

# COMPLETE SPECTRAL DATA FOR ANALYTIC ANOSOV MAPS OF THE TORUS

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ABSTRACT. Using analytic properties of Blaschke factors we construct a family of analytic hyperbolic diffeomorphisms of the torus for which the spectra of the associated transfer operator acting on a suitable Hilbert space can be computed explicitly. As a result, we obtain expressions for the decay of correlations of analytic observables without resorting to any kind of perturbation argument.

## 1. INTRODUCTION

Spectral theory constitutes one of the major approaches to study complex chaotic motion. Drawing on both functional analytic techniques and dynamical systems theory, it furnishes a powerful method to construct invariant measures with good statistical properties as well as a means to study the fine-structure of the corresponding correlation decay. The general theory is fairly well developed (see, for example, [KatH, Kel, Bal1, Bal2]) and has resulted in several major breakthroughs in the understanding of complex dynamical behaviour from an ergodic theoretic perspective. Despite this deep understanding there is still a considerable lack of exactly solvable models serving as paradigmatic examples illustrating the theory.

To date, examples of maps for which spectral properties of the corresponding transfer operator can be computed explicitly are essentially limited to the one-dimensional uniformly expanding case, with the first examples arising in the context of piecewise linear Markov maps, where spectral properties can be reduced to finite-dimensional matrix calculations (see [MorSO]; see also [SBJ1] for a more recent exposition). Exploiting the rich analytic structure of Blaschke products (see, for example, [Mar]) nonlinear examples of full-branch analytic expanding interval maps for which complete spectral data of the corresponding transfer operator is available have recently been introduced by the authors (see [SBJ2]; see also [BanJS] for examples of analytic expanding maps of the circle).

Trivial examples obtained by taking products of one-dimensional maps excepted, the situation in higher dimensions is even more challenging, which is unfortunate, since diffeomorphisms, in particular higher-dimensional symplectic maps, play a vital role for the dynamical foundations of nonequilibrium statistical mechanics, in particular regarding irreversibility and entropy production [AT, BaC, Dor, Gal]. Due to the lack of available models explicit calculations are normally limited to the linear case, including the celebrated Arnold cat map or baker-type transformations. To the best of our knowledge not a single properly nonlinear diffeomorphism is known for which the entire spectrum and the corresponding correlation decay rates

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have been computed explicitly.<sup>1</sup> We try to fill this gap by introducing a model where this spectral information is available.

For hyperbolic maps with expanding and contracting directions, progress was for a long time hampered by the lack of suitable function spaces on which the corresponding transfer operator can be shown to have good spectral properties. This changed with the publication of [BKL], where it was shown that by adapting the space to take into account expanding and contracting directions the spectral properties of transfer operators familiar from the uniformly expanding situation can be retained for Anosov diffeomorphisms of compact manifolds. Since then, quite a number of these ‘anisotropic’ Banach spaces have been constructed, capturing the behaviour of rather general hyperbolic diffeomorphisms with low regularity (see [GouL1, GouL2, BalT1, BalT2, BalG, FauRS], or [Bal2, Bal3] for recent reviews). The main thrust of these works has been to show that the associated transfer operator is quasicompact, that is, its peripheral spectrum is discrete like that of a compact operator, but lower-lying spectral points may (and usually will) be part of the essential spectrum, characterised by persistence under compact perturbations.

There are only few papers dealing with hyperbolic diffeomorphisms with very high regularity, where there is a chance of obtaining compact transfer operators forcing the essential spectrum to consist of the origin only. In the analytic setting, Rugh, in a paper predating [BKL], has constructed anisotropic Banach spaces of analytic functions on which the transfer operator of hyperbolic maps with rather special geometries can be shown to be trace class, and hence compact (see [Rug]). Our work fits into this category: by restricting to the analytic setting where transfer operators can be shown to be compact, a complete description of the spectrum is facilitated.

In the following, we will introduce an example of an analytic hyperbolic diffeomorphism of the torus, for which the entire spectrum of a properly defined compact transfer operator can be computed and linked with correlation decay of analytic observables. The underlying space is taken from a study of Faure and Roy [FauR], who were able to link the correlation decay of small analytic perturbations of linear automorphisms of the torus to spectral properties of a certain transfer operator. While we still base our analysis on an analytic deformation of the cat map, we do not need to resort to a perturbative treatment. The same space has recently been used by Adam [Ada] to show that transfer operators of generic analytic perturbations of hyperbolic linear automorphisms have non-trivial eigenvalues. His approach, however, only yields the existence of at least one non-zero eigenvalue (albeit generically), while our example exhibits infinitely many (explicitly known) eigenvalues. In passing we note that the generic existence of infinitely many eigenvalues for the transfer operators of analytic expanding maps on  $\mathbb{R}^n$  is shown in [Nau] and for transfer operators of analytic expanding circle maps in [BanN]. We also note that infinitely many Pollicott-Ruelle resonances have been shown to exist for contact Anosov flows (see [FauT]) and for certain compact hyperbolic surfaces (see [GuiHW]).

For any complex number  $\lambda$  smaller than one in modulus let us introduce the analytic map  $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  on the complex unit torus  $\mathbb{T}^2 = \{z \in \mathbb{C}^2 : |z_1| = 1, |z_2| = 1\}$

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<sup>1</sup> For flows the situation has changed recently with a series of articles by Dang and Rivière [DaR1, DaR2, DaR3] providing a complete description of Pollicott-Ruelle resonances as well as the corresponding resonant states for certain Morse-Smale flows.

defined by

$$T(z_1, z_2) = \left( z_1 \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} z_2, \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} z_2 \right). \quad (1)$$

Using canonical coordinates on the unit torus,  $z_\ell = \exp(2\pi i\phi_\ell)$ , the real representation of the map, considered as a map on  $\mathbb{R}^2/\mathbb{Z}^2$ , reads

$$(\phi_1, \phi_2) \mapsto (2\phi_1 + \psi(\phi_1) + \phi_2, \phi_1 + \psi(\phi_1) + \phi_2), \quad (2)$$

where the nonlinear part is given by

$$\psi(\phi) = \frac{1}{\pi} \arctan \left( \frac{|\lambda| \sin(2\pi\phi - \alpha)}{1 - |\lambda| \cos(2\pi\phi - \alpha)} \right). \quad (3)$$

Here,  $\lambda = |\lambda| \exp(i\alpha)$  denotes the polar representation of the parameter with  $|\lambda| < 1$ . Clearly, our family of maps contains the Arnold cat map for the choice  $\lambda = 0$ . The toral map (1) is a special case of a so-called two-dimensional Blaschke product which has already received some attention in the context of ergodic theory (see [PS]).

It is not difficult to see that the derivative of the map given by (2) maps the first and third quadrant of  $\mathbb{R}^2$  strictly inside itself and that the derivative of its inverse maps the second and fourth quadrant of  $\mathbb{R}^2$  strictly inside itself. Thus (1) yields a family of analytic uniformly hyperbolic toral diffeomorphisms, also known as Anosov diffeomorphisms (see [Mos, Lemma 4] or [Has, Chapter 2.1.b]).

Clearly, the map defined by (2) is area-preserving and thus provides an example of a chaotic Hamiltonian system. A few more empirical features, numerical simulations, and some basic results on correlation decay are presented in Appendix A.

Unlike the situation for one-dimensional non-invertible maps there is no clear distinction between Perron-Frobenius and Koopman operators as we are dealing with area preserving diffeomorphisms. The operator governing the dynamics of our system is essentially a composition operator  $\mathcal{C}$  defined by

$$(\mathcal{C}f)(z_1, z_2) = (f \circ T)(z_1, z_2) = f \left( z_1 \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} z_2, \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} z_2 \right) \quad (4)$$

where  $f : \mathbb{T}^2 \rightarrow \mathbb{C}$ . As alluded to earlier, the choice of a space of functions on which  $\mathcal{C}$  acts and has nice spectral properties is a delicate matter. We shall use a family of Hilbert spaces  $\mathcal{H}_a$  indexed by a positive real parameter  $a$  which contains all Laurent polynomials on the unit torus as a dense subset. Postponing the formal definition to the following section our main result can be stated as follows.

**Theorem 1.1.** *The composition operator  $\mathcal{C} : \mathcal{H}_a \rightarrow \mathcal{H}_a$  is a well-defined compact operator for any  $a > 0$  and any  $|\lambda| < 1$ . Its spectrum is given by*

$$\sigma(\mathcal{C}) = \{(-\lambda)^n : n \in \mathbb{N}\} \cup \{(-\bar{\lambda})^n : n \in \mathbb{N}\} \cup \{1, 0\}. \quad (5)$$

*Each non-zero element of the spectrum is an eigenvalue, the algebraic and geometric multiplicity of which coincide with the number of times the non-zero number occurs in (5).*

The above equality of algebraic and geometric multiplicity for each non-zero eigenvalue implies that the non-zero spectrum of  $\mathcal{C} : \mathcal{H}_a \rightarrow \mathcal{H}_a$  has no non-trivial Jordan blocks. In passing we note that non-trivial Jordan blocks can occur for piecewise linear expanding interval maps (see, for example, [AQ, Dae, Dri]) and for geodesic flows on hyperbolic surfaces (see [FF, GuiHW]).

Using the definition (4) and the invariance of Haar measure  $\mu$  on the unit torus it is straightforward to relate the spectral properties of the operator with correlation functions and to bound the decay of correlations for sufficiently nice observables.

**Corollary 1.2.** *For any functions  $g : \mathbb{T}^2 \rightarrow \mathbb{C}$  and  $h : \mathbb{T}^2 \rightarrow \mathbb{C}$  analytic in an open neighbourhood of the unit torus the corresponding correlation function*

$$C_{gh}(k) = \int g \cdot h \circ T^k d\mu - \int g d\mu \int h d\mu, \quad (6)$$

where  $\mu$  denotes the invariant Haar measure on  $\mathbb{T}^2$ , satisfies

$$|C_{gh}(k)| \leq K|\lambda|^k \quad (7)$$

for all  $k \in \mathbb{N}$  with  $K$  a suitable constant. In particular,  $T$  is strongly mixing with respect to  $\mu$ .

With a little bit more effort one can also derive asymptotic expansions for the correlation function. In particular, the estimate given in Corollary 1.2 is sharp as one can easily find cases where the upper bound is attained, see (73).

Another simple consequence of Theorem 1.1 is the following result on the location of the Pollicott-Ruelle resonances (see [Pol1, Pol2, Rue1, Rue2]) of  $T$ , that is, the poles of the meromorphic continuation of the Z-transform of the correlation function.

**Corollary 1.3.** *For any  $g : \mathbb{T}^2 \rightarrow \mathbb{C}$  and  $h : \mathbb{T}^2 \rightarrow \mathbb{C}$  analytic in an open neighbourhood of the unit torus the Z-transform  $\hat{C}_{gh}$  of the corresponding correlation function given by*

$$\hat{C}_{gh}(\zeta) = \sum_{k=0}^{\infty} \zeta^{-k} C_{gh}(k) \quad (8)$$

for  $\zeta \in \mathbb{C}$  with  $|\zeta| > 1$ , has a meromorphic continuation to  $\mathbb{C} \setminus \{0\}$  with no poles outside of

$$\{(-\lambda)^n : n \in \mathbb{N}\} \cup \{(-\bar{\lambda})^n : n \in \mathbb{N}\}. \quad (9)$$

This article is organised as follows. In Section 2 we will define the function space  $\mathcal{H}_a$  on which the transfer operator (4) will be defined. We will spend some effort on its motivation, as its structure is fundamentally linked to the physics of the underlying dynamical system. Compactness of the composition operator will be proven in Section 3 by establishing suitable bounds on the entries of a matrix representation of  $\mathcal{C}$  with respect to an orthonormal basis of  $\mathcal{H}_a$ . Using the fact that this matrix representation is lower-triangular we will then be able to obtain the entire spectrum of  $\mathcal{C}$  in closed form, thus completing the proof of our main result, Theorem 1.1.

Section 4 is devoted to proving the two corollaries, which involves a discussion of the properties of the invariant measure and the corresponding correlation decay for analytic observables.

Part of our presentation requires some basic knowledge of functional analysis, which, in spite of the fact that it can be found in standard textbooks, we cover in some detail so as to make the exposition accessible to a larger audience in applied dynamical systems theory.

In this article, we shall only be concerned with the particular example given in (1), postponing the discussion of possible generalisations to the conclusion and Appendix C.

## 2. HILBERT SPACE AND TRANSFER OPERATOR

The main purpose of this section is to introduce a family of Hilbert spaces and to show that the composition operator (4) is compact on each of these spaces.

We start by defining the family of Hilbert spaces. For  $\lambda = 0$  the map given by (1) or (2) induces a linear automorphism on the torus (viewed as  $\mathbb{R}^2/\mathbb{Z}^2$ ). The corresponding unstable/stable eigenvalues and eigenvectors are given by

$$\lambda_{u/s} = \varphi^{\pm 2}, \quad v_{u/s} = (\lambda_{u/s} - 1, 1), \quad (10)$$

where  $\varphi := (1 + \sqrt{5})/2$  denotes the golden mean. For brevity we will use multi-index notation  $n = (n_1, n_2) \in \mathbb{Z}^2$  with  $|n| = |n_1| + |n_2|$ , and we abbreviate the monomials of  $z = (z_1, z_2) \in \mathbb{C}^2$  by  $z^n = z_1^{n_1} z_2^{n_2}$ . Let us denote by

$$n_{u/s} = v_{u/s} n = (\lambda_{u/s} - 1)n_1 + n_2 \quad (11)$$

the components of  $n$  with respect to the stable and unstable direction of the cat map.

Before defining the family of spaces recall that a *Laurent monomial* is a map  $z \mapsto z^n$  from  $\mathbb{T}^2$  to  $\mathbb{C}$  where  $n \in \mathbb{Z}^2$ . Clearly, a Laurent monomial corresponds to a Fourier mode on  $\mathbb{R}^2/\mathbb{Z}^2$ . We call a finite linear combination of Laurent monomials a *Laurent polynomial* and denote the set of all Laurent polynomials by  $\mathcal{L}$ . Thus

$$\mathcal{L} = \{f : \mathbb{T}^2 \rightarrow \mathbb{C} : f(z) = \sum_{|n| \leq N} f_n z^n, \text{ with } f_n \in \mathbb{C}, N \in \mathbb{N}\}. \quad (12)$$

Following [FauR], we will define anisotropic Hilbert spaces adapted to the action of the transfer operator given by (4) as the completion of the set of Laurent polynomials with respect to a certain norm, which we shall define presently.

Given  $a > 0$  define an inner product on  $\mathcal{L}$  by

$$\langle f, g \rangle_a = \sum_{n \in \mathbb{Z}^2} f_n \bar{g}_n \exp(-2a|n_u| + 2a|n_s|), \quad (13)$$

where  $(f_n)_{n \in \mathbb{Z}^2}$  and  $(g_n)_{n \in \mathbb{Z}^2}$  denote the Fourier coefficients of the Laurent polynomials  $f$  and  $g$ , respectively, that is,

$$f(z) = \sum_{n \in \mathbb{Z}^2} f_n z^n \text{ and } g(z) = \sum_{n \in \mathbb{Z}^2} g_n z^n; \quad (14)$$

the corresponding norm will be denoted by  $\|\cdot\|_a$ , that is, we have

$$\|f\|_a^2 = \sum_{n \in \mathbb{Z}^2} |f_n|^2 \exp(-2a|n_u| + 2a|n_s|). \quad (15)$$

We are now ready to define the family of Hilbert spaces.

**Definition 2.1.** Let  $a > 0$  then  $\mathcal{H}_a$  is the completion of  $\mathcal{L}$  with respect to the norm  $\|\cdot\|_a$ .

It turns out that  $\mathcal{H}_a$  is a separable Hilbert space, which, by construction, contains all Laurent polynomials as a dense subset (see, for example, [RS, Theorem I.3]). However, it also contains functions analytic in a sufficiently large open neighbourhood of the torus as the following lemma shows.

**Lemma 2.2.** *If  $a > 0$  and  $f : \mathbb{T}^2 \rightarrow \mathbb{C}$  is analytic in an open neighbourhood of*

$$\{\exp(-\sqrt{5}a) \leq |z_1| \leq \exp(\sqrt{5}a)\} \times \{|z_2| = 1\}, \quad (16)$$

then  $f \in \mathcal{H}_a$ . In particular, any function analytic on an open neighbourhood of the torus belongs to  $\mathcal{H}_a$  for all sufficiently small  $a$ .

*Proof.* Suppose that  $f$  is analytic on an open neighbourhood of the poly-annulus (16). Then, using Cauchy's integral formula, the function  $f$  has a Laurent expansion of the form

$$f(z) = \sum_{n \in \mathbb{Z}^2} f_n z^n \quad (17)$$

with

$$\sum_{n \in \mathbb{Z}^2} |f_n|^2 \exp(2a\sqrt{5}|n_1|) < \infty. \quad (18)$$

Since, by (11), we have

$$|n_s| - |n_u| \leq |n_s - n_u| = \sqrt{5}|n_1|, \quad (19)$$

the bound (18) implies that

$$\begin{aligned} & \sum_{n \in \mathbb{Z}^2} |f_n|^2 \exp(-2a|n_u| + 2a|n_s|) \\ &= \sum_{n \in \mathbb{Z}^2} |f_n|^2 \exp(2a\sqrt{5}|n_1|) \exp(-2a|n_u| + 2a|n_s| - 2a\sqrt{5}|n_1|) < \infty, \quad (20) \end{aligned}$$

which shows that  $f$  is a limit of Laurent polynomials convergent in the norm  $\|\cdot\|_a$  and can thus be uniquely identified with an element in  $\mathcal{H}_a$ .  $\square$

While, as we have just seen, the space  $\mathcal{H}_a$  contains functions analytic on a sufficiently large neighbourhood of the torus, it also contains generalised functions, not interpretable as ordinary functions on the torus.

At first glance, the choice of weighting in the definition of the norm (15) appears peculiar. However, this choice is intimately linked with the underlying dynamics. Broadly speaking, the weighting requires that the Fourier coefficients  $(f_n)_{n \in \mathbb{Z}^2}$  of  $f \in \mathcal{H}_a$  decay exponentially in the stable direction whereas they are allowed to grow exponentially in the unstable direction. The corresponding function on the unit torus inherits this behaviour, that is, it is smooth in the stable but allowed to be rather rough in the unstable direction. It is precisely this property which makes it possible to capture the dynamics of the underlying map. For instance, the simple textbook example of the Arnold cat map shows that an initially smooth density remains smooth along the unstable direction but becomes jagged in the stable direction. If we keep in mind that a Perron-Frobenius operator governing the motion of densities involves the inverse of the map and thus interchanges stable and unstable direction, it is precisely the space defined above which is able to capture the ergodic properties of the dynamical system. In physics terms, the structure of this space breaks the time reversal symmetry of the dynamical system, capturing the macroscopically irreversible behaviour of the motion (see, for example, [AT]). In the mathematics literature, these ideas are precisely those underlying the construction of various anisotropic function spaces in [Bal3, BKL, BalT1, FauRS] as well as in [FauR], the main difference between the former groups of work and the latter being the restrictions imposed on the decay (respectively, growth) of the Fourier coefficients in the unstable (respectively, stable) direction, which is algebraic in the former and exponential in the latter.

As in [FauR], we could have used a slightly more general setup for the underlying space by giving different weights to the stable and unstable parts in Definition 2.1. The restricted case considered here will turn out to be sufficient for our purpose. We will revisit this issue in the conclusion.

For later use, we note that the normalised monomials

$$e_n(z) = z^n \exp(a|n_u| - a|n_s|) \quad (\forall n \in \mathbb{Z}^2), \quad (21)$$

yield an orthonormal basis for  $\mathcal{H}_a$  for every  $a > 0$ .

Having introduced the underlying Hilbert space we are now going to define a transfer operator associated with the map. The definition (4) makes perfect sense for Laurent polynomials, which form a dense subset of  $\mathcal{H}_a$ . Hence, it remains to show that  $\mathcal{C}$  is bounded with respect to the norm of  $\mathcal{H}_a$  on the set of Laurent polynomials. For this in turn, it is sufficient to evaluate the images (under  $\mathcal{C}$ ) of the basis elements (21) and to show that their norm decays sufficiently fast with  $|n|$ .

We start by observing that (4) and (21) yield

$$\begin{aligned} (\mathcal{C}e_n)(z) &= \exp(a|n_u| - a|n_s|) z_1^{n_1} z_2^{n_1+n_2} \left( \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} \right)^{n_1+n_2} \\ &= \exp(a|n_u| - a|n_s|) \sum_{m \in \mathbb{Z}^2} b_{m,n} z^m, \end{aligned} \quad (22)$$

where the expansion coefficients of the Laurent series in a neighbourhood of the unit torus are given by

$$b_{m,n} = \delta_{m_2, n_1+n_2} M_{-m_1+2n_1+n_2}(\lambda, n_1 + n_2), \quad (23)$$

where we have introduced the shorthand

$$M_\ell(\lambda, k) = \int_{|\zeta|=1} \zeta^\ell \left( \frac{1 - \lambda/\zeta}{1 - \bar{\lambda}\zeta} \right)^k \frac{d\zeta}{2\pi i \zeta} \quad (24)$$

for the expansion coefficient of a single Blaschke factor. Hence, using the definition of the norm in (15), we obtain

$$\begin{aligned} \|\mathcal{C}e_n\|_a^2 &= \sum_{m \in \mathbb{Z}^2} \exp(2a|n_u| - 2a|n_s|) \delta_{m_2, n_1+n_2} |M_{-m_1+2n_1+n_2}(\lambda, n_1 + n_2)|^2 \\ &\quad \times \exp(-2a|m_u| + 2a|m_s|). \end{aligned} \quad (25)$$

Before we proceed, let us first comment on the trivial case of the cat map, which corresponds to  $\lambda = 0$ . In this case, expression (24) simplifies to  $M_\ell(\lambda, k) = \delta_{\ell,0}$  and only the term  $m_1 = 2n_1 + n_2$ ,  $m_2 = n_1 + n_2$  contributes to the series in (25). Thanks to (11), that is, thanks to the stable and unstable directions of the cat map this gives  $m_{u/s} = \lambda_{u/s} n_{u/s}$  and so (25) becomes

$$\|\mathcal{C}e_n\|_a^2 = \exp(-2a(\lambda_u - 1)|n_u| - 2a(1 - \lambda_s)|n_s|). \quad (26)$$

Since all norms in  $\mathbb{R}^2$  are equivalent, we see that in this simple case there is a  $\delta > 0$ , such that

$$\|\mathcal{C}e_n\|_a \leq \exp(-\delta|n|) \quad (\forall n \in \mathbb{Z}^2), \quad (27)$$

that is, we end up with an upper bound, which is exponentially small in  $|n|$ . This in turn finally guarantees that the transfer operator  $\mathcal{C}$  is well-defined and compact on  $\mathcal{H}_a$ , using a simple summability argument (see the proof of Proposition 2.5).

The same observation together with a localisation argument for the expression (24) has been used in [FauR] to derive similar upper bounds for the transfer operators of maps which are small perturbations (in the  $C^1$  sense) of linear maps of the torus. Restricting to our particular choice of maps, we will obtain a slightly stronger result without resorting to any perturbative argument.

In order to do this, let us first focus on an estimate for the expression (24).

**Lemma 2.3.** *For any  $\lambda = |\lambda| \exp(i\gamma) \in \mathbb{C}$  with  $|\lambda| < 1$  the expression (24) obeys*

- i)  $M_\ell(\lambda, k) = M_{-\ell}(\bar{\lambda}, -k)$ ;
- ii)  $M_\ell(|\lambda| \exp(i\gamma), k) = \exp(i\ell\gamma) M_\ell(|\lambda|, k)$ ;
- iii)  $M_\ell(\lambda, 0) = \delta_{\ell,0}$ ;
- iv)  $M_\ell(\lambda, k) = 0$  if  $\ell > k > 0$ ;
- v)  $|M_\ell(\lambda, k)| \leq 1$ .

In addition, there exists  $\alpha > 0$  and  $\beta \in (0, 1)$  such that for  $k > 0$  and  $\beta k \leq \ell \leq k$  the estimate

$$|M_\ell(\lambda, k)| \leq \exp(-\alpha(\ell - \beta k)) \quad (28)$$

holds.

*Proof.* The symmetry properties i) and ii) can be obtained by appropriate substitutions in the integral (24), namely  $\zeta' = \zeta^{-1}$  and  $\zeta' = \zeta \exp(-i\gamma)$ , respectively. Property iii) is obvious. Since the integrand in (24) is holomorphic in the unit disk for  $\ell > k > 0$ , property iv) follows. Finally v) is obvious, as the integrand is bounded by one. Hence, the only non-trivial part which remains to be proven is the estimate (28).

Due to the phase symmetry ii) it is sufficient to prove (28) with  $|\lambda|$  instead of  $\lambda$ . By contour deformation we have for  $r \in (0, 1)$

$$\begin{aligned} |M_\ell(\lambda, k)| &= \left| \int_{|\zeta|=r} \zeta^\ell \left( \frac{1 - |\lambda|/|\zeta|}{1 - |\lambda|/|\zeta|} \right)^k \frac{d\zeta}{2\pi i \zeta} \right| \\ &\leq \frac{r^\ell}{2\pi} \int_0^{2\pi} \left| \frac{1 - |\lambda|/r \exp(-i\phi)}{1 - |\lambda|/r \exp(i\phi)} \right|^k d\phi. \end{aligned} \quad (29)$$

It is not difficult to see that the integrand takes its maximum at  $\phi = \pi$ , that is,

$$\left| \frac{1 - |\lambda|/r \exp(-i\phi)}{1 - |\lambda|/r \exp(i\phi)} \right| \leq \frac{1 + |\lambda|/r}{1 - |\lambda|/r} \quad (\forall \phi \in [0, 2\pi)) \quad (30)$$

so that for any  $\beta \in (0, 1)$  we have

$$|M_\ell(\lambda, k)| \leq r^{\ell - \beta k} \left( r^\beta \frac{1 + |\lambda|/r}{1 - |\lambda|/r} \right)^k. \quad (31)$$

The base  $F(r) = r^\beta (1 + |\lambda|/r)/(1 - |\lambda|/r)$  clearly obeys  $F(1) = 1$  and  $F'(1) > 0$  if  $\beta > 2|\lambda|/(1 + |\lambda|)$ . Hence, the assertion follows by first choosing  $\beta \in (2|\lambda|/(1 + |\lambda|), 1)$  and then choosing  $r = \exp(-\alpha) \in (0, 1)$  with  $F(r) \leq 1$ .  $\square$

Let us now return to (25). Using Lemma 2.3 it is fairly straightforward to establish the following.

**Lemma 2.4.** *For  $\lambda \in \mathbb{C}$  with  $|\lambda| < 1$  and  $a > 0$  there exists  $c > 0$  and  $\delta > 0$  such that*

$$\|\mathcal{C}e_n\|_a \leq c \exp(-\delta|n|) \quad (\forall n \in \mathbb{Z}^2). \quad (32)$$



*Proof.* Because of the symmetry relations i) and ii) in Lemma 2.3 the series (25) obeys  $\|\mathcal{C}e_n\|_a = \|\mathcal{C}e_{-n}\|_a$ . Hence it is sufficient to consider the case  $n_1 + n_2 \geq 0$ .

If  $n_1 + n_2 = 0$  then property iii) of Lemma 2.3 guarantees that only a single term with  $m_1 = 2n_1 + n_2$  and  $m_2 = n_1 + n_2$  contributes to (25), so that  $m_u = \lambda_u n_u$  and  $n_s = \lambda_s n_s$  by (11). Thus

$$\begin{aligned} \|\mathcal{C}e_n\|_a &= \exp(-a(\lambda_u - 1)|n_u| - a(1 - \lambda_s)|n_s|) \\ &\leq \exp(-a(1 - \lambda_s)(|n_u| + |n_s|)), \end{aligned} \quad (33)$$

where we have used  $\lambda_u \lambda_s = 1$  and  $\lambda_u > 1$ . As  $|n_1| + |n_2| \leq 2(|n_u| + |n_s|)$  relation (32) holds for any  $c \geq 1$  and any  $\delta \leq a(1 - \lambda_s)/2$ .

Let us now assume that  $n_1 + n_2 > 0$ . The sum in (25) only runs over  $m_1$ , as only  $m_2 = n_1 + n_2$  can give rise to non-zero terms. Making use of (28), we now split this sum into three parts.

$$\|\mathcal{C}e_n\|_a^2 = S_1 + S_2 + S_3, \quad (34)$$

where

$$S_i = \sum_{\substack{m_1 \in I_i \\ m_2 = n_1 + n_2}} |M_{-m_1 + 2n_1 + n_2}(\lambda, n_1 + n_2)|^2 \exp(-2a(|n_s| - |n_u| + |m_u| - |m_s|)) \quad (35)$$

with  $I_1 = \{m_1 : m_1 < n_1\}$ ,  $I_2 = \{m_1 : n_1 \leq m_1 \leq n_1 + (1 - \beta)(n_1 + n_2)\}$ , and  $I_3 = \{m_1 : m_1 > n_1 + (1 - \beta)(n_1 + n_2)\}$ . Note that  $S_1 = 0$  by iv) of Lemma 2.3.

For  $S_2$  and  $S_3$ , we first need to have a closer look at the exponential factor. Using (11), the exponent can be written as

$$2a(|n_s| - |n_u| + |m_u| - |m_s|) = a|n_1 + n_2| F\left(\frac{m_1}{n_1 + n_2}, \frac{n_1}{n_1 + n_2}\right) \quad (36)$$

where

$$F(x, y) = 2(|\varphi x + 1| - |\varphi^{-1}x - 1| + |\varphi y - 1| - |\varphi^{-1}y + 1|),$$

and, as before,  $\varphi = (\sqrt{5} + 1)/2$  denotes the golden mean. If we employ the basic lower bound for  $F$  derived in Lemma B.1 in Appendix B, then (36) yields

$$2a(|n_s| - |n_u| + |m_u| - |m_s|) \geq a(m_1 - n_1 + |n_1|\Theta_n/2) \quad (37)$$

with

$$\Theta_n = \begin{cases} 0 & \text{if } |n_1| < 2\varphi^{-1}|n_1 + n_2|, \\ 1 & \text{if } |n_1| \geq 2\varphi^{-1}|n_1 + n_2|. \end{cases} \quad (38)$$

With this lower bound we can now estimate  $S_2$  and  $S_3$  as we have essentially reduced the problem to a geometric series. For  $S_3$  we use the trivial estimate v) of Lemma 2.3 giving

$$|S_3| \leq \frac{\exp(-a(1 - \beta)(n_1 + n_2))}{1 - \exp(-a)} \exp(-a|n_1|\Theta_n/2). \quad (39)$$

Now, a short calculation shows that

$$|n_1 + n_2| + |n_1|\Theta_n \geq (|n_1| + |n_2|)/4, \quad (40)$$

so we can bound  $S_3$  from above by

$$|S_3| \leq \frac{\exp(-\delta'|n|)}{1 - \exp(-a)} \quad (41)$$

for any  $\delta' \leq a(1 - \beta)/8$ .

For  $S_2$ , the bound (28) yields

$$|S_2| \leq \sum_{0 \leq k \leq (1-\beta)(n_1+n_2)} \exp(-ak) \exp(-2\alpha((1-\beta)(n_1+n_2) - k)) \exp(-a|n_1|\Theta_n/2).$$

Estimating this finite sum by a simple bound for its largest term we can write

$$|S_2| \leq ((1-\beta)(n_1+n_2) + 1) \exp(-\min\{a, 2\alpha\}(1-\beta)(n_1+n_2) - a|n_1|\Theta_n/2).$$

Using (40) we see that

$$|S_2| \leq ((1-\beta)|n| + 1) \exp(-\delta'|n|), \quad (42)$$

for any  $\delta' \leq \min\{a/8, \alpha/4\}(1-\beta)$ . Putting the two bounds for  $S_2$  and  $S_3$  together, the assertion finally follows.  $\square$

Standard arguments now yield the following result.

**Proposition 2.5.** *For any  $\lambda \in \mathbb{C}$  with  $|\lambda| < 1$  and any  $a > 0$ , expression (4) gives rise to a bounded and compact operator  $\mathcal{C}: \mathcal{H}_a \rightarrow \mathcal{H}_a$ .*

*Proof.* Lemma 2.4 implies that

$$M := \left( \sum_{n \in \mathbb{Z}^2} \|\mathcal{C}e_n\|_a^2 \right)^{1/2} < \infty. \quad (43)$$

Thus the operator given by the expression (4) is bounded on the set of Laurent polynomials since, using the Cauchy-Schwarz inequality, we have

$$\|\mathcal{C}f\|_a \leq \sum_{n \in \mathbb{Z}^2} |f_n| \exp(-a|n_u| + a|n_s|) \|\mathcal{C}e_n\|_a \leq M \|f\|_a; \quad (44)$$

thus, by a standard result (see, for example, [RS, Theorem 1.7]) the operator  $\mathcal{C}$  has a unique bounded extension, which we denote by the same symbol, from  $\mathcal{H}_a$  to  $\mathcal{H}_a$ . In fact, inequality (43) implies that  $\mathcal{C}$  is Hilbert-Schmidt on  $\mathcal{H}_a$ , and therefore compact (see, for example, [RS, Theorem VI.22]).  $\square$

We have just seen that  $\mathcal{C}: \mathcal{H}_a \rightarrow \mathcal{H}_a$  is Hilbert-Schmidt. In fact, the exponential decay of the matrix elements of  $\mathcal{C}$  established in Lemma 2.4 implies that  $\mathcal{C}$  has even stronger compactness properties. It can be shown, for example by the argument used in the proof of [FauR, Theorem 7], that the singular values of  $\mathcal{C}$  decay at a stretched-exponential rate, so  $\mathcal{C}$  belongs to the exponential classes introduced in [Ban], in common with the transfer operators corresponding to higher-dimensional analytic expanding maps (see, for example, [BanJ1, BanJ2]).

### 3. SPECTRAL DATA

In order to complete the proof of Theorem 1.1, it remains to compute the spectrum of the operator  $\mathcal{C}$ . This can be achieved by considering suitable matrix representations of projections of this compact operator to finite-dimensional subspaces.

We start by observing that, by (21) and (22), the matrix representation  $\Gamma$  of  $\mathcal{C}$  with respect to the orthonormal basis  $(e_n)_{n \in \mathbb{Z}^2}$  of  $\mathcal{H}_a$  is of the form

$$\Gamma_{m,n} = \langle \mathcal{C}e_n e_m \rangle_a = b_{m,n} \exp(a|n_u| - a|n_s| - a|m_u| + a|m_s|). \quad (45)$$

A short calculation using (23), (24) and Lemma 2.3 i), iii), iv) yields the following cases:

$$m_2 \neq n_1 + n_2 : \quad b_{m,n} = 0 \quad (46)$$

$$m_2 = n_1 + n_2 = 0 : \quad b_{m,n} = \delta_{m_1, n_1} \quad (47)$$

$$m_2 = n_1 + n_2 > 0 : \quad b_{m,n} = \begin{cases} 0 & \text{if } m_1 < n_1 \\ (-\lambda)^{m_2} & \text{if } m_1 = n_1 \end{cases} \quad (48)$$

$$m_2 = n_1 + n_2 < 0 : \quad b_{m,n} = \begin{cases} 0 & \text{if } m_1 > n_1 \\ (-\bar{\lambda})^{-m_2} & \text{if } m_1 = n_1. \end{cases} \quad (49)$$

These properties will turn out to be sufficient to show that we can order the basis elements in such a way that the corresponding matrix is lower-triangular. We first arrange the basis  $(e_n)_{n \in \mathbb{Z}^2}$  as a sequence in the order of increasing norm  $|n|$ , with groups of elements with the same norm traversed in counter-clockwise direction, that is,

$$e_{0,0}, e_{1,0}, e_{0,1}, e_{-1,0}, e_{0,-1}, e_{2,0}, e_{1,1}, e_{0,2}, e_{-1,1}, e_{-2,0}, e_{-1,-1}, e_{0,-2}, e_{1,-1}, \dots$$

We then re-order this sequence as follows. We move along the sequence above from left to right. If we encounter a basis element  $e_{n_1, n_2}$  with  $n_1 n_2 < 0$  we move the element to the left-most position of the current sequence. We thus obtain the following order:

$$\dots, e_{1,-1}, e_{-1,1}, e_{0,0}, e_{1,0}, e_{0,1}, e_{-1,0}, e_{0,-1}, e_{2,0}, e_{1,1}, e_{0,2}, e_{-2,0}, e_{-1,-1}, e_{0,-2}, \dots$$

**Lemma 3.1.** *The matrix given by (45) is lower-triangular with respect to the basis re-ordered as above. Moreover, its only non-zero diagonal entries are  $\Gamma_{00,00} = 1$ ,  $\Gamma_{0k,0k} = (-\lambda)^k$  and  $\Gamma_{0-k,0-k} = (-\bar{\lambda})^k$  where  $k \in \mathbb{N}$ .*

*Proof.* We first prove that the entire upper-right block with  $n_1 n_2 \geq 0$  and  $m_1 m_2 < 0$  consists of zeros. Assume the contrary, that is, assume that there exists some non-vanishing matrix element  $\Gamma_{m,n}$  in this sector, that is,  $b_{m,n} \neq 0$ . From (46) we get  $m_2 = n_1 + n_2$ , which is non-zero as  $m_1 m_2 \neq 0$ . If  $n_1 + n_2 = m_2 > 0 > m_1$ , then  $n_1, n_2$  are non-negative, so  $n_1 > m_1$  and (48) results in the contradiction  $b_{m,n} = 0$ . A similar reasoning applies in the case  $n_1 + n_2 = m_2 < 0 < m_1$ .

Next, we confirm that the upper-left block matrix with  $n_1 n_2 < 0$  and  $m_1 m_2 < 0$  is an lower-triangular matrix with zeros on the diagonal. For this we assume that a matrix entry lying on or above the diagonal is non-zero. Note that, with the chosen ordering, the indices of a matrix element on or above the diagonal satisfy

$$|m_1| + |m_2| = |m| \geq |n| = |n_1| + |n_2|. \quad (50)$$

Since  $m_1 m_2 < 0$  we have  $m_2 \neq 0$ . If  $m_2 > 0$  then (48) implies  $m_1 \geq n_1$  and hence the condition (50) results in  $n_1 \leq m_1 < 0 < n_2 \leq m_2$ . In particular  $m_2 > n_1 + n_2$  so that (46) yields the contradiction  $b_{m,n} = 0$ . The case for  $m_2 < 0$  is analogous.

Finally, we show the claim for the most interesting case, the lower-right block matrix where  $n_1 n_2 \geq 0$  and  $m_1 m_2 \geq 0$ . In this case, the indices of a matrix element on or above the diagonal satisfy  $|n| \geq |m|$ . Since the components of  $n$  and  $m$  have equal signs, this condition can be written as  $|n_1 + n_2| \geq |m_1 + m_2|$ . If  $b_{m,n} \neq 0$ , then  $m_2 = n_1 + n_2$  by (46), so that  $|m_2| \geq |m_1 + m_2|$ . Since  $m_1 m_2 \geq 0$  we conclude  $m_1 = 0$ . Then one of the following three cases holds:

i)  $m_2 = 0$ : We get  $0 = m_2 = n_1 + n_2$  and  $n_1 n_2 \geq 0$  results in  $n_1 = n_2 = m_1 = m_2 = 0$  for which  $\Gamma_{00,00} = 1$  by (47).

ii)  $m_2 > 0$ : By (48) we have  $0 = m_1 \geq n_1$ . Since  $m_2 = n_1 + n_2 > 0$  and  $n_1$  and  $n_2$  have the same sign this implies  $m_1 = n_1 = 0$  and  $m_2 = n_2$ . The corresponding diagonal entry is given by  $\Gamma_{0m_2,0m_2} = (-\lambda)^{m_2}$  by (48).

iii)  $m_2 < 0$ : By (49) we have  $0 = m_1 \leq n_1$  and by the same argument as in the previous case we get  $n_1 = m_1 = 0$  and  $n_2 = m_2$ . The corresponding diagonal entry is given by  $\Gamma_{0m_2,0m_2} = (-\bar{\lambda})^{-m_2}$  by (49).  $\square$

**Lemma 3.2.** *The subspace of  $\mathcal{H}_a$  spanned by  $\{e_n : n_1 \cdot n_2 \geq 0\}$  decomposes as  $V^+ \oplus V_0^-$  into two invariant subspaces  $V^+$  and  $V_0^-$  spanned by  $\{e_n : n_1, n_2 \geq 0\}$  and  $\{e_n : n_1, n_2 \leq 0, n_1 + n_2 < 0\}$ , respectively, so that  $\mathcal{C}(V^+) \subseteq V^+$  and  $\mathcal{C}(V_0^-) \subseteq V_0^-$ .*

*Proof.* To show invariance of  $V^+$ , we inspect  $\mathcal{C}e_n = \sum_{m \in \mathbb{Z}^2} \Gamma_{m,n} e_m$  for  $e_n \in V^+$ , so that  $n_1, n_2 \geq 0$ . From (46) and (48) it follows that  $\Gamma_{m,n} = 0$ , unless  $m_2 = n_1 + n_2 \geq 0$  and  $m_1 \geq n_1 \geq 0$ . This implies that  $\mathcal{C}e_n \in V^+$  and hence  $\mathcal{C}(V^+) \subseteq V^+$ . Similarly we get  $\mathcal{C}(V_0^-) \subseteq V_0^-$ , completing the proof of the lemma.  $\square$

We are now able to finish the proof of the main result.

*Proof of Theorem 1.1.* Compactness of  $\mathcal{C} : \mathcal{H}_a \rightarrow \mathcal{H}_a$  was established in Proposition 2.5. Let  $N \in \mathbb{N}$  and let  $P_N : \mathcal{H}_a \rightarrow \mathcal{H}_a$  denote the orthogonal projection onto the subspace spanned by  $\{e_n : |n| \leq N\}$ . By Lemma 3.1, the spectrum of the finite rank operator  $P_N \mathcal{C} P_N$  is given by

$$\sigma(P_N \mathcal{C} P_N) = \{(-\lambda)^k : k \in \{1, \dots, N\}\} \cup \{(-\bar{\lambda})^k : k \in \{1, \dots, N\}\} \cup \{1, 0\}. \quad (51)$$

Moreover, each non-zero element of the spectrum of  $P_N \mathcal{C} P_N$  is an eigenvalue the algebraic multiplicity of which coincides with the number of times the non-zero number occurs in (51).

Since  $(-\lambda)^k \in \sigma(P_N(\mathcal{C}|_{V^+})P_N)$  and  $(-\bar{\lambda})^k \in \sigma(P_N(\mathcal{C}|_{V_0^-})P_N)$  for the invariant subspaces  $V^+, V_0^- \subset \mathcal{H}_a$  from Lemma 3.2, it follows that the geometric and algebraic multiplicities of these eigenvalues coincide, meaning they are 2 when  $(-\lambda)^k = (-\bar{\lambda})^k$ , and 1 otherwise.

Now, in order to finish the proof we only need to show that the non-zero spectrum (with algebraic and geometric multiplicities) of the transfer operator  $\mathcal{C}$  is captured by the non-zero spectra of the finite rank operators  $P_N \mathcal{C} P_N$ . This follows from a standard spectral approximation result (see, for example, [DS, XI.9.5]) together with the fact that  $P_N \mathcal{C} P_N$  converges to  $\mathcal{C}$  in the operator norm on  $\mathcal{H}_a$ , which in turn follows from the fact that  $\mathcal{C}$  is compact (see, for example, [ALL, Theorem 4.1]).  $\square$

#### 4. INVARIANT MEASURE AND CORRELATION DECAY

Since the map (2) is area preserving it is clear that the Haar measure  $\mu$  on the torus is invariant under  $T$ . This invariance can also be cast in terms of spectral properties of the transfer operator. In order to see this, we note that the constant function  $e_0$  is the eigenfunction of the transfer operator corresponding to the eigenvalue 1, since  $\mathcal{C}e_0 = e_0 \circ T = e_0$ . Furthermore, for  $f$  a Laurent polynomial we define the functional

$$\ell_0(f) = \int_{\mathbb{T}^2} f d\mu = \int_{\mathbb{T}^2} f(z) \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} = f_0. \quad (52)$$

Using the definition of the norm (15) we have  $|\ell_0(f)| = |f_0| \leq \|f\|_a$ . Thus the functional  $\ell_0$  is bounded on the dense subset of Laurent polynomials and thus

extends uniquely to a functional  $\ell_0 : \mathcal{H}_a \rightarrow \mathbb{C}$  on the entire space  $\mathcal{H}_a$ , which for simplicity we denote by  $\ell_0$  again.

Using the fact that the map  $T$  preserves Haar measure  $\mu$  on  $\mathbb{T}^2$ , we have for any Laurent polynomial  $f$  the relation  $\ell_0(\mathcal{C}f) = \ell_0(f \circ T) = \ell_0(f)$  and by continuity this identity carries over to the entire space as well. Hence  $\ell_0$  is the left-eigenfunctional of the transfer operator corresponding to the leading eigenvalue 1.

All in all, we can now define a bounded projection  $\mathcal{P}_0 : \mathcal{H}_a \rightarrow \mathcal{H}_a$  by setting

$$\mathcal{P}_0 f = \ell_0(f) e_0, \quad (53)$$

which, by what has been said above, satisfies

$$\mathcal{C}\mathcal{P}_0 = \mathcal{P}_0\mathcal{C} = \mathcal{P}_0, \quad (54)$$

which means that  $\mathcal{P}_0$  is the spectral projection corresponding to the eigenvalue 1.

We now turn to the study of correlation functions with respect to  $\mu$ . Let  $g : \mathbb{T}^2 \rightarrow \mathbb{C}$  be analytic in a neighbourhood of the unit torus so that  $|g_n| \leq c \exp(-\gamma|n|)$  for some  $\gamma > 0$ ,  $c > 0$ . Choose  $a$  sufficiently small so that  $\exp(-\gamma|n|) \exp(a|n_u| - a|n_s|)$  decays exponentially in  $|n|$ . Define the functional

$$\ell_g(f) = \int_{\mathbb{T}^2} fg \, d\mu = \int_{\mathbb{T}^2} g(z) f(z) \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} = \sum_{n \in \mathbb{Z}^2} g_{-n} f_n \quad (55)$$

on the dense subset of Laurent polynomials. Using the Cauchy-Schwarz inequality and the definition of the norm (15) we conclude that

$$|\ell_g(f)|^2 \leq \|f\|_a^2 \sum_{n \in \mathbb{Z}^2} |g_{-n}|^2 \exp(2a|n_u| - 2a|n_s|). \quad (56)$$

Hence  $\ell_g$  extends to a bounded functional on the entire space  $\mathcal{H}_a$ .

Thus, for any observable  $h \in \mathcal{H}_a$ , the correlation function (6) can, using (52), (53) and (55), be cast into spectral form as follows:

$$C_{gh}(k) = \ell_g(\mathcal{C}^k h) - \ell_g(e_0) \ell_0(h) = \ell_g(\mathcal{C}^k h - \mathcal{P}_0 h). \quad (57)$$

Since, by Theorem 1.1, the spectrum of  $\mathcal{C}$  is discrete and has no non-trivial Jordan blocks, we have a partial spectral decomposition (see, for example, [TL, Chapter V, Theorem 9.2]) of the form

$$\mathcal{C}^k h = \mathcal{P}_0 h + \lambda^k \mathcal{P}_1 + \bar{\lambda}^k \mathcal{P}_2 + \mathcal{Q}^k h. \quad (58)$$

Here  $\mathcal{P}_1$  and  $\mathcal{P}_2$  denote the one-dimensional spectral projections associated with the eigenvalues  $\lambda$  and  $\bar{\lambda}$ , respectively, while  $\mathcal{Q} : \mathcal{H}_a \rightarrow \mathcal{H}_a$  is a compact operator with

$$\sigma(\mathcal{Q}) = \sigma(\mathcal{C}) \setminus \{1, \lambda, \bar{\lambda}\}, \quad (59)$$

which implies that the spectral radius of  $\mathcal{Q}$  is equal to  $|\lambda|^2$ , that is,

$$\lim_{k \rightarrow \infty} \|\mathcal{Q}^k\|^{1/k} = |\lambda|^2. \quad (60)$$

Combining (57), (58) and (60) we obtain the desired bound

$$|C_{gh}(k)| \leq (|\ell_g(\mathcal{P}_1 h)| + |\ell_g(\mathcal{P}_2 h)|) |\lambda|^k + |\ell_g(\mathcal{Q}^k h)| \leq K |\lambda|^k, \quad (61)$$

where the constant  $K$  only depends on  $\ell_g$  and  $h$ . This furnishes the proof of Corollary 1.2.

It is quite easy to see that including lower-lying eigenvalues into the spectral decomposition (58) we can obtain asymptotic expansions for the correlation function.

For the proof of Corollary 1.3 we first note that (54) implies that for every natural number  $k$  we have

$$\mathcal{C}^k - \mathcal{P}_0 = \mathcal{R}^k, \quad (62)$$

where  $\mathcal{R} = \lambda\mathcal{P}_1 + \bar{\lambda}\mathcal{P}_2 + \mathcal{Q}$  is a compact operator with

$$\sigma(\mathcal{R}) = \sigma(\mathcal{C}) \setminus \{1\}. \quad (63)$$

Using a Neumann series for the resolvent of  $\mathcal{R}$  together with (62) and (57) we now obtain for all  $\zeta \in \mathbb{C}$  with  $|\zeta| > 1$

$$\hat{C}_{gh}(\zeta) = \sum_{k=0}^{\infty} \zeta^{-k} C_{gh}(k) \quad (64)$$

$$= \sum_{k=0}^{\infty} \zeta^{-k} \ell_g(\mathcal{C}^k h - \mathcal{P}h) \quad (65)$$

$$= \ell_g\left(\sum_{k=0}^{\infty} \zeta^{-k} \mathcal{R}^k h\right) \quad (66)$$

$$= \zeta \ell_g((\zeta I - \mathcal{R})^{-1} h). \quad (67)$$

The corollary now follows from (63) together with the observation that the resolvent of the compact operator  $\mathcal{R}$  is analytic on the punctured plane  $\mathbb{C} \setminus \{0\}$  except for poles at the non-zero eigenvalues of  $\mathcal{R}$  (see, for example, [TL, Chapter V, Corollary 10.3]).

## 5. CONCLUSION

Having access to explicitly solvable examples helps to understand dynamical features and to test conjectures. Our example demonstrates that in the analytic category hyperbolic diffeomorphisms exist for which the corresponding transfer operator has infinitely many distinct eigenvalues. In addition, eigenvalues can be arbitrarily close to one in modulus.

The Hamiltonian structure, that is, the fact that we have considered an area preserving diffeomorphism has simplified our arguments at a technical level. In addition, the model considered here does not show the generic decay of eigenvalues expected for two dimensional maps (see [Nau]). It is, however, rather straightforward to analyse more general models along the lines presented here to restore the generic behaviour and to investigate cases with a non-trivial invariant measure.

Our setup has been tailored for the model under consideration. We have chosen a special Hilbert space with equal weightings and components according to the eigendirections of the cat map, see (11). While these choices turned out to be successful their precise meaning remained somehow obscure. In addition, we were able to transform the matrix representation of the transfer operator to a triangular structure which gave us access to the entire spectrum. All these features are not entirely coincidental, in the sense that there is an underlying functional analytic structure. Uncovering this structure requires a more general approach based on more subtle functional analytic techniques. The focus of the present contribution has been on an elementary rigorous study of a particular example which should be accessible for a larger, non-specialised audience. The general theory for analytic diffeomorphism of the torus alluded to above will be presented elsewhere.

## APPENDIX A. SOME NUMERICAL FINDINGS

A visual impression of the hyperbolic structure can be obtained by the numerical computation of the unstable and stable manifold of the fixed point. Straightforward forward and backward iteration gives a fairly robust algorithm for the computation of a finite part of these manifolds, see Figure 1. Even though the invariant density is uniform the geometry of the hyperbolic structure is apparently non-uniform, but this non-uniformity is compensated for by a respective variation of the local expansion and contraction rates.

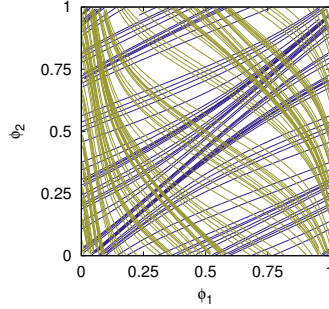


FIGURE 1. Numerical result for the unstable (blue) and stable (bronze) manifold of the map given by (2), for  $\lambda = 0.7 \exp(0.3i)$ .

For simple trigonometric observables, the correlation function can be computed directly. Consider, for instance, the case of the autocorrelation of  $\cos(\phi_2)$ , which corresponds to choosing  $g(z_1, z_2) = h(z_1, z_2) = (z_2 + z_2^{-1})/2$  in (6). Since the invariant density is constant the mean values obviously vanish. In order to compute the correlation integral we introduce the shorthand

$$w_{n_1 n_2}(z_1, z_2) = z_1^{n_1} z_2^{n_2} \quad (68)$$

for denoting monomials. By definition of the transfer operator, it follows that for a function  $f$  which is analytic in  $D_0 = \{|z_1| < 1, |z_2| < 1\}$  the expression  $\mathcal{C}f = f \circ T$  is also analytic in  $D_0$ . An analogous property holds for functions which are analytic in  $D_\infty = \{|z_1| > 1, |z_2| > 1\}$ . Hence, the correlation integral can be written as

$$C(k) = \frac{1}{4} \int_{\mathbb{T}^2} (z_2^{-1} (\mathcal{C}^k w_{01})(z_1, z_2) + z_2 (\mathcal{C}^k w_{0-1})(z_1, z_2)) \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2}. \quad (69)$$

As for the action of the transfer operator on monomials, we see that

$$(\mathcal{C}w_{0n_2})(z_1, z_2) = (-\lambda)^{n_2} z_2^{n_2} + \mathcal{O}(z_2^{n_2} z_1), \quad (n_2 \geq 0) \quad (70)$$

$$(\mathcal{C}w_{0n_2})(z_1, z_2) = (-\bar{\lambda})^{-n_2} z_2^{n_2} + \mathcal{O}(z_2^{n_2} z_1^{-1}), \quad (n_2 \leq 0) \quad (71)$$

$$(\mathcal{C}w_{n_1 n_2})(z_1, z_2) = \mathcal{O}(z_2^{n_1+n_2} z_1^{n_1}), \quad (n_1, n_2 \geq 0 \text{ or } n_1, n_2 \leq 0) \quad (72)$$

where the higher order terms, as mentioned above, are analytic either in  $D_0$  or  $D_\infty$ . Hence, only the leading term in (70) and (71) contributes to the integrals in (69) and we arrive at

$$C(k) = \frac{(-\lambda)^k + (-\bar{\lambda})^k}{4}. \quad (73)$$

As a by-product we obtain that, as expected, the upper bound given by Corollary 1.2 is sharp.

## APPENDIX B. A LOWER BOUND

This short appendix is devoted to proving a bound required in the proof of Lemma 2.4.

**Lemma B.1.** *Let  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by*

$$F(x, y) = 2(|\varphi x + 1| - |\varphi^{-1}x - 1| + |\varphi y - 1| - |\varphi^{-1}y + 1|). \quad (74)$$

Then for  $x \geq y$ , we have

$$F(x, y) \geq \begin{cases} x - y & \text{if } |y| < 2\varphi^{-1}, \\ x - y + |y|/2 & \text{if } |y| \geq 2\varphi^{-1}, \end{cases} \quad (75)$$

where, as before,  $\varphi = (1 + \sqrt{5})/2$  denotes the golden mean.

*Proof.* We can write

$$F(x, y) = G(x) - H(y), \quad (76)$$

where

$$G(x) = \begin{cases} -2x - 4 & \text{if } x \in (-\infty, -\varphi^{-1}), \\ 2\sqrt{5}x & \text{if } x \in [-\varphi^{-1}, \varphi], \\ 2x + 4 & \text{if } x \in (\varphi, +\infty), \end{cases} \quad (77)$$

$$H(y) = \begin{cases} 2y - 4 & \text{if } y \in (-\infty, -\varphi), \\ 2\sqrt{5}y & \text{if } y \in [-\varphi, \varphi^{-1}], \\ -2y + 4 & \text{if } y \in (\varphi^{-1}, +\infty). \end{cases} \quad (78)$$

Let  $x_m = -\varphi^{-1}$  denote the minimum of  $G$  and let  $y_m = -x_m$  denote the maximum of  $H$ . We start by showing that for  $x \geq y$  we have

$$F(x, y) \geq 2(x - y). \quad (79)$$

In order to see this, we first observe that  $G(t) - H(t) \geq 0$  for all  $t \in \mathbb{R}$  and that the minimal slope in modulus of  $G$  and  $H$  is 2. Moreover, we have  $G(x) \geq 2x$  for  $x \geq y_m$  and  $H(y) \leq 2y$  for  $y \leq x_m$ .

Now, for  $x \geq x_m$  we have  $G'(x) \geq 2$ , so that

$$F(x, y) = (G(x) - G(y)) + (G(y) - H(y)) \geq 2(x - y), \quad \text{for } x_m \leq y \leq x, \quad (80)$$

while for  $y \leq x \leq y_m$  we have  $H'(y) \geq 2$  so

$$F(x, y) = (G(x) - H(x)) + (H(x) - H(y)) \geq 2(x - y). \quad (81)$$

Finally, we note that if  $x \geq y_m$  and  $y \leq x_m$  then

$$F(x, y) = G(x) - H(y) \geq 2x - 2y = 2(x - y), \quad (82)$$

which proves (79) and the first part of the lemma.

In order to prove the second part, observe that for  $x, x_m \geq y$  we have

$$G(x) - H(y) \geq \frac{1}{2}(G(x) - H(y)) + \frac{1}{2}(G(x_m) - H(y)) \geq (x - y) + (x_m - y). \quad (83)$$

Thus, for  $2x_m \geq y$  we have  $x_m - y \geq -y/2 = |y|/2$ , and hence

$$F(x, y) = G(x) - H(y) \geq x - y + \frac{|y|}{2}. \quad (84)$$



Similarly, for  $x \geq y \geq 2y_m$  we have  $G(x) - H(y) \geq (x - y) + (y - y_m)$  and  $y - y_m > |y|/2$ , hence again  $F(x, y) \geq x - y + |y|/2$ , finishing the proof of the lemma.  $\square$

### APPENDIX C. A REMARK ON GENERAL BLASCHKE PRODUCTS

For our specific example defined in (1), the asymptotic decay of eigenvalues does not follow the generic pattern expected for two-dimensional maps (see [Nau]). It is nevertheless quite easy to come up with solvable models exhibiting this generic behaviour. If we recall that the cat map can be written as a composition of area preserving orientation reversing linear automorphisms, and if we deform this automorphism by introducing a Blaschke factor, that is, if we define

$$S_\lambda(z_1, z_2) = \left( \frac{z_1 - \lambda}{1 - \bar{\lambda}z_1} z_2, z_1 \right), \quad (85)$$

which is an area preserving diffeomorphism of the torus, then the composition

$$T = S_\lambda \circ S_\mu \quad (86)$$

yields a two-parameter area preserving family. With the tools introduced previously it is possible, but extremely tedious, to show that the corresponding transfer operator is compact on a suitably weighted Hilbert space. Even the spectrum, can be evaluated in closed form consisting of simple eigenvalues  $1, (-\lambda)^n, (-\bar{\lambda})^n, (-\mu)^n, (-\bar{\mu})^n, (-\lambda)^n(-\mu)^m, (-\lambda)^n(-\bar{\mu})^m, (-\bar{\lambda})^n(-\mu)^m,$  and  $(-\bar{\lambda})^n(-\bar{\mu})^m$  where  $n \geq 1$  and  $m \geq 1$ . A more conceptual proof of these assertions is possible, but requires fairly heavy machinery, to be presented elsewhere. Here we will simply illustrate this result by numerical means. For that purpose we compute a truncated matrix representation of the transfer operator by using the standard Fourier basis (see (21)), and apply a standard eigenvalue solver. The result is presented in Figure 2.

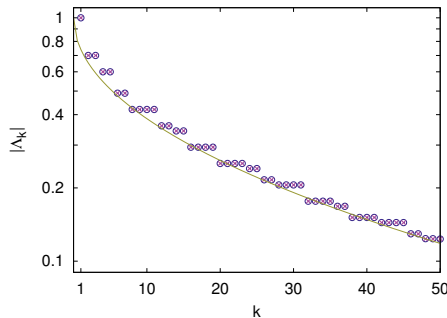


FIGURE 2. Eigenvalues  $\Lambda_k$  of the transfer operator corresponding to the map (86) for  $\lambda = 0.7$  and  $\mu = 0.6$ , ordered by size. The numerical diagonalisation is illustrated by circles and the exact analytic expression by crosses. The line indicates the generic asymptotic decay given by  $|\Lambda_k| \sim \exp(-c\sqrt{k})$  with  $c = (\ln \lambda \ln \mu/2)^{1/2}$ .

For simplicity, we have so far considered area preserving maps where the explicit expression for the invariant measure is known a priori. The invariant measures of two-dimensional Blaschke products exhibit a richer structure (see [PS]). The

tools introduced here allow for a detailed study of those measures. In a nutshell, maps where the determinant of the Jacobian is not constant, may have invariant measures exhibiting fractal properties. Toy examples based on piecewise linear maps are well established in the literature (see, for example, [Neu]). Blaschke products offer a systematic and analytic approach towards such features. For the purpose of illustration consider the simple model

$$T(z_1, z_2) = \left( z_1^2 \frac{z_2 - \mu}{1 - \bar{\mu}z_2}, z_1 \frac{z_2 - \mu}{1 - \bar{\mu}z_2} \right). \quad (87)$$

Our approach allows for a detailed investigation of the spectral structures, but details turn out to be quite cumbersome. Hence, to visualise the properties of the invariant measure we just compute a histogram by a suitable numerical simulation, see Figure 3.

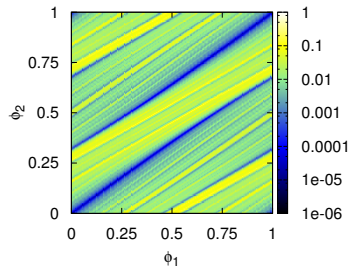


FIGURE 3. Density plot illustrating the invariant measure of the map given by (87) in real coordinates  $z_\ell = \exp(2\pi i\phi_\ell)$  for  $\mu = 0.4$ . The data show a histogram with resolution  $1/5000 \times 1/5000$  obtained from  $10^4$  time traces of length  $2 \times 10^7$  with uniformly distributed initial conditions.

## REFERENCES

- [Ada] A. Adam, *Generic non-trivial resonances for Anosov diffeomorphisms*, Nonlinearity **30** (2017) 1146–1164.
- [ALL] M. Ahues, A. Largillier and B. V. Limaye, *Spectral Computations for Bounded Operators* (Chapman & Hall/CRC, Boca Baton, 2001).
- [AQ] I. Antoniou and Bi Qiao, *Spectral decomposition of piecewise linear monotonic maps*, Chaos Solitons Fractals **7** (1996) 1895–1911.
- [AT] I. Antoniou and S. Tsasaki, *Generalized spectral decomposition of the  $\beta$ -adic baker’s transformation and intrinsic irreversibility*, Physica A **190**, (1992) 303–329.
- [BaC] O.F. Bandtlow and P.V. Coveney, *On the discrete time version of the Brussels formalism*, J. Phys. A **27** (1994) 7939–7955.
- [Ban] O.F. Bandtlow, *Resolvent estimates for operators belonging to exponential classes*, Integr. Equ. Oper. Theory **61** (2008) 21–43.
- [BanJ1] O.F. Bandtlow and O. Jenkinson, *Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions*, Adv. Math. **218** (2008) 902–925.
- [BanJ2] O.F. Bandtlow and O. Jenkinson, *On the Ruelle eigenvalue sequence*, Ergod. Th. & Dynam. Sys. **28** (2008) 1701–1711.
- [BanJS] O.F. Bandtlow, W. Just and J. Slipantschuk, *Spectral structure of transfer operators for expanding circle maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017) 31–43.

- [BanN] O.F. Bandtlow and F. Naud, *Lower bounds for the Ruelle spectrum of analytic expanding circle maps*, to appear in *Ergod. Th. & Dynam. Sys.*
- [Bal1] V. Baladi, *Positive Transfer Operators and Decay of Correlation* (World Scientific, Singapore, 2000).
- [Bal2] V. Baladi, *Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps* (Book manuscript available online, 2016).
- [Bal3] V. Baladi, *The quest for the ultimate anisotropic Banach space*. *J. Stat. Phys.* **166** (2017) 525–557.
- [BalG] V. Baladi and S. Gouëzel, *Banach spaces for piecewise cone-hyperbolic maps*, *J. Mod. Dyn.* **4** (2010) 91–137.
- [BalT1] V. Baladi and M. Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, *Ann. Inst. Fourier, Grenoble* **57** (2007) 127–154.
- [BalT2] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*; in K. Burns, D. Dolgopyat and Ya. Pesin (eds.), *Geometric and Probabilistic Structures in Dynamics* (Contemp. Math. **469**, American Mathematical Society, Providence, 2008), 29–68.
- [BKL] M. Blank, G. Keller and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, *Nonlinearity* **15** (2002) 1905–1973.
- [Dae] D. Daems, *Non-diagonalizability of the Frobenius-Perron operator and transition between decay modes of the time autocorrelation function*, *Chaos Solitons Fractals* **7** (1996) 1753–1760.
- [DaR1] N.V. Dang and G. Riviere, *Spectral analysis of Morse-Smale gradient flows*, preprint, [arXiv:1605.05516](https://arxiv.org/abs/1605.05516).
- [DaR2] N.V. Dang and G. Riviere, *Spectral analysis of Morse-Smale flows i: construction of the anisotropic spaces*, preprint, [arXiv:1703.08040](https://arxiv.org/abs/1703.08040).
- [DaR3] N.V. Dang and G. Riviere, *Spectral analysis of Morse-Smale flows ii: resonances and resonant states*, preprint, [arXiv:1703.08038](https://arxiv.org/abs/1703.08038).
- [Dor] J. Dorfman, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics* (Camb. Univ. Press, Cambridge, 1998).
- [Dri] D. Driebe, *Jordan blocks in a one-dimensional Markov map*, *Comput. Math. Appl.* **34** (1997) 103–109.
- [DS] N. Dunford and J.T. Schwartz, *Linear Operators Part 2: Spectral Theory* (Wiley-Interscience, New York, 1963).
- [FauR] F. Faure and N. Roy, *Ruelle-Pollicott resonances for real analytic hyperbolic maps*, *Nonlinearity* **19** (2006) 1233–1252.
- [FauRS] F. Faure, N. Roy and J. Sjöstrand, *Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances*, *Open Math. J.* **1** (2008) 35–81.
- [FauT] F. Faure and M. Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows*, *C. R. Math. Acad. Sci. Paris* **351** (2013) 385–391.
- [FF] L. Flaminio and G. Forni, *Invariant distributions and time averages for horocycle flows*, *Duke Math. J.* **119** (2003) 465–526.
- [Gal] G. Gallavotti, *Chaotic dynamics, fluctuations, nonequilibrium ensembles*, *Chaos* **8** (1998) 384–392.
- [GouL1] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, *Ergod. Th. & Dynam. Sys.* **26** (2006) 189–217.
- [GouL2] S. Gouëzel and C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, *J. Differential Geom.* **79** (2008) 433–477.
- [GuiHW] C. Guillarmou, J. Hilgert and T. Weich, *Classical and quantum resonances for hyperbolic surfaces*, preprint, [arXiv:1605.08801](https://arxiv.org/abs/1605.08801)
- [Has] B. Hasselblatt, *Hyperbolic dynamical systems*; in A. Katok and B. Hasselblatt (eds.), *Handbook of Dynamical Systems, Vol. 1A* (North-Holland, Amsterdam, 2002), 239–319.
- [KatH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems* (Cambridge University Press, Cambridge, 1996).
- [Kel] G. Keller, *Equilibrium States in Ergodic Theory* (Cambridge University Press, Cambridge, 1998).
- [Mar] N. Martin, *On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms*, *Bull. London Math. Soc.* **15** (1983) 343–348.

- [MorSO] H. Mori, B. S. So, and T. Ose, *Time-correlation functions of one-dimensional transformations*, Prog. Theor. Phys. **66** (1981) 1266–1283.
- [Mos] J. Moser, *On a theorem of Anosov*, J. Differential Equations **5** (1969) 411–440.
- [Nau] F. Naud, *The Ruelle spectrum of generic transfer operators*. Discrete Contin. Dyn. Syst. **32** (2012), 2521–2531.
- [Neu] J. Neunhäuserer, *Dimension theoretical properties of generalized Baker’s transformations*, Nonlinearity **15** (2002) 1299–1307.
- [Pol1] M. Pollicott, *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985) 413–426.
- [Pol2] M. Pollicott, *Meromorphic extensions of generalized zeta functions*, Invent. Math. **85** (1986) 147–164.
- [PS] E. Pujals and M. Shub, *Dynamics of two-dimensional Blaschke products*, Ergod. Th. & Dyn. Sys. **20** (2008) 575–585.
- [RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Volume I: Functional Analysis* (Academic Press, San Diego, 1980).
- [Rue1] D. Ruelle, *Resonances of chaotic dynamical systems*, Phys. Rev. Lett. **56** (1986) 405–407.
- [Rue2] D. Ruelle, *Locating resonances for Axiom A dynamical systems*, J. Stat. Phys. **44** (1986) 281–292.
- [Rug] H.-H. Rugh, *The correlation spectrum for hyperbolic analytic maps*, Nonlinearity **5** (1992) 1237–1263.
- [SBJ1] J. Slipantschuk, O.F. Bandtlow, and W. Just, *On the relation between Lyapunov exponents and exponential decay of correlations*, J. Phys. A: Math. Theor. **46** (2013) 075101 (16pp).
- [SBJ2] J. Slipantschuk, O.F. Bandtlow, and W. Just, *Analytic expanding circle maps with explicit spectra*, Nonlinearity **26** (2013) 3231–3245.
- [TL] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis* (John Wiley & Sons, New York, 1980).

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