both single and multi point functions. These latter ones are ultimately essential, as the fundamental non Markovian nature of these processes makes single point quantities insufficient to fully understand their properties. However, their complete theoretical characterisation is undoubtedly a challenging task [131, 132, 133, 134, 135, 136, 137, 138].

In addition, as in experiments one has often access to joint position-velocity data, it is necessary to develop a consistent framework to study both these quantities, which can then be employed to assess the nature of the microscopic processes underlying the observed dynamics. It is evident that this is a specific example of the more general joint description of a process and one of its observables, which are mathematically defined as functionals of its stochastic trajectories [139]. In the case of normal diffusive dynamics, this is provided by the celebrated Feynman-Kac (FK) equation, which will be presented later in this Chapter. The development of such a complete description for CTRWs, and in general for anomalous stochastic processes, is a second main theoretical challenge, which has been only solved recently in Ref. [140], whose content is here presented. This result provides the statistical properties of general observables of anomalous stochastic processes, which can be used as a mean to asses the nature of the observed dynamics in experimental datasets, on the one hand, and to investigate further yet unresolved theoretical issues for such processes, on the other hand. For instance, it would provide the framework for the analysis of their stochastic thermodynamics [141], where the work exerted by a dynamical particle is defined exactly as an observable of the position process, i.e., as a functional of the particle's trajectory.

In this Chapter, we will face both these two challenges by constructing a complete framework for the analysis of general anomalous stochastic processes, which are obtained by allowing for more general waiting time distributions in the CTRW model. Specifically, we will provide the description of their stochastic dynamics in terms of Langevin time-changed equations (similarly to those presented in Sec. 2.2.4) and derive from them both a Generalised Fractional Feynman-Kac (GFFK) equation and analytical formulas for their multi-point functions. We will finally apply our formalism to a few specific models of biological relevance and to the MSD data of mitochondria diffusing in S. Cerevisiae cells depleted of actin microfilaments [57] to support its relevance for experimental applications.

3.2 Brownian Functionals and the Feynman-Kac Equation

Let us consider a free diffusive brownian process $Y = (Y(t), t \geq 0)$, i.e., $\dot{Y}(t) = \sqrt{2\,\sigma}\,\xi(t)$, with ξ being a white Gaussian noise [with properties specified by Eqs. (2.98)], which describes the state of some physical system at time t. In order to characterise such system, one usually introduces suitable observables, which can provide information on its properties, and measures them on a finite time interval $[0, t]$. In mathematical terms, a general observable of Y can be defined as a functional of its stochastic paths [139]:

$$W(t) = \int_0^t U(Y(r))\,\mathrm{d}r\,, \tag{3.1}$$

where the function $U(x)$ is some prescribed arbitrary smooth function. Brownian functionals of the type of Eq. (3.1) have found several applications, as many different phenomena can be modelled by choosing the function U suitably. For instance, W can describe fluctuating interfaces ($U(x) = x^2$), local and occupation times [$U(x) = \delta(x)$, $U(x) = \Theta(x)$ respectively], advection of particles [$U(x) = x$ and Y interpreted as a velocity], like the Obukhov's model for particles in a turbulent flow [142, 143], and even stock prices' dynamics [$U(x) = e^{-\beta x}$, with β real positive parameter] [139, 144]. Thus, we review in this section how one can characterise the statistical properties of normal diffusive W and Y.

As discussed in Sec. 2.2, the statistics of W is completely determined by its PDF $P(w,t) = \langle \delta(w - W(t)) \rangle$, which is obtained by first computing the joint PDF $P(w,y,t) = \langle \delta(w - W(t))\, \delta(y - Y(t)) \rangle$ and secondly by integrating out the y variable. For later convenience, we consider its Fourier transform $\widehat{P}(p,y,t) = \left\langle e^{i\,p\,W(t)} \delta(y - Y(t)) \right\rangle$, where the average is over all the paths $Y(\tau)$ starting at a specified initial position y_0 at $\tau = 0$ and ending at the position y at time $\tau = t$. Different techniques, e.g., path-integral arguments [145, 139] or more mathematical ones based on the Itô formula [109], are employed to derive the evolution equation of $\widehat{P}(p,y,t)$, which is given by the Feynman-Kac (FK) formula:

$$\frac{\partial}{\partial t}\widehat{P}(p,y,t) = i\,p\,U(y)\,\widehat{P}(p,y,t) + \frac{\sigma}{2}\frac{\partial^2}{\partial y^2}\,\widehat{P}(p,y,t). \tag{3.2}$$

While its physical relevance has just been clarified, Eq. (3.2) also represents a milestone in the theory of stochastic processes, as it provides the stochastic representation of the solutions of a partial differential equation of the type of Eq. (3.2). In details, if $f(p,y,t)$ is a solution of Eq. (3.2), with initial condition $f(p,y,0) = g(y)$, then the stochastic representation holds: $f(p,y,t) = \left\langle g(Y(t)) e^{i\,p\,W(t)} \right\rangle$ with the dynamics of Y specified by the operator in the second term of the rhs of Eq. (3.2) (here a free diffusive BM) and W defined as in Eq. (3.1) with the function U defined by the first term in the rhs of of Eq. (3.2) [109]. In our following discussion, we will recover Eq. (3.2) as a special case.

3.3 The Generalised Feynman-Kac Equation

The derivation of the FK formula Eq. (3.2) generated an intense research activity along two main directions, i.e., (i) the derivation of its solution for relevant choices of U and (ii) the derivation of its extension to more general stochastic processes, specifically CTRWs. Indeed, a fractional extension of Eq. (3.2) has been recently derived in [146, 147, 148] by using a similar approach to that discussed in Sec. 2.1.2 to derive fractional evolution equation for the position PDF of a CTRW. In details, the authors first derived the analogue of the Montroll-Weiss Eq. (2.17) for the joint PDF $P(w,y,t)$, which accounts also for the displacement of W during the jumps of the walker, and then they took the diffusive limit of its Fourier-Laplace transform. They derived the following Fractional FK (FFK) equation:

$$\frac{\partial}{\partial t}\widehat{P}(p,y,t) = i\,p\,U(y)\,\widehat{P}(p,y,t) + K_\alpha\,\frac{\partial^2}{\partial y^2}\,\mathcal{D}_t^{1-\alpha}\,\widehat{P}(p,y,t), \tag{3.3}$$

where we introduce the fractional substantial derivative $\mathcal{D}_t^{1-\alpha}$ [149, 150], which is defined in Laplace space as $[\lambda + i\,p\,U(y)]^{1-\alpha}$ and K_α is a generalised diffusion coefficient. The FFK Eq. (3.3) is relevant, as it first provided a quantitative description of the statistical properties of the functionals of a CTRW. However, such derivation does not prove the link between their stochastic representation in terms of subordinated Langevin equations and the solutions of Eq. (3.3) (see Fig. 3.1). Thus, in order to have a comprehensive framework to describe functionals of the CTRW, including both the corresponding fractional evolution equation and their stochastic description, one needs to prove Eq. (3.3) directly from the subordinated Langevin equations, in analogy to what has been done for Eq. (3.2). We will provide such proof as a special case in the following discussion.

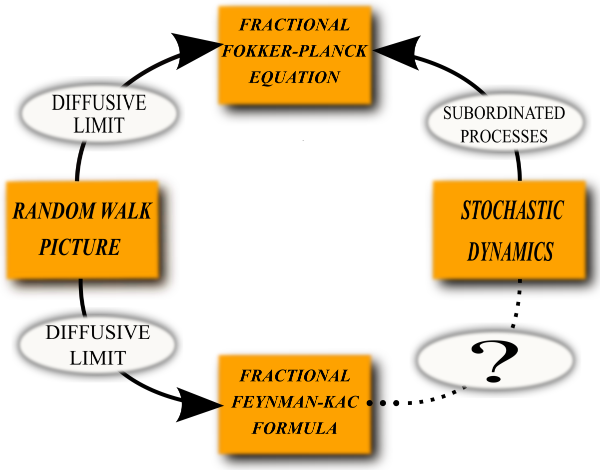

Figure 3.1: Equivalence between the random walk picture of CTRWs and their stochastic description in terms of subordinated Langevin equations before our work. While this equivalence has been recently proved for the FFP Eq. (2.29) [151, 152], this is not the case for the FFK Eq. (3.3). Indeed, its first derivation in Refs. [146, 147, 148] employs the random walk techniques presented in Sec. 2.1, but it does not discuss if it can be equivalently derived from the stochastic dynamical equations. This is obtained in [140].

3.3.1 Langevin Description of Anomalous Stochastic Processes

In this section, we extend the formulation of CTRWs in terms of subordinated Langevin equations [79], already discussed in Sec. 2.1.2, to arbitrary waiting time distributions. Specifically, this will be obtained by relaxing the hypothesis on its asymptotic tails. Instead, the waiting time distribution will be characterised in full generality by its characteristic function and corresponding Laplace exponent Eq. (2.156). We will further elucidate how the statistics of the waiting times is determined below. Finally, we will derive a FK formula for such anomalous processes and their functionals defined as in Eq. (3.1).

Let us focus on a process in 1-dim. As discussed in Sec. 2.1.2, the stochastic trajectory of a CTRW is determined by introducing two processes X and T with Langevin equations:

$$\dot{X}(s) = F(X(s)) + \sigma(X(s)) \cdot \xi(s), \tag{3.4a}$$
$$\dot{T}(s) = \eta(s). \tag{3.4b}$$

The CTRW $Y(t)$ is then obtained by time-changing the process X. Specifically, we define $Y(t) = X(S(t))$, with S specified by Eq. (2.47). The dynamics of X is that of a normal diffusive process in the operational time s, i.e., we require $\xi(s)$ to represent a white Gaussian noise with properties given by Eq. (2.98). Furthermore, we adopt the Itô convention for the multiplicative term of Eqs. (3.4a). In order to assure the existence and uniqueness of their corresponding time-changed solution Y, the functions $F(x)$ and $\sigma(x)$ need to satisfy (i) the *Lipschitz* condition, i.e., there exists a positive constant L such that $\forall\, x, y \in \mathbb{R}$ we have: $|F(x) - F(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|$ and (ii) the assumption that if X is càdlàg, then both $F(X)$ and $\sigma(X)$ are càglàd functions, i.e., they have left-continuous with right limits paths [120]. The noise $\eta(s)$ models the waiting times of the anomalous diffusion process in the operational time s, which we assume independent of the X process, i.e., the noises $\xi(s)$ and $\eta(s)$ are statistically independent. In the case of CTRWs, $\eta(s)$ is a one-sided stable Lévy noise of order $0 < \alpha \leq 1$ [101], whose integrated process T has characteristic function: $\left\langle e^{-\lambda T(s)} \right\rangle = e^{-s\lambda^{\alpha}}$ [see Eqs. (2.156, 2.158)]. In physical terms, T represents the elapsed physical time of the process (as already discussed in details in Sec. 2.1.2).

To extend this picture to arbitrary waiting times, we consider $\eta(s)$ as a more general type of noise, which can be modelled by a general one-sided (monotonically increasing) Lévy process with finite variation [101, 107]. As highlighted in Sec. 2.2.3, such a process satisfies the minimal assumptions needed to assure independent and stationary waiting times and causality of T. Thus, $\eta(s)$ is fully characterised by its characteristic functional:

$$G[u(s)] = \left\langle e^{-\int_0^{+\infty} u(s)\eta(s)\,\mathrm{d}s} \right\rangle = e^{-\int_0^{+\infty} \Phi(u(s))\,\mathrm{d}s}\,, \tag{3.5}$$

with the Laplace exponent Φ specified by Eq. (2.156). Different functional forms of Φ correspond to different distribution laws of the waiting times and of the renewal process T. The renewal nature of T is expressed by Eq. (3.4b) as $T(s) = \int_0^s \eta(\tau)\,\mathrm{d}\tau$. The characteristic function $\left\langle e^{-\lambda T(s)} \right\rangle$ is thus directly obtained from Eq. (3.5) by setting $u(s') = \lambda\Theta(s - \tau)$ leading to $\left\langle e^{-\lambda T(s)} \right\rangle = e^{-s\Phi(\lambda)}$ [see Eqs. (2.156)]. Therefore, T is a sum over waiting time increments $\Delta t = \int_0^{\Delta s} \eta(\tau)\,\mathrm{d}\tau$ over a small time step Δs with characteristic function $\left\langle e^{-\lambda \Delta t} \right\rangle = e^{-\Delta s \Phi(\lambda)}$, which can be used to simulate the process $Y(t)$ within a suitable discretization scheme [81]. Remarkably, the full multi-point statistics of T becomes easily accessible, because the functional Eq. (3.5) contains the information about the whole noise trajectory. By choosing Φ suitably, many different waiting time statistics can be captured, i.e., $Y(t)$ can be modelled according to the observed experimental dynamics. If we choose a power law $\Phi(\lambda) = \lambda^{\alpha}$ we recover the CTRW case with waiting times characterized by a diverging first moment. If instead $\Phi(\lambda) = \lambda$, T is simply a deterministic drift, $T = s$, and $Y(t)$ reduces to a normal diffusion (Brownian limit) with waiting times following an exponential distribution [9]. With these minimal assumptions, both the processes $T(s)$ and $S(t)$ are monotonically non decreasing, this also implying that S has paths of finite variation, as we discussed in Sec. 2.2.1. In addition, we can prove the relation [131]:

$$\Theta(s - S(t)) = 1 - \Theta(t - T(s)). \tag{3.6}$$

Moreover, recalling that $T(s)$ is strictly increasing, as it is defined in Eq. (3.4b) as an integral over a one-sided process, $S(t)$ can be shown to have continuous paths (see Sec. 2.2.4 and Ref. [104]), thus implying that the corresponding Itô formula does not have jump terms nor second order terms and it is given by Eq. (2.91) [for a general smooth function $f(s)$]:

$$f(S(t)) - f(0) = \int_0^t \frac{\partial f}{\partial s}(S(\tau))\,\mathrm{d}S(\tau)\,. \tag{3.7}$$

Consequently, if we choose $f(S(t)) = \Theta(s - S(t))$ and we use Eq. (3.6), we obtain:

$$\Theta(t - T(s)) = \int_0^t \delta(s - S(t))\, \mathrm{d}S(t)\,, \tag{3.8}$$

or equivalently in its corresponding differential form:

$$\delta(t - T(s)) = \delta(s - S(t))\dot{S}(t). \tag{3.9}$$

We note that $\dot{S}(t) = \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t}$ is a shorthand notation to denote an integration with respect to the time-change. With this notation, we can rewrite the coupled Langevin Eqs. (3.4a, 3.4b) as a time-changed stochastic differential equation [120]. In details, we first write Eq. (3.4a) in its integrated form as below (we assume $X(0) = y_0$):

$$X(s) - y_0 = \int_0^s F(X(\tau))\, \mathrm{d}\tau + \int_0^s \sigma(X(\tau))\, \mathrm{d}B(\tau)\,. \tag{3.10}$$

If we now apply directly the time change, we obtain the integrated equation for Y:

$$Y(t) - y_0 = \int_0^{S(t)} F(X(\tau))\, \mathrm{d}\tau + \int_0^{S(t)} \sigma(X(\tau))\, \mathrm{d}B(\tau)\,, \tag{3.11}$$

which can then be further simplified by employing the general result presented in Eq. (2.167a). In our specific case, we derive the following:

$$\begin{aligned} Y(t) - y_0 &= \int_0^t F(X(S(\tau))\, \mathrm{d}S(\tau) + \int_0^t \sigma(X(S(\tau)))\, \mathrm{d}B(S(\tau)) \\ &= \int_0^t F(Y(\tau))\, \mathrm{d}S(\tau) + \int_0^t \sigma(Y(\tau))\, \mathrm{d}B(S(\tau))\,, \end{aligned} \tag{3.12}$$

which can finally be written as a Langevin equation by taking its time derivative:

$$\dot{Y}(t) = F(Y(t))\dot{S}(t) + \sigma(Y(t)) \cdot \xi(S(t))\dot{S}(t). \tag{3.13}$$

This result is relevant as it expresses the evolution of the increments of Y directly in terms of those of the time change S. For clarity, we recall that $\xi(S(t))\dot{S}(t)$ denotes an increment over the time-changed brownian motion: $\xi(S(t))\dot{S}(t) = \lim_{\Delta t \to 0} \frac{B(S(t+\Delta t)) - B(S(t))}{\Delta t}$. This follows straightforwardly from Eq. (2.93), by recalling that $\Delta S(t) = S(t + \Delta t) - S(t) \to 0$ for $\Delta t \to 0$, as the paths of S are continuous and monotonically increasing. We note that the conditions on F and σ for the existence and uniqueness of the solution Y reported previously are a direct consequence of the fact that both Y and $S(t)$ are semimartingales [120] and can be proved from the time-changed Langevin Eq. (3.13) by employing Theorem 7, Chapter 5 of Ref. [106].

Let us now introduce a second stochastic process $W = (W(t), t \geq 0)$, which we define as a general functional of the anomalous process Y according to Eq. (3.1), and recall a few general properties of both the processes Y and W. First, Y can be shown to be a semimartingale, as long as the parent process X in Eq. (3.4a) is a semimartingale as well [119]. In our specific case, X is a Brownian diffusive process, i.e., it satisfies this property. Secondly, we note that $W(t)$ is a finite variation process and that the paths of both Y and W are continuous due to the continuity of those of S [120] and to the fact that the composition of two continuous function, i.e., of $X(s)$ and $S(t)$, is continuous. In order to use the Itô formula Eq. (3.15), we need the quadratic variation of Y. Recalling Eq. (2.114) and the formula for time-changed processes Eqs. (2.167a, 2.167b) [120], we obtain:

$$
\begin{aligned}
[Y,Y]_t = [X,X]_{S(t)} &= \int_0^{S(t)} \sigma^2(X(\tau))\,\mathrm{d}\tau \\
&= \int_0^t \sigma^2(X(S(\tau))\,\mathrm{d}S(\tau) \;=\; \int_0^t \sigma^2(Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau\,.
\end{aligned} \tag{3.14}
$$

We note that this same result was derived with a different approach earlier in Ref. [153].

3.3.2 Derivation of the Generalised Feynman-Kac Equation

Let us consider the joint process $Z(t) = (Y(t), W(t))$. As suggested in Sec. 2.2.4, the process Z is a semi-martingale with continuous paths, as well as Y and W. Thus, we can write its Itô formula (for a general smooth function f) by adapting Eq. (2.164) [102]:

$$
\begin{aligned}
f(Z(t)) = f(Z_0) &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y\,\partial w}(Z(\tau))\,\mathrm{d}[Y,W]_\tau + \int_0^t \frac{\partial}{\partial y} f(Z(\tau))\,\mathrm{d}Y(\tau) \\
+\int_0^t \frac{\partial}{\partial w} f(Z(\tau))\,\mathrm{d}W(\tau) &+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2} f(Z(\tau))\,\mathrm{d}[Y,Y]_\tau + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial w^2} f(Z(\tau))\,\mathrm{d}[W,W]_\tau
\end{aligned} \tag{3.15}
$$

In order to simplify this equation we need the following ingredients: (i) the time-discretised form of Eq. (3.13) that expresses the increments of Y in terms of the time-change increments $\mathrm{d}S(t)$; (ii) the explicit computation of the quadratic variation of Y Eq. (3.14); (iii) the quadratic variation $[W,W]_t$ and covariation $[Y,W]_t$, which are both null as W is a finite variation process. Further recalling from Eq. (3.1) that $\mathrm{d}W(t) = U(Y(t))\,\mathrm{d}t$, we obtain:

$$
\begin{aligned}
f(Z(t)) = f(Z_0) &+ \int_0^t \frac{\partial}{\partial w} f(Z(\tau)) U(Y(\tau))\,\mathrm{d}\tau + \int_0^t \frac{\partial}{\partial y} f(Z(\tau)) F(Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2} f(Z(\tau))\sigma^2(Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau + \int_0^t \frac{\partial}{\partial y} f(Z(\tau))\sigma(Y(\tau))\xi(S(\tau))\dot{S}(\tau)\,\mathrm{d}\tau\,.
\end{aligned} \tag{3.16}
$$

The equation for the double Fourier transform of the joint PDF $\widehat{P}(p,k,t)$ can be derived by evaluating Eq. (3.16) for $f(Z(t)) = e^{i\,k\,Y(t) + i\,p\,W(t)}$. Specifically, we obtain:

$$
\begin{aligned}
f(Z(t)) = f(Z_0) &+ i\,p \int_0^t f(Z(\tau)) U(Y(\tau))\,\mathrm{d}\tau + i\,k \int_0^t f(Z(\tau)) F(Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau \\
&- \frac{k^2}{2}\int_0^t f(Z(\tau))\sigma^2(Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau + i\,k \int_0^t f(Z(\tau))\sigma(Y(\tau))\xi(S(\tau))\dot{S}(\tau)\,\mathrm{d}\tau\,.
\end{aligned} \tag{3.17}
$$

Finally we need to take the ensemble average over the realisations of the noises ξ and η, the latter determining the realisations of the process S. Within the Itô prescription, the last integral in the rhs of Eq. (3.17) cancels out. Indeed, let us introduce a finite time-discretisation with step size Δt and let $N = t/\Delta t$. We denote: $Z_i = Z(t_i)$, $S_i = S(t_i)$ and $Y_i = Y(t_i)$. The stochastic integral then reads as:

$$
\int_0^t f(Z(\tau))\sigma(Y(\tau))\xi(S(\tau))\dot{S}(\tau)\,\mathrm{d}\tau = \lim_{\substack{N\to\infty \\ \Delta t\to 0}} \sum_{i=0}^{N-1} f(Z_i)\sigma(Y_i)[B(S_{i+1}) - B(S_i)]. \tag{3.18}
$$

Let us take the average over ξ first. For each fixed realisation of S we can then write: $\langle f(Z_i)\sigma(Y_i)[B(S_{i+1}) - B(S_i)]\rangle = \langle f(Z_i)\sigma(Y_i)\rangle \langle B(S_{i+1}) - B(S_i)\rangle = 0$ which is due to (i) the independence of the increments of B, that enables us to factorise the average because both Z_i, Y_i only depends on its previous increments, and (ii) to the zero first moment of B. Thus, the averaged Eq. (3.17) reduces to the following:

$$\begin{aligned}\langle f(Z(t))\rangle = f(Z_0) + i\,p \int_0^t \langle f(Z(\tau))U(Y(\tau))\rangle\,\mathrm{d}\tau \\ + \left\langle \int_0^t f(Z(\tau))\left[i\,k\,F(Y(\tau)) - \frac{k^2}{2}\sigma^2(Y(\tau))\right]\dot{S}(\tau)\,\mathrm{d}\tau \right\rangle,\end{aligned} \tag{3.19}$$

where in the second integral in its rhs we can recognise the Fourier transform of the FP operator of Eq. (3.4a): $\mathcal{L}_{FP}(y) = -\frac{\partial}{\partial y}F(y) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y)$. Consequently, if we now make the inverse Fourier transform of Eq. (3.19), we obtain the equation:

$$\frac{\partial}{\partial t}\widehat{P}(p,y,t) = i\,p\,U(y)\,\widehat{P}(p,y,t) + \mathcal{L}_{FP}(y)\frac{\partial}{\partial t}\left\langle \int_0^t e^{i\,p\,W(\tau)}\delta(y - Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau \right\rangle. \tag{3.20}$$

In order to close the equation, we still need to relate the averaged stochastic integral in the rhs of Eq. (3.20) to $\widehat{P}(p,y,t)$. We first note that W can be written as a subordinated process, by making the change of variables $\tau = S(r)$, i.e., $T(\tau) = r$, in Eq. (3.1):

$$W(t) = A(S(t)) \qquad A(s) = \int_0^s U(X(\tau))\eta(\tau)\,\mathrm{d}\tau\,, \tag{3.21}$$

where the noise $\eta(s)$ explicitly appears from Eq. (3.4b). Thus, by employing the property $1 = \int_0^{+\infty}\delta(s - S(t))\,\mathrm{d}s$ and then considering the same discretisation scheme and notation used in Eq. (3.18) to expand in series the stochastic integral, we obtain:

$$\begin{aligned}\int_0^t e^{i\,p\,W(\tau)}\delta(y - Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau &= \int_0^t \left[\int_0^{+\infty} e^{i\,p\,A(s)}\delta(y - X(s))\delta(s - S(\tau))\,\mathrm{d}s\right]\dot{S}(\tau)\,\mathrm{d}\tau \\ &= \lim_{\substack{N\to\infty \\ \Delta t\to 0}} \sum_{i=0}^{N-1}\left[\int_0^{+\infty} e^{i\,p\,A(s)}\delta(y - X(s))\delta(s - S_i)\,\mathrm{d}s\right](S_{i+1} - S_i) \\ &= \lim_{\substack{N\to\infty \\ \Delta t\to 0}} \sum_{i=0}^{N-1}\left[\int_0^{+\infty} e^{i\,p\,A(s)}\delta(y - X(s))\delta(t_i - T(s))\,\mathrm{d}s\right](\tau_{i+1} - \tau_i) \\ &= \int_0^t \left[\int_0^{+\infty} e^{i\,p\,A(s)}\delta(y - X(s))\delta(\tau - T(s))\,\mathrm{d}s\right]\mathrm{d}\tau\,,\end{aligned} \tag{3.22}$$

where (i) the continuity of the paths of S implies that no jump terms appear in the series expansion (see Sec. 2.2.1) and (ii) we used Eq. (3.9) to relate the stochastic increments of S to those of T. If we take the average over the realisations of the two noises η and ξ of both sides of Eq. (3.22) and then the time derivative of the resulting expression, we obtain:

$$\frac{\partial}{\partial t}\left\langle \int_0^t e^{i\,p\,W(\tau)}\delta(y - Y(\tau))\dot{S}(\tau)\,\mathrm{d}\tau \right\rangle = \int_0^{+\infty}\left\langle e^{i\,p\,A(s)}\delta(y - X(s))\delta(t - T(s))\right\rangle\mathrm{d}s \tag{3.23}$$

Remarkably, the rhs side of this equation can be related in Laplace space to the joint PDF $\widehat{P}(p,y,t)$. By using again the representation of W as a subordinated process in Eqs. (3.21) and the property used to derive Eq. (3.22), $\widehat{P}(p,y,t)$ can be rewritten as follows:

$$\widehat{P}(p,y,t)=\left\langle e^{i\,p\,A(S(t))}\delta(y-X(S(t)))\right\rangle=\int_0^{+\infty}\left\langle e^{i\,p\,A(s)}\delta(y-X(s))\delta(s-S(t))\right\rangle \mathrm{d}s \quad (3.24)$$

In this form, the Laplace transform of Eq. (3.24) can be computed straightforwardly. Indeed, recalling the dynamical equation of T Eq. (3.4b) and Eq. (3.6), we derive:

$$\begin{aligned}\int_0^{+\infty}\delta(s-S(t))e^{-\lambda\,t}\,\mathrm{d}t &= \frac{\partial}{\partial s}\int_0^{+\infty}\Theta(s-S(t))e^{-\lambda\,t}\,\mathrm{d}t \\ &= \frac{\partial}{\partial s}\int_0^{+\infty}\left[1-\Theta(t-T(s))\right]e^{-\lambda\,t}\,\mathrm{d}t = \frac{\partial}{\partial s}\int_0^{T(s)}e^{-\lambda\,t}\,\mathrm{d}t = \eta(s)e^{-\lambda\,T(s)}. \quad (3.25)\end{aligned}$$

In conclusion, the Laplace transform of Eq. (3.24) is readily given by:

$$\widehat{\widetilde{P}}(p,y,\lambda)=\int_0^{+\infty}\left\langle\left\langle e^{-\lambda\,T(s)+i\,p\,A(s)}\eta(s)\delta(y-X(s))\right\rangle\right\rangle \mathrm{d}s\,. \quad (3.26)$$

In Eq. (3.26), we explicitly highlighted that the ensemble average is made over the two different noises η and ξ, whose independence allows us to change arbitrarily the order in which these averages are performed. Such freedom can be readily employed to simplify Eq. (3.26) by expressing the $\eta(s)$-dependent part of the integrand as a derivative of the characteristic functional $G[k(l)]$ in Eq. (3.5). Indeed, by performing the average with respect to $\eta(s)$ first and recalling that X does not depend on it, this implying that the delta function can be taken out of such average, the only quantity needed to be computed is $\left\langle e^{-\lambda\,T(s)+i\,p\,A(s)}\eta(s)\right\rangle$. This can be obtained as follows:

$$\begin{aligned}\left\langle\eta(s)e^{-\lambda\,T(s)+i\,p\,A(s)}\right\rangle &= \left\langle\eta(s)e^{-\int_0^s[\lambda-i\,p\,U(X(r))]\eta(r)\,\mathrm{d}r}\right\rangle \\ &= -\frac{1}{\lambda-i\,p\,U(X(s))}\frac{\partial}{\partial s}\left\langle e^{-\int_0^s[\lambda-i\,p\,U(X(r))]\eta(r)\,\mathrm{d}r}\right\rangle \\ &= -\frac{1}{\lambda-i\,p\,U(X(s))}\frac{\partial}{\partial s}e^{-\int_0^s\Phi(\lambda-i\,p\,U(X(r)))\,\mathrm{d}r} \\ &= \frac{\Phi(\lambda-i\,p\,U(X(s)))}{\lambda-i\,p\,U(X(s))}\left\langle e^{-\int_0^s[\lambda-i\,p\,U(X(r))]\eta(r)\,\mathrm{d}r}\right\rangle \quad (3.27)\end{aligned}$$

where we used the characteristic functional Eq. (3.5) with $k(l)=(\lambda-i\,p\,U(X(l)))\,\Theta(s-l)$. Substituting Eq. (3.27) back into Eq. (3.26), we derive the following relation:

$$\widehat{\widetilde{P}}(p,y,\lambda)=\frac{\Phi\left[\lambda-i\,p\,U(y)\right]}{\lambda-i\,p\,U(y)}\int_0^{+\infty}\left\langle e^{-\lambda T(s)+i\,p\,A(s)}\delta(y-X(s))\right\rangle \mathrm{d}s\,, \quad (3.28)$$

where now the brackets denote again an average over both η, ξ. To conclude, we note that the Laplace transform of the rhs of Eq. (3.23) is equal to the integral in Eq. (3.28). Thus, by expressing it in terms of $\widehat{\widetilde{P}}(p,y,\lambda)$, taking its inverse Laplace transform and substituting in Eq. (3.20), we derive the Generalised FFK (GFFK) formula:

$$\begin{aligned}\frac{\partial}{\partial t}P(p,y,t) &= i\,p\,U(y)\,P(p,y,t) \\ &\quad + \mathcal{L}_{FP}(y)\left[\frac{\partial}{\partial t}-i\,p\,U(y)\right]\int_0^t K(t-\tau)\,e^{i\,p\,U(y)(t-\tau)}\,P(p,y,\tau)\,\mathrm{d}\tau\,, \quad (3.29)\end{aligned}$$

where the memory kernel is related to Φ trough the following relation (in Laplace space):

$$K(\lambda) = \Phi(\lambda)^{-1}. \tag{3.30}$$

We highlight that this derivation of Eq. (3.29) provides the generalization of the Feynman-Kac theorem to anomalous processes with arbitrary waiting time distributions.

3.3.3 Special Cases and Extensions

In this section we first show that the GFFK Eq. (3.29) provides as special cases several different results earlier presented in the literature. Specifically, we can derive:

- *The generalised Fractional Fokker-Planck Equation.*
  If we set $p = 0$, we find a Generalised Fractional FP (GFFP) equation for the position PDF [154]:
  $$\frac{\partial}{\partial t} P(y,t) = \mathcal{L}_{FP}(y) \frac{\partial}{\partial t} \int_0^t K(t-\tau)\, P(y,\tau)\, \mathrm{d}\tau\,. \tag{3.31}$$

- *The generalised Klein-Kramers Equation.*
  If we set $U(x) = x$ in Eq. (3.1), Y and W correspond respectively to the velocity and the position of an anomalously diffusing particle. Thus, after inverse Fourier transform, Eq. (3.29) yields a generalised Fractional Klein-Kramers (KK) equation [149, 150]:
  $$\begin{aligned}\frac{\partial}{\partial t} P(x,v,t) &= -\frac{\partial}{\partial x} v\, P(x,v,t) \\ &\quad + \mathcal{L}_{FP}(y) \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v\right] \int_0^t K(t-s) P(x - v(t-s), y, s)\, \mathrm{d}s\,. \end{aligned} \tag{3.32}$$
  We note that the shift in the position PDF in the memory integral elucidates the presence of the same retardation effects found for CTRWs [149, 150].

In addition, by suitably choosing the Laplace exponent Φ, we recover both normal diffusive processes and CTRWs. Specifically, we obtain:

- *Normal Diffusion.*
  This is obtained by setting $\Phi(\lambda) = \lambda$, i.e., $K(t) = 1$. In this case, Eqs. (3.29,3.31,3.32) reduces respectively to the normal FK Eq. (3.2), FP and KK equations.

- *CTRWs.*
  This is obtained by setting $\Phi(\lambda) = \lambda^\alpha$. In this case indeed, the integral operators in Eqs. (3.29,3.31) reduces to the fractional substantial derivative [149, 150, 146, 147, 155] and the RL operator respectively [156, 151].

We finally discuss two extensions of Eq. (3.29), i.e., (i) the case of a general prescription for the multiplicative term in Eq. (3.4a) and (ii) the case of time-dependent forces.

Extension to multiplicative processes with general prescription

This first generalisation is obtained by recalling that processes with general prescription can be written in terms of processes with the Itô one by suitably choosing the coefficients of the Langevin equation (see Sec. 2.2.2). We consider the subordinated equations:

$$\dot{X}(s) = F(X(s)) + \sigma(X(s)) \star \xi(s) \qquad\qquad \dot{T}(s) = \eta(s) \tag{3.33}$$

where we adopt the generalised α prescription defined in Eq. (2.101) for the multiplicative term. However, this process is equivalently described by Eqs. (3.4a, 3.4b), i.e. with the Itô prescription, by using the mapping proposed in Eqs.(2.130a, 2.130b). Thus, Eqs. (3.29, 3.30) still holds for the subordinated Eqs. (3.33) with the modified FP operator:

$$\mathcal{L}_{FP}(y) = -\frac{\partial}{\partial y}\left[F(y) + \kappa\, \sigma(y)\sigma'(y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y). \tag{3.34}$$

Extension to processes with time dependent force terms

In the presence of time dependent forces, Eq. (3.29) with the substitution $\mathcal{L}_{FP}(y) \rightarrow \mathcal{L}_{FP}(y,t) = -\frac{\partial}{\partial y}F(y,t) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y)$ was already proved in the specific case of CTRWs starting from a master equation approach in [147]. We recall that time dependence in the external force is introduced in the subordinated Langevin Eqs. (3.4a, 3.4b) by making the function F depend explicitly on T. Thus, the time dependent problem is written as below:

$$\dot{X}(s) = F(X(s), T(s)) + \sigma(X(s)) \cdot \xi(s), \qquad\qquad \dot{T}(s) = \eta(s). \tag{3.35}$$

The same conditions on F and σ, that we presented for Eqs. (3.4a, 3.4b), are also sufficient to ensure the existence and uniqueness of the time-changed solution $Y(t)$ of Eqs.(3.35) [120]. There are two main differences with the previous case of Eq. (3.4a). On the one hand, the processes X and T are no longer independent, such that the previous derivation of Eq. (3.28) does not hold any more. Specifically, the delta function in Eq. (3.26) needs to be kept inside the average over the realisations of the noise η. On the other hand, differently from the time independent case where the stochastic paths of X, and consequently those of F, have continuous paths, in the case of Eq. (3.35), due to the dependence of X on the Lévy process T, both its paths and those of F are generally discontinuous, with random jumps in correspondence to those of T. However, thanks to the finite variation of the paths of T, both X and the time-changed process Y are still semimartingales and additionally the paths of F are càdlàg. Thus, the Stiltjes integral of F is well defined and we can integrate Eq. (3.35) as follows:

$$X(s) - y_0 = \int_0^s F(X(\tau), T(\tau))\, \mathrm{d}\tau + \int_0^s \sigma(X(\tau))\, \mathrm{d}B(\tau)\,. \tag{3.36}$$

We note that the integral over F is done with respect to a process with finite variation and continuous paths, here specifically a deterministic drift, such that the contribution from the random jumps of T is still null. As before, we can then write:

$$\begin{aligned} Y(t) - y_0 &= \int_0^{S(t)} F(X(\tau), T(\tau))\, \mathrm{d}\tau + \int_0^t \sigma(X(\tau))\, \mathrm{d}B(\tau) \\ &= \int_0^t F(X(S(\tau), T(S(\tau)))\, \mathrm{d}S(\tau) + \int_0^t \sigma(X(S(\tau)))\, \mathrm{d}B(S(\tau)) \\ &= \int_0^t F(Y(\tau), \tau)\, \mathrm{d}S(\tau) + \int_0^t \sigma(Y(\tau))\, \mathrm{d}B(S(\tau))\,, \end{aligned} \tag{3.37}$$

where we employed again Eq. (2.167a). After taking its time derivative, we derive the following equation for the increments of Y [120]:

$$\dot{Y}(t) = F(Y(t), t)\dot{S}(t) + \sigma(Y(t)) \cdot \xi(S(t))\dot{S}(t). \tag{3.38}$$

This result elucidates that Y has still continuous paths and that Eq. (3.15) still holds. Similar arguments as in the previous derivation can be made, leading to the same Eq. (3.20) with $\mathcal{L}_{FP}(y,t)$ and the same averaged stochastic integral, which needs to be related to the joint PDF. As already highlighted, the proof Eq. (3.28) needs a detailed analysis, as both X and W depend on the realisations of the noise η. Starting from Eq. (3.26), we write:

$$\begin{aligned}
P(p,y,\lambda) &= \int_0^{+\infty} \left\langle e^{-\lambda T(s)+i\,p\,A(s)}\eta(s)\delta(y-X(s))\right\rangle \mathrm{d}s \\
&= \int_0^{+\infty} \left\langle \delta(y-X(s)) \left[\eta(s)\, e^{-\int_0^s \eta(r)[\lambda - i\,p\,U(X(r))]\,\mathrm{d}r}\right]\right\rangle \mathrm{d}s \\
&= -\frac{1}{\lambda - i\,p\,U(y)} \int_0^{+\infty} \left\langle \delta(y-X(s)) \frac{\partial}{\partial s} e^{-\int_0^s \eta(r)[\lambda - i\,p\,U(X(r))]\,\mathrm{d}r} \right\rangle \mathrm{d}s\,,
\end{aligned} \tag{3.39}$$

where we used again the definition Eq. (3.21) and the properties of the delta function to factorise the term $[\lambda - i\,p\,U(y)]$. However, as already highlighted, the factors inside the ensemble average can no longer be separated. Nevertheless, this expression can still be simplified if we look at its discretised form. We consider a partition $\pi = \{0 = s_0 < s_1 < \ldots < s_N = s\}$ of the interval $[0,s]$ with constant mesh Δs and $N = s/\Delta s$. We denote: $X(s_i) = X_i$ and $\eta(s_i) = \eta_i$. We recall that η_i are RVs with characteristic function specified by Φ, such that $\Delta T_i = \eta_i\,\Delta s$ is the corresponding T increment. Noting that the delta function only imposes a condition on the final point, we can write:

$$\begin{aligned}
&\left\langle \delta(y-X(s)) \frac{\partial}{\partial s} e^{-\int_0^s \eta(r)[\lambda - i\,p\,U(X(r))]\,\mathrm{d}r} \right\rangle \\
&\quad= \lim_{\substack{N\to\infty \\ \Delta s\to 0}} \left\langle \frac{e^{-\sum_{j=1}^{N+1}\eta_j\,[\lambda - i\,p\,U(X_{j-1})]\Delta s} - e^{-\sum_{j=1}^{N}\eta_j\,[\lambda - i\,p\,U(X_{j-1})]\Delta s}}{\Delta s} \right\rangle \Bigg|_{X_N=y} \\
&\quad= \lim_{\substack{N\to\infty \\ \Delta s\to 0}} \left\langle e^{-\sum_{j=1}^{N}\eta_j\,[\lambda - i\,p\,U(X_{j-1})]\Delta s} \left[\frac{e^{-\Delta s\,\eta_{N+1}[\lambda - i\,p\,U(X_N)]} - 1}{\Delta s}\right] \right\rangle \Bigg|_{X_N=y} \\
&\quad= \lim_{\substack{N\to\infty \\ \Delta s\to 0}} \left\langle e^{-\sum_{j=1}^{N}\eta_j\,[\lambda - i\,p\,U(X_j)]\Delta s} \right\rangle \left\langle \frac{e^{-\Delta s\,\eta_{N+1}[\lambda - i\,p\,U(y)]} - 1}{\Delta s} \right\rangle,
\end{aligned} \tag{3.40}$$

where in the first line we discretised the derivative in s and in the second line we factorised the average over the last increment $\Delta T = \Delta s\,\eta_{N+1}$. This is allowed because (i) $U(X_N)$ only depends on the increments of the process T up to N, which are independent on the final increment (as T is a Lévy process), and (ii) the end point value is conditioned to y, i.e., $U(X_N) = U(y)$, which is no longer a RV. The average is then computed with Eq. (3.5):

$$\left\langle \frac{e^{-\Delta s\,\eta_{N+1}[\lambda - i\,p\,U(y)]} - 1}{\Delta s} \right\rangle = \frac{e^{-\Delta s\,\Phi(\lambda - i\,p\,U(y))} - 1}{\Delta s}. \tag{3.41}$$

Substituting this term into Eq. (3.40) and taking the continuum limit $\Delta s \to 0$, we obtain:

$$\left\langle \delta(y - X(s)) \frac{\partial}{\partial s} e^{-\int_0^s \eta(r)[\lambda - i\,p\,U(X(r))]\,\mathrm{d}r} \right\rangle = \\ -\,\Phi(\lambda - i\,p\,U(y)) \left\langle \delta(y - X(s)) e^{-\int_0^s \eta(r)[\lambda - i\,p\,U(X(r))]\,\mathrm{d}r} \right\rangle , \quad (3.42)$$

leading with Eq. (3.39) to the same relation Eq. (3.28) also in the case of time dependent forces. The rest follows as in the earlier derivation. Thus, we have shown that Eq. (3.29) with the substitution $\mathcal{L}_{FP}(y) \to \mathcal{L}_{FP}(y,t)$ is the GFFKE of processes described by the subordinated Langevin Eqs. (3.35), where time dependent forces are present. Recalling that the only difference with Eq. (3.29) consists in the time dependence of the FP operator, the Brownian and CTRW special cases yield again the FK equation and its fractional extension, but with explicit time dependence in the force term. If we further set $p = 0$, we obtain the GFFP Eq. (3.31) with time dependent FP operator, which is placed on the left of the memory integral. Thus, our formalism naturally provides a solution to this issue, i.e., the position of the FP operator with respect to the memory integral, in a more general framework than CTRWs, in which specific case it has long been long debated [157, 151, 158, 152].

3.4 Multipoint Structure of Anomalous Processes

In this section, we provide a characterisation of the multipoint structure of the processes $Y(t)$ and $W(t)$. To this aim, we explicitly use their stochastic description Eqs. (3.4a-3.4b) and Eq. (3.1). Indeed, functions of both these processes evaluated at different times cannot be derived from Eq. (3.29), as this formula only describes functions evaluated at a single time. Within this discussion, we will also obtain useful properties of the time-change $S(t)$ and we will further obtain a formula for the two-point correlation function of Y, only involving integrals of single-time functions.

3.4.1 General Formalism

We first recall the standard technique used to compute ensemble averaged single point functions of Y and W. In the specific case of Eqs. (3.4a, 3.4b), (i) the dynamics of X is not coupled to T and (ii) the noises $\eta(s)$ and $\xi(s)$ are independent. Thus, the average of a general smooth function $f(y,t) = \langle f(Y(t)) \rangle$ can be written as below:

$$f(y,t) = \int_0^{+\infty} \langle f(X(s))\, \delta(s - S(t)) \rangle \,\mathrm{d}s \;= \\ \int_0^{+\infty} \langle f(X(s)) \rangle \, \langle \delta(s - S(t)) \rangle \,\mathrm{d}s \;= \int_0^{+\infty} \langle f(X(s)) \rangle \, h(s,t)\,\mathrm{d}s\,, \quad (3.43)$$

where we factorise the ensemble average on the realisations of the two noises and we introduce the single point PDF of S: $h(s,t) = \langle \delta(s - S(t)) \rangle$. This object can then be computed exactly by recalling Eq. (3.6) [131]. In details, we have: $h(s,t) = -\frac{\partial}{\partial s} \langle \Theta(t - T(s)) \rangle$. In Laplace space, this is readily computed. We have: $\int_0^{+\infty} e^{-\lambda t} \langle \Theta(t - T(s)) \rangle \,\mathrm{d}t \;= \left\langle e^{-\lambda T(s)} \right\rangle$, so that $h(s,t)$ is related in Laplace space to the characteristic function of the subordinator

T. If we use Eq. (3.5) with the test function $k(s') = \lambda\,\Theta(s - s')$, we obtain [159]:

$$\widetilde{h}(s,\lambda) = -\frac{1}{\lambda}\frac{\partial}{\partial s}\left\langle e^{-\lambda T(s)}\right\rangle = \frac{\Phi(\lambda)}{\lambda}e^{-s\Phi(\lambda)}. \tag{3.44}$$

Consequently, if $\langle f(X(s))\rangle$ is known, the combination of Eqs. (3.43, 3.44) provides the corresponding quantity for $Y(t)$, at least in Laplace space if an analytical inverse transform of Eq. (3.44) is not available. Specifically, we obtain the following result in Laplace space:

$$\widetilde{f}(y,\lambda) = \int_0^{+\infty} \langle f(X(s))\rangle\,\widetilde{h}(s,\lambda)\,\mathrm{d}s \tag{3.45}$$

In addition, we can compute straightforwardly the first moment of $W(t)$. Recalling its definition Eq. (3.1), we have: $\dot{W}(t) = U(Y(t))$, such that in Laplace space we find: $\lambda\,\widetilde{W}(\lambda) = \widetilde{U}(Y(\lambda)) = \int_0^{+\infty} e^{-\lambda t}\,U(Y(t))\,\mathrm{d}t$. If we now solve with respect to $\widetilde{W}$, take its ensemble average and use Eq. (3.45), we obtain the following equation:

$$\left\langle\widetilde{W}(\lambda)\right\rangle = \frac{1}{\lambda}\int_0^{+\infty} \langle U(X(s))\rangle\,\widetilde{h}(s,\lambda)\,\mathrm{d}s\,. \tag{3.46}$$

As a further example of application of Eq. (3.44), we provide a characterisation of the moments of the time-change S, which are given in Laplace space by the following formula:

$$\left\langle\left[\widetilde{S}(\lambda)\right]^n\right\rangle = \int_0^{+\infty} s^n\,\widetilde{h}(s,\lambda)\,\mathrm{d}s \;=\frac{n!}{\lambda\,[\Phi(\lambda)]^n}. \tag{3.47}$$

In particular, for $n = \{1,2\}$ we obtain respectively the first and second moments: $\left\langle\widetilde{S}(\lambda)\right\rangle = 1/[\lambda\,\Phi(\lambda)]$ and $\left\langle\left[\widetilde{S}(\lambda)\right]^2\right\rangle = 2/[\lambda\,[\Phi(\lambda)]^2]$. In the specific case of a CTRW subordinator, i.e., $\Phi(\lambda) = \lambda^\alpha$, we then recover the following results [131]: $\langle S(t)\rangle = t^\alpha/\Gamma(1+\alpha)$ and $\langle[S(t)]^2\rangle = 2\,t^{2\,\alpha}/\Gamma(1+2\,\alpha)$.

We now consider an averaged two point function of the process Y, i.e., a quantity of the type: $f(y_2,t_2;y_1,t_1) = \langle f(Y(t_1),Y(t_2))\rangle$ [131]. Following the same procedure, we find:

$$\begin{aligned} f(y_2,t_2;y_1,t_1) &= \int_0^{+\infty}\int_0^{+\infty} \langle f(X(S(t_2)),X(S(t_1)))\delta(s_2 - S(t_2))\delta(s_1 - S(t_1))\rangle\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_0^{+\infty}\int_0^{+\infty} \langle f(X(s_2),X(s_1))\rangle\,\langle\delta(s_2 - S(t_2))\delta(s_1 - S(t_1))\rangle\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_0^{+\infty}\int_0^{+\infty} \langle f(X(s_2),X(s_1))\rangle\,h(s_2,t_2;s_1,t_1)\,\mathrm{d}s_2\,\mathrm{d}s_1 \end{aligned} \tag{3.48}$$

where we factorise the ensemble average, thanks to the independence of the noises η and ξ and of the processes X and T, and we introduce the two point PDF of $S(t)$ as: $h(s_2,t_2;s_1,t_1) = \langle\delta(s_2 - S(t_2))\delta(s_1 - S(t_1))\rangle$. Again, we can exploit Eq. (3.6) to relate this object to the characteristic functional of the process T. In details, we find: $h(s_2,t_2;s_1,t_1) = \frac{\partial^2}{\partial s_2\,\partial s_1}\langle\Theta(t_2 - T(s_2))\Theta(t_1 - T(s_1))\rangle$, with the averaged quantity being related in Laplace space to the two point characteristic function of T: $\widetilde{Z}(\lambda_2,s_2;\lambda_1,s_1) = \left\langle e^{-\lambda_2 T(s_2)}e^{-\lambda_1 T(s_1)}\right\rangle$. Indeed, by making a double Laplace transform we obtain the following relation:

$$\widetilde{h}(s_2,\lambda_2;s_1,\lambda_1) = \frac{1}{\lambda_1\lambda_2}\frac{\partial^2}{\partial s_2\,\partial s_1}\widetilde{Z}(\lambda_2,s_2;\lambda_1,s_1). \tag{3.49}$$

We still need to find a closed equation for the two-point characteristic function Z. Indeed, its computation follows straightforwardly by distinguishing the two cases $t_2 > t_1$ and $t_2 < t_1$ and suitably separating the integrals over the noise η:

$$\widetilde{Z}(\lambda_2, s_2; \lambda_1, s_1) = \Theta(s_2 - s_1) \left\langle e^{-(\lambda_1+\lambda_2)\int_0^{s_1} \eta(\tau)\,\mathrm{d}\tau} e^{-\lambda_2 \int_{s_1}^{s_2} \eta(\tau)\,\mathrm{d}\tau} \right\rangle \\ + \Theta(s_1 - s_2) \left\langle e^{-(\lambda_1+\lambda_2)\int_0^{s_2} \eta(\tau)\,\mathrm{d}\tau} e^{-\lambda_1 \int_{s_2}^{s_1} \eta(\tau)\,\mathrm{d}\tau} \right\rangle . \quad (3.50)$$

Recalling that the increments of a Lévy process are independent and stationary, we can factorise the average in Eq. (3.50) and use Eq. (3.5) to compute each of the averages over distinct time intervals. In details, we obtain:

$$\widetilde{Z}(\lambda_2, s_2; \lambda_1, s_1) = \Theta(s_2 - s_1) e^{-s_1\,\Phi(\lambda_1+\lambda_2)} e^{-(s_2-s_1)\Phi(\lambda_2)} \\ + \Theta(s_1 - s_2) e^{-s_2\,\Phi(\lambda_1+\lambda_2)} e^{-(s_1-s_2)\Phi(\lambda_1)} . \quad (3.51)$$

If we then substitute this result in Eq. (3.49) and perform the time derivatives, we obtain:

$$\widetilde{h}(s_2, \lambda_2; s_1, \lambda_1) = \delta(s_2 - s_1) \frac{\Phi(\lambda_1) - \Phi(\lambda_1 + \lambda_2) + \Phi(\lambda_2)}{\lambda_1 \lambda_2} e^{-s_1 \Phi(\lambda_1+\lambda_2)} \\ + \Theta(s_2 - s_1) \frac{\Phi(\lambda_2)\left[\Phi(\lambda_1 + \lambda_2) - \Phi(\lambda_2)\right]}{\lambda_1 \lambda_2} e^{-s_1 \Phi(\lambda_1+\lambda_2)} e^{-(s_2-s_1)\Phi(\lambda_2)} \\ + \Theta(s_1 - s_2) \frac{\Phi(\lambda_1)\left[\Phi(\lambda_1 + \lambda_2) - \Phi(\lambda_1)\right]}{\lambda_1 \lambda_2} e^{-s_2 \Phi(\lambda_1+\lambda_2)} e^{-(s_1-s_2)\Phi(\lambda_1)} . \quad (3.52)$$

In the case of CTRWs, Eq. (3.52) recovers the result first presented in [131]. Consequently, if the corresponding two-point function of $X(s)$, i.e., $\langle f(X(s_2), X(s_1)) \rangle$, is known, the combination of Eqs. (3.48, 3.52) provides the corresponding quantity for Y, at least in 2D Laplace transform. This is indeed given by the formula below:

$$\widetilde{f}(y_2, \lambda_2; y_1, \lambda_1) = \int_0^{+\infty} \int_0^{+\infty} \langle f(X(s_2), X(s_1)) \rangle \, \widetilde{h}(s_2, \lambda_2; s_1, \lambda_1) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \quad (3.53)$$

Similarly to the single point case, we can employ Eq. (3.52) to compute the two-point correlation functions of W. Recalling that $\partial^2/(\partial t_1\, \partial t_2)\, W(t_1) W(t_2) = U(Y(t_1))\, U(Y(t_2))$, or in double Laplace transform: $\lambda_1\, \lambda_2\, \widetilde{W}(\lambda_1) \widetilde{W}(\lambda_2) = \widetilde{U}(Y(\lambda_1))\, \widetilde{U}(Y(\lambda_2))$, we find:

$$\left\langle \widetilde{W}(\lambda_1) \widetilde{W}(\lambda_2) \right\rangle = \frac{1}{\lambda_1 \lambda_2} \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda_1\, t_1} e^{-\lambda_2\, t_2} \langle U(Y(t_2)) U(Y(t_1)) \rangle \, \mathrm{d}t_1 \, \mathrm{d}t_2 \\ = \frac{1}{\lambda_1 \lambda_2} \int_0^{+\infty} \int_0^{+\infty} \langle U(X(s_2)), U(X(s_1)) \rangle \, \widetilde{h}(s_2, \lambda_2; s_1, \lambda_1) \, \mathrm{d}s_2 \, \mathrm{d}s_1 \quad (3.54)$$

where in the second line we use the same steps of Eq. (3.48).

As a further application of Eq. (3.52), we compute the two-point correlation function of S. Recalling its definition, we can write:

$$\langle S(t_1) S(t_2) \rangle = \int_0^{+\infty} \int_0^{+\infty} s_1\, s_2\, h(s_2, t_2; s_1, t_1) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \, , \quad (3.55)$$

which in Laplace space and by direct substitution of Eq. (3.52) becomes:

$$\left\langle \widetilde{S}(\lambda_1)\widetilde{S}(\lambda_2)\right\rangle = \frac{\Phi(\lambda_1)-\Phi(\lambda_1+\lambda_2)+\Phi(\lambda_2)}{\lambda_1\lambda_2}\int_0^{+\infty} s^2 e^{-s\,\Phi(\lambda_1+\lambda_2)}\,\mathrm{d}s \tag{3.56}$$
$$+\frac{\Phi(\lambda_2)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right]}{\lambda_1\lambda_2}\int_0^{+\infty}\!\!\int_0^{+\infty}\Theta(s_2-s_1)s_1\,s_2\,e^{-s_1\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\Phi(\lambda_2)}\,\mathrm{d}s_1\mathrm{d}s_2$$
$$+\frac{\Phi(\lambda_1)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_1)\right]}{\lambda_1\lambda_2}\int_0^{+\infty}\!\!\int_0^{+\infty}\Theta(s_1-s_2)s_1\,s_2\,e^{-s_2\,\Phi(\lambda_1+\lambda_2)}e^{-(s_1-s_2)\Phi(\lambda_1)}\,\mathrm{d}s_1\mathrm{d}s_2$$

where the remaining integrals can then be computed analytically. In particular, for the one in the first line of Eq. (3.56) we find: $\int_0^{+\infty} s^2 e^{-s\,\Phi(\lambda_1+\lambda_2)}\,\mathrm{d}s = 2/[\Phi(\lambda_1+\lambda_2)]^3$. The other two integrals can be solved with the change of variables: $(s_2', s_1') = (|s_2-s_1|, \min(s_2, s_1))$. In this new coordinates the integrals factorise, as we show below (for one of them):

$$\int_0^{+\infty}\!\!\int_0^{+\infty}\Theta(s_2-s_1)s_1\,s_2\,e^{-s_1\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\Phi(\lambda_2)}\,\mathrm{d}s_1\mathrm{d}s_2$$
$$=\int_0^{+\infty} s_2' e^{-s_2'\Phi(\lambda_2)}\,\mathrm{d}s_2' \int_0^{+\infty} s_1' e^{-s_1'\,\Phi(\lambda_1+\lambda_2)}\,\mathrm{d}s_1' + \frac{1}{\Phi(\lambda_2)}\int_0^{+\infty}(s_1')^2 e^{-s_1'\,\Phi(\lambda_1+\lambda_2)}\,\mathrm{d}s_1'$$
$$=\frac{1}{\Phi(\lambda_2)}\left[\frac{2}{[\Phi(\lambda_1+\lambda_2)]^3}+\frac{1}{\Phi(\lambda_1+\lambda_2)}\right] \tag{3.57}$$

Recalling that the third integral in Eq. (3.56) can be obtained by simply exchanging the indexes in Eq. (3.57), we derive the following result:

$$\langle S(\lambda_1)S(\lambda_2)\rangle = \frac{\Phi(\lambda_1)-\Phi(\lambda_1+\lambda_2)+\Phi(\lambda_2)}{\lambda_1\lambda_2}\frac{2}{[\Phi(\lambda_1+\lambda_2)]^3}$$
$$+\frac{\Phi(\lambda_2)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right]}{\lambda_1\lambda_2}\left\{\frac{1}{\Phi(\lambda_2)}\frac{2}{[\Phi(\lambda_1+\lambda_2)]^3}+\frac{1}{[\Phi(\lambda_2)\Phi(\lambda_1+\lambda_2)]^2}\right\}$$
$$+\frac{\Phi(\lambda_1)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_1)\right]}{\lambda_1\lambda_2}\left\{\frac{1}{\Phi(\lambda_1)}\frac{2}{[\Phi(\lambda_1+\lambda_2)]^3}+\frac{1}{[\Phi(\lambda_1)\Phi(\lambda_1+\lambda_2)]^2}\right\}, \tag{3.58}$$

which after simple algebra simplifies to the following formula:

$$\langle S(\lambda_1)S(\lambda_2)\rangle = \frac{1}{\lambda_1\lambda_2\Phi(\lambda_1+\lambda_2)}\left[\frac{1}{\Phi(\lambda_1)}+\frac{1}{\Phi(\lambda_2)}\right]. \tag{3.59}$$

In the CTRW case we recover: $\langle S(\lambda_1)S(\lambda_2)\rangle = \left[\lambda_1^{-1-\alpha}\,\lambda_2^{-1}+\lambda_2^{-1-\alpha}\,\lambda_1^{-1}\right]/(\lambda_1+\lambda_2)^{\alpha}$ [131].

Exact results in physical time are obtained by solving Eqs. (3.53, 3.54) in Laplace space and then by taking the Laplace inverse transform of the solutions found. However, these solutions often include terms containing the expression $1/\Phi(\lambda_1+\lambda_2)$, for instance in Eq. (3.59). This term can be manipulated by defining a two point operator $K\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)$, acting on a general two-point smooth function $g(t_1, t_2)$, whose Laplace transform reads:

$$\int_0^{+\infty} e^{-\lambda_1 t_1}\int_0^{+\infty} e^{-\lambda_2 t_2} K\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right) g(t_1,t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{g(\lambda_1,\lambda_2)}{\Phi\left(\lambda_1+\lambda_2\right)}, \tag{3.60}$$

which can be used to make the Laplace inverse transform of general expressions involving such factor, as it admits an exact integral expression in physical time. By recalling

Eq. (3.30), we can write the following:

$$\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2}\delta(t_2-t_1)K(t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2 = \int_0^{+\infty} e^{-(\lambda_1+\lambda_2)t}K(t)\,\mathrm{d}t$$
$$= \widetilde{K}(\lambda_1+\lambda_2) = \frac{1}{\Phi(\lambda_1+\lambda_2)}. \qquad (3.61)$$

Combining this result with the convolution theorem, we can define the two-point operator K as a two-fold convolution, i.e., $K\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)g(t_1,t_2) = K(t_1)\delta(t_2-t_1) **g(t_1,t_2)$. Introducing the notation $t^* = \min(t_1,t_2)$, the operator has the following integral form:

$$K\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)g(t_1,t_2) = \int_0^{t^*} K(s)\,g(t_1-s,t_2-s)\,\mathrm{d}s \qquad (3.62)$$

3.4.2 Two-point Correlation Function for Stationary Parent Processes

In this section, we compute the two-point correlation function of Y, when we consider a process X having correlation function $\langle X(s_1)X(s_2)\rangle = H(s_2-s_1)$ in the stationary regime, where H is a general smooth decreasing function. Thus, we will compute explicitly Eq. (3.53) with the further assumption $t_2 \geq t_1$. Recalling that the process S is increasing, but not strictly, such assumption induces the relation: $S(t_2) \geq S(t_1)$, or equivalently $s_2 \geq s_1$, i.e., we need to keep only the first and second term in Eq. (3.52). This condition also holds in the case $t_2 > t_1$, due to the presence of the trapping events in the trajectories of the anomalous diffusing particle. In this case indeed $S(t_2) = S(t_1)$. For later convenience, we treat the contributions from these two terms separately. First, the contribution of the term proportional to $\Theta(s_2-s_1)$ can be written in Laplace transform as follows:

$$\begin{aligned}
\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle &= \frac{\Phi(\lambda_2)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right]}{\lambda_1\,\lambda_2}\times \\
&\quad\times \int_0^{+\infty}\int_0^{+\infty} e^{-s_1\,\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\,\Phi(\lambda_2)}\Theta(s_2-s_1)\,H(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\
&= \frac{\Phi(\lambda_2)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right]}{\lambda_1\,\lambda_2}\int_0^{+\infty}\int_0^{+\infty} e^{-s_1'\,\Phi(\lambda_1+\lambda_2)}e^{-s_2'\,\Phi(\lambda_2)}H(s_2')\,\mathrm{d}s_1'\mathrm{d}s_2' \\
&= \frac{\Phi(\lambda_2)\left[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)\right]}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)}\,\widetilde{H}(\Phi(\lambda_2))\,,
\end{aligned} \qquad (3.63)$$

where we solved the corresponding integrals with the same coordinate change as in Eq. (3.57). If we further separate the terms in its rhs, we obtain:

$$\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle = \frac{\Phi(\lambda_2)}{\lambda_1\,\lambda_2}\widetilde{H}(\Phi(\lambda_2)) - \frac{\left[\Phi(\lambda_2)\right]^2}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)}\widetilde{H}(\Phi(\lambda_2))\,. \qquad (3.64)$$

If we now define auxiliary functions $f_{1,2}(t)$ by specifying their Laplace transform as below:

$$\widetilde{f}_1(\lambda) = \frac{\Phi(\lambda)}{\lambda}\widetilde{H}(\Phi(\lambda)) \qquad\qquad \widetilde{f}_2(\lambda) = -\frac{\left[\Phi(\lambda)\right]^2}{\lambda}\widetilde{H}(\Phi(\lambda)) \qquad (3.65)$$

the inverse Laplace transform of Eq. (3.64) can be written straightforwardly as below:

$$\langle Y(t_1)Y(t_2)\rangle = f_1(t_2) + \int_0^{t_1} K(s)f_2(t_2-s)\,\mathrm{d}s \qquad (3.66)$$

The specific form of the functions $f_{1,2}$ can be obtained once both the process X and the Laplace exponent of the subordinator Φ are specified. As a sanity check, we note that for $\Phi(\lambda) = \lambda$ (Brownian limit), we find: $\widetilde{f}_1(\lambda) = \widetilde{H}(\lambda)$ and $\widetilde{f}_2(\lambda) = -\lambda\,\widetilde{H}(\lambda)$, i.e., $f_1(t) = H(t)$ and $f_2(t) = \partial_t H(t)$, such that we have $\langle Y(t_1)\,Y(t_2)\rangle = H(t_2) + \int_0^{t_1} \frac{\partial}{\partial s} H(t_2 - s)\,\mathrm{d}s = H(t_2) + H(t_2 - t_1) - H(t_2) = H(t_2 - t_1)$, which recovers the assumption on the X-correlation. We here neglected the value at zero of H in the Laplace inverse transform of f_2, as it does not contribute to the correlation function. This point can be further shown by substituting $\widetilde{f}_{1,2}$ directly in Eq. (3.64) and make the Laplace inverse transform of the resulting expression. Secondly, the contribution of the term proportional to $\delta(s_2 - s_1)$ is given in Laplace transform as follows:

$$\begin{aligned}\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle &= \frac{\Phi(\lambda_1) - \Phi(\lambda_1+\lambda_2) + \Phi(\lambda_2)}{\lambda_1\,\lambda_2}\times \\ &\times \int_0^{+\infty}\int_0^{+\infty} e^{-s_1\,\Phi(\lambda_1+\lambda_2)}\delta(s_2-s_1)\,H(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= H_0\left[-\frac{1}{\lambda_1\,\lambda_2} + \frac{\Phi(\lambda_1)}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)} + \frac{\Phi(\lambda_2)}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)}\right], \end{aligned} \tag{3.67}$$

with the definition $H_0 = H(0)$. This result can be rewritten in the more compact form: $\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle = H_0\,\widetilde{C}(\lambda_1,\lambda_2)$ with the auxiliary function C defined as below:

$$\widetilde{C}(\lambda_1,\lambda_2) = \left[-\frac{1}{\lambda_1\,\lambda_2} + \frac{\Phi(\lambda_1)}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)} + \frac{\Phi(\lambda_2)}{\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)}\right]. \tag{3.68}$$

As a sanity check, we note that in the Brownian limit ($\Phi(\lambda) = \lambda$) Eq. (3.68) becomes: $\widetilde{C}(\lambda_1,\lambda_2) = -1/[\lambda_1\,\lambda_2] + [\lambda_1+\lambda_2]/[\lambda_1\,\lambda_2\,(\lambda_1+\lambda_2)] = 0$, i.e., the contribution from the trapping events is null, as required. Interestingly, the Laplace inverse transform of C can be done analytically, by introducing a third auxiliary function:

$$\widetilde{w}(\lambda) = \frac{\Phi(\lambda)}{\lambda}. \tag{3.69}$$

By recalling Eqs. (3.61, 3.62), we easily obtain [$t^* = \min(t_1,t_2)$]:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{\Phi(\lambda_1)}{\lambda_1\,\lambda_2}\frac{1}{\Phi(\lambda_1+\lambda_2)}\right\}(t_1,t_2) &= \int_0^{t^*} K(s)\,w(t_1-s)\,\mathrm{d}s \\ &= \Theta(t_1-t_2)\int_0^{t_2} K(s)\,w(t_1-s)\,\mathrm{d}s + \Theta(t_2-t_1)\int_0^{t_1} K(s)\,w(t_1-s)\,\mathrm{d}s \\ &= \Theta(t_1-t_2)\int_0^{t_2} K(s)\,w(t_1-s)\,\mathrm{d}s + \Theta(t_2-t_1), \end{aligned} \tag{3.70}$$

where the second integral is solved exactly by using the convolution theorem and Eq. (3.61). Clearly, the third term in Eq. (3.68) is obtained from the previous result by simply exchanging the indexes. With our assumption on the ordering of the times, we obtain:

$$C(t_1,t_2) = \int_0^{t_1} K(s)\,w(t_2-s)\,\mathrm{d}s \tag{3.71}$$

Thus, putting together Eqs. (3.66, 3.71), we obtain the final result:

$$\langle Y(t_1)Y(t_2)\rangle = f_1(t_2) + \int_0^{t_1} K(s) f_2(t_2-s)\,\mathrm{d}s + H_0 \int_0^{t_1} K(s)\, w(t_2-s)\,\mathrm{d}s \tag{3.72}$$

The result Eq. (3.72) indicates that the two-point correlation function of the anomalous process Y can be expressed in terms of an integral of *single-time* functions, thus highlighting a simple underlying structure.

As a second check, we consider the case of CTRW dynamics and show the consistency of our results with those presented in [86]. We first look at Eq. (3.66). In this case, we have: $\widetilde{f}_1(\lambda) = \lambda^{\alpha-1}\,\widetilde{H}(\lambda^\alpha)$ and $\widetilde{f}_2(\lambda) = -\lambda^{2\,\alpha-1}\,\widetilde{H}(\lambda^\alpha)$. We further consider the case of X being an Ornstein-Uhlenbeck (OU) process with friction coefficient γ and noise intensity D. Thus, $H(s) = \frac{D}{2\gamma} e^{-\gamma\, s}$, i.e., in Laplace space $\widetilde{H}(\lambda) = \frac{D}{2\gamma}\frac{1}{\lambda+\gamma}$, in the stationary regime, corresponding to the limit $s_2 > s_1 \to \infty$ with $|s_2 - s_1|$ finite [83]. Thus, we can compute: $f_1(\lambda) = \frac{D}{2\gamma}\frac{1}{\lambda+\gamma\,\lambda^{1-\alpha}}$, i.e., $f_1(t) = \frac{D}{2\gamma} E_\alpha(-\gamma\, t^\alpha)$, and $f_2(\lambda) = -\frac{D}{2\gamma}\frac{\lambda^{2\,\alpha-1}}{\gamma+\lambda^\alpha}$, whose Laplace inverse transform is obtained by using Eq. (A.10): $f_2(t) = -\frac{D}{2\gamma}\, t^{-\alpha} E_{\alpha,1-\alpha}(-\gamma\, t^\alpha)$. Thus, Eq. (3.72) becomes:

$$\begin{aligned}
\langle Y(t_1)Y(t_2)\rangle &= \frac{D}{2\gamma} E_\alpha(-\gamma\, t_2^\alpha) - \frac{D}{2\gamma}\frac{1}{\Gamma(\alpha)} \int_0^{t_1} s^{\alpha-1}\,(t_2-s)^{-\alpha}\, E_{\alpha,1-\alpha}(-\gamma\,(t_2-s)^\alpha)\,\mathrm{d}s \\
&= \frac{D}{2\gamma} E_\alpha(-\gamma\, t_2^\alpha) - \frac{D}{2\gamma}\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{+\infty} \frac{(-\gamma)^n}{\Gamma(1-\alpha+\alpha\, n)} \int_0^{t_1} s^{\alpha-1}\,(t_2-s)^{-\alpha+\alpha\, n}\,\mathrm{d}s \\
&= \frac{D}{2\gamma} E_\alpha(-\gamma\, t_2^\alpha) - \frac{D}{2\gamma}\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{+\infty} \frac{(-1)^n(\gamma\, t_2^\alpha)^n}{\Gamma(1-\alpha+\alpha\, n)} B\left(\frac{t_1}{t_2}, \alpha, 1-\alpha+\alpha\, n\right)
\end{aligned} \tag{3.73}$$

where we employed the series expansion of the ML function Eq. (A.9) and we introduce the incomplete beta function $B(z,a,b) = \int_0^z y^{a-1}(1-y)^{b-1}\,\mathrm{d}y$. If we now consider the limit $t_2 > t_1 \to \infty$ (with $\Delta t = t_2 - t_1$ finite), the incomplete beta functions have the scaling: $B\left(t_1/t_2, \alpha, 1-\alpha+\alpha\, n\right) \sim B\left(1, \alpha, 1-\alpha+\alpha\, n\right) = \frac{\Gamma(\alpha)\,\Gamma(1-\alpha+\alpha\, n)}{\Gamma(1+\alpha\, n)}$, such that the series in the rhs of Eq. (3.73) has the following asymptotic scaling:

$$\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{+\infty} \frac{(-1)^n(\gamma\, t_2^\alpha)^n}{\Gamma(1-\alpha+\alpha\, n)} B\left(\frac{t_1}{t_2}, \alpha, 1-\alpha+\alpha\, n\right) \sim \sum_{n=0}^{+\infty} \frac{(-1)^n(\gamma\, t_2^\alpha)^n}{\Gamma(1+\alpha\, n)} = E_\alpha(-\gamma\, t_2^\alpha). \tag{3.74}$$

Thus, the two terms in Eq. (3.73) cancel out in this limit and this first contribution converges to zero, consistently with [86] (in our case indeed the first moment with respect to the Boltzmann distribution is null, i.e., $\langle x\rangle_B = 0$). Regarding the second contribution, we obtain: $C(t_1, t_2) = \frac{B(t_1/t_2, \alpha, 1-\alpha)}{\Gamma(\alpha)\,\Gamma(1-\alpha)}$ and $H_0 = \frac{D}{2\gamma}$, i.e., we recover the scaling of [86]. This result suggests that the universal scaling behaviour of the two point correlation function of Y, found in Ref. [86] for the specific CTRW case, may hold for general waiting time distributions with the scaling function w. Further checks will be performed in future work.

The result presented in Eq. (3.72) is relevant as it describes the general behaviour of the correlation function of anomalous processes, obtained by subordination of a process X admitting a stationary regime. Indeed, transient terms eventually appearing in the correlation function of X can be shown to disappear for $t_2 > t_1 \to \infty$ using Tauberian theorems. We motivate this claim by elucidating the case already discussed before of X

being an OU process. In this specific case, the transient component of the correlation function has the temporal part described by the function $H(s_1+s_2) = H(s_1)H(s_2)$, with $H(s) = e^{-\gamma s}$. As before, we can use Eqs. (3.48, 3.52) to write (in Laplace space):

$$\begin{aligned}
\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle &= \frac{\Phi(\lambda_2)\,[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)]}{\lambda_1\,\lambda_2}\times \\
&\times \int_0^{+\infty}\int_0^{+\infty} e^{-s_1\,\Phi(\lambda_1+\lambda_2)}e^{-(s_2-s_1)\,\Phi(\lambda_2)}\Theta(s_2-s_1)\,H(s_2)H(s_1)\,\mathrm{d}s_1\;\mathrm{d}s_2 \\
&= \frac{\Phi(\lambda_2)\,[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)]}{\lambda_1\,\lambda_2}\int_0^{+\infty} e^{-s_1'\,\Phi(\lambda_1+\lambda_2)}H(2\,s_1')\mathrm{d}s_1'\int_0^{+\infty} e^{-s_2'\,\Phi(\lambda_2)}H(s_2')\mathrm{d}s_2' \\
&= \frac{\Phi(\lambda_2)\,[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)]}{2\,\lambda_1\,\lambda_2\,\Phi(\lambda_1+\lambda_2)}\,\widetilde{H}(\Phi(\lambda_2))\;\widetilde{H}\left(\frac{1}{2}\Phi(\lambda_1+\lambda_2)\right). \qquad (3.75)
\end{aligned}$$

In the specific OU case, $\widetilde{H}(\lambda) = 1/(\lambda+\gamma)$, i.e., we obtain:

$$\begin{aligned}
\left\langle \widetilde{Y}(\lambda_1)\widetilde{Y}(\lambda_2)\right\rangle &= \frac{\Phi(\lambda_2)[\Phi(\lambda_1+\lambda_2)-\Phi(\lambda_2)]}{\lambda_1\,\lambda_2}\frac{1}{\gamma+\Phi(\lambda_2)}\frac{1}{2\,\gamma+\Phi(\lambda_1+\lambda_2)} \\
&= \frac{\Phi(\lambda_2)}{\lambda_1\,\lambda_2}\left[\frac{1}{\gamma+\Phi(\lambda_2)}-\frac{1}{2\,\gamma+\Phi(\lambda_1+\lambda_2)}-\frac{\gamma}{[2\,\gamma+\Phi(\lambda_1+\lambda_2)][\gamma+\Phi(\lambda_2)]}\right]. \qquad (3.76)
\end{aligned}$$

We note that Eq. (3.76) does not have an explicit integral expression in time, as no suitable technique to make the Laplace inverse transform of the term $1/[\gamma+\Phi(\lambda_1+\lambda_2)]$ is available. Indeed, Eq. (3.61) is not useful in this case. However, for the purpose of our discussion, we can employ Tauberian theorems to show that Eq. (3.76) do not contribute to the correlation function in the limit $t_2 > t_1 \to \infty$. Indeed, recalling that $\Phi(0) = 0$, the leading order of the denominators in Eq. (3.76) are the following: $\gamma + \Phi(\lambda) \approx \gamma$, $2\,\gamma + \Phi(\lambda) \approx 2\,\gamma$ and $[2\,\gamma+\Phi(\lambda)][\gamma+\Phi(\lambda)] \approx 2\,\gamma^2$. Thus, in this scaling regime the term in square brackets in Eq. (3.76) is null and the transient term cancels out. A similar calculation shows that also the transient contribution from the flat intervals converges to zero in this scaling regime.

3.5 A Toolbox for Data Analysis

Earlier in this Chapter, we constructed a complete framework for the analysis of anomalous stochastic processes and of their general functionals, comprising (i) a Langevin description of their stochastic coarse-grained dynamics in terms of the subordinated Eqs. (3.4a, 3.4b), (ii) fractional evolution equations for their PDFs, in particular the GFFK Eq. (3.29) and GFFP Eq. (3.31), and (iii) the characterisation of their multipoint functions, among which correlation functions are the most relevant for application to experimental datasets. Thanks to the flexibility in the choice of both the waiting time distribution, i.e., of the function Φ in Eq. (3.5), and of the auxiliary process X, this formalism can generate several different dynamical processes, yet mainly unexplored. Consequently, our general results constitute an essential tool-kit of methods to interpret anomalous diffusive dynamics observed in experimental datasets, for instance of molecules' transport in biological systems.

In this section, we will look at specific examples to elucidate the potentialities of this formalism. Specifically, we will study two examples of dynamics for the process X in Eq. (3.4a): (i) the free diffusive case, i.e., $F(x) = 0$, $\sigma(x) = \sqrt{2\,\sigma}$ and (ii) the harmonic oscillator case, i.e., $F(x) = -\gamma x$, $\sigma(x) = \sqrt{2\,\sigma}$. Here, γ and σ are positive real constants.

Correspondingly, we will investigate its time averaged linear functional: $\overline{W}(t) = W(t)/t$, with $W(t)$ as in Eq. (3.1) with $U(x) = x$. In the case (i) we will study the MSD of both X and $\overline{W}(t)$; in the case (ii), we will study both their MSD and their two-point correlation function. In both cases, we will look at different examples of Φ.

General MSD

Let us first derive general equations for the MSD of X and W in both cases (i-ii). First, we start from the GFFP Eq. (3.31) in Laplace space ($P(y,0)$ is the PDF of the initial position):

$$\lambda\widetilde{P}(y,\lambda) - P(y,0) = \frac{\lambda}{\Phi(\lambda)}\left[-\frac{\partial}{\partial y}F(y) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(y)\right]\widetilde{P}(y,\lambda). \tag{3.77}$$

This equation can be employed to derive the moments of $Y(t)$ for general Φ. For instance, the first moment, which is given in Laplace space $\left\langle\widetilde{Y}(\lambda)\right\rangle = \int_{-\infty}^{+\infty} y\widetilde{P}(y,\lambda)\,\mathrm{d}y$, can be computed by multiplying Eq. (3.77) by y on both sides and performing the integral. This leads to the following result:

$$\lambda\left\langle\widetilde{Y}(\lambda)\right\rangle - \frac{\lambda}{\Phi(\lambda)}\left\langle\widetilde{F}(Y(\lambda))\right\rangle = y_0. \tag{3.78}$$

where we interpret the force dependent term as: $\left\langle\widetilde{F}(Y(\lambda))\right\rangle = \int_{-\infty}^{+\infty} F(y)\widetilde{P}(y,\lambda)\,\mathrm{d}y$. Similarly, recalling that $\left\langle\widetilde{Y}^2(\lambda)\right\rangle = \int_{-\infty}^{\infty} y^2\widetilde{P}(y,\lambda)\,\mathrm{d}y$, we obtain for the second moment:

$$\lambda\left\langle\widetilde{Y}^2(\lambda)\right\rangle - \frac{2\lambda}{\Phi(\lambda)}\left\langle\widetilde{Y}(\lambda)\widetilde{F}(Y(\lambda))\right\rangle - \frac{\lambda}{\Phi(\lambda)}\left\langle\widetilde{\sigma}^2(Y(\lambda))\right\rangle = y_0^2 \tag{3.79}$$

with the terms depending on F and σ being interpreted as in the previous case, i.e., $\left\langle\widetilde{Y}(\lambda)\widetilde{F}(Y(\lambda))\right\rangle = \int_{-\infty}^{+\infty} y\,F(y)\widetilde{P}(y,\lambda)\,\mathrm{d}y$ and $\left\langle\widetilde{\sigma}^2(Y(\lambda))\right\rangle = \int_{-\infty}^{+\infty}\sigma^2(y)\widetilde{P}(y,\lambda)\,\mathrm{d}y$. Similar formulas can be obtained for higher order moments. In the specific case (i), we find:

$$\left\langle\widetilde{Y}(\lambda)\right\rangle = \frac{y_0}{\lambda} \tag{3.80a}$$

$$\left\langle\widetilde{Y}^2(\lambda)\right\rangle = \frac{y_0^2}{\lambda} + \frac{2\,\sigma}{\lambda\,\Phi(\lambda)} \tag{3.80b}$$

or equivalently in time space, by recalling Eq. (3.30), $\langle Y(t)\rangle = y_0$ and $\left\langle Y^2(t)\right\rangle = y_0^2 + 2\,\sigma\int_0^t K(\tau)\,\mathrm{d}\tau$. These results indeed lead to the following MSD:

$$\widetilde{MSD}_Y(\lambda) = \frac{2\,\sigma}{\lambda\,\Phi(\lambda)} \qquad\qquad MSD_Y(t) = 2\,\sigma\int_0^t K(\tau)\,\mathrm{d}\tau\,. \tag{3.81}$$

As a sanity check, we note that for $K(t) = t^{\alpha-1}/\Gamma(\alpha)$ (CTRW case), we obtain $MSD(t) = 2\,\sigma\,t^{\alpha}/\Gamma(1+\alpha)$ [9]. In the case (ii), we obtain instead the following formulas:

$$\left\langle\widetilde{Y}(\lambda)\right\rangle = \frac{y_0}{\lambda}\frac{\Phi(\lambda)}{\Phi(\lambda)+\gamma} \tag{3.82a}$$

$$\left\langle\widetilde{Y}^2(\lambda)\right\rangle = \left[y_0^2 + \frac{2\,\sigma}{\Phi(\lambda)}\right]\frac{\Phi(\lambda)}{\lambda\,[\Phi(\lambda)+2\gamma]} \tag{3.82b}$$

An exact inverse Laplace transform can no longer be found, unless we specify $\Phi(s)$. The corresponding MSD is given in Laplace space by:

$$\widetilde{MSD}_Y(\lambda) = \frac{1}{2\gamma + \Phi(\lambda)} \left\{ \frac{2\,\sigma}{\lambda} + \frac{2\gamma^2 y_0^2}{\lambda\,[\gamma + \Phi(\lambda)]} \right\}. \tag{3.83}$$

Let us find now analogous results for $W(t)$. Form the definition Eq. (3.1), we trivially obtain that the first moment is given in Laplace space by: $\left\langle \widetilde{W}(\lambda) \right\rangle = \left\langle \widetilde{U}(Y(\lambda)) \right\rangle / \lambda$, with the definition: $\left\langle \widetilde{U}(Y(\lambda)) \right\rangle = \int_{-\infty}^{+\infty} U(y)\, \widetilde{P}(y, \lambda)\, \mathrm{d}y$. Further recalling that $\left\langle \widetilde{W^2}(\lambda) \right\rangle = - \frac{\partial^2}{\partial p^2} \int_{-\infty}^{+\infty} \widehat{\widetilde{P}}(p, y, \lambda)\, \mathrm{d}y \Big|_{p=0}$, one can derive the relation: $\left\langle \widetilde{W^2}(\lambda) \right\rangle = \frac{2}{\lambda} \left\langle \widetilde{W}(\lambda) \widetilde{Y}(\lambda) \right\rangle$, by taking the second order partial derivative of the GFFK formula Eq. (3.29) in Laplace-Fourier transform. We need to compute: $\left\langle \widetilde{W}(\lambda) \widetilde{Y}(\lambda) \right\rangle = -i \frac{\partial}{\partial p} \int_{-\infty}^{+\infty} y\, \widehat{\widetilde{P}}(p, y, \lambda)\, \mathrm{d}y \Big|_{p=0}$. This is done similarly to the case of Y, by multiplying both sides of the Laplace-Fourier transform of Eq. (3.29) by y and then first by integrating over such variable and secondly by taking the derivative in p. After evaluating this expression for $p = 0$, we obtain:

$$\begin{aligned} \lambda \int_{-\infty}^{+\infty} y\, \widehat{\widetilde{P}}(p, y, \lambda)\, \mathrm{d}y\, - Y_0 = ip \int_{-\infty}^{+\infty} y\, U(y)\, \widehat{\widetilde{P}}(p, y, \lambda)\, \mathrm{d}y \\ + \int_{-\infty}^{+\infty} y\, \mathcal{L}_{FP}(y)\, \frac{(\lambda - ipy) \widehat{\widetilde{P}}(p, y, \lambda)}{\Phi(\lambda - ipy)}\, \mathrm{d}y\,, \end{aligned} \tag{3.84}$$

where both the FP operator and U needs to be specified to obtain closed analytical results. In the case (i) and for a linear functional, i.e., $U(y) = y$, $\left\langle \widetilde{W}(\lambda) \widetilde{Y}(\lambda) \right\rangle = \left\langle \widetilde{Y}^2(\lambda) \right\rangle$, thus leading, by recalling Eq. (3.80b), to the following result:

$$\left\langle \widetilde{W}^2(\lambda) \right\rangle = \frac{2}{\lambda^2} \left\langle \widetilde{Y}(\lambda) \right\rangle = \frac{2\, y_0^2}{\lambda^3} + \frac{4\,\sigma^2}{\lambda^3\, \Phi(\lambda)}. \tag{3.85}$$

After Laplace inverse transform, we obtain:

$$\left\langle W^2(t) \right\rangle = y_0^2 t^2 + 2\,\sigma^2 \int_0^t (t - \tau)^2 K(\tau)\, \mathrm{d}\tau\,. \tag{3.86}$$

In the case (ii) instead we obtain the following result:

$$\left\langle W^2(\lambda) \right\rangle = \left[\frac{2}{\lambda^2} - \frac{2\gamma \Phi'(\lambda)}{\lambda \Phi(\lambda)[\gamma + \Phi(\lambda)]} \right] \left\langle Y^2(\lambda) \right\rangle, \tag{3.87}$$

with $\left\langle \widetilde{Y}^2(\lambda) \right\rangle$ specfied by Eq. (3.82b). As for the second moment of X, we cannot make its Laplace inverse transform, unless we specify the function Φ.

Two-Point Correlation Function of the Anomalous Harmonic Oscillator

We here consider the two point correlation functions of Y and W in the case (ii). We further assume $t_2 \geq t_1$. Recalling that correspondingly we have $S(t_2) \geq S(t_1)$, due to the monotonicity of S, we have also $s_2 \geq s_1$. The correlation function of $X(s)$ is given below:

$$\langle X(s_2) X(s_1) \rangle = \left(y_0^2 - \frac{\sigma}{\gamma} \right) e^{-\gamma(s_1 + s_2)} + \frac{\sigma}{\gamma}\, e^{-\gamma(s_2 - s_1)}. \tag{3.88}$$

We have two terms: (i) a transient contribution with time dependence $e^{-\gamma(s_1+s_2)}$ and (ii) the contribution surviving at equilibrium with time dependence $e^{-\gamma(s_2-s_1)}$. The transient term has been shown to converge asymptotically to zero for long times in Sec. 3.4.2. Thus, we will neglect it, by assuming $x_0^2 = \sigma/\gamma$. Thus, for $t_2 > t_1$, the correlation function of Y is given by Eq. (3.72) with w as in Eq. (3.69) and the following functions f_1, f_2:

$$\widetilde{f}_1(\lambda) = \frac{\sigma}{\gamma}\frac{\Phi(\lambda)}{\lambda[\gamma+\Phi(\lambda)]} \qquad \widetilde{f}_2(\lambda) = -\frac{\sigma}{\gamma}\frac{[\Phi(\lambda)]^2}{\lambda[\gamma+\Phi(\lambda)]} \tag{3.89}$$

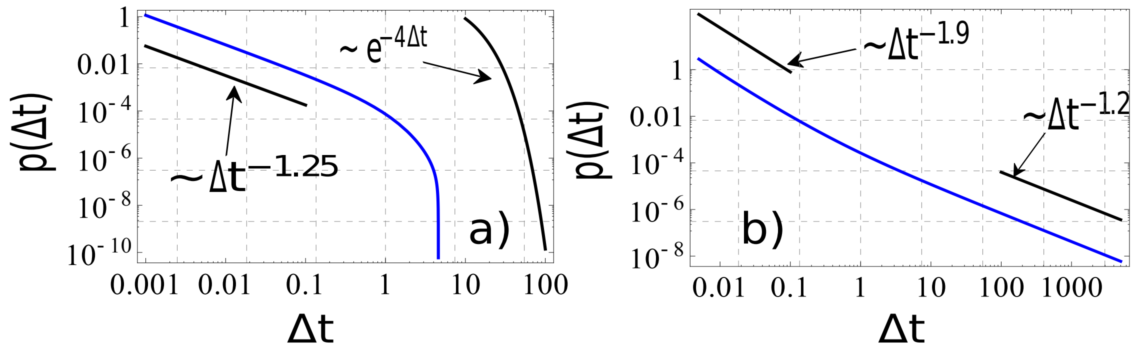

Figure 3.2: Probability of waiting time increments over an arc-length step $\Delta s = 0.001$ for different subordinators (blue lines). This is obtained by numerical inverse Laplace transform of their characteristic function $e^{-\Delta s\,\Phi(\lambda)}$. (a) Tempered Lévy stable subordinator of order parameter $\alpha = 0.25$ and tempering parameter $\mu = 1$. This subordinator interpolates between power-law distributed waiting times ($p(\Delta t) \sim \Delta t^{-\alpha-1}$) and exponentially distributed waiting times ($p(\Delta t) \sim e^{-[1/(\alpha\,\mu^{\alpha-1})]\,\Delta t}$) respectively for small/large increments, thus generating an hybrid dynamics between a CTRW and normal diffusion. (b) Mixture of two independent Lévy stable subordinators of parameters $\alpha_1 = 0.2$, $\alpha_2 = 0.9$, $B_1 = B_2 = 1$. This subordinator interpolates between waiting times distributed as power-laws with different exponents ($p(\Delta t) \sim \Delta t^{-\alpha_2-1}$ or $p(\Delta t) \sim \Delta t^{-\alpha_1-1}$ respectively for small/large Δt). Thus, the dynamics generated is an hybrid between different types of subdiffusive CTRWs.

3.5.1 The Tempered Stable Subordinator

As a first example, we consider η to be a tempered Lévy stable noise with tempering index μ and stability index $0 < \alpha \leq 1$, whose Lévy exponent is given by $\Phi(\lambda) = (\mu+\lambda)^\alpha - \mu^\alpha$, as we discussed in Sec. 2.2.3. Thus, we can compute the probability of waiting time increments $p(\Delta t)$ over a finite arc-length step Δs as the numerical inverse Laplace transform of the their characteristic function $e^{-\Delta s\,\Phi(\lambda)}$ (shown in Fig. 3.2a). This subordinator interpolates between exponentially distributed ($\mu \to \infty$) and power-law distributed ($\mu = 0$) waiting times, i.e., $p(\Delta t) \sim \Delta t^{-1-\alpha}$ or $p(\Delta t) \sim e^{-[1/(\alpha\,\mu^{\alpha-1})]\Delta t}$ respectively for small or large increments. Thus, we will expect it to generate an hybrid dynamics between a CTRW and normal diffusive behaviour [160, 161, 162, 159, 163]. According to Eq. (3.62), the corresponding memory kernel in the GFK Eq. (3.29) has Laplace transform: $\widetilde{K}(\lambda) = 1/[(\mu+\lambda)^\alpha - \mu^\alpha]$. By using Eq. (A.10) and $\mathcal{L}\{e^{-\mu\,t}f(t)\}(\lambda) = \widetilde{f}(\lambda+\mu)$, we derive its expression in time [164]:

$$K(t) = e^{-\mu t}t^{\alpha-1}E_{\alpha,\alpha}((\mu t)^\alpha)\,. \tag{3.90}$$

Substituting these formulas into Eqs. (3.81, 3.85), we can derive the MSD of Y and $\overline{W}$ for a free diffusive X. First, we study their asymptotic behaviour for short(long) times by taking the limit $\lambda \to \infty (\lambda \to 0)$ (Tauberian theorems). In the case of Y, we find:

$$\begin{aligned} \Phi(\lambda) \underset{\lambda\to\infty}{\sim} \lambda^{\alpha} \quad &\longrightarrow \quad MSD_Y(\lambda) \underset{\lambda\to\infty}{\sim} \tfrac{2\,\sigma}{\lambda^{1+\alpha}}, \\ \Phi(\lambda) \underset{\lambda\to 0}{\sim} \tfrac{\alpha}{\mu^{1-\alpha}}\lambda \quad &\longrightarrow \quad MSD_Y(\lambda) \underset{\lambda\to 0}{\sim} \tfrac{2\,\sigma}{\alpha\,\mu^{\alpha-1}}\tfrac{1}{\lambda^2}. \end{aligned} \tag{3.91}$$

Thus, we obtain the following scaling behaviour in time:

$$MSD_Y(t) \underset{t\to 0}{\sim} \frac{2\,\sigma}{\Gamma(1+\alpha)} t^{\alpha}, \qquad MSD_Y(t) \underset{t\to\infty}{\sim} \frac{2\,\sigma}{\alpha\,\mu^{\alpha-1}} t. \tag{3.92}$$

According to our prediction, the MSD of the anomalous process Y exhibits a crossover between subdiffusive behaviour for short times and normal one for long times. This crossover regime is verified by numerical simulations of Eqs. (3.4a, 3.4b) in Fig. 3.3a(main). In this case, by employing Eq. (3.81), we can obtain the following exact formula:

$$\begin{aligned} MSD_Y(t) &= 2\,\sigma \int_0^t e^{-\mu\,\tau}\,\tau^{\alpha-1}\,E_{\alpha,\alpha}((\mu\,\tau)^{\alpha})\,\mathrm{d}\tau \\ &= \sum_{n=0}^{\infty} \frac{2\,\sigma\,\mu^{\alpha\,n}}{\Gamma(\alpha+\alpha\,n)} \int_0^t e^{-\mu\,\tau}\,\tau^{\alpha(1+n)-1}\,\mathrm{d}\tau \;= \frac{2\,\sigma}{\mu^{\alpha}} \sum_{n=0}^{\infty} \frac{\gamma(\alpha(1+n),\mu\,t)}{\Gamma(\alpha+\alpha\,n)}, \end{aligned} \tag{3.93}$$

where we used the series expansion of the ML function Eq. (A.9) and we introduced an incomplete gamma function. As a sanity check, we note that for $\mu = 0$, only the first term in the series expansion survives, i.e., we obtain the MSD of an ordinary CTRW $MSD_Y(t) = 2\,\sigma\,t^{\alpha}/\Gamma(1+\alpha)$. In addition, for $\alpha = 1$, we recover the normal MSD of a free diffusion brownian motion. Similar calculations can be done in the case of W by using Eq. (3.85). In this case, we obtain the scaling behaviour:

$$MSD_W(t) \underset{t\to 0}{\sim} \frac{4\,\sigma^2}{\Gamma(3+\alpha)} t^{3+\alpha}, \qquad MSD_W(t) \underset{t\to\infty}{\sim} \frac{4\,\sigma^2}{3\,\alpha\,\mu^{\alpha-1}} t^3, \tag{3.94}$$

which shows that the time averaged linear functional $\overline{W}$ exhibits the same crossover of Y, as we also verify in numerical simulations [Fig. 3.3a(inset)]. Below, we calculate the exact MSD:

$$\begin{aligned} MSD_W(t) &= 2\,\sigma^2 \int_0^t e^{-\mu\,\tau}\,\tau^{\alpha-1}\,E_{\alpha,\alpha}((\mu\,\tau)^{\alpha})\,(t-\tau)^2\,\mathrm{d}\tau \\ &= 2\,\sigma^2 \sum_{n=0}^{\infty} \frac{\mu^{\alpha\,n}}{\Gamma(\alpha+\alpha\,n)} \int_0^t e^{-\mu\,\tau}\,\tau^{\alpha(1+n)-1}(t-\tau)^2\,\mathrm{d}\tau \\ &= 2\,\sigma^2\,t^{2+\alpha} \sum_{n=0}^{\infty} \frac{(\mu\,t)^{\alpha\,n}}{\Gamma(\alpha+\alpha\,n)} M(\alpha(1+n), 3+\alpha(1+n), -\mu\,t), \end{aligned} \tag{3.95}$$

where M is a confluent hypergeometric function (see Appendix A.1).

We next consider the case (ii) when $X(s)$ in Eq. (3.4a) is an OU process, such that $Y(t)$ intermediates between a fractional harmonic oscillator and a normal diffusive one. The MSD of the time-averaged W-process is given in closed form by Eq. (3.87), whose Laplace inverse transform can be performed numerically. Its plot as a function of time is presented in Fig. 3.3b. We find that the MSD exhibits an α-dependent plateau for $t \to \infty$ in the CTRW limit ($\mu = 0$) highlighting the ergodicity breaking of the process [146]. For

$\mu \neq 0$ instead the MSD shows the CTRW scaling for short times, but converges to zero for $t \to \infty$ as in the Brownian limit ($\mu \to \infty$, black solid line) confirming the ergodic nature of this anomalous process (Fig. 3.3b). Thus, the MSD of a general anomalous process needs to be observed for a sufficiently long time to properly assess ergodicity breaking.

We next consider the two-point correlation functions. To this aim, we need to compute the functions w, f_1 and f_2, whose Laplace transform is obtained by substituting Φ in Eqs. (3.69, 3.89). In this case, their Laplace inverse transform can be obtained exactly (see Appendix C), so that we obtain the following solutions ($\gamma \neq \mu^\alpha$):

$$w(t) = -\mu^\alpha + \frac{t^{-\alpha}\, e^{-\mu\, t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha, \mu\, t), \tag{3.96a}$$

$$f_1(t) = \frac{\sigma}{\gamma(\gamma - \mu^\alpha)}\left[-\mu^\alpha + \gamma\, g(\alpha, \gamma, \mu; t)\right], \tag{3.96b}$$

$$f_2(t) = -\frac{\sigma}{\gamma}\left[\frac{1}{\gamma - \mu^\alpha}\left(\mu^{2\alpha} - \gamma^2 g(\alpha, \gamma, \mu; t)\right) + \frac{t^{-\alpha}\, e^{-\mu t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha; \mu t)\right] \tag{3.96c}$$

with the function g specified as below:

$$g(\alpha, \gamma, \mu; t) = \sum_{n=0}^{\infty} \frac{(-1)^n((\gamma - \mu^\alpha)t^\alpha)^n}{\Gamma(1+\alpha n)} M(\alpha n, 1 + \alpha n, -\mu t). \tag{3.97}$$

The effect of $\mu \neq 0$ is evident (Fig. 3.3c,d), thus allowing to distinguish a CTRW from a process with waiting times distributed according to a tempered Lévy-stable law. We note that in the case $\gamma = \mu^\alpha$, the Laplace inverse transform of $f_{1,2}$ reduces to:

$$f_1(t) = \frac{\sigma}{\mu^\alpha}\left[1 - \frac{1}{\mu^\alpha\, \Gamma(\alpha)}\gamma(\alpha, \mu\, t)\right], \tag{3.98a}$$

$$f_2(t) = -\frac{\sigma}{\mu^\alpha}\left[-2\,\mu^\alpha + \frac{t^{-\alpha}\, e^{-\mu t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha; \mu t) + \frac{\mu^\alpha}{\Gamma(\alpha)}\,\gamma(\alpha, \mu\, t)\right] \tag{3.98b}$$

3.5.2 Weighted Sum of Lévy Stable Subordinators

As a second example, we assume η to be given by a mixture of two independent Lévy stable subordinators with different order parameters $0 < \alpha_1 < \alpha_2 \leq 1$. Thanks to their independence, the corresponding Laplace exponent of η is given by the sum of its two components: $\Phi(\lambda) = B_1\, \lambda^{\alpha_1} + B_2\, \lambda^{\alpha_2}$ with $B_1, B_2 \geq 0$. We plot the corresponding probability of waiting time increments over finite Δs in Fig. 3.2b. As one can infer from the plot, this subordinator interpolates between waiting times distributed according to power laws with different exponents. Specifically, the larger order parameter governs the scaling for small increments, i.e., $p(\Delta t) \sim \Delta t^{-1-\alpha_2}$; conversely, the smaller one determines the scaling exponent for large increments, i.e., $p(\Delta t) \sim \Delta t^{-1-\alpha_1}$. Thus, we will expect the generated dynamics to be an hybrid model between two different subdiffusive CTRWs. According to Eq. (3.62), the Laplace transform of its corresponding kernel in the GFFK Eq. (3.29) is given by $\widetilde{K}(\lambda) = 1/[B_1\, \lambda^{\alpha_1} + B_2\, \lambda^{\alpha_2}]$. Its Laplace inverse transform is obtained by

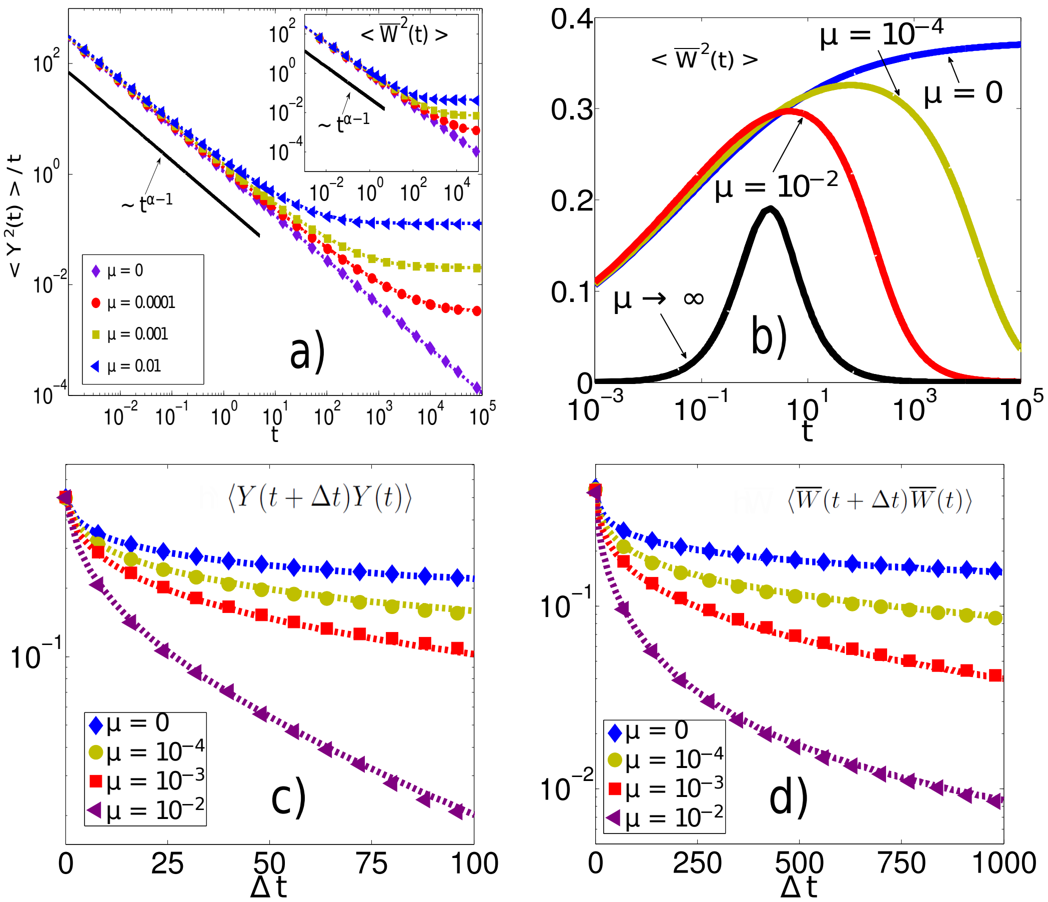

Figure 3.3: Here, $X(s)$ is (a) a free diffusion [$F(x) = 0$ and $\sigma(x) = 1$ in Eq. (3.4a)] or (b-d) an Ornstein-Uhlenbeck (OU) process [$F(x) = -x$ and $\sigma(x) = 1$ in Eq. (3.4a)], $\eta(s)$ is a tempered Lévy stable noise ($\alpha = 0.25$) and $U(x) = x$. (a) We plot the MSD of Y and $\overline{W} = W(t)/t$ (inset) for different values of the tempering parameter μ. The crossover from subdiffusive to normal scaling respectively for short and large times is evident. For $\mu = 0$, we recover a CTRW with power-law MSD for all times. (b) We plot the MSD of $\overline{W}$ for different values of μ and null initial condition ($y_0 = 0$). While for $\mu = 0$ (CTRW case), the MSD approaches an α-dependent plateau, signalling its ergodicity breaking, for $\mu \neq 0$ the MSD has its same scaling for short times but it converges to zero for long times, as in the Brownian limit (black solid line). This highlights that ergodicity is recovered by the tempered stable subordinated OU process. (c-d) Two-point correlation functions of Y and $\overline{W} = W(t)/t$ respectively for $y_0^2 = \sigma/\gamma$ and finite t.

employing the convolution theorem and recalling Eqs. (A.10, A.15). We obtain [75]:

$$K(t) = \frac{1}{B_2}\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\tau^{\alpha_2-\alpha_1-1}E_{\alpha_2-\alpha_1,\alpha_2-\alpha_1}\left(-\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\right)\mathrm{d}\tau$$
$$= \frac{t^{\alpha_2-1}}{B_2}E_{\alpha_2-\alpha_1,\alpha_2}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right), \tag{3.99}$$

where the integral of the three parameter Mittag-Leffler (ML) function is calculated in Eq. (A.15). Similarly to the case previously studied, we elucidate the MSD behaviour of both Y and $\overline{W}$. In the case of Y, by applying Tauberian theorems, we find the following scaling behaviour:

$$\begin{aligned}
\Phi(\lambda) &\underset{\lambda\to\infty}{\sim} B_2\,\lambda^{\alpha_2} &\longrightarrow\qquad MSD_Y(\lambda) &\underset{\lambda\to\infty}{\sim} \tfrac{2\,\sigma}{B_2}\tfrac{1}{\lambda^{1+\alpha_2}},\\
\Phi(\lambda) &\underset{\lambda\to 0}{\sim} B_1\,\lambda^{\alpha_1} &\longrightarrow\qquad MSD_Y(\lambda) &\underset{\lambda\to 0}{\sim} \tfrac{2\,\sigma}{B_1}\tfrac{1}{\lambda^{1+\alpha_1}},
\end{aligned} \tag{3.100}$$

or equivalently after inverse Laplace transform:

$$MSD_Y(t) \underset{t\to 0}{\sim} \frac{2\,\sigma}{B_2\,\Gamma(1+\alpha_2)}t^{\alpha_2}, \qquad MSD_Y(t) \underset{t\to\infty}{\sim} \frac{2\,\sigma}{B_1\,\Gamma(1+\alpha_1)}t^{\alpha_1}. \tag{3.101}$$

As predicted from the analysis of the waiting time increments statistics, this model interpolates between subdiffusive CTRWs with different characteristic exponents [Fig. 3.4a(main)]. The exact analytical formula can be obtained by simply integrating Eq. (3.99), which is realised by suitably adapting Eq. (A.15). We then find:

$$MSD_Y(t) = \frac{2\,\sigma}{B_2}\,t^{\alpha_2}\,E_{\alpha_2-\alpha_1,1+\alpha_2}\left(-\frac{B_2}{B_1}t^{\alpha_2-\alpha_1}\right). \tag{3.102}$$

In the case of W, by recalling Eq. (3.86), we obtain the following asymptotic behaviour:

$$MSD_W(t) \underset{t\to 0}{\sim} \frac{4\,\sigma^2}{B_2\Gamma(3+\alpha_2)}t^{2+\alpha_2}, \qquad MSD_W(t) \underset{t\to\infty}{\sim} \frac{4\,\sigma^2}{B_1\,\Gamma(3+\alpha_1)}t^{2+\alpha_1}. \tag{3.103}$$

Thus, as for the tempered stable subordinator, the time averaged linear functional $\overline{W}$ exhibits the same crossover of Y [Fig. 3.4a(inset)]. Below, we calculate the exact MSD:

$$MSD_W(t) = \frac{4\,\sigma^2}{B_2}\,t^{2+\alpha_2}\,E_{\alpha_2-\alpha_1,3+\alpha_2}\left(-\frac{B_2}{B_1}t^{\alpha_2-\alpha_1}\right). \tag{3.104}$$

We next consider the case (ii) of X in Eq. (3.4a) being an OU process. We plot the numerical Laplace inverse transform of Eq. (3.87) in Fig. 3.4b. The process approaches a plateau in the long-time limit with the same scaling of the CTRW case ($B_2 = 0$, black solid line), thus highlighting its weak ergodicity breaking [146]. This result, further supported by the opposite behaviour observed for the tempered Lévy stable subordinator, suggests that a major role in determining the weak ergodicity breaking of the process is played by the power-law scaling of the waiting time increments distribution. The corresponding scaling for small times is instead different from that of the CTRW.

We finally consider the two point correlation functions of both Y and $\overline{W}$. The inverse Laplace transform of Eqs. (3.69, 3.89) with this specific choice of Φ can be derived by applying the technique presented in Refs. [75] (see Appendix C for the detailed calculation). We find the following exact analytical results:

$$w(t) = \frac{B_1}{\Gamma(1-\alpha_1)}\,t^{-\alpha_1} + \frac{B_2}{\Gamma(1-\alpha_2)}\,t^{-\alpha_2}, \tag{3.105a}$$

$$f_1(t) = \frac{\sigma}{\gamma}\left[1 - \frac{\gamma}{B_2}\,t^{\alpha_2}\,H(t;\alpha_1,\alpha_2)\right], \tag{3.105b}$$

$$f_2(t) = \frac{\sigma}{\gamma}\left[-\gamma + \frac{B_1}{\Gamma(1-\alpha_1)}t^{-\alpha_1} + \frac{B_2}{\Gamma(1-\alpha_2)}t^{-\alpha_2} + \frac{\gamma^2}{B_2}\,t^{\alpha_2}\,H(t;\alpha_1,\alpha_2)\right], \tag{3.105c}$$

with H being the auxiliary function specified below:

$$H(t;\alpha_1,\alpha_2) = \sum_{n=0}^{\infty}(-1)^n \left(\frac{\gamma}{B_2}t^{\alpha_2}\right)^n E^{1+n}_{\alpha_2-\alpha_1,1+\alpha_2(1+n)}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right). \tag{3.106}$$

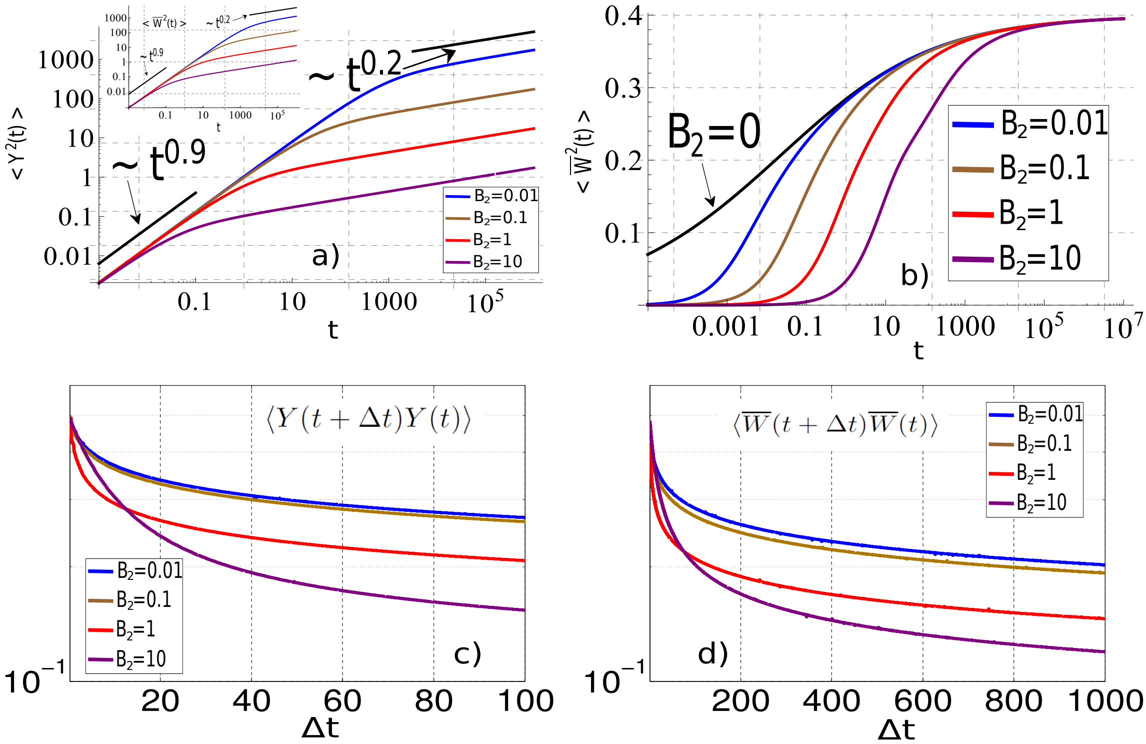

Figure 3.4: Here, $X(s)$ is (a) free diffusion [$F(x) = 0$ and $\sigma(x) = 1$ in Eq. (3.4a)] or (b) an Ornstein-Uhlenbeck process [$F(x) = -x$ and $\sigma(x) = 1$ in Eq. (3.4a)], η is the sum of two independent Lévy stable noises ($\alpha_1 = 0.2$, $\alpha_2 = 0.9$) with exponent specified in the main text ($B_1 = 1$) and $U(x) = x$. (a) MSD of Y (main) and $\overline{W} = W(t)/t$ (inset) for different values of the parameter B_2 (for initial position $y_0 = 0$). The crossover from subdiffusive dynamics with different scaling exponents ($\sim t^{\alpha_2}/t^{\alpha_1}$ respectively for short and long times) is evident. The same behaviour is displayed by the time-averaged linear functional $\overline{W}$ (inset). (b) MSD of $\overline{W}$ for different values of B_2 and null initial position ($Y_0 = 0$). The CTRW limiting case ($B_2 = 0$) is highlighted (black solid line). In all cases the MSD approaches the same plateau of the CTRW, signalling their ergodicity breaking [146]. The scaling behaviour is instead different for $B_2 \neq 0$ for short times. (c-d) Two-point correlation function of Y and $\overline{W}$ for $y_0^2 = \sigma/\gamma$ and finite t.

3.5.3 Curvature Modulation at the Crossover Region

In this section, we investigate a further choice of Φ, which will enable us to model MSD data exhibiting a crossover scaling between subdiffusive and normal diffusive regimes [124, 57, 126] in a more flexible way than with the tempered Lévy stable subordinator. In that case, indeed, the tuning of the parameter μ both changes the timescale with which the process approaches the normal diffusive regime, which becomes smaller as we increase its value, and the corresponding diffusion coefficient. This is proved by the $\mu-$dependence of the plateau for long times of the rescaled MSD in Fig. 3.3a. Here instead, our aim is to define a Φ, such that the curvature at the crossover between the two pure power laws, i.e., the timescale of the process approaching the normal diffusive regime, can be modulated without changing the final diffusion coefficient. Thus, we suggest the more flexible double power law form:

$$\Phi(\lambda) = d_1 \left(\frac{\lambda}{d_2}\right)^{\alpha_1} \left[1 + \left(\frac{\lambda}{d_2}\right)^{1/\beta}\right]^{(\alpha_2-\alpha_1)\beta} \tag{3.107}$$

where $0 < \alpha_1, \alpha_2 \leq 1$ and $d_1, d_2, \beta > 0$. Recalling Eq. (3.30), the corresponding memory kernel in the GFFK Eq. (3.29) is given in Laplace transform as follows:

$$\widetilde{K}(\lambda) = \frac{d_2^{\alpha_2}}{d_1} \frac{\lambda^{-\alpha_1}}{\left[d_2^{1/\beta} + \lambda^{1/\beta}\right]^{(\alpha_2-\alpha_1)\beta}}, \tag{3.108}$$

which leads via Eq. (A.10) to a three parameter ML function in time:

$$K(t) = \frac{d_2^{\alpha_2}}{d_1} t^{\alpha_2-1} E^{\beta(\alpha_2-\alpha_1)}_{1/\beta,\alpha_2}\left(-d_2^{1/\beta}\, t^{1/\beta}\right). \tag{3.109}$$

The parameters α_1, α_2 determine the scaling behaviour of the waiting time increments probability, and consequently of the MSD of Y and $\overline{W}$ in the case of X free diffusive. By applying Tauberian theorems, we obtain:

$$\begin{aligned}
\Phi(\lambda) \underset{\lambda\to\infty}{\sim} d_1\,\lambda^{\alpha_2}/d_2^{\alpha_2} \quad &\longrightarrow \quad MSD_Y(\lambda) \underset{\lambda\to\infty}{\sim} 2\,\sigma\, d_2^{\alpha_2}/[d_1\,\lambda^{1+\alpha_2}],\\
\Phi(\lambda) \underset{\lambda\to 0}{\sim} d_1\,\lambda^{\alpha_1}/d_2^{\alpha_1} \quad &\longrightarrow \quad MSD_Y(\lambda) \underset{\lambda\to 0}{\sim} 2\,\sigma\, d_2^{\alpha_1}/[d_1\,\lambda^{1+\alpha_1}],
\end{aligned} \tag{3.110}$$

corresponding to the following scaling in time:

$$MSD_Y(t) \underset{t\to 0}{\sim} 2\,\sigma\,\frac{d_2^{\alpha_2}}{d_1}\frac{t^{\alpha_2}}{\Gamma(1+\alpha_2)}, \qquad MSD_Y(t) \underset{t\to\infty}{\sim} 2\,\sigma\,\frac{d_2^{\alpha_1}}{d_1}\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}. \tag{3.111}$$

Thanks to the fact that K in Eq. (3.109) is given by a three parameter ML, the MSD of Y, according to Eq. (3.81), follows straightforwardly by adapting the formula Eq. (A.15):

$$MSD_Y(t) = 2\,\sigma\,\frac{d_2^{\alpha_2}}{d_1}\, t^{\alpha_2} E^{\beta(\alpha_2-\alpha_1)}_{1/\beta,1+\alpha_2}\left(-d_2^{1/\beta}\, t^{1/\beta}\right). \tag{3.112}$$

Similarly, recalling Eqs. (3.85, 3.86), the scaling behaviour of the linear functional W can be obtained straightforwardly:

$$MSD_W(t) \underset{t\to 0}{\sim} 4\,\sigma^2\,\frac{d_2^{\alpha_2}}{d_1}\frac{t^{2+\alpha_2}}{\Gamma(3+\alpha_2)}, \qquad MSD_W(t) \underset{t\to\infty}{\sim} 4\,\sigma^2\,\frac{d_2^{\alpha_1}}{d_1}\frac{t^{2+\alpha_1}}{\Gamma(3+\alpha_1)}, \tag{3.113}$$

together with its exact analytical expression [yet again obtained by using Eq. (A.15)]:

$$MSD_W(t) = 4\,\sigma^2\,\frac{d_2^{\alpha_2}}{d_1}\, t^{2+\alpha_2} E^{\beta(\alpha_2-\alpha_1)}_{1/\beta,3+\alpha_2}\left(-d_2^{1/\beta}\, t^{1/\beta}\right) \tag{3.114}$$

Thus, the rescaled functional $\overline{W}$ has the same asymptotic scaling behaviour of Y.

We now apply our formalism, with the Φ defined in Eq. (3.107), to MSD data of mitochondria diffusing in mating *S. cerevisiae* cells, depleted of actin microfilaments, which were obtained with Fourier imaging correlation spectroscopy in Ref. [57]. This MSD exhibits a crossover from a transient subdiffusive scaling of exponent $\alpha = 0.66$ to normal diffusion, which we cannot capture quantitatively by using a tempered Lévy stable subordinator, because the experimental curvature at the crossover between the two power laws is different from the predicted one. Instead, if we time-change a purely diffusive $X(s)$ process with the Φ in Eq. (3.107), we can tune the parameter β, such that the curvature between the power laws is well reproduced. By employing a least-squares method, with $\alpha_1 = 1$ and $\alpha_2 = 0.66$ to fix the asymptotic scaling respectively for long/small times, we determine that the choice of parameters: $d_1 = 4.53$, $d_2 = 0.02$ and $\beta = 0.85$ yields an excellent model of the experimental data across the double power-law region (see Fig. 3.5a). These specific parameters, along with the functional form of Φ, uniquely determine the waiting time increments probability, which can be employed to simulate the underlying anomalous diffusion process. We plot such quantity in Fig. 3.5b. In agreement with the asymptotic analysis in Eqs. (3.110), Φ interpolates between power-law distributed [$p(\Delta t) \sim \Delta t^{-1-\alpha_2}$, $\alpha_2 = 0.66$ in this case] and exponentially distributed (corresponding to a regime of normal diffusion) waiting time increments. In this plot, we also elucidate the role of the curvature parameter β, by plotting $p(\Delta t)$ for different values of β, but yet close to the optimal value obtained by the experimental fit. On the one hand, β directly determines the timescale needed by the process to reach the normal diffusive regime, which becomes larger as β increases. On the other hand, it does not affect the coefficient in the exponential function, describing the asymptotic scaling of the distribution in the normal diffusive regime, which is instead equal to d_2/d_1. This is different from the case of the tempered Lévy stable subordinator, whose corresponding coefficient depends explicitly on μ (see Fig. 3.2a).

We further investigate in Fig. 3.6a the MSD behaviour for X being an OU process and Φ being defined by Eq. (3.107) with the parameters of the experimental fit. We also compare it with three limiting cases: (i) normal diffusion (denoted "Brownian", black solid line), which is obtained from Eq. (3.107) by setting $\alpha_2 = \alpha_1 = 1$ and $d_1 = d_2$; (ii) subdiffusive CTRW of exponent $\alpha = 0.66$, which is obtained with the choice: $\alpha_2 = \alpha_1 = 0.66$ and $d_1 = d_2$ (black solid line); (iii) hybrid subdiffusive dynamics generated by the sum of two Lévy stable subordinators of exponents $\alpha_2 = 0.66$, $\alpha_1 = 1$, which is obtained by setting $\beta = 1/(\alpha_2 - \alpha_1)$ ($d_{1,2}$ as in the fit). On the one hand, we note that the MSD converges to the same plateau in the cases (ii, iii), which share the same power-law asymptotic behaviour of the waiting time increments probability for large Δt with exponent $\alpha = 0.66$. On the other hand, the hybrid model generated by the Φ in Eq. (3.107) with parameters specified by the experimental fit exhibits the same scaling behaviour of the CTRW for small times, whereas its MSD converges to zero for long times, thus recovering the same behaviour of the Brownian case. Thus, our hybrid model recovers ergodicity for long times, as in the case of the harmonic oscillator subordinated to a tempered Lévy stable process (see Fig. 3.3b). Recalling that these two hybrid models share the same exponential scaling of the waiting time increments, our result confirms that ergodicity breaking in anomalous stochastic dynamics is strictly related to the existence of heavy-tailed distributed waiting times, this corresponding to trapping events in the physical system. which corresponds to the physical scenario where the system gets trapped during its dynamics.

To conclude, we plot in Fig. 3.6b the predicted two-point correlation function of an OU process subordinated with the Φ in Eq. (3.107) with parameters specified by the fit in Fig. 3.5a. Despite the complex form of the waiting time distribution, the correlation function decays exponentially for long difference times, i.e., $\langle Y(t)\, Y(t+\Delta t)\rangle \sim a\, e^{-b\,\Delta t}$ for long Δt. The values of the parameters a, b are determined by a direct fit of the tail

($\Delta t \geq 10^2$) of the correlation function: $a = 0.4854, b = 0.004193$. Finally, we provide the Laplace inverse transform of w, f_1, f_2 in Eqs. (3.69, 3.89) (see Appendix C.3 for details):

$$w(t) = \frac{d_1}{d_2^{\alpha_2}} t^{-\alpha_2} E^{\beta(\alpha_1-\alpha_2)}_{1/\beta,1-\alpha_2}\left(-d_2^{1/\beta}\, t^{1/\beta}\right) \tag{3.115a}$$

$$f_1(t) = \frac{\sigma}{\gamma}\left[1 - \gamma\, \frac{d_2^{\alpha_2}}{d_1}\, t^{\alpha_2}\, G(t)\right], \tag{3.115b}$$

$$f_2(t) = -\frac{\sigma}{\gamma}\left[-\gamma + \frac{d_1}{d_2^{\alpha_2}} t^{-\alpha_2} E^{\beta(\alpha_1-\alpha_2)}_{1/\beta,1-\alpha_2}\left(-d_2^{1/\beta}\, t^{1/\beta}\right) + \gamma^2\, \frac{d_2^{\alpha_2}}{d_1}\, t^{\alpha_2}\, G(t)\right], \tag{3.115c}$$

with G being the auxiliary function specified below:

$$G(t) = \sum_{n=0}^{\infty}(-1)^n \left(\gamma\, \frac{d_2^{\alpha_2}}{d_1}\right)^n t^{\alpha_2\, n} E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,1+\alpha_2(1+n)}\left(-d_2^{1/\beta}\, t^{1/\beta}\right). \tag{3.116}$$

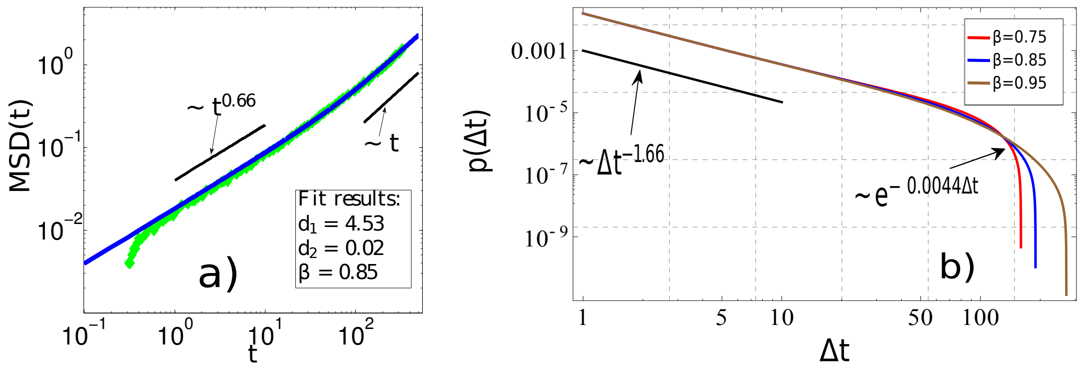

Figure 3.5: (a) Fit of the MSD data of mitochondria diffusing in *S. cerevisiae* cells depleted of actin microfilaments of Ref. [57]. Here, $X(s)$ is pure diffusion [$F(x) = 0$, $\sigma(x) = 1$ in Eq. (3.4a)] and Φ is given by Eq. (3.107) with $\alpha_1 = 1$, $\alpha_2 = 0.66$ [140]. (b) Waiting time increments probability $p(\Delta t)$, obtained by numerical inverse Laplace transform of $e^{-\Delta s\, \Phi(\lambda)}$ with $\Delta s = 0.001$ and Φ as in Eq. (3.107) with $\alpha_1 = 1$, $\alpha_2 = 0.66$, different sample values of β and the other parameters specified by the fit. Waiting time increments are either power-law [$p(\Delta t) \sim \Delta t^{-1-\alpha_2}$] or exponentially [$p(\Delta t) \sim e^{-(d_2/d_1)\Delta t}$] distributed respectively for small/large Δt. The resulting subordinated process is an hybrid between a CTRW of exponent $\alpha = 0.66$ and normal diffusion.

3.6 Outlook and Future Work

In this Chapter we fully characterised a class of anomalous stochastic processes, which generalise the CTRWs by accounting for more general waiting time distributions than the ordinary Lévy stable one with power-law decaying tails. Specifically, such distributions are determined by their characteristic functional Φ in Eq. (3.5), which is required to satisfy the minimal conditions given therein. We provided: (i) an explicit description of their stochastic dynamics in terms of subordinated Langevin equations, (ii) the GFFK Eq. (3.29), describing their joint statistics with one of their observables, and (iii) analytical formulas for the multipoint functions of both the process and its observables. All these results are derived for general Φ, which enables one to readily adapt them for any specific choice of it. We proved this point explicitly by applying our formalism to the case of both free diffusion and an OU process subordinated by either a tempered Lévy stable subordinator

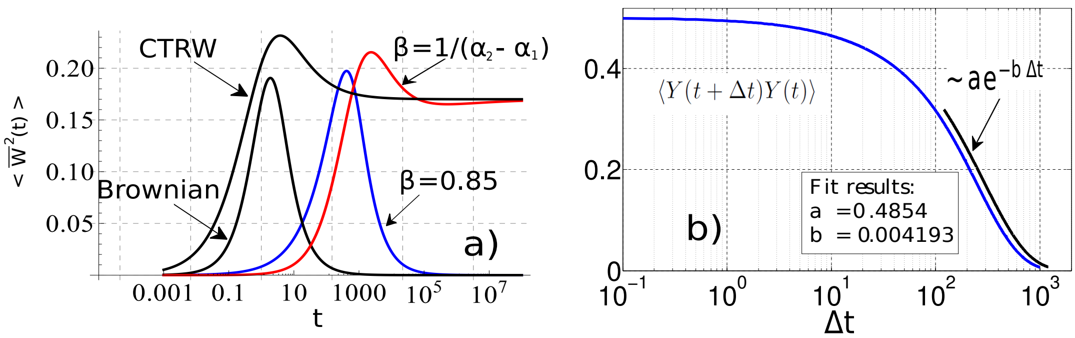

Figure 3.6: Here, $X(s)$ is an OU process $[F(x) = -x$, $\sigma(x) = 1$ in Eq. (3.4a)], $\eta(s)$ is specified by the Φ in Eq. (3.107) and $U(x) = x$. (a) MSD of $\overline{W}(t) = W(t)/t$ (for initial position $y_0 = 0$) for different sets of parameters: (i) $\alpha_1 = 1$, $\alpha_2 = 0.66$ and the other parameters as determined from the fit (blue solid line); (ii) $\alpha_1 = \alpha_2 = 1$ and $d_1 = d_2$ (Brownian limit, black solid line); (iii) $\alpha_1 = \alpha_2 = 0.66$ and $d_1 = d_2$ (CTRW limit, black solid line); (iv) $\alpha_1 = 1$, $\alpha_2 = 0.66$, $\beta = 1/(\alpha_2 - \alpha_1)$ and $d_{1,2}$ as in the fit (red solid line). Both (i-ii) correspond to exponentially distributed waiting time increments for large Δt and generate a subordinated process preserving ergodicity. Conversely, (iii-iv) correspond to power-law distributed waiting time increments and generate a subordinated process breaking ergodicity. This result confirms that heavy-tailed $p(\Delta t)$, i.e., trapping events in physical terms, causes ergodicity breaking. (b) Two point correlation function of Y for the fit parameters for $y_0^2 = \sigma/\gamma$ and finite t. The decay is exponential for large Δt.

or a mixture of two independent Lévy stable noises with different order parameters. The different properties observed for these two processes, for instance ergodicity breaking which may be either broken or preserved, clearly demonstrate the richness of our formalism, that can be employed to generate several different dynamical processes.

This flexibility in the choice of the waiting time distribution also renders our formalism of great relevance for the analysis of experimental data exhibiting anomalous diffusive behaviour, as we showed with the specific application to the position MSD data of diffusing mitochondria in S. Cerevisiae cells depleted of actin microfilaments of Ref. [57]. With the Φ given in Eq. (3.107), we obtained an excellent fit of the experimental MSD and characterised both the waiting time increments statistics and the two-point correlation function of the corresponding subordinated OU process. While the former one can be used together with Eqs. (3.4a, 3.4b) to simulate the underlying diffusion process, the second one constitutes a testable prediction to assess if the model correctly describes the observed biological system. In addition, our formalism would readily provide many other quantities of interest, for instance first passage times statistics, which could be used as further tests of the validity of the theoretical model. Furthermore, we note that several different parent processes X, which can be either additive or multiplicative (with generalised α-prescription), can also be chosen. This will indeed play a major role in determining the properties of the corresponding subordinated process, which we have not yet explored extensively.

For future work, it will be interesting to elucidate if our formalism can also generate superdiffusive dynamics. So far, only the integrated process W displayed such behaviour, whereas we would be interested in observing it either for the dynamics of Y or for that of $\overline{W}$. Relatedly, it would be relevant to verify if a suitable choice of X and Φ exists that could generate an hybrid process, whose MSD exhibits a crossover between subdiffusive and superdiffusive behaviour. The relevance of such dynamical behaviour is supported by recent experiments on the dynamics of *Escherichia Coli* chromosomal loci [130].

CHAPTER 4

Langevin Formulation of a Subdiffusive Continuous-Time Random Walk in Physical Time

In this Chapter, we propose an equivalent Langevin formulation of a force-free CTRW without subordination. By introducing a new type of white non-Gaussian noise, we express the CTRW dynamics in terms of a single Langevin equation in physical time with additive noise. We characterise such noise in terms of its full multi-point statistics and of its characteristic functional and compare it with the SBM, an alternative stochastic model of subdiffusive dynamics with Gaussian statistics. These two noises are identical up to the 2nd order correlation functions, but different in the higher order statistics. We extend our formalism to general waiting time distributions and force fields and compare our results with those of SBM. In the presence of external forces, our proposed noise generates a new class of stochastic processes, resembling CTRWs but with forces acting at all times. These results complement those in Chapter 3 within the general scope of formulating more general anomalous processes than CTRWs that can be compared with experiments.

4.1 Motivation

As extensively discussed in Chaps. 1 and 3, many systems in nature live in complex non-equilibrated or highly crowded environments, thus exhibiting anomalous diffusive patterns, which deviate from the well known Fick's law Eq. (1.2) of purely thermalised systems [18, 9, 19]. Their distinctive feature is instead a power-law scaling of the MSD, as given in Eq. (1.7) [54, 18, 165, 9, 19], with the value of its exponent characterising the different types of anomalous behaviour, i.e., subdiffusion ($0 < \alpha < 1$) or superdiffusion ($\alpha > 1$).

Examples of such anomalous processes are found in both physical and living systems. In particular these latter ones have recently been in the focus of experimental research, thanks to the huge improvements obtained in the experimental techniques employed in biology, which enabled researchers to provide joint position-velocity datasets of several different systems, from migrating cells to molecules and/or organelles moving within the cell, revealing such anomalous diffusive behaviour. We refer to Chaps. 1, 3 for a detailed overview of experimental evidence of anomalous diffusive behaviour in such systems.

Considering this wide variety of systems, which simultaneously exhibit anomalous scaling of the MSD and other anomalous features, for instance in their multipoint correlation functions [131, 132, 133, 134, 135, 136, 137], one needs to have a complete set of well studied models that can be employed to fit the experimental data and infer the specific microscopic processes underlying the observed dynamics. For subdiffusive processes, the most commonly used models are the CTRW [28, 9], which has already been discussed thoughtfully in Chapter 2, and the SBM [166, 167, 168, 169].

Contrarily to the CTRW that was introduced as a natural generalization of a random walk on a lattice [28] with waiting times between the jumps and their size being sampled from general and independent probability distributions, the SBM has been recently introduced as a Gaussian model of anomalous dynamics [166], which is able to provide the same anomalous scaling of the MSD in Eq. (1.7) for all its temporal evolution. If $B(t)$ is a usual Brownian motion, its scaled version is defined by making a power-law change of time with exponent α: $B(t^\alpha)$. Although being commonly used to fit data [170, 29, 167], it has recently been shown to be a non stationary process with paradoxical behaviour under confinement, i.e., in the presence of a linear viscous-like force, as its MSD unboundedly decreases towards zero. This is suggested to be ultimately caused by the time dependence of the environment, either of the temperature or of the viscosity. As a consequence, it has been ruled out as a possible alternative model of anomalous thermalized processes [169].

Thus, the aim of this Chapter is twofold. On the one hand, complementarily to the results presented in Chapter 3, (i) we generalise CTRWs to account for external forces that are exerted on the all temporal evolution of the system, i.e., both during the jumps, as in the CTRW, and during the waiting times. On the other hand, (ii) we clarify the behaviour of SBM under confinement, by comparing it with these new anomalous processes.

Specifically in the case of (i), we derive a novel type of noise, allowing us to express a free diffusive CTRW in terms of a single Langevin equation in physical time. We fully characterise its properties by providing its characteristic functional and the complete hierarchy of its multipoint correlation functions, that we then compare with those of the noise driving a SBM. We discuss both purely power-law waiting times and general waiting time distributions [159, 140]. We show that their correlation functions are identical up to the two point ones, but different for higher orders: the noise driving SBM is a Gaussian noise, while our new noise driving a CTRW is naturally non-Gaussian. Here, all odd correlation functions vanish, as for a typical Gaussian noise, but the even ones do not satisfy Wick's theorem [110, 111]. Finally, we show that the newly defined noise enables us to define a class of CTRW-like anomalous processes with forces acting at all times, thus being fundamentally different from their corresponding ordinary CTRWs.

Regarding (ii), we revisit the behaviour of the SBM under confinement and show that its MSD correctly converges to a plateau as it is typical of confined motion [92], provided that we use more general time changes with truncated power-law tails. This suggests that the anomaly observed in [169] is mainly due to the localizing effect of the external linear force, which is able to trap the particle in the zero position if we allow for infinitely long waiting times between the jumps to eventually occur in the long time limit.

4.2 Generalisation of CTRW and Scaled Brownian Motion

In this first section we recall definitions and properties of the free diffusive CTRW and of the SBM, which will be useful later in the discussion. We are mainly interested in their stochastic Langevin formulation and in both their FP equation and MSD. We then generalise these results to the case of arbitrary waiting time distributions and time transformations respectively for the CTRW and the SBM.

4.2.1 CTRW

As we discussed in Sec. 2.1.2, the Langevin representation of a CTRW is obtained by introducing two auxiliary processes $X(s)$ and $T(s)$, which we assume for now to be purely diffusive and Lévy stable with parameter α $(0 < \alpha \leq 1)$ respectively. Their dynamics is specified by the following Langevin equations [79] [adapted from Eqs. (2.46a, 2.46b)]:

$$\dot{X}(s) = \sqrt{2\sigma}\,\xi(s) \qquad\qquad \dot{T}(s) = \eta(s) \tag{4.1}$$

where $\xi(s)$ and $\eta(s)$ are two independent noises. For $X(s)$ to be a normal diffusion, we require $\xi(s)$ to be a white Gaussian noise with $\langle\xi(s)\rangle = 0$ and $\langle\xi(s_1)\xi(s_2)\rangle = \delta(s_2 - s_1)$ [see Eqs. (2.98)]. On the other hand, $\eta(s)$ is a stable Lévy noise with parameter α $(0 < \alpha \leq 1)$ [101]. The anomalous CTRW is then derived by making a randomization of time, i.e., by considering the time-changed (or subordinated) process: $Y(t) = X(S(t))$, with $S(t)$ being the inverse of $T(s)$ as defined in Eq. (2.47). The process $Y(t)$ is easily shown to satisfy Eq. (1.7) exactly for all its time evolution, by recalling that the PDF of $S(t)$ has the Laplace transform $\widetilde{h}(s,\lambda) = \lambda^{\alpha-1}e^{-s\lambda^\alpha}$ [131] and that $\left\langle X^2(s)\right\rangle = 2\,\sigma\, s$. Indeed, by recalling Eq. (3.45), we obtain in Laplace space:

$$\left\langle \widetilde{Y}^2(\lambda)\right\rangle = \int_0^{+\infty} \left\langle X^2(s)\right\rangle \widetilde{h}(s,\lambda)\,\mathrm{d}s = \frac{2\,\sigma}{\lambda^{1+\alpha}}, \tag{4.2}$$

whose inverse Laplace transform confirms its anomalous scaling [9]:

$$\left\langle Y^2(t)\right\rangle = \frac{2\,\sigma}{\Gamma(1+\alpha)}t^\alpha. \tag{4.3}$$

As expected, this same MSD is obtained by taking the diffusive limit of the microscopic random walk formulation of the CTRW, where we allow for asymptotically power-law distributed waiting times between the jumps of the walker, whose amplitudes are drawn from a distribution with finite variance [9]. In this limit, as we discussed in Sec. 2.1, the model provides the fractional diffusion Eq. (2.29) for the PDF of $Y(t)$. It is then natural to study if the set of Eqs. (4.1) can provide this same FP equation. This has been proved in [151, 55, 140], and in a more general set-up in Chapter 3, with the specification: $K_\alpha = \frac{\sigma}{\Gamma(1+\alpha)}$ in Eq. (2.29), thus confirming the equivalence in the diffusive limit of the single point statistics of the random walk picture and of the subordinated Langevin Eqs. (4.1).

4.2.2 Scaled Brownian Motion

If instead of a stochastic time change, we consider the deterministic time transformation $t \to t^* = t^\alpha$ in the normal diffusive process $X(t)$ (now in the physical time t), we obtain the SBM: $Y_*(t) = X(t^*)$. Its equivalent Langevin equation is given by [166, 167, 168, 169]:

$$\dot{Y}_*(t) = \sqrt{2\,\alpha\,\sigma\,t^{\alpha-1}}\,\xi(t), \tag{4.4}$$

with $\xi(t)$ being a white Gaussian noise (with the same properties as before, but in the physical time t). By using Eq. (4.4) we can prove straightforwardly that the MSD of $Y_*(t)$ is the same as Eq. (4.3) and that the corresponding FP equation is given by:

$$\frac{\partial}{\partial t}P(y,t) = \alpha\,\sigma\,t^{\alpha-1}\,\frac{\partial^2}{\partial y^2}P(y,t), \tag{4.5}$$

which has time dependent diffusion coefficient [171]. This process preserves all the properties of Brownian motion [166]: it is indeed Gaussian with time-dependent variance and Markov, as the monotonicity of the time change preserves the ordering of time. Furthermore, $Y_*(t)$ is self-similar and it has independent increments for non overlapping intervals. However, differently from Brownian motion, it is strongly non stationary [169]. Furthermore, $Y_*(t)$ turns out to be the mean-field approximation of the CTRW, as it describes the motion of a cloud of random walkers performing CTRW motion in the limit of a large number of walkers [168]. Recent investigation have also shown that SBM exhibits rich ageing properties, which strongly differentiates it from the standard BM [172].

4.2.3 Arbitrary Waiting Time Distributions and Time Transformations

In this section, we review the generalisation of Eqs. (4.1) to arbitrary waiting time distributions of the underlying random walk [173, 174, 154, 175, 159, 140] that we discussed in Chapter 3 and then derive the analogous extension of the SBM to general time transformations. We recall that such extension in the case of CTRWs is obtained naturally by choosing a different process $T(s)$ with the only assumption of it being strictly increasing in order to preserve the causality of time. Thus, we consider $\eta(s)$ in Eq. (4.1) to be an increasing Lévy noise with paths of finite variation and characteristic functional [101] given by Eq. (3.5), where $\Phi(u(s))$ is a non negative function with $\Phi(0) = 0$ and strictly monotone first derivative, while $u(s)$ is a test function. We recall that for $\Phi(s) = s^\alpha$ we recover the CTRW model. Under these assumptions, the integrated process $T(s)$ is a a one-sided strictly increasing Lévy process with finite variation. Furthermore, we assume $\eta(s)$ to be independent on the realizations of $\xi(s)$ in Eq. (4.1). As a consequence of the finite variation and the monotonicity of the paths of $T(s)$ respectively, $S(t)$ has continuous and monotone paths, with this second property implying the fundamental relation [131] Eq. (3.6). Similarly to Eq. (4.2), we can derive the corresponding MSD by recalling that $\widetilde{h}(s,\lambda) = \frac{\Phi(\lambda)}{\lambda}e^{-s\Phi(\lambda)}$ [159, 140], which is given by Eq. (3.81). Furthermore, the PDF of $Y(t)$ is obtained by solving the generalized FP equation [140] [adapted from Eq. (3.31)]:

$$\frac{\partial}{\partial t}P(y,t) = \sigma\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial t}\int_0^t K(t-\tau)\,P(y,\tau)\,\mathrm{d}\tau\,, \tag{4.6}$$

with K specified by Eq. (3.30). Its solution can be found for general $\Phi(s)$ in Laplace space:

$$\widetilde{P}(y,\lambda) = \frac{1}{\lambda}\sqrt{\frac{\Phi(\lambda)}{2\sigma}}e^{-\sqrt{\frac{\Phi(\lambda)}{2\sigma}}|y|}. \tag{4.7}$$

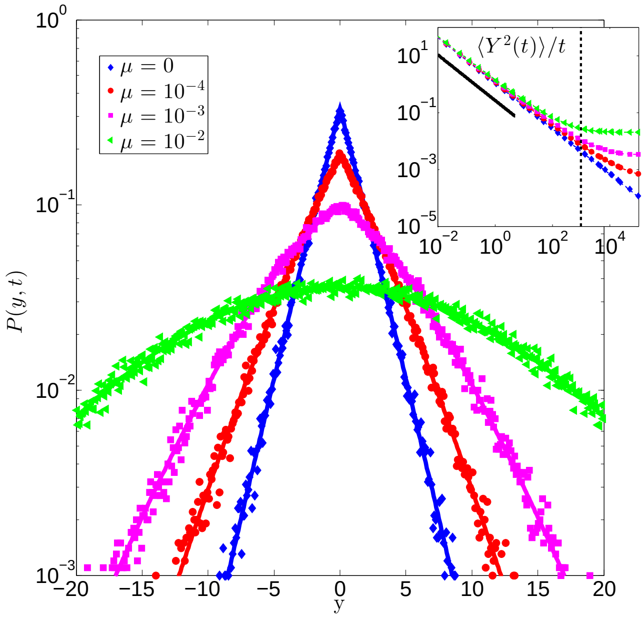

Figure 4.1: PDF (main) and MSD normalised to t (inset) of an anomalous process $Y(t)$ obtained by subordination of a free diffusion by a tempered Lévy stable process of tempering index μ and stability parameter $\alpha = 0.2$. The PDF is obtained by numerical Laplace inversion of Eq. (4.7) at $t = 10^3$ (black dotted lines in the inset) [178]. The smooth transition from the non-Gaussian PDF typical of CTRWs ($\mu = 0$) and the Gaussian one of normal diffusion ($\mu \to +\infty$) is evident. Note also the corresponding transition from anomalous to normal scaling of the MSD for increasing μ at time t. Simulations, obtained with the Algorithm 1 of Sec. 2.1.2 and the techniques of [81, 161] for the generation of the corresponding RVs (summarised in App. B), agree perfectly with the analytical results.

We look as an example at the case of a tempered stable Lévy noise with tempering index μ and stability index α [176], which is obtained by setting $\Phi(\lambda) = (\mu+\lambda)^\alpha - \mu^\alpha$, i.e., $K(t) = e^{-\mu t}\, t^{\alpha-1}\, E_{\alpha,\alpha}(\mu\, t)^\alpha$ [164]. As already pointed out, the CTRW case is recovered by setting $\mu = 0$, meaning that we do not truncate the long tails of the distribution, thus accounting for very long waiting times with a power-law decaying probability of occurrence. We plot in Figure 4.1 the numerical Laplace inverse of Eq. (4.7) (main) and the corresponding MSD (inset) at a fixed time $t = 1000$ (dotted line in the inset), which is given by Eq. (3.93) [177]. As expected, for $\mu = 0$ we recover the typical non Gaussian shape of the PDF of a free diffusive CTRW [9]. However, for increasing values of μ, the PDF of $Y(t)$, although still being non Gaussian, broadens, thus getting closer to a Gaussian. This has also evident consequences on the dynamical behaviour of the MSD, which for increasing values of μ goes from a pure subdiffusive scaling to a normal one (inset), as predicted by Eqs. (3.92).

We now discuss the corresponding extension of the SBM to arbitrary time transformations involving the kernel $K(t)$ obtained by Laplace inverse transform of Eq. (3.30). We then generalize Eq. (4.4) by adopting $K(t)$ as the time dependent coefficient of the white Gaussian noise ξ:

$$\dot{Y}_*(t) = \sqrt{2\sigma K(t)}\,\xi(t) = \sqrt{2\sigma}\zeta(t), \tag{4.8}$$

where we define the correlated noise $\zeta(t)$ with $\langle\zeta(t)\rangle = 0$ and two-point correlation function: $\langle\zeta(t_1)\zeta(t_2)\rangle = 2\sigma K(t_1)\delta(t_1 - t_2)$. This explicit time dependence manifestly signals that

$\zeta(t)$ is a non stationary noise. It is easily shown that the MSD of $Y_*(t)$ is identical to the one of $Y(t)$ given by Eq. (3.81). However, even if they share the same MSD, $Y(t)$ and $Y_*(t)$ provide different PDFs. Indeed, $Y_*(t)$ corresponds to a time rescaled Brownian motion $X(t^*)$ with transformation:

$$t^* = \int_0^t K(\tau)\,\mathrm{d}\tau\,. \tag{4.9}$$

In the case of the usual Brownian motion the corresponding diffusion equation has a Gaussian solution: $P(y,t) = \frac{1}{\sqrt{4\pi\sigma t}} e^{-\frac{(y-y_0)^2}{4\sigma t}}$ for the initial condition $P(y,0) = \delta(y-y_0)$. Since $Y_*(t)$ is just Brownian motion in the rescaled time t^*, we obtain similarly a Gaussian solution, provided we choose the same initial condition:

$$P(y,t) = \frac{1}{\sqrt{4\pi\sigma t^*}} e^{-\frac{(y-y_0)^2}{4\sigma t^*}}\,, \tag{4.10}$$

with t^* as in Eq. (4.9). We see that $P(y,t)$ is a solution of the diffusion equation:

$$\frac{\partial}{\partial t} P(y,t) = \sigma K(t) \frac{\partial^2}{\partial y^2} P(y,t), \tag{4.11}$$

with the time dependent diffusion constant: $D(t) = \sigma K(t)$. We remark that Eq. (4.4) can be recovered from these general results by setting $\Phi(\lambda) = \lambda^\alpha$, i.e., $K(t) = t^{\alpha-1}/\Gamma(\alpha)$ and $t^* = t^\alpha/\Gamma(1+\alpha)$. However, in order to have exact equivalence, we need to neglect the constant multiplicative factors in both $K(t)$ and t^* and make the substitution: $\sigma \to \alpha\,\sigma$.

4.3 Langevin Formulation of a CTRW in Physical Time

In this section, we formulate an alternative Langevin description of the process Y defined in Eqs. (4.1) directly in physical time, i.e., without formally employing the subordination technique. This will be obtained by defining a novel type of noise, whose characterisation will be provided both in terms of its higher order correlation functions, that will also be compared with those of the SBM, and of its characteristic functional [16].

4.3.1 Definition of the Noise

Starting from Eqs. (4.1) and using the property $1 = \int_0^{+\infty} \delta(s - S(t))\,\mathrm{d}s$, we obtain:

$$\begin{aligned} Y(t) &= \sqrt{2\,\sigma} \int_0^{+\infty} \delta(s - S(t)) \left[\int_0^s \xi(\tau)\,\mathrm{d}\tau \right] \mathrm{d}s \\ &= \sqrt{2\,\sigma} \int_0^{+\infty} \left[-\frac{\partial}{\partial s}\Theta(t - T(s)) \right] \left[\int_0^s \xi(\tau)\,\mathrm{d}\tau \right] \mathrm{d}s \\ &= \sqrt{2\,\sigma} \int_0^{+\infty} \Theta(t - T(s))\xi(s)\,\mathrm{d}s\,, \end{aligned} \tag{4.12}$$

where the fundamental relation between the paths of T and S Eq. (3.6) is used to obtain the second equality and we then get the third one with an integration by parts. We remark that the boundary term $\left[-\Theta(t - T(s)) \int_0^s \xi(\tau)\,\mathrm{d}\tau \right]\Big|_0^{+\infty}$ is zero trivially for $s = 0$, but it vanishes also for $s \to +\infty$ because $T(s)$ is increasing, thus always being larger than any fixed (and finite) time t. Written as in Eq. (4.12), $Y(t)$ is a differentiable (in a generalised sense) function of time, i.e., we can take its derivative and derive the Langevin equation:

$$\dot{Y}(t) = \sqrt{2\,\sigma}\,\overline{\xi}(t) \tag{4.13}$$

where we define the noise as below:

$$\overline{\xi}(t) = \int_0^{+\infty} \xi(s)\delta(t - T(s))\,\mathrm{d}s\,, \tag{4.14}$$

whose properties are fully determined by the choice of the waiting time distribution, or equivalently of the function $\Phi(s)$ in Eq. (3.5). By recalling the independence of $\xi(s)$ and $\eta(s)$, it is straightforward to show that $\overline{\xi}(t)$ has zero average. In addition, we can employ these properties to compute the two point correlation function of $\overline{\xi}$ in Laplace space:

$$\begin{aligned}
\left\langle \widetilde{\overline{\xi}}(\lambda_1)\widetilde{\overline{\xi}}(\lambda_2)\right\rangle &= \int_0^{+\infty}\int_0^{+\infty} \langle \xi(s_1)\xi(s_2)\rangle \left\langle e^{-\lambda_1\,T(s_1)}e^{-\lambda_2\,T(s_2)}\right\rangle \mathrm{d}s_1\;\mathrm{d}s_2 \\
&= \int_0^{+\infty}\int_0^{+\infty} \delta(s_2 - s_1)\left\langle e^{-\lambda_1\,T(s_1)}e^{-\lambda_2\,T(s_2)}\right\rangle \mathrm{d}s_1\;\mathrm{d}s_2 \\
&= \int_0^{+\infty} \left\langle e^{-(\lambda_1+\lambda_2)T(s)}\right\rangle \mathrm{d}s \;=\; \frac{1}{\Phi(\lambda_1+\lambda_2)}
\end{aligned} \tag{4.15}$$

where in the second line we use the property of the white Gaussian noise ξ and we compute in the third line the characteristic function of T by using Eq. (3.5). Further recalling Eq. (3.61), we obtain for its inverse Laplace transform the following result:

$$\left\langle \overline{\xi}(t_1)\overline{\xi}(t_2)\right\rangle = K(t_1)\delta(t_1 - t_2) \tag{4.16}$$

with $K(t)$ being specified by Eq. (3.30). Consequently, the character of the noise $\overline{\xi}(t)$ significantly depends on the choice of the function $\Phi(s)$ in Eq. (3.5). Thus, Eq. (4.13) defines a new Langevin model driven by a generalised and typically non Gaussian noise, except possibly for particular choices of the memory kernel $K(t)$. Such model is able to reproduce the dynamics of free diffusive anomalous processes with arbitrary waiting time distribution equivalently to the subordinated Langevin Eqs. (4.1).

The explicit time dependence of the two point correlation function of $\overline{\xi}$ via the function K highlights the non stationary character of such noise, which implies that the integrated process Y no longer satisfies the scaling properties of the ordinary Brownian motion, except for spatial reflection, that does not involve the time variable. For instance, let us consider (i) the process Y, obtained by integration of Eq. (4.13), whose two point correlation functions thus reads as $\langle Y(t_1)Y(t_2)\rangle = 2\,\sigma\int_0^{\mathrm{Min}(t_1,t_2)} K(\tau)\,\mathrm{d}\tau$ [obtained via double integration of Eq. (4.16)], and (ii) the process $Z(t) = Y(t_0 \pm t) - Y(t_0)$ with $t_0 \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$ or $t \in [0, t_0]$ respectively. It is a well known result that, if one considers a Brownian motion $B(t)$ instead of Y, the process $Z(t)$ is still a Brownian motion [104]. This result states the invariance of Brownian motion with respect to temporal translations (Z defined with $+$) and reflections (Z defined with $-$) On the contrary, in the present case the statistics of $Z(t)$ is different from that of the original process Y. As an example, we compute below its two point correlation function. On the one hand, for Z defined with $+$, we have ($t_1 < t_2$):

$$\begin{aligned}\langle Z(t_1)Z(t_2)\rangle &= \langle [Y(t_0+t_1) - Y(t_0)][Y(t_0+t_2) - Y(t_0)]\rangle \\ &= \langle Y(t_0+t_1)Y(t_0+t_2)\rangle - \langle Y(t_0+t_1)Y(t_0)\rangle - \langle Y(t_0+t_2)Y(t_0)\rangle + \langle Y(t_0)Y(t_0)\rangle \\ &= 2\,\sigma\left[\int_0^{t_0+t_1} - \int_0^{t_0} - \int_0^{t_0} + \int_0^{t_0}\right] K(\tau)\,\mathrm{d}\tau = 2\,\sigma \int_{t_0}^{t_0+t_1} K(\tau)\,\mathrm{d}\tau\,; \end{aligned} \tag{4.17}$$

on the other hand, for Z defined with $-$, we obtain:

$$\begin{aligned}\langle Z(t_1)Z(t_2)\rangle &= \langle [Y(t_0-t_1) - Y(t_0)][Y(t_0-t_2) - Y(t_0)]\rangle \\ &= \langle Y(t_0-t_1)Y(t_0-t_2)\rangle - \langle Y(t_0-t_1)Y(t_0)\rangle - \langle Y(t_0-t_2)Y(t_0)\rangle + \langle Y(t_0)Y(t_0)\rangle \\ &= 2\sigma\left[\int_0^{t_0-t_2} - \int_0^{t_0-t_1} - \int_0^{t_0-t_2} + \int_0^{t_0}\right] K(\tau)\,\mathrm{d}\tau = 2\sigma \int_{t_0-t_1}^{t_0} K(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{4.18}$$

Both these results differ from the corresponding one of Y, i.e., in this case the invariance with respect to temporal translations and reflections is broken. Note that, if we set $K(t) = 1$, i.e., the Brownian case, we recover $\langle Z(t_1)Z(t_2)\rangle = 2\,\sigma\,t_1$ in both cases. If we now define $Z(t) = t\,Y(1/t)$, corresponding to a temporal inversion, we obtain:

$$\langle Z(t_1)Z(t_2)\rangle = 2\,\sigma\,t_1\,t_2 \int_0^{\mathrm{Min}(1/t_1,1/t_2)} K(\tau)\,\mathrm{d}\tau\,, \tag{4.19}$$

which again shows that the invariance, holding for the Brownian motion, is not preserved. At last, we define $Z(t) = c^{-H}\,Y(c\,t)$, where $c > 0$ and H are constants. If Y is a Brownian motion, Z is again a Brownian motion with $H = 1/2$. This result states the invariance of Brownian motion with respect to a diffusive rescaling. In our case, this property is related to the self-similarity of the waiting time distribution, which only holds in the specific case of a Lévy stable subordinator, i.e., $K(t) = t^{\alpha-1}/\Gamma(\alpha)$. To support this argument, we compute the two point correlation function for this specific choice:

$$\begin{aligned}\langle Z(t_1)Z(t_2)\rangle = c^{-2\,H}\,\langle Y(c\,t_1)Y(c\,t_2)\rangle &= 2\,\sigma\,c^{-2\,H}\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{Min}(c\,t_1,c\,t_2)} \tau^{\alpha-1}\,\mathrm{d}\tau \\ &= 2\,\sigma\,c^{-2\,H+\alpha}\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{Min}(t_1,t_2)} \tau'^{\,\alpha-1}\,\mathrm{d}\tau'\,, \end{aligned} \tag{4.20}$$

which is equal to that of Y if we set $H = \alpha/2$. Thus, the time-changed process Y is still invariant with respect to diffusive rescaling if we choose a tempered Lévy stable waiting time distribution. In the general case, though, also this property no longer holds.

We highlight that so far the standard renewal picture underlying conventional CTRWs, which we discussed in Sec. 2.1.2, still applies. Eq. (4.13) is essentially the time derivative of Eq. (2.48) expressing the process in terms of stochastic increments. Differences will appear when external forces act on the diffusion processes. This case is discussed in Sec. 4.4.

4.3.2 Characterisation of the Multipoint Correlation Functions

The definition in Eq. (4.14) enables us to derive a complete characterization of the multipoint correlation structure of $\bar{\xi}(t)$. As a preliminary step, we need to compute the Laplace transform of the multipoint characteristic function of $T(s)$, i.e., the quantity

$Z(t_1, s_1; \ldots; t_N, s_N) = \left\langle \prod_{m=1}^{N} \delta(t_m - T(s_m)) \right\rangle$, $\forall N \in \mathbb{N}$. By using Eq. (4.1), we find:

$$\widetilde{Z}(\lambda_1, s_1; \ldots; \lambda_N, s_N) = \left\langle \prod_{m=1}^{N} e^{-\lambda_m \int_0^{s_m} \eta(s'_m)\, \mathrm{d}s'_m} \right\rangle. \tag{4.21}$$

Let us first assume an ordering for the sequence of times: $s_1 < s_2 < \ldots < s_N$ and compute the corresponding Eq. (4.21). If we rearrange the exponent by separating successive intervals, we obtain:

$$\begin{aligned}
\widetilde{Z}(\lambda_1, s_1; \ldots; \lambda_N, s_N) &= \left\langle e^{-\lambda_N \int_{s_{N-1}}^{s_N} \eta(s'_N)\, \mathrm{d}s'_N - \ldots - (\lambda_N + \ldots + \lambda_1) \int_0^{s_1} \eta(s'_1)\, \mathrm{d}s'_1} \right\rangle \\
&= \left\langle e^{-\sum_{m=0}^{N-1} \left[\left(\sum_{n=m+1}^{N} \lambda_n\right)\right] \int_{s_m}^{s_{m+1}} \eta(s'_m)\, \mathrm{d}s'_{m+1}} \right\rangle \\
&= \prod_{m=0}^{N-1} \left\langle e^{-\left(\sum_{n=m+1}^{N} \lambda_n\right) \int_{s_m}^{s_{m+1}} \eta(s'_m)\, \mathrm{d}s'_{m+1}} \right\rangle \\
&= \prod_{m=0}^{N-1} e^{-(s_{m+1} - s_m) \Phi\left(\sum_{n=m+1}^{N} \lambda_n\right)},
\end{aligned} \tag{4.22}$$

where we define $s_0 = 0$ to simplify the notation and we exploited the independence of the increments of $T(s)$ to factorise the ensemble average. Furthermore, the final result Eq. (4.22) is derived by using the stationarity of the increments of T and Eq. (3.5). However, in the general case where no a-priori ordering is assumed, we need to consider all the possible ordered sequences. We then introduce the group of permutations of N objects S_N, whose elements act on the sequence: $\boldsymbol{s} = (s_1, \ldots, s_N)$. When we make a permutation of $\boldsymbol{s}$, we obtain a new sequence with permuted indices: $\boldsymbol{s'} = \left(s_{\sigma(1)}, \ldots, s_{\sigma(N)}\right)$. All the possible orderings of $\boldsymbol{s}$ are thus obtained by summing over all the permutations in S_N. If we assume that $s_{\sigma(0)} = 0$, $\forall \sigma \in S_N$, i.e., the initial time is kept fixed by the permutations, and we use the result of Eq. (4.22), we derive:

$$\widetilde{Z}(\lambda_1, s_1; \ldots; \lambda_N, s_N) = \sum_{\sigma \in S_N} \prod_{m=0}^{N-1} \Theta\left(s_{\sigma(m+1)} - s_{\sigma(m)}\right) e^{-\left[s_{\sigma(m+1)} - s_{\sigma(m)}\right] \Phi\left(\sum_{n=m+1}^{N} \lambda_{\sigma(n)}\right)} \tag{4.23}$$

with the ordering of the permuted sequence being specified by the product of Heaviside functions. By factorising out the first term, we obtain the fundamental result:

$$\begin{aligned}
\widetilde{Z}(\lambda_1, s_1; \ldots; \lambda_N, s_N) = &\sum_{\sigma \in S_N} e^{-s_{\sigma(1)} \Phi\left(\sum_{m=1}^{N} \lambda_m\right)} \times \\
&\times \prod_{m=1}^{N-1} \Theta\left(s_{\sigma(m+1)} - s_{\sigma(m)}\right) e^{-\left[s_{\sigma(m+1)} - s_{\sigma(m)}\right] \Phi\left(\sum_{n=m+1}^{N} \lambda_{\sigma(n)}\right)}
\end{aligned} \tag{4.24}$$

As an example, we calculate the two-point case $\widetilde{Z}(\lambda_1, s_1; \lambda_2, s_2)$ [140]. Indeed, if we set $N = 2$ in Eq. (4.24) and consider the two possible permuted sequences: $\boldsymbol{s} = (s_1, s_2)$ and $\boldsymbol{s'} = (s_2, s_1)$, we recover Eq. (3.51). We can now use Eq. (4.24) to compute the correlation functions of $\overline{\xi}(t)$. Indeed, from Eq. (4.14) we obtain $\forall N \in \mathbb{N}$:

$$\left\langle \overline{\xi}(t_1) \ldots \overline{\xi}(t_{2N}) \right\rangle = \left[\prod_{m=1}^{2N} \int_0^{+\infty} \mathrm{d}s_m \right] \left\langle \prod_{m=1}^{2N} \xi(s_m) \right\rangle \left\langle \prod_{m=1}^{2N} \delta(t_m - T(s_m)) \right\rangle, \tag{4.25}$$

where we are allowed to factorise the ensemble average due to the independence of the noises ξ and η. This equation can be simplified by recalling the Wick theorem for the white Gaussian noise $\xi(s)$ [Eq. (2.98)] [110, 111]:

$$\begin{aligned}\left\langle \prod_{j=1}^{2N} \xi(t_j) \right\rangle &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \prod_{j=1}^{N} \langle \xi\big(t_{\sigma(2N-j+1)}\big)\xi\big(t_{\sigma(j)}\big)\rangle \\ &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \prod_{j=1}^{N} \delta\big(t_{\sigma(2N-j+1)} - t_{\sigma(j)}\big) \,. \end{aligned} \tag{4.26}$$

Consequently, we can substitute it in Eq. (4.25) and obtain:

$$\begin{aligned}\langle \overline{\xi}(t_1) \ldots \overline{\xi}(t_{2N}) \rangle &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \left[\prod_{m=1}^{2N} \int_0^{+\infty} \mathrm{d}s_m \right] \times \\ &\quad \times \prod_{j=1}^{N} \delta\big(s_{\sigma(2N-j+1)} - s_{\sigma(j)}\big) \left\langle \prod_{i=1}^{2N} \delta(t_i - T(s_i)) \right\rangle \\ &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \left[\prod_{m=1}^{N} \int_0^{+\infty} \mathrm{d}s_{\sigma(m)} \right] \times \\ &\quad \times \left\langle \prod_{j=1}^{N} \delta\big(t_{\sigma(2N-j+1)} - T\left(s_{\sigma(j)}\right)\big)\delta\big(t_{\sigma(j)} - T\left(s_{\sigma(j)}\right)\big) \right\rangle \end{aligned} \tag{4.27}$$

with N integrals being solved by using the delta functions obtained from the correlation functions of the white Gaussian noise $\langle \xi(s_1) \ldots \xi(s_{2N}) \rangle$. If we make a Laplace transform of Eq. (4.27), we obtain an expression involving $\widetilde{Z}(\lambda_1, s_1; \ldots; \lambda_N, s_N)$:

$$\begin{aligned}\left\langle \prod_{j=1}^{2N} \widetilde{\overline{\xi}}(\lambda_j) \right\rangle &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \left[\prod_{m=1}^{N} \int_0^{+\infty} \mathrm{d}s_m \right] \times \\ &\quad \times \widetilde{Z}\big(\lambda_{\sigma(1)} + \lambda_{\sigma(2N)}, s_1; \ldots; \lambda_{\sigma(N)} + \lambda_{\sigma(N+1)}, s_N\big) \,. \end{aligned} \tag{4.28}$$

Note that we changed the name of the integrated variables to simplify the notation. This formula can be further simplified by using Eq. (4.24). By substituting it and making a further permutation of the indices, we obtain:

$$\begin{aligned}\left\langle \prod_{j=1}^{2N} \widetilde{\overline{\xi}}(\lambda_j) \right\rangle &= \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \sum_{\sigma' \in S_N} \left[\prod_{m=1}^{N} \int_0^{+\infty} \mathrm{d}s_{\sigma'(m)} \right] \times \\ &\quad \times e^{-s_{\sigma'(1)} \Phi\left(\sum_{m=1}^{N} \lambda_m\right)} \prod_{m=1}^{N-1} \left[\Theta\big(s_{\sigma'(m+1)} - s_{\sigma'(m)}\big) \times \right. \\ &\quad \left. \times e^{-\left[s_{\sigma'(m+1)} - s_{\sigma'(m)}\right] \Phi\left(\sum_{n=m+1}^{N} \left(\lambda_{\sigma(\sigma'(n))} + \lambda_{\sigma(2N-\sigma'(n)+1)}\right)\right)} \right] , \end{aligned} \tag{4.29}$$

where the N integrals can then be solved by making suitable changes of variables. This leads to the following result for the Laplace transform of even multipoint functions of $\overline{\xi}(t)$:

$$\left\langle \widetilde{\overline{\xi}}(\lambda_1)\ldots\widetilde{\overline{\xi}}(\lambda_{2N})\right\rangle = \frac{1}{N2^N\Phi\left(\sum_{m=1}^{2N}\lambda_m\right)}\sum_{\sigma\in S_{2N}}\times$$
$$\times\sum_{\sigma'\in S_N}\prod_{m=1}^{N-1}\frac{1}{\Phi\left(\sum_{n=m+1}^{N}\left(\lambda_{\sigma(\sigma'(n))}+\lambda_{\sigma(2N-\sigma'(n)+1)}\right)\right)}. \tag{4.30}$$

We remark that odd multipoint correlation functions are zero; indeed, if we make the substitution $2N \to 2N+1$ in Eq. (4.25), we obtain an expression depending on the odd multipoint correlation functions of ξ, i.e., $\langle\xi(s_1)\ldots\xi(s_{2N+1})\rangle$, which vanish $\forall N\in\mathbb{N}$ [see Eq. (2.98)]. The corresponding quantities in time are derived by making the inverse Laplace transform of Eq. (4.30), which can be written as a $2N-$fold convolution:

$$\left\langle\overline{\xi}(t_1)\ldots\overline{\xi}(t_{2N})\right\rangle = \frac{1}{N2^N}K(t_1)\prod_{i=1}^{N-1}\delta(t_{i+1}-t_i)*_{2N}g(t_1,\ldots,t_{2N}) \tag{4.31}$$

where the function g is defined in Laplace space as follows:

$$\widetilde{g}(\lambda_1,\ldots,\lambda_{2N}) = \sum_{\sigma\in S_{2N}}\sum_{\sigma'\in S_N}\prod_{m=1}^{N-1}\frac{1}{\Phi\left(\sum_{n=m+1}^{N}\left(\lambda_{\sigma(\sigma'(n))}+\lambda_{\sigma(2N-\sigma'(n)+1)}\right)\right)}. \tag{4.32}$$

In Eq. (4.31) $K(t)$ is the memory kernel defined in Eq. (3.30). The set of Eqs. (4.31, 4.32) can be used to compute all the multipoint correlation functions of $\overline{\xi}(t)$ and consequently of $Y(t)$. It is straightforward to recover the two point case Eq. (4.30), whereas we compute below the four point function. First, we need to compute Eq. (4.32) in time space:

$$\begin{aligned} g(t_1,t_2,t_3,t_4) = [&K(t_1)\delta(t_2-t_1)\delta(t_3)\delta(t_4)+K(t_1)\delta(t_1-t_3)\delta(t_2)\delta(t_4)\\ &+K(t_2)\delta(t_2-t_4)\delta(t_1)\delta(t_3)+K(t_1)\delta(t_1-t_4)\delta(t_2)\delta(t_3)\\ &+K(t_2)\delta(t_2-t_3)\delta(t_1)\delta(t_4)+K(t_3)\delta(t_3-t_4)\delta(t_1)\delta(t_2)] \end{aligned} \tag{4.33}$$

and then solve the $\left.\frac{(2N)!N!}{N2^N}\right|_{N=2}=6$ convolution integrals of Eq. (4.31). This can be done explicitly, so that we derive:

$$\begin{aligned} \left\langle\overline{\xi}(t_1)\overline{\xi}(t_2)\overline{\xi}(t_3)\overline{\xi}(t_4)\right\rangle = [&K(\min(t_1,t_2))K(|t_1-t_2|)\delta(t_4-t_1)\delta(t_3-t_2)\\ &+K(\min(t_1,t_3))K(|t_1-t_3|)\delta(t_4-t_3)\delta(t_2-t_1)\\ &+K(\min(t_1,t_4))K(|t_1-t_4|)\delta(t_3-t_1)\delta(t_4-t_2)]\,. \end{aligned} \tag{4.34}$$

We verified that the same similar structure of the time dependent coefficients is shared by the six point correlation function. Considering the recursive structure evident from Eqs. (4.31, 4.32), we conjecture the following formula for the even correlation functions in time space (with $t_0 = 0$ kept fixed by the permutations):

$$\left\langle \prod_{j=1}^{2N} \overline{\xi}(t) \right\rangle = \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \prod_{m=1}^{N} \delta\big(t_{\sigma(2N-m+1)} - t_{\sigma(m)}\big) \times$$
$$\times \sum_{\sigma' \in S_N} \Theta\big(t_{\sigma(\sigma'(m))} - t_{\sigma(\sigma'(m-1))}\big)\, K\big(t_{\sigma(\sigma'(m))} - t_{\sigma(\sigma'(m-1))}\big)\,. \tag{4.35}$$

4.3.3 Comparison with the Scaled Brownian Motion

Once the underlying noise structure of the CTRW is revealed by Eqs. (4.31, 4.32, 4.35), a comparison with the corresponding multipoint correlation functions of the noise $\zeta(t)$ of the SBM reveals important common features of these two processes. Indeed, the correlation functions of $\zeta(t)$ are obtained straightforwardly by using the definition of Eq. (4.8) and the Wick theorem in Eq. (4.26):

$$\left\langle \prod_{j=1}^{2N} \zeta(t_j) \right\rangle = \frac{1}{N2^N} \sum_{\sigma \in S_{2N}} \prod_{m=1}^{N} K\big(t_{\sigma(m)}\big)\, \delta\big(t_{\sigma(2N-m+1)} - t_{\sigma(m)}\big)\,. \tag{4.36}$$

Odd correlation functions of $\zeta(t)$ are zero as for $\overline{\xi}(t)$. As an example to better clarify our discussion, we provide the four point correlation function:

$$\begin{aligned} \langle \zeta(t_1)\zeta(t_2)\zeta(t_3)\zeta(t_4)\rangle = {} & K(t_1)K(t_2)\delta(t_1-t_3)\delta(t_2-t_4) \\ & + K(t_1)K(t_3)\delta(t_1-t_2)\delta(t_3-t_4) \\ & + K(t_2)K(t_4)\delta(t_1-t_4)\delta(t_2-t_3). \end{aligned} \tag{4.37}$$

A first remark has to be done when we set $N = 2$, thus studying the two point correlation function. As already anticipated, this is the same for both the noises $\overline{\xi}(t)$ and $\zeta(t)$ and equal to Eq. (4.16), thus explaining why the corresponding integrated processes $Y(t)$ and $Y_*(t)$ share the same MSD. On the contrary, differences are evident only if we look at the higher order correlation functions. Thus, the two integrated processes are distinguishable only by looking at quantities that depend on them, e.g. their PDFs or their corresponding higher order correlation functions. Furthermore, by comparing Eqs. (4.35, 4.36), we can observe the same similar structure of the delta functions, typical of white Gaussian processes, but with a different correlated and mainly not factorizable time structure of the coefficients in the case of $\overline{\xi}(t)$, which depends on the difference between successive time in the ordered sequences. This ultimately causes its non Gaussian typical character. In fact, in the specific case of a constant memory kernel, for all times or in some scaling limit, the two noises coincide and reduce to a standard Brownian motion.

4.3.4 Characteristic Functional of the Noise

We conclude the characterisation of the noise $\overline{\xi}$ by deriving its characteristic functional: $G[u(s)] = \left\langle e^{i \int_0^{+\infty} u(s)\, \overline{\xi}(s)\, \mathrm{d}s} \right\rangle$. We remark that the brackets here denote an average over the realisations of both the noises ξ and η (or T equivalently). By substituting Eq. (4.14) into this expression, we obtain the following equation for a general test function $u(s)$:

$$\begin{aligned} G[u(s_1)] &= \left\langle \exp\left[i \int_0^{+\infty} u(s_1) \left(\int_0^{+\infty} \xi(s_2)\delta(s_1 - T(s_2))\,\mathrm{d}s_2 \right) \mathrm{d}s_1 \right] \right\rangle \\ &= \left\langle \exp\left[i \int_0^{+\infty} \xi(s_2) \left(\int_0^{+\infty} u(s_1)\delta(s_1 - T(s_2))\,\mathrm{d}s_1 \right) \mathrm{d}s_2 \right] \right\rangle \\ &= \left\langle \exp\left[i \int_0^{+\infty} \xi(s)\, h(s)\,\mathrm{d}s \right] \right\rangle . \end{aligned} \tag{4.38}$$

Here we changed the order of integration and introduced the auxiliary function h:

$$h(s) = \int_0^{+\infty} u(s')\delta(s' - T(s))\,\mathrm{d}s' , \tag{4.39}$$

which depends only on the process T, i.e., on the different realisations of the noise η. For each of these realisations, h is completely determined and it can be used as a test function in the characteristic functional of ξ. Thus, Eq. (4.38) can be simplified if we compute the average over ξ first. For a general Gaussian noise, either white or coloured, of correlation function $\langle \xi(s_1)\, \xi(s_2) \rangle = \sigma(s_2 - s_1)$, we obtain [179]:

$$\left\langle \exp\left[i \int_0^{+\infty} \xi(s)\, h(s)\,\mathrm{d}s \right] \right\rangle = \left\langle \exp\left[- \int_0^{+\infty} \int_0^{+\infty} h(s_1)h(s_2)\sigma(s_2 - s_1)\,\mathrm{d}s_1\ \mathrm{d}s_2 \right] \right\rangle . \tag{4.40}$$

We note that the average in the rhs of this equation is now only on the noise η. If we further substitute Eq. (4.39) back into Eq. (4.40), we obtain the final result:

$$G[u(r)] = \left\langle \exp\left[- \int_0^{+\infty} \int_0^{+\infty} u(r_1)u(r_2)\Lambda(r_1,\, r_2;\, T)\,\mathrm{d}r_1\ \mathrm{d}r_2 \right] \right\rangle , \tag{4.41a}$$

$$\Lambda(r_1,\, r_2;\, T) = \int_0^{+\infty} \int_0^{+\infty} \delta(r_1 - T(s_1))\delta(r_2 - T(s_2))\sigma(s_2 - s_1)\,\mathrm{d}s_1\ \mathrm{d}s_2 . \tag{4.41b}$$

We remark again that the ensemble average in the rhs of Eq. (4.41a) is only over the different stochastic paths of η. In the specific case of ξ being a white Gaussian noise, we have $\sigma(s_2 - s_1) = \delta(s_2 - s_1)$, so that Eq. (4.41b) simplifies as below:

$$\begin{aligned} \Lambda(r_1,\, r_2;\, T) &= \int_0^{+\infty} \int_0^{+\infty} \delta(r_1 - T(s_1))\delta(r_2 - T(s_2))\delta(s_2 - s_1)\,\mathrm{d}s_1\ \mathrm{d}s_2 \\ &= \int_0^{+\infty} \delta(r_1 - T(s))\delta(r_2 - T(s))\,\mathrm{d}s \\ &= \delta(r_2 - r_1) \int_0^{+\infty} \delta(r_1 - T(s))\,\mathrm{d}s , \end{aligned} \tag{4.42}$$

where we suitably employed the properties of the delta function. Substituting this result into Eq. (4.41a), we obtain the following characteristic functional:

$$\begin{aligned} G[u(r)] &= \left\langle \exp\left[- \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} u(r_1)u(r_2)\delta(r_2 - r_1)\delta(r_1 - T(s))\,\mathrm{d}s\ \mathrm{d}r_1\ \mathrm{d}r_2 \right] \right\rangle \\ &= \left\langle \exp\left[- \int_0^{+\infty} \int_0^{+\infty} [u(r)]^2\delta(r - T(s))\,\mathrm{d}s\ \mathrm{d}r \right] \right\rangle \\ &= \left\langle \exp\left[- \int_0^{+\infty} [u(T(s))]^2\,\mathrm{d}s \right] \right\rangle . \end{aligned} \tag{4.43}$$

Therefore, the characteristic functional of $\overline{\xi}$ can be expressed as a nonlinear functional of the process T. As a sanity check, we recover the PDF of a free diffusive CTRW. This is obtained by setting: $u(r) = \sqrt{2\,\sigma}\,k\,\Theta(t-r)$ in Eq. (4.43) and recalling Eq. (3.6):

$$\begin{aligned}\widehat{P}(k,t) &= \left\langle \exp\left[-2\,\sigma\,k^2 \int_0^{+\infty} \Theta(t-T(s))\,\mathrm{d}s\right]\right\rangle \\ &= \left\langle \exp\left[-2\,\sigma\,k^2 \int_0^{+\infty} [1-\Theta(s-S(t))]\,\mathrm{d}s\right]\right\rangle = \left\langle \exp\left[-2\,\sigma\,k^2\,S(t)\right]\right\rangle. \end{aligned} \tag{4.44}$$

As expected, we obtain the characteristic function of a time-changed free diffusion.

4.4 Models with External Forces

We now consider models of anomalous processes in the presence of external forces [78, 180, 157, 151, 181]. Let us first focus on the random walk picture of the CTRW and assume that external force fields, which depend on the position of the walker, only modify its dynamics during the jumps. In the continuum limit, these forces are naturally included in the Langevin equation of the process $X(s)$, i.e., Eq. (3.4a). Thus, if we consider the overdamped regime, we obtain the following equations [79]:

$$\dot{X}(s) = F(X(s)) + \sqrt{2\,\sigma}\,\xi(s), \tag{4.45a}$$

$$\dot{T}(s) = \eta(s), \tag{4.45b}$$

where the function $F(x)$ is required to satisfy the Lipschitz condition so that the time-changed solution Y exists and is unique (see Sec. 3.3.1). However, different scenarios may be observed in experiments where forces can modify the position of the walker also during the waiting times between different jumps, without changing the underlying waiting time distribution. For instance, we would expect this situation to occur for the motion of an organelle inside the cytoplasm of a freely migrating cell, or which is driven by an external field. This different situation turns out not to be easily described with the time-change technique, as it is not clear how to modify the Langevin subordinated equations in order to account for these further changes in the position variable. However, the characterization of the noise $\overline{\xi}(t)$ provided by Eqs. (4.31, 4.32), or equivalently by Eq. (4.35), enables us to describe it with a new class of models, defined with the Langevin equation:

$$\dot{Y}(t) = F(Y(t)) + \sqrt{2\,\sigma}\,\overline{\xi}(t). \tag{4.46}$$

As before, in order to prove the existence and uniqueness of a solution Y of this equation, we require F to satisfy the Lipschitz condition. This is proved by comparing Eq. (4.46) with the general time-changed stochastic differential Equation (4.1) of Ref. [120].

The difference between the dynamical behaviours generated by the two models becomes clear when we look at their simulated trajectories. In Figure 4.2 we plot the paths of $Y(t)$ obtained both via subordination of Eqs. (4.45a, 4.45b) (panel b) and via integration of Eq. (4.46) (panel a) for a linear viscous-like force $F(x) = -\gamma x$ with γ positive real constant. On the one hand, in the subordinated dynamics (dotted arrows, panel b) we observe time intervals where the corresponding anomalous process $Y(t)$ is constant, meaning that the walker, in the corresponding renewal picture, is waiting for the next jump to occur without any force being able to modify its position. On the other hand, during these same intervals the process $Y(t)$ generated by Eq. (4.46) is rapidly damped towards zero (dotted arrows, panel a), meaning that the walker is being driven by the external force. While indeed external forces act only during the jump times in the standard subordinated case, in the case of Eq. (4.46) they affect the dynamics of the system for all times, without intrinsically modifying the statistics of the waiting times and equivalently the relation between the number of steps and the physical time. We mention that another scenario involving external fields directly modifying the waiting time distribution of the random walk has been recently discussed in [182], but this formalism does not have an evident connection with ours.

Clearly, both in the standard subordinated picture and in the case of our new $\overline{\xi}-$driven processes the inclusion of a force changes the renewal picture for the position variable $Y(t)$, which can no longer be expressed as a superposition of i.i.d. position increments as in Eqs. (2.41, 2.48). These increments, indeed, now depend on the accumulated position up to the time before the jump, which also makes the two models discussed so far different. On the contrary, the process $T(s)$, i.e., the stochastic process of the jump times parametrized by the arc-length still represents a renewal process in both cases, since the waiting times are unaffected by the force.

In the following, we present a comparison of the MSD obtained from Eqs. (4.45a, 4.45b, 4.46) for a tempered Lévy stable subordinator as in Sec. 4.2.3 and for different choices of the external force $F(x)$. Specifically, we look at either a constant or a linear force. Except when explicitly stated we assume zero initial condition, so that the MSD coincides with the second order moment. We recall that the model of Eq. (4.46) defined with the time scaled noise $\zeta(t)$ instead of $\overline{\xi}(t)$ provides the same MSD.

4.4.1 Constant Force Case

We first look at the case of a constant homogeneous force field: $F(Y(t)) = F$ with $F \in \mathbb{R}^+$, for which Eq. (4.46) becomes:

$$\dot{Y}(t) = F + \sqrt{2\sigma}\,\overline{\xi}(t). \tag{4.47}$$

This equation can be solved formally for the exact trajectory of $Y(t)$:

$$Y(t) = F\,t + \sqrt{2\sigma}\int_0^t \overline{\xi}(\tau)\,\mathrm{d}\tau \tag{4.48}$$

and then used, together with Eq. (4.16), to derive the MSD:

$$\left\langle Y^2(t)\right\rangle = F^2\,t^2 + 2\,\sigma\int_0^t K(\tau)\,\mathrm{d}\tau \tag{4.49}$$

or equivalently in Laplace transform as a function of $\Phi(s)$:

$$\left\langle \widetilde{Y}^2(\lambda)\right\rangle = \frac{2F^2}{\lambda^3} + \frac{2\sigma}{\lambda\Phi\left(\lambda\right)}. \tag{4.50}$$

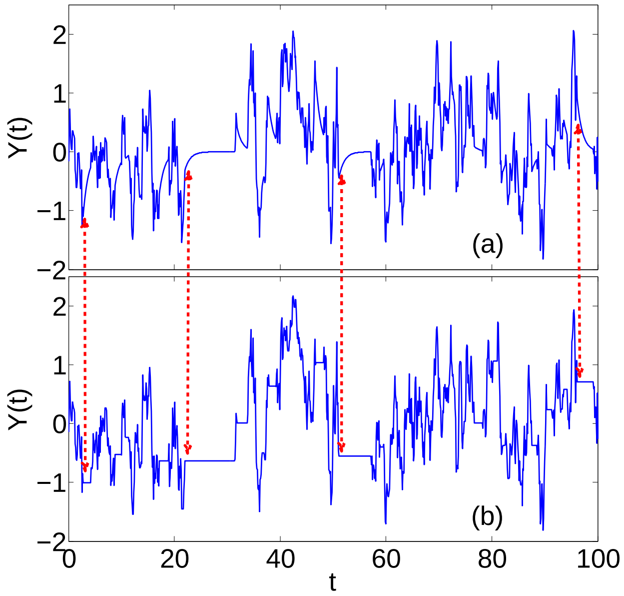

Figure 4.2: Simulated trajectories of a CTRW with a linear viscous-like force acting along its all time evolution [panel a, Eq. (4.46)] or acting only during the jumps [panel b, subordinated Eqs. (4.45a, 4.45b)]. Numerical algorithms are adapted from [81]. The difference on how the force affects the dynamics during trapping events is evident (dotted arrows): (a) the force acts on the particle, thus damping $Y(t)$ towards zero; (b) the force does not act, so that the particle gets physically stuck and $Y(t)$ is kept constant.

In the subordinated case, the MSD is computed with the same technique of Eq. (3.45) by using the specific variance $\langle X^2(s)\rangle = \left(F^2 s^2 + 2\,\sigma\, s\right)$. In Laplace space we obtain:

$$\left\langle \widetilde{Y}^2(\lambda)\right\rangle = \frac{2F^2}{\lambda\left[\Phi(\lambda)\right]^2} + \frac{\sigma^2}{\lambda\Phi(\lambda)}. \tag{4.51}$$

The Laplace inverse transform of both Eqs. (4.50, 4.51) is plotted, together with their corresponding scaling behaviours, in Figure 4.3 (main panel and inset respectively). In the small time limit, we find that they both share the same power-law scaling of Eq. (3.92). However, their scaling behaviour for long times is fundamentally different. On the one hand, Eq. (4.50) provides the long time scaling: $\langle Y^2(t)\rangle \sim F^2\, t^2$. Hence, the constant force in this limit induces a crossover from subdiffusive to ballistic dynamics. Examples of this nonlinear behaviour have been recently discovered in the dynamics of chromosomal loci, which exhibit rapid ballistic excursions from their fundamental subdiffusive dynamics, caused by the viscoelastic properties of the cytoplasm [31, 130]. Furthermore, it is evident that the exponential damping of the waiting time distribution does not affect the long time scaling, differently from the corresponding scaling of Eqs. (4.51), which turns out to be (Figure 4.3, inset):

$$\left\langle Y^2(t)\right\rangle \sim \begin{cases} \left(\frac{F\mu^{1-\alpha}}{\alpha}\right)^2 t^2 & \mu \neq 0 \\ \frac{2F^2}{\Gamma(1+2\alpha)} t^{2\alpha} & \mu = 0 \end{cases} \tag{4.52}$$

Thus, we find the same crossover to ballistic diffusion when $\mu \neq 0$, but with different μ-dependent scaling coefficients, whereas in the CTRW case ($\mu = 0$) this crossover pattern is lost and the power-law scaling is conserved, although with a different exponent.

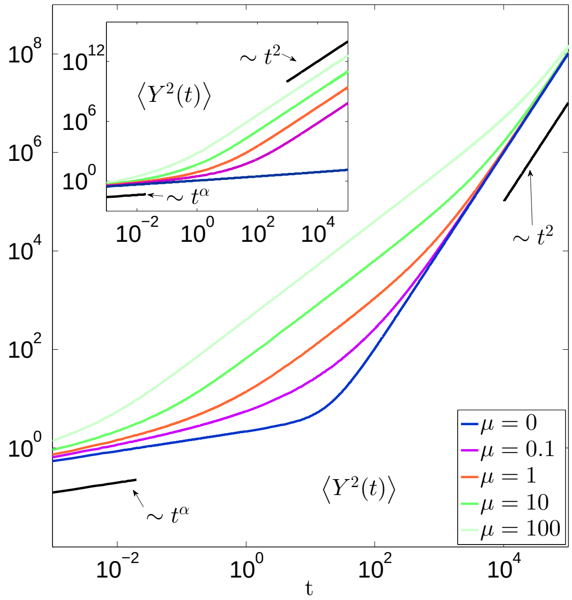

Figure 4.3: MSD of an anomalous process with tempered stable ($\alpha = 0.2$) distributed waiting times in the presence of a constant force acting throughout the all temporal evolution (main panel) or only during the jump times (inset). These two different scenarios are obtained with the $\bar{\xi}-$driven process or with the subordination technique, i.e., by numerical Laplace inverse transform of Eqs. (4.50, 4.51) respectively. The different long time scaling is evident: (main) the $\bar{\xi}-$driven process exhibits crossover to ballistic diffusion in all cases and without any dependence on the tempering parameter μ; (inset) the time-changed process exhibits crossover to ballistic diffusion with μ-dependent scaling coefficient when $\mu \neq 0$, whereas it still scales as a power-law with exponent $2\,\alpha$ for $\mu = 0$.

4.4.2 Harmonic Potential Case

We now consider an external harmonic potential, leading to a friction-like force: $F(Y(t)) = -\gamma Y(t)$ with γ real positive constant. Thus, Eq. (4.46) provides the following:

$$\dot{Y}(t) = -\gamma Y(t) + \sqrt{2\sigma}\,\bar{\xi}(t). \tag{4.53}$$

As before, we can solve formally Eq. (4.53) for the trajectory of $Y(t)$ and use it together with Eq. (4.16) to compute the Laplace transform of the corresponding MSD:

$$\left\langle \widetilde{Y}^2(\lambda) \right\rangle = \frac{2\sigma}{(\lambda + 2\,\gamma)\,\Phi\,(\lambda)}. \tag{4.54}$$

On the contrary, in the subordinated case we can proceed as in Eq. (3.45) by substituting: $\left\langle X^2(s) \right\rangle = \frac{\sigma}{\gamma}\left(1 - e^{-2\,\gamma\,s}\right)$. One can thus obtain the result below:

$$\left\langle \widetilde{Y}^2(\lambda) \right\rangle = \frac{\sigma}{\lambda\,[2\gamma + \Phi(\lambda)]}. \tag{4.55}$$

We plot in Figure 4.4 the numerical Laplace inverse transform of Eqs. (4.54, 4.55) (main panel and inset respectively), along with their scaling behaviour for small times. While the small time scaling is in both cases the same as in Eq. (3.92), we observe a different behaviour in the long time limit. Indeed, we find for Eq. (4.54) the following scaling laws:

$$\langle Y^2(t)\rangle \sim \begin{cases} \mu^{1-\alpha}/(\gamma\,\alpha) & \mu \neq 0 \\ \sigma\, t^{\alpha-1}/[\gamma\,\Gamma(\alpha)] & \mu = 0 \end{cases} \tag{4.56}$$

Thus, in the CTRW case the MSD decreases as a power-law towards zero. If we recall that this process is equal to the SBM up to the MSD, this is the same anomaly already reported in [169]. However, we also show that $Y(t)$ correctly converges to a plateau for $\mu \neq 0$, this being the expected dynamical behaviour of confined diffusion. By recalling that the waiting times are tempered Lévy stable distributed, the interpretation of the mentioned anomaly becomes clear. Indeed, the truncation of the power-law tails of the waiting time distribution is fundamental to let the system find a stationary state, so that the MSD can converge to a plateau, which is typical of confined diffusion. In fact, no damping of the tails is done in the CTRW case, meaning that very long trapping events may still happen with non zero, but small probability. Thus, if we wait long enough, i.e., in the long time limit, these events eventually occur. However, Eq. (4.53) establishes that the system is affected by the external linear force also during such events, which then damps all the oscillations of the system. This implies that the MSD should decrease to zero, because the system is not able to disperse and gets immobilized in $y = 0$. On the contrary, in the subordinated case the effect of the external force is stopped during the trapping events, so that the system does not get trapped in the zero position for long times. Indeed, the MSD for different values of μ share the same long-time plateau: $\langle Y^2(t)\rangle \sim \frac{\sigma}{\gamma}$.

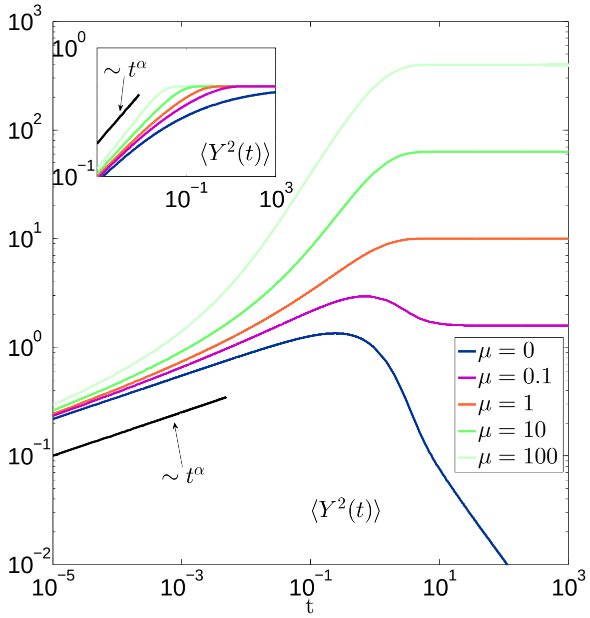

Figure 4.4: MSD of an anomalous process with tempered stable ($\alpha = 0.2$) distributed waiting times in the presence of a linear viscous force acting at all times (main panel) or only during the jump times (inset). These two cases are obtained with the $\overline{\xi}-$driven process or with the subordination technique, i.e., by numerical Laplace inversion of Eqs. (4.54, 4.55) respectively. Whereas for small times the two processes exhibit subdiffusive scaling, their long time behaviour differs: (main) the MSD of the $\overline{\xi}-$driven process decreases to zero in the CTRW case ($\mu = 0$), whereas it converges to a μ-dependent plateau for $\mu \neq 0$; (inset) in the subordinated case all the curves converge to the same plateau.

4.5 Outlook and Future Work

In this Chapter, we identified the underlying noise structure of a free diffusive CTRW with an arbitrary waiting time distribution and we defined its corresponding stochastic force. This enabled us to write a new Langevin equation, describing its dynamics directly in physical time and equivalently to the original formulation obtained with the subordination technique. We then derived a general formula, both in Laplace space and in physical time, providing all its multipoint correlation functions, which, although presenting the same time structure of white Gaussian processes, have time dependent coefficients with a non factorisable dependence on the memory kernel generated by the corresponding subordinator of the equivalent time-changed formulation. Thus, except for the specific choice of a constant kernel, which recovers the factorisability of these coefficients, but reduces the noise to a standard Brownian motion, our new $\bar{\xi}-$noise was shown to be naturally both non Gaussian and non Markov. We also provide its characteristic functional.

We then investigated the dynamics exhibited by processes driven by the $\bar{\xi}-$noise in the presence of external force fields and compared it with the one observed for usual subordinated processes. In general terms, we found that these processes belong to a new class of CTRW-like processes where external forces are exerted on the system at all times, i.e., both when the corresponding walker jumps or waits for the next jump to occur. Of course, this is different from the original subordinated model, where external forces are implicitly assumed to modify the dynamics only during the jump times. Consequently, during the typical trapping events of subdiffusive dynamics the anomalous process $Y(t)$ becomes constant on the one hand, when it is generated via subordination, or it is deterministically driven by the force on the other hand, when it is driven by the $\bar{\xi}-$noise in physical time.

Furthermore, we found that these processes have the same MSD of those obtained with the characteristic noise of the SBM with time dependent diffusion coefficient being a function of their memory kernel. This relation indeed both provides a better interpretation for the anomaly reported in Ref. [169] and show that the correct scaling of the MSD typical of confined motion can be obtained by choosing more general time transformations, which prevent an unbounded decay of the diffusion coefficient.

For future work it will be interesting to investigate the ageing and ergodicity breaking properties of our new class of processes [93]. This might further differentiate it from other anomalous processes such as the SBM. The properties of time-integrated observables of the $\bar{\xi}-$driven processes, which are expressed as functionals of their fluctuating trajectories, are also an open problem. For functionals of CTRWs, closed-form evolution equations can be derived that generalise the Feynman-Kac framework to anomalous processes [149, 150, 148, 147]. A further generalisation to anomalous processes with arbitrary waiting time distributions has been derived in Chapter 3 of this Thesis [140], which highlights the connection between the waiting time distribution and the memory kernel appearing in the fractional evolution equations [see Eq. (3.30)]. It will be relevant to study whether similar closed form equations can be formulated for functionals of processes driven by our $\bar{\xi}-$noise.

CHAPTER 5

Galilean Invariance of Anomalous Stochastic Processes

In this Chapter, we aim at understanding the role of Galilean invariance for stochastic diffusive processes, either normal or anomalous. In this context, a natural distinction between strong and weak Galilean invariance, i.e., invariance with respect to Galilean transformations of their characteristic Langevin equations of motion or of their corresponding PDFs respectively, needs to be introduced. By looking explicitly at the full Hamiltonian dynamics of the Mori-Zwanzig model [15, 183], we first clarify that the generalised Langevin equation naturally breaks strong Galilean invariance, because of the non commutativity of Galilean transformations and of the coarse-graining procedure. Due to its general character, this result also clarifies that strong Galilean invariance is not preserved by anomalous diffusive dynamics. On the contrary, we discuss how weak Galilean invariance is always preserved for normal diffusion, whereas one may or may not require it to be satisfied in the case of anomalous diffusive processes. This choice leads to different types of dynamical behaviour. On the one hand, if weak Galilean invariance is broken, we recover the ordinary CTRW; on the other hand, if weak Galilean invariance is preserved, we obtain a different dynamics that we characterise both in terms of fractional advection-diffusion equation (with a phenomenological argument first) and Langevin equations (for constant external force). Interestingly, we find that these equations correspond to those driven by the noise $\overline{\xi}$ of Chapter 4. We remark that our proof also holds in the superdiffusive regime. We conclude by discussing Fluctuation-Dissipation relations for these processes.

5.1 Galilean Invariance of Transport Processes

In this first section, we review the notion of Galilean invariance within the context of transport processes. Our general interest is to study how the equations of motion (EOMs) of a diffusive particle, either normal or anomalous, change in different inertial reference frames, which are related by suitable Galilean transformations [184]. In particular, we consider frames uniformly moving with a fixed constant velocity between themselves. Thus, we first review the specific form of the coordinate change connecting them. We then discuss how this general setup naturally leads to distinguish between different levels of invariance, namely a *strong* one, which occurs directly at the level of the EOMs, and a *weak* one, which instead only occurs at a statistical level, i.e., for averaged properties, like the position PDF.

Let us define two reference frames $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ with space, velocity and time coordinates (x, v, t) and $(\tilde{x}, \tilde{v}, \tilde{t})$ respectively. We note that clocks, i.e., the time variables, can always be synchronised, such that we have $\tilde{t} = t$. We further assume that $\widetilde{\mathcal{S}}$ is moving with constant velocity v_0 with respect to $\mathcal{S}$ and that its axes are oriented as those of $\mathcal{S}$, to which they coincide at the initial time $t_0 = 0$. Thus, the Galilean transformation connecting them is defined as:

$$\tilde{x} = x - v_0\, t, \tag{5.1}$$

$$\tilde{v} = v - v_0. \tag{5.2}$$

The corresponding transformation rules for the differentials follow straightforwardly:

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial x}{\partial \tilde{t}}\frac{\partial}{\partial x} + \frac{\partial v}{\partial \tilde{t}}\frac{\partial}{\partial v} + \frac{\partial t}{\partial \tilde{t}}\frac{\partial}{\partial t} = v_0\,\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \tag{5.3a}$$

$$\frac{\partial}{\partial \tilde{v}} = \frac{\partial x}{\partial \tilde{v}}\frac{\partial}{\partial x} + \frac{\partial v}{\partial \tilde{v}}\frac{\partial}{\partial v} + \frac{\partial t}{\partial \tilde{v}}\frac{\partial}{\partial t} = \frac{\partial}{\partial v}, \tag{5.3b}$$

$$\frac{\partial}{\partial \tilde{x}} = \frac{\partial x}{\partial \tilde{x}}\frac{\partial}{\partial x} + \frac{\partial v}{\partial \tilde{x}}\frac{\partial}{\partial v} + \frac{\partial t}{\partial \tilde{x}}\frac{\partial}{\partial t} = \frac{\partial}{\partial x}. \tag{5.3c}$$

Let us now consider the motion of a particle, whose position and velocity are described respectively by two functions of time $\widetilde{X}(t)$ and $\widetilde{V}(t)$ in $\widetilde{\mathcal{S}}$, with initial conditions given by: $\widetilde{V}(0) = V_0$ and $\widetilde{X}(0) = x_0$. If the physical system, composed of the moving particle and its environment, is specified, the Equations Of Motion (EOMs) of the particle can be derived in closed form. Let them be specified by two suitable functions $\widetilde{F}_{1,2}$, such that $\dot{\widetilde{V}}(t) = \widetilde{F}_1(t)$ and $\dot{\widetilde{X}}(t) = \widetilde{F}_2(t)$. Analogously in $\mathcal{S}$, the EOMs of the particle, whose velocity and position are described by the different functions $V(t)$ and $X(t)$, are given by the following: $\dot{V}(t) = F_1(t)$ and $\dot{X}(t) = F_2(t)$ for suitable $F_{1,2}$. Such EOMs are *strong Galilean Invariant* if the time evolution of $(\widetilde{X}(t), \widetilde{V}(t))$ and $(X(t), V(t))$ is the same in both $\widetilde{\mathcal{S}}$ and $\mathcal{S}$. Recalling the transformation rules of the particle's position and velocity Eqs. (5.1, 5.2), this is equivalent to the following relations being satisfied for all times: $\widetilde{F}_1(t) = F_1(t)$ and $\widetilde{F}_2(t) = F_2(t) + v_0$.

However, in experiments the full system is usually not known and only the statistical properties of the particle's position and velocity can be characterised. If we perform experimental measurements in the reference frame $\mathcal{S}$, quantities that can be fully determined are the position PDF $\widetilde{P}(\tilde{q}, t) = \langle \delta(\tilde{q} - \widetilde{X}(t)) \rangle$ and joint position-velocity PDF $\widetilde{P}(\tilde{q}, \tilde{r}, t) = \langle \delta(\tilde{q} - \widetilde{X}(t))\delta(\tilde{r} - \widetilde{V}(t)) \rangle$. Clearly, analogous quantities $P(q, t) = \langle \delta(q - X(t)) \rangle$ and $P(q, r, t) = \langle \delta(q - X(t))\, \delta(r - V(t)) \rangle$ can be determined if the measurements are performed in the other frame $\mathcal{S}$. We define the particle's motion, or equivalently the processes $(\widetilde{X}(t), \widetilde{V}(t))$, *weak Galilean Invariant* (GI) if these distributions are Galilean scalars, i.e., if they are invariant with respect to the change of reference frame $\tilde{\mathcal{S}} \to \mathcal{S}$. Clearly, imposing this condition induces a set of transformation rules also for the sample-space variables $\tilde{q}$

and $\tilde{r}$ [185]. Recalling Eqs. (5.1, 5.2) connecting $(X(t), V(t))$ to $(\widetilde{X}(t)\,, \widetilde{V}(t))$, we can write:

$$\begin{aligned}\widetilde{P}(\tilde{q},t) &= \Big\langle \delta\Big(\tilde{q} - \widetilde{X}(t)\Big)\Big\rangle \\ &= \Big\langle \delta\Big((\tilde{q} + v_0\, t) - \Big(\widetilde{X}(t) + v_0\, t\Big)\Big)\Big\rangle \\ &= \langle \delta(q - X(t))\rangle = P(q,t),\end{aligned} \tag{5.4}$$

which implies the transformation rule: $\tilde{q} = q - v_0\, t$ and the relation $P\,(q,t\,) = \widetilde{P}\,(q - v_0\, t, t\,)$, or equivalently in Fourier-Laplace space $P(k,\lambda) = \widetilde{P}(k, \lambda - i\, v_0\, k)$ [186]. Similarly, we can impose the invariance condition to the joint position-velocity PDF:

$$\begin{aligned}\widetilde{P}(\tilde{q},\tilde{r},t) &= \Big\langle \delta\Big(\tilde{q} - \widetilde{X}(t)\Big)\,\delta\Big(\tilde{r} - \widetilde{V}(t)\Big)\Big\rangle \\ &= \Big\langle \delta(q - X(t))\,\delta\Big((\tilde{r} + v_0) - \Big(\widetilde{V}(t) + v_0\Big)\Big)\Big\rangle \\ &= \langle \delta(q - X(t))\delta(r - V(t))\rangle = P(q,r,t),\end{aligned} \tag{5.5}$$

which now implies the additional transformation rule: $\tilde{r} = r - v_0$. In addition, we find the relation: $P(q,r,t) = \widetilde{P}(q - v_0\, t, r - v_0, t)$, or equivalently in double Fourier and Laplace space $P(p,k,\lambda) = e^{i\,k\,v_0}\widetilde{P}(p,k,\lambda - i\,p\,v_0)$. Thus, if the stochastic process observed in the experiments is weak GI, Eqs. (5.1, 5.2) both applies to the frame coordinates and to the sample-state variables of both the velocity and position [185]. Clearly, if the motion of the particle is strong GI, it is also weak GI. An example of this implication is represented by the Navier-Stokes equations [185]. In the rest of this Chapter, we will study the GI of both normal and anomalous diffusive processes in terms of the new definitions just presented.

5.2 Normal Diffusive Processes

In this section, we first look at the transformation properties of normal diffusive processes with respect to the Galilean coordinate change defined in Eqs. (5.1, 5.2). In this case, indeed, microscopic models of interacting particles are available, e.g., the Kac-Zwanzig model [15], from which an underdamped Langevin EOM of a tracer particle immersed in a heat bath can be formally derived. This enables us to study the problem of GI from first principles, i.e., by considering the system of bath particles and tagged one in a different Galilean frame and derive the corresponding transformed Langevin EOM. In this way, we show that the underdamped Langevin equation, which describes the dynamics of the tagged particle, cannot preserve strong GI (as already suggested by earlier studies [187, 188]), because the Galilean transformations and the coarse-graining procedure, which is employed to derive it, naturally do not commute. Nevertheless, we find that weak GI is instead always preserved for normal diffusive dynamics, because the change of reference frame does not modify the statistical properties of the stochastic driving force.

5.2.1 The underdamped Langevin equation

Let us first review the Kac-Zwanzig model of a thermal heat bath. This is a system of n harmonic oscillators with trajectories described in $\mathcal{S}$ by the curves $(x_i(t), v_i(t))$ and coupled with a tracer particle with trajectory $(X(t), V(t))$ [15, 183]. For simplicity, we neglect external forces and set all the masses equal to unity. This system is fully described in terms of its Hamiltonian function $H = H_B + H_T$, where $H_{B,T}$ are the Hamiltonian functions of the bath particles and of the tracer respectively, which are given as follows:

$$H_T = \frac{[V(t)]^2}{2} \qquad H_B = \sum_{j=1}^{n} \left\{ \frac{[v_j(t)]^2}{2} + \frac{\omega_j^2}{2} \left(x_j(t) - \frac{\gamma_j}{\omega_j^2} X(t) \right)^2 \right\}. \tag{5.6}$$

The corresponding EOMs of both the bath particles and the tracer can be derived straightforwardly by computing the corresponding Hamilton equations:

$$\dot{X}(t) = V(t) \qquad \dot{V}(t) = \sum_{j=1}^{n} \gamma_j \left(x_j(t) - \frac{\gamma_j}{\omega_j^2} X(t) \right), \tag{5.7}$$

$$\dot{x}_j(t) = v_j(t) \qquad \dot{v}_j(t) = -\omega_j^2 x_j(t) + \gamma_j X(t). \tag{5.8}$$

Our aim is then to integrate out the n variables of the bath particles and derive a closed EOM for the tracer velocity $V(t)$. We first solve the coupled Eqs. (5.8) for $x_j(t)$:

$$x_j(t) = x_j(0) \cos(\omega_j t) + v_j(0) \frac{\sin(\omega_j t)}{\omega_j} + \gamma_j \int_0^t \frac{\sin(\omega_j (t-s))}{\omega_j} X(s)\, \mathrm{d}s, \tag{5.9}$$

perform an integration by parts of the integral in the rhs of Eq. (5.9) and substitute the result obtained back into Eq. (5.7). We finally derive the following EOM [15, 183]:

$$\dot{V}(t) = -\int_0^t K(t-s)\, V(s)\, \mathrm{d}s + F(t) \tag{5.10}$$

where the memory kernel K and F are the deterministic functions defined below [15]:

$$K(t) = \sum_{j=1}^{n} \frac{\gamma_j^2}{\omega_j^2} \cos(\omega_j t), \tag{5.11a}$$

$$F(t) = \sum_{j=1}^{n} \gamma_j v_j(0) \frac{\sin(\omega_j t)}{\omega_j} + \sum_{j=1}^{n} \gamma_j \left(x_j(0) - \frac{\gamma_j}{\omega_j^2} X(0) \right) \cos(\omega_j t). \tag{5.11b}$$

The bath trajectories only enter the definition of F Eq. (5.11b) through their initial conditions, so that, if they were known, $F(t)$ could be determined exactly at each time. However, in realistic experiments it is not possible to obtain such information. Instead, through repeated independent measurements, each corresponding to a different set of initial conditions, one can determine statistically the properties of F. Thus, one usually consider a coarse-grained model, i.e., where we do not specify the details of the microscopic bath of particles, and assume F to be a stochastic driving force with specified statistical properties: (i) $\langle F(t) \rangle = 0$ and (ii) $\langle F(t_1)\, F(t_2) \rangle = k_B\, T\, K(t_1 - t_2)$, where k_B is the Boltzmann constant and T is the temperature of the system. These specific properties are determined by considering in the Kac-Zwanzig model an ensemble of initial states $(x_j(0), v_j(0))$ for the bath particles drawn from a canonical distribution at the same temperature T [15].

Let us now investigate this same system, before the coarse-graining, in the moving frame $\widetilde{\mathcal{S}}$. Let $(\widetilde{x}_i(t), \widetilde{v}_i(t))$ and $(\widetilde{X}(t), \widetilde{V}(t))$ be the position and velocity variables of the bath particles and of the tracer respectively. Clearly, an observer in this frame would repeat the same calculations before in the new coordinates and obtain the following equation:

$$\dot{\widetilde{V}}(t) = -\int_0^t K(t-s)\ \widetilde{V}(s)\ \mathrm{d}s\ + \widetilde{F}(t)\,, \tag{5.12}$$

$$\widetilde{F}(t) = \sum_{j=1}^{n} \gamma_j\, \widetilde{v}_j(0) \frac{\sin(\omega_j\, t)}{\omega_j} + \sum_{j=1}^{n} \gamma_j \left(\widetilde{x}_j(0) - \frac{\gamma_j}{\omega_j^2} \widetilde{X}(0) \right) \cos(\omega_j\, t), \tag{5.13}$$

where (i) $\widetilde{F}$ depends on the initial conditions in the new coordinates and (ii) K is the same as in Eq. (5.11a). According to the definition given in the previous section, strong GI is here preserved if the EOMs of the particle, i.e., Eqs. (5.12, 5.10), are the same in both reference frames. Recalling the transformation rules Eqs. (5.1, 5.2), which imply that the relations $\widetilde{x}_j(0) = x_j(0)$, $\widetilde{X}(0) = X(0)$ and $\widetilde{v}_j(0) = v_j(0) - v_0$ hold, we find, on the one hand, the following transformation rule for $\widetilde{F}$:

$$\widetilde{F}(t) = F(t) - v_0 \sum_{j=1}^{n} \frac{\gamma_j}{\omega_j} \sin(\omega_j\, t). \tag{5.14}$$

On the other hand, the frictional term explicitly transforms as $\int_0^t K(t-s)\ \widetilde{V}(s)\ \mathrm{d}s = \int_0^t K(t-s)\ V(s)\ \mathrm{d}s\ - v_0 \sum_{j=1}^{n} \frac{\gamma_j^2}{\omega_j^3} \sin(\omega_j\, t)$, thus leading to the transformed equation:

$$\dot{V}(t) = -\int_0^t K(t-s)\, V(s)\ \mathrm{d}s\ + F(t) - v_0 \sum_{j=1}^{n} \frac{\gamma_j}{\omega_j} \left(1 - \frac{\gamma_j}{\omega_j^2} \right) \sin(\omega_j\, t). \tag{5.15}$$

The third term appearing in the rhs of Eq. (5.15) quantitatively estimates the breaking of GI of the underdamped Langevin equation and it accounts for both a contribution from the frictional term, damping the frame velocity, and one from the bath particles trough the dependence on their initial velocities of the function F. However, in the specific case $\gamma_j = \omega_j^2$, this term disappears and the GI of the equation is restored. We show below that this is ultimately due to the GI of the Newton EOMs of both the bath particles and the tracer, which only holds if this condition is satisfied. Indeed, we can express the Hamiltonian functions in $\widetilde{\mathcal{S}}$ in terms of the coordinates of $\mathcal{S}$ (apart from constant terms not contributing to the EOMs) as below:

$$\widetilde{H}_T = H_T - v_0\, V(t) \tag{5.16a}$$

$$\widetilde{H}_B = H_B - v_0 \sum_{j=1}^{n} v_j(t) - v_0\, t \sum_{j=1}^{n} \omega_j^2 \left(1 - \frac{\gamma_j}{\omega_j^2} \right) \left(x_j(t) - \frac{\gamma_j}{\omega_j^2}\, X(t) \right). \tag{5.16b}$$

Here, GI is broken by both (i) the kinetic term, which is velocity dependent, and by (ii) the interaction potential between the tracer and the bath particles, which contributes to the third term in the rhs of Eq. (5.16b). Interestingly, this term is null when $\gamma_j = \omega_j^2$. The corresponding Hamiltonian EOMs are reported below:

$$\dot{X}(t) = V(t) - v_0 \quad \dot{V}(t) = \sum_{j=1}^{n} \gamma_j \left(x_j(t) - \frac{\gamma_j}{\omega_j^2} X(t) \right) - v_0\, t \sum_{j=1}^{n} \gamma_j \left(1 - \frac{\gamma_j}{\omega_j^2} \right), \tag{5.17}$$

$$\dot{x}_j(t) = v_j(t) - v_0 \quad \dot{v}_j(t) = -\omega_j^2\, x_j(t) + \gamma_j\, X(t) + v_0\, t\, \omega_j^2 \left(1 - \frac{\gamma_j}{\omega_j^2} \right). \tag{5.18}$$

As suggested, the condition $\gamma_j = \omega_j^2$ eliminates the time dependent term in the Newton Eqs. (5.17, 5.18), thus making them strong GI. Our calculation shows that the Mori-Zwanzig formalism leads to a strong GI underdamped Langevin equation if the interaction potential between the tracer and the bath particles is pairwise and dependent on the difference between their positions, i.e., it has the functional form $V(|x_j(t) - X(t)|)$ for $j = 1, \ldots, n$. In addition, it elucidates that both the transformation rule of F and the GI of the underdamped Langevin equation depend on the specific microscopic model of the heat bath. Thus, when $F(t)$ is described by a stochastic driving force, i.e., we neglect the details of the underlying heat bath, (i) an explicit transformation rule, like Eq. (5.14), for the stochastic random force cannot be defined and (ii) the resulting underdamped Langevin equation always breaks strong GI, due to both the contribution of the friction term damping the frame velocity and of the stochastic force. However, as the constant shift in Eq. (5.14) do not depend on the initial conditions of the bath particles, the stochastic forces in $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ have the same statistical properties, this suggesting that the processes $(X(t), V(t))$, described by the underdamped Langevin Eq. (5.10), satisfy weak GI.

5.2.2 Transport evolution equations

We consider Eq. (5.12) when $K(t) = \gamma\, \delta(t)$ [15] and $\widetilde{F}(t)$ is described by the stochastic force $\widetilde{\xi}(t)$, i.e., the EOMs of the particle's velocity and position $\left(\widetilde{X}(t), \widetilde{V}(t)\right)$ are given by:

$$\dot{\widetilde{X}}(t) = \widetilde{V}(t) \qquad \dot{\widetilde{V}}(t) = -\gamma\, \widetilde{V}(t) + \sqrt{2\,\sigma}\, \widetilde{\xi}(t)\,. \tag{5.19}$$

According to the earlier discussion, $\widetilde{\xi}(t)$ is a white Gaussian noise with $\left\langle \widetilde{\xi}(t) \right\rangle = 0$ and $\left\langle \widetilde{\xi}(t_1)\, \widetilde{\xi}(t_2) \right\rangle = \delta(t_2 - t_1)$. Moreover, we have $\sigma = \gamma\, k_B\, T$ (T is the temperature of the thermal bath). The frame independence of this coefficient is understood by recalling that $\sigma = \gamma\, v_{th}^2$, where v_{th} is the thermal velocity of a Brownian particle with mass $m = 1$, i.e., the square root of its averaged velocity without potential in the stationary state [189]. However, the uniform motion of the frame only contributes to the two-point velocity correlation function by a term $\propto e^{-\gamma\,(t_1+t_2)}$, i.e., null in the long-time limit. Thus, $v_{th} = \sqrt{k_b\, T}$ in both frames. The evolution equation of the joint position-velocity PDF of this process is given by the KK equation [189]:

$$\frac{\partial}{\partial t}\, \widetilde{P}(\tilde{q}, \tilde{r}, t) = -\frac{\partial}{\partial \tilde{q}}\, \tilde{r}\, \widetilde{P}(\tilde{q}, \tilde{r}, t) + \gamma \frac{\partial}{\partial \tilde{r}} \tilde{r}\, \widetilde{P}(\tilde{q}, \tilde{r}, t) + \sigma \frac{\partial^2}{\partial \tilde{r}^2} \widetilde{P}(\tilde{q}, \tilde{r}, t)\,. \tag{5.20}$$

We now want to understand how Eq. (5.20) transforms when we move to the reference frame $\mathcal{S}$ and verify that the processes described by Eqs. (5.19) are weak GI. By employing

Eqs. (5.1, 5.2), we obtain in the frame $\mathcal{S}$ the following Langevin equations:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -\gamma\, V(t) + \gamma\, v_0 + \sqrt{2\,\sigma}\,\xi(t). \tag{5.21}$$

Here, according to the previous discussion, $\xi(t)$ is a Gaussian white noise with the same statistical properties of $\widetilde{\xi}(t)$. As already suggested, this different noise term accounts for the effect of the Galilean transformation on the thermal bath particles and elucidates that the set of Langevin Eqs. (5.19) are not strong GI [187]. Starting from these equations, one can derive the corresponding KK equation, which is given as below:

$$\frac{\partial}{\partial t}P(q,r,t) = -\frac{\partial}{\partial q}\, r\, P(q,r,t) + \gamma\frac{\partial}{\partial r}(r - v_0)\, P(q,r,t) + \sigma\frac{\partial^2}{\partial r^2}P(q,r,t). \tag{5.22}$$

Clearly, if we perform the inverse change of variables Eqs. (5.1, 5.2) on Eq. (5.22), we find that Eq. (5.5) needs to be satisfied in order to consistently recover Eq. (5.20). This implies that weak Galilean invariance is here preserved. On the contrary, strong GI is ultimately broken by the frictional term damping the frame velocity.

We now pose this same question for the overdamped limit of Eqs. (5.19), which reads in $\widetilde{\mathcal{S}}$ as follows (with external position-dependent or constant force):

$$\gamma\,\dot{\widetilde{X}}(t) = \widetilde{F}\Big(\widetilde{X}(t)\Big) + \sqrt{2\,\sigma}\,\widetilde{\xi}(t)\,. \tag{5.23}$$

The corresponding position PDF is described by the Smoluchowski equation [189]:

$$\frac{\partial}{\partial t}\,\widetilde{P}(\tilde{q},t\,) = \left[-\frac{1}{\gamma}\frac{\partial}{\partial \tilde{q}}\,\widetilde{F}(\tilde{q}) + \frac{\sigma}{\gamma}\frac{\partial^2}{\partial \tilde{q}^2}\right]\widetilde{P}(\tilde{q},t\,)\,, \tag{5.24}$$

which can be obtained explicitly via an adiabatic expansion of Eq. (5.20) [189, 83]. As before, we want to study how Eq. (5.24) transforms with respect to the Galilean transformation defined by Eqs. (5.1, 5.2) and verify if also the overdamped process is weak GI. Similarly to the underdamped case, if we Galilean transform Eq. (5.23), we obtain:

$$\gamma\,\dot{X}(t) = \gamma\, v_0 + F(X(t)) + \sqrt{2\,\sigma}\,\xi(t), \tag{5.25}$$

where $F(q) = \widetilde{F}(q - v_0\, t)$ is the transformed force. Its corresponding evolution equation reads as follows:

$$\frac{\partial}{\partial t}P(q,t) = -v_0\frac{\partial}{\partial q}P(q,t) + \left[-\frac{1}{\gamma}\frac{\partial}{\partial q}F(q) + \frac{\sigma}{\gamma}\frac{\partial^2}{\partial q^2}\right]P(q,t). \tag{5.26}$$

Similarly to the KK Eq. (5.22), if we make the change of variables Eqs. (5.1,5.2) on Eq. (5.26), we find that Eq. (5.4) needs to hold in order to consistently recover Eq. (5.24).

In conclusion, for single particle motion both the Langevin equation, either underdamped or overdamped, and the corresponding evolution equations are not strong GI, this being fundamentally rooted in the stochastic description of the thermal bath. Indeed, by neglecting the details of the bath particles' motion, one automatically breaks strong GI, as the term generated by a Galilean transformation of the friction cannot be compensated by the bath. However, in both cases weak GI is still preserved.

5.3 Anomalous Diffusive Processes

In this Section we study the transformation properties of anomalous diffusive processes, based on the CTRW model. Differently from the case of normal processes, rigorous derivations of the subordinated Langevin Eqs. (3.4a, 3.4b) starting from models of interacting particles or Boltzmann-type equations are not available, so that we cannot make an analysis from first principles as in Sec. 5.2.1. Nevertheless, the non commutativity of the Galilean transformation and of the coarse-graining procedure still holds, as it does not depend on the type of stochastic dynamics exhibited by the tracer particle. Thus, similarly to normal processes, also anomalous subordinated Langevin equations and their corresponding fractional evolution equations do not preserve strong GI. However, the remarkable feature of such anomalous diffusive processes is that they can either preserve weak Galilean invariance or break it, thus leading to completely different dynamical behaviour. In the latter case we recover the ordinary CTRW, while in the former case we obtain a new class of anomalous stochastic processes, that we characterise in terms of their fractional evolution equations. Later in Sec. 5.5 we will show (for the specific case of an external constant force) that such processes are those of Chapter 4, described in terms of $\overline{\xi}$-driven Langevin equations. This elucidates that the weak Galilean invariance of fractional evolution equations and their transformation rules with respect to the Galilean transformation defined in Eqs. (5.1, 5.2) are intimately related to the role played by external forces in the CTRW dynamics.

Here instead, we first present a phenomenological derivation of a weak GI FFP equation, both in the subdiffusive and in the superdiffusive regime, which will naturally involve the fractional substantial derivative [149, 150], i.e., a fractional extension of the material derivative, instead of the usual RL operator. Within this discussion, we suggest the need to fix the equations describing weak GI anomalous processes earlier discussed in the literature. Indeed, these earlier equations generate non-physical PDFs in the subdiffusive regime and do not possess a Langevin formulation of their corresponding microscopic dynamics (discussed in details in Sec. 5.5). We conclude by discussing the transformation rules of the FKK equation under the coordinate change of Eqs. (5.1, 5.2) and the assumption of Galilean invariant PDFs. We remark that, in the following section, we will suppress the explicit notation of the Fourier-Laplace transforms used in the previous Chapters and we will denote them by simply using the independent variables (k, λ) respectively. The symbol $\sim$ will denote quantities evaluated in the moving reference frame $\widetilde{\mathcal{S}}$.

5.3.1 Subdiffusive Processes

In the earlier Refs. [78, 190, 9], a FFP equation for the position PDF of a weak GI anomalous subdiffusive walker was built ad-hoc, by exploiting the fact that the position PDFs in the two reference frames $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are related by Eq. (5.4), i.e., in Fourier-Laplace space by a linear shift of the Laplace variable. In this section, we first review this derivation and then show that the proposed equation do not transform correctly under the Galilean transformation of Eqs. (5.1, 5.2). Let us assume that the process $\widetilde{X}(t)$ in the comoving frame $\widetilde{\mathcal{S}}$ is described by a CTRW, i.e., its position PDF is given in Fourier-Laplace transform by the Montroll-Weiss equation with power-law tailed waiting time distribution [9]:

$$\widetilde{P}(k,\lambda) = \frac{\left\langle e^{ikX_0}\right\rangle}{\lambda + K_\alpha\, k^2\, \lambda^{1-\alpha}}, \tag{5.27}$$

where K_α is a generalised diffusion coefficient independent on the frame considered. This PDF is the solution of the FFP Eq. (2.29). For convenience, we rewrite it here:

$$\frac{\partial}{\partial t}\widetilde{P}(\tilde{q},t) = K_\alpha \frac{\partial^2}{\partial \tilde{q}^2}\, {}_0D_t^{1-\alpha}\widetilde{P}(\tilde{q},t)\,. \tag{5.28}$$

together with the RL operator ${}_0D_t^{1-\alpha}P(q,t)$, which is given by Eq. (2.30):

$${}_0D_t^{1-\alpha}P(q,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{P(q,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,. \tag{5.29}$$

According to Eq. (5.4), the GI position PDF $P(k,\lambda)$ in the laboratory frame $\mathcal{S}$ is obtained by shifting the Laplace variable λ:

$$P(k,\lambda) = \frac{\left\langle e^{ikX_0}\right\rangle}{\lambda - i\,v_0\,k + K_\alpha k^2\,(\lambda - i\,v_0\,k)^{1-\alpha}}. \tag{5.30}$$

In order to find the corresponding evolution equation, the standard technique consists in (i) taking the diffusive limit $(k,\lambda)\to(0,0)$ of Eq. (5.30) and (ii) making the inverse Fourier-Laplace transform of the result. In this limit, one can make the approximation: $(\lambda - i\,v_0\,k)^{1-\alpha} \approx \lambda^{1-\alpha}$ at the denominator of Eq. (5.30) and obtain the following PDF:

$$P(k,\lambda) = \frac{\left\langle e^{ikX_0}\right\rangle}{\lambda - i\,v_0\,k + K_\alpha k^2\,\lambda^{1-\alpha}}, \tag{5.31}$$

corresponding to the fractional evolution equation (denoted as MK in the following) [9]:

$$\frac{\partial}{\partial t}P(q,t) + v_0\,\frac{\partial}{\partial q}P(q,t) = K_\alpha \frac{\partial^2}{\partial q^2}\, {}_0D_t^{1-\alpha}P(q,t). \tag{5.32}$$

This same technique has been later applied to the case of an arbitrary asymptotic scaling behaviour of the waiting time distribution in the Montroll-Weiss equation [191]. Therein, a more general fractional advection-diffusion equation was derived and shown to coincide to Eq. (5.32) in the case of a power-law tailed waiting time distribution.

However, Eq. (5.32) do not transform as required under the coordinate change of Eqs. (5.1, 5.2), ultimately originating distributions with peculiar and non-physical features (see Sec. 5.4). Let us first see how to modify it, such that Eqs. (5.1, 5.2) transform it into Eq. (5.28). On the one hand, by using explicitly the transformations of Eqs. (5.1, 5.2), the lhs of Eq. (5.32) changes into:

$$\frac{\partial}{\partial t}P(q,t) + v_0\,\frac{\partial}{\partial q}P(q,t) = \frac{\partial}{\partial t}\widetilde{P}(\tilde{q},t)\,. \tag{5.33}$$

On the other hand, we obtain for the RL operator:

$${}_0D_t^{1-\alpha}P(q,t) = \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t} - v_0\frac{\partial}{\partial \tilde{q}}\right]\int_0^t \frac{P(\tilde{q}+v_0\,t,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,. \tag{5.34}$$

The result Eq. (5.34) differs from Eq. (5.29) because of (i) a further integral term with a spatial derivative at its front and (ii) the function $P(\tilde{q}+v_0\,t,\tau)$ inside the integral operator, which cannot be directly transformed into the rest frame PDF $\widetilde{P}(\tilde{q},\tau)$. These issues are intimately related to both the transformation of the sample-space variable and the non locality of the RL operator. Indeed, the sample-space variable q in $\mathcal{S}$ transforms into $\tilde{q}+v_0\,\tilde{t}$ in $\widetilde{\mathcal{S}}$, with the v_0-dependent term being evaluated at a later time than τ. Thus, the Galilean transformation naturally induces a retardation effect, i.e., $P(\tilde{q}+v_0\,t,\tau)=P(\tilde{q}+v_0\,(t-\tau)+v_0\,\tau,\tau)=\widetilde{P}(\tilde{q}+v_0\,(t-\tau)\,,\tau)$. The transformation rule of the RL operator then follows:

$$ {}_0D_t^{1-\alpha}P(q,t)=\frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial\tilde{q}}\right]\int_0^t\frac{\widetilde{P}(\tilde{q}+v_0\,(t-\tau)\,,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,. \tag{5.35}$$

Putting together the transformation rules Eqs. (5.33 5.35), the transformed FFPE becomes:

$$\frac{\partial}{\partial t}\widetilde{P}(\tilde{q},t\,)=\frac{K_\alpha}{\Gamma(\alpha)}\frac{\partial^2}{\partial\tilde{q}^2}\left[\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial\tilde{q}}\right]\int_0^t\frac{\widetilde{P}(\tilde{q}+v_0\,(t-\tau)\,,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,, \tag{5.36}$$

which is different from the previously derived Eq. (5.28), both for the presence of the third order space derivative in front of the fractional integral and for the shift in the integrand PDF. Indeed, we can postulate the FFPE correctly transforming under Eqs. (5.1, 5.2), if we account for the rules Eqs. (5.33, 5.35) just derived. Such equation, which will be called weak GI FFPE from now, is given by:

$$\frac{\partial}{\partial t}P(q,t)+\frac{\partial}{\partial q}v_0\,P(q,t)=K_\alpha\frac{\partial^2}{\partial q^2}\mathcal{D}_t^{1-\alpha}P(q,t), \tag{5.37}$$

where $\mathcal{D}_t^{1-\alpha}P(q,t)$ denotes the fractional substantial derivative [149, 150], which is defined in Fourier-Laplace space as $\mathcal{F}\left\{\mathcal{L}\left\{\mathcal{D}_t^{1-\alpha}P(q,t)\right\}\right\}(k,\lambda)=[\lambda+i\,v_0\,k]^{1-\alpha}P(k,\lambda)$, or equivalently in (q,t) space:

$$\mathcal{D}_t^{1-\alpha}P(q,t)=\frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial q}\right]\int_0^t\frac{P(q-v_0(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,. \tag{5.38}$$

As a sanity check, Eq. (5.37, 5.38) can be shown to transform into Eq. (5.28, 5.29) by using Eqs. (5.33, 5.35) and to reduce to the usual FP Eq. (5.26) in the Brownian limit $\alpha=1$ (with $F(q)=0$ and $K_1=\sigma/\gamma$). Indeed, as the time derivative in Eq. (5.38) also operates on the shift term, we obtain: $\left[\frac{\partial}{\partial t}+v_0\,\frac{\partial}{\partial q}\right]\int_0^t P(q-v_0\,(t-\tau),\tau)\,\mathrm{d}\tau\;=P(q,t)+\int_0^t\left[\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial q}\right]P(q-v_0(t-\tau),\tau)=P(q,t)$, because $\frac{\partial}{\partial t}P(q-v_0(t-\tau),\tau)=-v_0\,\frac{\partial}{\partial q}P(q-v_0(t-\tau),\tau)$.

Since its first derivation in the context of the generalised KK equation of a system of weakly damped inertial particles [149, 150], where it was caused by the Galilean invariance of the resulting evolution equation, the fractional substantial derivative Eq. (5.38) has been related in several different systems to the existence of a strong space-time coupling in the observed dynamical processes. For instance, its generalised version appeared in the derivation of the FFK Eq. (3.3) [146, 147, 148, 140], which describes the time evolution of the joint PDF of a CTRW and one of its observables, i.e., a general functional of its stochastic path, for which such a coupling arises naturally. Indeed, if τ is the waiting time between two successive jumps of a CTRW, placed at the position x after the earlier jump, its corresponding functional $U(x)$ changes of $\tau\, U(x)$ during this interval. Thus, the value of the functional after the second jump will depend on both the corresponding position displacement and the waiting time τ, this clearly elucidating the existence of such a natural space-time coupling in the dynamics of CTRWs' functionals. Similar fractional substantial derivatives are also used in the case of Lévy walks [192, 193, 194], where this coupling is imposed to avoid the instantaneous jumps characterising Lévy flights. Differently from all these cases, in our derivation of Eq. (5.38) the space-time coupling is naturally induced by the assumption of a weak GI position PDF. We note that no Langevin description of the microscopic dynamics leading to the Eqs. (5.32, 5.37) has been found so far. Indeed, only the microscopic dynamics of CTRWs can be formulated in terms of subordinated equations [79], though the corresponding FFPE is different from both Eqs. (5.32, 5.37).

We conclude this section by investigating the transformation rule of the fractional KKE [149, 150, 146, 148, 147, 140]. We will both confirm the GI of its fractional derivative and show that, similarly to the case of normal diffusive processes, the breaking of GI of the equation is determined by the frictional term damping the frame velocity. We consider two auxiliary processes $\widetilde{Y}(s)$ and $\widetilde{T}(s)$, satisfying the equations [79, 140]:

$$\dot{\widetilde{Y}}(s) = -\gamma\, \widetilde{Y}(s) + \sqrt{2\,\sigma}\, \widetilde{\xi}(s) \qquad\qquad \dot{\widetilde{T}}(s) = \widetilde{\eta}(s) \tag{5.39}$$

with $\widetilde{\xi}(s)$ being a Gaussian white noise with correlation function $\left\langle \widetilde{\xi}(s_1)\, \widetilde{\xi}(s_2) \right\rangle = \delta(s_2 - s_1)$ and $\widetilde{\eta}(s)$ being a Lévy stable noise of parameter $0 < \alpha < 1$ [101, 107]. The anomalous process is obtained by time-changing the process $\widetilde{Y}$, i.e., by setting $\widetilde{V}(t) = \widetilde{Y}\left(\widetilde{S}(t)\right)$, where we define $\widetilde{S}$ as the inverse of the process $\widetilde{T}$ in terms of the first passage time problem below:

$$\widetilde{S}(t) = \inf_{s>0} \left\{ s : \widetilde{T}(s) > t \right\}. \tag{5.40}$$

As the process $\widetilde{V}(t)$ is the velocity of the anomalous particle, we can define the corresponding position as a functional of the $\widetilde{V}-$paths:

$$\widetilde{X}(t) = \int_0^t \widetilde{V}(\tau)\, \mathrm{d}\tau\, . \tag{5.41}$$

The corresponding fractional KKE for the joint position-velocity PDF can be derived by using functional methods and the stochastic calculus of time-changed processes [140]:

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{P}(\tilde{q}, \tilde{r}, t) = &-\frac{\partial}{\partial \tilde{q}}\tilde{r}\, \widetilde{P}(\tilde{q}, \tilde{r}, t) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\gamma \frac{\partial}{\partial \tilde{r}}\tilde{r} + \sigma \frac{\partial^2}{\partial \tilde{r}^2}\right] \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \tilde{q}}\tilde{r}\right] \int_0^t \frac{\widetilde{P}(\tilde{q} - \tilde{r}\,(t-\tau)\,, \tilde{r}, \tau)}{\left(\tilde{t} - \tau\right)^{1-\alpha}}\, \mathrm{d}\tau\, . \end{aligned} \tag{5.42}$$

Similar arguments as before can be used to make a Galilean transformation of this equation.

Let us look in details at the transformation rule for the fractional substantial derivative:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial \tilde{q}}\tilde{r}\right]\int_0^t \frac{\widetilde{P}(\tilde{q}-\tilde{r}\,(t-\tau)\,,\tilde{r},\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau = \\
&\qquad\left[\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial q}+\frac{\partial}{\partial q}(r-v_0)\right]\int_0^t \frac{\widetilde{P}(q-v_0\,t-(r-v_0)\,(t-\tau)\,,r-v_0,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau = \\
&\qquad\qquad\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial q}r\right]\int_0^t \frac{P(q-r(t-\tau),r,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau
\end{aligned}
\tag{5.43}
$$

where in the second line we used the fact that $\widetilde{P}(q-v\,t-(r-v_0)\,(t-\tau)\,,r-v_0,\tau) = \widetilde{P}(q-r(t-\tau)-v_0\,\tau,r-v_0,\tau) = P(q-r(t-\tau),r,\tau)$ due to Eq. (5.5). If we use this result, we obtain the transformed equation:

$$
\begin{aligned}
\frac{\partial}{\partial t}P(q,r,t) &= -\frac{\partial}{\partial q}r\,P(q,r,t) \\
&+\frac{1}{\Gamma(\alpha)}\left[\gamma\frac{\partial}{\partial r}(r-v_0)+\sigma\frac{\partial^2}{\partial r^2}\right]\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial q}r\right]\int_0^t \frac{P(q-r(t-\tau),r,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau
\end{aligned}
\tag{5.44}
$$

which, similarly to the normal KKE, is not GI due to the frictional term. Furthermore, Eq. (5.44) is the KKE of the process $(V(t),X(t))$, where $V(t)=Y(S(t))$ is an anomalous process determined by the auxiliary processes $V(s)$ and $T(s)$, satisfying the equations:

$$
\dot{V}(s) = -\gamma\,V(s)+\gamma\,v_0+\sqrt{2\,\sigma}\,\xi(s), \qquad\qquad \dot{T}(s)=\eta(s), \tag{5.45}
$$

with the time-change S being defined as before in terms of the inverse of the process T

$$
S(t)=\inf_{s>0}\{s:T(s)>t\}, \tag{5.46}
$$

and the position process X is a functional of the stochastic paths of Y:

$$
X(t)=\int_0^t Y(\tau)\,\mathrm{d}\tau\,. \tag{5.47}
$$

Similarly to the case of normal processes, the noises $\xi(t)$ and $\eta(s)$ have the same statistical properties of the corresponding stochastic forces in $\widetilde{\mathcal{S}}$, though being defined differently. This explicit transformation of the noise terms signals that the subordinated Langevin equations, together with the corresponding fractional evolution equations, do not preserve GI. However, our calculation provides explicit transformation rules for both the fractional KK Eq. (5.42) and the fractional FP Eq. (5.28) under the Galilean transformation in Eqs. (5.1-5.2) and elucidates the role played by the fractional substantial derivative Eq. (5.38) in preserving GI of the corresponding PDFs.

5.3.2 Superdiffusive Processes

Similarly to the subdiffusive case just discussed, the MK Eq. (5.32), here with characteristic parameter $1<\alpha<2$, has been first proposed as a GI fractional evolution equation for superdiffusive dynamics by using an approximation scheme first proposed by Balescu [195, 196, 197], which, however, do not account for the correct transformation rule of the RL operator Eq. (5.34). In this section, we will first review this approximation procedure

[195, 196] and then show how to appropriately modify it in order to derive Eq. (5.37) ($1 < \alpha < 2$). By doing so, we will clarify important issues regarding the derivation of evolution equation for processes driven by general, either Gaussian or non Gaussian, coloured noise. We remark that the validity of Eq. (5.37) is further supported by the fact that its solution Eq. (5.30) is a well defined PDF also in the superdiffusive regime (Sec. 5.4).

Let us consider the overdamped Langevin equation in $\mathcal{S}$:

$$\dot{X}(t) = v_0 + \sqrt{2\,\sigma}\,\xi(t) \tag{5.48}$$

with $\xi(t)$ being a general coloured noise with two-point power-law correlation function:

$$\langle \xi(t_1)\xi(t_2)\rangle = \frac{1}{\Gamma(\alpha-1)}\frac{1}{|t_1-t_2|^{2-\alpha}}. \tag{5.49}$$

If $\xi(t)$ is further assumed to be Gaussian, then Eq. (5.49) is sufficient to specify its properties and consequently those of X. For instance, in this case one can compute the position PDF of X by using the characteristic functional of $\xi(t)$ [198, 199], which is defined as [179]:

$$\begin{aligned} G[k(s)] &= \left\langle \exp\left[i\int_0^\infty k(s)\,\xi(s)\,\mathrm{d}s\right]\right\rangle \\ &= \exp\left[-\frac{1}{\Gamma(\alpha-1)}\int_0^\infty\int_0^\infty k(s_2)k(s_1)|s_2-s_1|^{\alpha-2}\,\mathrm{d}s_1\,\mathrm{d}s_2\right] \end{aligned} \tag{5.50}$$

with $k(s)$ being an arbitrary test function. By recalling that Eq. (5.48) can be integrated analytically, one can use Eq. (5.50) to compute the characteristic function of X, i.e., $P(k,t) = e^{i\,k\,X(t)}$. In details, we find:

$$\begin{aligned} P(k,t) &= \exp\left[i\,k\,(X_0+v_0\,t)\right]\left\langle \exp\left[i\,k\sqrt{2\,\sigma}\int_0^t \xi(\tau)\,\mathrm{d}\tau\right]\right\rangle \\ &= \exp\left[i\,k\,(X_0+v_0\,t)\right]G[\sqrt{2\,\sigma}\,k\,\Theta(t-s)] \\ &= \exp\left[i\,k\,(X_0+v_0\,t)\right]\,\exp\left[-k^2\frac{2\,\sigma}{\Gamma(\alpha-1)}\int_0^t\mathrm{d}s_1\int_0^t\mathrm{d}s_2\,|s_2-s_1|^{\alpha-2}\right] \\ &= \exp\left[i\,k\,(X_0+v_0\,t)-k^2 D_\alpha t^\alpha\right] \end{aligned} \tag{5.51}$$

with $D_\alpha = \frac{2\,\sigma}{\Gamma(\alpha+1)}$ being a generalized diffusion coefficient. By taking its Fourier inverse transform, we obtain a Gaussian PDF with time-dependent variance:

$$P(q,t) = \frac{1}{\sqrt{8\,\pi\,D_\alpha\,t^\alpha}}e^{-\frac{(q-X_0-v_0\,t)^2}{8\,D_\alpha\,t^\alpha}} \tag{5.52}$$

This result is further confirmed if we compute its corresponding FP equation. Specifically, we want to derive the time evolution of the stochastic function: $F(q,t) = \delta(q-X(t))$. We then take its time derivative and assume the Stratonovich interpretation. We obtain:

$$\frac{\partial}{\partial t}F(q,t) = -\frac{\partial}{\partial q}\delta(q-X(t))\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -[v_0+\sqrt{2\,\sigma}\,\xi(t)]\frac{\partial}{\partial q}F(q,t) \tag{5.53}$$

after substituting Eq. (5.48). If we now take the ensemble average of Eq. (5.53) and recall that $P(q,t) = \langle F(q,t)\rangle$, we obtain the following equation:

$$\frac{\partial}{\partial t}P(q,t) = -v_0\frac{\partial}{\partial q}P(q,t) - \sqrt{2\,\sigma}\frac{\partial}{\partial q}\left\langle \xi(t)F(q,t)\right\rangle, \tag{5.54}$$

where the quantity $\langle \xi(t)F(q,t)\rangle$ can be computed by recalling Novikov's theorem [112]:

$$\begin{aligned}\langle \xi(t)F(q,t)\rangle &= \int_0^t \langle \xi(t)\xi(t')\rangle \left\langle \frac{\delta F(q,t)}{\delta\xi(t')}\right\rangle \mathrm{d}t' \\ &= \frac{-1}{\Gamma(\alpha-1)}\int_0^t |t-t'|^{\alpha-2}\left\langle\frac{\partial}{\partial q}F(q,t)\frac{\delta X(t)}{\delta\xi(t')}\right\rangle \mathrm{d}t' = -\sqrt{2\,\sigma}D(t)\frac{\partial}{\partial q}P(q,t),\end{aligned} \tag{5.55}$$

where we used the relation $\frac{\delta X(t)}{\delta\xi(t')} = \sqrt{2\,\sigma}\,\Theta(t-t')$ and we defined the time-dependent diffusion coefficient $D(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-t')^{\alpha-2}\,\mathrm{d}t' = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. If we substitute this result in Eq. (5.54), we obtain the following FP equation:

$$\frac{\partial}{\partial t}P(q,t) = -v_0\frac{\partial}{\partial q}P(q,t) + 2\,\sigma D(t)\frac{\partial^2}{\partial q^2}P(q,t), \tag{5.56}$$

whose solution is consistently given by Eq. (5.52). Interestingly, the FP Eq. (5.56) is the same recently derived for the SBM [169, 200].

On the contrary, if $\xi(t)$ is non Gaussian, Eq. (5.55) does not hold and one needs to either compute $\langle \xi(t)\,F(q,t)\rangle$ explicitly by summing all the higher order correlations of the noise [112] or employ a suitable approximation scheme to derive the corresponding evolution equation. In the specific case of Eq. (5.48), we can derive the exact equation [195, 196, 197]:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial q}\right)P(q,t) = \frac{\partial^2}{\partial q^2}\int_0^t \langle \xi(t)\xi(t')\rangle\, P(q-\Delta(t,t'),t')\,\mathrm{d}t' \tag{5.57}$$

where the following fluctuating term explicitly appears in the integral in the rhs of Eq. (5.57):

$$\Delta(t,t') = v(t-t') + \int_{t'}^t \xi(t'')\,\mathrm{d}t''\,. \tag{5.58}$$

The presence of this term makes Eq. (5.57) difficult to be solved exactly. Thus, following Balescu, we can consider a local approximation, which consists in neglecting the fluctuating term Eq. (5.58). Differently from the original literature [201], we keep the shift term, which is a contribution that cannot be neglected for general fields. We obtain:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial q}\right)P(q,t) &= \frac{\partial^2}{\partial q^2}\int_0^t \langle \xi(t)\xi(\tau)\rangle\, P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \\ &= \frac{K_\alpha}{\Gamma(\alpha-1)}\frac{\partial^2}{\partial q^2}\int_0^t \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{2-\alpha}}\,\mathrm{d}\tau\end{aligned} \tag{5.59}$$

However, this equation is not yet in the form of Eq. (5.37). To this aim, we need to rewrite it in terms of the RL operator Eq. (5.29). This indeed requires to adopt a regularization scheme for the memory integral [150]. We then rewrite the integral as follows:

$$\int_0^t \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{2-\alpha}}\,\mathrm{d}\tau = \lim_{\Delta\to 0}\int_0^t K_\Delta(t-\tau)P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \tag{5.60}$$

where we introduce the auxiliary function:

$$K_\Delta(t) = \begin{cases} K_1(t) & t < \Delta \\ |t|^{\alpha-2} & t > \Delta \end{cases} \tag{5.61}$$

With these definitions, we are allowed to treat separately the two integrals:

$$\begin{aligned}\int_0^t K_\Delta(t-\tau)P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \,&= \int_{t-\Delta}^t K_1(t-\tau)P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \\ &+ \int_0^{t-\Delta} \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{2-\alpha}}\,\mathrm{d}\tau\end{aligned} \tag{5.62}$$

By recalling that $(t-\tau)^{\alpha-2} = \frac{1}{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}(t-\tau)^{\alpha-1}$, we can rewrite the second integral as

$$\begin{aligned}\int_0^{t-\Delta} \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{2-\alpha}}\,\mathrm{d}\tau \,&= \frac{1}{\alpha-1}\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial q}\right]\int_0^{t-\Delta} \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \\ &- \frac{1}{\alpha-1}\frac{1}{\Delta^{1-\alpha}}P(q-v\Delta,t-\Delta)\end{aligned} \tag{5.63}$$

If we substitute Eq.(5.63) into Eq. (5.62) and assume $K_1(t) = K_1$ (constant), we obtain:

$$\begin{aligned}\int_0^t K_\Delta(t-\tau)P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \,&= \frac{1}{\alpha-1}\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial q}\right]\int_0^{t} \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \\ + K_1\int_{t-\Delta}^t P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \,&- \frac{1}{\alpha-1}\frac{1}{\Delta^{1-\alpha}}P(q-v\Delta,t-\Delta) \\ &- \frac{1}{\alpha-1}\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial q}\right]\int_{t-\Delta}^{t} \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\end{aligned} \tag{5.64}$$

Finally, we take the limit $\Delta \to 0$ to recover the fractional integral. If we assume smoothness of the PDF in this limit, i.e., $P(q-v\Delta,t-\Delta) \approx P(q,t)$, we find:

$$\int_{t-\Delta}^t P(q-v(t-\tau),\tau)\,\mathrm{d}\tau \,\approx \Delta\, P(q,t), \tag{5.65a}$$

$$\int_{t-\Delta}^t \frac{P(q-v(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \,\approx P(q,t)\Delta^\alpha. \tag{5.65b}$$

In the rhs side of Eq. (5.64) only the first term, i.e., the fractional substantial derivative, survives in the limit, together with the term: $K_1\Delta P(q,t) - \frac{1}{\alpha-1}\frac{1}{\Delta^{1-\alpha}}P(q,t)$. This term always converges to zero in the regime $1 < \alpha < 2$, whereas it diverges in the subdiffusive regime. However, in this second case we can set $K_1 = \frac{1}{\alpha-1}\frac{1}{\Delta^{2-\alpha}}$, so that the overall contribution still converges to zero. In conclusion, our calculation provides a justification of Eq. (5.37) in the superdiffusive regime, but not its correct Langevin description.

5.4 Comparison between MK and weak GI processes

In this section, we investigate the effect of the approximation proposed in the earlier Refs. [78, 190, 9], i.e., we will compare both moments and position PDFs of MK and weak GI processes, respectively described by the FFP Eqs. (5.32, 5.37) with corresponding solutions in Laplace-Fourier space given by Eqs. (5.30, 5.31). In details, we will (i) show that such approximation causes dramatic changes in the statistical properties of weak GI processes, thus leading to the peculiar, even unphysical in the subdiffusive regime, features of MK processes, and (ii) identify a suitable parameter and its corresponding scaling limit for which the position PDFs Eqs. (5.30, 5.31) asymptotically coincide. Except in this particular regime, we remark that the transformation rule Eq. (5.35) needs to be taken into account in order to define well-behaved anomalous weak GI processes. For simplicity, we will denote in the following $X_i(t)$ with $i = 1, 2$ MK and weak GI processes respectively.

Moments

Let us first compare the moments of MK and weak GI processes. To this aim, we note that Eqs. (5.30, 5.31) have the general form: $P_i(k, \lambda) = \langle e^{ikx_0} \rangle / D_i(k, \lambda)$, with $D_i(k, \lambda)$ being:

$$D_i(k,\lambda) = \begin{cases} (\lambda - i\,v\,k) + K_\alpha\, k^2\, \lambda^{1-\alpha} & i = 1 \\ (\lambda - i\,v\,k) + K_\alpha\, k^2\, (\lambda - i\,v\,k)^{1-\alpha} & i = 2 \end{cases} \tag{5.66}$$

This simple common structure enables us to compute general results for the moments of both these processes in terms of $D_i(k, \lambda)$, as we can use the well-known relation:

$$\langle X_i^n(\lambda) \rangle = \left(i\,\frac{\partial}{\partial k} \right)^n P_i(k,\lambda) \bigg|_{k=0} = \left(i\,\frac{\partial}{\partial k} \right)^n \frac{\langle e^{ikx_0} \rangle}{D_i(k,\lambda)} \bigg|_{k=0}. \tag{5.67}$$

Explicit results for the first two moments are given below:

$$\langle X_i(\lambda) \rangle = \frac{\langle x_0 \rangle\, D_i(0,\lambda) + i\, D_i'(0,\lambda)}{D_i^2(0,\lambda)}, \tag{5.68a}$$

$$\left\langle X_i^2(\lambda) \right\rangle = \frac{\left\langle x_0^2 \right\rangle D_i(0,\lambda) + D_i''(0,\lambda)}{D_i^2(0,\lambda)} + 2\frac{D_i'(0,\lambda)\left[\langle x_0 \rangle\, i\, D_i(0,\lambda) - D_i'(0,\lambda)\right]}{D_i^3(0,\lambda)}. \tag{5.68b}$$

In Eqs. (5.68a, 5.68b), only derivative up to the second order of the function $D_i(k, \lambda)$ appear. However, if we compute them for both cases and evaluate them at $k = 0$, we find identical results, i.e., $D_i(0, \lambda) = \lambda$, $D_i'(0, \lambda) = -i\,v$ and $D_i''(0, \lambda) = 2\, K_\alpha \lambda^{1-\alpha}$. Thus, both processes have the same first and second moment and consequently the same MSD [9]:

$$\left\langle (X_i(t) - x_0)^2 \right\rangle = \frac{2\, K_\alpha}{\Gamma(1+\alpha)} t^\alpha + v^2\, t^2. \tag{5.69}$$

In both cases, we find either subdiffusive ($0 < \alpha < 1$) or superdiffusive ($1 < \alpha < 2$) behaviour for short times and ballistic drift for long times, which is due to the advection of the moving laboratory frame. We remark that these results do not depend on the specific initial condition, as the equivalence is shown also for the x_0-dependent terms. Intuitively, MK and weak GI processes have the same MSD because the corresponding FFP Eqs. (5.32, 5.37) differ only by a third order spatial derivative, whose effect cannot be captured by the MSD. On the contrary, we expect deviations between the two processes, once we look at higher order moments. For instance, if we compute the third order one, we find that only one term depends on the third order derivative of $D_i(k, \lambda)$, which is different in the two cases: $D_1'''(0, \lambda) = 0$ and $D_2'''(0, \lambda) = 6\,i\,v\left(\frac{\alpha-1}{\lambda^\alpha}\right)$. Contrarily, all the other terms depend on its lower order derivatives, thus being equal. The deviation between the third order moments reads $\left\langle Y_2^3(\lambda) \right\rangle - \left\langle Y_1^3(\lambda) \right\rangle = -6\,v\left(\frac{\alpha-1}{\lambda^{2+\alpha}}\right)$, which is independent on x_0.

Position PDFs

A complete understanding of the deviations between MK and weak GI processes is obtained by looking at their position PDFs Eqs. (5.30, 5.31). To avoid artefacts due to the numerical Fourier-Laplace inverse transform, we will first derive it analytically in terms of Fox H-functions and then evaluate them numerically, thus reducing the problem to the numerical approximation of a Mellin-Barnes complex integral [202]. In the case of weak GI processes, this is obtained by making the Galilean transformation Eqs. (5.1, 5.2) of the exact solution in the comoving frame Eq. (5.27) [9]:

$$P_2(q,t) = \frac{1}{\sqrt{4\,K_\alpha\,t^\alpha}}\,H^{10}_{11}\left[\frac{|q-v_0\,t|}{\sqrt{K_\alpha t^\alpha}}\,\middle|\,\begin{matrix}(1-\frac{\alpha}{2},\frac{\alpha}{2})\\(0,1)\end{matrix}\right]. \tag{5.70}$$

On the contrary, the derivation of an exact formula for the PDF of MK processes has long been an open problem. So far, only its corresponding Fourier inverse transform has been derived [201, 197], whereas a full solution in (x,t)-space is still missing. We propose such solution in space-time of the position PDF for MK processes as follows:

$$P_1(q,t) = \frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}Q_1(q,t), \tag{5.71}$$

where $Q(q,t)$ is given as an infinite series of Fox H-functions ($\forall x \neq 0$):

$$\begin{aligned}Q_1(q,t) = \Theta(q)&\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{v_0\,t}{\sqrt{K_\alpha\,t^\alpha}}\right)^n \overline{H}^{2,1}_{2,3}\left(\overline{q}^2;\alpha,n\right)\right]\\ &+\Theta(-q)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{v_0\,t}{\sqrt{K_\alpha\,t^\alpha}}\right)^n \overline{H}^{2,1}_{2,3}\left(\overline{q}^2;\alpha,n\right)\right],\end{aligned} \tag{5.72}$$

where $\overline{q}^2 = \frac{q^2}{4\,K_\alpha t^\alpha}$ and the auxiliary function $\overline{H}^{2,1}_{2,3}\left(\overline{q}^2;\alpha,n\right)$ is defined as follows:

$$\overline{H}^{2,1}_{2,3}(z;\alpha,n)=\begin{cases}(-1)^\nu H^{2,1}_{2,3}\left[z\,\middle|\,\begin{matrix}\left(\frac{1-2\nu}{2},1\right),\left(\frac{(2-\alpha)(1+2\nu)}{2},\alpha\right)\\(0,1),\left(\frac{1+2\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right] & n=2\,\nu\\ (-1)^\nu H^{2,1}_{2,3}\left[z\,\middle|\,\begin{matrix}(-\nu,1),((2-\alpha)(1+\nu),\alpha)\\\left(\frac{1}{2},1\right),(1+\nu,1),(0,1)\end{matrix}\right] & n=1+2\,\nu\end{cases} \tag{5.73}$$

Finally, its value at the origin is specified by the three-parameter Mittag-Leffler function:

$$P_1(0,t) = \frac{1}{\sqrt{4\,K_\alpha\,t^\alpha}}E^{1/2}_{2-\alpha,(2-\alpha)/2}\left(-\frac{v_0^2\,t^{2-\alpha}}{4\,K_\alpha}\right). \tag{5.74}$$

The derivation of Eqs. (5.71-5.74) is presented in the next paragraph. We further note that the MK PDF is expressed as an expansion in the force field v_0, i.e., the velocity of the moving frame. As a sanity check, we compute the zero-th order term, which must be equal to the solution in the comoving frame. This is confirmed below (note that the corresponding terms in the two series in Eq. (5.71) are equal, i.e., Eq. (5.75) holds $\forall\, x \in \mathbb{R}$):

$$
\begin{aligned}
P_1(q,t) &= \frac{1}{\sqrt{4\,\pi\,K_\alpha\,t^\alpha}} H^{2,1}_{2,3}\left[\frac{q^2}{4\,K_\alpha t^\alpha}\;\middle|\; \begin{matrix}\left(\frac{1}{2},1\right),\left(\frac{2-\alpha}{2},\alpha\right)\\ (0,1)\,,\left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]\\
&= \frac{1}{\sqrt{4\,\pi\,K_\alpha\,t^\alpha}} H^{2,0}_{1,2}\left[\frac{q^2}{4\,K_\alpha t^\alpha}\;\middle|\; \begin{matrix}\left(\frac{2-\alpha}{2},\alpha\right)\\ (0,1)\,,\left(\frac{1}{2},1\right)\end{matrix}\right]
\end{aligned}
\tag{5.75}
$$

which solves Eq. (5.27) [9]. We used the property Eq. (A.20) of the Fox H-function [202].

We compare in Fig. 5.1 the MK PDF, obtained by numerical evaluation of Eqs. (5.71-5.73) (left panels) with the weak GI PDF obtained by direct evaluation of Eq. (5.70) (right panels) for different times and both in the subdiffusive and the superdiffusive regime, i.e., $\alpha = \{0.5, 0.8, 1.2, 1.5\}$ (top to bottom). The two solutions exhibit significant differences that cannot be neglected. On the one hand, the MK PDFs are characterised by a point of non differentiability in $x = 0$ for all times, which is intuitively due to the particular structure of Eq. (5.71), where the two series over the positive/negative axis have different signs. We further note that in the subdiffusive regime this distribution has negative values in the region $x < 0$, which seriously questions its physical relevance. To confirm that these values are not due to numerical artefacts in the evaluation of Eqs. (5.71-5.73), we compute analytically the integral of $P_1(x,t)$ on both the regions $x \gtrless 0$:

$$
I^+(t) = \int_0^\infty P_1(q,t)\,\mathrm{d}q = \frac{1}{2} + \frac{v_0\,t}{\sqrt{16\,K_\alpha\,t^\alpha}} F(t) \tag{5.76a}
$$

$$
I^-(t) = \int_{-\infty}^0 P_1(q,t)\,\mathrm{d}q = \frac{1}{2} - \frac{v_0\,t}{\sqrt{16\,K_\alpha\,t^\alpha}} F(t) \tag{5.76b}
$$

where we introduce the auxiliary function:

$$
F(t) = E^{1/2}_{2-\alpha,(4-\alpha)/2}\left(-\frac{v_0^2}{4\,K_\alpha} t^{2-\alpha}\right). \tag{5.77}
$$

We plot $I^-(t)$ and the value of the PDF at the origin Eq. (5.74) in Fig. 5.2 (right/left panel respectively). Both $I^-(t)$ and $P_1(0,t)$ exhibit negative values for finite time in the subdiffusive regime, thus confirming that our result on the negativity of the MK PDF in the subdiffusive regime is reliable and not due to numerical artefacts. Furthermore, due to (i) the non zero value of this integral for finite time and (ii) the existence of the critical point, the MK PDF lacks a symmetry axis. On the other hand, the PDF of weak GI processes still presents a critical point in $x = v_0\,t$, which is expected considering that the process resides in a frame moving with velocity v_0. Moreover, we can easily check that the integrals of $P_2(x,t)$ over the intervals $[-\infty, v_0\,t]$ and $[v_0\,t, +\infty]$ are equal for all times, thus implying that this PDF is symmetric with respect to its critical point.

Derivation of the position PDF of MK Processes

We present in this section the derivation of the Fourier-Laplace inverse transform of Eq. (5.31). Our approach will be similar to that recently discussed in [203]. For simplicity, we assume $X_0 = 0$. We first write Eq. (5.31) as follows:

$$
P_1(k,\lambda) = \frac{1}{\lambda^{\alpha'} + b(k)\lambda^\beta + c(k)}, \tag{5.78}
$$

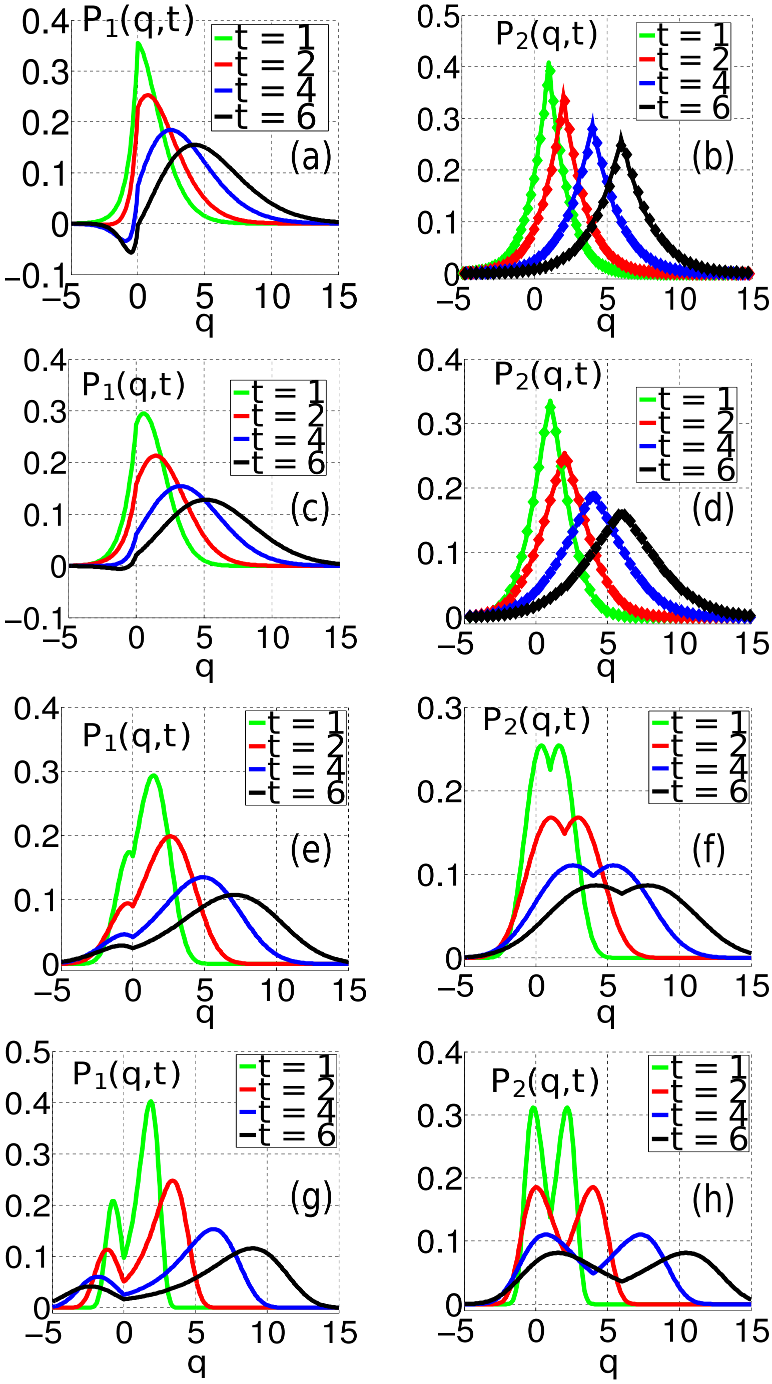

Figure 5.1: Plot of the position PDFs of MK (left panels) and GI processes (right panels) at different times in both the subdiffusive ($\alpha = \{0.5, 0.8\}$, panels (a,b) and (c,d) respectively) and the superdiffusive regime ($\alpha = \{1.2, 1.5\}$, panels (e,f) and (g,h) respectively). Other parameters are: $v_0 = 1$, $K_\alpha = 1$, $Y_0 = 0$. $P_1(q,t)$ is obtained by numerical evaluation of Eqs. (5.71-5.73), whereas $P_2(q,t)$ is obtained by numerical evaluation of Eq. (5.70). We find that MK PDFs exhibit a point of non differentiability in $x = 0$ for all times, which breaks the symmetry of the corresponding weak GI PDFs. Indeed, these still exhibit a critical point, which moves with velocity v_0 in time, but they are symmetric with respect to an axis passing through it. Contrarily to MK PDFs, they do not exhibit any negative value in the subdiffusive regime (further confirmed by the results in Fig. 5.2). Perfect agreement of the weak GI PDFs with simulations of the Langevin Eq. (5.102) (coloured markers) for a constant force v_0 is found in the subdiffusive regime.

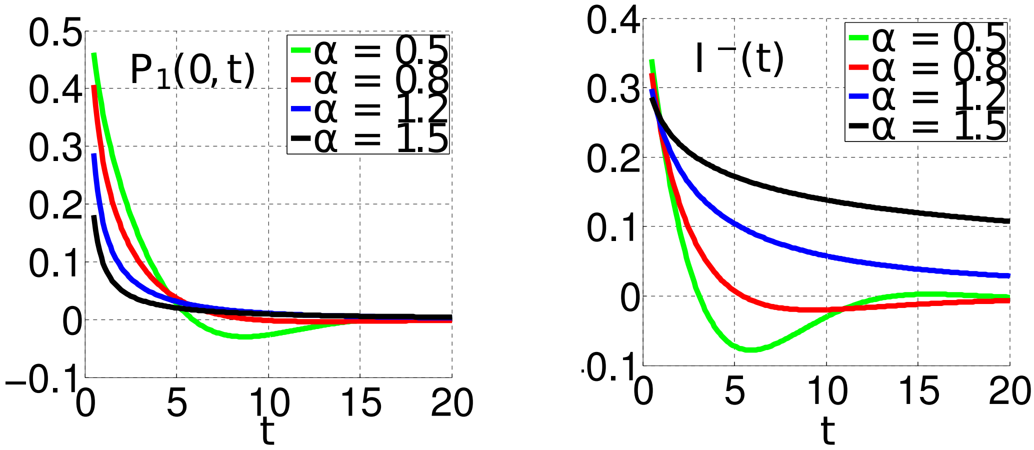

Figure 5.2: Plot of $P_1(0,t)$, i.e., the value at zero of the MK PDF, Eq. (5.74) (left panel) and of $I^-(t)$, i.e., its integral on the negative axis, Eq. (5.76b) (right panel) both in the subdiffusive ($\alpha = \{0.5, 0.8\}$) and in the superdiffusive regime ($\alpha = \{1.2, 1.5\}$). Other parameters are: $v_0 = 1$, $K_\alpha = 1$, $Y_0 = 0$. Both $P_1(0,t)$ and $I^-(t)$ exhibit negative values in the subdiffusive regime, thus supporting the discussion of the main text.

where we introduce the auxiliary parameters: $\alpha' = 1$, $\beta = 1 - \alpha$, $b(k) = K_\alpha\, k^2$ and $c(k) = -i\, v_0\, k$. We remark that $\alpha' > \beta\ \forall\, \alpha \in (0,2)$. We use the series expansion method first discussed in [75] to write:

$$\begin{aligned}
P_1(k,\lambda) &= \frac{1}{c(k)}\frac{1}{1+\frac{\lambda^{\alpha'}+b(k)\lambda^{\beta}}{c(k)}} = \frac{1}{c(k)}\frac{\lambda^{-\beta}c(k)}{\lambda^{\alpha'-\beta}+b(k)}\frac{1}{1+\frac{\lambda^{-\beta}c(k)}{\lambda^{\alpha'-\beta}+b(k)}} \\
&= \frac{1}{c(k)}\frac{\lambda^{-\beta}c(k)}{\lambda^{\alpha'-\beta}+b(k)}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{-\beta\, n}[c(k)]^n}{[\lambda^{\alpha'-\beta}+b(k)]^n} \\
&= \sum_{n=0}^{\infty}(-1)^n[c(k)]^n\frac{\lambda^{-\beta-\beta\, n}}{[\lambda^{\alpha'-\beta}+b(k)]^{n+1}}
\end{aligned} \tag{5.79}$$

We can now make a term by term Laplace inverse transform of Eq. (5.79) by recalling Eq. (A.10). Thus, $P_1(k,t)$ is given as a series of three-parameter Mittag-Leffler functions:

$$\begin{aligned}
P_1(k,t) &= \sum_{n=0}^{\infty}(-1)^n[c(k)]^n t^n E^{1+n}_{\alpha,1+n}\left(-t^{\alpha}\, b(k)\right) \\
&= \sum_{n=0}^{\infty}(i\, v_0\, t)^n\, k^n\, E^{1+n}_{\alpha,1+n}\left(-K_\alpha\, t^\alpha\, k^2\right).
\end{aligned} \tag{5.80}$$

We now need to make a term by term inverse Fourier transform of Eq. (5.80). To this aim, we first rewrite it in terms of Fox H-functions by using Eq. (A.11). In our case, we obtain:

$$E^{1+n}_{\alpha,1+n}\left(-K_\alpha\, t^\alpha\, k^2\right) = \frac{1}{\Gamma(1+n)} H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-n,1) \\ (0,1),(-n,\alpha)\end{matrix}\right]. \tag{5.81}$$

Secondly, we note that the Fourier inverse transform of the expression just derived can be written explicitly in terms of the following cosine and sine transforms of Fox H-functions:

$$\mathcal{F}^{-1}\left\{k^n H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\right\}(q,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(k\,q)k^n H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\mathrm{d}k - \frac{i}{2\pi}\int_{-\infty}^{\infty}\sin(k\,q)k^n H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\mathrm{d}k\,. \tag{5.82}$$

Let us first assume $q > 0$. We remark that (i) the first/second integral in the rhs of Eq. (5.82) is not null only for even/odd indices, i.e., for $n = 2\nu/1{+}2\nu\ \forall\,\nu \in \mathbb{N}_0$ respectively, due to the parity of the Fox H-function appearing in Eq. (5.82) and that (ii) they are equal to twice the corresponding integral on the semi-half positive line, once not null. Thus, we can use Eqs. (A.25) to explicitly compute these integral transforms :

$$\int_0^{\infty}\cos(k\,q)k^{2\,\nu} H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-2\,\nu,1)\\(0,1),(-2\,\nu,\alpha)\end{matrix}\right]\mathrm{d}k = \frac{\sqrt{\pi}\,2^{2\nu}}{|q|^{1+2\nu}}H^{1,2}_{3,2}\left[\frac{4\,K_\alpha\,t^\alpha}{q^2}\,\middle|\, \begin{matrix}\left(\frac{1}{2}-\nu,1\right),(-2\nu,1),(-\nu,1)\\(0,1),(-2\nu,\alpha)\end{matrix}\right], \tag{5.83a}$$

$$\int_0^{\infty}\sin(k\,q)k^{1+2\,\nu} H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-(1+2\,\nu),1)\\(0,1),(-(1+2\,\nu),\alpha)\end{matrix}\right]\mathrm{d}k = \frac{\sqrt{\pi}\,2^{1+2\nu}}{|q|^{2+2\nu}}H^{1,2}_{3,2}\left[\frac{4\,K_\alpha\,t^\alpha}{q^2}\,\middle|\, \begin{matrix}\left(-\frac{1}{2}-\nu,1\right),(-(1+2\,\nu),1),(-\nu,1)\\(0,1),(-(1+2\nu),\alpha)\end{matrix}\right]. \tag{5.83b}$$

Finally, by using the relation in Eq. (A.23) we obtain:

$$\mathcal{F}^{-1}\left\{k^n H^{1,1}_{1,2}\left[K_\alpha\, t^\alpha\, k^2 \,\middle|\, \begin{matrix}(-n,1)\\(0,1),(-n,\alpha)\end{matrix}\right]\right\}(q,t) = \frac{1}{\sqrt{\pi}}\begin{cases}\frac{2^{2\nu}}{|q|^{1+2\nu}}H^{2,1}_{2,3}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\, \begin{matrix}(1,1),(1+2\nu,\alpha)\\\left(\frac{1}{2}+\nu,1\right),(1+2\nu,1),(1+\nu,1)\end{matrix}\right] & n=2\nu\\[2ex] \frac{(-i)2^{1+2\nu}}{|q|^{2+2\nu}}H^{2,1}_{2,3}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\, \begin{matrix}(1,1),(2+2\nu,\alpha)\\\left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right] & n=1+2\nu\end{cases} \tag{5.84}$$

These results enable us to write Eq. (5.78) explicitly in (x,t)-space in terms of two infinite series of Fox H-functions (corresponding to the original series over odd and even indices):

$$P_1(q,t) = \frac{1}{\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(v_0\,t)^{2\nu}}{(2\,\nu)!}\frac{2^{2\nu}}{|q|^{1+2\nu}}H^{2,1}_{2,3}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\, \begin{matrix}(1,1),(1+2\nu,\alpha)\\\left(\frac{1}{2}+\nu,1\right),(1+2\nu,1),(1+\nu,1)\end{matrix}\right] + \frac{1}{\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(v_0\,t)^{1+2\nu}}{(1+2\nu)!}\frac{2^{1+2\nu}}{|q|^{2+2\nu}}H^{2,1}_{2,3}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\, \begin{matrix}(1,1),(2+2\nu,\alpha)\\\left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right] \tag{5.85}$$

Finally, we can exploit Eq. (A.24) to absorb the q-dependent multiplicative factors into the Fox H-functions. For each term separately, we obtain:

$$\frac{2^{2\nu}}{|q|^{1+2\nu}} H_{2,3}^{2,1}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\,\begin{matrix}(1,1),(1+2\,\nu,\alpha)\\ \left(\frac{1}{2}+\nu,1\right),(1+2\,\nu,1),(1+\nu,1)\end{matrix}\right]=$$
$$\frac{1}{\sqrt{4\,K_\alpha\,t^\alpha}}\left(\frac{1}{\sqrt{K_\alpha\,t^\alpha}}\right)^{2\nu} H_{2,3}^{2,1}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\nu,1\right),\left(1+2\nu-\alpha\left(\frac{1}{2}+\nu\right),\alpha\right)\\ (0,1),\left(\frac{1}{2}+\nu,1\right),\left(\frac{1}{2},1\right)\end{matrix}\right] \quad (5.86a)$$

$$\frac{2^{1+2\nu}}{|q|^{2+2\nu}} H_{2,3}^{2,1}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\,\begin{matrix}(1,1),(2+2\nu,\alpha)\\ \left(\frac{3}{2}+\nu,1\right),(2+2\nu,1),(1+\nu,1)\end{matrix}\right]=$$
$$\frac{1}{\sqrt{4\,K_\alpha\,t^\alpha}}\left(\frac{1}{\sqrt{K_\alpha\,t^\alpha}}\right)^{1+2\,\nu} H_{2,3}^{2,1}\left[\frac{q^2}{4\,K_\alpha\,t^\alpha}\,\middle|\,\begin{matrix}(-\nu,1),(2+2\nu-\alpha(1+\nu),\alpha)\\ \left(\frac{1}{2},1\right),(1+\nu,1),(0,1)\end{matrix}\right] \quad (5.86b)$$

In the opposite case $q<0$ the second term in the rhs of Eq. (5.82) changes sign, so that the sum over odd indices in Eq. (5.85) has an opposite sign. If we take this into account and substitute Eqs. (5.86a, 5.86b) into Eq. (5.85), we obtain the result of Eqs. (5.71-5.73). Also Eq. (5.74), specifying the value of the PDF in $x=0$, can be computed similarly. In this case, only the sum over even indices contributes to the PDF in Eq. (5.80) with coefficients defined by solving the correspondent integral of Fox function with Eqs. (A.26-A.17):

$$\int_0^\infty k^{2\nu} H_{1,1}^{1,2}\left[\sqrt{K_\alpha t^\alpha}|k|\,\middle|\,\begin{matrix}\left(-2\nu,\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right),\left(-2\nu,\frac{\alpha}{2}\right)\end{matrix}\right]\mathrm{d}k=\left(\frac{1}{\sqrt{K_\alpha t^\alpha}}\right)^{1+2\nu}\frac{\left[\Gamma\left(\frac{1}{2}+\nu\right)\right]^2}{\Gamma\left((1+2\nu)\left(1-\frac{\alpha}{2}\right)\right)}. \quad (5.87)$$

By substituting such coefficients into the series over even indices, we obtain:

$$P_1(0,t)=\frac{1}{\sqrt{4K_\alpha t^\alpha}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(2\nu)!}{4^\nu(\nu!)^2}\frac{1}{\Gamma\left(\left(1-\frac{\alpha}{2}\right)(1+2\nu)\right)}\left(\frac{v^2\,t^{2-\alpha}}{4K_\alpha}\right)^\nu, \quad (5.88)$$

which is the series expansion of Eq. (5.74). As a sanity check of our result, we check the normalisation of Eqs. (5.71-5.73), which is expected as we have from Eq. (5.78) that $P_1(k,\lambda)|_{k=0}=1/\lambda$. Due to the different sign of the sums over odd indices, only those over even ones contribute to the normalisation of the PDF. Due to the parity of the Fox H-function, the integral can be restricted to the semi-half positive line, i.e., we can write:

$$\int_{-\infty}^{+\infty}P_1(q,t)\,\mathrm{d}q=\frac{1}{\sqrt{\pi\,K_\alpha\,t^\alpha}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(2\,\nu)!}\left(\frac{v\,t}{\sqrt{K_\alpha\,t^\alpha}}\right)^{2\nu}\times$$
$$\int_0^\infty H_{2,3}^{2,1}\left[\frac{q^2}{4K_\alpha t^\alpha}\,\middle|\,\begin{matrix}\left(\frac{1-2\,\nu}{2},1\right),\left((2-\alpha)\left(\frac{1+2\,\nu}{2}\right),\alpha\right)\\ (0,1),\left(\frac{1+2\,\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{matrix}\right]\mathrm{d}q\,. \quad (5.89)$$

At last, we compute the integral of the Fox H-function by recalling Eqs. (A.22-A.26). We find:

$$\int_0^\infty H_{2,3}^{2,1}\left[\frac{q^2}{4K_\alpha t^\alpha}\,\middle|\,\begin{array}{l}\left(\frac{1-2\nu}{2},1\right),\left((2-\alpha)\left(\frac{1+2\nu}{2}\right),\alpha\right)\\(0,1),\left(\frac{1+2\nu}{2},1\right),\left(\frac{1}{2},1\right)\end{array}\right]\mathrm{d}q=$$
$$\sqrt{K_\alpha\,t^\alpha}\int_0^\infty H_{2,3}^{2,1}\left[q\,\middle|\,\begin{array}{l}\left(\frac{1-2\nu}{2},\frac{1}{2}\right),\left((2-\alpha)\left(\frac{1+2\nu}{2}\right),\frac{\alpha}{2}\right)\\(0,\frac{1}{2}),\left(\frac{1+2\nu}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\end{array}\right]\mathrm{d}q=\sqrt{K_\alpha\,t^\alpha}\Theta(-1),\tag{5.90}$$

where the function Θ is defined in Eq. (A.17), which reads in this specific case as:

$$\Theta(s)=\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\nu+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)\Gamma\left((2-\alpha)\left(\frac{1+2\nu}{2}\right)+\frac{\alpha}{2}s\right)}=\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\nu+\frac{1}{2}s\right)}{\Gamma\left((2-\alpha)\left(\frac{1+2\nu}{2}\right)+\frac{\alpha}{2}s\right)}\prod_{i=0}^{\nu-1}\left(\frac{1}{2}-\frac{s}{2}+i\right).\tag{5.91}$$

We note that for $s=-1$ all terms, except the one for $\nu=0$ which is equal to $\sqrt{\pi}$, cancel out. Eq. (5.90) is then equal to $\sqrt{\pi K_\alpha t^\alpha}$, i.e., the PDF is correctly normalised. Finally, we recall that, in order to compute Eqs. (5.76a-5.76b), we also need the integral over the odd series, which can be computed in the same way as Eq. (5.90). Indeed, we find:

$$\int_0^\infty H_{2,3}^{2,1}\left[\frac{q^2}{4K_\alpha t^\alpha}\,\middle|\,\begin{array}{l}(-\nu,1),((2-\alpha)(1+\nu),\alpha)\\\left(\frac{1}{2},1\right),(1+\nu,1),(0,1)\end{array}\right]\mathrm{d}q=$$
$$\sqrt{K_\alpha\,t^\alpha}\int_0^\infty H_{2,3}^{2,1}\left[q\,\middle|\,\begin{array}{l}\left(-\nu,\frac{1}{2}\right),\left((2-\alpha)(1+\nu),\frac{\alpha}{2}\right)\\\left(\frac{1}{2},\frac{1}{2}\right),\left(1+\nu,\frac{1}{2}\right),\left(0,\frac{1}{2}\right)\end{array}\right]\mathrm{d}q=\sqrt{K_\alpha\,t^\alpha}\Theta(-1),\tag{5.92}$$

where the function Θ in this case is given by:

$$\Theta(s)=\frac{\Gamma\left(\frac{1}{2}(1-s)\right)\Gamma\left(1+\nu-\frac{s}{2}\right)\Gamma\left(1+\nu+\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left((2-\alpha)(1+\nu)-\frac{\alpha}{2}s\right)}=\frac{\Gamma\left(\frac{1}{2}(1-s)\right)\Gamma\left(1+\nu-\frac{s}{2}\right)}{\Gamma\left((2-\alpha)(1+\nu)-\frac{\alpha}{2}s\right)}\prod_{i=0}^{\nu-1}\left(1+\frac{s}{2}+i\right).\tag{5.93}$$

If we evaluate this expression for $s=-1$, we obtain the following result:

$$\Theta(-1)=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{2}+\nu\right)\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma\left((2-\alpha)(1+\nu)+\frac{\alpha}{2}\right)},\tag{5.94}$$

which can finally be used to compute Eqs. (5.76a-5.76b). For instance, let us compute explicitly Eq. (5.76a). Recalling $(1+2\,n)!=\Gamma(2+2\,n)=\frac{1}{\sqrt{\pi}}2^{1+2\,n}\,\Gamma(1+n)\,\Gamma(\frac{3}{2}+n)$ and

Eqs. (5.92-5.94), we have:

$$\int_0^\infty \mathrm{d}q\, P_1(q,t) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+2\,n)!} \left(\frac{v_0\, t}{\sqrt{K_\alpha\, t^\alpha}}\right)^{1+2\,n} \times$$
$$\times \int_0^\infty H_{2,3}^{2,1}\left[q \,\middle|\, \begin{matrix} \left(-n, \frac{1}{2}\right), \left((2-\alpha)\,(1+n), \frac{\alpha}{2}\right) \\ \left(\frac{1}{2}, \frac{1}{2}\right), \left(1+n, \frac{1}{2}\right), \left(0, \frac{1}{2}\right) \end{matrix} \right] \mathrm{d}x$$
$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{4\,2^n\,\Gamma(1+n)} \left(\frac{v_0\, t}{\sqrt{K_\alpha\, t^\alpha}}\right)^{1+2\,n} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left((2-\alpha)(1+n)+\frac{\alpha}{2}\right)}$$
$$= \frac{1}{2} + \frac{v_0\, t}{\sqrt{16\,\pi\, K_\alpha\, t^\alpha}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(1+n)} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left((2-\alpha)(1+n)+\frac{\alpha}{2}\right)} \left(\frac{v_0\, t}{\sqrt{4\,K_\alpha\, t^\alpha}}\right)^{2\,n}, \tag{5.95}$$

where in the rhs we obtain the series expansion of Eq. (5.77).

Asymptotic comparison of MK and weak GI distributions

In this section, we investigate if a characteristic scaling parameter exists, such that the MK and weak GI PDFs asymptotically coincide. To this aim, we first need to express Eq. (5.70) in the form of Eq. (5.71-5.72), i.e., as a series expansion in the parameter v_0. Let us consider separately the case $x \gtrless v_0\, t$. In order to Taylor expand the Fox H-function in Eq. (5.70), we need its hierarchy of derivatives. If we use the properties in Eqs. (A.28-A.24-A.20), we find:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} H_{1,1}^{1,0}\left[z \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right] = \left(\frac{1}{z}\right)^n H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (0,1), \left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \\ (0,1), (n,1) \end{matrix} \right]$$
$$= H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (-n,1), \left(1-\frac{\alpha}{2}(1+n), \frac{\alpha}{2}\right) \\ (0,1), (-n,1) \end{matrix} \right] = H_{1,1}^{1,0}\left[z \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}(1+n), \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right]. \tag{5.96}$$

With this result, we find the following Taylor expansion (for instance for $x > v_0\, t$):

$$H_{1,1}^{1,0}\left[\frac{q-v_0\, t}{\sqrt{K_\alpha t^\alpha}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v_0\, t}{\sqrt{K_\alpha t^\alpha}}\right)^n H_{1,1}^{1,0}\left[\frac{q}{\sqrt{K_\alpha t^\alpha}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}(1+n), \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right]. \tag{5.97}$$

We note that for $x < v_0\, t$, the Taylor expansion is obtain from Eq. (5.97) by substituting $q \to -q$ and $v_0 \to -v_0$. We finally obtain the following expansion of Eq. (5.70):

$$P_2(q,t) = \frac{1}{\sqrt{4\,K_\alpha\, t^\alpha}} Q_2(q,t), \tag{5.98}$$

where $Q_2(q,t)$ is given as the following series of Fox H-functions:

$$Q_2(q,t) = \Theta(q+v\,t) \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{v\,t}{\sqrt{K_\alpha\, t^\alpha}}\right)^n H_{1,1}^{1,0}\left[\frac{q}{\sqrt{K_\alpha t^\alpha}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}(1+n), \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right] \right]$$
$$+ \Theta(-q-v\,t) \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{v\,t}{\sqrt{K_\alpha\, t^\alpha}}\right)^n H_{1,0}^{1,1}\left[\frac{-q}{\sqrt{K_\alpha t^\alpha}} \,\middle|\, \begin{matrix} \left(1-\frac{\alpha}{2}(1+n), \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right] \right] \tag{5.99}$$

By comparing Eqs. (5.72,5.99), a natural choice of the scaling parameter is the quantity $\epsilon = q/\sqrt{K_\alpha\, t^\alpha}$, governing the scaling behaviour of the Fox H-functions. Let us further assume $v_0 \ll 1$, such that we can keep only the first order terms in the series expansions Eqs. (5.72,5.99). Let us focus on the half-positive line, specifically $q > v\, t$ (in the opposite case similar arguments hold). From Eq. (5.99) we find immediately the first order term:

$$P_2^{(1)}(x,t) = \frac{v_0\, t}{2\, K_\alpha\, t^\alpha} H_{1,1}^{1,0}\left[\frac{q}{\sqrt{K_\alpha t^\alpha}} \middle| \begin{matrix} \left(1-\alpha, \frac{\alpha}{2}\right) \\ (0,1) \end{matrix} \right]. \tag{5.100}$$

On the other hand, the first order term of Eq. (5.72) can be simplified as follows:

$$P_1^{(1)}(q,t) = \frac{v\, t}{4\, K_\alpha\, t^\alpha} H_{1,1}^{1,0}\left[\frac{q}{\sqrt{K_\alpha t^\alpha}} \middle| \begin{matrix} \left(2-\alpha, \frac{\alpha}{2}\right) \\ (1,1) \end{matrix} \right]. \tag{5.101}$$

Interestingly, these two Fox H-functions have the same scaling behaviour for large values of the parameter ϵ, i.e., $P_{1,2}^{(1)}(q,t) \sim \epsilon^{\frac{2\,\alpha-1}{2-\alpha}}\, e^{-\frac{2-\alpha}{2}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}} \epsilon^{\frac{2}{2-\alpha}}}$ for $\epsilon \gg 1$ [202]. Thus, in this scaling regime we expect the two solutions to agree. We remark that this result only holds in the limit of small velocity. Indeed, for general v_0 further terms of the series expansions need to be considered, whose asymptotic behaviour for $\epsilon \gg 1$ will be left for future work.

5.5 Langevin Formulation of Weak Galilean Invariant Anomalous Processes

In this section, we provide a characterisation of the microscopic stochastic dynamics of weak GI anomalous processes, whose position PDF is described by the weak GI FFP Eq. (5.37). Interestingly, it turns out that these Langevin equations are driven by the noise $\overline{\xi}$ introduced in Chapter 4 [200]. We will discuss both the subdiffusive regime ($0 < \alpha < 1$), where we will conveniently employ the subordination technique to formally derive Eq. (5.37), and the superdiffusive regime, where instead we will extend Novikov's theorem [see Eq. (2.99)] by summing over the hierarchy of correlation functions of the noise [200].

Subdiffusive regime

Let us consider a process $X(t)$ described by the Langevin equation:

$$\dot{X}(t) = F(X(t)) + \sqrt{2\,\sigma}\,\overline{\xi}(t), \tag{5.102}$$

where the stochastic noise $\overline{\xi}(t)$ is defined formally as in Eq. (4.14) [200], with the process T therein being a Lévy stable process of parameter $0 < \alpha < 1$. Recalling that $\overline{\xi}(t)$ is the derivative of a time-changed Brownian motion [200], the integrated process $Y(t)$ is a semi-martingale, so that we can write its Itô formula as follows [102]:

$$f(X(t)) = f(X_0) + \int_0^t f'(X(s))\, \mathrm{d}X(s) + \frac{1}{2}\int_0^t f''(X(s))\, \mathrm{d}[X,X]_s\,, \tag{5.103}$$

which does not exhibit jump terms as X has continuous paths. If we evaluate Eq. (5.103) for the function $f(X(t)) = e^{i\,k\,X(t)}$ and recall that $[X,X]_s = 2\,\sigma\int_0^s \mathrm{d}S(\tau)$ [153], we obtain:

$$\begin{aligned}
e^{i\,k\,X(t)} &= e^{i\,k\,X_0} + i\,k\int_0^t e^{i\,k\,X(s)}\,\mathrm{d}X(s) \,-\sigma\,k^2\int_0^t e^{i\,k\,X(s)}\,\mathrm{d}S(s) \\
&= e^{i\,k\,X_0} + i\,k\int_0^t e^{i\,k\,X(s)} F(X(s))\,\mathrm{d}s \\
&\quad + i\,k\sqrt{2\,\sigma}\int_0^t e^{i\,k\,X(s)}\overline{\xi}(s)\,\mathrm{d}s \,-\sigma\,k^2\int_0^t e^{i\,k\,X(s)}\,\mathrm{d}S(s)\,,
\end{aligned} \tag{5.104}$$

where in the second line we used explicitly Eq. (5.102). We note that the term dependent on $\overline{\xi}(t)$ is null once we take the ensemble average of Eq. (5.104), due to the fact that $\overline{\xi}$ has null first moment. We further note that the ensemble average of $f(X(t))$ is equal to the characteristic function of X. Thus, by (i) taking the ensemble average of Eq. (5.104), (ii) making its Fourier inverse transform and (iii) taking the time derivative of the resulting equation, we derive the FP equation:

$$\frac{\partial}{\partial t}P(q,t) = -\frac{\partial}{\partial q}F(q)P(q,t) + \sigma\frac{\partial^2}{\partial q^2}\frac{\partial}{\partial t}\left\langle \int_0^t \delta(q-X(s))\,\mathrm{d}S(s)\right\rangle. \tag{5.105}$$

However, we still need to express the averaged stochastic integral in the rhs of Eq. (5.105) in terms of $P(q,t)$. For simplicity of notation, let us define the auxiliary function:

$$Q(q,t) = \left\langle \int_0^t \delta(q-X(s))\,\mathrm{d}S(s)\right\rangle. \tag{5.106}$$

For a constant drift, i.e., $F(X(t)) = v_0$, we need to prove the relation in Fourier space:

$$Q(k,t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t\left[\int_0^\tau \frac{e^{i\,v_0\,k\,(\tau-s)}}{(\tau-s)^{2-\alpha}}P(k,s)\,\mathrm{d}s\right]\mathrm{d}\tau\,, \tag{5.107}$$

or equivalently in Laplace transform $\lambda\,Q(k,\lambda) = (\lambda - i\,v_0\,k)^{1-\alpha}\,P(k,\lambda)$ $(Q(y,0) = 0)$. Indeed, by taking the time derivative of Eq. (5.107) and performing its Fourier inverse transform, we obtain the same integral as in Eq. (5.59), i.e., the correct fractional substantial derivative after regularization. Consequently, Eq. (5.105) is the same as Eq. (5.37).

We here propose a derivation of Eq. (5.107), i.e., valid for the specific case of a constant drift, whereas the general treatment of Eq. (5.106) will be discussed in future work. For $F(X(t)) = v_0$, Eq. (5.102) can be integrated and written as a subordinated process:

$$\begin{aligned}
X(t) &= v_0\,t + \sqrt{2\,\sigma}\int_0^{S(t)} \xi(\tau)\,\mathrm{d}\tau \\
&= v_0\,T(S(t)) + \sqrt{2\,\sigma}\int_0^{S(t)} \xi(\tau)\,\mathrm{d}\tau \,= Y(S(t))
\end{aligned} \tag{5.108}$$

where we introduce the auxiliary process:

$$Y(s) = v_0\,T(s) + \sqrt{2\,\sigma}\int_0^s \xi(\tau)\,\mathrm{d}\tau\,. \tag{5.109}$$

If we use Eqs. (5.108,5.109) in Eq. (5.106), we can write:

$$Q(q,t) = \left\langle \int_0^t \left[\int_0^\infty \delta(q - X(s))\delta(s - S(\tau))\,\mathrm{d}s \right] \mathrm{d}S(\tau) \right\rangle \tag{5.110}$$

Let us now take the Fourier transform of the previous result, change the orders of the integrals and substitute the definition of the process Y in Eq. (5.109). We find:

$$\begin{aligned} Q(k,t) &= \left\langle \int_0^t \left[\int_0^\infty e^{i\,k\,Y(s)} \delta(s - S(\tau))\,\mathrm{d}s \right] \mathrm{d}S(\tau) \right\rangle \\ &= \left\langle \int_0^\infty \left[\int_0^t e^{i\,k\,v_0\,T(s)} e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r}\, \delta(s - S(\tau))\,\mathrm{d}S(\tau) \right] \mathrm{d}s \right\rangle \\ &= \int_0^\infty \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \left[\left\langle \int_0^t e^{i\,k\,v_0\,T(s)} \delta(s - S(\tau))\,\mathrm{d}S(\tau) \right\rangle \right] \mathrm{d}s\,, \end{aligned} \tag{5.111}$$

where in the third line we use the independence of the processes $\xi(s)$ and $T(s)$ to factorise the ensemble average. We further rewrite the stochastic integral in the rhs of Eq. (5.111) in terms of time increments by using the relation Eq. (3.9) [140]:

$$\begin{aligned} Q(k,t) &= \int_0^\infty \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \left[\left\langle \int_0^t e^{i\,k\,v_0\,T(s)} \delta(\tau - T(s))\,\mathrm{d}\tau \right\rangle \right] \mathrm{d}s \\ &= \int_0^\infty \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \left\langle \Theta(t - T(s))\, e^{i\,k\,v_0\,T(s)} \right\rangle \mathrm{d}s\,. \end{aligned} \tag{5.112}$$

This result can be further simplified if we take its Laplace transform. By recalling that $\mathcal{L}\left\{\Theta(t - T(s)\right\}(\lambda) = \frac{1}{\lambda} e^{-\lambda\,T(s)}$ and the characteristic functional of T, we obtain:

$$\begin{aligned} Q(k,\lambda) &= \frac{1}{\lambda} \int_0^\infty \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \left\langle e^{-(\lambda - i\,k\,v_0)T(s)} \right\rangle \mathrm{d}s \\ &= \frac{1}{\lambda} \int_0^\infty \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle e^{-s\,(\lambda - i\,k\,v_0)^\alpha}\,\mathrm{d}s\,. \end{aligned} \tag{5.113}$$

On the other hand, we can manipulate directly the position PDF by using (i) the relation $1 = \int_0^\infty \delta(s - S(t))\,\mathrm{d}s$ and the independence of the processes $\xi(s)$ and $T(s)$ to write:

$$\begin{aligned} P(k,t) &= \int_0^\infty \left\langle \delta(s - S(t))\, e^{i\,k\,Y(s)} \right\rangle \mathrm{d}s \\ &= \int_0^\infty \left\langle \delta(s - S(t))\, e^{i\,v_0\,k\,T(s)} \right\rangle \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \mathrm{d}s \end{aligned} \tag{5.114}$$

and (ii) the relation in Laplace space $\int_0^\infty \delta(s - S(t)) e^{-\lambda\,t}\,\mathrm{d}t = \eta(s)\, e^{-\lambda\,T(s)}$ [140]. We obtain:

$$P(k,\lambda) = \int_0^\infty \left\langle \eta(s)\, e^{-(\lambda - i\,v_0\,k)\,T(s)} \right\rangle \left\langle e^{i\,k\,\sqrt{2\,\sigma}\int_0^s \xi(r)\,\mathrm{d}r} \right\rangle \mathrm{d}s \tag{5.115}$$

Finally, the η-dependent average can be computed in exact terms by using the characteristic functional of T. By setting the test function $k(r) = \Theta(s-r)(\lambda - i\, v_0\, k)$, we derive:

$$\begin{aligned}\left\langle \eta(s)\, e^{-(\lambda - i\, v_0\, k)\, T(s)} \right\rangle &= \left\langle \eta(s)\, e^{-\int_0^\infty k(r)\eta(r)\,\mathrm{d}r} \right\rangle \\ &= \frac{-1}{\lambda - i\, v_0\, k} \frac{\mathrm{d}}{\mathrm{d}s} \left\langle e^{-\int_0^\infty k(r)\eta(r)\,\mathrm{d}r} \right\rangle \\ &= \frac{-1}{\lambda - i\, v_0\, k} \frac{\mathrm{d}}{\mathrm{d}s} e^{-s\,(\lambda - i\, v_0\, k)^\alpha} = (\lambda - i\, v_0\, k)^{\alpha-1} e^{-s\,(\lambda - i\, v_0\, k)^\alpha}. \quad (5.116)\end{aligned}$$

Substituting Eq. (5.116) in Eq. (5.115) and rearranging the terms, we find the relation:

$$\int_0^\infty \left\langle e^{i\, k\, \sqrt{2\,\sigma}\, \int_0^s \xi(r)\,\mathrm{d}r} \right\rangle e^{-s(\lambda - i\, v_0\, k)^\alpha}\,\mathrm{d}s = (\lambda - i\, v_0\, k)^{1-\alpha} P(k,\lambda) \quad (5.117)$$

However, the lhs of Eq. (5.117) coincides with the integral at the rhs of Eq. (5.113). By eliminating it, we finally prove Eq. (5.107). To further support this result, we note that numerical simulations of Eq. (5.102) ($v_0 = 1$, coloured markers) in Fig. 5.1 (panels b,d) exhibit perfect agreement with the exact position PDF Eq. (5.70) (solid coloured lines).

Superdiffusive regime

In the superdiffusive regime, Eq. (5.102) is still the correct description of the microscopic dynamics of GI processes. However, no suitable process T can be found to define Eq. (4.14), i.e., $\overline{\xi}(t)$ is characterised in terms of its hierarchy of correlation functions [200]. Thus, the corresponding FPE is given by Eq. (5.54), where $\left\langle \overline{\xi}(t)\, F(q,t) \right\rangle$ needs to be computed explicitly, as Novikov's theorem does not hold, due to the non Gaussian character of $\overline{\xi}$. For simplicity, let us compute its Fourier transform $\left\langle \overline{\xi}(t)\, F(k,t) \right\rangle$, where in the specific case of Eq. (5.102) $F(k,t) = e^{i\, k\, v_0\, t} e^{i\, k\, \sqrt{2\,\sigma}\, \int_0^t \overline{\xi}(s)\,\mathrm{d}s}$. The second exponential is a functional of the noise path, that can be expanded by using functional Taylor series [112, 204]:

$$\begin{aligned}e^{-i\, k\, v_0\, t} F(k,t) &= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \mathrm{d}s_1 \ldots \int_0^\infty \mathrm{d}s_n\, H^{(n)}(k, s_1, \ldots, s_n)\, \overline{\xi}(s_1) \ldots \overline{\xi}(s_n) \\ &= \left[1 + \sum_{n=1}^\infty \frac{(i\, k\, \sqrt{2\,\sigma})^n}{n!} \int_0^t \mathrm{d}s_1 \ldots \int_0^t \mathrm{d}s_n\, \overline{\xi}(s_1) \ldots \overline{\xi}(s_n) \right] \quad (5.118)\end{aligned}$$

where we computed exactly the variational derivatives $H^{(n)}(k, s_1, \ldots, s_n)$:

$$H^{(n)}(k, s_1, \ldots, s_n) = \left. \frac{\delta^{(n)} e^{i\, k\, \sqrt{2\,\sigma}\, \int_0^t \overline{\xi}(s)\,\mathrm{d}s}}{\delta\overline{\xi}(s_1) \ldots \delta\overline{\xi}(s_n)} \right|_{\overline{\xi}=0} = (i\, k\, \sqrt{2\,\sigma})^n \Theta(t - s_1) \ldots \Theta(t - s_n). \quad (5.119)$$

Let us take (i) the ensemble average of Eq. (5.118) and (ii) its time derivative. As the odd correlation functions of $\overline{\xi}$ are null, only the terms with even indices survive. We obtain:

$$\begin{aligned}\left[-i\, v_0\, k + \frac{\partial}{\partial t} \right] \langle F(k,t) \rangle &= e^{i\, k\, v_0\, t} \left[\frac{(i\, k\, \sqrt{2\,\sigma})^2}{2} K(t) + \frac{(i\, k\, \sqrt{2\,\sigma})^4}{4} \int_0^t K(t-s) K(s)\,\mathrm{d}s \right] + \\ &+ e^{i\, k\, v_0\, t} \sum_{n=3}^\infty \frac{(i\, k\, \sqrt{2\,\sigma})^{2\,n}}{2^n} \int_0^t \mathrm{d}s_{n-2}\, K(t - s_{2\,n-2}) \ldots \int_0^{s_2} K(s_2 - s_1) K(s_1)\,\mathrm{d}s_1. \quad (5.120)\end{aligned}$$

This result can be understood by recalling that the n-th $\overline{\xi}$-correlation function contains $\frac{(2\,n)!}{2^n\,n!}$, each corresponding to a different structure of the delta functions. In addition, as we are integrating over time, we need to consider all the $n!$ possible orderings of the n distinct times. After integrating, we obtain $\frac{(2\,n)!}{2^n}$ integrals of the type in Eq. (5.120), thus leading to the final result. At the same time, let us multiply Eq. (5.118) by $\overline{\xi}(t)$ and take its ensemble average. By eliminating the null terms, we derive the following equation:

$$\left\langle\overline{\xi}(t)\,F(k,t)\right\rangle=\sum_{n=0}^{\infty}\frac{(i\,k\,\sqrt{2\,\sigma})^{1+2\,n}}{(1+2\,n)!}e^{i\,k\,v_0\,t}\int_0^t \mathrm{d}s_1\ldots\int_0^t \mathrm{d}s_{1+2\,n}\left\langle\overline{\xi}(t)\overline{\xi}(s_1)\ldots\overline{\xi}(s_{1+2\,n})\right\rangle \quad (5.121)$$

We then need to find a relation between Eqs. (5.120,5.121) by solving the integrals over the $\overline{\xi}$-correlations. We start from the integral term of Eq. (5.121). Recalling Eq. (4.35), we find (i) $\int_0^t \mathrm{d}s_1\,\left\langle\overline{\xi}(t)\overline{\xi}(s_1)\right\rangle = K(t)$ for $n=0$, (ii) $\int_0^t \mathrm{d}s_1\,\int_0^t \mathrm{d}s_2\,\int_0^t \mathrm{d}s_3\,\left\langle\overline{\xi}(t)\overline{\xi}(s_1)\overline{\xi}(s_2)\overline{\xi}(s_3)\right\rangle = 3\int_0^t \mathrm{d}s_1\,K(t-s_1)\,K(s_1)\,\mathrm{d}s_1$ for $n=1$ and for general $n>1$:

$$\begin{aligned}e^{i\,k\,v_0\,t}\int_0^t \mathrm{d}s_1\,\ldots\int_0^t \mathrm{d}s_{1+2\,n}\,\left\langle\overline{\xi}(t)\overline{\xi}(s_1)\ldots\overline{\xi}(s_{1+2\,n})\right\rangle &= \frac{(1+2\,n)!}{2^n}\times\\ \times\int_0^t \mathrm{d}s_n\,K(t-s_n)e^{i\,k\,v_0\,(t-s_n)}&\left[e^{i\,k\,v_0\,s_n}\prod_{m=2}^{n}\int_0^{s_m}\mathrm{d}s_m\,K(s_m-s_{m-1})K(s_1)\right] \quad (5.122)\end{aligned}$$

Substituting these results into Eq. (5.121), we find ($s_n=s$):

$$\begin{aligned}&\left\langle\overline{\xi}(t)\,F(k,t)\right\rangle = (i\,k\,\sqrt{2\,\sigma})e^{i\,k\,v_0\,t}K(t)+(i\,k\,\sqrt{2\,\sigma})\left\{\int_0^t \mathrm{d}s\,K(t-s)\,e^{i\,k\,v_0\,t}\,\times\right.\\ &\left.\left[\frac{(i\,k\,\sqrt{2\,\sigma})^2}{2}\,K(s)+\sum_{n=2}^{\infty}\frac{(i\,k\,\sqrt{2\,\sigma})^{2\,n}}{2^n}\prod_{m=2}^{n}\int_0^{s_m}\mathrm{d}s_{m-1}\,K(s_m-s_{m-1})\,K(s_1)\right]\right\}. \quad (5.123)\end{aligned}$$

Comparing Eqs. (5.120,5.123), we obtain the equation:

$$\begin{aligned}\left\langle\overline{\xi}(t)\,F(k,t)\right\rangle &= (i\,k\,\sqrt{2\,\sigma})e^{i\,k\,v_0\,t}K(t)\\ &\quad+\left(i\,k\,\sqrt{2\,\sigma}\right)\int_0^t \mathrm{d}s\,K(t-s)\,e^{i\,k\,v_0\,(t-s)}\left[-i\,v_0\,k+\frac{\partial}{\partial s}\right]\langle F(k,s)\rangle\\ &=\left(i\,k\,\sqrt{2\,\sigma}\right)\left[-i\,v_0\,k+\frac{\partial}{\partial t}\right]\int_0^t \mathrm{d}s\,K(t-s)\,e^{i\,k\,v_0\,(t-s)}\,\langle F(k,s)\rangle\,, \quad (5.124)\end{aligned}$$

If we take the Fourier inverse transform of Eq. (5.124) and plug it in Eq. (5.54), we obtain:

$$\frac{\partial}{\partial t}P(q,t) = -v_0\frac{\partial}{\partial q}P(q,t)+2\,\sigma\frac{\partial^2}{\partial q^2}\left[\frac{\partial}{\partial t}+v_0\,\frac{\partial}{\partial q}\right]\int_0^t K(t-s)\,P(q-v_0\,(t-s),s), \quad (5.125)$$

which is equal to Eq. (5.37) for the specific case $K(t)=t^{\alpha-1}/\Gamma(\alpha)$. As a sanity check, we remark that we are free to choose $0<\alpha<1$, thus providing a derivation of Eq. (5.37) in the subdiffusive regime, which is equivalent to the one presented earlier and exploiting the subordination technique. This calculation further confirms the validity of Eq. (5.37) in the superdiffusive regime $1<\alpha<2$, first shown with the Balescu local approximation, and simultaneously provide the Langevin equation of the microscopic dynamical processes.

5.6 Fluctuation-Dissipation Relation of Weak Galilean Invariant Processes

In this section, we investigate the existence of a Fluctuation-Dissipation Relation of the first kind (FDRI) for weak GI anomalous processes. We will show that these processes do not satisfy a normal FDRI, due to the strong space-time coupling required to satisfy the assumption of weak statistical Galilean invariance. We will then conclude by defining new $\overline{\xi}-$driven processes preserving FDRI, that indeed are no longer weak GI.

We first review the case of Brownian processes, which are weak GI as the anomalous ones driven by the $\overline{\xi}$-noise. Let us consider the overdamped process $X(t)$ in the reference frame $\mathcal{S}$, where a constant external force F_0 is present. Its FPE is obtained from Eq. (5.24):

$$\frac{\partial}{\partial t}P(q,t) = \left[-\frac{1}{\gamma}\frac{\partial}{\partial q}F_0 + \frac{\sigma}{\gamma}\frac{\partial^2}{\partial q^2}\right]P(q,t). \tag{5.126}$$

By using Eq. (5.126), we can easily compute the first moment of X. Indeed, by recalling that $\langle X(t)\rangle = \int_{-\infty}^{+\infty} \mathrm{d}q\; q\, P(q,t)$ and taking its time derivative, we obtain:

$$\frac{\partial}{\partial t}\langle X(t)\rangle = \int_{-\infty}^{+\infty}\mathrm{d}q\, q\frac{\partial}{\partial t}P(q,t) = \int_{-\infty}^{+\infty}\mathrm{d}q\, q\left[-\frac{1}{\gamma}\frac{\partial}{\partial q}F_0 + \frac{\sigma}{\gamma}\frac{\partial^2}{\partial q^2}\right]P(q,t) = \frac{1}{\gamma}F_0, \tag{5.127}$$

where the boundary terms left by the integration by parts cancel out due to the property of the PDF for $|q| \gg 1$. If we integrate this equation, we find: $\langle X(t)\rangle = X_0 + \frac{F_0}{\gamma}t$. In a similar way, we can compute the variance of the process X in the absence of external force. For simplicity, we denote with a pedix 0 all the quantities that are computed for null external force. Recalling that $\left\langle X^2(t)\right\rangle_0 = \int_{-\infty}^{+\infty}\mathrm{d}q\; q^2\, P_0(q,t)$, we obtain the equation:

$$\frac{\partial}{\partial t}\left\langle X^2(t)\right\rangle_0 = \int_{-\infty}^{+\infty}\mathrm{d}q\; q^2\,\frac{\partial}{\partial t}P_0(q,t) = \frac{\sigma}{\gamma}\int_{-\infty}^{+\infty}\mathrm{d}q\; q^2\,\frac{\partial^2}{\partial q^2}P_0(q,t) = 2\,\frac{\sigma}{\gamma}, \tag{5.128}$$

whose time integration yields: $\left\langle X^2(t)\right\rangle_0 = X_0^2 + 2\,\frac{\sigma}{\gamma}\,t$. Putting these results together and setting $X_0 = 0$, we find that the FDRI holds:

$$\langle X(t)\rangle = \frac{F_0}{2\,\sigma}\left\langle X^2(t)\right\rangle_0. \tag{5.129}$$

How does this relation transform in a different Galilean frame? According to the transformation rule of the FP Eq. (5.126), the effect of the external flow, induced by the motion of the frame, is equivalent to that of a constant external force driving the process [see Eq. (5.26)]. Thus, calling v_0 the velocity of the new frame and exploiting the mentioned equivalence, the same Eq. (5.129) with F_0 substituted by $F_0 - \gamma\, v_0$ holds.

In the anomalous case we need to distinguish between (i) usual subordinated processes, i.e., CTRWs, which are not weak GI, and (ii) weak GI anomalous processes whose PDF is described by Eq. (5.37). Let us first consider the case of a subordinated process $X(t)$ in the presence of the external constant force F_0. Its FFPE reads reads as below [9]:

$$\frac{\partial}{\partial t}P(q,t) = \left[-\frac{\partial}{\partial q}F_0 + K_\alpha\frac{\partial^2}{\partial q^2}\right]{}_0D_t^{1-\alpha}P(q,t). \tag{5.130}$$

Similarly to the case of Brownian dynamics, we can compute both $\langle X(t)\rangle$ and $\left\langle X^2(t)\right\rangle_0$. In details, we find:

$$\frac{\partial}{\partial t}\langle X(t)\rangle = \int_{-\infty}^{+\infty} \mathrm{d}q\; q \left[-\frac{\partial}{\partial q}F_0 + K_\alpha \frac{\partial^2}{\partial q^2}\right] {}_0D_t^{1-\alpha} P(q,t) = F_0\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{5.131}$$

whose time integration provides the following result: $\langle X(t)\rangle = X_0 + \frac{F_0}{\Gamma(1+\alpha)}t^\alpha$. Analogously, we obtain for the variance in the absence of external force: $\left\langle X^2(t)\right\rangle_0 = X_0^2 + \frac{2\,K_\alpha}{\Gamma(1+\alpha)}t^\alpha$. Putting these results together and setting $X_0 = 0$, we obtain again Eq. (5.129), with σ substituted by the generalised diffusion coefficient K_α. Thus, a normal FDRI still holds for CTRWs and in general for processes obtained by subordination of an auxiliary process satisfying Eq. (5.129). Conversely, in the case of weak GI processes, Eq. (5.37) specifies to:

$$\frac{\partial}{\partial t}P(q,t) = -F_0\frac{\partial}{\partial q}P(q,t) + \frac{\sigma}{\Gamma(\alpha)}\frac{\partial^2}{\partial q^2}\left[\frac{\partial}{\partial t} + F_0\,\frac{\partial}{\partial q}\right]\int_0^t \frac{P(q-F_0(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,, \tag{5.132}$$

whose solution is in the form of Eq. (5.30). Thus, we can use Eqs. (5.68a-5.68b) to find: $\langle X(t)\rangle = X_0 + F_0\,t$ and $\left\langle X^2(t)\right\rangle_0 = X_0^2 + \frac{2\,\sigma}{\Gamma(1+\alpha)}t^\alpha$, leading to the modified relation:

$$\langle X(t)\rangle = t^{1-\alpha}\,\Gamma(1+\alpha)\,\frac{F_0}{2\,\sigma}\,\left\langle X^2(t)\right\rangle_0 \tag{5.133}$$

As a sanity check, we note that for $\alpha = 1$ we recover the FDR Eq. (5.129). Thus, weak GI anomalous processes violates the normal FDRI Eq. (5.129), as a time dependent factor naturally appears in Eq. (5.133). As in the case of Brownian dynamics, Eq. (5.133) also holds in different Galilean reference frames, with the proper measured external force, accounting for the effect of the external flow, due to the transformation rule of Eq. (5.132). This is different from what we find for CTRWs. As before, if we consider a new frame moving with velocity v_0 with respect to the one where Eq. (5.130) holds and we use the correct transformation rules Eqs. (5.33, 5.35), we find the transformed equation:

$$\begin{aligned}\frac{\partial}{\partial t}P(q,t) = v_0\frac{\partial}{\partial q}P(q,t) + \left[-F_0\,\frac{\partial}{\partial q} + K_\alpha\frac{\partial^2}{\partial q^2}\right] \times \\ \times\left[\frac{\partial}{\partial t} - v_0\,\frac{\partial}{\partial q}\right]\frac{1}{\Gamma(\alpha)}\int_0^t \frac{P(q+v_0(t-\tau),\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau\,,\end{aligned} \tag{5.134}$$

where now the external flow, induced by the moving frame, is no longer equivalent to a constant external force. In this case, Eq. (5.129) is no longer satisfied as we show below. Indeed, whereas the quantity $\left\langle X^2(t)\right\rangle_0$ is equal to the one computed before in the case of Eq. (5.130), the other quantity $\langle X(t)\rangle$ is different in general and we need to compute it. To this aim, we first need to compute the solution of Eq. (5.134) and then exploit Eq. (5.67). In Fourier-Laplace transform, the solution of Eq. (5.134) is given below:

$$P(k,\lambda) = \frac{\left\langle e^{i\,k\,x_0}\right\rangle}{\lambda + i\,k\,v_0 - i\,k\,F_0\,(\lambda + i\,k\,v_0)^{1-\alpha} + K_\alpha\,k^2\,(\lambda + i\,k\,v_0)^{1-\alpha}}. \tag{5.135}$$

The first moment corresponding to this equation is given in general by Eq. (5.68a) with $D(k,\lambda) = \lambda + i\,k\,v_0 - i\,k\,F_0\,(\lambda + i\,k\,v_0)^{1-\alpha} + \sigma\,k^2\,(\lambda + i\,k\,v_0)^{1-\alpha}$. Specifically, we obtain: $D(0,\lambda) = \lambda$, $D'(0,\lambda) = i\,v_0 - i\,F_0\,\lambda^{1-\alpha}$, such that $\langle X(\lambda)\rangle = \frac{-v_0 + F_0\,\lambda^{1-\alpha}}{\lambda^2}$, or back in time space $\langle X(t)\rangle = Y_0 - v_0\, t + F_0 \frac{t^\alpha}{\Gamma(1+\alpha)}$. Recalling that in the absence of force the quantity $\left\langle X^2(t)\right\rangle_0$ is the same as in the previous cases, we obtain the following modified FDR:

$$\langle X(t)\rangle = \frac{F_0}{2\,\sigma}\left\langle X^2(t)\right\rangle_0 - \frac{v_0}{2\,\sigma}\left\langle X^2(t)\right\rangle_0 \Gamma(1+\alpha)\, t^{1-\alpha}, \tag{5.136}$$

which is composed of two different terms: (i) the usual FDR Eq. (5.129) and (ii) a time dependent correction, with the same structure of Eq. (5.133). As a sanity check, we note that for $\alpha = 1$, we find: $\langle X(t)\rangle = \frac{F-v_0}{2\,\sigma}\left\langle X^2(t)\right\rangle_0$, which is of the form of Eq. (5.129) where the force is equal to the difference between the constant force exerted on the tracer particle and the external flow, due to the change of reference frame. We further note that if we choose $v_0 = F$ in the Brownian case, the rhs of Eq. (5.136) is null, which is expected as the Galilean transformation brings the system in its comoving frame, where no resulting external force acts on the tracer. The result presented in Eq. (5.136) elucidates that (i) the validity of the FDRI for anomalous processes, and CTRWs in particular, is restricted to the specific reference frame where Eq. (5.130) holds and that (ii) the long-range interactions between the bath particles and the tracer break the equivalence between a constant force and an external flow, which instead holds for Brownian and weak GI anomalous processes.

We conclude this section by showing that we can define $\overline{\xi}-$driven processes satisfying FDRI Eq. (5.129) at the expenses of weak Galilean invariance. Let us consider the following generalized Langevin equation driven by the $\overline{\xi}$ noise:

$$\int_0^t \gamma(t,s)\dot{X}(s)\,\mathrm{d}s \, = F_0 + \sqrt{2\,K_\alpha}\,\overline{\xi}(t) \tag{5.137}$$

where K_α is a generalised diffusion coefficient and $\gamma(t_1, t_2)$ is a memory kernel, chosen such that the FDRI is established for the resulting process X. We remark that the non stationarity of $\overline{\xi}$ implies that the kernel γ depends explicitly on both the times t and s, and not only on their difference as in the usual generalized Langevin equation [205]. Despite this fact, we show that if a FDR of the second kind is established, also a FDRI holds, thus confirming results already presented for Gaussian stationary noises [206]. Recalling the two-point correlation function of $\overline{\xi}$ [200], we make the following assumption:

$$\gamma(t_1, t_2) = \left\langle\overline{\xi}(t_1)\overline{\xi}(t_2)\right\rangle = K(t_1)\delta(t_2 - t_1). \tag{5.138}$$

Substituting this into Eq. (5.137) we find the following equation:

$$K(t)\,\dot{X}(t) = F_0 + \sqrt{2\,K_\alpha}\,\overline{\xi}(t). \tag{5.139}$$

This equation can now be easily integrated to obtain the stochastic path (we set $X_0 = 0$):

$$X(t) = F_0 \int_0^t \frac{\mathrm{d}s}{K(s)} + \sqrt{2\,K_\alpha}\int_0^t \frac{\mathrm{d}s}{K(s)}\overline{\xi}(s), \tag{5.140}$$

which can finally be used to compute first and second moment:

$$\langle X(t)\rangle = F_0 \int_0^t \frac{\mathrm{d}s}{K(s)} \tag{5.141a}$$

$$\left\langle X^2(t)\right\rangle = F_0^2 \int_0^t \frac{\mathrm{d}s_1}{K(s_1)} \int_0^t \frac{\mathrm{d}s_2}{K(s_2)} + 2\,K_\alpha \int_0^t \frac{\mathrm{d}s}{K(s)} \tag{5.141b}$$

Evaluating Eq. (5.141b) for $F_0 = 0$ and comparing it with Eq. (5.141a), we find:

$$\langle Y(t)\rangle = \frac{F_0}{2\,K_\alpha} \left\langle Y^2(t)\right\rangle_0 . \tag{5.142}$$

As a sanity check, we note that in the Brownian limit we have $K(t) = 1$ and $K_\alpha = \sigma$ so that Eq. (5.129) is recovered. As an example, let us consider the case of a power-law kernel, i.e., $K(t) = t^{\alpha-1}/\Gamma(\alpha)$. In this case, if we set $K_\alpha = \frac{\sigma}{\Gamma(\alpha)}$, we recover the normal FDRI Eq. (5.129). In this specific case, Eq. (5.139) reads as follows:

$$\dot{X}(t) = F_0\, t^{1-\alpha} + \sqrt{2\,K_\alpha}\, t^{1-\alpha}\, \bar{\xi}(t). \tag{5.143}$$

5.7 Outlook and Future Work

In this Chapter, we clarified the role of Galilean invariance for both normal and anomalous diffusive processes. Specifically, we show that in this context a natural distinction arises between strong and weak Galilean invariance. While the former is satisfied if the EOMs of the observed diffusive process are the same in different inertial reference frames in constant uniform motion between themselves, the latter is satisfied at a coarse-grained, statistical level, namely when the PDFs of either the position or the velocity in different Galilean frames are obtained by performing the same Galilean transformation connecting the frame coordinates on the sample state variables of the distribution. Such transformation of the PDFs has been recently proposed in Ref. [185] in the framework of the Navier-Stokes equations. This result agrees with our description, as these equations are strong GI, which also implies their being weak GI. On the contrary, this is not the case for diffusive processes.

Indeed, starting from the Mori-Zwanzig description of the motion of a particle in a heat bath [15] and considering such model in different Galilean frames, we showed that (i) the exact EOM of the tracer particle is strong GI if its interaction with the bath particles is pairwise and dependent on the difference between their positions and (ii) that the coarse-graining procedure employed to derive the underdamped Langevin equation, which introduces a stochastic random force with specified statistical properties to describe the overall effect of the bath, naturally breaks strong GI. This is caused by the fact that a transformation rule for such noise term cannot be determined.

Nevertheless, at least for normal diffusive processes, weak Galilean invariance still holds. On the contrary, in the case of anomalous diffusive processes, different scenarios may be encountered. In particular, we discussed that CTRWs do not satisfy this property, whereas the processes introduced in Chapter 4, and described by Langevin equations driven by the noise defined in Eq. (4.14), do instead. Within this discussion, we derived their fractional evolution equations (for a constant external force), thus improving the corresponding fractional advection-diffusion equation (MK) earlier presented in the Literature [78, 190, 9].

Our new proposed equation involves the fractional substantial derivative [150, 149], as the requirement of weak Galilean invariance naturally induces a strong space-time coupling in the dynamical evolution of the process, and it holds both in the subdiffusive and in the superdiffusive regime. Thus, our $\bar{\xi}$-driven processes represent the Langevin formulation of weak GI anomalous processes, and of weak GI CTRWs [207] as a special case.

We remark that the characterisation that we propose here of weak GI anomalous processes exclusively holds in the specific case of a constant external force. For future work, it will be interesting to extend our results for general position dependent forces and to derive a complete characterisation of their functionals, similarly to what discussed in Chapter 3 for anomalous processes with general waiting time distribution. These results would provide a complete frame-independent framework, that could be employed to investigate the stochastic thermodynamics of anomalous stochastic processes of this type [208]. In addition, it will be necessary to develop efficient numerical techniques for the simulation of $\bar{\xi}$ in the superdiffusive regime. Indeed, contrarily to the subdiffusive case, where the subordination picture can still be employed to simulate it by using the Algorithm 1 in Sec. 3.3.1, in the superdiffusive case no subordinator T can be defined, such that the formal Eq. (4.14) holds, to the extent of our knowledge. Consequently, one can only employ the characterisation of the noise in terms of higher order correlation functions given in Eq. (4.35), for which however no suitable numerical techniques have so far been derived. Finally, it will be interesting to assess the experimental relevance of weak GI anomalous processes, by checking for the occurrence of time-shifted PDFs in experimental datasets.

CHAPTER 6

Conclusions

In this Doctoral Dissertation we conducted an extensive investigation of the CTRW model, which aims at providing a complete toolbox of methods and techniques, which can potentially be employed by experimental researchers in physics, chemistry and biology to interpret experimental data of systems exhibiting anomalous diffusive behaviour. Our theoretical analysis is focused on two main different aspects, covered by the research Chaps. 3-5.

On the one hand, in Chap. 3 we employed the general description of the waiting time distribution of a CTRW by means of its Laplace exponent to study more general anomalous diffusive processes, that can account for a wider range of MSD scaling behaviour than the pure power law, which is obtained in the specific case of Lévy stable distributed waiting times characteristic of ordinary CTRWs. We then derived the complete characterisation of these processes and of their observables in terms of (i) the description of their microscopic dynamics in terms of subordinated Langevin equations, (ii) fractional evolution equations, specifically the GFFK equation for the joint PDF of the process and its observables, and (iii) their multipoint correlation functions. In addition, we showed the relevance of our formalism for experimental applications, by successfully fitting the MSD of diffusing mitochondria in S. Cerevisiae cells and by deriving the two point correlation function of the corresponding subordinated process, which can be readily tested in the experiments.

On the other hand, in Chaps. 4, 5 we formulated a new class of anomalous stochastic processes, which share the same renewal picture of CTRWs for the elapsed physical time, but not for the position variable, when external forces are present. Indeed, differently from the original model, where such forces affect the dynamics only during the jumps, in the case of our new processes they are also exerted during the waiting times, thus implying that the position can no longer be expressed as a sum of i.i.d random variables. We characterise the microscopic dynamics of these new anomalous processes in terms of Langevin equations driven by a novel non Gaussian noise, whose properties are determined both in terms of its characteristic functional and of its hierarchy of correlation functions, which is able to reproduce the typical fluctuations of a free diffusive CTRW. We further characterised their fractional evolution equation in the case of a constant external force and show that they satisfy weak Galilean invariance, contrarily to the case of ordinary CTRWs.

Despite being calibrated on the specific case of the CTRW model, the detailed formalism that we presented in this Thesis comprises techniques and general ideas that could well be applied to other commonly used models of anomalous diffusion, for instance the fractional Brownian motion, Lévy flights and walks, or models of diffusion in viscoelastic medium, thus laying the foundation of a comprehensive framework for the analysis of general anomalous diffusive systems. Such an ambitious scope will be of relevance for both experimental and theoretical researchers directly working on such systems and/or more generally interested in the experimental study of diffusive processes in biological systems.

In addition to this general scope, our results pose several questions still unresolved, that we suggest as possible future work. One major relevant challenge will be to further test the applicability of the general formulas and techniques provided in Chapter 3 for the modelling of experimental datasets and their effectiveness in assessing the microscopic processes underlying the observed experimental dynamics. Even though we discussed in details one interesting experimental dataset therein, the capability of our formalism in reproducing other different dynamical behaviours, for instance MSD displaying crossover from subdiffusion to superdiffusion or ballistic motion recently found in moving chromosomal loci of Escherichia Coli [130], needs to be investigated. A second major challenge will be to test the experimental relevance of the weak GI anomalous stochastic processes, which could be applicable to experimental systems exhibiting time shifted PDF.

On the theoretical side, the general formulation derived for both the anomalous processes with general waiting time distribution and weak GI anomalous processes can be potentially employed to define toy-models to further elucidate the fundamental nature of specific properties of CTRWs, for instance weak ergodicity breaking or ageing, with the aim at understanding if either the choice of the waiting time distribution or how external forces are included in the dynamical process play a major role in determining them. Another major challenge will be to derive the characterisation of the functionals of weak GI anomalous processes, that was not discussed in the present Thesis. This will also represent the first preliminary step towards a throughout study of the stochastic thermodynamics of anomalous processes, which is still an outstanding challenge for ordinary CTRWs, on which our weak GI anomalous processes will shed some light by providing a consistent frame-invariant framework for its analysis.

APPENDIX A

Special Functions: Definitions and Useful Relations

In this Appendix, we present definitions and useful relations for several special functions, which have been used throughout the thesis.

A.1 The Confluent Hypergeometric Function

In this section, we discuss definition, integral representations and scaling behaviour of the confluent Hypergeometric function. We refer to Refs. [209, 210] for the proof of the results presented here and for further useful properties. The hypergeometric function is the solution of the differential equation:

$$z\,\frac{\mathrm{d}^2}{\mathrm{d}z^{\,2}}w + (c-z)\frac{\mathrm{d}}{\mathrm{d}z}w - a\,w = 0 \tag{A.1}$$

where a, c are parameters and both a regular singularities for $z = 0$ of exponents 0 and $1-c$ and an irregular singularity at infinity of rank 1 are present. A series solution of such equation, corresponding to the exponent 0 at the origin, is given as follows:

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!} \qquad (b \neq 0, -1, -2, \ldots) \tag{A.2}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. This function is equivalently denoted as $M(a, b, z)$ or $\Phi(a, b; z)$. This series converges for all z, thus defining an entire function in the complex plane. On the contrary, it is only a meromorphic function in b due to the presence of the poles. In order to eliminate these singularities and have an entire function both in a and b for fixed z, the following modified function is often used:

$$\boldsymbol{M}(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(b+n)}\frac{z^n}{n!}. \tag{A.3}$$

The solution for the second exponent at the origin and its regularised form are the following:

$$N(a,b,z) = z^{1-c}\,{}_1F_1(1+a-b;2-b;z) \qquad (b \neq 2,3,4,\ldots) \tag{A.4a}$$

$$\boldsymbol{N}(a,b,z) = \frac{1}{\Gamma(2-c)}N(a,b,z) = z^{1-c}\boldsymbol{M}(a,b,z) \tag{A.4b}$$

The following useful integral expression can be found:

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 s^{a-1}(1-s)^{b-a-1}\,e^{z\,s}\,\mathrm{d}s \qquad (\operatorname{Re} c > \operatorname{Re} a > 0) \tag{A.5}$$

In particular, let us set $b = 1 + a$. The following relation is easily shown:

$${}_1F_1(a;1+a;-z) = a\int_0^1 s^{a-1}\,e^{-z\,s}\,\mathrm{d}s \;=\; \frac{a}{z^a}\int_0^z s^{a-1}\,e^{-s}\,\mathrm{d}s \;=\; \frac{a}{z^a}\gamma(a,z). \tag{A.6}$$

We conclude this section by reporting the asymptotic behaviour of $\boldsymbol{M}$, which are relevant when exploring MSD behaviour. In this case, $\boldsymbol{M}$ is a function of time and we are interested in the limits $t \to 0$ or $t \to \infty$. We find:

$$\boldsymbol{M}(a,b,t) \sim \frac{1}{\Gamma(a)}e^t\,t^{a-b} \qquad t \to \infty \tag{A.7}$$

$$\boldsymbol{M}(a,b,t) \sim \frac{1}{\Gamma(b)} \qquad t \to 0 \tag{A.8}$$

A.2 The Three Parameter Mittag-Leffler Function

In this section, we discuss definition and properties of the three parameter Mittag-Leffler function. We refer to [211] and references therein for the proof of the following relations and for other useful properties. The three parameter Mittag-Leffler function is defined as:

$$E^{\delta}_{\alpha,\beta}\left(z\right) = \sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma\left(\beta+\alpha n\right)}\frac{z^n}{n!} \tag{A.9}$$

where $(\delta)_n = \Gamma(\delta+n)/\Gamma(\delta)$ is the Pochhammer symbol. The two and one parameter Mittag-Leffler functions $E_{\alpha,\beta}\left(z\right)$ and $E_{\alpha}\left(z\right)$ can be obtained as special cases of Eq. (A.9) by setting $\delta = 1$, and also $\beta = 1$ in the latter case. Its Laplace transform is given by:

$$\mathcal{L}\left\{z^{\beta-1}E^{\delta}_{\alpha,\beta}\left(\pm c\,z^{\alpha}\right)\right\}(\lambda) = \frac{\lambda^{\alpha\,\delta-\beta}}{(\lambda^{\alpha}\mp c)^{\delta}} \tag{A.10}$$

with $\operatorname{Re}(\lambda) > |c|^{1/\alpha}$. The three parameter Mittag-Leffler function can be expressed as a Fox H-function (see below for its definition). The exact relation is given by [202, 211]:

$$E^{\delta}_{\alpha,\beta}\left(\pm\,z\right) = \frac{1}{\Gamma(\delta)}H^{1,1}_{1,2}\left[\mp\,z\,\middle|\,\begin{matrix}(1-\delta,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right]. \tag{A.11}$$

This formula can be derived by solving the corresponding integral Eq. (A.16) with the residue theorem. In several physical systems, displaying anomalous diffusive behaviour, this function plays a major role, as it often describes their corresponding MSD (in this case then z is the time variable). It is then important to study its asymptotic scaling for both short and large times. In the latter case, the function $E^{\delta}_{\alpha,\beta}\left(-t^{\alpha}\right)$ behaves as a stretched exponential. Indeed, by looking at the definition Eq. (A.9) we can write:

$$\begin{aligned} E^{\delta}_{\alpha,\beta}\left(-t^{\alpha}\right) &\sim \frac{1}{\Gamma(\beta)} - \delta\,\frac{t^{\alpha}}{\Gamma(\alpha+\beta)} \\ &\sim \frac{1}{\Gamma(\beta)}\exp\left(-\delta\,\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}t^{\alpha}\right) \end{aligned} \tag{A.12}$$

In the former case, it is convenient to look at the equivalent definition [212, 213, 214]:

$$E^{\delta}_{\alpha,\beta}\left(-z\right) = \frac{z^{-\delta}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)}{\Gamma\left(\beta-\alpha(\delta+n)\right)}\frac{-z^{n}}{n!}, \tag{A.13}$$

which elucidates that in the long time limit the function $E^{\delta}_{\alpha,\beta}\left(-t^{\alpha}\right)$ behaves as a power-law:

$$E^{\delta}_{\alpha,\beta}\left(-t^{\alpha}\right) \sim \frac{t^{-\alpha\,\delta}}{\Gamma(\beta-\alpha\,\delta)} \tag{A.14}$$

We recall that further extensions of the definition Eq. (A.9) has been derived in Refs. [215, 216], which have proved useful to describe the MSD behaviour of processes subordinated with a mixture of Lévy stable distributions with different characteristic exponents [203]. In conclusion, we report below the calculation of the convolution integral of a generalised ML function with a power-law, which is used extensively throughout the main text:

$$\begin{aligned} \int_0^t (t-\tau)^{m_1}\,\tau^{m_2}\,E^{\delta}_{\alpha,\beta}\left(c\,\tau^{\alpha}\right)\mathrm{d}\tau &= \sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma\left(\beta+\alpha n\right)}\frac{c^n}{n!}\int_0^t (t-\tau)^{m_1}\,\tau^{m_2+\alpha\,n}\,\mathrm{d}\tau \\ &= \sum_{n=0}^{\infty}\frac{(\delta)_n\,m_1!\,\Gamma(1+m_2+\alpha\,n)}{\Gamma\left(\beta+\alpha n\right)}\frac{c^n}{n!}\mathcal{L}^{-1}\left\{\frac{1}{\lambda^{2+m_1+m_2+\alpha\,n}}\right\}(t) \\ &= m_1!\,t^{1+m_1+m_2}\sum_{n=0}^{\infty}\frac{(\delta)_n\,\Gamma(1+m_2+\alpha\,n)}{\Gamma\left(\beta+\alpha n\right)\Gamma(2+m_1+m_2+\alpha\,n)}\frac{(c\,t^{\alpha})^n}{n!}. \end{aligned} \tag{A.15}$$

A.3 The Fox H-Function

In this section, we study the properties of the Fox H-function. All the results presented here, except where specified, are adapted from Ref. [202]. The Fox H-function is a special function, which is formally defined in terms of the Mellin-Barnes type integral:

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}[a_p, A_p]\\ [b_q, B_q]\end{matrix}\right] = \frac{1}{2\,\pi\,i}\int_{\Omega}\Theta(s)\,z^{-s}\,\mathrm{d}s\,, \tag{A.16}$$

where $i=(-1)^{-1/2}$ is the imaginary unity, $z\neq 0$ and $z^{-s}=\exp\left[-s\left(\ln|z|+i\arg z\right)\right]$. Here, $\ln|z|$ stands for the natural logarithm of $|z|$, whereas $\arg z$ is not necessarily its principal value. The function $\Theta(s)$ is defined in terms of gamma functions as follows:

$$\Theta(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+B_j\,s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-A_j\,s)\right\}}{\left\{\prod_{j=1+m}^{q}\Gamma(1-b_j-B_j\,s)\right\}\left\{\prod_{j=1+n}^{p}\Gamma(a_j+A_j\,s)\right\}}, \tag{A.17}$$

where $m, n, p, q \in \mathbb{N}_0$ with $0 \le n \le p$ and $1 \le m \le q$; $A_j, B_j \in \mathbb{R}_+$; $a_j, b_j \in \mathbb{R}$ (or alternatively $\mathbb{C}$) with $i = 1, \dots, p$ and $j = 1, \dots, q$. Any empty product in Eq. (A.17) is to be interpreted as unity. The contour Ω in Eq. (A.17) is suitably chosen to separate the poles $\xi_{j\,\nu} = -\left(\frac{\nu + b_j}{B_j}\right)$, with $j = 1, \dots, m$ and $\nu \in \mathbb{N}_0$, of $\Gamma(b_j + B_j s)$ from the poles $\chi_{i\,\nu} = \left(\frac{1 - a_i + \nu}{A_i}\right)$, with $i = 1, \dots, n$ and same ν, of $\Gamma(1 - a_j - A_j s)$. Thus, the condition $A_i\,(b_j + \nu) \neq B_j\,(a_i - 1 - \nu)$ ensures the existence of the contour Ω and consequently the convergence of the integral of Eq. (A.17). A popular choice for the contour Ω consists in a path running parallel to the imaginary axis from $\gamma - i\,\infty$ to $\gamma + i\,\infty$, where $\gamma \in \mathbb{R} = (-\infty, +\infty)$ is chosen arbitrarily such that it separates all the poles $\xi_{j\,\nu}$ from all the poles $\chi_{i\,\nu}$. If we choose such a contour, the convergence of the Mellin-Barnes integral Eq. (A.16) is obtained if $a^* > 0$ and $|\arg z| < \frac{1}{2}\,\pi\,a^*$, $z \neq 0$, with a^* being the following parameter:

$$a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{A.18}$$

The integral also converges if $a^* = 0$, $\gamma\,\mu + \mathrm{Re}(\delta) < -1$, $\arg z = 0$ and $z \neq 0$, where

$$\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \tag{A.19}$$

Other equivalent choices of Ω, with the corresponding convergence conditions for the integral of Eq. (A.17), can be found. A first useful property of the H-function regards its symmetry under exchange of the pairs of parameters $[a_p, A_p]$ and/or $[b_p, B_p]$. Specifically, the Fox H-function is symmetric under permutations of the pairs (a_i, A_i) for $i = 1, \dots, n$ or separately for $i = n+1, \dots, p$; likewise it is symmetric if we make a permutation of the pairs (b_j, B_j) for $j = m+1, \dots, q$ or separately for $j = 1, \dots, m$. A second property enables us to reduce the order of the H function if some of the pairs $[a_p, A_p]$ or $[b_p, B_p]$ are the same. Indeed, this reduction property holds if one of the pairs (a_i, A_i) for $i = 1, \dots, n$ is equal to one of the pairs (b_j, B_j) for $j = 1+m, \dots, q$, or alternatively for $i = 1+n, \dots, p$ and $j = 1, \dots, m$. In these different cases, the Fox H-function reduces to one of lower order with p, q and n (or m respectively) decreased by one. In formulas, we have:

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (a_1, A_1) \end{matrix}\right] = H^{m,n-1}_{p-1,q-1}\left[z \,\middle|\, \begin{matrix} (a_2, A_2), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}) \end{matrix}\right], \tag{A.20}$$

provided $n \ge 1$ and $q > m$; and alternatively:

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (b_1, B_1) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix}\right] = H^{m-1,n}_{p-1,q-1}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}) \\ (b_2, B_2), \dots, (b_q, B_q) \end{matrix}\right], \tag{A.21}$$

provided $m \ge 1$ and $p > n$. The Fox H-function satisfies the following scaling relation:

$$H^{m,n}_{p,q}\left[z^r \,\middle|\, \begin{matrix} [a_p, A_p] \\ [b_q, B_q] \end{matrix}\right] = \frac{1}{r} H^{n,m}_{q,p}\left[z \,\middle|\, \begin{matrix} [a_p, A_p/r] \\ [b_q, B_q/r] \end{matrix}\right] \qquad \forall r \in \mathbb{R}_+/\{0\} \tag{A.22}$$

Other two formulas are relevant for the calculations presented in the thesis. A first relation enables us to invert the independent variable inside the H-function:

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]=H_{q,p}^{n,m}\left[\frac{1}{z}\,\middle|\,\begin{matrix}[1-b_q,B_q]\\ [1-a_p,A_p]\end{matrix}\right]; \tag{A.23}$$

The second one instead enables one to absorb powers of the independent variable of exponent $\sigma\in\mathbb{C}$ inside the H-function:

$$z^{\sigma}H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]=H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}[a_p+\sigma A_p,A_p]\\ [b_q+\sigma B_q,B_q]\end{matrix}\right]. \tag{A.24}$$

The following two properties have been used thoughtfully in Chapter 5. On the one hand, we report the Mellin-cosine(sine) transform of the Fox H-function [217]:

$$\begin{aligned}&\int_0^{+\infty} z^{\rho-1}\left\{\begin{matrix}\sin(\kappa z)\\ \cos(\kappa z)\end{matrix}\right\}H_{p,q}^{m,n}\left[az^r\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]\mathrm{d}z=\\ &\frac{2^{\rho-1}\sqrt{\pi}}{\kappa^{\rho}}H_{p+2,q}^{m,n+1}\left[a\left(\frac{2}{\kappa}\right)^r\,\middle|\,\begin{matrix}\left(\left(\frac{3\mp1-2\rho}{4}\right),\frac{r}{2}\right), & [a_p,A_p], & \left(\left(\frac{3\pm1-2\rho}{4}\right),\frac{r}{2}\right)\\ & [b_q,B_q] & \end{matrix}\right]\end{aligned} \tag{A.25}$$

where the following conditions must be satisfied: (i) $\alpha,r,\kappa>0$, (ii) $|\arg(a)|<\alpha\,\frac{\pi}{2}$, (iii) $\mathrm{Re}\,(\rho)+r\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{B_j}\right)>\frac{(-1\mp1)}{2}$, (iv) $\mathrm{Re}\,(\rho)+r\max_{1\le j\le n}\mathrm{Re}\left(\frac{a_j-1}{A_j}\right)<1$. On the other hand, we report the Mellin transform of a general H-function:

$$\int_0^{\infty} z^{\xi-1}H_{p,q}^{m,n}\left[az\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]\mathrm{d}z=a^{-\xi}\Theta(\xi) \tag{A.26}$$

where the function Θ is defined in Eq. (A.17). Another useful formula enables us to evaluate a general convolution integral of the Fox H-function. This is given below:

$$\begin{aligned}&\int_0^{a} z^{\alpha-1}(a-z)^{\beta-1}H_{p,q}^{m,n}\left[\omega\,z^r(a-z)^{\rho}\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]\mathrm{d}z=\\ &a^{\alpha+\beta-1}H_{p+2,q+1}^{m,n+2}\left[\omega\,a^{r+\rho}\,\middle|\,\begin{matrix}(1-\alpha,r), & (1-\beta,\rho), & [a_p,A_p]\\ [b_q,B_q] & (1-\alpha-\beta,r+\rho) & \end{matrix}\right],\end{aligned} \tag{A.27}$$

which holds under the conditions: (i) $m,n\neq0$, (ii) $r,\rho\geq0$, (iii) $a,a^*>0$, (iv) $|\arg(\omega)|<a^*\,\frac{\pi}{2}$, (iii) $\mathrm{Re}\,(\alpha)+r\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{B_j}\right)>0$, (iv) $\mathrm{Re}\,(\alpha)+\rho\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{B_j}\right)>0$. Finally, we provide the following formula to compute the derivatives of the H-function:

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}H_{p,q}^{m,n}\left[(c\,x+d)^h\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]=\left(\frac{c}{c\,x+d}\right)^rH_{1+p,1+q}^{m,1+n}\left[(c\,x+d)^h\,\middle|\,\begin{matrix}(0,h),[a_p,A_p]\\ [b_q,B_q],(r,h)\end{matrix}\right] \tag{A.28}$$

We conclude this section by presenting a MATHEMATICA code for the evaluation of the Fox H-function, which is based on the numerical evaluation of the Mellin-Barnes integral in Eq. (A.16). We note that W is a large parameter ($W=100$ in the evaluations presented in the thesis), whereas $R\in\mathbb{R}$ is chosen such as the poles $\xi_{j\,\nu}$ and $\chi_{i\,\nu}$ are separate.

Algorithm 2 MATHEMATICA Code for Numerical Evaluation of the H-function

```
FoxH[m_, n_, p_, q_, A_, B_, z_, R_, W_]:=Module[
(*-- Local Variables --*)
{AL, AR, BL, BR, V, s},
AL = A[[All, 1]]; AR = A[[All, 2]];
BL = B[[All, 1]]; BR = B[[All, 2]];
(*-- Auxiliary Functions--*)
FL[s_, aL_, aR_]:=1 - aL - aR s;
FR[s_, aL_, aR_]:=aL + aR s;
(*-- Moment Generating Function --*)
MT[s_]:=Module[{F}, F = 1;
Do[F*= Gamma[FR[s, BL[[l]], BR[[l]]]], {l, 1, m}];
Do[F*= Gamma[FL[s, AL[[l]], AR[[l]]]], {l, 1, n}];
Do[F/= Gamma[FL[s, BL[[l]], BR[[l]]]], {l, 1 + m, q}];
Do[F/= Gamma[FR[s, AL[[l]], AR[[l]]]], {l, 1 + n, p}];
Return[F];
];
(*-- Contour Integration (Summation of poles )--*)
V = 1/(2πi) Quiet [NIntegrate [MT[s] z^-s, {s, R - i W, R + i W}]] ;
(*-- Return Value of FoxH--*)
Return[V];
]
```

APPENDIX B

Numerical Generation of Random Variables

In this Appendix, we present the numerical algorithms used to generate: (i) Lévy stable RVs; (ii) tempered Lévy stable RVs. These are used to numerically integrate the Langevin Eq. (3.4b), which is needed for the simulation of the stochastic trajectories of a CTRW, as we explained in the Algorithm 1. Extensive Monte Carlo simulations of the trajectories of a CTRW with either tempered or not Lévy stable waiting time distribution are used to confirm analytical results in all the research Chapters 3-5.

B.1 Lévy Stable Random Variables

We first focus on how to generate a Lévy stable RV $\eta_\alpha(s_i, \Delta s)$, where i denotes different sample values and $0 < \alpha \leq 1$ is the order parameter of the stable distribution. We present the algorithm of Refs. [218, 81]. The steps to be performed are the following:

1. we generate a RV V_i uniformly distributed on the interval $]-\pi/2, \pi/2[$. This can be realised by sampling a uniformly distributed RV u_i^1 on $]0,1[$, which can be done straightforwardly (for instance with the techniques given in [219]), and then setting $V_i = \pi\left(u_i^1 - 1/2\right)$. However, equivalent optimised techniques are available;

2. we generate a second independent RV W_i exponentially distributed with mean 1. This is obtained similarly by sampling another uniformly distributed RV u_i^2 on $[0,1]$, independent on u_i^1, and setting $W_i = -\log\left(u_i^2\right)$ or with other equivalent techniques.

3. we finally generate the Lévy distributed RV by setting:

$$\eta_\alpha(s_i, \Delta s) = (\Delta s)^{1/\alpha} \frac{\sin\left[\alpha\left(V_i + \frac{\pi}{2}\right)\right]}{\left[\cos\left(V_i\right)\right]^{1/\alpha}} \left\{ \frac{\cos\left[V_i - \alpha\left(V_i + \frac{\pi}{2}\right)\right]}{W_i} \right\}^{(1-\alpha)/\alpha}. \tag{B.1}$$

We refer to the original Refs. [218, 81] for the numerical verification that the RVs $\eta_\alpha(s_i, \Delta s)$ have the expected distribution. By using adequate random number generators for V_i and W_i, the resulting Lévy RVs are uncorrelated as required. For $\alpha = 1$, we obtain: $\eta_\alpha(s_i, \Delta s) = \Delta s$ as expected.

B.2 Tempered Lévy Stable Random Variables

We now focus on how to generate RVs distributed according to a tempered Lévy stable distribution with parameters μ (tempering) and α (stability), with $0 < \alpha \geq 1$ as before. We will use the rejection method of Ref. [161]. The steps to be performed are the following:

1. we generate a first Lévy stable RV $\eta_\alpha(s_i, \Delta s)$ of order parameter α with the algorithm presented in the earlier section;

2. we then draw a second RV Y_i with exponential distribution of parameter μ^{-1}, again as discussed in the previous section;

3. if $Y_i < \eta_\alpha(s_i, \Delta s)$, we reject both and repeat $(1-2)$; otherwise, we keep $\eta_\alpha(s_i, \Delta s)$.

A proof that this algorithm provides RVs distributed according to a tempered Lévy stable distribution is found in [161]. We conclude by plotting exemplary stochastic paths of CTRWs with tempered Lévy stable distributed waiting times with $\alpha = 0.5$ and $\mu = \{0, 0.1, 1, 100\}$. The transition from subdiffusion to normal diffusion, as μ increases, is evident.

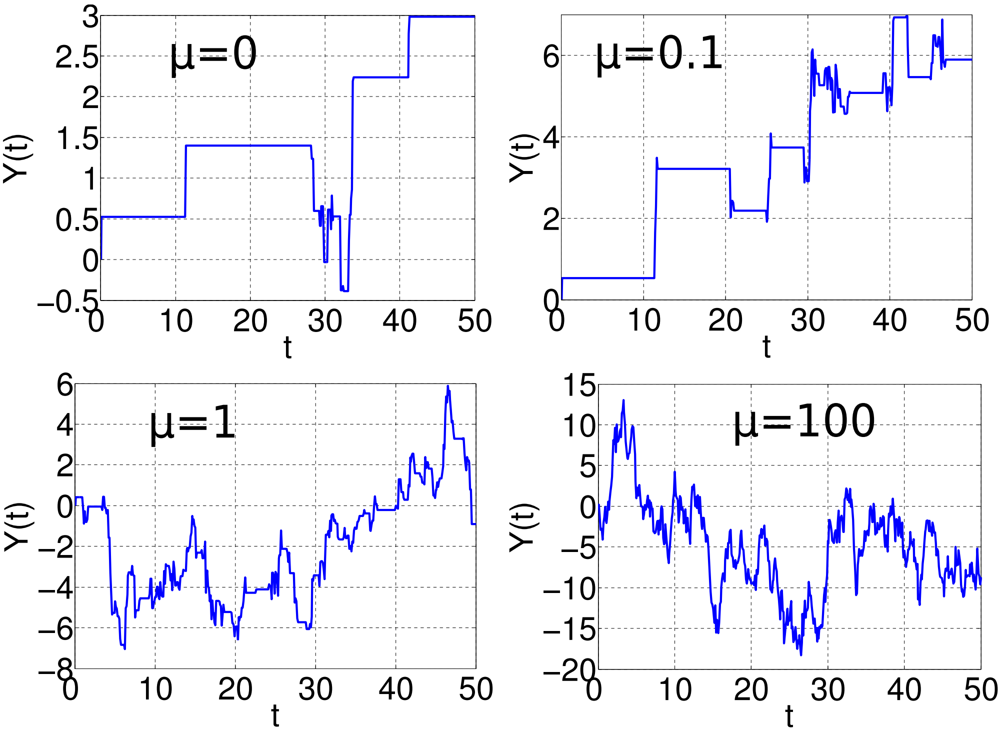

Figure B.1: (Colors Online) Exemplary stochastic trajectories of CTRWs with tempered Lévy stable distributed waiting times with order parameter $\alpha = 0.5$ and different values of the tempering parameter μ. Simulations of the subordinated Langevin Eqs. (2.46a, 2.46b) are obtained with the Algorithm 1 and with waiting time increments generated by the rejection method explained in the main text.

APPENDIX C

Calculation of Two-Point Correlation Functions

In this Appendix we present the calculations of the inverse Laplace transform of the functions w, f_1, f_2 defined in Eqs. (3.69, 3.65) for the specific examples presented in the main text. Specifically, we will choose the process X in Eq. (3.4a) to be an OU process [$F(x) = -\gamma\, x$ and $\sigma(x) = \sqrt{2\,\sigma}$ with $\gamma, \sigma \in \mathbb{R}^+$], in which case the Laplace transform of these functions is given by Eqs. (3.89). We will then specify η in Eq. (3.4b) as (i) a tempered Lévy stable noise with stability parameter $0 < \alpha < 1$ and tempering index μ, (ii) a mixture of two Lévy stable noises with different exponents $0 < \alpha < \beta < 1$ and (iii) a one sided Lévy noise with characteristic functional defined by the function Φ in Eq. (3.107).

C.1 Tempered Lévy Stable Subordinator

We first consider the case (i) where η is a tempered Lévy stable noise with parameters $0 < \alpha < 1$ (stability) and μ (tempering). As discussed in Sec. 2.2.3, its Laplace exponent is given by: $\Phi(\lambda) = (\lambda + \mu)^\alpha - \mu^\alpha$. We can then substitute it in Eqs. (3.89) and rearrange the terms in a convenient way to make the Laplace inverse transform. For f_1, we obtain:

$$\widetilde{f}_1(\lambda) = \frac{\sigma}{\gamma}\frac{1}{\lambda}\left[1 - \frac{\gamma}{(\lambda+\mu)^\alpha + \gamma - \mu^\alpha}\right] \tag{C.1}$$

Clearly, the first term in the brackets transforms back in time as a constant. For the second term, we need to employ the convolution theorem and the shifting property of the Laplace transform. In details, for $\gamma \neq \mu^\alpha$ we can write:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\lambda}\frac{1}{[(\lambda+\mu)^\alpha + \gamma - \mu^\alpha]}\right\}(t) &= \int_0^t e^{-\mu\,\tau}\mathcal{L}^{-1}\left\{\frac{1}{\lambda^\alpha + \gamma - \mu^\alpha}\right\}(\tau)\,\mathrm{d}\tau \\ &= \int_0^t e^{-\mu\,\tau}\tau^{\alpha-1}E_{\alpha,\alpha}(-(\gamma-\mu^\alpha)\,\tau^\alpha)\,\mathrm{d}\tau \end{aligned} \tag{C.2}$$

where we employed Eq. (A.10) to find the Laplace inverse transform of the integrand function. Putting everything together, we obtain:

$$f_1(t) = \frac{\sigma}{\gamma}\left[1 - \gamma \int_0^t e^{-\mu\,\tau}\tau^{\alpha-1} E_{\alpha,\alpha}(-\,(\gamma-\mu^\alpha)\;\tau^\alpha)\,\mathrm{d}\tau\right] \tag{C.3}$$

The integral can then be solved exactly by series expanding the generalised Mittag-Leffler function (see Eq. (A.9)). In details, we obtain:

$$\begin{aligned}\int_0^t e^{-\mu\,\tau}\tau^{\alpha-1} E_{\alpha,\alpha}(-\,(\gamma-\mu^\alpha)\;\tau^\alpha)\,\mathrm{d}\tau &= \sum_{n=0}^{\infty}\frac{(-1)^n(\gamma-\mu^\alpha)^n}{\Gamma(\alpha+\alpha\,n)}\int_0^t e^{-\mu\,\tau}\tau^{\alpha(1+n)-1}\,\mathrm{d}\tau\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{\mu^{\alpha(1+n)}}\frac{(\gamma-\mu^\alpha)^n}{\Gamma(\alpha+\alpha\,n)}\int_0^{\mu\,t} e^{-s}s^{\alpha(1+n)-1}\,\mathrm{d}s\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{\mu^{\alpha(1+n)}}\frac{(\gamma-\mu^\alpha)^n}{\Gamma(\alpha+\alpha\,n)}\gamma(\alpha(1+n),\mu\,t),\end{aligned} \tag{C.4}$$

where we made the change of variables $s = \mu\,\tau$ in the second line to write the integral as an incomplete gamma function. If we recall the relation Eq. (A.6):

$$\gamma(\alpha(1+n),\mu\,t) = \frac{(\mu\,t)^{\alpha+\alpha\,n}}{\alpha(1+n)} M(\alpha(1+n), 1+\alpha(1+n), -\mu\,t), \tag{C.5}$$

with M being the confluent hypergeometric function (see Appendix A.1), we can write:

$$\begin{aligned}&\int_0^t e^{-\mu\,\tau}\tau^{\alpha-1} E_{\alpha,\alpha}(-\,(\gamma-\mu^\alpha)\;\tau^\alpha)\,\mathrm{d}\tau =\\ &\qquad\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma-\mu^\alpha)^n}{\Gamma(1+\alpha(1+n))}t^{\alpha(1+n)}M(\alpha(1+n),1+\alpha(1+n),-\mu\,t) =\\ &\qquad\qquad\frac{-1}{\gamma-\mu^\alpha}\left[-1+\sum_{n=0}^{\infty}\frac{(-1)^n[(\gamma-\mu^\alpha)t^\alpha]^n}{\Gamma(1+\alpha\,n)}M(\alpha\,n,1+\alpha\,n,-\mu\,t)\right].\end{aligned} \tag{C.6}$$

In the rhs of this equation, we now find the function g defined in Eq. (3.97). Thus, by substituting g in Eq. (C.6) and then in Eq. (C.3), we derive the final result that was presented in Eq. (3.96b):

$$f_1(t) = \frac{\sigma}{\gamma}\left[1 - \frac{\gamma}{\gamma-\mu^\alpha} + \frac{\gamma}{\gamma-\mu^\alpha}\,g(t;\alpha,\gamma,\mu)\right] = \frac{\sigma}{\gamma(\gamma-\mu^\alpha)}\left[-\mu^\alpha + \gamma\,g(t;\alpha,\gamma,\mu)\right]. \tag{C.7}$$

In the case $\gamma = \mu^\alpha$, the calculation simplifies. We obtain indeed the following:

$$\widetilde{f}_1(\lambda) = \frac{\sigma}{\gamma}\frac{1}{\lambda}\left[1 - \frac{\gamma}{(\lambda+\mu)^\alpha}\right] \tag{C.8}$$

whose Laplace inverse transform follows straightforwardly from the convolution theorem:

$$\mathcal{L}\left\{\frac{1}{\lambda\,(\lambda+\mu)^\alpha}\right\}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\mu\,\tau}\,\tau^{\alpha-1}\,\mathrm{d}\tau \;= \frac{1}{\Gamma(\alpha)\,\mu^\alpha}\gamma(\alpha,\mu\,t) \tag{C.9}$$

We now consider the function f_2 for $\gamma \neq \mu^\alpha$. By substituting Φ in its Laplace transform in Eqs. (3.89) and rearranging terms, we find the following expression:

$$\widetilde{f}_2(\lambda) = -\frac{\sigma}{\gamma}\frac{1}{\lambda}\left[-\gamma - \mu^\alpha + (\lambda+\mu)^\alpha + \frac{\gamma^2}{(\lambda+\mu)^\alpha + \gamma - \mu^\alpha}\right] \tag{C.10}$$

The Laplace inverse transform of the first and third term in brackets can be done as before. We need to focus on the second term. If we employ the shifting property, we can write:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{(\lambda+\mu)^\alpha}{\lambda}\right\}(t) = &= \frac{e^{-\mu\, t}}{\Gamma(1-\alpha)}\int_0^t \tau^{-\alpha}\mathcal{L}^{-1}\left\{\frac{\lambda}{\lambda-\mu}\right\}(t-\tau)\,\mathrm{d}\tau \\
&= \frac{e^{-\mu\, t}}{\Gamma(1-\alpha)}\int_0^t \tau^{-\alpha}[\delta(t-\tau) + \mu\, e^{\mu(t-\tau)}]\,\mathrm{d}\tau \\
&= \frac{t^{-\alpha}\, e^{-\mu\, t}}{\Gamma(1-\alpha)} + \frac{\mu}{\Gamma(1-\alpha)}\int_0^t \tau^{-\alpha}e^{-\mu\,\tau}\,\mathrm{d}\tau \\
&= \frac{t^{-\alpha}\, e^{-\mu\, t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha,\mu\, t)
\end{aligned} \tag{C.11}$$

Thus, recalling the result in Eq. (C.6), we obtain the result presented in Eq. (3.96c):

$$\begin{aligned}
f_2(t) &= -\frac{\sigma}{\gamma}\left[-(\gamma+\mu^\alpha) + \frac{t^{-\alpha}\, e^{-\mu\, t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha,\mu\, t) - \frac{\gamma^2}{\gamma-\mu^\alpha}[-1+g(t;\alpha,\gamma,\mu)]\right] \\
&= -\frac{\sigma}{\gamma}\left[\frac{\mu^{2\,\alpha}}{\gamma-\mu^\alpha} - \frac{\gamma^2}{\gamma-\mu^\alpha}g(t;\alpha,\gamma,\mu) + \frac{t^{-\alpha}\, e^{-\mu\, t}}{\Gamma(1-\alpha)} + \frac{\mu^\alpha}{\Gamma(1-\alpha)}\gamma(1-\alpha,\mu\, t)\right]
\end{aligned} \tag{C.12}$$

We note that the resul for the case $\gamma = \mu^\alpha$ follows straightforwardly from Eq. (C.9). In addition, the expression in time of w is easily derived by using Eq. (C.11).

C.2 Mixture of Two Lévy Stable Subordinators

In this second section, we consider the case (ii), i.e. we assume $\eta = \eta_1 + \eta_2$, where $\eta_{1,2}$ are independent Lévy stable subordinator with exponents $0 < \alpha_1 < \alpha_2 < 1$. Due to their independence, the Laplace exponent of η is given by the sum of the exponents of each of its component, i.e. $\Phi(\lambda) = B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}$ with $B_1, B_2 \geq 0$. As before, we substitute this function in Eqs. (3.89) and compute their Laplace inverse transform. On the one hand for f_1 we obtain the following expression:

$$\widetilde{f}_1(\lambda) = \frac{\sigma}{\gamma}\frac{1}{\lambda}\left[1 - \frac{\gamma}{\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}}\right]; \tag{C.13}$$

on the other hand for f_2 we find similarly:

$$\widetilde{f}_2(\lambda) = -\frac{\sigma}{\gamma}\frac{1}{\lambda}\left[-\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2} + \frac{\gamma^2}{\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}}\right]. \tag{C.14}$$

The expression in time of both these equations can be derived, if we compute the Laplace inverse transform of the term $1/[\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}]$. This is obtained by adopting the technique of [75, 203]. Specifically, we write:

$$
\begin{aligned}
\frac{1}{\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}} &= \frac{1}{\lambda^{\alpha_1}[B_1 + B_2\lambda^{\alpha_2-\alpha_1}]}\frac{1}{1+\frac{\gamma\,\lambda^{-\alpha_1}}{B_1+B_2\,\lambda^{\alpha_2-\alpha_1}}}\\
&= \frac{\lambda^{-\alpha_1}}{[B_1 + B_2\lambda^{\alpha_2-\alpha_1}]}\sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n\,\lambda^{-\alpha_1\,n}}{[B_1 + B_2\,\lambda^{\alpha_2-\alpha_1}]^n}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n\,\lambda^{-\alpha_1\,(1+n)}}{[B_1 + B_2\,\lambda^{\alpha_2-\alpha_1}]^{1+n}} \qquad \text{(C.15)}
\end{aligned}
$$

This expression is now in a convenient form to be Laplace inverse transformed term by term. Recalling Eq. (A.10), we obtain $\forall n \in \mathbb{N}_0$:

$$
\mathcal{L}^{-1}\left\{\frac{\lambda^{-\alpha_1\,(1+n)}}{[B_1 + B_2\,\lambda^{\alpha_2-\alpha_1}]^{1+n}}\right\}(t) = \frac{t^{\alpha_2(1+n)-1}}{B_2^{1+n}}E^{1+n}_{\alpha_2-\alpha_1,\alpha_2(1+n)}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right). \qquad \text{(C.16)}
$$

Substituting this result back into Eq. (C.15), we obtain the final inverse Laplace transform:

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{\gamma + B_1\,\lambda^{\alpha_1} + B_2\,\lambda^{\alpha_2}}\right\}(t) = \\
&\qquad \frac{t^{\alpha_2-1}}{B_2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\gamma}{B_2}t^{\alpha_2}\right)^n E^{1+n}_{\alpha_2-\alpha_1,\alpha_2(1+n)}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right). \qquad \text{(C.17)}
\end{aligned}
$$

For simplicity, we introduce the auxiliary function:

$$
l(t) = t^{\alpha_2-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\gamma}{B_2}t^{\alpha_2}\right)^n E^{1+n}_{\alpha_2-\alpha_1,\alpha_2(1+n)}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right). \qquad \text{(C.18)}
$$

Thus, by applying the convolution theorem, we obtain the following results:

$$
f_1(t) = \frac{\sigma}{\gamma}\left[1 - \frac{\gamma}{B_2}\int_0^t l(\tau)\,\mathrm{d}\tau\right] \qquad \text{(C.19a)}
$$

$$
f_2(t) = -\frac{\sigma}{\gamma}\left[-\gamma + B_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)} + B_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{\gamma^2}{B_2}\int_0^t l(\tau)\,\mathrm{d}\tau\right] \qquad \text{(C.19b)}
$$

To get a solution in closed form, we still need to solve the integral over l. This can be derived by writing the function l in terms of Fox H-functions (see Appendix A.3) trough Eq. (A.11) and by using the property Eq. (A.27). In details, we can write:

$$
\begin{aligned}
\int_0^t l(\tau)\,\mathrm{d}\tau &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\gamma}{B_2}\right)^n \times\\
&\times \int_0^t \tau^{\alpha_2(1+n)-1}H^{1,1}_{1,2}\left[\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\left|\begin{matrix}(-n,1)\\(0,1),(1-\alpha_2(1+n),\alpha_2-\alpha_1)\end{matrix}\right.\right]\mathrm{d}\tau \qquad \text{(C.20)}
\end{aligned}
$$

The analytical solution of the integral term by term is then obtained by employing Eq. (A.27):

$$\int_0^t \tau^{\alpha_2(1+n)-1} H_{1,2}^{1,1}\left[\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\left|\begin{matrix}(-n,1)\\(0,1),(1-\alpha_2(1+n),\alpha_2-\alpha_1)\end{matrix}\right.\right]\mathrm{d}\tau =$$
$$t^{\alpha_2(1+n)} H_{3,3}^{1,3}\left[\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\left|\begin{matrix}(1-\alpha_2(1+n),\alpha_2-\alpha_1),(0,0),(-n,1)\\(0,1),(1-\alpha_2(1+n),\alpha_2-\alpha_1),(-\alpha_2(1+n),\alpha_2-\alpha_1)\end{matrix}\right.\right] =$$
$$t^{\alpha_2(1+n)} H_{2,3}^{1,2}\left[\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\left|\begin{matrix}(1-\alpha_2(1+n),\alpha_2-\alpha_1),(-n,1)\\(0,1),(1-\alpha_2(1+n),\alpha_2-\alpha_1),(-\alpha_2(1+n),\alpha_2-\alpha_1)\end{matrix}\right.\right] =$$
$$t^{\alpha_2(1+n)} H_{1,2}^{1,1}\left[\frac{B_1}{B_2}\tau^{\alpha_2-\alpha_1}\left|\begin{matrix}(-n,1)\\(0,1),(-\alpha_2(1+n),\alpha_2-\alpha_1)\end{matrix}\right.\right], \tag{C.21}$$

where we also first simplified the Fox H-function in the first line by computing explicitly the function Θ in Eq. (A.16) and then employed the reduction formula Eq. (A.20) to further simplify it in the second line. Substituting this result in Eq. (C.20) and using the relation Eq. (A.11), we can write:

$$\int_0^t l(\tau)\,\mathrm{d}\tau = \sum_{n=0}^{\infty}(-1)^n\left(\frac{\gamma}{B_2}\right)^n t^{\alpha_2(1+n)} E^{1+n}_{\alpha_2-\alpha_1,1+\alpha_2(1+n)}\left(-\frac{B_1}{B_2}t^{\alpha_2-\alpha_1}\right), \tag{C.22}$$

which is the function H defined in Eq. (3.106), except for the time dependent prefactor. Thus, from Eqs. (C.19a, C.19b) we obtain directly Eqs. (3.105b, 3.105c).

C.3 One Sided Lévy Noise with Exponent Φ in Eq. (3.107)

In this last section, we consider the case (iii), where the characteristic functional of η is defined by Eq. (3.107) with $d_1, d_2, \beta > 0$ and $0 < \alpha_1, \alpha_2 \leq 1$. As before, we substitute this function in Eqs. (3.89) and compute their Laplace inverse transform. On the one hand for f_1 we obtain the following expression:

$$\widetilde{f}_1(\lambda) = \frac{\sigma}{\gamma}\frac{1}{\lambda}\left\{1-\gamma\left[\gamma+\frac{d_1}{d_2^{\alpha_1}}\lambda^{\alpha_1}\left(1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right)^{\beta(\alpha_2-\alpha_1)}\right]^{-1}\right\} \tag{C.23}$$

The Laplace inverse transform of the second term in its rhs is derived by writing it as:

$$\left[\gamma+\frac{d_1}{d_2^{\alpha_1}}\lambda^{\alpha_1}\left(1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right)^{\beta(\alpha_2-\alpha_1)}\right]^{-1} =$$
$$\frac{d_2^{\alpha_1}}{d_1}\frac{\lambda^{-\alpha_1}}{\left[1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right]^{\beta(\alpha_2-\alpha_1)}}\left[1+\gamma\frac{d_2^{\alpha_1}}{d_1}\frac{\lambda^{-\alpha_1}}{\left(1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right)^{\beta(\alpha_2-\alpha_1)}}\right]^{-1}. \tag{C.24}$$

We can then expand it in series as below:

$$\left[\gamma + \frac{d_1}{d_2^{\alpha_1}}\lambda^{\alpha_1}\left(1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right)^{\beta(\alpha_2-\alpha_1)}\right]^{-1} = \frac{d_2^{\alpha_1}}{d_1}\sum_{n=0}^{\infty}(-1)^n\left(\gamma\,\frac{d_2^{\alpha_1}}{d_1}\right)^n \frac{\lambda^{-\alpha_1(1+n)}}{\left[1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right]^{\beta(\alpha_2-\alpha_1)(1+n)}}, \quad \text{(C.25)}$$

whose term by term Laplace inverse transform is done by recalling Eq. (A.10). We obtain:

$$\mathcal{L}\left\{\frac{\lambda^{-\alpha_1(1+n)}}{\left[1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right]^{\beta(\alpha_2-\alpha_1)(1+n)}}\right\}(t) = d_2^{(1+n)(\alpha_2-\alpha_1)}\,t^{\alpha_2(1+n)-1}E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,\alpha_2(1+n)}\left(-d_2^{1/\beta}\,t^{1/\beta}\right) \quad \text{(C.26)}$$

After substituting this result in Eq. (C.25), we obtain the following expression in time:

$$t^{\alpha_2-1}\frac{d_2^{\alpha_2}}{d_1}\sum_{n=0}^{\infty}(-1)^n\left(\gamma\,\frac{d_2^{\alpha_2}}{d_1}\right)^n t^{\alpha_2\,n}E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,\alpha_2(1+n)}\left(-d_2^{1/\beta}\,t^{1/\beta}\right). \quad \text{(C.27)}$$

We note that Eq. (C.23) also contains a factor $1/\lambda$, which corresponds to a time integration via the convolution theorem. If we perform such integral term by term, we obtain:

$$\int_0^t \tau^{\alpha_2(1+n)-1}E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,\alpha_2(1+n)}\left(-d_2^{1/\beta}\,\tau^{1/\beta}\right)\mathrm{d}\tau\; = t^{\alpha_2(1+n)}E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,1+\alpha_2(1+n)}\left(-d_2^{1/\beta}\,t^{1/\beta}\right). \quad \text{(C.28)}$$

Putting everything together, we obtain the following result:

$$\mathcal{L}\left\{\frac{1}{\lambda}\left[\gamma + \frac{d_1}{d_2^{\alpha_1}}\lambda^{\alpha_1}\left(1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right)^{\beta(\alpha_2-\alpha_1)}\right]^{-1}\right\}(t) = \frac{d_2^{\alpha_2}}{d_1}t^{\alpha_2}\sum_{n=0}^{\infty}(-1)^n\left(\gamma\,\frac{d_2^{\alpha_2}}{d_1}\right)^n t^{\alpha_2\,n}E^{\beta(1+n)(\alpha_2-\alpha_1)}_{1/\beta,1+\alpha_2(1+n)}\left(-d_2^{1/\beta}\,t^{1/\beta}\right), \quad \text{(C.29)}$$

which is the series expansion of the function G in Eq. (3.116), except for the time dependent prefactor. Thus, Eq. (3.115b) follows directly from Eqs. (C.23, C.29). In the case of f_2, we need to determine the term $\Phi(\lambda)/\lambda$. In our specific case, we can rewrite it as below:

$$\frac{\Phi(\lambda)}{\lambda} = \frac{d_1}{d_2^{\alpha_1}}\,\lambda^{\alpha_1-1}\left[1+\left(\frac{\lambda}{d_2}\right)^{1/\beta}\right]^{\beta(\alpha_2-\alpha_1)} = \frac{d_1}{d_2^{\alpha_2}}\,\frac{\lambda^{\alpha_1-1}}{\left[d_2^{1/\beta}+\lambda^{1/\beta}\right]^{\beta(\alpha_1-\alpha_2)}}, \quad \text{(C.30)}$$

whose inverse Laplace transform can again be computed with Eq. (A.10):

$$\mathcal{L}^{-1}\left\{\frac{\lambda^{\alpha_1-1}}{\left[d_2^{1/\beta}+\lambda^{1/\beta}\right]^{\beta(\alpha_1-\alpha_2)}}\right\}(t) = t^{-\alpha_2}E^{\beta(\alpha_1-\alpha_2)}_{1/\beta,1-\alpha_2}\left(-d_2^{1/\beta}\,t^{1/\beta}\right). \quad \text{(C.31)}$$

By employing Eqs. (C.29, C.31), the inverse Laplace transform of Eq. (3.89) leads to the result presented in Eq. (3.115c).

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