

Two-loop scattering amplitudes from the Riemann sphere

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The scattering equations give striking formulas for massless scattering amplitudes at tree level and, as shown recently, at one loop. The progress at loop level was based on ambitwistor-string theory, which naturally yields the scattering equations. We proposed that, for ambitwistor strings, the standard loop expansion in terms of the genus of the world sheet is equivalent to an expansion in terms of nodes of a Riemann sphere, with the nodes carrying the loop momenta. In this paper, we show how to obtain two-loop scattering equations with the correct factorization properties. We adapt genus-two integrands from the ambitwistor string to the nodal Riemann sphere and show that these yield correct answers, by matching standard results for the four-point two-loop amplitudes of maximal supergravity and super-Yang-Mills theory. In the Yang-Mills case, this requires the loop analogue of the Parke-Taylor factor carrying the color dependence, which includes nonplanar contributions.

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I. INTRODUCTION

The Cachazo-He-Yuan (CHY) formulas provide remarkable tree-level expressions for scattering amplitudes in theories of massless particles, written as an integral over marked points on the Riemann sphere. The integral localizes as a sum over the solutions to the scattering equations [1]. This formalism generalizes earlier work of Roiban, Spradlin and Volovich [2] based on Witten's twistor string theory [3]. The CHY formulas themselves originate in ambitwistor-string theory [4]: this provided a loop-level formulation [5,6] giving new formulas at genus one (torus) [5,7] and two [8] for type II supergravities in 10 dimensions. In [9,10], we showed how the torus formulas reduce to formulas on a nodal Riemann sphere, by means of integration by parts in the moduli space of the torus. The node carries the loop momentum. We proposed that an analogous reduction was possible at any genus, leading to a new formalism that could become a practical tool in the computation of scattering amplitudes. In the one-loop case, our explicit analysis provided a proof that the formulas from ambitwistor strings reproduce the correct answer. Furthermore, on the nodal Riemann sphere, the formalism is more flexible than on the torus, and the formulas could be extended to a variety of theories with or without supersymmetry. An alternative approach to the one-loop scattering equations was pursued in [11,12].

However, one loop is not such a stringent test of the framework, as many difficulties arise only at higher loops. The Feynman tree theorem, for example, shows how to construct one-loop integrands from tree formulas, if massive legs are allowed, and massive legs had already been considered in this context [13]; an example of our formulas has been reproduced following such an approach [14]. However, the situation is more difficult at higher loops despite recent progress inspired by the tree theorem [15].

In [9], we gave a brief sketch as to how the loop-level scattering equations are obtained by reduction to the nodal Riemann sphere. In this article, we give a precise formulation at two loops. To fix the details of the reduction to the sphere, we use a factorization argument that leads to new off-shell scattering equations. An alternative approach [16] applies higher-dimensional tree-level rules for the integration of the scattering equations to give diagrams for a scalar theory; however, our aim here is to give a framework that yields loop integrands on a nodal Riemann sphere for complete amplitudes. With this, we adapt genus-two supergravity integrands (type II, $d = 10$) to a doubly nodal sphere, leading to the correct integrand for the four-point amplitude in maximal supergravity. We then conjecture an adjustment that gives instead a super-Yang-Mills integrand. These are checked both by factorization and numerically. Nonsupersymmetric integrands require certain degenerate solutions to the scattering equations (on which the supersymmetric integrands vanish). We characterize these degenerate solutions here, but leave the subtler nonsupersymmetric integrands for the future.

II. FROM HIGHER GENUS TO THE SPHERE

The higher genus scattering equations were formulated in the ambitwistor-string framework on Riemann surfaces

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Σ_g of genus g [5,6], in terms of a meromorphic 1-form P^μ , $\mu = 1, \dots, d$ (the momentum of the string) that solves

$$\bar{\partial}P = \sum_{i=1}^n k_i \delta^2(z - z_i) dz \wedge d\bar{z}, \quad (1)$$

where z_i are n marked points on Σ_g . The solution is written as

$$P = \sum_{i=1}^n k_i \omega_{z_i, z_0}^{(g)}(z) + \sum_{r=1}^g \ell_r \omega_r^{(g)}, \quad (2)$$

where $\omega_r^{(g)}$, $r = 1, \dots, g$ span a basis of holomorphic 1-forms on Σ_g dual to a choice of a cycles a_r , and $\omega_{z_i, z_0}^{(g)}(z)$ are meromorphic differentials with simple poles of residues ± 1 at z_i and z_0 and vanishing a -cycle integrals. The dependence on the auxiliary point z_0 drops by momentum conservation. The $\ell_r \in \mathbb{C}^d$ parametrize the zero modes of P and will play the role of the loop momenta.

The genus- g scattering equations are a minimal set of conditions on the z_i and moduli of Σ_g required for P^2 to vanish globally. They include n conditions

$$k_i \cdot P(z_i) = 0, \quad i = 1, \dots, n \quad (3)$$

that set the residues of the simple poles of P^2 at the z_i to zero. These fix the locations of z_i on the surface. Once these are imposed, the quadratic differential P^2 is holomorphic, so it has $3g - 3$ further degrees of freedom, corresponding to the moduli of the surface (its shape). We therefore impose another $3g - 3$ scattering equations to reach $P^2 = 0$. This can be done by writing

$$P^2 = \sum_{r,s=1}^g u_{rs} \omega_r^{(g)} \omega_s^{(g)}, \quad (4)$$

where the u_{rs} only depend on the moduli of Σ_g and on the kinematics, and setting an independent $3g - 3$ subset of the u_{rs} to zero. In total, the scattering equations localize the full moduli space integral to a discrete set of points. For $g = 2, 3$, there are precisely $3g - 3$ u_{rs} 's, thus we simply set them all to zero and the ambitwistor-string loop integrand reads

$$\mathcal{M}_n^{(g)} = \int_{\mathcal{M}_{g,n}} \mathcal{I} d^{3g-3} \mu \prod_{r \leq s} \bar{\delta}(u_{rs}) \prod_{i=1}^n dz_i \bar{\delta}(k_i \cdot P(z_i)), \quad (5)$$

where \mathcal{I} is a correlator depending on the theory and the holomorphic δ functions are $2\pi i \bar{\delta}(f(z)) = \bar{\partial}(1/f)$. We stress that this is a formula for the loop integrand, and the n -point g -loop amplitude is $\int d^D \ell_1 \dots d^D \ell_g \mathcal{M}_n^{(g)}$.

To reduce this expression to one on nodal Riemann spheres, the heuristics described in [9], based on the explicit genus-one calculation, was to integrate by parts (or use residue theorems) in the moduli integral g times. This relaxes the delta functions $\bar{\partial}(1/u_{rr})$ to give measure factors $\prod_r 1/u_{rr}$, with the integration by parts yielding residues at the boundary of moduli space where all the chosen a cycles contract to give double points, leaving a Riemann sphere with g pairs of double points. This leaves $2g - 3$ moduli that can be identified with the moduli of $2g$ points on the Riemann sphere, corresponding to the g nodes, modulo Möbius transformations. These moduli are fixed by $2g - 3$ remaining scattering equations.

On the nodal Riemann sphere Σ , the basis of 1-forms descending from the $\omega_r^{(g)}$'s dual to the pinched a cycles, given by the pairs of double points $\sigma_{r\pm}$, is

$$\omega_r = \frac{(\sigma_{r^+} - \sigma_{r^-}) d\sigma}{(\sigma - \sigma_{r^+})(\sigma - \sigma_{r^-})}, \quad r = 1, \dots, g \quad (6)$$

so now

$$P = d\sigma \sum_{i=1}^n \frac{k_i}{\sigma - \sigma_i} + \sum_{r=1}^g \ell_r \omega_r. \quad (7)$$

From (4) the coefficient of the double poles at $\sigma_{r\pm}$ identifies u_{rr} as $u_{rr} = \ell_r^2$. Thus the measure factor becomes $\prod_r 1/\ell_r^2$. Furthermore, the quadratic differential

$$\mathcal{S}_g = P^2 - \sum_{r=1}^g \ell_r^2 \omega_r^2 \quad (8)$$

now only has simple poles at the σ_i and $\sigma_{r\pm}$. The $n + 2g$ off-shell scattering equations were then proposed in [9] to be $\text{Res}_{\sigma_A} \mathcal{S} = 0$. There are three relations between these equations so that only $n + 2g - 3$ of them need to be imposed to enforce $\mathcal{S} = 0$. The three relations follow from the vanishing of the sum of residues of \mathcal{S} multiplied by three independent tangent vectors to the sphere.

There is an ambiguity at two loops and higher, however. We could equally well have defined \mathcal{S} as

$$\tilde{\mathcal{S}}_g = P^2 - \sum_{r=1}^g \ell_r^2 \omega_r^2 + \sum_{r < s} a_{rs} \omega_r \omega_s, \quad (9)$$

where the a_{rs} are linear combinations of the u_{rs} . We will see that $a_{rs} = \alpha(u_{rr} + u_{ss}) = \alpha(\ell_r^2 + \ell_s^2)$ is a better choice where $\alpha = \pm 1$ at two loops. This does not change the heuristic argument as it corresponds to replacing the original $u_{rs} = 0$ scattering equations for the moduli by nondegenerate linear combinations thereof. The choice $\alpha = 1$ (or equivalently -1) at two loops will be forced upon us by requiring correct factorization channels.

Thus our formula on the nodal Riemann sphere is

$$\mathcal{M}_n^{(g)} = \frac{1}{\prod_{r=1}^g \ell_r^2} \int_{\mathcal{M}_{0,n+2g}} \mathcal{I}_0 \frac{d^{n+2g} \sigma}{\text{vol}SL(2, \mathbb{C})} \prod_{A=1}^{n+2g} \delta(E_A), \quad (10)$$

where $E_A = \text{Res}_{\sigma_A} \tilde{S}_g$, with the index A spanning the n marked points and the $2g$ double points. The delta functions enforce the off-shell scattering equations [17]. In the first instance, \mathcal{I}_0 will be taken to be the nodal limit of the higher-genus world sheet correlator from the ambitwistor-string type II supergravity in $d = 10$ (together with a cross ratio motivated by factorization). This can be extended to theories for which no higher-genus expression is known, as we will demonstrate explicitly for super-Yang-Mills theory.

III. THE 2-LOOP SCATTERING EQUATIONS

We now take $g = 2$ with σ_{1^\pm} and σ_{2^\pm} the double points corresponding to ℓ_1 and ℓ_2 . The two-loop scattering equations are the vanishing of the residues of

$$S := P^2 - \ell_1^2 \omega_1^2 - \ell_2^2 \omega_2^2 + \alpha(\ell_1^2 + \ell_2^2) \omega_1 \omega_2. \quad (11)$$

We adopt the shorthand notation $(AB) = \sigma_A - \sigma_B$. The scattering equations $2E_A = \text{Res}_{\sigma_A} S(\sigma)$ are then given by

$$\begin{aligned} \pm E_{1^\pm} &= \frac{1}{2} \frac{L(2^+2^-)}{(1^\pm 2^+)(1^\pm 2^-)} + \sum_i \frac{\ell_1 \cdot k_i}{(1^\pm i)}, \\ \pm E_{2^\pm} &= \frac{1}{2} \frac{L(1^+1^-)}{(2^\pm 1^+)(2^\pm 1^-)} + \sum_i \frac{\ell_2 \cdot k_i}{(2^\pm i)}, \\ E_i &= \frac{k_i \cdot \ell_1(1^+1^-)}{(i1^+)(i1^-)} + \frac{k_i \cdot \ell_2(2^+2^-)}{(i2^+)(i2^-)} + \sum_{j \neq i} \frac{k_i \cdot k_j}{(ij)}, \end{aligned} \quad (12)$$

where $L = \alpha(\ell_1^2 + \ell_2^2) + 2\ell_1 \cdot \ell_2$. In particular, for $\alpha = \pm 1$, $L = \pm(\ell_1 \pm \ell_2)^2$. The equations are not independent, since there are three linear relations between them,

$$\sum_A E_A = 0, \quad \sum_A \sigma_A E_A = 0, \quad \sum_A \sigma_A^2 E_A = 0. \quad (13)$$

We will see that $\alpha = \pm 1$ follows from the correct factorization.

A. Poles and factorization

Factorization channels of the integrand are related by the scattering equations to the boundary of the moduli space of the Riemann surface, where a subset of the marked points coalesce. Conformally, these configurations are equivalent to keeping the marked points at a finite distance, but pinching them off on another sphere, connected to the original one at the coalescence point σ_I .

When Σ degenerates in this way, the scattering equations force a kinematic configuration where an intermediate momentum goes on shell [18], corresponding to a potential pole in the integrand. The pole can thus be calculated as $\sigma_A \rightarrow \sigma_I$ for $A \in I$ from

$$\sum_{A \in I} (\sigma_A - \sigma_I) E_A = 0. \quad (14)$$

Note however that whether this singularity is realized in a specific theory depends on the integrand \mathcal{I}_0 .

Let $K_I = \sum_{i \in I} k_i$ (with external particles only). The location of the singularities in terms of the external and loop momenta can then be characterized as follows:

- (i) When $\sigma_{1^\pm}, \sigma_{2^\pm} \notin I$, (14) simply gives $K_I^2 = 0$, the standard factorization channel as for tree amplitudes, where a pole can appear in some intermediate propagator in massless scattering.
- (ii) When $\sigma_{1^\pm} \in I$, but $\sigma_{2^\pm} \notin I$, we find the (potential) pole $2\ell_1 \cdot K_I \pm K_I^2$ as at 1-loop [9], where such poles arise from certain partial fraction relations and shifts in the loop momenta, and coincide with the ‘‘Q-cut’’ poles [15].
- (iii) The crucial new configuration at two loops is given by $\sigma_{1^+}, \sigma_{2^\pm} \in I$, corresponding to the condition

$$L + 2(\ell_1 \pm \ell_2) \cdot K_I + K_I^2 = 0, \quad (15)$$

with L as above. As detailed in [15], the partial fraction identities and shifts always give a quadratic propagator of the form $(\ell_1 \pm \ell_2 + K_I)^2$ at two loops; see also Appendix A. Therefore, requiring the correct behavior under factorization determines $\pm = +$ and $\alpha = 1$, or $\pm = -$ and $\alpha = -1$. While both options are fully equivalent up to reparametrization of the loop momenta, we will choose the former for the rest of the paper.

For $\sigma_{1^+}, \sigma_{2^+} \in I$, this choice leads to a potential pole at $(\ell_1 + \ell_2 + K_I)^2$. However, for $\sigma_{1^+}, \sigma_{2^-} \in I$, we are left with an unphysical potential pole at $(\ell_1 + \ell_2)^2 + 2(\ell_1 - \ell_2) \cdot K_I + K_I^2 = 0$. The requirement that this pole is absent from the final answer will give important restrictions on the integrand \mathcal{I}_0 .

- (iv) Let us briefly comment on the only other new scenario at 2-loops—to have both $\sigma_{1^\pm} \in I$. Since their contributions cancel in (14), this just leads to $K_I^2 = 0$, although now associated to two 1-loop diagrams joined by an on-shell propagator.

The criterion for an integrand \mathcal{I}_0 to give a simple kinematic pole in the final formula at one of these potential singularities is that \mathcal{I}_0 should have a pole of order $2|I| - 2$ as the marked points coalesce, as described in detail in Sec. 4.1 of [10]. If the pole has lower degree, the final formula will not have a factorization pole in this channel. This gives an important criterion for determining the precise forms of possible integrands \mathcal{I}_0 .

B. Degenerate and regular solutions

The off-shell scattering equations $\pm E_{r^\pm}(\sigma_{r^\pm}) = 0$ associated to the node r have the same functional form, when seen as functions of $\sigma = \sigma_{r^+}$ and σ_{r^-} respectively. We distinguish between regular solutions, when σ_{r^\pm} are different roots of $E_{r^\pm}(\sigma) = 0$, and degenerate solutions with $\sigma_{r^+} = \sigma_{r^-}$ the same root, for generic momenta. These degenerate solutions can be summarized by the factorization diagrams given in Fig. 1, and can be understood as forward limits of the $(d + g)$ -dimensional tree-level scattering equations; see Appendix B for details and [14, 19] for a discussion at one loop.

However, not all solutions of the $d + 2$ -dimensional scattering equations survive in the forward limit. Consider a degeneration parameter τ which vanishes in the forward limit. While there are degenerate solutions with $\sigma_{r^+ r^-} \sim \tau^2$, the zero locus of (12) excludes them, and thus the two-loop integrands localize on the degenerate solutions with $\sigma_{r^+ r^-} \sim \tau$ and on

$$N_{\text{reg}} = (n + 1)! - 4n! + 4(n - 1)! + 6(n - 3)! \quad (16)$$

regular solutions (with $\sigma_{r^+ r^-} \sim 1$). Moreover, we shall see that the supersymmetric integrands only receive contributions from the regular solutions. As an important consequence, unphysical poles in the form of Gram determinants arising from double roots in $E_{r^\pm}(\sigma)$ are absent for supersymmetric theories: as discussed in [14] and Appendix B, these poles can be localized on the degenerate solutions. For nonsupersymmetric theories, however, degenerate solutions may contribute, and one must check that contributions with unphysical poles vanish upon loop integration, as detailed in [14] at one loop.

IV. SUPERSYMMETRIC TWO-LOOP AMPLITUDES

We now consider explicit expressions at four points for maximal supergravity and super-Yang-Mills. These expressions are examples of (10) for $g = 2$, $n = 4$. The

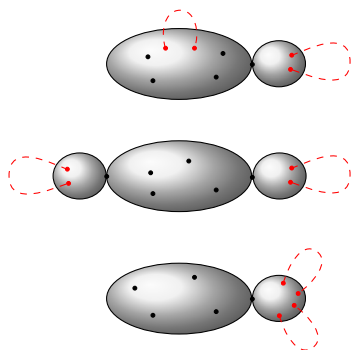


FIG. 1. Different possible world sheet degenerations: case A (top), case B (middle), case C (bottom).

representation of loop integrands that arises can be connected to a standard Feynman-like representation, after use of partial fractions and shifts in the loop momenta as in [9] at one loop; see Appendix A.

A. Four-point supergravity integrand

We use the integrand that arises directly from the degeneration of the genus-two (ambitwistor) string [8, 20–22]. Define

$$\Delta_{i,j} = \omega_1(\sigma_i)\omega_2(\sigma_j) - \omega_1(\sigma_j)\omega_2(\sigma_i) \quad (17)$$

and

$$\hat{\mathcal{Y}} = \frac{(k_1 - k_2) \cdot (k_3 - k_4) \Delta_{1,2} \Delta_{3,4} + \text{cyc}(234)}{3(1^{+2^+})(1^{+2^-})(1^{-2^+})(1^{-2^-})}, \quad (18)$$

where $\text{cyc}(234)$ is a sum over cyclic permutations. Our prescription for the four-point supergravity integrand is

$$\mathcal{I}_0^{\text{SUGRA}} = (\mathcal{K}\tilde{\mathcal{K}})\hat{\mathcal{Y}}^2 \frac{(1^{+2^-})(1^{-2^+})}{(1^{+1^-})(2^{+2^-})}, \quad (19)$$

where $\mathcal{K}\tilde{\mathcal{K}}$ is the standard kinematical supersymmetry prefactor $\mathcal{K} = (k_1 \cdot k_2)(k_2 \cdot k_3)A_{\text{tree}}^{\text{SYM}}(1, 2, 3, 4)$ and $\tilde{\mathcal{K}} = (k_1 \cdot k_2)(k_2 \cdot k_3)\tilde{A}_{\text{tree}}^{\text{SYM}}(1, 2, 3, 4)$, so that the supergravity states in the scattering are the direct product of super-Yang-Mills untilded (left) and tilded (right) states.

The cross ratio is inserted by hand to remove poles in the unphysical factorization channels discussed in the previous section. This is an important aspect of our prescription. We set the relative sign between ℓ_1 and ℓ_2 to be +, consistently with the $(\ell_1 + \ell_2)^2$ factors in the scattering equations. This choice implies that the degenerations of the worldsheet at $(1^{+2^+}) \rightarrow 0$ or $(1^{-2^-}) \rightarrow 0$ occur when $(\ell_1 + \ell_2 \pm K_I)^2 \rightarrow 0$, where K_I is a partial sum of the external momenta. These physical poles can be realized in the formula. However, the numerator in the cross ratio suppresses unphysical poles of the type $(\ell_1 + \ell_2)^2 \pm 2(\ell_1 - \ell_2) \cdot K_I + K_I^2 \rightarrow 0$, which might have arisen when $(1^{+2^-}) \rightarrow 0$ or $(1^{-2^+}) \rightarrow 0$.

We have evaluated our formula numerically and checked that it matches the known result for this amplitude [23],

$$\frac{\mathcal{M}_4^{(2)}}{\mathcal{K}\tilde{\mathcal{K}}} = (k_1 \cdot k_2)^2 [I_{12,34}^{\text{planar}} + I_{34,21}^{\text{planar}} + I_{1,2,34}^{\text{non-planar}} + I_{3,4,21}^{\text{non-planar}}] + \text{cyc}(234). \quad (20)$$

The planar and nonplanar double-box integrands are written down in the ‘‘shifted’’ representation in Appendix A. It is also possible to check the factorization of this formula explicitly.

B. Four-point super-Yang-Mills integrand

There is no fully well-defined ambitwistor model that would give a first principle derivation of a super-Yang-Mills integrand, however the tree and one-loop results motivated us to postulate the following expression:

$$\mathcal{I}_0^{\text{SYM}} = \mathcal{K} \hat{\mathcal{Y}} \mathcal{I}_{\text{PT}(2)}, \quad (21)$$

where \mathcal{K} is again the standard kinematical supersymmetry prefactor. The new and crucial ingredient is the extension to two loops of the Parke-Taylor factor, $\mathcal{I}_{\text{PT}(2)}$. In [9], we presented the analogous object at one loop. We will comment later on the general form, and first focus on the explicit formula for the four-point two-loop case, including the nonplanar (NP) contributions,

$$\begin{aligned} \mathcal{I}_{\text{PT}(2)} = & [N_c^2 C_{1234}^{\text{P}} + C_{1234}^{\text{NP},1}] [\text{tr}(1234) + \text{tr}(4321)] \\ & + N_c C_{12,34}^{\text{NP},2} \text{tr}(12)\text{tr}(34) + \text{cyc}(234), \end{aligned} \quad (22)$$

where N_c is the rank of the gauge group and $\text{tr}(12\dots) \equiv \text{tr}(T^{a_1} T^{a_2} \dots)$ denote the color traces. We have, for the planar part

$$\begin{aligned} C_{1234}^{\text{P}} = & c_{1234}^{\text{P}}(1^+, 1^-, 2^+, 2^-) + c_{1234}^{\text{P}}(1^-, 1^+, 2^-, 2^+) \\ & + c_{1234}^{\text{P}}(2^+, 2^-, 1^+, 1^-) + c_{1234}^{\text{P}}(2^-, 2^+, 1^-, 1^+), \end{aligned}$$

with

$$\begin{aligned} c_{1234}^{\text{P}}(a, b, c, d) = & \frac{1}{(abcd1234)} + \frac{1}{(ab1dc234)} \\ & + \frac{1/2}{(ab12dc34)} + \frac{1}{(acdb1234)} \\ & + \text{cyc}(1234), \end{aligned}$$

where $(123\dots m)$ stands for $(12)(23)\dots(m1)$. For the double-trace contribution, we have

$$\begin{aligned} C_{12,34}^{\text{NP},2} = & c_{12,34}^{\text{NP}}(1^+, 1^-, 2^+, 2^-) + c_{12,34}^{\text{NP}}(1^-, 1^+, 2^-, 2^+) \\ & + c_{12,34}^{\text{NP}}(2^+, 2^-, 1^+, 1^-) + c_{12,34}^{\text{NP}}(2^-, 2^+, 1^-, 1^+), \end{aligned}$$

with

$$\begin{aligned} c_{12,34}^{\text{NP}}(a, b, c, d) = & \frac{1/2}{(ac12db34)} + \frac{2}{(acd12b34)} \\ & + \frac{1}{(a1cd2b34)} + \text{perm}(12, 34), \end{aligned}$$

where $\text{perm}(12,34)$ denotes the eight permutations $(1 \leftrightarrow 2)$, $(3 \leftrightarrow 4)$ and $(12) \leftrightarrow (34)$. The remaining contribution is determined by the ones already given, as seen in [24],

$$C_{1234}^{\text{NP},1} = 2[C_{1234}^{\text{P}} + C_{1342}^{\text{P}} + C_{1423}^{\text{P}}] - C_{13;24}^{\text{NP},2}. \quad (23)$$

We checked numerically that our proposal matches the known result of [25]. For instance, using the color decomposition of Eq. (22), we get for the planar part

$$\frac{\mathcal{M}_{1234}^{(2)\text{P}}}{\mathcal{K}} = k_1 \cdot k_2 I_{12,34}^{\text{planar}} + k_4 \cdot k_1 I_{41,23}^{\text{planar}}. \quad (24)$$

The two-loop Parke-Taylor formula $\mathcal{I}_{\text{PT}(2)}$ is nontrivial, and may seem hard to extend for higher multiplicity or loop order. We propose, however, that in general it can be computed from the correlator of a current algebra on the Riemann sphere, which was our procedure at four points. This extends the tree-level result of [26] and, more generally, follows by analogy to the heterotic string [27], where gauge interactions have a closed stringlike nature as in ambitwistor-string theory [4]. The sum over states at a node of the Riemann sphere translates into a sum over the Lie algebra index of two additional operator insertions per loop momentum. To eliminate the contributions from the unwanted poles, we drop Parke-Taylor terms that have orderings where loop momentum insertions appear with alternate signs as in $(1^+ \cdot 1^- \cdot 2^+ \cdot 2^- \cdot \dots)$, keeping only terms with orderings of the type $(1^+ \cdot 1^- \cdot 2^- \cdot 2^+ \cdot \dots)$; here \cdot denotes any external particles. For instance, we keep contributions such as $1/(1^+ 1^- 2^- 2^+ 1234)$, but discard terms like $1/(1^+ 1^- 2^+ 2^- 1234)$. This achieves the same effect as the cross ratio appearing in the supergravity integrand. Moreover, we only include contributions with a single cyclic structure, e.g. (123456) , and discard contributions with subcycles, e.g. $(123)(456)$. These properties can be verified in the expressions above. Our two-loop Parke-Taylor expressions should be applicable to Yang-Mills theories with or without supersymmetry.

V. DISCUSSION

We have obtained scattering equation formulas for two-loop integrands on the Riemann sphere, following the heuristic reduction of genus-two ambitwistor-string formulas by integration by parts on the moduli space of Riemann surfaces as in [9]. Our analysis is not a rigorous derivation from the genus-two ambitwistor-string formulas of [8], and in particular does not fix the parameter α in the off-shell scattering equations. Nevertheless, we have seen that factorization fixes the ambiguity in α , and this choice leads to correct two-loop integrands for maximally supersymmetric theories. A more refined analysis of the ambitwistor-string degeneration should uniquely fix the scattering equations and the details of the integrands (such as the cross ratio in the supergravity case). It would also give us the tools to address the higher-loop case, where we expect different boundary contributions (e.g. at three loops “mercedes” vs “ladder” graphs) in the integration by parts

on the moduli space associated to different classes of scattering equations.

There are clearly many other challenges. It should also be possible to obtain integrands for nonsupersymmetric theories, as we did in [10] at one loop. These will in principle also have support on the degenerate solutions to the two-loop scattering equations, which we studied here; see [28–30] and references therein for recent work on the scattering equations. For both supersymmetric and non-supersymmetric Yang-Mills and gravity at higher points, we need to understand the higher-loop analogues of the CHY Pfaffians on the Riemann sphere with and without supersymmetry. The extent of supersymmetry should be determined by the particular sum over spin structures, as in [10] at one loop. More generally, we would like to extend our results to a new formalism, where our formulas arise directly as correlation functions of vertex operators on the nodal Riemann sphere. A natural question is then what type of quantum field theories admit such a formulation; there are CHY formulas and ambitwistor-string models for a variety of theories [6,31,32]. Finally, it would be important to clarify the relation of these ideas to full string theory, which has been the subject of recent works [33–35].

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APPENDIX A: SHIFTED INTEGRANDS FOR PLANAR AND NONPLANAR DOUBLE BOXES

In this section, we give the representation of double-box integrands with shifted loop momenta that appear in the formalism of the loop-level scattering equations [9]. Consider the planar double-box integrand with standard quadratic propagators,

$$\frac{1}{\ell_1^2(\ell_1 - k_1)^2(\ell_1 + k_2)^2\ell_2^2(\ell_2 - k_4)^2(\ell_2 + k_3)^2(\ell_{12} + k_{23})^2}, \quad (\text{A1})$$

where we use the notation $\ell_{12} = \ell_1 + \ell_2$ and $k_{ij} = k_i + k_j$. The choice of canonical loop momenta arising from the shifts requires that this integrand is split into two different contributions—one for which $\ell_1 + \ell_2$ is in the middle of the box, (A), and one for which ℓ_1 or ℓ_2 is (B). In the case (A), we obtain 9 terms, corresponding to applying the partial fraction identity and shifts to the 3

factors containing only ℓ_1 , and to the 3 factors containing only ℓ_2 . This gives

$$I_{12,34}^A(\ell_1, \ell_2) = \frac{1}{\ell_1^2\ell_2^2} V_1^a M_{ab} V_2^b, \quad (\text{A2})$$

with

$$V_1 = \left(\frac{1}{(-2\ell_1 \cdot k_1)(2\ell_1 \cdot k_2)}, \frac{1}{(2\ell_1 \cdot k_1)(2\ell_1 \cdot k_{12} + k_{12}^2)}, \frac{1}{(-2\ell_1 \cdot k_2)(-2\ell_1 \cdot k_{12} + k_{12}^2)} \right),$$

$$V_2 = \left(\frac{1}{(-2\ell_2 \cdot k_4)(2\ell_2 \cdot k_3)}, \frac{1}{(2\ell_2 \cdot k_4)(2\ell_2 \cdot k_{34} + k_{34}^2)}, \frac{1}{(-2\ell_2 \cdot k_3)(-2\ell_2 \cdot k_{34} + k_{34}^2)} \right),$$

$$M = \begin{pmatrix} \frac{1}{(\ell_{12} + k_{23})^2} & \frac{1}{(\ell_{12} - k_1)^2} & \frac{1}{(\ell_{12} + k_2)^2} \\ \frac{1}{(\ell_{12} - k_4)^2} & \frac{1}{\ell_{12}^2} & \frac{1}{(\ell_{12} + k_{12})^2} \\ \frac{1}{(\ell_{12} + k_3)^2} & \frac{1}{(\ell_{12} + k_{34})^2} & \frac{1}{\ell_{12}^2} \end{pmatrix}.$$

In case (B), take ℓ_1 to be in the middle propagator of the box first, as in

$$\frac{1}{\ell_1^2\ell_2^2(\ell_2 - k_4)^2(\ell_2 + k_3)^2(\ell_{12} + k_3)^2(\ell_{12} - k_4)^2(\ell_{12} + k_{23})^2}. \quad (\text{A3})$$

There are again 9 terms, because there are 3 factors with ℓ_2 and 3 factors with ℓ_{12} . After symmetrizing also in the choice of ℓ_2 versus ℓ_{12} , we get

$$I_{12,34}^B(\ell_1, \ell_2) = \frac{1}{\ell_1^2\ell_2^2} V_{12a} V_2^a, \quad (\text{A4})$$

with V_2 given as above and

$$V_{12} = (v(\ell_{12}), v(\ell_{12} + k_4), v(\ell_{12} - k_3)),$$

$$v(\ell_{12}) = \frac{1}{(\ell_{12} + k_{23})^2(-2\ell_{12} \cdot k_2 - k_{23}^2)(2\ell_{12} \cdot k_1 - k_{23}^2)} + \frac{1}{(\ell_{12} + k_3)^2(2\ell_{12} \cdot k_2 + k_{23}^2)(-2\ell_{12} \cdot k_{34})} + \frac{1}{(\ell_{12} - k_4)^2(-2\ell_{12} \cdot k_1 + k_{23}^2)(2\ell_{12} \cdot k_{34})}.$$

The total contribution from the planar double box is obtained after the further symmetrization of the loop momentum choices:

$$I_{12,34}^{\text{planar}} = \frac{1}{12} (I_{12,34}^A + I_{12,34}^B + I_{34,12}^B + [\ell_1 \leftrightarrow \ell_2] + [\ell_1 \rightarrow -\ell_1, \ell_2 \rightarrow -\ell_2]), \quad (\text{A5})$$

where the numerical factor takes into account the symmetrizations over three types of shifts and the four types of loop momentum choices.

The nonplanar double box is analogous. Starting with case (C) where there are three propagators with ℓ_{12} ,

$$\frac{1}{\ell_1^2(\ell_1 - k_2)^2 \ell_2^2(\ell_2 - k_1)^2 \ell_{12}^2(\ell_{12} + k_4)^2(\ell_{12} + k_{34})^2}, \quad (\text{A6})$$

we obtain

$$I_{1,2,34}^C(\ell_1, \ell_2) = \frac{1}{\ell_1^2 \ell_2^2 (2\ell_1 \cdot k_2)(2\ell_2 \cdot k_1)} (t(\ell_{12}) + t(\ell_{12} + k_{12}) - t(\ell_{12} + k_1) - t(\ell_{12} + k_2)), \quad (\text{A7})$$

with

$$U_1 = \left(\frac{1}{(-2\ell_1 \cdot k_2)}, \frac{1}{(2\ell_1 \cdot k_2)} \right),$$

$$U_2 = \left(\frac{1}{(-2\ell_2 \cdot k_4)(-2\ell_2 \cdot k_{34} + k_{34}^2)}, \frac{1}{(2\ell_2 \cdot k_4)(-2\ell_2 \cdot k_3)}, \frac{1}{(2\ell_2 \cdot k_{34} + k_{34}^2)(2\ell_2 \cdot k_3)} \right),$$

$$N = \begin{pmatrix} u(\ell_{12}) & u(\ell_{12} + k_4) & u(\ell_{12} + k_{34}) \\ u(\ell_{12} + k_2) & u(\ell_{12} + k_{24}) & u(\ell_{12} - k_1) \end{pmatrix},$$

$$u(\ell_{12}) = \left(\frac{1}{\ell_{12}^2} - \frac{1}{(\ell_{12} + k_1)^2} \right) \frac{1}{(2\ell_{12} \cdot k_1)^2}.$$

Finally, the total contribution from the nonplanar double box is

$$I_{1,2,34}^{\text{non-planar}} = \frac{1}{12} (I_{12,34}^C + I_{12,34}^D + I_{34,12}^D + [\ell_1 \leftrightarrow \ell_2] + [\ell_1 \rightarrow -\ell_1, \ell_2 \rightarrow -\ell_2]). \quad (\text{A10})$$

APPENDIX B: ANALYSIS OF THE DEGENERATE SOLUTIONS

In this section, we give a more detailed analysis of the two-loop scattering equations and their solutions. As at one loop [14,19], the key is to study the $d+g$ -dimensional (massless) tree-level scattering equations for $2g$ additional particles, then reduce to d dimensions and take the forward limit:

$$d\mu_{0,n+4}^{(d+2)} \rightarrow d\mu_{2,n}^{(d)} \equiv \frac{\prod_A \sigma_A}{\text{vol}(\text{SL}(2, \mathbb{C}))} \prod_A \bar{\delta}(E_A). \quad (\text{B1})$$

In particular, while this procedure reconstructs the two-loop scattering equations E_A on the nodal Riemann sphere (12),

$$t(\ell_{12}) = \frac{1}{\ell_{12}^2(2\ell_{12} \cdot k_4)(2\ell_{12} \cdot k_{34} + k_{34}^2)} + \frac{1}{(\ell_{12} + k_4)^2(-2\ell_{12} \cdot k_4)(2\ell_{12} \cdot k_3 + k_{34}^2)} + \frac{1}{(\ell_{12} + k_{34})^2(-2\ell_{12} \cdot k_{34} - k_{34}^2)(-2\ell_{12} \cdot k_3 - k_{34}^2)}.$$

The other case, (D), is when there three propagators with ℓ_1 or ℓ_2 , say ℓ_2 :

$$\frac{1}{\ell_1^2(\ell_1 - k_2)^2 \ell_2^2(\ell_2 - k_4)^2(\ell_2 - k_{34})^2 \ell_{12}^2(\ell_{12} + k_1)^2}. \quad (\text{A8})$$

Then we get

$$I_{1,2,34}^D(\ell_1, \ell_2) = \frac{1}{\ell_1^2 \ell_2^2} U_{1ab}^a N_{ab} U_{2b}^b, \quad (\text{A9})$$

with

it retains enough information of the massive scattering equations (in d dimensions) to analyze the different classes of solutions. The main incentive for this study is an unphysical pole arising from double roots in the loop scattering equations. We will see explicitly how this pole can be reduced to a specific subset of the solutions, which do not contribute for the supersymmetric theories discussed in this article.

At two loops, our starting points are thus the $d+2$ -dimensional massless scattering equations for $n+4$ particles with momenta $\{k_{1^\pm}, k_{2^\pm}, k_i\}$,

$$\mathcal{E}_A^{(d+2)} = \sum_B \frac{k_A \cdot k_B}{(AB)}, \quad (\text{B2})$$

with $A \in \{1^\pm, 2^\pm, 1, \dots, n\}$, where we have suggestively indexed the particles that will give rise to the loop momentum under the forward limit by 1^\pm and 2^\pm . In particular, we take the external particles k_i to only have components in d dimensions, and we denote this d -dimensional part of k_{1^\pm}, k_{2^\pm} by $\tilde{\ell}_{1^\pm}, \tilde{\ell}_{2^\pm}$ respectively.

It is now always possible to choose the remaining components of k_{1^\pm} , k_{2^\pm} such that the scattering equations reduce to

$$\begin{aligned} \mathcal{E}_{1^\pm}^{(d)} &= \frac{\tilde{\ell}_{1^+} \cdot \tilde{\ell}_{1^-} + m_1^2}{(1^\pm 1^\mp)} + \sum_{r=2^\pm} \frac{\tilde{\ell}_{1^\pm} \cdot \tilde{\ell}_r \pm \frac{1}{2}(m_1^2 + m_2^2)}{(1^\pm r)} \\ &\quad + \sum_{i=1}^n \frac{\tilde{\ell}_{1^\pm} \cdot k_i}{(1^\pm i)} \end{aligned} \quad (\text{B3a})$$

$$\begin{aligned} \mathcal{E}_{2^\pm}^{(d)} &= \frac{\tilde{\ell}_{2^+} \cdot \tilde{\ell}_{2^-} + m_2^2}{(2^\pm 2^\mp)} + \sum_{r=1^\pm} \frac{\tilde{\ell}_{2^\pm} \cdot \tilde{\ell}_r \pm \frac{1}{2}(m_1^2 + m_2^2)}{(2^\pm r)} \\ &\quad + \sum_{i=1}^n \frac{\tilde{\ell}_{2^\pm} \cdot k_i}{(2^\pm i)} \end{aligned} \quad (\text{B3b})$$

$$\mathcal{E}_i^{(d)} = \sum_{j=1}^n \frac{k_i \cdot k_j}{(ij)} + \sum_{r=1^\pm, 2^\pm} \frac{k_i \cdot \tilde{\ell}_r}{(ir)}. \quad (\text{B3c})$$

Note in particular that in the forward limit (governed by a parameter $\tau \rightarrow 0$), where

$$\tilde{\ell}_{r^\pm} = \pm \ell_r + \frac{\tau}{2} q_r, \quad \ell_r^2 = m_r^2, \quad (\text{B4})$$

these equations smoothly limit onto the two-loop scattering equations (12). However, as first pointed out in [19] at one loop, not all their solutions have a smooth limit as well—the zero locus of (12) excludes a subset of the solutions. To see this, first recall that we distinguish two different classes of solutions: since the two-loop scattering equations $E_r \equiv \pm E_{r^\pm}$ have the same functional form as functions of $\sigma = \sigma_{r^\pm}$ respectively, there are both “regular solutions” with σ_{r^\pm} localizing on different roots of E_r and “degenerate solutions,” where $\sigma_{r^+} = \sigma_{r^-}$. Moreover, perturbing around the soft limit, the degenerate solutions come in three variations, see Fig. 1:

case A: $\sigma_{1^+} = \sigma_{1^-}$, but $\sigma_{2^+} \neq \sigma_{2^-}$ (or $\sigma_{2^+} = \sigma_{2^-}$, but $\sigma_{1^+} \neq \sigma_{1^-}$)

case B: $\sigma_{1^+} = \sigma_{1^-}$ and $\sigma_{2^+} = \sigma_{2^-}$, but $\sigma_{1^+} \neq \sigma_{2^+}$

case C: $\sigma_{1^+} = \sigma_{1^-} = \sigma_{2^+} = \sigma_{2^-}$.

For each degeneration of the nodal Riemann sphere, we distinguish furthermore between two types of solutions, depending on the rate of coalescence of σ_{r^\pm} . For the soft limit parameter τ as above, they behave as

regular: $\sigma_{r^+r^-} \sim 1$

type I: $\sigma_{r^+r^-} \sim \tau$

type II: $\sigma_{r^+r^-} \sim \tau^2$.

While the type II solutions contribute for the $d+2$ -dimensional tree-level scattering equations $\mathcal{E}_A^{(d+2)}$, the zero

locus of (12) excludes them, and thus the two-loop integrands localize on the type I solutions and the regular solutions.

Case A. To see this explicitly, let us perturb around the forward limit and focus on the case A. Both these solutions and the type B solutions bear a close resemblance to one loop [19], and our discussion will proceed in analogy. We take the forward limit (B4), where q is a fixed vector with $q^2 \neq 0$, and moreover

$$\sigma_{2^\pm} = \sigma_1 \pm \varepsilon + \mathcal{O}(\varepsilon^2), \quad (\text{B5})$$

to restrict to the degenerate solutions only, and then study the scattering equations perturbatively in ε and τ . To the relevant order, the scattering equations become

$$\mathcal{E}_i^A = \frac{\ell_1 \cdot k_i}{(i1^+)} - \frac{\ell_1 \cdot k_i}{(i1^-)} + \sum_{j=1}^n \frac{k_i \cdot k_j}{(ij)} + \mathcal{O}(\varepsilon, \tau), \quad (\text{B6a})$$

$$\mathcal{E}_{1^\pm}^A = \pm \sum_{i=1}^n \frac{\ell_1 \cdot k_i}{(1^\pm i)} + \mathcal{O}(\varepsilon, \tau), \quad (\text{B6b})$$

$$\begin{aligned} \mathcal{E}_{2^\pm}^A &= \pm \frac{\tau^2 q_2^2}{2\varepsilon} \pm \sum_{i=1}^n \frac{\ell_2 \cdot k_i}{(Ii)} + \frac{\tau}{2} \sum_i \frac{q_2 \cdot k_i}{(Ii)} - \varepsilon \sum_i \frac{\ell_2 \cdot k_i}{(Ii)^2} \\ &\quad + \frac{\pm \frac{1}{2}(\ell_1 + \ell_2)^2 + \frac{\tau}{2}(\pm q_1 \cdot \ell_2 + q_2 \cdot \ell_1) + \frac{\tau^2}{4} q_1 \cdot q_2}{(I1^+)} \\ &\quad + \frac{\mp \frac{1}{2}(\ell_1 + \ell_2)^2 + \frac{\tau}{2}(\mp q_1 \cdot \ell_2 - q_2 \cdot \ell_1) + \frac{\tau^2}{4} q_1 \cdot q_2}{(I1^-)} \\ &\quad + \mathcal{O}(\varepsilon\tau, \varepsilon^2). \end{aligned} \quad (\text{B6c})$$

To leading order, the first two equations are the scattering equations at one loop, while the equations \mathcal{E}_{2^\pm} are best understood in their polynomial form, $\mathcal{F}_0 \equiv \mathcal{E}_{2^+} + \mathcal{E}_{2^-}$ and $\varepsilon \mathcal{F}_1 \equiv \sigma_{2^+} \mathcal{E}_{2^+} + \sigma_{2^-} \mathcal{E}_{2^-}$;

$$\mathcal{F}_0^A = \tau \sum_{i=1}^n \frac{q_2 \cdot k_i}{(Ii)} - \varepsilon \sum_{i=1}^n \frac{\ell_2 \cdot k_i}{(Ii)^2} + \mathcal{O}(\varepsilon\tau, \tau^2) \quad (\text{B7a})$$

$$\mathcal{F}_1^A = \sum_{i=1}^n \frac{\ell_2 \cdot k_i}{(Ii)} + \frac{\tau^2 q_2^2}{\varepsilon} + \mathcal{O}(\tau, \varepsilon). \quad (\text{B7b})$$

Evidently, there are two dominant balances for ε in terms of τ : $\varepsilon \sim \tau$ from (B7a) (type I in the notation above), and $\varepsilon \sim \tau^2$ from (B7b) (type II). We can now confirm that the zero locus of the two-loop scattering equations excludes the type II solutions with $\varepsilon \sim \tau^2$, since this dominant balance includes a term $\tau^2 q_2^2 / \varepsilon \sim 1$, which is absent in (12).

Moreover, each case gives n solutions for the coalescence point σ_I , and one solution for ε . Together with the

$(n-1)! - 2(n-2)!$ (nondegenerate) solutions to the one-loop scattering equations on the nodal sphere, this leads to $n((n-1)! - 2(n-2)!)$ solutions of each type. This is as expected: since the momentum flowing through the node connecting the two spheres is soft due to the back-to-back loop momenta $\pm \ell_2$, the location of the connecting node σ_I does not affect the other $n+2$ marked points. The prefactor n originates from the equation determining the location of the soft node in terms of the other marked points. This reflects the known results for a soft particle scattering with N hard particles—the scattering equations for the N particles decouple, giving $(N-3)!$ solutions, while the equation determining the location of the soft particle is of degree $N-2$.

Case B. Case B follows by analogy. Starting from the forward limit and restricting to the degenerate solutions

$$\sigma_{1^\pm} = \sigma_{I_1} \pm \varepsilon_1 + \mathcal{O}(\varepsilon_1^2), \quad \sigma_{2^\pm} = \sigma_{I_2} \pm \varepsilon_2 + \mathcal{O}(\varepsilon_2^2), \quad (\text{B8})$$

the tree-level scattering equations \mathcal{E}_i decouple. To leading order, the remaining scattering equations in their polynomial form have the same functional form as in case A, $\mathcal{F}_{0,1}^A(\sigma_I, \varepsilon) = \mathcal{F}_{0,1}^B(\sigma_{I_2}, \varepsilon_2) = \mathcal{F}_{0,1}^B(\sigma_{I_1}, \varepsilon_1)|_{\ell_2 \rightarrow \ell_1, q_2 \rightarrow q_1}$. Therefore, there are four types of degenerate solutions, with $\sigma_{1+1-} \sim \sigma_{2+2-} \sim \tau$; $\sigma_{1+1-} \sim \tau$ and $\sigma_{2+2-} \sim \tau^2$ (and with $\sigma_{1^\pm}, \sigma_{2^\pm}$ interchanged); and $\sigma_{1+1-} \sim \sigma_{2+2-} \sim \tau^2$. However, only $\sigma_{1+1-} \sim \sigma_{2+2-} \sim \tau$ is a solution to the two-loop scattering equations for the same reason discussed above. When counting the number of solutions, there are now $(n-2)$ solutions for each of the coalescence points $\sigma_{I_{1,2}}$, and $(n-3)!$ solutions to the tree-level scattering equations, thus $(n-2)^2(n-3)!$ solutions for type I in total.

Case C. While the cases A and B work in close analogy to one loop due to the decoupling of the scattering equations on the main Riemann sphere, the case C degeneration is a new feature at two loops. In contrast to the discussion above, there are now four remaining equations on the Riemann sphere containing the loop momenta, determining the location of the connecting node as well as the separations of the loop nodes. The most convenient parametrization for the marked points is

$$\sigma_{r^\pm} = \sigma_I + \varepsilon x_{r^\pm}, \quad \text{with } x_{r^+} = x_r + y_r \text{ and } x_{r^-} = x_r. \quad (\text{B9})$$

Möbius invariance on the sphere guarantees that we can always fix the locations of two of these, e.g. x_r (with the connecting node taken to be at $x_I = \infty$ on the sphere containing the loop momenta). After imposing both the forward limit and (B9) to restrict to case C, the scattering equations are most concise in the polynomial form

$$\mathcal{F}_a^C = \sum_{r=1^\pm, 2^\pm} x_r^a \mathcal{E}_r^C, \quad a = 0, 1, 2, 3. \quad (\text{B10a})$$

In particular, we find that \mathcal{F}_3^C contains a term of the form

$$\mathcal{F}_3^C = \frac{(\ell_1 + \ell_2)^2}{2\varepsilon} y_1 y_2 + \mathcal{O}(1), \quad (\text{B11})$$

with all other terms of order $\mathcal{O}(1)$. Thus, we conclude that we only obtain solutions for

$$y_1 \sim \tau \quad \text{and} \quad y_2 \sim 1 \quad \text{or} \quad y_1 \sim 1 \quad \text{and} \\ y_2 \sim \tau \quad \text{or} \quad y_1 \sim y_2 \sim \tau.$$

A closer investigation reveals that in fact neither of the former cases gives a consistent dominant balance, and thus our solutions are of the form $y_1 \sim y_2 \sim \tau$,

$$\sigma_{r^\pm} = \sigma_I + \varepsilon x_{r^\pm}, \quad \text{with } x_{r^+} = x_r + \tau w_r \text{ and } x_{r^-} = x_r. \quad (\text{B12})$$

This leads to only solutions of type I, with $\varepsilon \sim \tau$. Moreover, since w_r and ε appear linearly in the equations, there is exactly one solution for each of them. The remaining polynomial in σ_I is of degree $6(2n-3)$ after eliminating w_r and ε , and thus we find $6(2n-3)(n-3)!$ solutions.

Number of solutions. This discussion can be summarized in Table I. As discussed, the type II solutions are excluded by the two-loop limit. The factor of 2 in the case A comes from interchanging ℓ_1 and ℓ_2 . In particular, this implies that the two-loop scattering equations have N_{reg} solutions, with

$$N_{\text{reg}} = (n+1)! - 4n! + 4(n-1)! + 6(n-3)!. \quad (\text{B13})$$

Degenerate solutions in supersymmetric theories. Following the discussion in Sec. 4.1 of [10], the measure for the degenerate solutions becomes

- (i) Case A: $d\mu_{2,n} = (\bar{\delta}(\varepsilon - \tau F_A) d\varepsilon d\tilde{\mu}_I) d\mu_{1,n}$
- (ii) Case B: $d\mu_{2,n} = (\bar{\delta}(\varepsilon_1 - \tau F_{B_1}) d\varepsilon_1 d\tilde{\mu}_{I_2}) (\bar{\delta}(\varepsilon_2 - \tau F_{B_2}) d\varepsilon_2 d\tilde{\mu}_{I_2}) d\mu_{0,n}$
- (iii) Case C: $d\mu_{2,n} = \varepsilon^5 (\bar{\delta}(\varepsilon - \tau F_C) d\varepsilon d\tilde{\mu}_I) d\mu_{0,n}$

Here, we have extracted the ε dependence explicitly, so $d\tilde{\mu}_I$ represents the remaining measure on the loop-momentum Riemann sphere. For the super-Yang-Mills (19) and super-gravity integrands (21) at two loops, we find that

TABLE I. Number of solutions corresponding to various degenerations, as shown in Fig. 1.

| | Type I | Type II |
|--------|------------------------|------------------------|
| Case A | $2n((n-1)! - 2(n-2)!)$ | $2n((n-1)! - 2(n-2)!)$ |
| Case B | $(n-2)^2(n-3)!$ | $3(n-2)^2(n-3)!$ |
| Case C | $6(2n-3)(n-3)!$ | |

$$\text{Case A: } \mathcal{I}^{\text{SUGRA}} \sim \varepsilon^3 \quad \mathcal{I}^{\text{sYM}} \sim \varepsilon^1$$

$$\text{Case B: } \mathcal{I}^{\text{SUGRA}} \sim \varepsilon_1^3 \varepsilon_2^3 \quad \mathcal{I}^{\text{sYM}} \sim \varepsilon_1^1 \varepsilon_2^1$$

$$\text{Case C: } \mathcal{I}^{\text{SUGRA}} \sim 1 \quad \mathcal{I}^{\text{sYM}} \sim \varepsilon^{-2}.$$

Therefore, all supersymmetric two-loop amplitudes behave as $\mathcal{O}(\varepsilon)$ for all degenerate solutions. The supersymmetric integrands thus only receive contributions from the regular solutions with $\sigma_{r^+r^-} \sim 1$.

Absence of the unphysical pole. As pointed out in [14] at one loop, the loop-level scattering equations on the Riemann sphere contain unphysical poles. To see this, recall that the scattering equations at the nodes have the same functional form $E_r(\sigma)$, and the regular solutions are characterized by localizing σ_{r^\pm} on different roots. A special case thus arises when $E_r(\sigma)$ develops degenerate roots, and thus regular solutions become degenerate. As discussed above, when a solution is degenerate $\sigma_{r^+} = \sigma_{r^-}$, the scattering equations separate into lower loop-order equations, and the remaining marked points are independent of σ_{r^\pm} . Thus, denoting the sphere containing the marked points (and possibly one pair of nodes associated to the

loop momenta in case A) by Σ_r , each unphysical pole is given by the discriminant of E_r [14],

$$\Delta = \prod_{\text{sol for } \sigma_r \in \Sigma_r} \text{Disc}(E_r). \quad (\text{B14})$$

Note that this argument does not make any reference to the form of the other scattering equations (apart from requiring that they decouple for degenerate solutions), and thus the treatment of the unphysical pole at two loops proceeds exactly as at one loop. More specifically, integrating by parts localizes the contribution of the unphysical pole to the degenerate solutions. In particular, since the integrands for super-Yang-Mills and supergravity vanish on these solutions, this implies that the degenerate pole never occurs.

For nonsupersymmetric theories, more care is needed since the degenerate solutions will contribute in general. Just as at one loop, it is thus necessary to show that the contribution from the pole (and the integrand in this channel) is homogeneous in the loop momenta and thus vanishes upon integration.

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