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OPTIMISATION OF CONDITION NUMBER FOR EIGENSTRUCTURE PROBLEMS

By

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To my wife and daughter

Abstract

This thesis deals with a new solution for the problem of eigenstructure assignment in control systems design. A wide range of challenging issues is examined involving the problem of eigenstructure assignment and associated system properties through different forms of complexity which are strongly related to control system design. In this thesis, specific attention has been given to the issue of skewness of the closed-loop eigenframe of the state matrix. In fact, the aim is to develop a new methodology for determining the best angle between closed-loop eigenvectors by optimising the minimal condition number of the closed-loop eigenvector matrix. This problem is strongly linked to sensitivity of eigenvalues to parameter uncertainty, perturbations to model parameter uncertainty. The importance of this methodology can be expressed in terms of results related to the Sensitivity of eigenvalues, Relative measures of controllability and observability and also deviations from strong stability to overshooting behaviour. Among this variety of eigenstructure assignment methods, special consideration has been paid to Geometric Theory [4], [5], which introduces an alternative solution to the assignability of spectrum of controllability subspaces (cs) based on an eigenvector approach and then develops a new pole assignment algorithm based on open-loop/closed-loop spectra as a practical application of this approach. In order to tackle the problem of measuring the skewness of angles between closed-loop eigenvalues, some measures for Eigenframe skewness have been defined in general and so the necessary and efficient conditions have been derived for the angle between some subspaces in a direct sum decomposition to be maximized. This has been done via three metrics; Condition Number, Determinant of Gram Matrix and Singular Values. The thesis presents the parametrisation of closed-loop eigenframes result by the method generated in [4]. Within this thesis, a non-smooth algorithm has been developed in order to select the most orthogonal closed-loop eigenframe and so the influence of selected closed-loop spectra. Also, the parametrisation of controllability subspaces in a standard direct sum decomposition using *matrix fraction description* (*MFD*) has been derived. Within this thesis the construction and

the existence of controllability subspaces connected to (A,B)-invariant subspaces, has also been studied. In addition, an algebraic description of the total system behaviour which leads to an algebraic characterisation of the total input, state and output behaviour in an implicit formulation is given based on properties of MFD descriptions, a topic which remains open for future studies.

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London, United Kingdom   Ali A. Sayyad

September 2016

CHAPTER 1

INTRODUCTION

1.1. Introduction

The purpose of this thesis is to study the problem of eigenstructure assignment in control systems design. This is achieved via a new methodology which relies on the minimisation of the skewness of the eigenframe of the state matrix. This reduces the sensitivity of the system's eigenvalues to model perturbations and has as a result improvement of robustness of the system. The problem of eigenstructure assignment has a long history going back to the works done in 1975 [40], [98], [99].

Despite considerable work in this area, a number of fundamental issues still remain open and a number of key problems have not been properly addressed. In this work, the main emphasis has been on defining frames which are as orthogonal as possible. This guarantees minimal eigenvalue sensitivity due to a classical result by Wilkinson [1], applicable to the case of real distinct eigenvalues. The result in [6] has been the main motivation behind the minimisation of sensitivity of eigenvalues to parametric uncertainty.

The property of minimal eigenvalue sensitivity is important both for Numerical Analysis and for Control. In Numerical Analysis, uncertainty arises due to finite precision effects and the accumulation of numerical errors during the execution of various steps of an algorithm. In Control, uncertainty normally arises due to model errors and is in the form of model-structured uncertainty (i.e. uncertainty arising due to incorrect model structure), parametric uncertainty (e.g. imprecise knowledge of coefficients of a differential equation) or unstructured uncertainty (e.g. due to ignoring high-frequency dynamics or ignoring high-frequency modes when approximating a distributed system by a finite-dimensional model). In such cases,

insensitivity of the eigenvalues of the (closed-loop) state matrix is a highly desirable property of the feedback system since it can provide robust stability and robust performance properties to the design, i.e. avoiding loss of stability of performance characteristics due to the effects of model uncertainty.

The overall area of research in this field may be grouped to a number of areas, where particular aspects of the problem are considered. We may distinguish:

Problem Area I - System Properties and Eigen-frames: This area is crucial for design and relates to the problem of distribution, selection of poles and eigenstructure and overall system performance. Robustness, seen in a partial way, has been the dominant driving force in recent works done in the area of control.

Problem Area II - Parameterization of Eigen-frames: Additional work is needed in providing alternative descriptions that may be suitable for addressing different design issues. Evaluation of existing parameterizations, Minimal bases parameterisation of eigen-structures, Open-closed loop spectra parameterisation and related topics belong to this area.

Problem Area III - Design of Compensation Schemes for Achieving Desired Frame Spectrum: The design of compensation schemes involves the selection of spectra and eigen-frames which satisfy certain criteria and their realization via specific feedback scheme configurations. In particular the following two problems can be distinguished:

   (i)   Selection of Eigen-frames

   (ii) Feedback Realisation of Selected Eigen-frames

The key issue in the first area is the development of measures for orthogonality and other properties of eigenframes and then the definition of appropriate optimization problems that can produce the "best" - in some sense - frames.

Problem Area IV - System Potential for Delivering Eigenstructure Solutions and Integrated Design: The basic properties of the parameterisation schemes, as well as

the nature of optimal solutions and the feasibility of their realisation depends on the properties of the system model on which the various problems are posed. Investigating the links between system model and its potential for a design solution is the subject of the work here.

The essence of the work is to use this knowledge at the stage of design, redesign of system model, when there is the possibility to enhance the potential of the model to deliver solutions for multi objective and multi constraint type problems.

The above four areas are strongly interrelated. Note that most of the work so far has been in areas (II), (III), some in (I), but little in (IV) [3]. Within the current research and from the above problem areas, various problems related to the orthogonality of eigen-frames and robustness have been studied. These include the optimal choice of eigen-frames [6] which guarantees minimal eigenvalue sensitivity via a generalization of a classical result by Wilkinson [1], the definition of appropriate measures of "*skewness*" and the optimal choice of controllability subspaces in a standard direct sum decomposition using *matrix fraction description* (*MFD*).

This present research aims to develop a new methodology for determining the best angle between closed-loop eigenframes by computing the minimal condition number of the closed-loop eigenvector matrix. The importance of this methodology can be expressed in terms of results related to the Sensitivity of eigenvalues, Relative measures of controllability and observability and also deviations from strong stability to overshooting behaviour.

1.2. Objectives

Our main focus within this presented thesis has boon on the following objectives:

Objective (1): Review Existing Methodologies for Eigenstructure Assignment.

Objective (2): Develop Eigenstructure Parameterisations suitable for Eigenstructure Assignment.

Objective (3): Define measures for Eigenframe skewness.

Objective (4): Develop optimisation algorithms for selection of the most orthogonal frame and consider the influence of selected closed-loop spectra in such optimisations.

In fact, similar to any other research, we first will review some of existing methodologies around Eigenstructure assignment in order to explore the fundamentals and also to evaluate their pros and cons.

Among this variety of eigenstructure assignments and within this current research, the main attention has been paid to method introduced by Prof. Karcanias [4]. He first presents an alternative solution to the assignability of spectrum of controllability subspaces (cs) based on an eigenvector approach and then a new pole assignment algorithm has been derived as a practical application of this approach.

In order to tackle the second objective, we then have developed a parametrisation on the closed- loop eigenvectors obtained from the method in [4]. The purpose here is to then work on this parametrisation and to improve the stability of the closed-loop control system with the direct attention to the condition number of the closed-loop eigenvector matrix. This in fact will result in a better closed-loop system performance based on the fact that the overall sensitivity of the system or the skewness of closed-loop eigenframes is minimized.

In general, the problem that frequently emerges in the study of performances of linear systems is the issue of *"skewness"* of eigenframes. This problem is linked to sensitivity of eigenvalues to parameter uncertainty, perturbations, as well as sensitivity of Nyquist diagrams to model parameter uncertainty. These skewness properties are also linked to measures of controllability and observability, when these are assessed in their model setting.

So far, the measure of skewness has been considered for eigenframes corresponding to distinct eigenvalues via standard tools which include the Gramian, Singular Value Decomposition and Condition Number. For eigenframes corresponding to repeated eigenvalues or complex eigenvalues, however, there are no uniquely-determined basis-sets, despite the fact that the corresponding subspaces are uniquely defined.

Consequently, prior to determination of objective Two, we have addressed a general problem of development of measures of *"skewness"* between subspaces, defined via a direct sum decomposition of the state space and by developing the concept of angle between subspaces. Three measurement tools have been applied in order to develop desirable outcomes for this objective. These tools are: Condition Number, Determinant of Gram Matrix and Singular Value Decomposition (SVD). Using these measurement tools, the objective is to derive necessary and sufficient conditions in each case, such that, the overall angle between subspaces in a direct sum decomposition is minimised.

Finally, the last objective will be to select the most orthogonal closed-loop eigenframe through development of a non-smooth optimisation algorithm and so considering the impact of selected closed-loop spectra in such optimisations. In fact, the problem will be to use one of those measurement tools applied to previous objective and generate the most orthogonal closed-loop eigenvector matrix such that error is minimised.

1.3. Main achievements

This thesis provides a structured and powerful new approach based on the results from Geometric Theory [5], [100], [101] which is an alternative to the work of Kaustky et al., [6], on the problem of perturbation of the eigenvalues and corresponding eigenvectors, by optimizing the condition number of the closed-loop eigenvector matrix.

The main achievement of the thesis is the development of a closed-loop eigenframe based on the following main principles:

i. A parameterisation of controllability subspaces (cs) leads to a family of direct sum decompositions of the state space.
ii. In every controllability subspaces (cs), a set of possible closed loop eigenvectors associated with a given spectrum can be defined.

iii.   Define measures for the skewness of direct sum decompositions of the state space.

iv.   Develop necessary conditions for minimising the skewness of the direct sum decomposition of the state space and algorithms for selection of the most orthogonal set of closed-loop eigenvectors in a cs.

In fact controllability subspaces are (*A, B)*-invariant subspaces with the property that any two points may be connected by some appropriate trajectory generated by a control input such that the trajectory always remains in the given space [9]. Their spectra are not fixed, and so the question arises as to whether or not such subspaces may assume any given spectrum. An alternative to the solution, already established based on an eigenvector approach, [10], is proposed here and involves the construction of characteristic bases having as a spectrum the set of assignable frequencies. We, however, have focused on the alternative parameterisation of eigenframes, [4], based on the property that such frames are arbitrarily assignable spectra that are characteristic bases of controllability subspaces, where we are searching for stabilisation, rather than exact eigenvector selection based only on frequency assignment.

Then, by defining and using three measurement tools which have been introduced earlier during this chapter, we have provided detailed and necessary conditions for the skewness of direct sum decompositions of the state space to be minimised.

Finally, a non-smooth optimisation algorithm has been developed in order to select the most orthogonal set of closed-loop eigenvectors in a cs. Numerical examples have been presented in order for accuracy of the results to be tested where it has been necessary. The selection of the most appropriate stable spectrum leading to optimization of the orthogonality of the eigenframe has been considered for small dimension systems and the development of a procedure for the general dimension case remains an open question.

1.4. Summary of upcoming chapters

The structure of the thesis is as follows:

Chapter Two shall give a brief account of some of the mathematical and control topics which will be required in the reminder of this present research. Through its mathematical section, the definitions and/or theorems corresponding to some key concepts such as Condition number, Gram matrix, Singular value decomposition, Jordan Block, Vandermond matrix, etc are presented and proofed whenever it is strongly linked to the practical techniques which are used subsequently. The same work has been also done on some control-related fundamentals which are well connected to our main objectives of this presented thesis. These elements include Matrix Fraction Description (MFD), Minimal Bases of Matrix Pencils and their structures. These will lead to the study of controllability subspaces, which includes the important topic of matrix parameterisation of controllability subspaces through later chapters.

Chapter Three contains the literature review on Eigenstructure assignment including the relative basic concepts and background results. Throughout this chapter, a brief review of the concept of eigenvalues and eignevectors and their corresponding theorems is given. Then by reviewing the link between closed-loop eigenvalues and feedback, the problem of eigenstructure assignment has been reminded and reviewed through state and output feedbacks. Then some of main results achieved around state and output feedbacks have been presented including parametric cases.

The main part of this chapter is to study the theory of perturbation, introduced by Wilkinson, [1], and the corresponding results achieved by Kautsky et al., on robust eigenstructure assignment, [6], which have been presented in details.The importance of this part is under the fact that our main achievement of the presented thesis is in parallel to the work done by Kaustky et al., where in our presented method, we consider a set of possible closed loop eigenvectors associated with a given spectrum in a controllability subspaces (cs) and use a new parameterisation of closed-loop eigenvectors based on the polynomial characterisation of controllability subspaces [8], [96]. Finally, this chapter will review some other important results in the area of

eigenstructure assignmnets in order for a reader to obtain some more useful information around this subject.

Chapter Four presents the concepts of *A*- and (*A*,*B)* –invariant subspaces, followed by the construction and the existence of controllability subspaces connected to (*A*,*B)*-invariant subspaces. Through this chapter and from the geometric theory's point of view, firstly the motion of eigenvectors within an *A*- invariant subspace of a linear system is reviewed and then by extending this matter to the (*A*,*B)*-invaraint subspaces, the Rectilinear motion of Nonautonomous system in the input output and state spaces is studied and presented. This will follow by the study of link between Simple Rectilinear Motion and one or R-dimensional (*A*,*B)*- invariant subspaces. Finally the concept of controllability subspaces is discussed in depth as the result of characteristic decomposition of (*A*,*B)*- invariant subspaces.

Within **Chapter Five,** the problem of eigenvector frame parameterisations is considered via two different parameterisations: first is the parameterization of closed-loop eigenframes based on the open and closed loop spectra and the second is based on the algebraic characterization and parameterization of controllability subspaces. These methods are in fact produced by Prof. Karcanias, [4], through his tremendous works done on the area of control system design theory.

Through this chapter, the relative fundamentals of each method are well-explained and then by introducing the Minimal Dimension Controllability Subspaces, assigning the spectrum of a controllability subspace is discussed and reviewed. This is then led to the Eigenvalue Placement algorithm based on mobility of Open to Closed Loop Spectra introduced by Prof. Karcanias [4].

In this presented thesis we have used this algorithm and as the result have generated a general parametrisation of corresponding closed-loop eigenvectors and so have introduced an optimisation problem in order to select the most orthogonal closed-loop eigenvectors and so to obtain a closed-loop system with the minimum sensitivity. The solution to this optimisation problem itself will then be presented in

Chapter Eight where this problem has been tackled and solved by a non-smooth optimisation method.

Chapter Six contains one of the main topics of this research: This is the problem of minimising the angle between subspaces in a direct sum decomposition which will be used in order to obtain the solution for the optimisation problem introduced at the final part of Chapter Five. During this chapter we tend to overview the issue of *"skewness"* of eigenframs in general which is linked to sensitivity of eigenvalues to parameter uncertainty, perturbations, as well as sensitivity of Nyquist diagrams to model parameter uncertainty. The objective of this chapter is to develop some measures of "skewness" between subspaces defining a direct sum decomposition of the state space and thus develop a concept of angle between these sets of subspaces. In fact our main aim in this chapter is to provide the required new concept of the relative positioning between subspaces that can be used in quantifying Sensitivity of eigenvalues as well as Relative measures of controllability and observability and also Deviations from strong stability to overshooting behaviour and for this purpose we will use The Gramian (determinant to be maximized), The Condition number (to be minimized) and The Spread of singular values or a deviation measure of the singular values (which is minimized), as our three measurement tools. In this chapter we will proof that no matter which measurement is used, in all these cases the optimal condition arises when orthogonality conditions apply. The theoretical results are also supported by a number of numerical examples.

Chapter Seven presents an algebraic description of the total system behaviour which allows the study of closed loop eigenvectors in a systematic way by providing new parameterisations. This will then leads to an algebraic characterisation of the total input, state and output behaviour in an implicit formulation and it is given based on properties of MFD descriptions which will remain open for future studies.

During this chapter, a description of an implicit system is given followed by its relative issues such as Duality and corresponding system behaviour. Then the computation of input-state generator pairs, [4], is reviewed followed by Closed Loop Eigenvectors and their link to the frequency Transmission. Then we will study the

impact of pole assignment using output feedback on closed-loop eigenvectors, this time via MFD's. This will follow by a brief review of the link between system frequency and poles and zeros of the system since each of these is strongly link to the overall behaviour of the system. We then introduce new route for designing of State Feedback Controllers by using Eigenvector parametrisation and through ordered minimal bases, [33], [7], as the use of minimal bases suggests a simple procedure for selection of an independent eigenframe.

In fact in this chapter, the selection of the full rank eigenframe and its relation to the definition of state feedback along with the problem of minimal basis parameterization and its role to the shaping of frames are considered and deeply reviewed. This comes from the fact that each full rank closed-loop eigenframe can be written as the product of a matrix of ordered minimal bases (of matrix pencil) and a matrix containing all the existing eignevalues of the system in the form of Vandermonde matrix. As one can see, the matrix of ordered minimal bases has such importance as this is used to produce the controllability subspaces of closed-loop system and so if the optimal matrix is chosen then the stability of the system is guaranteed. This factor has led to the parametrisation of these minimal bases which is presented towards the end of Chapter Seven via Toeplitz form. In fact by exploiting the general parameterisation of the minimal bases, the general formula for any other minimal bases with the same degree can be computed using a Toeplitz matrix construction and the controllability subspaces separated by the largest possible angles can be identified. In each case, the theoretical results are illustrated with a numerical example. One interesting topic could be to use this parametrization and find a best choice of minimal bases such that the angle between these bases and consequently between the resulted controllability subspaces is maximized.

Chapter Eight presents the main achievement of this thesis, i.e. to obtain the minimum angle between closed-loop eigenvectors via a non-smooth optimisation method in order for the angel between closed-loop eigenvectors (Refer to Chapter Five) to be maximised. This is done through finding the solutions of minimizing the condition number of the gram matrix as one of the measurement tools introduced in

Chapter Six to assign and to reduce the Sensitivity of closed-loop eigenvetors for any controllable system(s), such that the system is robust to any perturbation injected to the eigenvalues or their relative eigenvectors, i.e. the error is minimized. The development of a robust and efficient optimization algorithm needs further consideration due to the non-convex nature of the optimization problem.

For this purpose, firstly, the concept of non-smooth optimisation has been derived [11]. Then by introducing the generalised gradient of condition number of any square symmetric matrix or its Gram matrix, the concepts of smoothing approximation following by non-smooth optimisation is described along with nay essential theorem and corresponding proof. Then by introducing an algorithm of relative optimisation method, our conditioning problem presented in Chapter Five is solved and some numerical results have been also derived.

Finally, **Chapter Nine** contains the main conclusions of the work and suggestions for future research. Within this chapter, a deep summery of what has been studied and established as the result of this research is presented. In addition, some important open issues have been presented and discussed which could be used as future topics of study in this field. These include finding the best choice of ordered minimal bases matrix such that the angle between these bases is maximised. This for instance will guarantee the best choice of resulted controllability subspaces and so improves the stability of the closed-loop system as much as possible.

CHAPTER 2

MATHEMATICS AND CONTROL BACKGROUND

2.1. Introduction

In this section we give a brief account of some of the mathematical and control concepts which are required in future chapters of this work. Note that no attempt has been made to give complete proofs of all the fundamental theorems, unless these are strongly linked to the theory developed.

2.2. Mathematics background

2.2.1. Vector space. This is a set $\mathcal{V}$ of vectors over a field $\mathcal{F}$ which is closed under the operation of addition (associative and commutative) and has an identity (0) and additive inverse included within the set. The set is also closed under an operation of left multiplications of the vectors by any scalar of $\mathcal{F}$ with the following properties:

For $a, b \in \mathcal{F}$ and $\underline{x}, \underline{y} \in \mathcal{V}$:

$$a(\underline{x} + \underline{y}) = a\underline{x} + b\underline{y}$$
$$(a + b)\underline{x} = a\underline{x} + b\underline{x}$$
$$a(b\underline{x}) = (ab)\underline{x}$$
$$e\underline{x} = \underline{x}$$

where $e \in \mathcal{F}$ and it is the multiplicative identity. ([12], [13], [14], [15]).

2.2.2. Subspace. Let X be a nonempty subset of a vector space $\mathcal{V}$ over field $\mathcal{F}$, that is: $X \subseteq \mathcal{V}$. If X is also a vector space over Field $\mathcal{F}$ for the same addition and multiplication operations, then X is called a subspace of $\mathcal{V}$, i.e. if and only if:

$$\underline{x}, \underline{y} \in X \Rightarrow \underline{x} + \underline{y} \in X$$

and

$$\underline{x} \in X \Rightarrow a\underline{x} \in X \text{ for all } a \in \mathcal{F}. \quad ([12], [14])$$

2.2.3. Orthogonal and Orthonormal vectors. Let the angle θ between each two vectors of set of vectors $\underline{x}_1, \underline{x}_2, ..., \underline{x}_k \in \mathbb{C}^n$ is defined as:

$$\cos\theta = \frac{\underline{x}_i^t \underline{x}_j}{\left\| \underline{x}_i^t \right\| \left\| \underline{x}_j \right\|}$$

where $\underline{x}_i^t \underline{x}_j = \sum_{i=1}^{n} \overline{x}_i \underline{x}_i$ is the inner product of these two vectors. Then the whole set is said to be a orthogonal set if $\theta = 90^{\text{o}}$ or in other word $\underline{x}_i^t \underline{x}_j = 0$ where $1 \le i \le j \le k$. Additionally, these vectors are said to be orthonormal if each of them is normalized, $\underline{x}_i^t \underline{x}_i = 1$, $i = 1, \cdots, k$. Note that an orthonormal set of vectors is linearly independent. [Refer to Horn and Johnson, 1985, Theorem 2.1.2, for complete proof]. ([12], [14], [15], [16], [17])

2.2.4. Unitary Matrix. [12] A matrix $U \in \mathcal{M}_n$ is said to be Unitary if and only if $U^t U = I$, where U^t is the conjugate transpose of matrix U.

2.2.5. Vector norms. For any vector $\underline{x} = [x_1, x_2, \cdots, x_n]$ where $\underline{x} \in \mathbb{C}^n (or \mathbb{R}^n)$, then the p-norm of vector X is given by $\left\| \underline{x} \right\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ where $p \ge 1$. For this vector $\underline{x}$, there are some very famous norms, which are used frequently and are defined as follow:

(i) *Euclidean norm (2-norm)*, is defined as :

$$\left\| \underline{x} \right\| := \sqrt{x_1^2 + \cdots + x_n^2} \text{ or } \left\| \underline{x} \right\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

(ii) *1-norm* , is represented as: $\quad \|\underline{x}\|_1 := \sum_{i=1}^{n} |x_i|$

(iii) ∞ *-norm (Maximum norm)*, is defined as:

$\|\underline{x}\|_\infty := \max(|x_1|, \cdots, |x_n|)$ ([12], [14], [17]).

2.2.6. Matrix norms. Because $\mathbb{C}^{m \times n}$ is a vector space of dimension $m.n$, magnitudes of matrices $A \in \mathbb{C}^{m \times n}$ can be "measured" by employing any vector norm on $\mathbb{C}^{m \times n}$ ([14]).

In this work the following two matrix norms are used ([12], [14], [15], [17]):

(i) *Frobenius matrix norm* is defined by

$$\|A\|_F \equiv \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^2} = trace(A^t A) = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2}$$

where α_{ij} denotes any element of matrix A, A^t is the conjugate transpose of matrix A and σ_i defines any singular values of A.

(ii) *Matrix 2-norm*

The matrix norm induced by the Euclidean norm is

$\|A\|_2 = \max_{\|X\|_2 = 1} \|A\underline{x}\|_2 = \sqrt{\lambda_{\max}}$

where $\lambda_{\max}$ is the largest eigenvalue such that $A^t A - \lambda I$ is singular. In other words:

$\|A\|_2 = \sqrt{\lambda_{\max}(A^t A)}$,

A^t is the conjugate transpose of matrix A.

(iii) *Matrix 1-norm* is defined as follow:

$\|A\|_1 = \max_{\|X\|_1 = 1} \|A\underline{x}\|_1 = \max_j \sum_i |a_{ij}|$

which is in fact the largest absolute column sum of the matrix.

(iv) *Matrix ∞ -norm*

The matrix norm induced by vector ∞-norm is as follows:

$$\|A\|_\infty = \max_{\|X\|_\infty=1} \|A\underline{x}\|_\infty = \max_i \sum_j |a_{ij}|.$$

i.e. the largest absolute row sum of a matrix.

2.2.7. *Hadamard's inequality theorem.* Let $(\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n)$ be columns (vectors) in $\mathbb{R}^n$ and $A = (\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n)$ be the corresponding $n \times n$ matrix. The Hadamard's inequality states that: $|\det A| \leq \prod_{n=1}^n \|\underline{a}_n\|$, where $\|.\|$ is the Euclidean norms on vectors in $\mathbb{R}^n$. ([18], [19]).

2.2.8. *Gramian (Gram) Matrix.* [17] Consider vectors $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n \in \mathbb{R}^n$. The *Gram matrix* of the collection is the $m \times m$ matrix G with elements $G_{ij} = \underline{x}_i^t \underline{x}_j$. The matrix can be expressed compactly in terms of the matrix $X = [\underline{x}_1, \cdots, \underline{x}_n]$, as

$$G = X^t X = \begin{pmatrix} \underline{x}_1^t \\ \vdots \\ \underline{x}_m^t \end{pmatrix} \begin{pmatrix} \underline{x}_1 & \cdots & \underline{x}_m \end{pmatrix}.$$

By construction, a Gram matrix is always symmetric, meaning that $G_{ij} = G_{ji}$, for every pair (i, j). It is also positive semi-definite, meaning that $\underline{u}^T G u \geq 0$, for every vector $\underline{u} \in \mathbb{R}^n$ (this comes from the identity $\underline{u}^t G \underline{u} = \|G\underline{u}\|_2^2$).

Assume that each vector $\underline{x}_i$ is normalized: $\|\underline{x}_i\|_2 = 1$. Then the coefficient G_{ij} can be expressed as $G_{ij} = \cos\theta_{ij}$, where θ_{ij} is the angle between the vectors $\underline{x}_i$ and $\underline{x}_j$. Thus G_{ij} is a measure of how similar $\underline{x}_i$ and $\underline{x}_j$ are ([12], [20]).

Geometrically, the Gram determinant is the square of the volume of the parallelotope formed by the vectors. Obviously by having all the vectors orthogonal to each other, the Gram determinant will be maximum.

2.2.9. _Condition Number._ For any $n \times n$ matrix A, the condition number $\kappa(A) \geq 1$, based on matrix 2-norms, is defined to be : $\kappa(A) = \|A\| \cdot \|A^{-1}\|$. Similarly, the condition number could be also defined using the singular values of matrix A as follows: $\kappa(A) = \dfrac{\sigma_1(A)}{\sigma_n(A)}$ where σ_1 is the largest and σ_n is the smallest singular value of matrix A ([12], [21]).

A matrix is so called: "well-conditioned" if its value of condition number is low and is called to be "ill-conditioned" if its condition number is high.

2.2.10. _Vandermonde matrix._ A $n \times n$ Vandermonde matrix A is a matrix of form

$$A = A(\underline{x}_0, \cdots, \underline{x}_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

where $x_1, \cdots, x_n \in \mathcal{F}$, that is, $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ with $a_{ij} = x_i^{j-1}$. The determinant of this matrix can be computed as:

$$\det(A) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$$

A is nonsingular if and only if all the parameters $x_1, \cdots, x_n \in \mathcal{F}$ are distinct.

Note that the Vandermonde matrix evaluates a polynomial at a set of points; formally, it transforms coefficients of a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ to the values the polynomial takes at the points α_i. The non-vanishing of the Vandermonde determinant for distinct points α_i shows that, for distinct points, the map from coefficients to values at those points is a one-to-one correspondence, and thus that the polynomial interpolation problem is solvable with unique solution.

([12], [13], [17], [21], [22]).

2.2.11. The Jordan Canonical Form of a Matrix. For an $n \times n$ complex matrix A, there exists a non-singular matrix T such that $T^{-1}AT = J = diag(J_1, \cdots, J_k)$, where:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}$$

is $m_i \times m_i$ and $m_1 + \cdots + m_k = n$. The matrices J_i are called Jordan matrices or Jordan blocks, J is called the Jordan Canonical Form (JCF) of A and $\lambda_i, i = 1, 2, \cdots, k$ is the eigenvalue of A with multiplicity of m_i. The same eigenvalue can appear in more than one block ([12], [17], [21]).

2.3. Control background

This part of the chapter describes a very important tool which is used to the study of controllability subspaces. This leads to one of the main topics of the current research which involves the best parameterization of the controllability subspaces in a direct sum decomposition in order to compute the minimal basis of the whole controllability space.

2.3.1. Matrix Fraction Description (MFD)

We first introduce the basics of *MFD* and then review the problem statement related to the controllability subspaces by studying some basic concepts and definitions.

The first step when studying and designing a control strategy for a physical system is the development of mathematical equations that describe the system. The linear equations used to describe linear systems are generally limited either to:

-    input-output description (external description)

or

-    the state-variable equation description (internal description)

Prior to 1960, the design of control systems had been mostly carried out by using transfer functions. However, the design had been limited to the single variable, or single-input-single-output (SISO) case. Its extension to the multivariable, or multi-input-multi-output (MIMO) case had not been successful.

The state-variable approach was developed in the sixties, and a number of new results were established in the SISO and MIMO cases. At that time, these results were not available in the transfer-function, or polynomial approach, so the interest in this approach was renewed in the seventies. Now most of the results are available both in the state-space and polynomial settings. Refer to [2], [5], [8], [13] and [71] for some of these results.

2.3.1.1. Scalar systems

- Rational transfer function (RTF)

Although there are too many different interpretations related to Rational Transfer Function, here we will present the fundamental summary of RTF including major definitions and the way that RTF is constructed ([13], [23], [30]).

Assuming that the knowledge of the internal structure of the system is not available, the transfer function description of a system gives a mathematical relation between the input and output signals of the system.

Assuming zero initial conditions, the relationship between the input u and the output y of a system can be written as

$$y(s) = G(s)u(s)$$

where s is the Laplace transform variable in continuous-time (for discrete-time systems, we use the z-transform) and $G(s)$ is the scalar *transfer function* of the system. $G(s)$ is a rational function of the indeterminate s that can be written as a ratio of two polynomials where $n(s)$ is a *numerator polynomial* and $d(s)$ is a *denominator polynomial* in the indeterminate s.

In the above description of a transfer function, it is assumed that polynomials $n(s)$ and $d(s)$ are *relatively prime*. The degree of denominator polynomial $d(s)$ is the order of the linear system.

Here we give some of useful definition relative to the matrix fraction description area:

Monic polynomial: The polynomial with leading coefficient equal to one ([13], [23], [30]).

Nominal transfer function: When the denominator polynomial is *monic*, then the transfer function is *normalized or nominal (*[97]).

Column-reduced: A polynomial matrix with non-singular column leading coefficient matrix ([13], [23], [30]).

Greatest common divisor: A highest degree common factor that can be extracted from two polynomials ([13], [23], [30]).

Hermite form: A triangular canonical form of a polynomial matrix ([13]).

Irreducible: A transfer function is irreducible when its numerator and denominator polynomials are relatively prime ([13], [23], [30]).

Leading coefficient matrix: The constant matrix whose entries are built from coefficients of highest powers of the entries of a polynomial matrix ([13], [23]).

Left MFD: An MFD where the denominator polynomial matrix enters from the left ([13], [23], [30]).

Matrix fraction description: Ratio of two polynomial matrices describing a matrix transfer function ([13], [23], [30]).

Minimal: A state-space realization is minimal if it has the lowest possible dimension ([13], [23])

Proper: A matrix transfer function is proper if the degree of the denominator polynomial of each entry is greater than or equal to the degree of the numerator

polynomial. If the degree is greater, then the transfer function called the strictly proper ([13], [23], [30]).

Rational vector space: The linear vector space over $\mathbb{R}(s)$ contains all the rational functions in s having real coefficients ($\in \mathbb{R}^{m \times n}(s)$) ([33]).

Relatively prime: Two polynomials or polynomial matrices are relatively prime if they have no common factor ([13], [23], [30]).

Right MFD: An MFD where the denominator polynomial matrix enters from the right ([7], ([13], [23], [30]).

Row-reduced: A polynomial matrix with non-singular row leading coefficient matrix ([13]).

Unimodular: A polynomial matrix with a non-zero constant determinant ([13], [23], [30]).

For a system:

$$E\underline{\dot{x}} = A\underline{x} + B\underline{u}, \, y = C\underline{x}$$

with n variables, m inputs, and p outputs, the transfer function $G(s) = C(sE - A)^{-1}B$

is a $p \times m$ matrix whose elements are rational functions.

A closer analogy to the SISO case is the matrix fraction description:

$$G(s) = N_R(s)D_R^{-1}(s) \text{ or } G(s) = D_L^{-1}(s)N_L(s)$$

where N_R, D_R, N_L, D_L are polynomial.

We can define a *right matrix fraction description*, or right MFD for short, $G(s) = N_R(s)D_R^{-1}(s)$

Alternatively, we can also define a *left MFD*

$$G(s) = D_L^{-1}(s)N_L(s)$$

It is always possible to go from right MFD's to the left and vice versa.

There can be many right and left matrix fraction description (MFDs) of $G(s)$. Indeed, given a right MFD, an infinity of others can be obtained by choosing any non-singular polynomial matrix $U(s)$ such that:

$$\bar{N}_R(s) = N_R(s)U(s), \quad \bar{D}_R(s) = D_R(s)U(s).$$

We call $U(s)$ a right divisor of $N_R(s)$ and $D_R(s)$. Moreover, since deg det $\bar{D}_R(s)$ = deg det$D_R(s)$+deg det$U(s)$, it holds:

$$\text{deg det } \bar{D}_R(s) \geq \text{deg det} D_R(s)$$

which means that the degree of a MFD can be reduced by removing right divisors of the numerator and denominator matrices.

Obviously, we will get a minimum-degree right MFD by extracting a *greatest common right divisor* (gcrd) of $N_R(s)$ and $D_R(s)$. In other words, a gcrd from $N_R(s)$ and $D_R(s)$ is extracted if and only if deg det $\bar{D}_R(s)$ = deg det$D_R(s)$.

If $N_R(s)$ and $D_R(s)$ have only unimodular right *gcrds*, then these two matrices are *right coprime* and the right MFD $G(s) = N_R(s)D_R^{-1}(s)$ is *irreducible*. Similar statements can be given for left MFDs ([13], [23], [30]).

For every transfer function

$$G(s) = \frac{n(s)}{d(s)}$$

there is an unlimited number of state-space realizations. Some of them such as *canonical* realizations, known as the controllable and observable forms which are commonly used depend on the application and requirement.

If $V(s) \in \mathbb{R}^{n \times p}[s]$ is a polynomial matrix basis for the rational vector space, then we have the following definitions for $V(s) \in \mathbb{R}^{n \times p}[s]$:

Least degree basis: $V(s) \in \mathbb{R}^{n \times p}[s]$ is a least degree basis if the greatest common divisor of all its $m \times m$ minors is equal to 1 ([25]).

Minimal basis: The basis $V(s) \in \mathbb{R}^{n \times p}[s]$ is called minimal if it is row proper and it has least degree ([7], [33]).

The degree of $V(s)$: The degree d of the basis $V(s) \in \mathbb{R}^{n \times p}[s]$ is the highest degree among all the $m \times m$ minors of the matrix $V(s)$ ([25], [33],).

The complexity of $V(s)$: ([13], [23], [30]). The complexity c of the basis $V(s)$ is the sum of the degrees of its row vectors, i.e..,

$$c = \sum_{i=1}^{m} \delta_i .$$

Note that normally $c \geq d$, $V(s)$ is so called **row-proper** if and only if $c = d$. For any full rank polynomial matrix $M(s)$ which is $m \times p$ $(m > p)$, we can create a module $\mathcal{M}_M$, by multiplying from the right to a uni-modular matrix $Q(s)$, $p \times p$, and $|Q(s)| = c \neq 0$, such that $M(s)Q(s) = \mathcal{M}_M$ (The degree of $\mathcal{M}_M$ is the maximum degree of $C_p(M(s))$ where $C_p(M(s))$ is the exterior product of minor p of $M(s)$).

So for any other bases $M'(s) = M(s)Q(s)$, the degree of $M(s)$ & $M'(s)$ is the same. If we introduce $M(s)$ to be:

$$M(s) = \left[\underline{m}_1(s), \cdots, \underline{m}_p(s) \right]$$

and $\delta_1, \cdots, \delta_p$ are the degrees of each row polynomial vector $\underline{m}_i(s) = m_i^{\delta_i} s_i^{\delta_i} + \cdots + m_i^1 s + m_i^0$, where each $m_i^{\delta_i}$ contains the coefficients related to the highest degree of its polynomial .

Denote vector $\underline{m}_i^{\delta_i}$ as $\underline{m}_i^h$; then the row highest coefficient matrix of $M(s)$ is the matrix $M_h = \left[\underline{m}_1^h \quad \underline{m}_2^h \quad \cdots \quad \underline{m}_p^h \right]$ which contains just the constants value and $M(s)$ can be written as:

$$M(s) = \begin{bmatrix} \underline{m}_1^h & \underline{m}_2^h & \cdots & \underline{m}_p^h \end{bmatrix} \begin{bmatrix} s^{\delta_1} & & & 0 \\ & s^{\delta_2} & & \\ & & \ddots & \\ 0 & & & s^{\delta_p} \end{bmatrix} + \cdots.$$

Note that $M(s)$ is row proper if and only if, M_h is full rank. ([25])

2.3.2. Minimal bases of matrix pencils and coprime matrix descriptions

In this part, we review the problem of constructing coprime MFD's from a minimal state space realization of a transfer function matrix. This part is widely used in the analysis and design of linear systems described by transfer function matrices using MFD's. The main results in this section are taken from Karcanias (2002) [7], [8], [100].

Consider the minimal system described by

$$\begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{cases} \tag{2.3.1}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{m \times n}$ and with transfer function matrix $G(s) \in \mathbb{R}^{m \times p}(s)$.

The problem here is to find the right and left coprime MFD's for $G(s)$, i.e. to express $G(s)$ as :

$$G(s) = N(s)D(s)^{-1} = D'(s)^{-1}N'(s) \tag{2.3.2}$$

where $N(s), N'(s) \in \mathbb{R}^{m \times p}(s)$, $D(s) \in \mathbb{R}^{p \times p}(s)$, $D'(s) \in \mathbb{R}^{m \times m}(s)$ and *(N(s),D(s))* are right coprime and $(D'(s), N'(s))$ are left coprime.

It can be shown [7] that the computation of coprime MFD's from state space description is reduced to computation of coprime MFD's of the transfer functions:

$$H_{IS}(s) = (sI - A)^{-1}B, \quad H_{SO}(s) = C(sI - A)^{-1} \tag{2.3.3}$$

where $H_{IS}(s)$ is the input-state description and $H_{SO}(s)$ is the state-output description.

It is evident that the above descriptions are dual and thus treatment of the first provides also solution for second. Let us consider $H_{IS}(s)$ and right MFD's, i.e.

$$(sI - A)^{-1}B = \bar{N}(s)\bar{D}(s)^{-1} \Rightarrow [sI - A, -B]\begin{bmatrix} \bar{N}(s) \\ \bar{D}(s) \end{bmatrix} = 0, \quad \bar{T}(s) = \begin{bmatrix} \bar{N}(s) \\ \bar{D}(s) \end{bmatrix} \tag{2.3.4}$$

Given that $(\bar{N}(s), \bar{D}(s))$ is right coprime, from the full rank property of pencil $[sI - A, -B]$, we have the following results:

Proposition 2.1: Every least degree polynomial basis of:

$$\mathcal{N}_r\{sI - A, -B\}, \ Z_r(s) = [X(s)^t; U(s)^t]^t, \text{ i.e. } \quad [sI - A, -B]\begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = 0 \tag{2.3.5}$$

Defines a right coprime MFD for $H_{IS}(s) = \bar{N}(s)\bar{D}(s)^{-1}$, where

$$\bar{N}(s) = X(s), \bar{D}(s) = U(s).$$

Proof: See [7] for full proof.

∎

The above result stated for least degree base is also valid for minimal bases. If $N \in \mathbb{R}^{((n-p)\times n)}$, $B^\dagger \in \mathbb{R}^{p \times n}$ is a pair of left annihilator, left inverse of B ($NB = 0$, rank *rank* $N = n - p$, $B^\dagger B = I_p$), then

$$
\begin{aligned}
(sN - NA)X(s) &= 0 \\
U(s) &= B^\dagger(sI - A)X(s)
\end{aligned}
\tag{2.3.6}
$$

The pencil $sN - NA$ has $(n - p) \times n$ dimension and its significance is emphasised by the following result.

Proposition 2.2: The pair $(X(s), U(s))$ defines a minimal basis $Z_r(s)$ for $\mathcal{N}_r\{[sI - A, -B]\}$, if and only if $X(s)$ is a minimal basis for $\mathcal{N}_r[sN - NA]$ ([7], see p.247 for proof).

∎

During next section, we will review the methodologies used to obtain the minimal bases of matrix pencil in more detail.

2.3.3 Construction of minimal bases of matrix pencils and coprime matrix fraction descriptions using Toeplitz matrix

Since coprime MFDs have been used widely in linear control theory, several methods for their computation have been developed ([7], [13]). A first approach was developed in Rosenbrock (1970) [26].

This method requires elementary row operation on polynomial matrices and is therefore difficult to implement on a computer and can be numerically unstable. There are two other main approaches that have emerged so far: the reduction to the block Hessenberg form (BHF) and the minimal design problem (MDP).

The BHF approach uses state space realisations starting from controllable realizations and then by suitable transformations that reduces to the Hessenberg form. In other words, the approach consists of a transformation from state space model $S(A,B,C,D)$ to reduced model $S(F,G,H)$ where the matrix F is the lower (upper) block Hessenberg matrix. Orthogonal transformations are preferred for the reduction to BHF because of their attractive numerical stability properties.

Reduction to the block Frobenius form (BFF) can also be employed, i.e. the model $S(F,G,H)$ can be reduced to the form $S(\bar{F},\bar{G},\bar{H})$, where $\bar{F}$ is in BFF. By permutation of the state space variables of $S(\bar{F},\bar{G},\bar{H})$ a canonical representation $S(F_c,G_c,H_c)$ similar to the Lueberger canonical form is obtained.

The MDP approach starts form a left, not necessarily coprime, MFD and computes a right coprime MFD using the theory of minimal bases of rational vector spaces, Forney (1975). This approach uses the properties of Sylvester matrices, obtained by the equivalent formulation of the algebraic problem into real matrix computations. ([27], [28], [29]).

The main disadvantage of these methods is that they involve operations on matrices whose orders may often be much higher that the dimension of the state space of the given system.

Another method for obtaining relative prime MFD from a state space representation of the transfer function is given by Wolovich (1974) [30], [31]; Wolovich & Falb (1969) [32] and Datta & Gangopadhyay (1992) [28].

This method is most commonly used for computation MFD's since it is the easiest to implement on a computer and can be used for high-order system. The disadvantage is that the state space system must first be transformed to the Lueberger canonical form which can be quite sensitive numerically.

Here we give two methods of computation of coprime MFD's using Toeplitz matrix by Karcanias (2002) ([7], [96], [100]). These are the indirect and direct methods which are based on the construction of minimal bases for the kernels of the state-space-based pencils $[sI - A, -B]$, $[sI - A^t, -C^t]$.

2.3.3.1. Computation of minimal bases for $\mathcal{N}_r\{[sI - A, -B]\}$ using the Toeplitz structure: the indirect method [7], [100]

First essentials notation and definitions are introduced:

For the pencil $R(s) = sNA - NA = sF - G$ we first note the following:

If $T(s) = [sI - A, -B]$ and $R(s) = sNA - NA$ and $\mathcal{Z}$ and $\mathcal{X}$ respectively their rational, right null spaces respectively, then:

The ordered set of Forney dynamical indices [33] of $\mathcal{X}$, or column minimal indices (cmi) of the corresponding pencils are denoted by:

$$\mathcal{I}(\mathcal{X}) = \left\{(\varepsilon_i, \rho_i), i \in \underset{\sim}{\mu}, 0 \leq \varepsilon_1 \leq \ldots \leq \varepsilon_\mu\right\} \tag{2.3.7}$$

$$\mathcal{I}(\mathcal{Z}) = \left\{(\tilde{\varepsilon}_i, \tilde{\rho}_i), i \in \underset{\sim}{\mu}, 0 \leq \tilde{\varepsilon}_1 \leq \ldots \leq \tilde{\varepsilon}_\mu\right\} \tag{2.3.8}$$

Note that $\rho_i = \tilde{\rho}_i$, $\tilde{\varepsilon}_i = \varepsilon_i + 1, \forall i \in \underset{\sim}{\mu}$. In the above, $\rho_i, \tilde{\rho}_i$ denote the multiplicity of the distinct value of $\varepsilon_i, \tilde{\varepsilon}_i$. The $\tilde{\varepsilon}_i$ are known as the controllability indices of the $S(A,B)$ system.

- Any minimal basis matrix of $\mathcal{X}$, $X(s)$ is called an ordered minimal basis matrix (OMBM) if it may be expressed as :

$$X(s) = \left[X_1(s); \cdots ; X_\mu(s) \right], \quad X_i(s) = s^{\varepsilon_i} X^i_{\varepsilon_i} + \cdots + s X^i_1 + X^i_0 \in \mathbb{R}^{n \times \rho_i}[s] \tag{2.3.9}$$

- If the Toeplitz matrix related to X_i for every $k \geq \varepsilon_i$ with $f_i = k - \varepsilon_i \geq 0$, is as :

$$T^k_{\varepsilon_i}(X_i) = \begin{bmatrix} X^i_{\varepsilon_i} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ X^i_1 & & \ddots & & \vdots \\ X^i_0 & & & & 0 \\ 0 & \ddots & & & X^i_{\varepsilon_i} \\ & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & X^i_1 \\ 0 & \cdots & \cdots & 0 & X^i_0 \end{bmatrix} \in \mathbb{R}^{(k+1)n \times (f_i+1)\rho_i} \tag{2.3.10}$$

also known as the *kth* Toeplitz matrix of $X_i(s)$.

Then the *kth* Toeplitz matrix of (F,G) pair is defined as:

$$T_k(F,G) = \begin{bmatrix} F & 0 & \cdots & \cdots & 0 \\ -G & F & & & \vdots \\ 0 & -G & & & 0 \\ \vdots & \vdots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & F \\ 0 & \cdots & \cdots & 0 & -G \end{bmatrix} \in \mathbb{R}^{(k+2)q \times (k+1)n}, \quad k = 0,1,2,\cdots. \tag{2.3.11}$$

So above notations and results provide the means for the construction of special basis matrices $\mathcal{N}^k{}_r$ spaces, which in turn lead to the construction of minimal bases for space $\mathcal{N}_r\{[sI-A,-B]\}$.

Here is the indirect algorithm ([7], [96]):

Part (I): Computation of $\mathcal{I}(\mathcal{X})$. For all the $k \geq 0$ compute $\operatorname{rank} T_k(F,G) = \sigma_k$, $k = 0,1,2,\cdots$.

The right nullity of $T_k(F,G)$ defined by $r_k = (k+1)n - \sigma_k, k = 1,2,\cdots$. This defines the sequence $\mathcal{C}_r(F,G) = \{r_k : k \geq -2, r_{-2} = r_{-1} = 0, \forall k \geq 0\}$.

By computing the gap sequence

$$\delta_k = r_k + r_{k-2} - 2r_{k-1}, k \geq 0, r_{-2}, r_{-1} = 0 \tag{2.3.12}$$

the distinct values of ε_i are defined by those values of k for which $\delta_k > 0$ and the corresponding multiplicity ρ_i is the value of $\delta_k > 0$.

This defines $\mathcal{I}(\mathcal{X}) = \left\{(\varepsilon_i, \rho_i), i \in \underset{\sim}{\mu}, 0 \leq \varepsilon_1 \leq \ldots \leq \varepsilon_\mu\right\}$.

Part(II): Computation of a set of state generator $\Omega_{\mathcal{X}}^{\mathcal{I}}$. Having $\mathcal{I}(\mathcal{X}) = \left\{(\varepsilon_i, \rho_i), i \in \underset{\sim}{\mu}, 0 \leq \varepsilon_1 \leq \ldots \leq \varepsilon_\mu\right\}$ the computation of a set $\Omega_{\mathcal{X}}^{\mathcal{I}} = \left\{N_{\varepsilon_i}, i \in \underset{\sim}{\mu}\right\}$ involves the following steps:

Step(1): For $k = \varepsilon_1$, compute any basis matrix for $\mathcal{N}_r^{\varepsilon_1}$ and let this be $N_{\varepsilon_1} \in \mathbb{R}^{(\varepsilon_1+1)n \times \rho_1}$.

Step(2): For $k = \varepsilon_2$, compute $T_{\varepsilon_1}^{\varepsilon_2}(N_{\varepsilon_1})$ and complete it to a basis for $\mathcal{N}_r^{\varepsilon_2}$. This defines $N_{\varepsilon_2} \in \mathbb{R}^{(\varepsilon_2+1)n \times \rho_2}$.

Step(i) (General step): Let $\left\{N_{\varepsilon_1}, \cdots, N_{\varepsilon_{i-1}}\right\}$ be the matrices defined by the previous step for $k = \varepsilon_{i-1}$. For $k = \varepsilon_i$ compute $T^{\varepsilon_i}(N_{\varepsilon_1}, \cdots, N_{\varepsilon_{i-1}})$ and complete it to a basis of $\mathcal{N}_r^{\varepsilon_i}$. This defines $N_{\varepsilon_i} \in \mathbb{R}^{(\varepsilon_i+1)n \times \rho_i}$.

The procedure is completed in μ steps and this leads to a set of state generators

$$\Omega_{\mathcal{X}}^{\mathcal{I}} = \left\{ N_{\varepsilon_i}, N_{\varepsilon_i} \in \mathbb{R}^{(\varepsilon_i+1)n \times \rho_i}, i \in \underset{\sim}{\mu} \right\}.$$

Part (III): Computation of the OMBM of $\mathcal{X}$. Each of the matrices N_{ε_i} in 0

$\Omega_{\mathcal{X}}^{\mathcal{I}} = \left\{ N_{\varepsilon_i}, N_{\varepsilon_i} \in \mathbb{R}^{(\varepsilon_i+1)n \times \rho_i}, i \in \underset{\sim}{\mu} \right\}$ can be partitioned as shown below:

$$N_{\varepsilon_i} = \begin{bmatrix} X_{\varepsilon_i}^i \\ X_{\varepsilon_{i-1}}^i \\ \vdots \\ X_1^i \\ X_0^i \end{bmatrix}, X_j^i \in \mathbb{R}^{n \times \rho_i}, \forall j = 0,1,\cdots,\varepsilon_i, \forall i \in \underset{\sim}{\mu}$$

and thus we define the OMBM by:

$$X(s) = \left[X_1(s); \cdots; X_\mu(s) \right], \quad X_i(s) = s^{\varepsilon_i} X_{\varepsilon_i}^i + \cdots + s X_1^i + X_0^i \in \mathbb{R}^{n \times \rho_i}[s].$$

2.3.3.2 Direct method for computing a minimal basis for the right null space $[sI - A, -B]$

Opposite to the indirect method, in this method we will consider the direct method introduced by Karcanis in his published paper in 2002 [7], which although it involves larger dimension pencil, $[sI - A, -B]$ instead of $sN - NA$, it has the more convenient form, which avoid Toeplitz matrix computation.

Let $(X(s), U(s))$ be a pair of polynomial matrices such that:

$$(sI - A)X(s) = BU(s) \tag{2.3.13}$$

where $X(s) \in \mathbb{R}^{n \times q}(s), U(s) \in \mathbb{R}^{p \times q}(s)$, and the scalar degree of $U(s)$ is k and thus that of $X(s)$ is $k-1$. We may call such pair as (k, q)-order right pair and write them as

$$X(s) = X_0^k + s X_1^k + \cdots + s^{k-1} X_{k-1}^k \tag{2.3.14}$$

$$U(s) = U_0^k + s U_1^k + \cdots + s^{k-1} U_{k-1}^k + s^k U_k^k. \tag{2.3.15}$$

By substituting these two in equation (2.3.13), we will get the following equation

$$
\underbrace{\begin{bmatrix}
-I & 0 & 0 & 0 & & & & & & \\
A & B & -I & 0 & 0 & 0 & & & & \\
0 & 0 & A & B & -I & 0 & & & & \\
& & & & \ddots & & & & & \\
& & & & & \ddots & & & & \\
& & & & & & \ddots & & & \\
& & & & & & 0 & 0 & 0 & 0 \\
& & & & & & A & B & -I & 0 \\
& & & & & & 0 & 0 & A & B
\end{bmatrix}}_{=T_k(A,B)}
\underbrace{\begin{bmatrix}
0 \\
U_k^k \\
X_{k-1}^k \\
U_{k-1}^k \\
X_{k-2}^k \\
U_{k-2}^k \\
\vdots \\
X_1^k \\
U_1^k \\
X_0^k \\
U_0^k
\end{bmatrix}}_{=Z_k}
= 0.
\tag{2.3.16}
$$

The Toeplitz matrix $T_k(A,B)$ characterizes the minimal bases. This equation even can be made simpler which it has been made simplified during the work done by Karcanias (2002) [7]. By some more calculations he shows that the whole solution can be done in the easier way as the following proposition:

Proposition 2.3. A (k,q)-order right pair of $S(A,B)$, for which $\rho(B)=p$, described as:

$$X(s) = X_0^k + sX_1^k + \cdots + s^{k-1}X_{k-1}^k$$

$$U(s) = U_0^k + sU_1^k + \cdots + s^{k-1}U_{k-1}^k + s^k U_k^k.$$

is defined from the set $\left\{U_0^k, U_1^k, \ldots, U_k^k\right\}$ which is the solution of the equation

$$
\underbrace{\left[A^k B, A^{k-1}B, \ldots, AB, B\right]}_{=Q_k(A,B)}
\underbrace{\begin{bmatrix}
U_k^k \\
U_{k-1}^k \\
\vdots \\
U_1^k \\
U_0^k
\end{bmatrix}}_{=\hat{U}_k}
= 0.
\tag{2.3.17}
$$

Proof: Refer to [7] for full details. ∎

For any solution of above equation the parameters $\left[X_0^k, X_1^k, \ldots, X_{k-1}^k\right]$ are defined by

$$
\hat{X}_{k-1} = \begin{bmatrix} X_{k-1}^k \\ U_{k-2}^k \\ \vdots \\ U_1^k \\ U_0^k \end{bmatrix} = \begin{bmatrix} B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{k-2}B & A^{k-3}B & \cdots & B & 0 \\ A^{k-1}B & A^{k-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} U_k^k \\ U_{k-1}^k \\ \vdots \\ U_2^k \\ U_1^k \end{bmatrix}.
$$ (2.3.18)

$$= R_k(A, B)$$

The matrix that determines the overall solvability is the kth order controllability matrix of the system and it is defined by

$$Q_k(A, B) = \left[A^k B, A^{k-1}B, \ldots, AB, B \right]$$

Here is the algorithm for finding OMBMs of $\mathcal{Z}$:

Part (l): Computation of $\mathcal{I}(\mathcal{Z})$. For all the $k \geq 0$ compute rank $q_k = Q_k(A, B)$, $k = 0, 1, 2, \cdots$, and thus the sequence

$$C_r^*(A, B) = \left\{ q_k : \forall k \geq -2, q_k = rank\ Q_k(A, B), \forall k \geq 0 \text{ and } q_{-2} = p, q_{-1} = 0 \right\}.$$

By computing the gap sequence:

$$\tilde{\delta}_k = 2q_{k-1} - q_k - q_{k-2}, \forall k \geq 0$$ (2.3.19)

the distinct values of $\tilde{\varepsilon}_i$ are defined by those values of k for which $\tilde{\delta}_k > 0$ and the corresponding multiplicity ρ_i is the value of $\tilde{\delta}_k > 0$. This defines $\mathcal{I}(\mathcal{Z}) = \left\{ (\tilde{\varepsilon}_i, \rho_i), 1 \leq \tilde{\varepsilon}_1 \leq \ldots \leq \tilde{\varepsilon}_\mu, i \in \underset{\sim}{\mu} \right\}.$

Part(II): Computation of a set of state generator $\Omega_{\mathcal{U}}^{\tilde{\mathcal{I}}}$. Having

$\mathcal{I}(\mathcal{Z}) = \left\{ (\tilde{\varepsilon}_i, \rho_i), 1 \leq \tilde{\varepsilon}_1 \leq \ldots \leq \tilde{\varepsilon}_{\underset{\sim}{\mu}}, i \in \underset{\sim}{\mu} \right\}$. the computation of a set $\Omega_{\mathcal{U}}^{\tilde{\mathcal{I}}} = \left\{ Q_{\tilde{\varepsilon}_i}, i \in \underset{\sim}{\mu} \right\}$

involves the following steps:

Step(1): For $k = \tilde{\varepsilon}_1$, compute any basis matrix for $\mathcal{Q}_{\tilde{\varepsilon}_1} = \mathcal{N}_r \left\{ Q_{\varepsilon_1}(A, B) \right\}$ and let this be

$Q_{\tilde{\varepsilon}_1} \in \mathbb{R}^{(\tilde{\varepsilon}_1 + 1) p \times \rho_1}$.

Step(2): For $k = \tilde{\varepsilon}_2$, compute $T_{\tilde{\varepsilon}_1}^{\tilde{\varepsilon}_2}(Q_{\tilde{\varepsilon}_1})$ and complete it to a basis for $\mathcal{Q}_r^{\tilde{\varepsilon}_2}$. This defines

$Q_{\tilde{\varepsilon}_2} \in \mathbb{R}^{(\tilde{\varepsilon}_2 + 1) p \times \rho_2}$.

Step(i) (General step): Let $\left\{ Q_{\tilde{\varepsilon}_1}, \cdots, Q_{\tilde{\varepsilon}_{i-1}} \right\}$ be the matrices defined by the previous step

for $k = \tilde{\varepsilon}_{i-1}$. For $k = \tilde{\varepsilon}_i$ compute $T^{\tilde{\varepsilon}_i}(Q_{\tilde{\varepsilon}_1}, \cdots, Q_{\tilde{\varepsilon}_{i-1}})$ and complete it to a basis of $\mathcal{Q}_r^{\tilde{\varepsilon}_i}$. This

defines $Q_{\tilde{\varepsilon}_i} \in \mathbb{R}^{(\tilde{\varepsilon}_i + 1) p \times \rho_i}$.

The procedure is completed in μ steps and this leads to a set of state generators

$\Omega_{\mathcal{U}}^{\tilde{\mathcal{I}}} = \left\{ Q_{\tilde{\varepsilon}_i}, Q_{\tilde{\varepsilon}_i} \in \mathbb{R}^{(\tilde{\varepsilon}_i + 1) p \times \rho_i}, i \in \underset{\sim}{\mu} \right\}$.

Part (III): Computation of a set of state generator $\Omega_{\mathcal{X}}^{\mathcal{I}}$. Having computed

$\Omega_{\mathcal{U}}^{\tilde{\mathcal{I}}} = \left\{ Q_{\tilde{\varepsilon}_i}, i \in \underset{\sim}{\mu} \right\}$ we may compute a set $\Omega_{\mathcal{X}}^{\mathcal{I}} = \left\{ N_{\varepsilon_i}, \varepsilon_i = \tilde{\varepsilon}_i - 1, \forall i \in \underset{\sim}{\mu} \right\}$.

Part (IV): Computation of the OMBM of $\mathcal{Z}$. Having computed a set of compatible

input, state generator $\Omega_{\mathcal{Z}}^{\mathcal{I}} = \left\{ (Q_{\tilde{\varepsilon}_i}, N_{\varepsilon_i}), \forall i \in \underset{\sim}{\mu} \right\}$, then we construct the associated

polynomial matrices $X_{\Omega}^{\mathcal{I}}(s), U_{\Omega}^{\mathcal{I}}(s)$ and $Z(s)_{\Omega}^{\mathcal{I}}(s) = [X_{\Omega}^{\mathcal{I}}(s)^t, U_{\Omega}^{\mathcal{I}}(s)^t]^t$ is an OMBM of

$\mathcal{Z}$.

2.4. Conclusion

During this chapter, the essential and required mathematical and control backgrounds reviewed and presented. One of the major applications of the mathematical notations such as condition number, SVD, Gram determinant will be

during the study of the measurement of the angle between subspaces in a direct sum decomposition when these will be used as measurement tools (Chapter Six).

In other side, the computation of the minimal basis of pencils $[sI - A, -B]$ nd $[sN - NA]$ leads to the computation of the controllability subspaces of the system defined by (A, B), which is strongly connected to the current presented thesis and will be Studied in chapter Four and Five.

CHAPTER 3

LITERATURE REVIEW ON EIGENSTRUCTURE ASSIGNMENT: BASIC CONCEPTS AND BACKGROUND RESULTS

3.1. Introduction

There has been a substantial amount of work performed in the field of control theory over the past three decades that examines the control of systems through the restructuring of the eigenvalues and eigenvectors, namely eigenstructure assignment. More recently, these techniques have been successfully applied to the control of flexible structures, especially in the area of enhancing modern flight control systems where existing systems are often hampered by the limitations exhibited by the classical control methods. The eigenstructure assignment problem therefore has a very important role to play in order to guarantee successful controller design in the sense of stability and robustness. It must be stated however, that eigenstructure assignment can only be carried out if the system is described by state space equations, which are made up from physical variables. In this case, it makes sense to impose conditions on the eigenframe which is linked to variables with a physical significance. The countenance of this thesis exhibits issues concerning robustness. A desired effect of a closed loop system is that its response is impervious to modelling errors and external disturbances. Close attention has to be paid to sensitivity minimisation and control system robustness. Therefore it is necessary to devise an algorithm that reduces the sensitivity of the closed loop eigenvalues to such undesired features.

In view of the problems of stability and robustness that arise in an open loop configuration, it may be necessary to reassign, or shift, certain modes and reshape

the eigenframe of a system by [4],[36] implementing some kind of feedback, so as to improve the dynamical response and properties of the system. This chapter will start off by a brief review of the basics related to rectilinear motions and so *A*- and *A,B*-Invariant subspaces. We should point out to the fact that due to the fact that (*A,B*)-invariant subspaces hold controllability property, a study on the concept of (*A,B*)-invariant subspaces has been presented in next chapter and so we will review only some of the basics and properties of these two subspaces and will reserve the main discussion on these subspaces for later. The main focus in this chapter will be on the background on eigenvalues and eigenvectors, especially the relationship with rectilinear motions. The theoretical analysis will then go on to examine the notion of transmission subspaces, and the association of closed loop eigenvalues with feedback. Finally there will be a review of the results in the literature concerned with methods of assigning the eigenstructure of a system.

3.2. Background on Eigenvalues and Eigenvectors

3.2.1 Rectilinear motions

To begin with, it will be necessary to examine the theory related to rectilinear motions in the state space for free motions, which is primarily concerned with the internal workings of a linear system. Subspaces of the state space that are of a one dimensional nature which have the property of retaining any free motion for every $t \geq 0$, are in fact the eigenvectors of the dynamic map *A*. The corresponding motions are of the exponential type $e^{\lambda t} \underline{x}(0)$, where λ is the eigenvalue related to the corresponding eigenvector. Such motions are called *rectilinear*. The ensuing problem is thus to restrict the free motion in a one-dimensional subspace with a view to find the pairs of a vector and a frequency satisfying the eigenvalue-eigenvector relationship.

The problem of keeping the state trajectory of a linear system within a given subspace of the state space is of great importance in a number of control problems.

This section will concentrate on the restriction of the free motion in a given subspace, and will begin with by stating the following theorem.

Theorem 3.1: [34], [35] Let $\mathcal{S}(A,\,B,\,C,\,D)$ be a linear system and $\mathcal{V}$ an *r*-dimensional subspace of the state space $\mathcal{X}$. A necessary and sufficient condition for the free motion part of the state trajectory $\underline{x}(t)$ to be kept within $\mathcal{V}$ $\forall t \geq 0$ whenever the state is released from any initial condition $\underline{x}(0) \in \mathcal{V}$ is

(i)   For every trajectory $\underline{x}(t) \in \mathcal{V}$ there exists another trajectory $\underline{x}'(t) \in \mathcal{V}$ such that

$$A\underline{x}(t) = \underline{x}'(t) \quad \forall t \geq 0 \tag{3.2.1.a}$$

(ii) $A\mathcal{V} \in \mathcal{V}$ (3.2.1.b)

Proof: Refer to [34] for full details.   ∎

The subspace $\mathcal{V}$ satisfying the above conditions is called an *A*-invariant subspace. The above theorem provides links with the fundamental notion of rectilinear motions [36]. We will study the motion along eigenvectors and the *A*- invariant subspaces of a linear system more in depth within Chapter Four (refer to section 4.2). *A*-invariance is strongly linked to the study of the problem of restricting the free motion of a system inside a subspace $\mathcal{V}$ for any initial condition $\underline{x}_0 \in \mathcal{V}$ [36].

Such subspaces are also linked to the zero input problem whilst the state and output trajectories are rectilinear. This can be illustrated in the following diagram. Both $\underline{x}(t)$ and $\underline{x}(0)$ exist within the subspace $\mathcal{V}$. But what happens to the frequencies and their associated rectilinear motions when $\underline{u}(t) \neq 0$? This is where *A*-invariance is extended to (*A,B*)-invariance, and will be dealt with later in this chapter and more precisely within next chapter.

$$x(0) = u$$

$$u(t) = 0$$

S

$$\underline{y}(t) = \underline{y}_0 e^{\lambda t}$$

$$x(t) = x(0)e^{\lambda t}$$
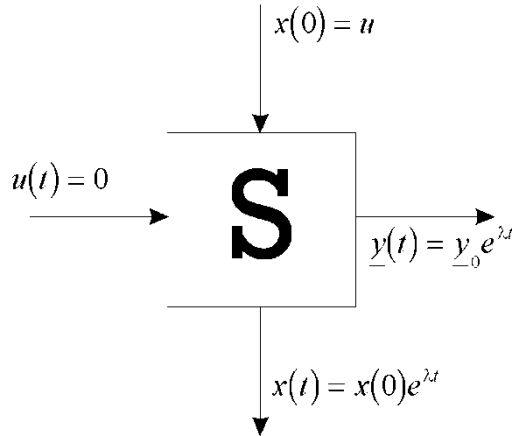
Figure 3.1: Zero input problem

The above notions have shown that the free motion of a system starting from an initial condition is called rectilinear, which is in fact a motion along an eigenvector. The frequency corresponding to this motion is called an eigenvalue. *A*-invariance is a condition for the free motion part of the trajectory to be kept within the boundaries of a certain subspace when released from an initial point. The definition of *A*-invariance is given by equation (3.2.1).

3.2.2 Summary of spectral characterisation

As a recollection from earlier, an eigenvector $\underline{u}_i$ that corresponds to an eigenvalue λ_i is a nontrivial solution of

$$[\lambda_i I_n - A]\underline{u}_i = 0 \tag{3.2.3}$$

The *spectral decomposition* of A in the case of distinct eigenvalues is of the form

$$A = U\Lambda V$$
$$VA = \Lambda V \tag{3.2.4}$$

where U is the matrix of eigenvectors and $V = U^{-1}$ is the matrix of dual eigenvectors and $\Lambda = \mathrm{diag}(\lambda_i)$. If $\mathcal{B}$ and $\mathcal{B}'$ represent the eigenbasis and dual eigenbasis described by $\{\underline{u}_1, \ldots, \underline{u}_n\}$ and $\{\underline{v}_1, \ldots, \underline{v}_n\}$ respectively, then

$$VU = \begin{bmatrix} \underline{v}_1^t \\ \vdots \\ \underline{v}_n^t \end{bmatrix} [\underline{u}_1, \quad \ldots, \quad \underline{u}_n] = I_n$$

(3.2.5).

$$\Rightarrow \underline{v}_i^t \underline{u}_j = \delta_{ij}$$

Consider an $n \times n$ linear multivariable system, described by the following state space model

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x}$$

(3.2.6)

The system transfer function matrix is given by

$$G(s) = C(sI_n - A)^{-1}B$$

(3.2.7)

If U and V satisfy condition (3.2.4), and A is of simple structure and $\Lambda = \text{diag}(\lambda_i)$, the transfer function matrix can be expressed in the *dyadic* form below

$$G(s) = \sum_{i=1}^{n} \frac{C\underline{u}_i \underline{v}_i^t B}{s - \lambda_i}$$

(3.2.8)

As can be seen from equation (3.2.8), eigenvalues, eigenvectors and dual eigenvectors have an important role to play in the formulation of the system transfer function.

3.2.3 Controllability and Observability issues

One of the issues arising frequently in the area of control design is related to the problems concerning controllability and observability, so a link between these two qualitative properties and the eigenstructure of a system has to be established. Take the system described by equation (3.2.6), where A has distinct eigenvalues, and the modes of interest are λ_i, $\underline{u}_i$ and $\underline{v}_i^t$. Accroding to [37], the complete mode $\left(\lambda_i, \quad \underline{u}_i, \quad \underline{v}_i^t \right)$ is uncontrollable if

$$\underline{\beta}_i^t = \underline{v}_i^t B = 0$$
$$\underline{v}_i^t (\lambda_i I_n - A) = 0$$

(3.2.9)

The mode $\left(\lambda_i, \quad \underline{u}_i, \quad \underline{v}_i^t \right)$ is unobservable if

$$\underline{\gamma}_i = C\underline{u}_i = 0$$
$$(\lambda_i I_n - A)\underline{u}_i = 0$$
(3.2.10)

A mode $\left(\lambda_i, \quad \underline{u}_i, \quad \underline{v}_i^t \right)$ is said to be:

♦ Controllable and observable if $\underline{\beta}_i^t \neq 0$ and $\underline{\gamma}_i \neq 0$

♦ Controllable and unobservable if $\underline{\beta}_i^t \neq 0$ and $\underline{\gamma}_i^t = 0$

♦ Uncontrollable and observable if $\underline{\beta}_i^t = 0$ and $\underline{\gamma}_i \neq 0$

♦ Uncontrollable and unobservable if $\underline{\beta}_i^t = 0$ and $\underline{\gamma}_i^t = 0$

The conditions $\underline{\beta}_i^t = \underline{v}_i^t B = 0$ and $\underline{\gamma}_i = C\underline{u}_i = 0$ provide the basis for a geometric characterisation of uncontrollability and unobservability. In fact condition (3.2.10) implies that the left eigenvector $\underline{v}_i$ satisfies the geometric condition

$$\underline{v}_i \in N_l(B) \equiv \mathcal{N}$$
(3.2.11)

Likewise, equation (3.2.10) implies that the right eigenvector satisfies the geometric condition

$$\underline{u}_j \in N_r(C) \equiv \mathcal{M}$$
(3.2.12)

The above geometric conditions are expressed as conditions on spaces and thus they may be used to provide measures of the "degree" on controllability and of observability by measuring the proximity of the left eigenvector to the $\mathcal{N}$ space and the proximity of the right eigenvector to the $\mathcal{M}$ space. Although controllability is invariant under state feedback and observability invariant under output injection [34], [36], [38], their respective degrees are not. Thus in shaping the closed loop eigenframe by feedback, the degree of controllability and observability due to positioning of the resulting closed loop eigenframes is an important indicator that can be considered as a design parameter.

3.3. Forced rectilinear motions and closed-Loop Eigenstructure

3.3.1 Physical problems

In Subsection 3.2.1, the problem of rectilinear motions for zero input conditions was examined. An extension of this problem can be stated as follows [4], [98]:

Problem 3.1: [36] Given the system $\mathcal{S}(A, B, C, D)$ and a subspace of $\mathcal{X}$, $\mathcal{V}$, find under what conditions, for any $\underline{x}_0 \in \mathcal{V}$ there exists a control input which restricts the state trajectory in $\mathcal{V}$, $\forall t \geq 0$.

Here, the case when $\underline{u}(t) \neq 0$ will be looked at. So the question that must be posed is whether the rectilinear motion problem can be extended to forced systems or not, i.e. when $\underline{u}(t) \neq 0$. In order to examine this, it is necessary to make use of the input-state pencil [36]

$$\begin{bmatrix} sI_n - A, & -B \end{bmatrix} \begin{bmatrix} \hat{\underline{x}} \\ \hat{\underline{u}} \end{bmatrix} = \underline{x}_0 \tag{3.3.1}$$

From equation (3.3.1), the input-state pencil is given by $\mathcal{C}(s) = [sI_n - A, -B]$, and is used to help describing the coupling between the input and the state. Taking into consideration the initial condition $\underline{x}(0) = \underline{x}_0$, and the system description of equation (3.2.6), the problem of forced rectilinear motions can be formulated as follows

Problem 3.2: [36] Is it possible to find a specific $\underline{x}_0$ and $\underline{u}(t)$ such that $\underline{x}(t) = e^{\lambda_i t} \underline{x}_0$, $\forall t \geq 0$, for some $\lambda_i \in \mathcal{C}$?

In order to tackle this problem, it is necessary to look back at Section 3.2, where the study of A-invariant subspaces and rectilinear motions within them was introduced. For the case of forced systems a more general situation arises. Apart from the internal mechanism characterised by the A matrix, and the way it is coupled to the environment via the output map C, the way in which the outside is coupled to the system via the input map B is taken into consideration. Thus the initial concept of A-invariance is now extended to (A,B)-invariance. Unlike the case of A-invariant

subspaces, (A,B)-invariant subspaces may not be described in terms of a frequency only, and this is associated with a *generalised* eigenvalue-eigenvector problem. As we have mentioned before, more in detailed properties of (A,B)-invariant subspaces will be give within next chapter.

3.3.2 Characterisation of Transmission

The difference between the frequency and vector correspondence for the two cases of A and (A,B)-invariance can be summarised in the following way. The spectrum $\sigma\left[\{\underline{x}_0\}\right] = s_0$ can uniquely characterise a 1-dimensional A-invariant subspace $\{\underline{x}_0\}$. Each spectral frequency s_0 has a unique characteristic vector $\underline{w}[s_0] = \underline{x}_0$. For (A,B)-invariant cases, each subspace $\{\underline{x}_0\}$ (for $\{\underline{x}_0\} \cap \mathcal{B} = \underline{0}$) is uniquely characterised by a generalised spectral frequency s_0, but unlike A-invariance, there is no unique corresponding characteristic vector $\underline{x}_0$. Any vector x_0 satisfying

$$N\left(sI - A\right)\underline{x}_0 = 0$$
$$NB = 0$$

(3.3.2)

where N is a basis matrix for $N_l\left(B\right)$, is (A, B)-invariant and is uniquely characterised by s_0. However, equation (3.3.2) has more than one solution for $\underline{x}_0$. In order to be able to distinguish between the correspondence of frequencies and characteristic subspaces for the two cases of A- and (A,B)-invariance it is necessary to introduce concepts relating to the frequency transmission through forced systems.

The first concept is the *transmission subspace* of s_0, $\mathcal{T}(s_0)$ [35], to be the subspace spanned by the totality of the solutions of $\underline{x}_0$ for the same frequency s_0. The second concept is that the frequency s_0 corresponding to $\mathcal{T}(s_0)$ is called the *frequency content* of the frequency subspace.

The concept of $\mathcal{T}(s_0)$ is quite an important one. In order for the successful transmission of a particular frequency s_0, the initial condition x_0 and the associated trajectory $\underline{x}(t)$ must remain within $\mathcal{T}(s_0)$. Furthermore, because the transmission

subspace is uniquely characterised by a frequency, rectilinear motions sustained in any subspace of $\mathcal{T}(s_0)$ will enable the transmission of the frequency s_0 only. It must be noted that these statements only hold true for (A,B)-invariant subspaces that do not intersect $\mathcal{B}$.

Remark 3.1: [35] An (A,B)-invariant subspace that intersects with $\mathcal{B}$ has part of its spectrum arbitrarily assignable and contains a controllability subspace.

Proposition 3.1: [35] All transmission subspaces of a system $\mathcal{S}(A, B)$, where A and B are of sizes $n{\times}n$ and $n{\times}l$ respectively, for which $l > n/2$ have an intersection with $\mathcal{B}$. Otherwise, when $l \leq n/2$, then such an intersection generally does not exist.

Proof: Before looking at a way to compute the transmission subspace, it is necessary to look at its physical significance with respect to frequency propagation. The transmission of the frequency s_0 only takes place in subspaces of the transmission subspace $\mathcal{T}(s_0)$. Conversely, every subspace of $\mathcal{T}(s_0)$ only allows the transmission of the frequency s_0. This focuses attention solely on the behaviour of the state vector without taking into consideration the type of input vector required to initialise a frequency transmission. This can be justified by looking at equation (3.3.2), where the existence of a solution for $\underline{x}_0$ immediately implies a solution for $\underline{u}_0$, which is indicated by

$$\underline{u}_0 = B^+ \left(s_0 I - A \right) \underline{x}_0 \tag{3.3.3}$$

where $\left(B^+ B = I_l \right)$. An interesting exercise would be to identify the particular subspace in the input space $\mathcal{U}$ from which $\mathcal{T}(s_0)$ in $\mathcal{X}$ may be reached. The subspace in $\mathcal{U}$ is defined as the *input transmission subspace*, and is denoted by $\mathcal{T}_u(s_0)$.

■

With this in mind, the following proposition can be made:

Proposition 3.2: [35] $\mathcal{T}_u(s_0)$ coincides with the whole input space $\mathcal{U}$ for all frequencies if $\mathcal{B}$ does not intersect with any $\mathcal{T}(s_0)$. If there is an intersection, then

the same applies to all frequencies s_0 again, except for those that belong to the controllable part of the spectrum of A.

Proof: All vectors in $\mathcal{T}(s_0)$ for any frequency s_0 which is not an eigenvalue of A are given as

$$\underline{x}_0 = (s_0 I - A)^{-1} B \underline{u}_0 \tag{3.3.4}$$

From the above condition, any vector $\underline{u}_0$ leads to a vector $\underline{x}_0 \in \mathcal{T}(s_0)$. But if $s_0 = \lambda_i$, $\lambda_i \in \sigma(A)$,

$$A \underline{x}_0 = \lambda_i \underline{x}_0 - B \underline{u}_0 \tag{3.3.5}$$

when this is projected onto the eigenframe of A, the following condition arises

$$\underline{v}_i^t B \underline{u}_0 = 0 \tag{3.3.6}$$

where $\underline{v}_i^t$ is the left eigenvector of A corresponding to the eigenvalue λ_i. Thus it is still possible to use any vector as so long as $\underline{v}_i^t B = 0$, that is $\mathcal{T}_u(\lambda_i) = \mathcal{U}$ if λ_i is an uncontrollable mode. If $\underline{v}_i^t B \neq 0$, then $\underline{u}_0$ may not assume values for which $B\underline{u}_0 \in \{\underline{w}_i\}$, where $\underline{w}_i$ is the eigenvector of A that corresponds to λ_i.

Equation (3.3.6) gives the totality of vector solutions for $\underline{x}_0$, where $\underline{x}_0 \in \mathcal{T}(s_0)$ for any frequency s_0 such that s_0 is not in the spectrum of A, i.e. $s_0 \notin \sigma(A)$. Therefore $\mathcal{T}(s_0)$ can be expressed as

$$\mathcal{T}(s_0) = \text{range}\left\{(s_0 I - A)^{-1} B\right\} \tag{3.3.7}$$

∎

Remark 3.2: [35] For the general case, $s_0 \in \mathcal{C}$, where $s_0 \in \sigma(A)$, the transmission subspace is defined as the $\underline{x}_0$ vector solutions of $(s_0 N - NA)\underline{x}_0 = 0$. ∎

3.3.3. Feedback and closed-loop Eigenvalues

The transmission of the frequency s_0 is generally affected from any input $\underline{u}_0$ in the input space $\mathcal{U}$. However it may only be propagated along a direction belonging to a given subspace of the output space $\mathcal{Y}$. It is required that such transmissions are only

possible if the state vector is restricted to the transmission subspace $\mathcal{T}(s_0)$, and that the ensuing trajectories in the input, state and output spaces are all of the rectilinear type. The rectilinear motions in $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ all share the same frequency s_0. The need for an external excitation in the form of a controlled input $\underline{u}(t)$ could be eliminated by deploying suitable feedback connections from either the states or the outputs back to the inputs. Therefore applying an appropriate state feedback operator K_s, or output feedback operator K_0 such that

$$K_s \underline{x}_0 = \underline{u}_0 \tag{3.3.8}$$

or

$$K_o \underline{y}_0 = \underline{u}_0 \tag{3.3.9}$$

it is possible to generate the control input $\underline{u}(t)$ needed to sustain the rectilinear motions by closing the loops around the system $\mathcal{S}$ in the manner indicated below:
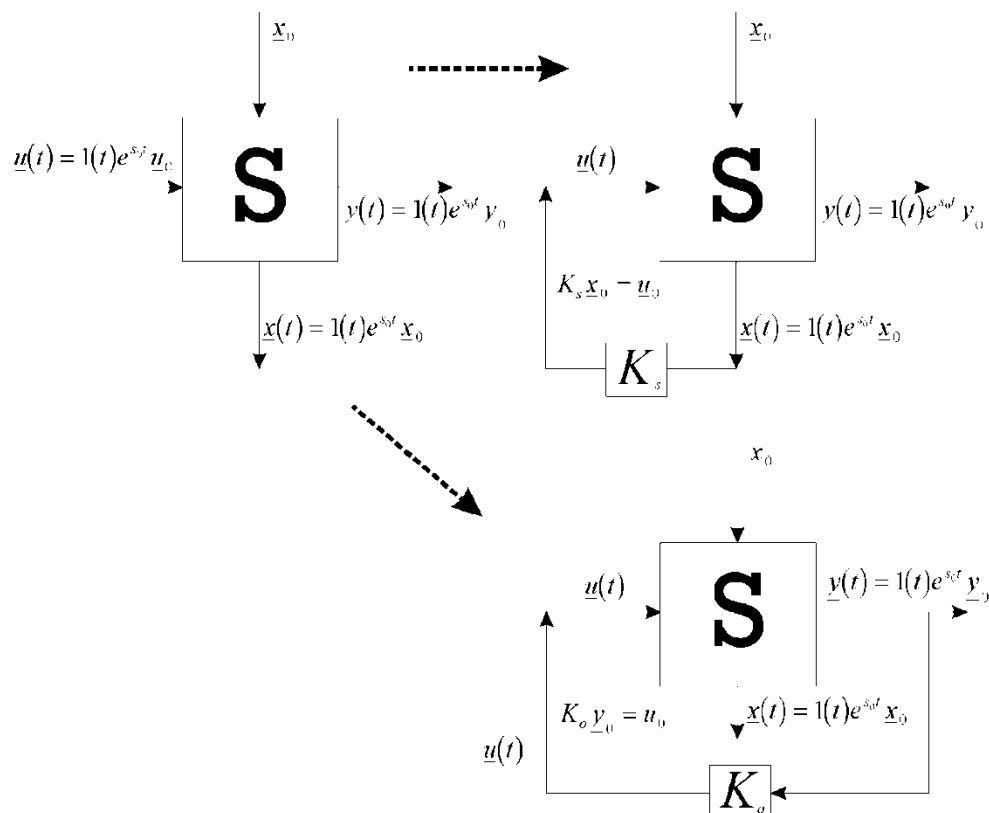
Figure 3.2: Feedback systems

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The top right of the diagram shows a state feedback configuration, and the bottom part shows an output feedback one. The actual physical interpretations of these diagrams do not need an external input, and can be considered as free responding systems. The associated problem can be linked to restricting the state vector (and output vector) of an autonomous system. So now the vector $\underline{x}_0$ which originally was seen as a member of a transmission subspace becomes a member of a closed-loop eigenspace. Then, it can be shown that

$$\left(s_0 I - A - B K_s\right)\underline{x}_0 = 0 \tag{3.3.10}$$

$$\left(s_0 - A - B K_o C\right)\underline{x}_0 = 0 \tag{3.3.11}$$

are obtained. $\left(A + B K_s\right)$ and $\left(A + B K_o C\right)$ are the closed-loop state matrices under state and output feedback respectively. These play a huge part in the problem of the placement of closed-loop poles and eigenvectors.

3.3.4. The problem of eigenspace assignment

An adequate way of summarising the above subsection would be to say that frequency transmissions along (*A*,*B*)-invariant directions could be self generated by the utilisation of an appropriate family of feedback operators connecting the states (or outputs) back to the inputs. Therefore, rectilinear motions of the type $\exp(s_0 t)\underline{x}_0$ stimulated by an input $\exp(s_0 t)\underline{u}_0$ could be made to be self perpetuating if the input signal was generated by a combination of the state (or output) variables and the action of a feedback operator. To keep things simple, only state feedback will be considered, such that $K_s \underline{x}_0 = \underline{u}_0$.

It has been documented that any motion in a general *r*-dimensional (*A*,*B*)-invariant subspace may be broken into a number of simple and higher order rectilinear motions, each linked to a specific spectral frequency s_{0_i} that take place along the generalised eigenspace defined by the vectors $\left[\underline{x}_{0_i}^0, \quad \underline{x}_{0_i}^1, \quad \ldots, \quad \underline{x}_{0_i}^\mu\right]$. Such motions are sustained by inputs that consist of rectilinear type components, of which each are associated with one particular frequency s_{0_i}. These frequencies take place along the

input characteristic vectors, defined by $\left[\underline{u}_{0_i}^0, \quad \underline{u}_{0_i}^1, \quad \ldots, \quad \underline{u}_{0_i}^\mu\right]$. Therefore the state feedback law of (3.3.8) may be restated in order to satisfy the conditions of r-dimensional (A,B)-invariant subspaces as follows

$$K_s \underline{x}_0^j = \underline{u}_{0_0}^j \tag{3.3.12}$$

and in matrix form

$$K_s V_0 = U_0 \tag{3.3.13}$$

The action of the matrix K_s as a feedback operator has already been illustrated in Figure 3.2. The diagram shows how the restriction of the state trajectory $\underline{x}(t) \in \mathcal{V}$ can be achieved by a closed-loop system without the necessity of a control input $\underline{u}(t)$. Thus the concept of (A,B)-invariance can be extended to $(A+BK_s)$-invariance, which leads onto the problem of eigenspace assignment. The equivalence between the two can be investigated by first considering a derivation of equation $A\underline{x}_0 = s_0 \underline{x}_0 - B\underline{u}_0$ which is needed to be solved in order for $\underline{x}_0$ to be found [36],

$$AV_0 = V_0 J_R - BU_0 \tag{3.3.14}$$

where J_R is the Jordan block diagonal canonical form of $\Lambda_R = \text{diag}\{s_{0_i}\}$. If U_0 from (3.3.13) is substituted into (3.3.14), then

$$\left(A + BK_s\right)V_0 = V_0 J_R \tag{3.3.15}$$

which in turn can be expressed in vector space notation by

$$(A + BK_s)\mathcal{V} \subset \mathcal{V} \tag{3.3.16}$$

The following theorem states under what circumstances the assignment of an eigenspace can be considered:

Theorem 3.2: [36] The sufficient and necessary condition for the assignability of a given vector as a closed-loop eigenvector is that it belongs to a transmission subspace, of which the frequency content designates the corresponding closed loop eigenvalue. ∎

With this in mind, the general form of the eigenstructure assignment problem can be formulated as follows:

Problem 3.3: Given the system $\mathcal{S}(A,B)$, find a set of independent vectors associated with the frequencies $\{\lambda_i\}$, $i = 1, \ldots, r$, find an appropriate feedback operator (either K_s for state feedback or K_0 for output feedback) that makes the frequencies λ_i closed-loop eigenvalues, and the corresponding closed loop eigenvectors while at the same time the resulting eigenframe satisfies some given properties.

Basically the point of eigenstructure assignment is to shift certain undesirable characteristic frequencies to new locations and to exercise some control over the resulting eigenvectors. The latter, in tandem with the input and output maps *B* and *C* respectively, are vital for the problem of well conditioning controllability and observability properties. It is well known that the controllability and observability properties have certain invariance properties under feedback/output injection as stated below.

Theorem 3.3: [13], [36] Given the system $\mathcal{S}(A,B,C,D)$, the following hold true:
  (i)   The controllability properties are invariant under state feedback.
  (ii)   The observability properties are invariant under output injection.

■

The above suggests that state feedback cannot make a controllable system uncontrollable, but it can affect the degrees of controllability when these are suitably defined. However, state feedback can make the system unobservable, if the system has zeros and a suitable feedback is selected [36],[40]. Similar arguments can be made for the output injection. Thus the general eigenstructure assignment involves a simultaneous selection of a suitable closed loop set of frequencies and a suitable eigenframe that can guarantee some additional properties.

More details on the above argument will be given in next chapter, where by using the properties of *(A,B)*- Invariant subspaces, necessary conditions for the selection of a suitable state feedback has been presented based on open-loop/ closed-loop spectra, in order for eigenstructurs to be assigned.

In the next section, a review of some of the literature dealing with some of the methods formulated to tackle the problem of eigenstructure assignment is made.

3.4. Review of results on Eigenstructure assignment

3.4.1 Early results

The progression of work done in formulating methods that attempt to solve the eigenstructure assignment problem will now be reviewed. The response of a control system is largely dependent on its eigenvalues and eigenvectors, namely its eigenstructure. The eigenvalue assignment problem was first addressed by Wonham [39] in 1967. The author proved that a system was controllable if and only if state feedback could be applied and calculated so as to make the newly formed closed loop system have an arbitrary set of self-conjugate scalars as its poles.

Since this paper, there have been hundreds of publications dedicated to the subject of pole placement and its applications, which go on to discuss the assignment of eigenvectors as well. A handful set has been selected in order to give an insight into some of the methodologies that have been developed for both output and state feedback cases.

The problem of using eigenvectors and assigning them was first considered by Karcanias [4], [98] and was used by Shaked and Karcanias [40] as part of the wider issues of model reduction of linear systems. The problem was also defined independently by Moore (1976) [99]. The aim of their work was to find a state feedback matrix such that the closed loop system had the maximum number of eigenvectors possible in the kernel of the output matrix *C*. An algorithm was developed whereby the maximum number of newly assigned eigenvectors, which corresponded to stable modes, lay in the kernel of *C*. This meant that the maximum possible number of stable modes became unobservable. This took advantage of the fact that the observability properties of a system are not invariant under state feedback.

At around the same time, Moore [41] established the fact that state feedback could be used to assign the closed loop system and desired self -conjugate set of eigenvalues, if and only if the open loop system was controllable.

The purpose of his paper was to identify the freedom (other than the choice of eigenvalues to be assigned) offered by state feedback. It was shown that the freedom available was a choice of one particular set from the class of "allowable" sets of closed loop eigenvectors. Porter and D'Azzo [42] presented a set of results for closed loop eigenstructure assignment by state feedback in multivariable linear systems which took advantage of the freedom available due to the pole placement method by Moore [41]. The results provided a method for the direct computation of the state feedback matrix which can be used to assign prescribed Jordan canonical forms, eigenvectors and generalised eigenvectors to the plant matrices of closed loop systems.

Also it is pointed out that even in the case of systems for which the pair (A,B) is uncontrollable, certain prescribed eigenvectors of the feedback system $(A + BK_s)$ can be assigned by state feedback. In the case of systems with asymptotically stable but uncontrollable modes, they state that it is often possible to achieve significant improvements in the dynamical behaviour of such systems by the introduction of appropriate state feedback controllers. The results from this paper led to a further development by Porter and D'Azzo [43]. The algorithm presented is based along solving

$$\left[A + BK_s - \lambda_i I\right]\underline{u}_i = 0 \tag{3.4.1}$$

for K_s by arbitrarily assigning a vector ω_i to find the set of eigenvectors $\underline{u}_i$ which satisfy the relationship $K_s\underline{u}_i = \omega_i$. The nature of the computations is simple due to the case of the elementary column operations involved.

The early results of eigenstructure assignment described here pioneered further investigations into this novel control problem. These early studies opened a new channel in control design that steered away from standard classical techniques

(second order PID controllers) to allow more complex feedback controllers to be designed and implemented.

3.4.2. State feedback results

The poles of a system are also the roots of the characteristic equation that gives rise to the eigenvalues of a system. Therefore the term "pole-shifting" means the same as relocating the eigenvalues of a system to obtain improved behavioural patterns. In view of this, Retallack and MacFarlane [44] derived a straightforward state feedback pole-shifting algorithm, which relates the open and closed loop characteristic frequencies of multivariable feedback systems to the Bode return difference of the system.

The useful algorithm developed provided an interesting link between state-space and transfer function matrix representations in the treatment of pole shifting. Although many algorithms exist for the solution of the pole placement problem using state feedback, it can generally be concluded that most of them are numerically unstable, yet the paper by Minimis and Paige [45] attempted to prove that their algorithm was numerically stable.

They suggested a direct algorithm for the computation of the linear state feedback matrix for multi-input systems such that the resultant closed-loop system matrix has specified eigenvalues. This method has the added advantage of an extra degree of freedom which can be used in different ways, for example to decrease some norm of the feedback matrix and hence the control effort or to improve the condition of some eigenvalues of the closed loop matrix. The algorithm devised uses unitary transformations for numerical reliability, and its stability results from the use of explicit shifting for the allocation of each eigenvalue.

Another numerically stable and efficient computational algorithm for pole assignment of linear multi-input systems was proposed by Petkov et al. [46]. The preliminary stage of the algorithm involves the reduction of the state matrices into an orthogonal transformation of the closed loop system matrix into an upper quasi-

triangular form whose diagonal blocks correspond to the desired poles. The computed gain matrix, due to its numerical stability, is also exact for a system with slightly perturbed matrices. It works equally well with real and complex, distinct and multiple poles and is also applicable to ill-conditioned and high order problems.

The problem with using state feedback is that the states of a system are not always readily available. This creates the problem of the inability of the designer to shift all the states of a system. This is where output feedback has an advantage, where the states can be fed back as functions of the output.

3.4.3. Output feedback results

In 1978, Porter and Bradshaw [47] derived a method for entire eigenstructure assignment which was applicable to the design of multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. The method was illustrated by designing an error-actuated dynamic controller which caused the output of a second order continuous time plant to track a constant command input in the presence of an unmeasurable constant disturbance input. The feedback matrix K_0 is solved using the eigenvalue-eigenvector relationship

$$\left[A + BK_o C - \lambda_i I\right]\underline{u}_i = 0 \qquad\qquad (3.4.2)$$

where A, B and C are the state, input and output matrices respectively. λ_i represents the eigenvalues to be assigned, and $\underline{u}_i$ is the corresponding eigenvector set of the new system.

A new approach was developed by Alexandridis and Parakevopoulos [48], which identifies the eigenspaces for the desired set of all the closed loop eigenvalues. In order for the desired set of eigenvalues to be successfully assigned, necessary and sufficient conditions are established and met.

The proposed approach is based on the idea of breaking down the problem of the output feedback pole assignment in the following two steps. In the first step, an expression for K_0 is derived which relates the output feedback gain matrix to the

eigenstructure assignment for the set Λ_1 of the closed loop eigenvectors. In the second step, the remaining closed loop eigenvectors are assigned to be in the set Λ_2 without affecting the assignment of the first set of Λ_1 eigenvalues.

The problem of determining the free parameters in K_0 either to a bilinear system of real algebraic equations in the general case or to a linear system is achieved by algebraic manipulations. Sobel et al [49] also presented a comprehensive use of eigenstructure assignment design methodology using output feedback. The implementation of their technique is applicable to the design of advanced flight control systems. Their method enables the designer to satisfy damping, settling time and mode decoupling specifications by directly choosing the eigenvalues and eigenvectors. They also tackle the problem of eigenvalue sensitivity, which arises due to the incremental change in the eigenvalues as a result of incremental changes in the stability of the aircraft and control derivatives. Duan [50] proposed a simple and effective algorithm for robust pole assignment in multivariable linear systems via output feedback.

The presented method gives a robust solution in the sense that the closed loop eigenvalues are as insensitive as possible to perturbations in the system coefficient matrices. The solution to the problem involves three steps, the first of which is aimed at trying to find a proper eigenvalue sensitivity index. The second step involves stating the freedom of the control system and in the final step, the freedom of the system is optimised by minimising the proposed eigenvalue sensitivity index.

The eigenvalue sensitivity index can be described appropriately by the condition number of the eigenvector matrix of the closed loop system. The algorithm conveniently includes closed loop eigenvalues as optimising parameters and it also possesses stable numerical properties, as well as being fairly simple to implement. Kabamba and Longman [51] produced a note addressing the problem of the assignability of the eigenvalues of the matrix $A + BK_0C$ by the choice of the feedback matrix K_0. This mathematical problem corresponds to pole assignment in the direct

output feedback problem, and by proper changes of variables it also represents the pole assignment problem with dynamic feedback controllers.

The key to the solution presented by the authors is the introduction of the concept of local assignability which in loose terms is the arbitrary perturbability of the eigenvalues of $A + BK_0C$ by the perturbations of K_0. If n is the order of the system, they show that if $A + BK_0C$ has distinct eigenvalues, a necessary and sufficient condition for local complete assignability at K_o is that the matrices $C[A + BK_oC]^{i-1}$ be linearly independent for $1 \leq i \leq n$.

In special cases, this condition can be reduced to known criteria for controllability and observability. Although such properties are necessary conditions for assignability, the paper also addresses the question of assignability of uncontrollable and unobservable systems, both by direct output feedback and dynamic compensation. Fletcher et al [52] presented a set of necessary and sufficient conditions for closed loop eigenvector assignment by output feedback in time invariant linear multivariable control systems. The basis of the paper is a simple condition on a square matrix, which is necessary and sufficiently adequate for it to be the closed loop plant matrix of a given system.

It is employed to obtain a condition concerning the assignment of an eigenstructure consisting of the eigenvalues with a mixture of left and right eigenvectors. Thus their arguments suggest that the analysis of the closed loop eigenstructure should be carried out in terms of a mixture of left and right eigenvectors.

The disadvantage of the output feedback approach is that it is limited by a lack of degree of freedom. The output feedback matrix is restricted by the size of the output matrix, C, whereas state feedback is not. The nature of the control problem dictates whether state or output feedback is used.

3.4.4. Combined state and output feedback approach

An interesting result was produced by Lovass-Nagy et al [53] where the output feedback matrix can be calculated from knowledge of the state feedback matrix. A

method using matrix generalised inverses is developed for the computation of the matrix K_s (state feedback) such that the matrix $A + BK_s$ has prescribed eigenvalues which need satisfy only the condition that a certain number of them are distinct and real.

A feedback law of the form $\underline{u} = \underline{v} + K_s \underline{x}$ is used to achieve the desired eigenvalue placement. The method does not require the solution of sets of non-linear equations or manipulation of polynomial matrices, and no knowledge of the eigenvalues and/or the eigenvectors of A is necessary. If the computed matrix K_s and the given matrix C satisfy a consistency condition, then the output feedback matrix K_0 can be found from the relationship $K_o C = K_s$, and the desired eigenvalue placement can be realised by the output feedback law $\underline{u} = \underline{v} + K_o \underline{y}$.

This interesting result allows direct information of the state space to be used to calculate an output feedback controller. It is worth further investigation in order to check system responses that indicate just how valid the approach is.

3.4.5. Approach that reduces controllers complexity

A note dealing with the use of feedback to approximate the closed loop eigenstructure of a system to a prescribed set of values was proposed by Calvo-Ramon [54] in order to reduce the controller complexity based on eigenvalue sensitivity concepts. Output feedback is used to approximate the closed loop eigenstructure of the system to a desired set of values. The method is quite systematic and the design of a constrained output feedback system from a prescribed eigenstructure is well established. Residue analysis (based on left and right eigenvectors) is used to estimate the effect on the eigenvalues of constraints in the feedback gains. The numerical results show that some eigenvectors can be approximately preserved, although eigenvector sensitivities have not been considered. The main drawback of this method is that the eigenvector sensitivities are estimated, which may lead to inaccurate controller designs as stronger poles may be mistakenly overlooked.

3.4.6 Results obtained from a subspace theme

The problem with Wonham's [39] fundamental state feedback result is that in most practical situations the state is not available directly. Kwon and Youn [55] attempted to find a condition under which the system is eigenvalue assignable despite the system having incomplete state observation.

They presented a generalisation of an entire eigenstructure assignment method for linear time-invariant multivariable systems, without using assumptions and with the eigenvalues of the closed-loop system being distinct or different from any of the eigenvalues of the open-loop system. The presented method has sufficient conditions that show that the closed loop eigenstructure assignment by output feedback is constrained by the requirement that the generalised right and left eigenvectors lie in certain subspaces.

Following on from the subspace theme, Søgaard, Trostmann and Conrad [56] presented a method whereby all the residuals assignable by state feedback must be characterised geometrically in terms of subspaces. These subspaces are defined by the freely selectable closed loop eigenvalues. Any desired residual may be selected from these subspaces. The applicability of this result is complimented by the fact that basic control design objectives like I/O response and robustness can be expressed in terms of the residuals.

The approach here stimulates further analysis into the assignable spectra of controllability subspaces, and will be studied in greater detail in Chapter 8.

3.4.7. Parametric state feedback results

Roppenecker [57] derived an explicit parametric expression for the controller gain matrix of a linear state-variable feedback system. It is based on a modal analysis of the input control vector $\underline{u}(t)$ under linear state-variable feedback conditions. The parameterisation of the class of all state feedback controllers that assign a prescribed set of distinct eigenvalues was found in terms of certain parameter vectors which are functions of the gain matrix and the new eigenvectors to be derived.

The same algorithm, provided the prescribed eigenvalues are distinct and that the system is completely controllable, can always calculate the controller gain matrix. The method for deriving the controller parameters is also applicable to the case where all the open-loop eigenvalues are required to be shifted by an appropriate control action. Fahmy and O'Reilly [58] devised another parametric solution for closed-loop eigenstructure assignment via state feedback in a linear multivariable system with n states and r control inputs. This was achieved by introducing a lemma on the differentiation of the determinant of the matrix $\left[I_r - K_s (\lambda_i I_n - A)^{-1} B \right]$, the class of assignable eigenvectors and generalised eigenvectors associated with the assigned eigenvalues can be explicitly described by a set of free parameter vectors.

Fahmy and O'Reilly followed this up in another paper [59], where a general eigenstructure assignment (EA) problem for linear multivariable systems was formulated and solved within the framework of the parametric eigenstructure assignment methodology derived earlier [58]. It was shown that EA control is achievable by means of a family of classes of state feedback controllers. The number of classes is equal to the number of admissible Jordan forms of the closed loop system. Each class is characterised by a specific minimum number of free parameters (degrees of freedom) in the parametric form of the feedback gain matrix. The class of EA controllers with the greatest value of free parameters is used for the assignment of the eigenstructure.

A significant advantage of this method occurs when not all of the eigenvalues need to be shifted, thus releasing extra free parameters for other design purposes.

3.4.8 Parametric output feedback results

There have also been methodologies for the output feedback case that follow the parametric approaches devised under state feedback conditions. Fahmy and O'Reilly [60] proposed the development of an effective multistage parametric approach for eigenstructure assignment in linear multivariable systems by output feedback control.

The sets of closed loop eigenvalues and associated eigenvectors are suitably divided into subsets and the entire eigenstructure is constructed by parts in two (or more) consecutive stages. The eigenvalue-vector subset assigned in a certain stage is intermediately protected, i.e. made invariant under output feedback, so that another subset can be assigned in a subsequent stage without disturbing the former subset. This allows the subsets of right and left eigenvectors to be assigned in separate stages, which relaxes the computational algorithm from the orthogonality conditions.

The number of effective free parameters beyond the eigenvalue assignment is also determined, and the notion of redistributing these parameters among the assignable right and left eigenvectors is introduced. The approach as a whole is remarkably simple and systematic, and it provides much insight into the mechanism of eigenstructure assignment by output feedback control. Duan [61] introduced another complete parametric approach for eigenstructure assignment by decentralised output feedback. By using a complete parametric solution of a generalised Sylvester matrix equation, parametric representations of both the left and right closed loop eigenvectors and generalised eigenvectors and two series of partially free parameter vectors are established.

The whole problem is therefore divided into two subproblems. The first is concerned with the solution of two generalised Sylvester matrix equations, and solved by using a complete parametric solution to the generalised Sylvester matrix equation. The second subproblem is concerned with the solution of a series of real matrices satisfying two sets of linear matrix equations. The obtained algorithm does not require any conditions on the closed loop eigenvalues, and provides a high number of degrees of design freedom for the eigenstructure assignment problem.

3.4.9. Perturbation Theory

So far the problem of stabilization of the control systems through the eigenvalue and eigenstructure assignment by state/output feedback has been defined and reviewed. Of course there are so many different algorithms relative to these theories which can

be used in practice in order to satisfy the various specifications asked by different designers.

But the important point to be considered is that there are the practical problems involved in computation such these solutions on a digital computer and in determining the accuracy of the computed eigensystems.

A major problem will be that of the estimating the effort of the various errors which are inherent in the formulation of the problem and its solution. Wiliknson in 1965 [1] has grouped these kinds of errors in three major parts:

(i)   The elements of the given matrix in the initial computation of the mathematical model may be determined directly from physical measurements, and therefore be subject to the errors inherent in all observations. In this case, the state matrix A corresponding to these measurements, is in fact an approximation to the original matrix A.

In fact if it can be asserted that the error in every element of A is bounded by δ ,then the true matrix is $(A+E)$, where E is some matrix for which $\left| e_{ij} \right| < \delta.$ A complete solution of the practical problem then involves not only the determination of the eigenvalues of A , but also an assessment of the range of the variation of the eigenvalues of all matrices of the class $(A+E)$. We are thus led to the consideration of the perturbations of the eigenvalues of the matrix corresponding to the perturbations in its elements.

(ii)   The elements of the matrix may be defined exactly by the mathematical formula but we may still be prevented by the practical considerations from presenting a digital computer with this exact matrix. The matrix A, for instance, may be defined as the product of several matrices each of which has elements with the full number of digits normally used on the

computer. In this case, we are facing with much the same as the first case and $(A+E)$ is the true state matrix.

(iii)   Even if we can regard the matrix presented to the digital computers as exact, the same will not be true, in general, of the computed solution. Most commonly the solution is computed will involve the calculation of a sequence of similarity transforms $A_1, A_2, \cdots,$ of the original matrix A, and rounding errors will be made in carrying out each of the transformations.

Frequently we shall be able to show that computed matrices A_i from the original matrix A, are exactly similar to matrices $(A+E_i)$, where E_i have small elements which are functions of the rounding errors. Thus again we are led to consider the perturbations of the eigenvalues and eigenvectors of a matrix corresponding to perturbations in its elements.

It should be mention that the major work regarding the perturbation ,done during the current research is the extend to the original work by Wlikinson on the real distinct eigenvalues and their relative eigenvectors, which will be studied for the complex eigenvalues and the space of the direct sum decomposition of the eigenvectors with more than one dimension.

The importance of the theory of the perturbation is that leads to the calculation of the sensitivity of the eigenvectors of any system which is strongly connected to the angle between these eigenectors which is a way to the robustness of the system by maximize the angle or in other word by minimize the sensitivity of the eigenvalues and their relative eigenvectors.

3.4.9.1. Perturbation theory for simple eigenvalue

Consider two matrices A and B such that for each element of them:

$$\left| a_{ij} \right| < 1, \qquad \left| b_{ij} \right| < 1$$

And let λ_1 be a simple eigenvalue of A. We wish to examine the corresponding eigenvalue of $(A + \varepsilon B)$, where εB is the error (ε too small) added to the original matrix A. If the characteristic equation of A is:

$$\det(A) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_0 = 0$$

Then the characteristic equation of $(A + \varepsilon B)$ is given by

$$\det(A) = \lambda^n + c_{n-1}(\varepsilon)\lambda^{n-1} + c_{n-2}(\varepsilon)\lambda^{n-2} + \cdots + c_0(\varepsilon) = 0,$$

Where $c_r(\varepsilon)$ is a polynomial of degree $(n-r)$ in ε such that

$$c_r(0) = c_r.$$

This is immediately obvious of we examine the explicit expression for $\det(\lambda I - A - \varepsilon B)$. We may write

$$c_r(\varepsilon) = c_r + c_{r1}\varepsilon + c_{r2}\varepsilon^2 + \cdots + c_{r.n-r}\varepsilon^{n-r}.$$

Now since λ_1 is a simple root of $\det(\lambda I - A - \varepsilon B)$ for sufficiently small ε there is a simple root $\lambda_1(\varepsilon)$ of $\det(\lambda I - A - \varepsilon B)$ given by a convergent power series $\lambda_1(\varepsilon) = \lambda_1 + k_1\varepsilon + k_2\varepsilon^2 + \cdots.$

Clearly $\lambda_1(\varepsilon) \to \lambda_1$ as $\varepsilon \to 0$. Note that $|\lambda_1(\varepsilon) - \lambda_1| = 0(\varepsilon)$ independent of the multiplicities of other eigenvalues ([1]).

3.4.9.2. Perturbation of corresponding eigenvector

It has been shown by Wilkinson [1] that: " if the eigenvector $\underline{x}_1$ is corresponding to a simple eigenvalue λ_1 of matrix A, then the eigenvector of $(A + \varepsilon B)$ will be $\underline{x}_1(\varepsilon)$. Clearly the elements of $\underline{x}_1(\varepsilon)$ are polynomials in $\lambda_1(\varepsilon)$ and ε, and since the power series for $\lambda_1(\varepsilon)$ is convergent for all sufficiently small ε, we see that element of $\underline{x}_1(\varepsilon)$

is represented by a convergent power series in ε, the constant term in which is the corresponding element of $\underline{x}_1$. We may write

$$\underline{x}_1(\varepsilon) = \underline{x}_1 + \varepsilon \underline{x}_1 + \varepsilon^2 \underline{x}_1 + \cdots$$

where each component of the vector series on the right is a convergent power series in ε. Corresponding to the result we had for the perturbation of simple eigenvalue, for the eigenvector, we have :

$$\left| \underline{x}_1(\varepsilon) - \underline{x}_1 \right| = 0(\varepsilon) ,$$

An again there are no fractional powers of ε.

3.4.9.3. First-order perturbations of eigenvalues and eigenvectors

Based on what has been presented in section 3.4.9 so far, the first order perturbations of eigenvalues and eigenvectors have been computed [1] as followings:

We let $\underline{x}_1(\varepsilon) = \underline{x}_1 + \varepsilon \underline{x}_1 + \varepsilon^2 \underline{x}_1 + \cdots$ and $\underline{y}_j$, j = 1,2, ... , n, be the right and left eigenvectors of the closed-loop system matrix $M = A + BF$, corresponding to eigenvalue $\in \Lambda$ and $\Lambda = \{ \lambda_1, \lambda_2, \cdots, \lambda_n \}$ $\lambda_j \in \Lambda$ and $\Lambda = \{ \lambda_1, \lambda_2, \cdots, \lambda_n \}$ is the set of closed-loop eigenvalues of the system, that is:

$$M \underline{x}_j = \lambda \underline{x}_j, \underline{y}_j^t M = \lambda_j \underline{y}_j^t$$

If M is *non-defective*, that is, M has n linearly independent eigenvectors, then M is diagonalizable and it can be shown, [1], that the sensitivity of the eigenvalue λ_j to perturbations in the components of A, B and state feedback F depends upon the magnitude of the *sensitivity* c_j, where

$$c_j = 1/s_j = \frac{\left\| \underline{y}_j \right\|_2 \left\| \underline{x}_j \right\|_2}{\left| \underline{y}_j^t \underline{x}_j \right|} \geq 1 ,$$

In the case of multiple eigenvalues, a particular choice of eigenvectors is assumed. (For real λ_j the sensitivity s_j is just the cosine of the angle between the right and left eigenvectors corresponding to λ_j).

More precisely, if a perturbation $O(\varepsilon)$ is made in the coefficients of the matrix M, then the corresponding first-order perturbation in the eigenvalue λ_j of M is of the order of $\varepsilon n c_j$.

If M is defective, then the corresponding perturbation in some eigenvalue is at least an order of magnitude worse in ε, and therefore, system matrices which are defective are necessarily less robust than those which are non-defective.

We observe that a bound on the sensitivities of the eigenvalues is given by

$$\max_j c_j \leq \kappa_2(X) \equiv \|X\|_2 \|X^{-1}\|_2$$

where $\kappa_2(X)$ is the *condition number* of the matrix $X = [\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n]$ of eigenvectors. Furthermore, the condition numbers take minimum value $c_j = 1$, for all $j = 1, 2, \cdots, n$, if and only if M is a normal matrix, that is $M'M = MM'$. In this case the eigenvectors of M may be scaled to give an orthonormal basis for $\mathbb{R}^n$ and then matrix X is perfectly conditioned with $\kappa_2(X) = 1$ ([1], [6]).

We expect the eigenvectors corresponding to the simple eigenvalue λ_1 to be very sensitive to perturbations in A if λ_1 is close to any of the other eigenvalues, and this is indeed true. When λ_1 is well separated from the other eigenvalues and none of the s_j (j=2,3,….,n) is small we can certainly say that the eigenvector $\underline{x}_1$ is comparatively insensitive to perturbations in A ([1]).

3.4.9.4. Robust pole placement

The state-feedback pole assignment problem in control system design is essentially an inverse eigenvalue problem. The solution is, in general, underdetermined, with many degrees of freedom.

A desirable property of any system design is that the poles should be insensitive to perturbations in the coefficient matrices of the system equations. This criterion may be used to restrict the degrees of freedom in the assignment problem, and to produce a *well-conditioned* or *robust* solution to the inverse eigenproblem. Based what has been said about the perturbation, the problem of the robust pole placement can be defined [6] as follows:

Given (A, B) *and* Λ *(as in 2.6.3), find real matrix* F *and non-singular matrix* X *satisfying* $(A + BF)X = X\Lambda$ *where* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *such that some measure* v *of the conditioning, or robustness, of the eigenproblem is optimized.*

We remark that the measure v could, for example, be chosen to be $v_1 = \|c\|_\infty$ where $c^T = [c_1, c_2, \cdots, c_n]$ is the vector of condition numbers corresponding to the selected matrix X of eigenvectors.

Alternatively, we could take as a measure of robustness $v_2 = \kappa_2(X)$ the condition number of matrix X which has been considered as one of the main tools of measurements within this presented thesis. The measure v_2 then gives an upper bound on the measure v_1 and both measures attain their (common) minimum value simultaneously. There also exist some other measures which will not be discussed in this presented chapter and can be found in [6].

The degrees of freedom available in the choice of the feedback F are reflected precisely by the degrees of freedom available in the selection of the matrix X of eigenvectors. In the case $m = 1$, if F exists, X is uniquely determined (up to scaling), and the condition numbers c_j cannot be controlled. In the case $m = n$ and X may always be chosen to be orthogonal, $(X = I$ suffices) and hence to be such that $c_j = 1, \forall j$.

For a general multi-input system ($1 < m < n$) we may control the sensitivities of the assigned poles to a restricted extent by an appropriate choice of the eigenvectors comprising X. We observe that in the *robust* pole placement problem, the choice of eigenvectors which may be assigned is restricted such that the resulting system matrix $A + BF$ is *non-defective.* This restriction implies certain simple conditions on the multiplicity of the poles which may be assigned.

3.4.9.5. Robust eigenstructure assignment

Given real matrix pair (A, B) and eigenvalue set Λ, our objective is to choose eigenvectors given by X satisfying $(A + BF)X = X\Lambda$ and such that the conditioning of the eigenproblem is minimized.

No restriction on the controllability of (A, B) is made, and we remark that although the uncontrollable modes of the system cannot be affected by the feedback F.

The corresponding eigenvectors may be modified and the conditioning of uncontrollable modes may be improved by an appropriate choice of X.

Kautsky and et al. (1985) proofed that for a given non-singular matrix X and $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, there exists F, a solution to $(A + BF)X = X\Lambda$ if and only if $U_1^T(AX - X\Lambda) = 0$ where $B = [U_0, U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix}$ with $U = [U_0, U_1]$ orthogonal and Z non-singular. Then F is given *explicitly* by $F = Z^{-1}U_0^T(X\Lambda X^{-1} - A)$.

Robust pole assignment problem can be reduced [1] to the problem of selecting *independent* vectors x_j, $j = 1, 2, \cdots, n$ corresponding to the assigned eigenvalues $\lambda_j \in \Lambda$ such that eigenproblem $(A + BF)X = X\Lambda$ is as well-conditioned as possible.

3.4.10. Other approaches

To conclude the review, a couple of unconventional assignment methods will be looked at. Datta [62] proposed a conceptually simple algorithm to assign eigenvalues in a Hessenberg matrix. The method is based on the evaluation of a simple recursive

relation. A matrix $H = \left(h_{ij} \right)$ is an upper Hessenberg matrix if $h_{ij} = 0$ whenever $i > j+1$. Such a matrix is unreduced if $h_{i,i-1} \neq 0$.

Datta considered the problem of replacing the first row of a given unreduced upper Hessenberg matrix such that the resulting matrix has the desired spectrum of eigenvalues. Murdoch and Shriba considered the same problem [63], however, one disadvantage of their method is that the case of the assignment of repeated eigenvalues cannot be considered without considerable alterations to the algorithm.

Yet it does have a couple of advantages, the first of those being that the required first row elements are yielded by the solution of a set of linear equations for which reliable algorithms exist in program libraries. The second advantage is that the effect of each assigned eigenvalue on the solution is easily identified, as each is associated with one row of respective equations. Olbrot [64] considered arbitrary robust eigenvalue placement by static state feedback.

The author demonstrated that robust eigenvalue placement in the disk of an arbitrary radius *r* centred at –2*r*, can be achieved by a static state feedback controller for systems with so called matched perturbations of uncertain parameters in the state coefficient matrix *A* (i.e. with perturbations of *A* in the range of the input matrix *B*). This implies that such systems can be robustly stabilised with an arbitrarily fixed degree of exponential decay.

Next chapter will deal with the significance of eigenvectors with a view to robust eigenvector assignment using open/close loop eigenvalues. It is well known that due to the presence of uncertainty or the variation of parameters, there is a need to an accurate mathematical model of a control system in order to have the best approximation of its corresponding physical problem, especially when it comes to the problem of robust control. The analysis of stability robustness or performance robustness has been very important for control systems under perturbations. From a practical point of view, the analysis of robustness is one of the most important problems that attempts to obtain a quantitative measure of the perturbations under which the systems still maintain the desired performance.

A condition for robustness is the orthogonality of the eigenframe, which was examined primarily by Wilkinson [1] in 1965. Since then, several papers have been dedicated to the issue of assigning the eigenstructure to satisfy robustness criteria. Juang, Hong and Wang [65] based their robust pole assignment method upon the Lyapunov approach [66], where the upper bounds of the perturbations are obtained to retain the system eigenvalues located within an arbitrarily chosen region in the complex plane.

The bounds derived by the proposed method provide useful quantitative measures in consideration of both the stability robustness and performance robustness of uncertain systems. However Wang and Lin [67] argued that the robustness bounds for eigenvalue assignment could be obtained without the need to solve the Lyapunov equation.

The analysis of the problem of eigenvalue assignment is based on some essential properties of the induced norms and certain matrix measures, which eliminate the heavy computational burden of the Lyapunov approach. However the Lyapunov approach was taken a step further by Wilson, Cloutier and Yedavalli [68].

They presented a generalised eigenstructure assignment procedure for designing a controller which has the best eigenstructure achievable while simultaneously maintaining stability robustness to time varying parametric variations. The problem was approached by constraining the minimisation of the difference between the actual and desired eigenstructure. This minimisation is made subject to the constraints of the eigenstructure equation and the closed loop Lyapunov equation.

3.5. Summary and conclusion

Eigenstructure assignment has attracted a lot of attention but it has focussed on a standard parameterisation of possible eigenstructures and has addressed mainly the robustness of performance using as a test the orthogonality of the eigenframe. Other features and implications of the eigenstructure have not been considered with the exception of the effect of the eigenstructure on the degrees of controllability and observability. In this thesis the above robustness criteria are extended by introducing

a new property that demonstrates the effect of the eigenstructure on the state overshoots of corresponding systems.

Most of the techniques on eigenstructure assignment deal with ways to maximise the orthogonality of the eigenframe, which is one particular problem and is indeed only one issue within the eigenstructure design problem family. Issues such as the best selection of closed loop spectrum that guarantees the most orthogonal solution are not sufficiently addressed. However, in this thesis, a new result on this problem will be introduced and reviewed through non-smooth optimisation (Chapter Eight) where the closed loop eigenframes are obtained through the method introduced by Karcanias [4] using open/close spectra.

Eigenstructure assignment algorithms which can handle a multitude of performance criteria require more flexible parameterisations. Specifically, what is required, are parameterisations tuned to the needs of the specific criteria. The new algebraic criterion to be introduced in Chapter Five seems to be the most flexible since it provides an explicit description of the structure of the eigenframe based on the properties of the open loop/closed loop spectra. This new form has the potential to study problems such as specification of closed loop spectra that can guarantee the most orthogonal closed loop eigenstructure. The alternative test based on open loop and closed loop spectra is also important since it permits the linking of state feedback design to pole mobility using energy considerations or norm of the feedback matrix used.

In light of the literature review that examined numerous methodologies for the application of procedures that assign the eigenframe of a system to a new predetermined state so as to enhance its performance, it is evident that such techniques can be split into the following categories:

☞ Effect of the eigenstructure on system performance
☞ Eigenstructure assignment using a *state feedback* approach
☞ Eigenstructure assignment using an *output feedback* approach
☞ Eigenstructure assignment by *parameterising* the eigenvectors

Before examining the way the eigenstructure can be changed by certain forms of compensation, it is important to examine the role of the eigenstructure on different aspects of system performance. The issues that are fundamental to this are:

(i)   Eigenstructure and system properties such as controllability, observability, robustness, stability, etc.

(ii)   Measuring the degree of orthogonality of the eigenframe and its effects on system properties.

(iii) The selection of desirable spectra and its effect on resulting orthogonality.

(iv)   Alternative forms for parameterising eigenframes.

Such properties are very important and have not been paid the appropriate attention in the study of eigenstructure assignment problems.

The state feedback approach is centred on the solutions for $\underline{u}_i$ and K_s of equation (3.4.1). Pivotal to the method that uses output feedback is equation (3.4.2), which is used to find solutions for $\underline{u}_i$ and K_0. The third procedure is the parametric approach, whereby either of the relationships for state or output feedback are used to formulate methods that make use of parametric equations to determine solutions for the respective feedback matrices and corresponding eigenvectors. Generally, feedback has an effect on the closed-loop characteristic polynomial of a system, and thus affects stability and system performance. The advantage of state feedback is that it presents the designer with extra freedom with which multivariable control systems can be successfully applied. However, there are systems in which the states are not measurable, and so the use of full state feedback is impractical. Therefore eigenstructure assignment by output feedback is used.

It is essential that the solutions obtained are such that the sensitivity of the assigned eigenvalues to system modelling discrepancies and external disturbances is minimised. In this thesis it will be shown that a degree of closed loop system robustness can be achieved by ensuring that the eigenvector matrix is as orthogonal as possible. This presents another hurdle with respect to measuring the orthogonality of a matrix, or a frame.

Another criterion central to the theme of the work carried out in this thesis is the requirement to accommodate system controllability (and observability). It is desired to maintain these two properties when assigning the eigenstructure of a system. As discussed earlier, this is achieved by ensuring that the eigenvectors are in the left null space of the input matrix and the right null space of the output matrix for controllability and observability respectively. Therefore the fundamental problem to be considered is that given the system matrices A and B and a set $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of stable, controllable eigenvalues, find an appropriate feedback matrix F, and an eigenvector matrix U such that a measure of the conditioning, or robustness, is minimised. With regards to feedback, because open and closed loop systems have the same restricted input-state pencil ($sN - NA$), the controllability properties of a system are invariant under state feedback, yet the observability properties change.

CHAPTER 4

BASIC CONCEPTS OF EIGENSTRUCTURE ASSIGNMENT FROM GEOMETRIC THEORY

4.1. Introduction

The concept of *A*-invariant and (*A*,*B*)-invariant subspaces has been given considerable attention in recent years. This along with the concept of controllability subspaces established a geometric setting which has suggested new methods of attacking synthesis problems in many fields of linear system theory [38], such methods have proved to be intuitive and economical.

It is believed that new insight could be gained in a number of problems of this efficient geometric setting introduced from a physical viewpoint rather that an abstract mathematical one; the vehicle for such an apparatus is the concept of rectilinear motions in the input and state spaces, a generalization of the motions along eigenvectors.

One-dimensional subspaces of the state space having the property of retaining any free motion for every $t \geq 0$ turns out to be eigenvectors of the dynamic map *A*- the corresponding motions are of the exponential type $\exp(\lambda t)$, where λ is the eigenvalue corresponding to the eigenvector, and are called rectilinear. Thus, the problem of restricting the free motion in a one dimensional subspace reveals the existence of pairs consisting of a vector and a frequency satisfying an eigenvalue-eigenvector problem. These elementary results provide a modification for the search of subspaces having a spectrum associated with them and characterized by rectilinear motions.

Generalizing the eigenvector - eigenvalue results on the lines discussed above, we are led to the concept of *A*-invariant subspaces; this is briefly presented in next section, where the characteristic decomposition, the spectrum and the general rectilinear motions are also discussed.

This will be followed by studying the problem of finding control inputs to restrict the trajectory in a given subspace which yields the concept of the (*A*,*B*) – invariant subspaces. Such subspaces are distinguished into two categories: those with an intersection with the range of input matrix *B*, (𝓑) and those having no intersection with (𝓑). For the first class, it is shown that there exists a unique characteristic decomposition and spectrum and that the associated control inputs are of the exponential type; furthermore, it is proved that there is an infinite number of one dimensional (*A*,*B*) – invariant subspaces with an arbitrary spectrum in any subspace of this class. The results derived from the study of (*A*,*B*) – invariance are used for the study of the eigenvector shifting problem; necessary and sufficient conditions for the assignability of a set of independent vectors as closed–loop eigenvectors of the system by state and output feedback are given.

Since (*A*,*B*)-invariant subspaces having controllability property, so controllability subspaces (c.s.) are discussed as the result of having such property. It will be shown that they belong to the second class of (*A*,*B*) – invariant subspaces and that they have the minimal property. The construction of characteristic bases with an arbitrary defined spectrum for a controllability subspace also will be discussed to give an alternative proof to the Wonham and Morse theorem for the assignability of the spectrum of c.s. [39].

For a certain class of cs having intersection with (𝓑), an eigenvector approach to pole assignment by state feedback based on the construction of the closed-loop eigenvectors using open-loop eigenvalues and their corresponding eigenvectors along with the minimization of the relative closed-loop eigenvector matrix will be formulated and studied in later chapters.

4.2. Motion along eigenvectors and the *A*- invariant subspaces of a linear system

The problem of keeping the state trajectory of a linear system in a given subspaces of the state space is of great importance in a number of control problems. Two versions of this problem may be considered. First, the problem of restricting the free motion in a given subspace and the second the problem of keeping the total state trajectory in a given subspace by making use of the control input. We will study the first case in this chapter.

Theorem 4.1: Let $\mathcal{S}(A, B, C, D)$ be a linear system and $\mathcal{V}$ an r-dimensional subspace of $\mathcal{X}$. Necessary and sufficient condition for the free motion part of the state trajectory $\underline{x}(t)$ to be kept in $\mathcal{V}$ for $\forall t \geq 0$ whenever the state is released from $\underline{x}(0) \in \mathcal{V}$ is

i)   For every trajectory $\underline{x}(t) \in \mathcal{V}$ there exists another trajectory $\underline{x}'(t) \in \mathcal{V}$ such that:

$$A\underline{x}(t) = \underline{x}'(t) \qquad \forall t \geq 0 \tag{4.2.1}$$

ii) $A\mathcal{V} \subset \mathcal{V}$ \hspace{4cm} (4.2.2)

The subspace $\mathcal{V}$ satisfying these conditions is called A-invariant subspace [5].

Proof: See [4] for full details. \hspace{3cm} ∎

For one-dimensional subspace $\mathcal{V}$, the free motion of the system starting on $\underline{x}_0$ is then

$$\underline{x}(i) = 1(t)\underline{x}_0 e^{s_0 t} \tag{4.2.3}$$

and is called a simple rectilinear motion or a motion along an eigenvector. We conclude the following:

Result 4.1: One dimensional *A*-invariant subspaces of $\mathcal{X}$ are the simple eigenspaces of *A*. Each of them is characterized by a unique frequency s_0 which is the eigenvalue corresponding to the simple eigenvector.

Let now $\mathcal{V}$ be an r-dimensional *A*-invariant subspace and $\{\underline{v}_i\}$ $i = 1, \cdots, r$ be a basis for $\mathcal{V}$. By definition (cond. (4.2.2)) we can find vectors $\underline{w}_i \in \mathcal{V}$ $i = 1, \cdots, r$ such that

$$A\underline{v}_i = \underline{w}_i \quad i = 1, \cdots, r$$

or

$$AV = W$$

Where $V = [\underline{v}_1 \vdots \cdots \vdots \underline{v}_r], \quad W = [\underline{w}_1 \vdots \cdots \vdots \underline{w}_r]$. Because $\{\underline{v}_i\}$ is a basis for $\mathcal{V}$ we may write $W = V\overline{A}$ or

$$AV = V\overline{A} \tag{4.2.4}$$

The matrix *A* is an $r \times r$ having $Q\Lambda Q^{-1}$ as characteristic decomposition. If we define a new basis by the transformation $U = VQ$ then we have

$$AU = U\overline{\Lambda}. \tag{4.2.5}$$

The matrices $\overline{A}, \overline{\Lambda}$ are called restrictions of *A* on the subspace $\mathcal{V}$ with respect to the bases *V* or *U* [38]. The matrix $\overline{\Lambda}$ might have a simple or non-simple structure including Jordan blocks, thus for the defined bases $\{\underline{u}_j\}$ called the characteristic basis of $\mathcal{V}$, the following conditions hold

$$A\underline{u}_i = \lambda_i \underline{u}_i \qquad i = 1, \dots, r \tag{4.2.6}$$

or, if $\overline{A}$ has ,say, one Jordan block:

$$A\underline{u}_j = \lambda_j \underline{u}_j \qquad j = 1, \dots, \mu$$

$$A\underline{u}_i = \lambda_\mu \underline{u}_i + \underline{u}_{i-1} \qquad i = \mu + 1, \dots, r \tag{4.2.7}$$

The basis is unique (unless we have repeated λ_i's and simple structure) and it is spanned by eigenvectors or pseudo-eigenvectors of the matrix *A*. The subspaces corresponding to the Jordan blocks are called Jordan eigenspaces.

The set of frequencies $\{\lambda_i\}$ taking into account their multiplicity as this is expressed by the dimensions of the Jordan blocks, is called the spectrum of $\mathcal{V}$. If we denote the characteristic basis of $\mathcal{V}$ by $\left\{\underline{x}_0^1, \ldots, \underline{x}_0^r\right\}$ and if for the sake of simplicity we assume that A has a simple structure, then transform of the state trajectory $\underline{x}(s) \in \mathcal{V}$ is $\sum_{i=1}^{r} \phi_i(s) \underline{x}_0^i$.

For an initial condition $\underline{x}_0 = \sum_{i=1}^{r} \alpha_i \underline{x}_0^i$, condition (4.2.7) yields

$$\sum_{i=1}^{r}\left\{-\alpha_i + s\phi_i(s)\right\}\underline{x}_0^i = \sum_{i=1}^{r} \phi_i(s) A \underline{x}_0^i.$$

4.3. Rectilinear motion of Non-autonomous system in the input output and state spaces and the concept of (A,B) – invariance

The concept of rectilinear motions, introduced in the previous section, was found to be strongly related to the concept of *A*-invariance.

The existence of directions characterised by a frequency and an associated rectilinear motion for autonomous systems stimulates our interest for the search for similar directions for non-autonomous systems. Thus, *A*-invariance is extended to (*A,B*)-invariance and the eigenvector problem to the generalized eigenvector eigenvalue problem. Finally, the structural similarities of *A*-invariant and (*A,B*)- invariant subspaces are demonstrated by introducing the characteristic decomposition of an (*A,B*)-invariant subspace and by defining the spectrum associated with that decomposition.

As we discussed in previous chapter (Problem 3.1.), we are looking for conditions such that for a given the system $\mathcal{S}(A,B,C,D)$ and a subspace of $\mathcal{X}$ and $\mathcal{V}$, for any $\underline{x}_0 \in \mathcal{V}$, there exists a control input which restricts the state trajectory in $\mathcal{V}$ for $\forall t \geq 0$. An initial answer to this, together with a general characterisation of subspaces $\mathcal{V}$ with such properties is given by the following theorem:

Theorem 4.2: [9], [35] Let $\mathcal{S}(A,B,C,D)$ be a linear system and $\mathcal{V}$ an r-dimensional linear subspace of $\mathcal{X}$. Necessary and sufficient condition for the existence of an input $\underline{u}(t)$ such that any state trajectory released from $\underline{x}_0 \in \mathcal{V}$ is kept in $\mathcal{V}$ for $\forall t \geq 0$, is

    i)   For every trajectory $\underline{x}(t) \in \mathcal{V}$ there exists $\underline{x}'(t) \in \mathcal{V}$ and $\underline{u}(t)$ such that

$$A\underline{x}(t) = \underline{x}'(t) - B\underline{u}(t) \tag{4.3.1}$$

    ii) $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ (4.3.2)

where $\mathcal{B}$ is the range space of B.

Proof: The proof can be seen in [38] ∎

Subspaces $\mathcal{V}$ satisfying condition (4.3.2) are called (*A,B*)- invariant and they have been introduced by Wonham [9].

4.4 Simple rectilinear motion and the one-dimensional (*A,B*)- invariant subspaces

The problem of restricting the state trajectory in one dimensional subspaces for non autonomous systems is now studied in the following theorems.

Theorem 4.3: Let S be a linear system described by

$$\dot{x} = A\underline{x} + B\underline{u} \tag{4.4.1}$$

and $\{\underline{x}_0\}$ is a subspace of $\mathcal{X}$ such that $\{\underline{x}_0\} \cap \mathcal{B} = \underline{0}$. There exists a uniquely defined control input such that for any initial condition $\underline{x}_0 \in \{\underline{x}_0\}$ the state trajectory $\underline{x}(t)$ remains in $\{\underline{x}_0\}$ for $\forall t \geq 0$ iff

$$A\underline{x}_0 = s_0\underline{x}_0 - B\underline{k}_0 \qquad s_0 \in \mathbb{C}, \underline{k} \in \mathbb{C}^l \tag{4.4.2}$$

$$A\{\underline{x}_0\} \subset \{\underline{x}_0\} + \mathcal{B} \tag{4.4.3}$$

It can be proofed that the control input and the state trajectory are then uniquely expressed by

$$\underline{u}(t) = 1(t)\underline{k}e^{s_0 t} \tag{4.4.4a}$$

$$\underline{x}(t) = 1(t)\underline{x}_0 e^{s_0 t} \tag{4.4.4b}.$$

∎

Thus in fact the above equations proof that in subspace $\mathcal{X}/\mathcal{B}$: complementary of $\mathcal{B}$ with respect to $\mathcal{X}$, there exist directions characterized by a uniquely defined frequency s_0 , input direction $\underline{k}$, and an associated rectilinear motion; condition (4.4.3) under which such directions exist, now clearly defines a generalized directions in the subspace $\mathcal{B}$ is given by the following theorem.

Theorem 4.4: [4] Let S be a linear system described by

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

and $\{\underline{x}_0\}$ a subspace of $\mathcal{X}$ such that $\{\underline{x}_0\} \cap \mathcal{B} \neq \underline{0}$ or that $\underline{x}_0 = B\underline{m}$ where $\underline{m}$ is a non - zero constant vector. A unique form of input exists

$$\underline{u}(t) = 1(t)\underline{k}e^{s_0 t} \qquad \forall s_0 \in \mathbb{C}, \underline{k} \in \mathbb{C}^l \tag{4.4.5}$$

such that the state trajectory $\underline{x}(t)$ remains in $\{\underline{x}_0\}$ for $\forall t \geq 0$ and for any initial condition $\underline{x}_0 \in \{\underline{x}_0\}$ if and only if for $\forall s_0 \in \mathbb{C}$

$$A\underline{\dot{x}} = s_0\underline{x}_0 - B\underline{k} \tag{4.4.6}$$

$$A\{\underline{x}_0\} \subset \{\underline{x}_0\} + \mathcal{B}$$

Under such conditions the state trajectory is expressed by

$$\underline{x}(t) = 1(t)\underline{x}_0 e^{s_0 t} \tag{4.4.7}$$

Proof: The proof can be seen in [4], [35]. ∎

The frequency s_0 and the associated input vector $\underline{k}$ are uniquely characterized whenever $\{\underline{x}_0\} \cap \mathcal{B} = \underline{0}$; however, the pair $\{s_0, \underline{k}\}$ in only one solution pair out of the whole class that exist when $\{\underline{x}_0\} \cap \mathcal{B} \neq \underline{0}$

Corollary 4.1: Let $\{\underline{x}_0\}$ be a one dimensional linear subspace of $\mathcal{X}$. If a control input $\underline{u}(t)$ which restricts the state trajectories in $\{\underline{x}_0\}$ for $\forall t \geq 0$ exists then it is always of the form $1(t)\underline{k}e^{s_0 t}$, the resulting state trajectories also expressed by $1(t)\underline{x}_0 e^{s_0 t}$ where they satisfy the necessary and sufficient condition

$$\left[s_0 I - A \vdots -B \right] \begin{bmatrix} x_0 \\ \underline{k} \end{bmatrix} = 0 \tag{4.4.8}$$

s_0 and $\underline{k}$ are unique if $\{\underline{x}_0\} \cap \mathcal{B} = \underline{0}$ and there is an infinite number of pairs $\{s_0, \underline{k}\}$ if $\{\underline{x}_0\} \cap \mathcal{B} \neq \underline{0}$. ∎

In general, for $\forall s_0 \in \mathbb{C}$, we can find $\underline{k}$ such that for a matrix L and: $L\underline{x}_0 = \underline{k}$ [35], [36], we see that there exists a class of feedback schemes which assign s_0 as an eigenvalue of the corresponding closed-loop system. This simple observation leads to the study the assignability of the spectrum of controllability subspaces, a subject which will be studied in the following section.

4.5. Rectilinear motions in *r*-dimensional (A,B)- invariant subspaces, characteristic decomposition of (A,B)- invariant subspaces

The study of one dimensional subspaces undertaken in the previous section has revealed the strong association of the concept of (*A,B*) – invariant with rectilinear motions in the input , state and output spaces. It was also shown that the subspaces with no intersection with $\mathcal{B}$ are characterized by a unique frequency s_0, called spectrum of the subspace, and that subspaces with an intersection with $\mathcal{B}$ may have any desirable spectrum corresponded to them.

Our main proposition here is to show how the association of rectilinear motion tie up with (A,B)- invariant subspaces and how the concept of the spectrum can be generalized to any dimension (A,B)- invariant subspace. Investigation of this kind leads to the definition of the characteristic decomposition of an (A,B)- invariant subspace in a manner similar to the one introduced for A-invariant subspaces.

We start off by establishing the existence of simple (A,B)-invariant subspaces contained in an r-dimensional (A,B)- invariant subspace.

Lemma 4.1: Let $\mathcal{V}$ be an r-dimensional (A,B)- invariant subspace of $\mathcal{X}$. There always exists a basis $\left\{\underline{x}_0^1, ..., \underline{x}_0^r\right\}$ and set of frequencies $s_i \in \mathbb{C}$, $i = 1, ..., r$ such that for some $k_i \in \mathbb{C}^l$ the following conditions hold:

$$A\underline{x}_0^i = s_i \underline{x}_0^i - B\underline{k}_i \qquad i = 1, ..., r \tag{4.5.1}$$

Proof: Let $\left\{\underline{v}_1, ..., \underline{v}_r\right\}$ be any basis for $\mathcal{V}$. By definition there exist vectors $\underline{w}_i \in \mathcal{V}$ and $k'_i \in \mathbb{C}^l$, $i = 1, ..., r$ such that

$$A\underline{v}_i = B\underline{k}'_i + \underline{w}_i \qquad i = 1, ..., r$$

Since $\{v_i\}$ are linearly independent, there exists a matrix L such that

$$L\underline{v}_i = -\underline{k}_i' \tag{4.5.2}$$

Substituting for $\underline{k}_i'$ in the last condition we have

$$(A + BL)\underline{v}_i = \underline{w}_i \quad i = 1, ..., r$$

Or, in matrix form

$$(A + BL)V = W \tag{4.5.3}$$

where V, W are matrices formed by the column vectors $\underline{v}_i$ and $\underline{w}_i$. Expressing the matrix W with respect to the basis $\{\underline{v}_i\}$ condition (4.5.3) yields

$$(A + BL)V = V\overline{A} \tag{4.5.4}$$

If $\bar{A} = Q\bar{\Lambda}Q^{-1}$ is the characteristic decomposition of $\bar{A}$ and if we define by X_0 , a new matrix we may write $X_0 = \left[\underline{x}_0^1 \vdots \ldots \vdots \underline{x}_0^r \right] = VQ$ we may write

$$(A + BL)X_0 = X_0\bar{\Lambda} \tag{4.5.5}$$

If $\bar{A}$ has a simple structure then conditions (4.5.1) immediately follow where by $\underline{k}_i$ we denote the $L\underline{x}_0^i$ vectors. ∎

In the proof of the lemma above it was assumed that $\bar{A}$ has simple structure. In general, however, some of the eigenvealues of $\bar{A}$ may be repeated and may be associated with Jordan blocks. In such cases an inspection of equation (4.5.2) and (4.5.5) readily establishes that the vectors $\underline{x}_i$ and $\underline{k}_i$ and the frequencies s_i that correspond to the general decomposition will be given by

$$A(\underline{x}_0)_\sigma^{i_\sigma} = s_\sigma(\underline{x}_0)_\sigma^{i_\sigma} + (\underline{x}_0)_\sigma^{i_\sigma - 1} - B\underline{k}_\sigma^{i_\sigma} \tag{4.5.6}$$

where $i_\sigma = 1, \cdots, r_\sigma,$ $\sigma = 1, \cdots, \rho,$ $(\underline{x}_0)_\sigma^0 = \underline{0}$ and $\sum_{\sigma=1}^{\rho} r_\sigma = r = \dim \mathcal{V}.$

Obviously, for frequencies s_σ with $r_\sigma = 1$ vectors, conditions (4.5.1) are satisfied.

There are two important roles played by this lemma. First it bridges the concept of (A,B)-invariance to the concept of A-invariance and second it provides the tool for the generalization of the results derived for A-invariant subspaces to (A,B)- invariant subspaces.

Thus, condition (4.5.1) implies the following remark:

Remark 4.1: (A,B)- invariant subspaces $\mathcal{V}$ are A-invariant subspaces of the system formed after the application of state feedback through appropriate operator L, yields

$$(A + BL)\mathcal{V} \subset \mathcal{V} \tag{4.5.7}$$

Another important conclusion derived from this lemma is the following:

Remark 4.2: Every (*A*,*B*)- invariant subspace contains a number of simple, or Jordan structure (*A*,*B*)-invariant subspaces with the property that each of them is characterised by a frequency. Each of the simple subspace $\{\underline{x}_0^{\,i}\}$ satisfies the condition

$$A\underline{x}_0^i = s_i \underline{x}_0^i - B\underline{k}_i \tag{4.5.8a}$$

while each of the vectors spanning a Jordan structure subspace with an associated frequency s_σ are defined by

$$A(\underline{x}_0)_\sigma^{i_\sigma} = s_\sigma (\underline{x}_0)_\sigma^{i_\sigma} + (\underline{x}_0)_\sigma^{i_\sigma - 1} - B\underline{k}_\sigma^{i_\sigma}, \quad i_\sigma = 1, \quad (\underline{x}_0)_\sigma^0 = 0 \ . \tag{4.5.8b}$$

Pre-multiplying the last two conditions by N (left annihilator of *B*), we have a new set of necessary and sufficient condition which basis $\{\underline{x}_0^{\,i}\}$ and the associated frequencies s_i must satisfy. These conditions are independent of the control inputs and can be given as

$$s_i N\underline{x}_0^i = NA\underline{x}_0^i \tag{4.5.9a}$$

or

$$s_\sigma N(\underline{x}_0)_\sigma^{i_\sigma} + N(\underline{x}_0)_\sigma^{i_\sigma - 1} = NA(\underline{x}_0)_\sigma^{i_\sigma}, \quad i_\sigma = 1,...,r_\sigma \ , \ (\underline{x}_0)_\sigma^0 = \underline{0}. \tag{4.5.9b}$$

The problem remaining unanswered by the discussion above is the uniqueness of the established basis. If this basis is unique, then the simple or Jordan subspaces and the frequencies associated with them are characteristics of the (*A*,*B*)-invariant subspaces $\mathcal{V}$. One dimensional (*A*,*B*)-invariant subspaces which intersect with $\mathcal{B}$ have an arbitrary spectrum associated with them. That suggests distinguishing the (*A*,*B*)-invariant subspaces into two categories: subspaces which intersect and subspaces which do not intersect with the range space of *B*, $\mathcal{B}$. Our attention is first focussed on subspaces for which

$$\mathcal{V} \cap \mathcal{B} = \underline{0} \tag{4.5.10}$$

The vectors of the basis (4.5.1) and the associated frequencies s_i are uniquely defined.

Now let $\mathcal{V}$ be an (A,B)-invariant subspace such that $\mathcal{V} \cap \mathcal{B} = \underline{0}$, $\left\{ \underline{x}_0^{\ i} \right\}$ the uniquely defined basis of $\mathcal{V}$ and $\left\{ \underline{k}_i \right\}$ the set of input directions $(i = 1,...,r)$, then the linear independence of the $\underline{x}_0^{\ i}$ implies that a class of matrices L exists which satisfies the following conditions:

$$L\underline{x}_0^{\ i} = \underline{k}_i \qquad i = 1,...,r \tag{4.5.11}$$

This class $\mathcal{L}(\mathcal{V})$ of matrices L defines the totality of state feedback schemes generating the control inputs associated with $\mathcal{V}$ as combinations of the states. Making use of condition (4.5.11), conditions (4.5.1) or (4.5.2) yields

$$(s_i I - A - BL)\underline{x}_0^i = 0 \qquad i = 1,...,r \tag{4.5.12}$$

or

$$(s_\sigma I - A - BL).(\underline{x}_0)_\sigma^{i_\sigma} = (\underline{x}_0)_\sigma^{i_\sigma - 1} \tag{4.5.13}$$

where $i_\sigma = 1,...,r_\sigma$, $\sigma = 1,...,\rho$, $(\underline{x}_0)_\sigma^0 = \underline{0}$ and $\sum_{\sigma=1}^{\rho} r_\sigma = r = \dim \mathcal{V}$

Thus, the vectors of characteristic basis $\left\{ \underline{x}_0^{\ i} \right\}$ become closed-loop eigenvectors for any L satisfying condition (4.5.11). It is easy to see that the control input $u(t)$ can be generated from state vector $\underline{x}(t)$ if a feedback operator L satisfying condition (4.5.11) is used.

Such connection clearly converts a controlled open-loop system to a free-responding closed-loop system. Thus through the feedback connection L it is possible to establish the equivalence between (A,B)- invariance and $(A - BL)$ –invariance. Thus any (A,B)-invariant subspace $\mathcal{V}$ maybe thought as A_c -invariant subspace

corresponding to a matrix $A_c = A - BL$ derived by applying an appropriate state feedback.

Such considerations together with the fact that an (A,B)-invariant subspace $\mathcal{V}$ of dimension r can be composed by a number of simple (A,B)-invariant subspaces, brings new light to the problem of assigning a set of independent vectors, as closed-loop eigenvectors of the pair (A,B). The following theorem gives the necessary and sufficient conditions for the set of vectors $\{\underline{u}_i\}$ $i = 1,...,r$ to be assignable as eigenvectors of some closed-loop matrix A_c by state feedback.

Theorem 4.5: Necessary and sufficient conditions for the set of independent vectors $\{\underline{u}_i\}$ $i = 1,...,r$ to be assignable as eigenvectors by state feedback is that each of the vectors $\underline{u}_i$'s or groups of the other vectors to be simple (A,B)- invariant subspaces of the pair (A,B).

Proof: Let us assume that the vectors $\{\underline{u}_i\}$ $i = 1,...,r$ are closed-loop eigenvectors of some matrix $A_c = A\text{-}BL$. Then each of them should satisfy the condition

$$(A - BL)\underline{u}_i = \lambda_i \underline{u}_i \tag{4.5.14}$$

Or in the case of Jordan structures

$$(A - BL)\underline{u}_i^j = \lambda_i \underline{u}_i^j + \underline{u}_i^{j-1} \tag{4.5.15}$$

With $j = 1,...,\sigma$ and $\underline{u}_i^0 = \underline{0}$. By defining $L\underline{u}_i = \underline{k}_i$ or $L\underline{u}_i^j = \underline{k}_i^j$, it is easily seen that $\underline{u}_i$'s should be simple (A,B)-invariant subspace of the pair (A,B); this proves the necessity of the theorem.

The sufficiency follows easily if given the pairs $\{\underline{u}_i, \underline{k}_i\}$ we define a matrix L such that

$$L\underline{u}_i = \underline{k}_i \quad i = 1,...,r \tag{4.5.16}$$

Because of the linear independent of the $\underline{u}_i$'s, matrix L exists and the vectors $\{\underline{u}_i\}$ $i = 1,...,r$ become eigenvectors of the closed-loop matrix $A_c = A\text{-}BL$. ∎

It becomes clear that it is not always possible to assign as closed loop eigenvectors by any given set of linearly independent vectors. It is for this reason that the approximate eigenvector shifting problem becomes important. This problem is formulated as follows: Given the set of linearly independent vectors $\{\underline{u}_i\}$ find a assert of linearly independent simple (A,B)-invariant subspaces approximating the $\underline{u}_i$'s in the best possible way.

4.6. Controllability Subspaces

For the solution of a number of control problems, it is important to know whether or not a given system has the property that it may be steered from any given state to any other given state. This leads to the concepts of controllability and of controllable subspaces which first introduced by Wonham & Morse in 1970 [9], [10]. The study of the relation between the concept of controllability and the concept of $(A,B)-$ invariance forms the main objective of this section.

An $(A,B)-$ invariant subspace $\mathcal{V}$ has the property that for any $\underline{x}_0 \in \mathcal{V}$ a control input $\underline{u}(t)$ can be found such that $\underline{x}(t) \in \mathcal{V}$ for all $t \geq 0$. If further, the condition that every $\underline{x}_\sigma \in \mathcal{V}$ is reachable from the origin in finite time and the associate trajectory belonging to $\mathcal{V}$ is imposed, we are then led to a special structure of $(A,B)-$ invariant subspaces; the controllability subspaces. For such space the generic symbol $\mathcal{R}$ is used. We will present a detailed discussion of controllability subspaces within the next chapter.

Two different approaches may be adopted for the introduction of such a class of subspaces; the one is to give the mathematical definition first [4] and then derive the physical properties; the other is to base the definition of physical arguments and from this derive the necessary and sufficient conditions for the characterization of the subspaces. Here the second approach will be adopted.

Definition 4.1: [9] Let $\mathcal{R}$ be a subspace of the state space $\mathcal{X}$ having the following two properties. First, for every $\underline{x}_0 \in \mathcal{R}$ there exists an input such that the state

trajectory is restricted in $\mathcal{R}$ for $\forall t \geq 0$. Secondly, every state $\underline{x} \in \mathcal{R}$ is reachable from the origin in finite time and the associated trajectory lies wholly in $\mathcal{R}$. Such a subspace is called a controllability subspace. ∎

The first condition implies that $\mathcal{R}$ is an $(A, B)-$ invariant subspace of the system or that for some $L \in \mathcal{L}(\mathcal{R})$

$$(A + BL)\mathcal{R} \subseteq \mathcal{R} \text{ for } \forall L \in \mathcal{L}(\mathcal{R}). \tag{4.6.1}$$

The second condition implies that $\mathcal{R}$ must have the properties of a controllable subspace of the system, if every state in $\mathcal{R}$ is reachable from the origin in finite time. The subspace $\mathcal{R}$ must be also the largest controllable subspace influenced by a control vector since the trajectory bringing the system from $\underline{0}$ state to any state $\underline{x} \in \mathcal{R}$ does not leave $\mathcal{R}$. Obviously if $\mathcal{R}$ is not the largest subspace for some class of inputs there is no reason why the trajectory from $\underline{0}$ to $\underline{x} \in \mathcal{R}$ should be restricted in $\mathcal{R}$ rather than move in larger subspace controllable from the same control inputs. Thus, as $\mathcal{R}$ is both an $(A, B)-$ invariant and a controllable subspace, it must have the general form $\{A + BL/\hat{\mathcal{B}}\}$ where $\hat{\mathcal{B}}$ is an appropriate subspace of $\mathcal{B}$ associated with the set of the control inputs for which $\mathcal{R}$ becomes the maximal subspace satisfying the condition

$$\mathcal{R} = \{A + BL/\hat{\mathcal{B}}\}, \quad \hat{\mathcal{B}} \subset \mathcal{B}. \tag{4.6.2}$$

In other words, a controllability subspace $\mathcal{R}$ is associated with the existence of a control input

$$\underline{u} = L\underline{x} + G\underline{v} \tag{4.6.3}$$

where L is a state feedback matrix and G is an $l \times r, (r \leq l)$ **"gain"** matrix connected at the inputs of the system and defining a subset of the inputs BG such that $\{BG\} = \hat{\mathcal{B}}$. This will be followed by below definition:

Definition 4.2: **[9],** [10] A subspace $\mathcal{R}$ of $\mathcal{X}$ is a controllability subspace (c.s.) of the pair (A, B) if there exist maps $L: \mathcal{X} \to \mathcal{U}$ and $G: \mathcal{U} \to \mathcal{U}$ such that

$$\mathcal{R} = \{A + BL / \{BG\}\}.$$
(4.6.4)

Now it becomes clear that $\mathcal{R}$ is the controllable subspace of the pair $(A+BL, BG)$ where L belongs to a certain class $\mathcal{L}(\mathcal{R})$ matrices.

■

For the single-input system corresponding to a pair $(A, \underline{b})$ the family of c.s. obviously comprises simply the o and $\{A/b\}$. However, in the multi-input situation where $\dim(\mathcal{B}) \geq 2$, this family is in general not trivial. Wonham and Morse [10] have proved that if a pair of (L, G) exist such that condition (4.6.4) is satisfied then the following definition is equivalent to the one given previously.

Definition 4.3: **[9],** [10] A subspace $\mathcal{R}$ of $\mathcal{X}$ is a controllability subspace of the system if there exists a map $L: \mathcal{X} \rightarrow \mathcal{U}$ such that
$$\mathcal{R} = \{A + BL / \mathcal{B} \cap \mathcal{R}\},$$
(4.6.5)

if $\mathcal{R}$ is a controllability subsapce then there exists a class $\mathcal{L}(\mathcal{R})$ of L matrices for which condition (4.6.5) is satisfied. ■

The last definition yields the following more general criterion for the characterisation of the c.s. [70].

Theorem 4.6: **[9],** [10], [4] The necessary and sufficient condition for a subspace $\mathcal{R}$ of $\mathcal{X}$ to be a controllability subspace of the pair (A, B) are
$$A\mathcal{R} \subseteq \mathcal{R} + \mathcal{B}$$
(4.6.6)
$$\mathcal{R} \cap \mathcal{B} \neq o$$
(4.6.7)

and there is no proper subspace of $\mathcal{R}$ satisfying these two conditions. ■

It is interesting to note that theorem (4.6.1.) provides the means for an alternative definition of c.s. the main difference lying in the fact that the minimal property excludes the zero subspace o. This minimal property differentiates the concepts of

$(A, B)-$ invariant and of controllability subspaces considerably. It simply implies that a c.s. $\mathcal{R}$ can never be written as a direct sum of simpler controllability subspaces; however, the same subspace $\mathcal{R}$ considered as $(A, B)-$ invariant is always expressed as a direct sum of simpler $(A, B)-$ invariant subspaces.

4.7. Conclusion

This chapter discussed the concept of (A,B)-invariant subspaces (one and higher dimensional) and their impact on input, output and state feedbacks. The main emphasis was given to (A,B)-invariant subspaces which do not intersect with $\mathcal{B}$; subspaces of this kind are characterised by the uniqueness of their spectrum and their decomposition into a number of simple or Jordan subspaces. Considering this type of (A,B)-invariant subspaces, the construction of controllability subspaces c.s. as a property of this case of (A,B)-invariant subspaces is presented in details.

CHAPTER 5:

DECOMPOSITION OF STATE SPACE INTO CONTROLLABILITY SUBSPACES

5.1. Introduction

In this chapter the problem of eigenvector frame parameterisations is considered and new parameterisations are presented having different advantages from the design point of view. We consider two different parameterizations; one is a parameterization of closed-loop eigenframes based on the open and closed loop spectra and the other is based on the algebraic characterization and parameterization of controllability subspaces [8], [71], [102]. Those two alternative parameterizations have different advantages and these will be exploited in subsequent chapters demonstrating their potential for design. The development of the two parameterizations uses results of the geometric theory and especially the notions of (A, B)-invariant and controllability subspaces [9], [103] and in particular their algebraic characterization [7], [71], [102]. The fundamental notions of the geometric theory are reviewed first and then we develop the main results on the parameterizations.

5.2. *(A, B) – invariant subspaces intersecting with* $\mathscr{B}$

The discussion in the previous chapters has shown how important it is for an (A, B)−invariant subspace to be disjoint from $\mathscr{B}$; it was shown that for this case their spectrum, their decomposition into simpler (A, B)−invariant subspaces and the

associated control inputs are uniquely defined; subspaces of this kind have a fixed spectrum. It was also shown that (A, B)−invariant subspaces of an open loop system become $(A + BL)$−invariant subspaces of a closed loop system whenever we close the loops with a state feedback matrix $L \in \mathcal{L}(\mathcal{V})$; thus by closing the loops through L the spectrum and the simple subspaces of an (A, B)−invariant subspace become closed loop poles and closed loop eigenspaces of the derived system. Arguments of this kind clearly provide the motivation for the study of (A, B)−invariant subspaces with an assignable spectrum. A strong lead to the study has already been provided by the properties of (A, B)−invariant subspaces that intersect with $\mathcal{B}$. For this reason we initiate our investigation by concentrating on (A, B)−invariant subspaces $\mathcal{V}$ for which $\mathcal{V} \cap \mathcal{B} \neq o$. The following two lemmas prove the existence of infinite number of simple and Jordan (A, B)−invariant subspaces contained in an (A, B)−invariant subspace $\mathcal{V}$ which intersects with $\mathcal{B}$.

Lemma 5.1: [4] Let $\mathcal{V}$ be an r−dimensional (A, B)−invariant subspace of the pair (A, B) and let $\mathcal{V} \cap \mathcal{B} \neq o$. For any frequency $\mu \in \mathbb{C}, \mu \neq \lambda_i$ where $\{\lambda_i\}$ is the spectrum of a characteristic basis of $\mathcal{V}$ there exist vectors $\underline{\tilde{u}}(\mu) \in \mathbb{C}^n$ and $\underline{\tilde{k}}(\mu) \in \mathbb{C}^l$ such that:

$$A\underline{\tilde{u}}(\mu) = \mu\underline{\tilde{u}}(\mu) + B\underline{\tilde{k}}(\mu) \tag{5.2.1}$$

Proof: Let as assume that $\{\underline{u}_i\}$ is a characteristic basis of $\mathcal{V}$ and let us make the further assumption that $\{\underline{u}_i\}$ has a simple structure. The vectors $\underline{u}_i$ satisfy the following conditions

$$A\underline{u}_i = \lambda_i \underline{u}_i + B\underline{k}_i, i = 1, ..., r \quad . \tag{5.2.2}$$

Any vector $\underline{u}' \in \mathcal{V}$ may then be expressed with respect to that basis as $\underline{u}' = \sum_{j=1}^{r} a'_j \underline{u}_j$ and using the equation (5.2.1) it is easy to see that the following identity holds for $\forall \mu \in \mathbb{C}$

$$(\mu I - A)\underline{u}' = \sum_{j=1}^{r} a_j' \left(\mu - \lambda_j \right) \underline{u}_j - B \sum_{j=1}^{r} a_j' \underline{k}_j \tag{5.2.3}$$

Let us now consider another vector $\underline{u} \in \mathcal{V} \cap \mathcal{B}$ which is expressed by

$$\underline{u} = \sum_{j=1}^{r} a_j \underline{u}_j = B \underline{m} . \tag{5.2.4}$$

By choosing $\tilde{\underline{u}}(\mu)$ such that $a_j' = a_j / \left(\mu - \lambda_j \right), j = 1, ..., r$ where $\mu \neq \lambda_i$ and substituting into identity (5.2.3) it follows that:

$$(\mu I - A)\tilde{\underline{u}}(\mu) = \sum_{j=1}^{r} a_j \underline{u}_j - B \sum_{j=1}^{r} a_j / \left(\mu - \lambda_j \right) \underline{k}_j$$

$$= B \left\{ \underline{m} - \sum_{j=1}^{r} a_j / \left(\mu - \lambda_j \right) \underline{k}_j \right\} = -B \tilde{\underline{k}}(\mu)$$

from which

$$A \tilde{\underline{u}}(\mu) = \mu \tilde{\underline{u}}(\mu) + B \tilde{\underline{k}}(\mu) \qquad \blacksquare$$

This first Lemma shows that in $(A, B)-$invariant subspace $\mathcal{V}$ which intersects with $\mathcal{B}$, there is an infinite number of one-dimensional $(A, B)-$invariant subspaces having an arbitrary spectrum; equivalently we may say, that for any frequency $s_0 \in \mathbb{C}$ we can find at least a $1-$dimensional $(A, B)-$invariant subspace which belongs to $\mathcal{V}$ and has s_0 as its spectrum. If we choose any vector $\underline{u} \in \mathcal{V} \cap \mathcal{B}$ expressed by equation (5.2.4) then the vectors $\tilde{\underline{u}}(\mu)$ and $\tilde{\underline{k}}(\mu)$ are given by

$$\tilde{\underline{u}}(\mu) = \sum_{j=1}^{r} \frac{a_j}{\mu - \lambda_j} \underline{u}_j \tag{5.2.5a}$$

$$\tilde{\underline{k}}(\mu) = -\underline{m} + \sum_{j=1}^{r} \frac{a_j}{\mu - \lambda_j} \underline{k}_j . \tag{5.2.5b}$$

The following Lemma deals with the problem of finding Jordan subspaces of an arbitrary spectrum in an $(A, B)-$invariant subspace, which intersects with $\mathcal{B}$.

Lemma 5.2: [4] Let $\mathcal{V}$ be an r-dimensional (A, B)-invariant subspace of the pair (A, B) and let $\mathcal{V} \cap \mathcal{B} \neq o$. For any frequency $\mu \in \mathbb{C}, \mu \neq \lambda_i$ where $\{\lambda_i\}$ is the spectrum of a characteristic basis of $\mathcal{V}$ there exist vectors $\tilde{u}_1(\mu), \tilde{u}_2(\mu), ..., \tilde{u}_\tau(\mu)$ and $\tilde{k}_1(\mu), \tilde{k}_2(\mu), ..., \tilde{k}_\tau(\mu), \tau \leq r$ such that:

$$A\tilde{u}_i(\mu) = \mu\tilde{u}_i(\mu) + \tilde{u}_{i-1} + B\tilde{k}_i(\mu), i = 1, ..., \tau, \ \tilde{u}_0(\mu) = \underline{0}. \tag{5.2.6}$$

Proof: Making the same assumptions as in the proof of the previous Lemma and repeating the same arguments, we derive against identity (5.2.3). The first of conditions (5.2.6) follows from Lemma (5.1). The proof of the remaining conditions (5.2.6) follows immediately if we define an appropriate set of vectors $\tilde{u}_2, \tilde{u}_3, \tilde{u}_4, ..., \tilde{u}_\tau$.

Thus to begin with, choose $\tilde{u}_2(\mu)$ as $\tilde{u}_2(\mu) = \sum_{j=1}^{r} a_j^{(2)} \tilde{u}_j$ where $a_j^{(2)} = a_j \left\{ 1/(\mu - \lambda_j) - 1/(\mu - \lambda_j)^2 \right\}$; then using identity (5.2.3) we have:

$$(\mu I - A)\tilde{u}_2(\mu) = \sum_{j=1}^{r} a_j \underline{u}_j - \sum_{j=1}^{r} \frac{a_j}{(\mu - \lambda_j)} \underline{u}_j - B \left\{ \sum_{j=1}^{r} a_j^{(2)} \underline{k}_j \right\}$$

$$= B \left\{ \underline{m} - \sum_{j=1}^{r} a_j^{(2)} \underline{k}_j \right\} - \tilde{u}_1(\mu)$$

from which

$$A\tilde{u}_2(\mu) = \mu\tilde{u}_2(\mu) + \tilde{u}_1(\mu) + B\tilde{k}_2(\mu)$$

In general by defining the vectors $\tilde{u}_\tau(\mu), \tilde{k}_\tau(\mu)$ as

$$\tilde{u}_\tau(\mu) = \sum_{j=1}^{r} a_j^{(\tau)} \underline{u}_j, \quad \tilde{k}_\tau(\mu) = -m + \sum_{j=1}^{r} a_j^{(\tau)} \underline{k}_j$$

with

$$a_j^{(\tau)} = a_j \left\{ \frac{1}{\mu - \lambda_j} - \frac{1}{(\mu - \lambda_j)^2} + \frac{1}{(\mu - \lambda_j)^3} - .... + \frac{(-1)^{\tau-1}}{(\mu - \lambda_j)^\tau} \right\}$$

the general condition (5.2.6) is satisfied. ∎

Thus, assuming that the characteristic basis of $\mathcal{V}$ is simple, the vectors $\tilde{\underline{u}}_\rho(\mu), \rho=1,\ldots,\tau$ where $\tau \leq r$ that span the Jordan subspace of $\mathcal{V}$ which is characterised by the frequency μ and the corresponding input directions $\tilde{\underline{k}}_\rho(\mu)$ are given by

$$\tilde{\underline{u}}_\rho(\mu) = \sum_{j=1}^{r} a_j^{(\rho)} \underline{u}_j, \tilde{\underline{k}}_\rho(\mu) = -\underline{m} + \sum_{j=1}^{r} a_j^{(\rho)} \underline{k}_j \tag{5.2.7a}$$

$$a_j^{(\rho)} = a_j \left\{ \frac{1}{\mu-\lambda_j} - \frac{1}{(\mu-\lambda_j)^2} + \frac{1}{(\mu-\lambda_j)^3} - \ldots + \frac{(-1)^{\rho-1}}{(\mu-\lambda_j)^\rho} \right\}, \rho=1,\ldots,\tau. \tag{5.2.7b}$$

Because of the extensive use of the vectors $\tilde{\underline{u}}, \tilde{\underline{k}}$ in the sequel we summarise the expressions for these vectors, when the more general basis $\{\underline{u}_i\}$ having a Jordan block of dimension v is considered. Thus, if the vectors $\{\underline{u}_i\}$ satisfy the following conditions

$$A\underline{u}_i = \lambda_1 \underline{u}_i + \underline{u}_{i-1} + B\underline{k}_i, i=1,\ldots,v, \underline{u}_0 = \underline{0}$$

$$A\underline{u}_j = \lambda_j \underline{u}_j + B\underline{k}_j, j=v+1,\ldots,r \tag{5.2.8}$$

The vectors $\tilde{\underline{u}}_\rho(\mu), \tilde{\underline{k}}_\rho(\mu)$ satisfying the general condition

$$A\tilde{\underline{u}}_\rho(\mu) = \mu\tilde{\underline{u}}_\rho(\mu) + \tilde{\underline{u}}_{\rho-1} + B\tilde{\underline{k}}_\rho(\mu), \rho=1,\ldots,\tau, \tau \leq r \tag{5.2.9}$$

with $\tilde{\underline{u}}_0(\mu)=\underline{0}$ are given by conditions (5.2.7a) with the $a_j^{(\rho)}$ defined by

$$a_1^{(\rho)} = \sum_{j=1}^{v} a_j \delta_{j,\rho}(\mu,\lambda_1), a_2^{(\rho)} = \sum_{j=2}^{v} a_j \delta_{j-1,\rho}(\mu,\lambda_1),\ldots$$

$$\ldots, a_{v-1}^{(\rho)} = \sum_{j=v-1}^{v} a_j \delta_{j-v+2,\rho}(\mu,\lambda_1), a_v^{(\rho)} = \sum_{j=1}^{v} a_v \delta_{1,\rho}(\mu,\lambda_1) \tag{5.2.10}$$

$$a_i^{(\rho)} = a_i \delta_{1,\rho}(\mu,\lambda_1), i=v+1,\ldots r$$

The constants a_j are defined by condition (5.2.4) while the functions $\delta_{\tau,\rho}(\mu,\lambda_i)$ are given by

$$\delta_{\tau,\rho}\left(\mu,\lambda_i\right)=\frac{s_\tau\left(1\right)}{\left(\mu-\lambda_i\right)^\tau}-\frac{s_\tau\left(2\right)}{\left(\mu-\lambda_i\right)^{\tau+1}}+....+\left(-1\right)^{\rho-1}\frac{s_\tau\left(\rho\right)}{\left(\mu-\lambda_i\right)^{\tau+\rho-1}} \tag{5.2.11}$$

where by $s_\tau\left(v\right),\tau,v=1,...$ we denote the elements of the sequence

$$s_\tau\left(v+1\right)-s_\tau\left(v\right)=s_{\tau-1}\left(v+1\right) \tag{5.2.12}$$

These results highlight a very important structural property of $\left(A,B\right)-$invariant subspace $\mathcal{V}$ which intersects $\mathcal{B}$; namely that such subspaces contain an infinite number of simple or higher order $\left(A,B\right)-$invariant subspaces which may assume an arbitrary spectrum. In conclusion, therefore, one may assert that unlike the case $\mathcal{V}\cap\mathcal{B}=o$, $\left(A,B\right)-$invariant subspaces which intersect $\mathcal{B}$ do not possess a unique characterisation. Finally, for such a class of subspaces, the problem of finding the form of control inputs restricting the state trajectory in $\mathcal{V}$ may be tackled in a manner similar to the one used for non-intersecting subspaces; the control inputs are again expressed as sums of vector exponentials. However the frequencies associated with the vector exponentials and the corresponding input directions are no longer unique.

5.3. Algebraic characterization of Controllability Subspaces

The concept of controllability subspaces (c.s.) introduced by Wonham and Morse (1970) [10] has emerged as a powerful tool in the theory of decoupling, pole assignment, disturbance rejection etc. It has been shown [10] that the solution of problems relating to the above topics depends on the existence of suitable sets of controllability subspaces.

In this section the input state pencil $sN-NA$, a matrix pencil which emerges from the study of generalized free response and generalized forced response problems [35] is used to simplify the presentation of the Warren & Eckberg (1975) [71] results. It is also shown that when the pair $\left(A,B\right)$ is expressed in the controllable companion

form $\left(\tilde{A}, \tilde{B} \right)$ the corresponding pencil $s\tilde{N} - \tilde{N}\tilde{A}$, takes the Kronecker canonical form [17].

5.3.1. Problem statement

If the state-space realization of any system is as form:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times l}$, by considering $\mathcal{X}$ to be the controllability space of (A, B), we have the following.

Theorem 5.1: [96], [102]**, [104]** A subspace $\mathcal{R} \subset \mathcal{X}$ of dimension r is a c.s. if and only if there exist polynomial vectors $\underline{p}(s) \in \mathbb{R}^n[s]$ and $\underline{u}(s) \in \mathbb{R}^l[s]$ such that for some initial value $\underline{p}^t \in \mathcal{R}$:

 (i) *deg* $\underline{u}(s) = k$ and *deg* $\underline{p}(s) = k - 1$ for some $k \geq r$.

 (ii) $(sI - A)\underline{p}(s) = B\underline{u}(s) - \underline{p}^t$

 (iii) $\underline{p}(s) = s^{k-1}\underline{p}_{k-1} + \cdots + s\underline{p}_1 + \underline{p}_0$,   (5.3.1)

In this case $\mathcal{R} = span\left\{ \underline{p}_{k-i}, i \in \tilde{k} \right\}$. In particular, if $k = r$, $\underline{p}(s)$ can be chosen such that the set of vectors $\left\{ \underline{p}_{r-i}, i \in \tilde{r} \right\}$ form a basis for $\mathcal{R}$. These conditions also satisfied with $\underline{p}^t = 0$.   ■

Considering the above results, the set of $\left\{ \underline{p}_{r-i}, i \in \tilde{r} \right\}$ can be selected to form a basis for $\mathcal{R}$. A characterization of the c.s. $\mathcal{R}$ in terms of the $\underline{p}(s)$ alone can be derived if we apply the full rank transformation:

$$\begin{bmatrix} N \\ B^+ \end{bmatrix} (NB = 0, B^+ B = I_l)$$ (5.3.2)

such that

$$N(sI - A)\underline{p}(s) = 0$$

$$\underline{u}(s) = B^{\dagger}(sI - A)\underline{p}(s)$$

(5.3.3)

where $B^{\dagger}$ is a left inverse of B.

Generally for a c.s. $\mathcal{R}$ of dimensions r, there is no unique minimal degree polynomial that corresponds to $\mathcal{R}$. So there exists a minimal degree polynomial solution to the equations in (5.3.3.) for each impulse direction $\underline{b}$ where, $\left\{ \underline{p}_{r-i}, i \in \tilde{r}, \underline{p}_{r-1} = \underline{b} \right\}$. These solutions can form a basis which can be written in matrix form as:

$$P = \left[\underline{p}_0 \vdots \underline{p}_1 \vdots \ldots \vdots \underline{p}_{r-1} \right], \underline{p}_{r-1} = \underline{b}$$

(5.3.4)

and the polynomial $\underline{p}(s)$ from above basis P corresponding to the generator $\underline{p}_{r-1} = b$ can be written as

$$\underline{p}(s) = p_0 + sp_1 + \cdots + s^{r-1}p_{r-1}$$

(5.3.5)

or

$$\underline{p}(s) = \left[\underline{p}_0 \vdots \underline{p}_1 \vdots \ldots \vdots \underline{p}_{r-1} \right] \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{r-1} \end{bmatrix}$$

(5.3.6)

So we can introduce minimal basis $P(s) = [\underline{p}_1(s), \underline{p}_2(s), \cdots, \underline{p}_l(s)]$, with the set of indices $\varepsilon_1, \cdots, \varepsilon_l$ respectively. These indices are so called the controllability indices, since by this basis, the whole controllability space can be generated or, basically the set $\{ \underline{p}_0, \underline{p}_1, \ldots, \underline{p}_{r-1} \}$ of any $P_{r-1}(s)$ spans the whole controllability space:

$$sp\left\{ \underline{p}_0, \underline{p}_1, \ldots, \underline{p}_{r-1} \right\} \subset \mathcal{X}.$$

In other words, because the impulse directions of these vectors are linearly independent, each of these polynomial vectors defines a minimal c.s. $\mathcal{R}$, such that

$\mathcal{R} = \left\{ \underline{p}_0, \underline{p}_1, \cdots, \underline{p}_{r-1} \right\}$ is of dimension r, and the whole controllability subspace will be:

$$\mathcal{X} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots \oplus \mathcal{R}_r \qquad (5.3.7)$$

where in general, the dimension of subspaces are from δ_1 to δ_r respectively.

Note that the set of minimal polynomials $p_{r-1}(s)$ dictates a decomposition on the state space $\mathcal{X}$ which, however, owing to the non-uniqueness of the fundamental series $\left\{ \underline{p}_{r-i}, i \in \tilde{r}, \underline{p}_{r-1} = b \right\}$, is itself not unique, then the different parameterisation can be written for this controllability space.

Also the input-state pencil $sN - NA$ corresponding to the pair (A, B) and its Kronecker canonical form [104], is invariant under state/output feedback, that is, for the closed-loop pair $(A - BL, B)$ we have: $sN - N(A - BL) = sN - NA$, since $NB = 0$.

Thus by finding any $\underline{p}_{r-1}(s)$ from the minimal basis matrix $P(s) = [\underline{p}_1(s), \underline{p}_2(s), \cdots, \underline{p}_l(s)]$, basically the closed-loop eigenvector matrix of the original $(A - BL, -B)$ for any certain set of closed-loop eigenvalues $\{s_0, s_1, \cdots, s_{r-1}\}$ has been found.

Any relative bases $\underline{p}_{r-1}(s) \in P(s)$ will be of form

$$p(s) = [\underline{p}(s_1) \vdots \underline{p}(s_2) \vdots \cdots \vdots \underline{p}(s_{r-1})] = [\underline{p}_0, \underline{p}_1, \cdots, \underline{p}_{r-1}] \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_{r-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_1^{r-1} & s_2^{r-1} & \cdots & s_{r-1}^{r-1} \end{bmatrix} \qquad (5.3.8)$$

where $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_{r-1} \\ \vdots & \vdots & \cdots & \vdots \\ s_1^{r-1} & s_2^{r-1} & \cdots & s_{r-1}^{r-1} \end{bmatrix}$ is the Vandermonde matrix generated by the distinct

frequencies of the system. Clearly, and as we discussed earlier, $\left[\underline{p}_0, \underline{p}_1, \cdots, \underline{p}_{r-1} \right]$ is full rank.

It can be seen that the optimal choice of the Vandermonde matrix can have a big effect on the properties of the choice of minimal bases for the kernel of $sN - NA$, for any choice of the closed-loop poles.

By finding the optimal choice of the bases and also by considering the fact that the dimensions of all the controllability subspaces are assumed to be fixed, within our current research, one result has been derived achieving the algebraic parametrisation of (5.3.7) using Toeplitz matrix representation.

This parametrization is useful where the best (minimal) selection of controllability subspaces is needed in order to form a controllability space $\mathcal{X}$, i.e. the dimension of each subspace remains fixed but the angle between subspaces to be maximised.

5.3.2 Minimal dimension Controllability Subspaces

Consider a set of linearly independent polynomial vectors $\underline{\tilde{p}}_{\sigma_i - 1}(s), i = 1, 2, ..., l$ of minimal degrees that span the kernel of the input-state pencil $s\tilde{N} - \tilde{N}\tilde{A}$. Because the impulse directions of these polynomial vectors are linearly independent, each of the polynomial vectors $\underline{\tilde{p}}_{\sigma_i - 1}(s)$ defines a minimal c.s. $\mathcal{R}_{\sigma_i}$ of dimension σ_i. The set of subspaces $\left\{ \mathcal{R}_{\sigma_i}, i \in l \right\}$ is a linearly independent set and

$$\mathcal{X} = \mathcal{R}_{\sigma_i} \oplus \mathcal{R}_{\sigma_2} \oplus ... \oplus \mathcal{R}_{\sigma_l}. \tag{5.3.9}$$

Thus, the set of minimal polynomials $\underline{\tilde{p}}_{\sigma_i - 1}(s)$ dictates a decomposition on the state space $\mathcal{X}$ which, however, owing to the non-uniqueness of the fundamental series $\left\{ \underline{\tilde{p}}_{\sigma_i - 1}(s), i \in \tilde{l} \right\}$, is not unique. It can be readily verified that each of the subspaces $\mathcal{R}_{\sigma_i}$ covers the basis vector $\hat{e}_{\sigma_i}$ of $\hat{\mathcal{B}}$ where:

$$\hat{e}_{\sigma_i} = \left[\underbrace{0...0}_{\sigma_1} \vdots ... \vdots \underbrace{0...01}_{\sigma_i} \vdots 0...0 \right]^T, i = 1, ..., l \tag{5.3.10}$$

The problem of finding the minimal c.s. which is constrained to contain a given subspace of $\mathcal{B}$ has been discussed by Warren and Eckberg (1975) [71] and it was

developed as far as geometric theory in [96]; it is our intension here to use the structure of the basis matrix to derive these results in a simpler way.

The study of the polynomial vectors $\underline{p}(s)$ restricted in the kernel space of the pencil $(sN - NA)$, has used the Kronecker canonical form of the pencil. It was shown that if T is the coordinate transformation bringing the pair (A, B) in the controllable companion form, then a mere multiplication of $(sN - NA)$ on the right by T^{-1} brings the pencil in the Kronecker canonical form. The transformation T^{-1} belongs to the class of strict equivalent transformations and, as such, does not affect the Kronecker canonical form.

In fact, it has been shown [4] that any state transformation T that brings the pair (A, B) to its Luenberger form $(\tilde{A}, \tilde{B})$ has a corresponding pencil $s\tilde{N} - \tilde{N}\tilde{A}$ in its Kronecker form. This result will be used for the parameterization of controllability subspaces. Another important set of transformations on the pair (A, B), is the set of state/output feedback transformations; the input-state pencil that corresponds to a closed loop pair $(A - BL, B)$ is $sN - N(A - BL) = sN - NA$, since $NB = 0$. Thus we are led to the following theorem.

Theorem 5.2: [96], [104] The input-state pencil $(sN - NA)$ corresponding to the pair (A, B) and its Kronecker canonical form are invariant under state/output feedback.

∎

We note finally that for a given polynomial vector $\underline{p}(s)$ satisfying condition:

$$(sN - NA)\underline{p}(s) = 0 \tag{5.3.11}$$

the input polynomial vector corresponding to $\underline{p}(s)$ is defined by:

$$\underline{u}(s) = B^{\dagger}(sI - A)\underline{p}(s). \tag{5.3.12}$$

By using the Luenberger form for (A,B) [104], the pencil $sN\text{-}NA$ is expressed in the Kronecker canonical form $s\tilde{N} - \tilde{N}\tilde{A}$ and this has the following canonical description that allows parameterization of the controllability subspaces.

$$
s\tilde{N} - \tilde{N}\tilde{A} = \begin{bmatrix}
\begin{array}{cccccc}
s & -1 & 0 & \dots & 0 & 0 \\
0 & s & -1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & s & -1
\end{array} & & \mathbf{O} \\
& \ddots & \\
\mathbf{O} & & \begin{array}{cccccc}
s & -1 & 0 & \dots & 0 & 0 \\
0 & s & -1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \dots & s & -1
\end{array}
\end{bmatrix}
\tag{5.3.13}
$$

The canonical description of the restriction pencil $s\tilde{N} - \tilde{N}\tilde{A}$ above provides the means for parameterizing all polynomial vectors $\tilde{p}(s)$ which satisfy the equivalent conditions (5.3.11) and this in turn provides the means to refer back the results to the original frame by using the inverse transformation T^{-1}. In fact, note that (5.3.11) is equivalent to

$$
(sN - NA)T^{-1}T\underline{p}(s) = 0 \Leftrightarrow (s\tilde{N} - \tilde{N}\tilde{A})\underline{\tilde{p}}(s) = 0,\; \underline{\tilde{p}}(s) = T\underline{p}(s)
\tag{5.3.14}
$$

and thus any parameterization defined on $s\tilde{N} - \tilde{N}\tilde{A}$ may be transferred back to the original frame.

The polynomial vector $\underline{u}(s)$ described above is one of the input polynomials associated with the c.s. defined by $\underline{p}(s)$; it has a further property that if the state feedback operator L defined by the set of conditions

$$
L\underline{p}_{d-i} = \underline{u}_{d-i},\, i \in \tilde{d}
\tag{5.3.15}
$$

is applied around the system, then the c.s. $\mathcal{R}_d$, becomes cyclic under $(A+BL)$ with as generator the vector $\underline{p}_{d-i} = B\underline{u}_{d-i}$ for which $(A+BL)^d \underline{p}_{d-i} = 0$.

The Kronecker canonical form of the input-state pencil $(sN - NA)$, $s\tilde{N} - \tilde{N}\tilde{A}$ used in the derivation of the basis matrix description of the polynomial vectors that lie in the $Ker(s\tilde{N} - \tilde{N}\tilde{A})$ and its parameterization [8], **[96]** is described in chapter Seven.

The decomposition into minimal dimension controllability subspaces also introduces an alternative algorithm for pole assignment that results directly into a full rank state feedback which is an issue treated in the following section.

5.4. Assignability of the spectrum of a Controllability Subspace

5.4.1. Problem Statement

The family of controllability subspaces [9], [103] are special types of (A,B)-invariant subspaces that intersect with the range space $\mathcal{B}$. In fact controllability subspaces are (A,B)-invariant subspaces with the property that any two points may be connected by some appropriate trajectory generated by a control input with the property that the trajectory always remains in the given space [9]. Their spectra are not fixed, and so the question arises as to whether or not such subspaces may assume any given spectrum. An alternative to the solution already established, [4], based on an eigenvector approach is proposed here and involves the construction of characteristic bases having as a spectrum the set of assignable frequencies. This section provides an alternative parameterisation of eigenframes based on the property that such frames are arbitrarily assignable spectra that are characteristic bases of controllability subspaces explained in [4]. The results in this section provide an eigenvalue assignment algorithm that conveniently follows the approach mentioned above.

5.4.2. Assigning the spectrum of a Controllability Subspace

An alternative establishment of the classical result of the geometric theory is considered here [4]. Consider first the following lemma:

Lemma 5.3: [4], [36] Let $\mathcal{R}$ be a c.s. of the pair (*A,B*) and $\{\underline{u}_j\}$ a characteristic basis for $\mathcal{R}$. A vector control input $\underline{u} \in \mathcal{R} \cap \mathcal{B}$ can always be found such that

$$\underline{u} = \sum_{j=1}^{r} a_j \underline{u}_j = B\underline{m}$$

$$a_j \neq 0 \ \ \forall j, \ j = 1, \ldots, r, \ r = \dim \mathcal{R}$$

$$(5.4.1)$$

Proof: [4] With respect to the basis $\{\underline{u}_j\}$, the vector $\underline{u}$ may be written as

$$\underline{u} = \begin{bmatrix} \underline{u}_1 & \vdots & \underline{u}_2 & \vdots & \ldots & \vdots & \underline{u}_r \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix} = BG\underline{r} = \hat{B}\underline{r}$$

$$(5.4.2)$$

where *G* (*p×r*) (*p≤r*) is the input transformation gain matrix such that the space $\mathcal{R}$ is generated by vectors in the range of $\hat{B}$, $\hat{\mathcal{B}}$, i.e. $\mathcal{R} = \left\{ (A + BL) \middle/ \hat{\mathcal{B}} \right\}$. For some state feedback matrix *L* the vectors of the basis $\{\underline{u}_j\}, j = 1, \cdots, r$, become a subset of the eigenvectors of the matrix $(A + BL)$ defined by the columns of the matrix *U*. If *V* defines the dual eigenvector frame to *U* and if $\mathcal{R}$ is the controllable subspace of the pair $(A + BL, \hat{B})$, then matrix $V\hat{B}$ has no row that contains all zero elements. Multiplying (5.4.2) on the left by *V* gives

$$a_i = v_i^t \hat{B}\underline{r}$$

$$(5.4.3)$$

where v_i^t denotes the rows of *V*. Since none of the $v_i^t \hat{B}$ rows are zero, $\underline{r}$ can always be chosen such that $a_i \neq 0$. Then $\underline{m} = G\underline{r}$. Having established this lemma the main results of this section will now be stated, which is the assignment to $\mathcal{R}$ of a characteristic basis having any given spectrum. ∎

Theorem 5.3: [4], [36] Let $\mathcal{R}$ be a c.s. of the pair (*A,B*) and $\{\underline{u}_j\}$, *j* = 1, ..., *r* a characteristic basis of $\mathcal{R}$, *r* = $\dim \mathcal{R}$. A new characteristic basis $\left\{ \underline{u}_{\mu_i} \right\}$ of $\mathcal{R}$ can

always be found such that the spectrum associated with $\{\underline{u}_{\mu_i}\}$ is any given $\{\mu_i\}$, $i = 1, \ldots, r$.

Proof: For the sake of simplicity it is assumed that $\{\underline{u}_i\}$ is a characteristic basis of $\mathcal{R}$ and has a simple structure that corresponds to eigenvalues with a diagonalisable Jordan form. Then

$$A\underline{u}_i = \lambda_i \underline{u}_i + B\underline{k}_i \tag{5.4.4}$$

Making the further assumption that the assignable spectrum $\{\mu_i\}$, $i = 1, \ldots, r$ consists of distinct frequencies, then

(i)   Assume that $\{\mu_i\} \cap \{\lambda_j\} = \varnothing$ $\forall$ $i, j, i, j = 1, \ldots, r$, where 0 is the zero space.

Making use of Lemma 5.3, a vector $\underline{u} = \sum\limits_{j=1}^{r} a_j \underline{u}_j = B\underline{m}$ with $a_i \neq 0$ and vectors $\underline{u}_{\mu_i}$, $\underline{k}_{\mu_i}$ can be found such that

$$A\underline{u}_{\mu_i} = \mu_i \underline{u}_{\mu_i} + B\underline{k}_{\mu_i} \tag{5.4.5}$$

where

$$\underline{u}_{\mu_i} = \begin{bmatrix} \underline{u}_1 & \vdots & \underline{u}_2 & \vdots & \ldots & \vdots & \underline{u}_r \end{bmatrix} \begin{bmatrix} a_1(\mu_i) \\ a_2(\mu_i) \\ \vdots \\ a_r(\mu_i) \end{bmatrix} = \begin{bmatrix} \underline{u}_1 & \vdots & \underline{u}_2 & \vdots & \ldots & \vdots & \underline{u}_r \end{bmatrix} \operatorname{diag}\{a_i\} \begin{bmatrix} \dfrac{1}{\mu_i - \lambda_1} \\ \dfrac{1}{\mu_i - \lambda_2} \\ \vdots \\ \dfrac{1}{\mu_i - \lambda_r} \end{bmatrix}.$$

$$\tag{5.4.6}$$

The set of r vectors defined this way can be written in a matrix form as follows

$$U_\mu = U D_a M_{\mu,\lambda} \tag{5.4.7}$$

where U_μ designates the matrix having as columns the vectors $\underline{u}_{\mu_i}$, U is the matrix having as columns the vectors $\underline{u}_i$, D_a the diagonal matrix of the a_i elements, and finally by $M_{\mu,\lambda}$ the matrix with its entries defined by $\delta_{i,j}(\mu,\lambda) = 1/(\mu_j - \lambda_i)$. Because the elements of D_a are nonzero, it always has full rank. Furthermore the matrices $M_{\mu,\lambda}$ always have full rank

whenever the sets $\{\mu_i\}$, $\{\lambda_i\}$ have no common element between them. Thus the matrix U_μ has full column rank and the vectors $\{\underline{u}_{\mu_i}\}$ form a basis for R with the desirable spectrum.

(ii)   Now assume that the $\{\mu_i\}$, $\{\lambda_i\}$ sets have some common elements. In that case a new distinct spectrum, $\{\xi_i\}$, may be defined such that $\{\xi_i\} \cap \{\lambda_j\} = \varnothing$ and $\{\xi_i\} \cap \{\mu_k\} = \varnothing$ $\forall\, i, j, k$. To the spectrum $\{\xi_i\}$, there will correspond a new basis $\{\underline{u}_{\xi_i}\}$ which according to condition (5.4.7) can be derived from:

$$U_\xi = U D_a M_{\xi,\lambda} \tag{5.4.8}$$

The vector $\underline{u} \in \mathcal{R} \cap \mathcal{B}$ is now expressed with respect to the new basis $\{\underline{u}_{\xi_i}\}$ as

$$\underline{u} = U\underline{a} = U_\xi M_{\xi,\lambda}^{-1} D_a^{-1} \underline{a} = U_\xi M_{\xi,\lambda}^{-1} \underline{e} = \sum_{i=1}^{r} a_{i\xi} \underline{u}_{i\xi} \tag{5.4.9}$$

with

$$\underline{a}^t = \begin{bmatrix} a_1, & a_2, & \ldots, & a_r \end{bmatrix}$$
$$\underline{e}^t = \begin{bmatrix} 1, & 1, & \ldots, & 1 \end{bmatrix} \tag{5.4.10}$$
$$\underline{a}_\xi = M_{\xi,\lambda}^{-1} \underline{e}$$

By Lemma 5.3 it is evident that $a_{i\xi} \neq 0\, \forall\, i$. The new basis $\{\underline{u}_{\xi_i}\}$ with the desired spectrum $\{\mu_i\}$ can be easily determined using (5.4.7) with the assumption that $\{\xi_i\} \cap \{\mu_i\} = \varnothing$.

The above theorem implies that, given the characteristic basis $\{\underline{u}_i\}$ for a c.s., $\mathcal{R}$, all that is needed to generate a new characteristic basis $\{\underline{u}_{\mu_i}\}$ which will have as its spectrum the prescribed set of frequencies $\{\mu_i\}$ is a vector $\underline{u} = B\underline{m} = \sum_{j=1}^{r} a_j \underline{u}_j$. It thus appears appropriate to refer to the vector $\underline{u}$ as the "generator" of the c.s. $\mathcal{R}$. It is worth noting that due to the minimal property of a c.s. $\mathcal{R}$, that the generator $\underline{u}$ can

be chosen to be any vector $\underline{u} \in \mathcal{R} \cap \mathcal{B}$. The characterisation of the basis $\left\{\underline{u}_{\mu_i}\right\}$ in terms of its spectrum is given in matrix form by the following condition,

$$AU_\mu = U_\mu \Lambda_\mu + BK_\mu$$

(5.4.11)

where for generality the matrix Λ_μ is assumed to have a Jordan block structure. Since U_μ has full column rank, a state feedback matrix L can always be found such that

$$LU_\mu = -K_\mu \ .$$
(5.4.12)

Then (5.4.11) and (5.4.12) yield

$$(A + BL)U_\mu = U_\mu \Lambda_\mu$$
(5.4.13)

■

These results may be summarised in the following corollary.

Corollary 5.1: [4], [36] Given a c.s. $\mathcal{R}$ and a set of frequencies $\{\mu_i\}$, $i = 1, \ldots, r, r = dim\,\mathcal{R}$, there always exists a state feedback matrix L such that the restriction $(A + BL)/\mathcal{R}$ has the set $\{\mu_i\}$ as its spectrum. ■

If the pair (A, B) is controllable, then the whole state space $\mathcal{X}$ is a c.s. since $\mathcal{X} \cap \mathcal{B} = \mathcal{B}$ and $\mathcal{X} = \{A/\mathcal{X} \cap \mathcal{B}\} = \{A/\mathcal{B}\}$. Thus the theorem for the assignability of the poles by state feedback stated [4] follows immediately if Corollary (5.1) is used. This theorem is stated as follows.

Theorem 5.4: [4], [36] Let (A, B) be a controllable pair and let $\{\mu_i\}$, $i = 1, \ldots, n$, be a set of complex numbers symmetrically distributed along the real axis. There always exists a state feedback matrix L which assigns the frequencies μ_i's as closed loop eigenvalues of the dynamic map $A_c = A - BL$. ■

The above theorem provides a closed loop eigenvector based alternative proof to the assignability of the spectrum of a c.s. Unlike the original approach in [4] which was based on the definition of characteristic polynomials of cyclic subspaces, the treatment given in this section constitutes an eigenvector approach in as far as it is based on the construction of characteristic bases. Next, a pole assignment algorithm is proposed which is based on the concepts outlined.

5.4.3 Eigenvalue placement algorithm based on mobility of open to closed loop spectra

The above eigenvector approach to the fundamental theorem of assignability of the closed loop eigenvalues yields an algorithm for eigenvalue placement that involves the following fundamental steps based on [4]:

PROCEEDURE FOR EIGENVALUE PLACEMENT

The assignment problem follows the subsequent steps:

(i) Given *A*, the set of eigenvalues and the corresponding eigenvectors $\{\lambda_i, \underline{u}_i\}$ are first found. The vectors $\underline{u}_i$ form a basis for the c.s. $\mathcal{R} = \mathcal{X}$ with the corresponding input directions $\underline{k}_i = 0 \; \forall \; i = 1, \ldots, n$.

(ii) If $\{\mu_i\}$ is the assignable spectrum it is safe to always assume that $\{\mu_i\} \cap \{\lambda_i\} = \varnothing$. This is admissible since if $\{\mu_i\} \cap \{\lambda_i\} = \varnothing$ then it may necessary to resort to the technique suggested by equations (5.2.8) and (5.2.9) and thus define a new basis with spectrum $\{\xi_i\}$ for which $\{\mu_i\} \cap \{\lambda_i\} = \varnothing$. Alternatively it is possible to initially apply an arbitrary state feedback which without changing the controllability properties of the pair (*A*, *B*) that scatters the closed loop poles to a new spectrum $\{\lambda_i'\}$ such that $\{\lambda_i'\} \cap \{\mu_i\} = \varnothing$.

(iii) A generator $\underline{u}$ of the c.s. is in the form $\underline{u} = \sum\limits_{i=1}^{n} a_i \underline{u}_i = B\underline{m}$. If $\underline{v}_i^t$ denotes the eigenvectors dual to $\underline{u}_i$, then the set a_i is given by

$$a_i = \underline{v}_i^t B\underline{m} \quad i = 1,\ldots,n \tag{5.4.14}$$

Since the pair (A,B) is controllable, none of the $\underline{v}_i^t B$ vectors are zero and the vector $\underline{m}$ may be chosen such that each a_i is non zero.

(iv) Given the sets of the frequencies $\{\mu_i\}$, $\{\lambda_i\}$ such that $\{\mu_i\} \cap \{\lambda_i\} = \varnothing$ and having found the coefficients of a_i, the basis $\{\underline{u}_{\mu_i}\}$ may be defined by using the following conditions

$$U_\mu = U D_a M_{\mu,\lambda}$$

$$\underline{u}_p(\mu_i) = \sum_{j=1}^{r} a_j^{(p)}(\mu_i)\underline{u}_j \tag{5.4.15}$$

$$\underline{k}_p(\mu_i) = -\underline{m} + \sum_{j=1}^{r} a_j^{(p)}(\mu_i)\underline{k}_j$$

where the coefficients $a_j^{(p)}(\mu_i)$ are defined by the following expressions

$$a_1^{(p)} = \sum_{j=1}^{v} a_j \delta_{j,p}(\mu,\lambda_1),$$

$$a_2^{(p)} = \sum_{j=2}^{v} a_j \delta_{j-1,p}(\mu,\lambda_1),\ldots$$

$$a_{v-1}^{(p)} = \sum_{j=v-1}^{v} a_j \delta_{j-v+2,p}(\mu,\lambda_1), \tag{5.4.16}$$

$$a_v^{(p)} = a_v \delta_{1,p} 0(\mu_1,\lambda_1)$$

$$a_i^{(p)} = a_i \delta_{1,p}(\mu,\lambda_1),$$

$$i = v+1,\ldots,r$$

where the functions $\delta_{\tau,p}(\mu,\lambda_i)$ are given by

$$\delta_{\tau,p}(\mu,\lambda_i) = \frac{s_\tau(1)}{(\mu - \lambda_i)^\tau} - \frac{s_\tau(2)}{(\mu - \lambda_i)^{\tau+1}} + \cdots + (-1)^{p-1} \frac{s_\tau(p)}{(\mu - \lambda_i)^{\tau+p-1}} \tag{5.4.17}$$

where $s_\tau(v)$, τ, $v = 1,\ldots$ denote the elements of the sequence

$$s_\tau(v+1) - s_\tau(v) = s_{\tau-1}(v+1) \tag{5.4.18}$$

(v) The input directions corresponding to the vectors $\{\underline{u}_{\mu_i}\}$ are then

$$k_{\underline{\mu}_i} = -\underline{m} + \sum_{j=1}^{n} a_j(\mu_i)\underline{k}_j = -\underline{m} \tag{5.4.19}$$

Since every $\underline{k}_j = 0 \ \forall \ j = 1, \ldots, n$ every A-invariant subspace is also (A,B)-invariant with zero input directions.

(vi) The state feedback matrix is now defined to be

$$L\left[\underline{u}_{\mu_1} \mid \underline{u}_{\mu_2} \mid \ldots \mid \underline{u}_{\mu_n}\right] = -\left[\underline{m} \mid \underline{m} \mid \ldots \mid \underline{m}\right]. \tag{5.4.20}$$

Because $\underline{u}_{\mu_i}$ is linearly independent the matrix $V_\mu = U_\mu^{-1}$ exists and

$$L = -\left[\underline{m} \mid \underline{m} \mid \ldots \mid \underline{m}\right]V_\mu \tag{5.4.21}$$

(vii) The closed loop dynamic map A_c is then given by

$$A_c = A - BL = U_\mu \text{diag}\{\mu_i\}V_\mu \tag{5.4.22}$$

and thus can be computed without needing to work out the state feedback matrix L. ∎

It is worth noting that the eigenvalue assignment algorithm presented here yields a unity rank state feedback matrix L. This is due to the fact that the matrix of the input directions corresponding to the closed loop eigenvectors is of unity rank. An alternative approach leading to a full rank state feedback matrix can be formulated as follows.

Given A, an arbitrarily state feedback with a matrix L_0 having full rank may be applied. For the new matrix $A_0 = A - BL_0$, the previously described algorithm may be applied, yielding a unity rank state feedback matrix L_u assigning the poles of A_0 at the desirable locations. The controller $L = L_0 + L_u$ is in general in full rank and assigns the eigenvalues of A at the desired locations.

The essence of the proposed modification is that instead of using the eigenframe of A as a characteristic basis of $\mathcal{X}$ with an associated set of input directions zero, any other characteristic basis of $\mathcal{X}$ may be used with a full rank set of corresponding input directions.

Such a basis may be defined as the eigenframe U_0 of some closed loop matrix $A_0 = A - BL_0$, where L_0 is a state feedback matrix having full rank. The input directions corresponding to this new eigenframe are given by $L_0 u_{i_0}$ and the resulting matrix formed, K_0 is of full rank. The successive application of the steps detailed above yield a full rank state feedback matrix L in general.

5.4.4. Numerical examples

Example 5.1: Let us assume that for an open-loop state-state model, we have the following A and B matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Then the set of eigenvalues of A and their corresponding eigenvectors are:

$$\lambda_1 = 1 \, , \, \lambda_2 = 2 \, , \, \lambda_3 = 3$$

$$\underline{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} , \, \underline{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} , \, \underline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system is controllable and therefore is pole assignable.

    i)   Assume the desirable closed-loop frequencies to be

          $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$, then we have that $\{\mu_i\} \cap \{\lambda_i\} = \phi$ and hence no modification to the A matrix is needed.

    ii)   Considering the fact that any generator $\underline{u}$ of the c.s. is in the form of

          $\underline{u} = \sum_{i=1}^{n} a_i \underline{u}_i = B\underline{m}$ and using equation (5.4.14): $a_i = \underline{v}_i^t B\underline{m} \; i = 1,\ldots,n$, where $\underline{v}_i^t$ denotes the eigenvectors dual to u_i, a non-zero vector $\underline{m}$ will be chosen such that any coefficients a_i is non-zero.

$$\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or } \underline{m} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is easily seen that

$$\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sum_{i=1}^{3} \alpha_i u_i = (1) \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + (4) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or that $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 4$.

iii)   From conditions (5.4.15) we have that

$$\underline{u}_{\mu_i} = \frac{1}{\mu_i - 1} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{\mu_i - 2} \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \frac{1}{\mu_i - 3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore the vectors $\underline{u}_{\mu_i}$ are

$$\underline{u}_{\mu_1} = \frac{1}{60} \begin{bmatrix} 10 \\ -10 \\ -10 \end{bmatrix}, \; \underline{u}_{\mu_2} = \frac{1}{60} \begin{bmatrix} 5 \\ -10 \\ -3 \end{bmatrix}, \; \underline{u}_{\mu_3} = \frac{1}{60} \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

iv)   The input directions then are given

$$\underline{k}_{\mu_1} = \underline{k}_{\mu_2} = \underline{k}_{\mu_3} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

v)   So that the state feedback matrix L can be derived as follow

$$L = 60 \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 & 5 & 3 \\ -10 & -10 & -9 \\ -10 & -3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 21 & 12 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

vi)   Finally the matrix $A_c = A - BL$

$$A_c = A - BL = \begin{bmatrix} 0 & 1 & 0 \\ -23 & -9 & -15 \\ 5 & 1 & 3 \end{bmatrix}$$

having -1, -2, -3 as eigenvalues and the $\underline{u}_{\mu_i}$'s as eigenvectors.

Example 5.2: Let A and B be

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ -4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ -6 & -3 & 3 \\ 4 & -4 & 1 \end{bmatrix} \cdot \tfrac{1}{9}$$

$$\underset{\equiv U}{} \qquad \underset{\equiv \Lambda}{} \qquad \underset{\equiv V}{}$$

and let the assignable spectrum be $\mu_1 = \mu_2 = \mu_3 = -5$. The above pair is controllable

and $\{\mu_i\} \cap \{\lambda_i\} = \varnothing$.

1.   Choosing the generator vector $\underline{u}$ as

$$\underline{u} = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 9 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

   $m^t = [0\ 9]$ and $a_1 = 1$, $a_2 = -3$, $a_3 = -4$.

2.   The closed loop eigenvectors are given by

$$\underline{u}_\mu^{(1)} = a_1^{(1)}\underline{u}_1 + a_2^{(1)}\underline{u}_2 + a_3^{(1)}\underline{u}_3$$

$$\underline{u}_\mu^{(2)} = a_1^{(2)}\underline{u}_1 + a_2^{(2)}\underline{u}_2 + a_3^{(2)}\underline{u}_3$$

$$\underline{u}_\mu^{(3)} = a_1^{(3)}\underline{u}_1 + a_2^{(3)}\underline{u}_2 + a_3^{(3)}\underline{u}_3$$

where

$$a_1^{(1)} = \frac{a_1}{(\mu-\lambda_1)} + \frac{a_2}{(\mu-\lambda_1)^2} = -\frac{7}{16},$$

$$a_2^{(1)} = \frac{a_2}{(\mu-\lambda_1)} = \frac{12}{16}, \quad a_3^{(1)} = \frac{a_3}{(\mu-\lambda_3)} = \frac{64}{16}$$

$$a_1^{(2)} = a_1\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2}\right\} + a_2\left\{\frac{1}{(\mu-\lambda_1)^2} - \frac{2}{(\mu-\lambda_1)^3}\right\} = \frac{-19}{32}$$

$$a_2^{(2)} = a_2\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2}\right\} = \frac{30}{32}, \quad a_3^{(2)} = a_3\left\{\frac{1}{\mu-\lambda_3} - \frac{1}{(\mu-\lambda_3)^2}\right\} = \frac{256}{32}$$

$$a_1^{(3)} = a_1\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2} + \frac{1}{(\mu-\lambda_1)^3}\right\} +$$

$$a_2\left\{\frac{1}{(\mu-\lambda_1)^2} - \frac{2}{(\mu-\lambda_1)^3} + \frac{3}{(\mu-\lambda_1)^4}\right\} = \frac{-165}{256}$$

$$a_2^{(3)} = a_2\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2} + \frac{1}{(\mu-\lambda_1)^3}\right\} = \frac{252}{256}$$

$$a_3^{(3)} = a_3\left\{\frac{1}{\mu-\lambda_3} - \frac{1}{(\mu-\lambda_3)^2} + \frac{1}{(\mu-\lambda_3)^3}\right\} = \frac{3072}{256}$$

and

$$\underline{u}_\mu^{(1)} = \frac{1}{16}\begin{bmatrix} 45 \\ -90 \\ 36 \end{bmatrix}, \quad \underline{u}_\mu^{(2)} = \frac{1}{32}\begin{bmatrix} 207 \\ -324 \\ 180 \end{bmatrix}, \quad \underline{u}_\mu^{(3)} = \frac{1}{256}\begin{bmatrix} 2655 \\ -3654 \\ 2412 \end{bmatrix}$$

3. The input directions corresponding to the vectors $\underline{u}_\mu^{(i)}$ are

$$\underline{k}_\mu^{(1)} = \underline{k}_\mu^{(2)} = \underline{k}_\mu^{(3)} = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$$

4. The state feedback matrix is given by

$$L = 256\begin{bmatrix} 0 & 0 & 0 \\ -9 & -9 & -9 \end{bmatrix}\begin{bmatrix} 720 & 1656 & 2655 \\ -1440 & -2592 & -3654 \\ 576 & 1440 & 2412 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 39 & 9 & -30.25 \end{bmatrix}$$

5. For the matrix L the closed loop matrix $A_c = A - BL$ becomes

$$A_c = \begin{bmatrix} -3 & 1 & 0 \\ -39 & -12 & 31.25 \\ -4 & 0 & 0 \end{bmatrix}$$

The eigenvalues of A_c are at $\mu_1 = \mu_2 = \mu_3 = -5$ while $\underline{u}_\mu^{(1)}$ is an eigenvector and $\underline{u}_\mu^{(2)}$ and $\underline{u}_\mu^{(3)}$ are pseudo-eigenvectors if A_c.

The final example given here is intended to illustrate the modified algorithm which yields a full rank matrix L.

Example 5.3: Let A and B be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

By applying an arbitrary state feedback by the matrix L_0

$$L_0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

a closed loop matrix $A_c = A - BL_0$ is obtained having the following set of eigenvalues and eigenvectors

$$\lambda_1 = 3,\ \underline{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \lambda_2 = 3,\ \underline{u}_2 = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix},\ \lambda_3 = 6,\ \underline{u}_3 = \begin{bmatrix} 3 \\ 18 \\ 17 \end{bmatrix}$$

The input directions corresponding to the $\underline{u}_i$ set are defined by $\underline{k}_i = L_0\underline{u}_i$ or

$$\underline{k}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \underline{k}_2 = \begin{bmatrix} -6 \\ 0 \end{bmatrix},\ \underline{k}_3 = \begin{bmatrix} -6 \\ -18 \end{bmatrix}$$

The sets $\underline{u}_i$ and $\underline{k}_i$ define the new characteristic basis of the controllable state space of $\mathcal{X}$. Choosing the generator $\underline{u}_m$ as $\underline{u}_m = B\underline{m}$ with $\underline{m}^t = [1\ 8\ 0]$ yields

$$\underline{u}_m = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix} = (-22)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 18 \\ 17 \end{bmatrix}$$

such that $a_1 = -22, a_2 = -1, a_3 = 1$. If the desired closed loop spectrum is $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$ then a new characteristic basis of X having the set μ_i as a spectrum is defined as follows

$$\underline{u}_{\mu_1} = \begin{bmatrix} 36 \\ -36 \\ -27 \end{bmatrix},\ \underline{k}_{\mu_1} = \begin{bmatrix} -324 \\ 36 \end{bmatrix},\ \underline{u}_{\mu_2} = \begin{bmatrix} 45 \\ -90 \\ -9 \end{bmatrix},\ \underline{k}_{\mu_2} = \begin{bmatrix} -810 \\ 90 \end{bmatrix},\ \underline{u}_{\mu_3} = \begin{bmatrix} 6 \\ -18 \\ 1 \end{bmatrix},\ \underline{k}_{\mu_3} = \begin{bmatrix} -174 \\ 18 \end{bmatrix}$$

The state feedback matrix L assigning the set of frequencies $\{\mu_i\}$ as eigenvalues of the closed loop matrix $A_c = A - BL$ is

$$L = \begin{bmatrix} -324 & -810 & -174 \\ 36 & 90 & 18 \end{bmatrix} \begin{bmatrix} 36 & 45 & 6 \\ -36 & -90 & -18 \\ -27 & -9 & 1 \end{bmatrix}^{-1}$$

which has full rank.

5.5. Selection of the most orthogonal eigenvectors and sensitivity of closed-loop system based on the open loop -closed loop spectra method

Choosing the best possible eigenvectors for a closed-loop control system, is generally one of the most important issues whenever it comes to the area of strong stability and robustness.

So based on this fact, there have been lots of works done in this area in order to achieve a closed-loop control system with the set of eigenvectors with the minimum condition number which results to low sensitivity of the whole system.

Considering above descriptions, our intention in this part is to formulate an optimization problem regarding closed-loop eigenvectors and represent the main variables of the formula which will be involved for the optimization problem. Note that the solution for the optimization problem will be presented in chapter 8.

5.5.1. Problem statement

Considering equation (5.4.6), then for any open-loop controllable system with n eigenvalues and relative eigenvectors, each closed-loop eigenvector $\underline{u}_{\mu_i}$ corresponding to a closed-loop eigenvalue μ_i, will be given by the following:

$$\underline{u}_{\mu_i} = \left[\underline{u}_1 \vdots \underline{u}_2 \vdots \cdots \vdots \underline{u}_n\right] \begin{bmatrix} a_1(\mu_i) \\ a_2(\mu_i) \\ \vdots \\ a_n(\mu_i) \end{bmatrix} = \left[\underline{u}_1 \vdots \underline{u}_2 \vdots \cdots \vdots \underline{u}_n\right] \begin{bmatrix} a_1 & & & \\ & a_2 & & 0 \\ & 0 & \ddots & \\ & & & a_n \end{bmatrix} \begin{bmatrix} \dfrac{1}{\mu_i - \lambda_1} \\[2mm] \dfrac{1}{\mu_i - \lambda_2} \\ \vdots \\ \dfrac{1}{\mu_i - \lambda_n} \end{bmatrix} \tag{5.5.1}$$

Let's take U_μ to be the matrix of all the eigenvectors $\underline{u}_{\mu_i}$'s computed from above equation, U to be the matrix having as columns the vectors $\underline{u}_i$, D_a the diagonal matrix of the a_i elements, and finally by $M_{\mu,\lambda}$ the matrix with its entries defined by $\delta_{i,j}(\mu,\lambda) = 1/(\mu_j - \lambda_i)$.

So equations (5.5.1) can be written as follows:

$$U_\mu = U \begin{bmatrix} \dfrac{a_1}{\mu_1 - \lambda_1} & \dfrac{a_1}{\mu_2 - \lambda_1} & \cdots & \dfrac{a_1}{\mu_n - \lambda_1} \\[3mm] \dfrac{a_2}{\mu_1 - \lambda_2} & \dfrac{a_2}{\mu_2 - \lambda_2} & \cdots & \dfrac{a_2}{\mu_n - \lambda_2} \\[3mm] \vdots & \vdots & \cdots & \vdots \\[3mm] \dfrac{a_n}{\mu_1 - \lambda_n} & \dfrac{a_n}{\mu_2 - \lambda_n} & \cdots & \dfrac{a_n}{\mu_n - \lambda_n} \end{bmatrix} \tag{5.5.2}$$

Based on conditions (5.4.14) : $a_i = \underline{v}_i^t B \underline{m} \quad i = 1, \ldots, n$.

This along with (5.5.2) will construct the following equation:

$$U_\mu = U. \begin{bmatrix} \dfrac{v_1^t B\underline{m}}{\mu_1 - \lambda_1} & \dfrac{v_1^t B\underline{m}}{\mu_2 - \lambda_1} & \cdots & \dfrac{v_1^t B\underline{m}}{\mu_n - \lambda_1} \\[3mm] \dfrac{v_2^t B\underline{m}}{\mu_1 - \lambda_2} & \dfrac{v_2^t B\underline{m}}{\mu_2 - \lambda_2} & \cdots & \dfrac{v_2^t B\underline{m}}{\mu_n - \lambda_2} \\[3mm] \vdots & \vdots & \cdots & \vdots \\[3mm] \dfrac{v_n^t B\underline{m}}{\mu_1 - \lambda_n} & \dfrac{v_n^t B\underline{m}}{\mu_2 - \lambda_n} & \cdots & \dfrac{v_n^t B\underline{m}}{\mu_n - \lambda_n} \end{bmatrix} \tag{5.5.3}$$

In above equation (5.5.3) U_μ represents the closed-loop eigenvector matrix, U is the open-loop eigenvector matrix, B is the input matrix of the system, left eigenvector v_i^t

is corresponding to any open-loop eigenvalue λ_i, $\underline{m}$ is a nonzero vector and $\mu_i\text{'}s$ are the desired closed-loop poles.

Now the problem arises here is to select the most orthogonal closed-loop eigenvector, so that the condition number of the U_μ matrix (or its Gramian matrix) is minimized (or equally, the determinant of its Gramian matrix is maximized).

Clearly the optimization problem will be dependent on the choice of vector $\underline{m}$ and also closed-loop poles μ_i's, since other parameters of equation (5.5.3) are fixed and are not to be changed.

So based on above description, there are two optimization problems to be solved. One is to optimize by taking vector $\underline{m}$ as the only degree of freedom, considering a fixed set of desirable closed-loop poles and the other case is to optimize by considering an affixed vector $\underline{m}$ and the closed-loop poles to be varied.

For both the above cases, one needs to use non-smooth optimization method, as for some points, the function of condition number of the closed-loop eigenvector matrix (or it's Gramian) may not be differentiable.

In this thesis, the condition number of the Gramian of closed-loop eigenvector matrix has been considered to be optimized over vector $\underline{m}$ (or equivalently the determinant of Gram matrix to be maximized) assuming to have fixed closed-loop eigenvalues. So the optimization formula will be as following:

- Suppose each entry of $V(\underline{x})$ is a continuously differentiable function of $\underline{x} \in \mathbb{R}^m$, then each entry of $A(\underline{x}) = U_\mu^t(\underline{x}).U_\mu(\underline{x})$ is also a continuously differentiable function of x. We consider the following minimization problem:

$$\text{minimize} \qquad \mathcal{K}(A(\underline{x}))$$

$$\text{subject to} \qquad \underline{x} \in \mathcal{X} \qquad\qquad (5.5.4)$$

where $\mathcal{X}$ is a convex set in $\mathbb{R}^m$.

5.6. Conclusion

The purpose of this chapter was to introduce alternative parameterizations of potential closed loop eigenstructures based on the properties of the fundamental concepts of $(A, B)-$invariant, $(A, B)-$invariant and controllability subspaces, c.s. Such results enable the derivation of a new approach based on the assignment of eigenvectors corresponding to the assignable spectrum rather than the assignment of characteristic polynomials.

The alternative formulation of the problem in terms of eigenvectors not only deepens the insight into the significance of c.s. but also yields solutions to the problem of pole assignment by state feedback and allows the introduction of new criteria related to the properties of the eigenframe, such as maximization of degree of orthogonality of the frame, role of pole mobility in the shaping of eigenframe properties etc.

This work led to the parametrization of the closed-loop eigenvectors obtained from this method which was also presented along with the optimization problem which will be studied in detail in chapter Eight, where by using non-smooth optimization algorithm, the best choice of closed-loop eigenframes will be computed by optimizing the condition number of Gram matrix related to the closed-loop eigenvector matrix.

CHAPTER 6

MEASUREMENT OF ANGLE BETWEEN SUBSPACES IN DIRECT SUM DECOMPOSITION

6.1. Introduction

The problem that frequently emerges in the study of performances of linear systems is the issue of *"skewness"* of eigenframes. This problem is linked to sensitivity of eigenvalues to parameter uncertainty, perturbations, as well as sensitivity of Nyquist diagrams to model parameter uncertainty.

These skewness properties are also linked to measures of controllability and observability, when these are assessed in their model setting. So far, the measure of skewness has been considered on eigenframes corresponding to distinct eigenvalues and thus standard tools such as the Gramian, Singular Value Decomposition, Condition Number, Sdur compliment can be used. However, frequently, we have eigenframes corresponding to repeated eigenvalues, complex eigenvalues, where a vector basis set is not uniquely defined, although the corresponding subspaces are.

The problem that is addressed here is the development of measures of *"skewness"* between subspaces defining a direct sum decomposition of the state space and thus developing a concept of angle between sets of subspaces.

The aim of the chapter is to provide the required new concept of the relative positioning between subspaces that can be used in quantifying:

- Sensitivity of eigenvalues
- Relative measures of controllability and observability.

- Deviations from strong stability to overshooting behaviour.

Our work is based on developing:

1. General properties of positioning of subspaces in direct sum decomposition.
2. Development of measures of skewness using:
   - The Gramian
   - Condition number
   - Spread of singular values

Our intention in this chapter is to produce some results which could provide the bases for the computation of the most orthogonal decomposition of the state space into controllability spaces. This is considered as a first step in selecting a set of closed –loop eigenvectors which are nearly orthogonal and thus achieve reduced sensitivity. This discussion involves parametrising the family of controllability subspaces using results on the parameterisation of minimal bases. The solution to the problem of finding the most orthogonal decomposition still remains open (future research).

6.2. Problem statement and preliminary results

Let us consider the direct sum decomposition of $\mathbb{R}^n$ in terms of some spaces $\mathcal{V}_i$ such that $\mathcal{V}_i \subset \mathbb{R}^n$, $\dim \mathcal{V}_i = \rho_i$, $i = 1, 2, \cdots, k$. i.e.

$$\mathbb{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k \tag{6.2.1}$$

The set of such spaces $\{\mathcal{V}_i, i \in k\}$ will be referred to as a decomposing set of $\mathbb{R}^n$. Clearly these spaces are linearly independent. What we want to investigate is the relative "degree" of independency between these spaces.

The spaces $\mathcal{V}_i$ are assumed given and may represent the generalised eigenspaces associated with repeated eigenvalues, or the two dimensional space associated with a pair of complex conjugate eigenvalues, or the higher order spaces associated with repeated complex eigenvalues.

Let V be a basis of $\mathbb{R}^n$ defined as:

$$V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \tag{6.2.2}$$

where V_i is a basis of $\mathcal{V}_i$. We can always assume that the columns of V_i are normalised to unit length. Clearly $V_i \in \mathbb{R}^{n \times \rho_i}$ and so for any square matrix Q such that:

$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix} , \quad Q_i \in \mathbb{R}^{\rho_i \times \rho_i} \text{ and } |Q_i| \neq 0, \quad i = 1, 2, \cdots, k,$$

any other basis of $\mathbb{R}^n$, consistent with the (6.2.1) decomposition is given by:

$$\tilde{V} = \left[\underline{\tilde{V}}_1 \middle| \underline{\tilde{V}}_2 \middle| \cdots \middle| \underline{\tilde{V}}_k \right] = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}. \tag{6.2.3}$$

Bases such as those defined above, will be referred to as $\{V_i\}_k$-structured bases of $\mathbb{R}^n$.

Of special interest are the so called normal-$\{V_i\}_k$-structured bases which are defined by the property that the columns of each $V_i \in \mathbb{R}^n$ are orthonormal, i.e.

$$V_i^t V_i = I_{\rho_i} \qquad i = 1, 2, \cdots, k. \tag{6.2.4}$$

Normal-$\{V_i\}_k$-structured bases are examined first. One may preliminary compare the structure of singular values corresponding to any two normal-$\{V_i\}_k$-structured bases as following:

Proposition 6.1. Let V_i and $\tilde{V}_i$ be two normal-$\{V_i\}_k$-structured bases, then V_i and $\tilde{V}_i$ have the same singular values.

Proof:

If V and $\tilde{V}$ are normal-$\{V_i\}_k$-structured bases, then they are related as:

$$\tilde{V} = \left[\tilde{\underline{V}}_1 \middle| \tilde{\underline{V}}_2 \middle| \cdots \middle| \tilde{\underline{V}}_k \right] = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}$$

where Q_i are orthogonal i.e. $Q_i^t Q_i = I_\rho$, $i = 1, 2, \cdots, k.$

Clearly: $\tilde{V}^t \tilde{V} = diag\{Q_i^t\}.V^t V.diag\{Q_i\}$ and since $diag\{Q_i\}$ are orthogonal, then V and $\tilde{V}$ have the same singular values. ∎

As we expected the above result suggests that any selection of orthogonal bases leads to the same singular values (all equal to one).

However, the main question arises when one of the bases is not necessarily orthogonal. We will investigate this as following:

Problem 6.1: Given the direct sum decomposition as in (6.2.1), where $\dim(\mathcal{V}) = e_i$, $V_i = \left[\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}\right]$ an orthonormal basis of $\mathcal{V}_i$ generates alternative bases for $\mathcal{V}_i$, $\tilde{V}_i = \left[\tilde{\underline{v}}_{1i}, \tilde{\underline{v}}_{2i}, \cdots, \tilde{\underline{v}}_{e_i i}\right]$, not necessary orthonormal such that $\left\|\tilde{\underline{v}}_{ji}\right\| = 1$, $i = 1, \cdots, k$, $j = 1, 2, \cdots, e_i$.

Proposition 6.2: If $V_i = \left[\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}\right]$, $i = 1, \cdots, k$ are orthonormal bases of $\mathcal{V}_i$, then $\tilde{V}_i = \left[\tilde{\underline{v}}_{1i}, \tilde{\underline{v}}_{2i}, \cdots, \tilde{\underline{v}}_{e_i i}\right]$ is also a basis with $\left\|\tilde{\underline{v}}_{ji}\right\| = 1$, if and only if $\tilde{V}_i = V_i Q_i$, $Q_i = \left[\underline{q}_{1i}, \underline{q}_{2i}, \cdots, \underline{q}_{e_i i}\right]$, in which $\left\|\underline{q}_{ji}\right\| = 1$, $j = 1, 2, \cdots, e_i$, $i = 1, \cdots, k$.

Proof: $\tilde{V}_i$ and V_i are linked as:

$$\tilde{V}_i = \left[\underline{\tilde{v}}_{1i}, \underline{\tilde{v}}_{2i}, \cdots, \underline{\tilde{v}}_{e_i i}\right] = \left[\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}\right] \begin{bmatrix} q_{1i}^1 & \cdots & q_{e_i i}^1 \\ \vdots & & \vdots \\ q_{1i}^{e_i} & \cdots & q_{e_i i}^{e_i} \end{bmatrix}.$$

Hence,

$$\underline{\tilde{v}}_{ji} = \underline{v}_{1i} q_{ji}^1 + \underline{v}_{2i} q_{ji}^2 + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i}, \qquad j = 1, 2, \cdots, e_i,$$

and

$$\left\|\tilde{v}_{ji}\right\|^2 = \underline{\tilde{v}}_{ji}^T \underline{\tilde{v}}_{ji} = (\underline{v}_{1i} q_{j1}^i + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i})^T (\underline{v}_{1i} q_{j1}^i + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i}) =$$

$$(q_{ji}^i)^2 + \cdots + (q_{ji}^{e_i})^2 = \underline{q}_{ji}^T \underline{q}_{ji} = \left\|q_{ji}\right\|^2$$

$, \; j = 1, 2, \cdots, e_i$

due to orthogonality and hence, $\left\|\underline{\tilde{v}}_{ji}\right\| = 1$ if and only if $\left\|\underline{q}_{ji}\right\| = 1$. ∎

6.3. Measuring the degree of orthogonality

In this part and based on the above result, we will use different type metrics to define the degree of orthogonality of the decomposition, or alternatively to measure the skewness of the direct sum decomposition.

6.3.1. The Gramian

A standard test for checking the degree of orthogonality is that based on the *volume* or the *Gramian* and so based on its description which is presented in Chapter Two, the Gramian of the $\tilde{V}$ matrix, in (6.2.3), is given by:

$$G(\tilde{V}) = \det(\tilde{V}^t \tilde{V}) = \left|diag\left\{Q_i^t\right\}\right| \det(\tilde{V}^t \tilde{V}) \left|diag\left\{Q_i\right\}\right| \tag{6.3.1}$$

and since V is orthogonal with unit length, then

$$G(\tilde{V}) = \det(V^t V).\det(diag\left\{Q_i^t\right\} diag\left\{Q_i\right\}) \tag{6.3.2}$$

According to the *Hadamard's inequality theorem* [18], [19], the determinant of a matrix, when it's restricted to real numbers, can be bounded in terms of the lengths of it's vectors. Specifically, Hadamard's inequality states that if N is the matrix having columns $\underline{v}_i$, $i = 1, 2, \cdots, n$, then

$$\left| \det(N) \right| \le \prod_{i=1}^{n} \left\| \underline{v}_i \right\| \tag{6.3.3}$$

Clearly, in our case, since the length of the vectors belong to $G(\tilde{V})$ can be varied between zero to 1, so as the result the value of determinant of $G(\tilde{V})$ will be also varied between zero to 1.

What we are interested in, is to see what is the condition in which this value is maximum or in other hand, the vectors in $\tilde{V}$, has maximum angle.

Proposition 6.3: If $V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right]$ is any basis corresponding to the $\mathbb{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k$ decomposition where V_i is an orthogonal basis of V with unit length vectors, then:

    (i) The singular values of $V'V$ are invariant of any selection of orthogonal basis.

    (ii) The value of $\det(V'V)$ is invariant of any selection of the orthogonal basis.

Proof:

Any two orthonormal bases $V, \tilde{V}$ are related by (6.3.1) as:

$$\tilde{V} = \left[\underline{\tilde{V}}_1 \middle| \underline{\tilde{V}}_2 \middle| \cdots \middle| \underline{\tilde{V}}_k \right] = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}$$

where Q_i are orthogonal bases i.e. $Q_i^t Q_i = I_{\rho_i}$. Thus

$$\tilde{V}^t \tilde{V} = diag\{Q_i^t\} V'V . diag\{Q_i\}$$

and since Q_i are orthogonal, the result follows. ∎

Assume now that the $\{V_i, i \in k\}$ bases are orthogonal but we select another arbitrary bases $\tilde{V}_i = V_i Q_i$ with unit length vectors, but not necessarily orthogonal. Inspection of equation (6.2.3) and the latest result suggest that the value of $G(\tilde{V})$ really depends on the property of the matrix T where T is as following:

$$T = \begin{bmatrix} Q_1^t & 0 & \cdots & 0 \\ 0 & Q_2^t & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & Q_k^t \end{bmatrix} \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & Q_k \end{bmatrix} = diag\{Q_i^t Q_i\} \tag{6.3.4}$$

or its determinant $|T|$ defined as

$$|T| = \prod_{i=1}^{k} \left| Q_i^t Q_i \right|. \tag{6.3.4a}$$

For any matrix $Q \in \mathbb{R}^{\rho_i \times \rho_i}$, $Q_i = \left[\underline{q}_1, \underline{q}_2, \cdots, \underline{q}_{\rho_i} \right]$ with $\left| \underline{q}_i \right| = 1$, we have that:

$$p = Q^t Q = \begin{bmatrix} \underline{q}_1^t \\ \underline{q}_2^t \\ \vdots \\ \underline{q}_k^t \end{bmatrix} \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_k \end{bmatrix} = \begin{bmatrix} \underline{q}_1^t \underline{q}_1 & \underline{q}_1^t \underline{q}_2 & \cdots & \underline{q}_1^t \underline{q}_k \\ \underline{q}_2^t \underline{q}_1 & \underline{q}_2^t \underline{q}_2 & \cdots & \underline{q}_2^t \underline{q}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{q}_k^t \underline{q}_1 & \underline{q}_k^t \underline{q}_2 & \cdots & \underline{q}_k^t \underline{q}_k \end{bmatrix} \tag{6.3.5}$$

or

$$p = \begin{bmatrix} 1 & \underline{q}_1^t \underline{q}_2 & \cdots & \underline{q}_1^t \underline{q}_k \\ \underline{q}_2^t \underline{q}_1 & 1 & \cdots & \underline{q}_2^t \underline{q}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{q}_k^t \underline{q}_1 & \underline{q}_k^t \underline{q}_2 & \cdots & 1 \end{bmatrix}. \tag{6.3.6}$$

Note that the matrix P is positive definite. Furthermore:

$$|T| = \sum_{i=1}^{k} \det\{Q_i^t Q_i\} \tag{6.3.7}$$

The main issue is now the properties of the $\det\{Q_i^t Q_i\}$ and the investigations of the conditions under which we can maximise det{T}. We note first the following lemma.

Lemma 6.1: For any $n \times n$ matrix positive definite X with constant trace, $tr[X] = \alpha$, the determinant is maximised when:

$$X = \frac{\alpha}{n} I_n. \tag{6.3.8}$$

Proof:

Applying *Hadamard inequality* (6.3.3), the determinant of an $n \times n$ matrix X is maximized when the matrix is diagonal, that is, eigenvalues of the matrix are the diagonal elements. If $\underline{a} = (a_1, a_2, \cdots, a_n), \sum_i a_i = \alpha$ is the vector of eigenvalues of X, from majorization theory [72], the vector $a^t = \left(\dfrac{\alpha}{n}, \dfrac{\alpha}{n}, \cdots, \dfrac{\alpha}{n} \right)$, with all elements equal, is majorized by any other vector a.

Also, a majorization result says that if g is a continues nonnegative function on $I \subset \mathbb{R}$, a function $\phi(X) = \prod_{i=1}^{n} g(x_i)$ is schur-concave (convex) on I^n if and only if $log(g)$ is concave (convex) in I. In our case, $log(x)$ is a concave function on $\mathbb{R}^+$ and $\det(X) = \prod_{i=1}^{n} a_i$ is a schur-concave function and it's maximum is attained for a^t. Having all eigenvalues equal is equivalent to saying that X is a scaled identity matrix, under it's trace constraint [73]. ∎

For our case the matrix P which has: $tr[P] = k$, will have its determinant maximised when $P=I$, i.e. the transformation Q_i are orthogonal. This then leads to the following main result.

Theorem 6.1: Let us consider the decomposition of $\mathbb{R}^n$ as:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \ dim(V_i) = \rho_i, \ i = 1, 2, \cdots, k.$$

and let V_i be a basis for each of the $\mathcal{V}_i$ spaces of vectors with unit length. Then the Gramian of the basis $V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right]$ is:

$$G(V) = \det(\begin{bmatrix} \underline{V}_1^t \\ \underline{V}_2^t \\ \vdots \\ \underline{V}_k^t \end{bmatrix} V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right])$$

and it is maximised if and only if the bases V_i for the $\mathcal{V}_i$ subspaces are orthogonal and unit length.

Proof:

The invariant of *G(V)* for the selection of different bases has been established .

This together with *lemma 6.1.* established the result. ∎

The above establishes *G(V)*, where $\{V_i\}$ are any orthogonal ,unit length , as a measure of the angle between a set of subspaces ,that will be defined as the *Gramian angle* of the $\{V_i, i \in k\}$ decomposition.

6.3.2. Condition Number

Condition Number could be considered to be used as another measurement tool in order to measure the "*skewness*" of eigenframes.

As we presented the condition number before (refer to Chapter Two for full details), for any matrix *A*, the condition number is defined as:

$$\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \tag{6.3.9}$$

where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are maximal and minimal singular values of *A* respectively.

In general: $\kappa(A) \geq 1$ and hence for any normal matrix $\kappa(A) = 1$ as all the singular values of the normal matrix are equal to 1.

Considering the above description of condition number, we will define another measure of the degree of orthogonality or another measure of skewness of the decomposition.

Definition 6.1: If $A \in M^{m \times n}$ is a given matrix, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma q \geq 0$ (q=min[m,n]), then $\|A\| = \sigma_1$ [12].

Lemma 6.2: For any matrix $A \in F^{n \times n}$, there are n singular values such that:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \; ; \text{Where } \sigma_{max} = \sigma_1 \text{ and } \sigma_{min} = \sigma_n \text{ [12].} \qquad \blacksquare$$

Corollary 6.1: Let $A \in F^{n \times m}$ and $B \in F^{m \times m}$, then for all $i = 1, \cdots, \min\{n, m\}$, we will have

$$\sigma_i(A)\sigma_{min}(B) \leq \sigma_i(AB) \leq \sigma_i(A)\sigma_{max}(B) \text{ [74].} \qquad \blacksquare$$

Corollary 6.2: Let $A \in F^{n \times m}$. If $n=m$ and A is non-singular, then:

$$\|A^{-1}\| = \sigma_{min}(A^{-1}) = \frac{1}{\sigma_{max}(A)} \text{ [74].}$$

Proposition 6.4: We consider the direct sum decomposition on $\mathbb{R}^n$ which is :

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k \qquad (6.3.10)$$

where $dim(\mathcal{V}_k) = \rho_i$, $i = 1, 2, \cdots, k$. and all bases $V_i \in \mathcal{V}_i$ in the decomposition (6.3.10) to have unit length vectors. Then the Condition number of the basis $V = \left[\underline{V}_1 \big| \underline{V}_2 \big| \cdots \big| \underline{V}_k \right]$ is:

$$\kappa(V) = \frac{\sigma_{max}(V)}{\sigma_{min}(V)}$$

and it is minimized if and only if the bases V_i for the $\mathcal{V}_i$ subspaces are orthonormal.

Proof:

Let the columns of V_i form an orthonormal basis of $\mathcal{V}_i$. Then all other bases of $\mathcal{V}_i$ consisting of vectors of unit length are given as $V_i Q_i$ where $\det(Q_i) \neq 0$ and all the

columns of Q_i have unit length. Thus all bases of $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ can be written

$$as: \tilde{V} = \left[\tilde{\underline{V}}_1 \middle| \tilde{\underline{V}}_2 \middle| \cdots \middle| \tilde{\underline{V}}_k \right] = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix} = V.Q \ ; \ Q \in \Phi$$

where Φ is defined as the set of all block diagonal matrices $Q = diag(Q_1, \cdots, Q_k)$, such that $\det(Q_i) \neq 0$ and all the columns of Q_i have unit length.

We should show that $\min\limits_{Q \in \Phi} \kappa(VQ) = \kappa(V)$ and that, the minimum is attained for $Q = diag(Q_1, \cdots, Q_k)$ with all Q_i orthogonal. First by using corollary 6.1., we have:

$$\sigma_i(V)\sigma_{\min}(Q) \leq \sigma_i(VQ) \leq \sigma_i(V)\sigma_{\max}(Q) \ . \tag{6.3.11}$$

Hence and based on *Lemma 6.2.* and *Definition 6.1.*, we can have from (6.3.11) that:

For i=1:

$$\frac{\|V\|}{\max\limits_{j \in \underline{k}}\left\|Q_j^{-1}\right\|} \leq \|VQ\| \leq \|V\| \max\limits_{j \in \underline{k}}\left\|Q_j\right\| , \ \underline{k} = \{1,2,\cdots,k\} \tag{6.3.12}$$

Now if the minimum singular value named as σ_n, then:

for i=n

$$\frac{\sigma_n(V)}{\max\limits_{j \in \underline{k}}\left\|Q_j^{-1}\right\|} \leq \sigma_n(VQ) \leq \sigma_n(V) \max\limits_{j \in \underline{k}}\left\|Q_j\right\| \ . \tag{6.3.13}$$

In order to obtain the condition number of $\tilde{V}$ or equally, VQ , we have:

$$\kappa(\tilde{V}) = \kappa(VQ) = \frac{\sigma_{\max}(VQ)}{\sigma_{\min}(VQ)} = \frac{\|VQ\|}{\sigma_n(VQ)} \ .$$

So from (6.3.11) and (6.3.13), we have:

$$\frac{\dfrac{\|V\|}{\max_{j\in\underline{k}}\|Q_j^{-1}\|}}{\sigma_n(V)\max_{j\in\underline{k}}\|Q_j\|} \le \kappa(VQ) \le \frac{\|V\|\max_{j\in\underline{k}}\|Q_j\|}{\dfrac{\sigma_n(V)}{\max_{j\in\underline{k}}\|Q_j^{-1}\|}} \tag{6.3.14}$$

or equivalently,

$$\frac{\kappa(V)}{\max_{j\in\underline{k}}\|Q_j\|\cdot\max_{j\in\underline{k}}\|Q_j^{-1}\|} \le \kappa(VQ) \le \kappa(V).\max_{j\in\underline{k}}\|Q_j\|.\max_{j\in\underline{k}}\|Q_j^{-1}\| . \tag{6.3.15}$$

Note that:

$$\|Q_j^{-1}\| = \sigma_{n_j}^{-1}(Q_j) = \frac{1}{\sigma_{n_j}(Q_j)} \text{ and so: } \max_{j\in\underline{k}}\|Q_j^{-1}\| = \max_{j\in\underline{k}}\frac{1}{\sigma_{n_j}(Q_j)} = \frac{1}{\min_{j\in\underline{k}}\sigma_{nj}(Q_j)}$$

and so that (6.3.14) is equivalent to:

$$\frac{\kappa(V)}{\sigma(Q)} \le \kappa(VQ) \le \kappa(V).\sigma(Q) \tag{6.3.16}$$

where:

$$\sigma(Q) = \frac{\max_{j\in\underline{k}}\|Q_j\|}{\min_{j\in\underline{k}}\sigma_{n_j}(Q_j)} \ge 1 . \tag{6.3.17}$$

From (6.3.16) we get that:

$$\frac{1}{\sigma(Q)} \le \frac{\kappa(VQ)}{\kappa(V)} \text{ for every } Q\in\Phi . \tag{6.3.18}$$

Also from (6.3.17) we have that:

$$\min_{Q\in\varphi}\sigma(Q) = \frac{1}{\max_{Q\in\varphi}\sigma(Q)} = 1 . \tag{6.3.19}$$

So by using (6.3.19), in (6.3.18) we have:

$$\min_{Q \in \varphi} \frac{\kappa(VQ)}{\kappa(V)} \geq 1.$$ (6.3.20)

Using (6.3.16) and notify that $\sigma(Q) = 1$ if and only if $Q = diag(Q_1, \cdots, Q_k)$ with Q_k orthogonal [see *Lemma 6.2.* for proof], thus $\min_{Q \in \varphi} \frac{\kappa(VQ)}{\kappa(V)} = 1$. Since the condition number of V is fixed and assumed to be minimum (which is 1) so it means that the condition number of $\tilde{V}$ is minimum if and only if for all the $Q \in \Phi$, Qs are orthonormal. ∎

Theorem 6.2: Let us consider the decomposition of $\mathbb{R}^n$ as:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$ (6.3.21)

$dim(\mathcal{V}_i) = \rho_i, i = 1, 2, \cdots, k$ and all bases $V_i \in \mathcal{V}_i$ in the decomposition (6.3.21) to have unit length vectors. Then the Condition number of the basis $V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right]$ is:

$$\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)}$$

and it is minimized if and only if the bases V_i for the $\mathcal{V}_i$ subspaces are orthonormal.

Proof:

The invariant of $\kappa(V)$ for the selection of different bases has been established. This together with *Lemma 6.2., Corollaries 6.1.* and *6.2.*, established the result. ∎

The above establishes $\kappa(V)$ where $\{\underline{V}_i\}$ are of orthogonal unit length, also as a measure of the angle between a set of subspaces, defined as the *Condition Number* of the $\{V_i, i \in k\}$ decomposition.

6.3.3 The Spread of Singular Values

So far we have seen two different tools in order to measure the degree of orthogonality or to measure the skewness of the decomposition.

One other way to measure the skewness of the decomposition is to use so called the spread of singular values of a space. Note that by "spread of singular values", we mean the difference between the values of singular values of any decomposition. What we are interested to, is to show that the spread of singular values of decomposition is minimized when the space is orthonormal.

Problem 6.2: Let us consider the direct sum decomposition of $\mathbb{R}^n$ in terms of subspaces V_i and $V_i \in \mathbb{R}^n$, $\dim V_i = \rho_i$, $i = 1, 2, \cdots, k$. i.e.

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k \tag{6.3.22}$$

If $V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right]$ is a normal-$\{V_i\}_k$-structured bases of $\mathbb{R}^n$ and for any V_i, $V_i = \left[v_{1,i}, \cdots, v_{\rho_i,i} \right]$, then any other is expressed as

$$\tilde{V} = \left[\underline{\tilde{V}}_1 \middle| \underline{\tilde{V}}_2 \middle| \cdots \middle| \underline{\tilde{V}}_k \right] = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix} = V.Q \; ; Q \in \Phi$$

where Φ is defined as the set of all block diagonal matrices $Q = diag(Q_1, \cdots, Q_k)$, such that $\det(Q_i) \neq 0$ and all the columns of Q_i have unit length. Show that for all the singular values of $\tilde{V}$, $\sigma_i(\tilde{V}) = \sigma_i(V) = 1$, $i = 1, 2, \cdots, k$. if and only if $\tilde{V}$ is a normal-$\{V_i\}_k$-structured bases.

Solution:

Let $V_i \in \mathbb{R}^{n \times 2}$, then $V = \left[\underline{V}_1 \middle| \underline{V}_2 \right]$, $\underline{V}_i = \left[v_{1,i}, \cdots, v_{\rho_i,i} \right]$, $i = 1, 2$.

Since V is a orthonormal bases, then $\left\| v_{\rho_i,i} \right\| = 1$, $i = 1,2$, and V has full column rank and for any θ:

$$V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Since V has orthogonal columns hence $\sigma_1(\tilde{V}) = \sigma_2(V) = 1$. Now for any other bases $\tilde{V} \in \mathbb{R}^{n \times 2}$, we have that:

$$\tilde{V} = VQ \tag{6.3.23}$$

where $|Q| \neq 0$ and Q is a square matrix ,that is, $Q = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 \end{bmatrix}$ and $\left\| \underline{q}_i \right\| = 1$, i=1,2, but Q is not necessarily orthogonal. Based on these specifications, let's choose Q as following: $Q = \begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix}$, $0 \leq \varepsilon \leq 1$ and $0 \leq \delta \leq 1$.

Then from (6.3.23):

$$\tilde{V} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix}.$$

Now in order to obtain the singular values of $\tilde{V}$, the following procedure can be done:

$$\sigma^2(\tilde{V}) = \sigma^2(VQ) = \lambda_i(Q'V'VQ) = \lambda_i(Q'Q) \text{ , i=1,2.} \tag{6.3.24}$$

The orthogonality of V gives that: $V'V = I_2$ \tag{6.3.25}

then (6.3.23) becomes:

$$\sigma^2(\tilde{V}) = \lambda_i(Q'Q) \text{, i=1,2.} \tag{6.3.25a}$$

Now to obtain the eigenvalues of $(Q'Q)$, we have:

$$Q^tQ = \begin{bmatrix} \varepsilon & \sqrt{1-\varepsilon^2} \\ \delta & \sqrt{1-\delta^2} \end{bmatrix} \begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix} = \begin{bmatrix} 1 & X \\ X & 1 \end{bmatrix}$$

where $X = \varepsilon\delta + \left(1-\varepsilon^2\right)^{\frac{1}{2}}\left(1-\delta^2\right)^{\frac{1}{2}}$. (6.3.26)

And finally: $\det[\lambda I - Q^tQ] = 0$ gives the eigenvalues of Q^tQ or in fact the eigenvalues of $\tilde{V}$.

$$\begin{vmatrix} \lambda-1 & -X \\ -X & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 - X^2 = 0 .$$ (6.3.27)

From (6.3.27), the values of two eigenvalues of Q^tQ will be calculated as following:

$\lambda_{1,2} = \{1-X, 1+X\}$. Obviously based on (6.3.25a), the singular values of $\tilde{V}$ will be:

$\sigma_{1,2} = \left\{\sqrt{1-X}, \sqrt{1+X}\right\}$.

We can clearly see that since $X \geq 0$ is always true, then

$\sigma_1 = \sqrt{1+X} \geq 1$ and $\sigma_2 = \sqrt{1-X} \leq 1$.

The inequalities will be changed to equalities if and only if Q is also orthogonal. ∎

The above example simply shows that for any combinations of bases rather than those of orthonormals, some of the singular values will be greater than 1 and some others less than 1. The above result arises another interesting issue which is strongly relative to the above problem and that is, to find the value of minimum singular value of any $\{V_i\}_k$-structured bases chosen from (6.3.22) as following;

Theorem 6.3: Let $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ and suppose that the columns of $V_j \in \mathbb{R}^{n \times n_i}$ form an orthonormal basis of V_i, $i = 1, 2, \cdots, k$, so that $n = \sum_{j=1}^{k} n_j$ and hence: $V = \left[\underline{V}_1 | \underline{V}_2 | \cdots | \underline{V}_k\right]$ is a square, invertible matrix. Then $\sigma_{\min}(V) \leq 1$; furthermore, this is equality if and only if $V_i \perp V_j$ for all $i \neq j$, so that V is an orthogonal matrix.

Proof:

Assume that the singular values of V are introduced as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ (Since V is square, invertible matrix, then all the singular values are positive). A direct evaluates gives:

$$V^tV = \begin{bmatrix} \underline{V}_1^t \\ \underline{V}_2^t \\ \vdots \\ \underline{V}_k^t \end{bmatrix} \begin{bmatrix} \underline{V}_1 & \underline{V}_2 & \cdots & \underline{V}_k \end{bmatrix} = \begin{bmatrix} I_{n_1} & \underline{V}_1^t\underline{V}_2 & \cdots & \underline{V}_1^t\underline{V}_k \\ \underline{V}_2^t\underline{V}_1 & I_{n_2} & \cdots & \underline{V}_2^t\underline{V}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{V}_k^t\underline{V}_1 & \underline{V}_k^t\underline{V}_2 & \cdots & I_{n_k} \end{bmatrix}$$

and hence:

$$trace(V^tV) = \sum_{i=1}^{n}\lambda_i(V^tV) = \sum_{i=1}^{n}\sigma_i^2(V) = \sum_{j=1}^{k}n_j = n.$$

Thus $n\sigma_n^2(V) \leq \sum_{i=1}^{n}\sigma_i^2(V) = n \Rightarrow \underline{\sigma_n(V) \leq 1}$.

Next let $\sigma_n(V) = 1$. Then we have that $n\sigma_n^2(V) = n$ and hence $n\sigma_n^2(V) = \sum_{i=1}^{n}\sigma_i^2(V)$, so that $\sigma_1(V) = \sigma_2(V) = \cdots = \sigma_n(V) = 1$. This implies immediately that V is orthogonal .Conversely if V is orthogonal then all the singular values of V (including $\sigma_n(V)$) are equal to 1. ∎

Corollary 6.3: Let $V = \begin{bmatrix} \underline{V}_1 | \underline{V}_2 | \cdots | \underline{V}_k \end{bmatrix}$ with $V_j \in \mathbb{R}^{n \times n_i}$ and $n = \sum_{j=1}^{k}n_j$, be a non-singular matrix with all columns of V_j normalized to 1, (j=1,2,…,k). Then $\sigma_{min}(V) \leq 1$; furthermore $\sigma_n(V) = 1$ if and only if *V* is orthogonal.

Proof: Consider that *V* contains n-one dimensional subspaces as following:

$V = \left\{ col_1(V_1) \quad \cdots \quad col_{n_1}(V_1) \quad col_1(V_2) \quad \cdots \quad col_{n_2}(V_2) \quad \cdots \quad col_{n_k}(V_k) \right\}$, then Theorem 6.3 applies and hence $\sigma_{min}(V) \leq 1$. Furthermore, based on the same theorem $\sigma_n(V) = 1$ if and only if *V* is orthogonal. ∎

6.4. Numerical example

So far and within this chapter, it has been proofed that for any space $V = \left[\underline{V}_1 \middle| \underline{V}_2 \middle| \cdots \middle| \underline{V}_k \right]$ with $V_j \in \mathbb{R}^{n \times n_i}$, where the angle between the subspaces is fixed, the space can be in it's best condition if and only if, all the subspaces V_j (normalized to 1) are internally orthogonal.

In this section and by using *MATLAB* tools programming, a simple numerical example will be used in order to achieve the optimum conditions for the space.

Consider the following space H :

$$H = L1 \oplus L2 = \left[L1 \middle| L2 \right] = \left[\underline{v}_1, \underline{v}_2 \middle| \underline{v}_3, \underline{v}_4 \right]$$

$$H = \begin{bmatrix} -0.5366 & -0.4434 & -0.6896 & 0.1873 \\ -0.5542 & -0.4297 & -0.4885 & -0.3240 \\ -0.4253 & 0.3194 & -0.4017 & -0.5697 \\ -0.4733 & 0.7188 & -0.3526 & 0.7317 \\ \underbrace{}_{L1} & & \underbrace{}_{L2} & \end{bmatrix}$$

where it's subspaces *L1* and *L2* are internally orthonormal. For the matrix H , the following measurements are true:

Condition number of H	*12.9930*
Gramian determinant of H	*0.0197*

What we are looking for is to investigate and observe the effect of changing the internal angles of both subspaces *L1* and *L2* on the sensitivity of the whole space *H*.

Let's take $\underline{w}_1 = a\underline{v}_1 + b\underline{v}_2$ and $\underline{w}_2 = a\underline{v}_3 + b\underline{v}_4$. Note that *a,b,c,d* are arbitrary scalars.

In another word, $\underline{w}_1$ and $\underline{w}_2$ can be introduced as the internal weights for the both subspaces, as it has been shown in Figure 6.1:
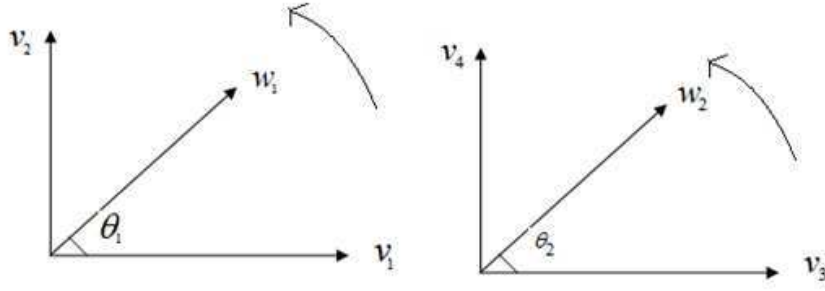
Figure 6.1: Internal weights $\underline{w}_1$ and $\underline{w}_2$

Obviously θ_1 and θ_2 are the internal angles of *L1* and *L2* respectively. As the angles change within each internal subspace, the relative weight will move and as the result the overall sensitivity's measurements will be affected.

In order to normalize w_1, we have:

$$\underline{w}_1^t.\underline{w}_1 = 1 \Rightarrow (a\underline{v}_1 + b\underline{v}_2)^t.(a\underline{v}_1 + b\underline{v}_2) = 1$$

Since $\|\underline{v}_1\| = \|\underline{v}_2\| = 1$ and $\underline{v}_1^t.\underline{v}_2 = 0$, then

$$\underline{w}_1^t.\underline{w}_1 = a^2 + b^2 = 1. \tag{6.4.1}$$

Also based on Figure.1:

$$w_1^T.v_1 = (av_1 + bv_2)^T.v_1 = a = \cos(\theta_1) \tag{6.4.2}$$

Then from (6.4.1) and (6.4.2) we can re-define w_1 to be:

$$\underline{w}_1 = a\underline{v}_1 + b\underline{v}_2 = \underline{v}_1 \cos(\theta_1) + \underline{v}_2 \sin(\theta_1). \tag{6.4.3}$$

Obviously by the same calculation we are able to show that:

$$\underline{w}_2 = c\underline{v}_3 + d\underline{v}_4 = \underline{v}_3 \cos(\theta_2) + \underline{v}_4 \sin(\theta_4). \tag{6.4.4}$$

Next, in order to investigate the effect of the angles on the sensitivity, one can define the weighted space *H1* to be:

$$H1 = \left[\underline{v}_1, \underline{w}_1 \middle| \underline{v}_3, \underline{w}_2\right]. \tag{6.4.5}$$

For this new space and within the interval of $[0.3, \pi - 0.3]$ for the both internal angles of both new subspaces, where the interval has been divided to 100 equal points, the following graphs can be obtained for both the condition number and Gram determinant of $H1$:
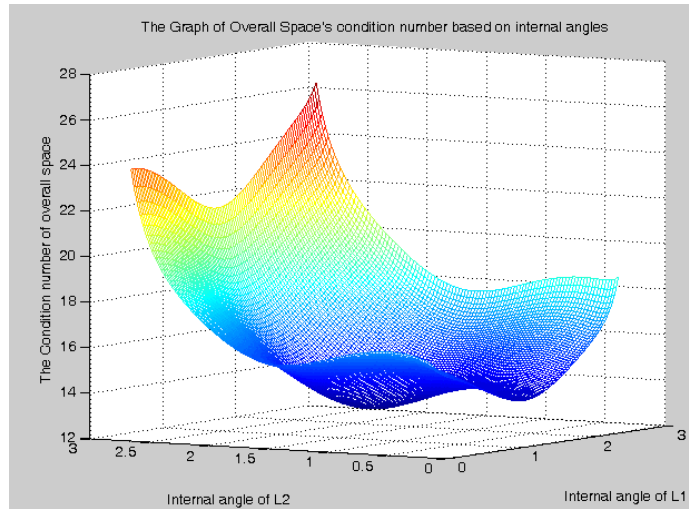

Figure 6.2: The Graph of the Overall Space's Condition number based on different values for internal angels θ_1 and θ_2 (in Radians)
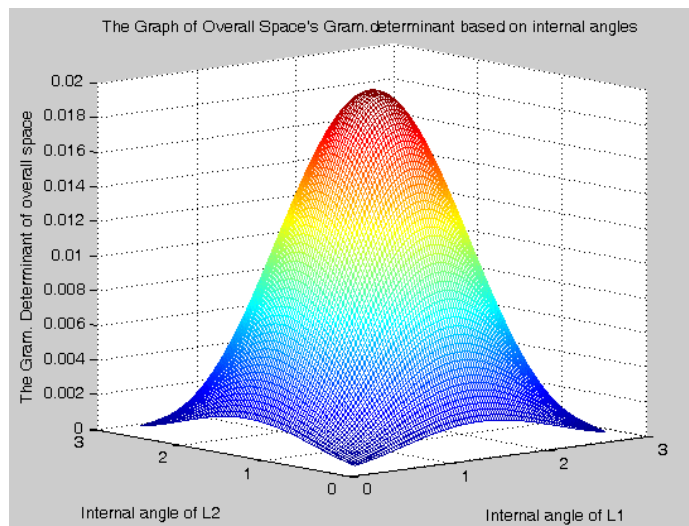

Figure 6.3: The Graph of the Overall Space's Gram. determinant based on different values for θ_1 and θ_2 (in Radians)

It is very clear that the minimum condition number of the whole space and also the maximum Gramian determinant accrue when both internal angles θ_1 and θ_2 get very

close to the value of $\frac{\pi}{2}$ Radians. The results might be clearer if we use the relative contour graphs of the both measurement tools. Within the following figures (Figures 6.4 and 6.5), the optimum values of condition number and Gramian determinant through different values of θ_1 *and* θ_2 can be observed:
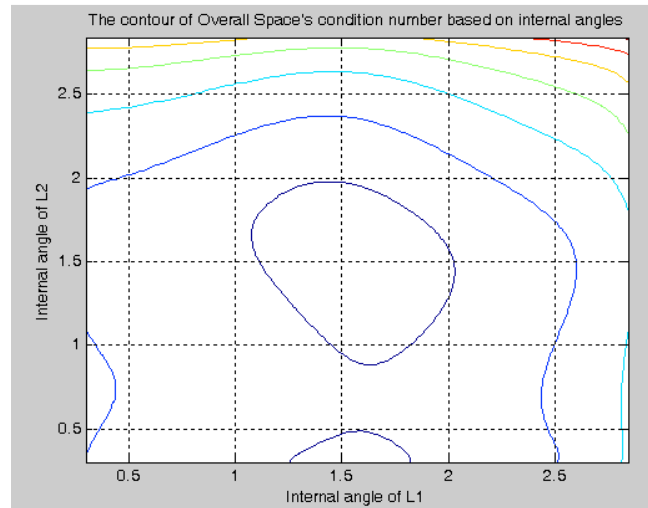

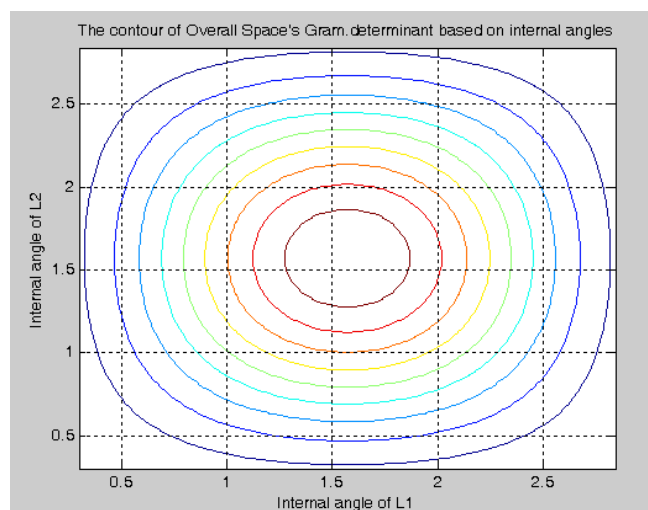Figure 6.4: The Graph of the contour of Overall Space's Condition number based on different values for θ_1 and θ_2 (in Radians)

Figure 6.5: The Graph of the contour of the Overall Space's Gram. determinant based on different values for θ_1 and θ_2 (in Radians)

Note that the above graphs have been obtained by dividing the interval of $[0.3, \pi - 0.3]$ to 100 equal points, then as the result, both condition number and Gramian determinant of matrix $H1$ will be matrices of dimensions 100×100.

In the following table the values of the the optimum points of both matrices will be the same as we had it for the original matrix H .i.e.

$\theta_1 = \theta_2 \cong \dfrac{\pi}{2}$	The minimum value of the matrix of Condition number of ($H1$)	*12.9930*
$\theta_1 = \theta_2 \cong \dfrac{\pi}{2}$	The maximum value of the matrix of Gramian determinant of ($H1$)	*0.0197*

6.5. Conclusion

The problem of the skewness of the eigenframes in a direct sum decomposition, has been studied during this chapter and by using two different kind of measurement tools: Condition number, Gramian determinant and it has been proved that for a fixed angle between subspaces of a direct sum decomposition, the optimum values of both condition number and Gramian determinant of the whole space are obtained if and only if any individual subspace contains orthonormal vectors. It also has been shown that within this situation, the spread of the singular values of the whole space is minimized.

Finally, by using a numerical example, the effect of the internal angle of each individual subspace on the value of condition number and Gramian determinant of the overall space has been studied and by observing the relative graphs, it has been concluded that the optimum points for both of the measurements has been obtained when the internal angles of every single subspace is quiet close to $\dfrac{\pi}{2}$.

By considering the achievements from this chapter, we will study the effect of the angle between controllability subspaces in direct sum decomposition where each subspace contains orthonormal vectors.

CHAPTER 7:

PARAMETERISATION OF ORDERED MINIMAL BASES OF CONTROLLABILITY SUBSPACES

7.1. Introduction

Within this chapter, an alternative algebraic approach that can provide a characterisation of the closed loop eigenvectors will be considered, as well as introducing a new way of characterising system properties such as poles and zeros based on an algebraic characterisation of the behaviour of linear systems.

An algebraic description of the total system behaviour is presented which in turn allows the study of closed loop eigenvectors in a systematic way by providing new parameterisations which leads to an algebraic characterisation of the total input, state and output behaviour in an implicit formulation and it is given based on properties of MFD descriptions.

This framework allows a novel unifying characterisation of poles and zeros based on input and output zeroing problems. The analysis also provides explicit algebraic means for characterising the zero structure and providing a new algebraic characterisation of the family of closed loop eigenvectors and related input and output directions.

The approach which is introduced here enables the derivation of a new method of eigenstructure assignment via state feedback, using minimal basis theory, and this is demonstrated via an example. We also develop some ideas how to optimise the eigenframe, which contains as parameters the closed loop eigenvalues in order to guarantee maximum system robustness by making it as close to orthogonality as

possible. This will be done by construction of parametrisation of ordered minimal bases [96].

7.2. Implicit system description

For the system $\mathcal{S}(A,B,C,E)$, which will be assumed to be minimal, i.e. controllable and observable, the total behaviour solution of system equations under zero initial conditions is expressed by [2]

$$\begin{bmatrix} sI - A & -B \\ -C & -E \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -\underline{y}(s) \end{bmatrix} \tag{7.2.1a}$$

or equivalently [75]

$$\begin{bmatrix} sI - A & -B & 0 \\ -C & -E & -I \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = 0 \tag{7.2.1b}$$

where the composite vector $\underline{\xi}(s) = \begin{bmatrix} \underline{x}(s)^t, & \underline{u}(s)^t, & \underline{y}(s)^t \end{bmatrix}^t$ will be referred to as the *total behaviour vector* of the system.

Also, it should be noted that if $s = \lambda$ and $\underline{x}(\lambda)$, $\underline{u}(\lambda)$, $\underline{y}(\lambda)$ denote corresponding constant vectors, then (7.2.1a) or (7.2.1b) denotes vector solutions of the rectilinear motion problem for the given λ. The problem we address is the solution of (7.2.1b) in parametric form using the system model structure. Note that the system equations are

$$(sI - A)\underline{x}(s) = B\underline{u}(s), \quad \underline{y}(s) = C\underline{x}(s) + E\underline{u}(s) \tag{7.2.2}$$

Consider now the relationship

$$\underline{x}(s) = (sI - A)^{-1} B\underline{u}(s) \tag{7.2.3a}$$

and let us consider its right MFD factorization

$$(sI - A)^{-1} B = \overline{N}(s)\overline{D}(s)^{-1} \tag{7.2.3b}$$

of the input state transfer function.

Then

$$\underline{x}(s) = (sI - A)^{-1} B\underline{u}(s) = \overline{N}(s)\overline{D}(s)^{-1}\underline{u}(s).$$
(7.2.4)

By defining

$$\underline{h}(s) = \overline{D}(s)^{-1}\underline{u}(s) \ \text{ or } \ \underline{u}(s) = \overline{D}(s)\underline{h}(s)$$
(7.2.5a)

the output relationship may be expressed as

$$\underline{y}(s) = C\underline{x}(s) + E\underline{u}(s) = \left\{ C(sI-A)^{-1} B + E \right\}\underline{u}(s) =$$
$$= \left\{ C\overline{N}(s)\overline{D}(s)^{-1} + E \right\}\underline{u}(s) = \left\{ C\overline{N}(s) + E\overline{D}(s) \right\}\overline{D}(s)^{-1}\underline{u}(s)$$
(7.2.5b)

Proposition 7.1: [38] If the system $\mathcal{S}(A,B,C,E)$ is controllable and observable and the state input factorisation in (7.2.3b) is coprime, then a right MFD is defined by

$$G(s) = C(sI-A)^{-1}B + E = N(s)D(s)^{-1}$$
(7.2.6a)

where

$$N(s) = C\overline{N}(s) + E\overline{D}(s), \ \ D(s) = \overline{D}(s)$$
(7.2.6b)

and is a right coprime MFD.

Proof: If the system is minimal, then $n = \deg\left|\overline{D}(s)\right|$. From equation (7.2.6b), it is obvious that $\left\{ C\overline{N}(s) + E\overline{D}(s), \ \overline{D}(s) \right\}$ defines an MFD since $\deg\left|\overline{D}(s)\right| = n = \delta_M(G(s))$, the factorisation is minimal. Substituting (7.2.6a) into (7.2.6b)

$$\underline{y}(s) = \left\{ C\overline{N}(s) + E\overline{D}(s) \right\}\underline{h}(s)$$
(7.2.6c)

and assembling (7.2.6a), (7.2.6b) and (7.2.6c), the following result is obtained. ∎

Proposition 7.2: The total behaviour vector of the system is defined in parametric form as

$$
\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} \underline{h}(s) = Q_r(s)\underline{h}(s)
\tag{7.2.7}
$$

where $\left(\overline{N}(s),\ \overline{D}(s) \right)$ is a coprime right MFD pair of the input state transfer function and $h(s) \in \mathbb{R}(s)$ is an arbitrary vector parameter for the rational behaviour. ■

The matrix $Q_r(s) \in \mathbb{R}(s)^{(r+m+p)\times p}$ is referred to as the *behavioural representation*, and contains as a submatrix the input-output behavioural representation $T_r(s)$ which is defined below as

$$
Q_r(s) = \begin{bmatrix} \overline{N}(s) \\ \overline{\overline{D}(s)} \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{T_r(s)} \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ D(s) \\ N(s) \end{bmatrix} .
\tag{7.2.8a}
$$

The rational vector

$$
\mathcal{Q} \equiv \mathrm{colsp}_{\mathbb{R}(s)} \left\{ Q_r(s) \right\}
\tag{7.2.8b}
$$

characterises the total behaviour and has as a complete invariant a corresponding Plücker matrix, or the Grassman Representative of $\mathcal{Q}$ [76].

Remark 7.1: The expression of the total behaviour as in (7.2.8a) suggests that the whole theory of transformations and invariants may be expressed in terms of properties of the $Q_r(s)$ matrix. Furthermore, for minimal $S(A,B,C,E)$ systems all aspects of behavioural structures are generated by the input-state factorisation, i.e.

$$
Q_r(s) = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} = \begin{bmatrix} I & & \\ \hline & I & 0 \\ \hline & E & C \end{bmatrix} \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ \overline{N}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ N(s) \end{bmatrix}
\tag{7.2.9a}
$$

which clearly denotes how MFDs are generated from the input-state transfer function, which has implications for their computation. ∎

Remark 7.2: Given that the Smith structure of $N(s)$ defines the zeros, the zero structure formation may be considered as a *model projection problem* [77] defined in polynomial terms by

$$N(s) = \begin{bmatrix} E, & C \end{bmatrix} \begin{bmatrix} \overline{D}(s) \\ \overline{\overline{N}}(s) \end{bmatrix} = E\overline{D}(s) + C\overline{N}(s) \; . \tag{7.2.9b}$$

∎

The problem of "squaring down" [76] is thus a special case of the above problem of selecting (E, C) to assign the structure of $N(s)$. The important issue here is the problem of transformation of the controllability indices, that is the Forney indices of $\begin{bmatrix} \overline{D}(s)^t, & \overline{N}(s)^t \end{bmatrix}^t$, to those of $N(s)$. Note that "squaring down" corresponds to the boundary case where all Forney indices of $N(s)$ are zero.

The framework already developed on zero assignment [76] may be extended to model projection using the above formulation. This, however, is now a more complex problem since now controllability indices are transformed to Forney dynamical orders and possible zeros and this is a topic for future research.

It is clear that the MFD pair $\left(\overline{D}(s), \overline{N}(s) \right)$ emerges as a crucial element for the overall analysis and shall be referred to as an *input-state generator pair*. Such pairs will always be assumed to be coprime.

7.2.1. Duality issues and behaviour

Consider now the solutions of

$$\begin{bmatrix} \underline{z}(s)^t, & \underline{v}^t(s) \end{bmatrix} \begin{bmatrix} sI - A \\ -C \end{bmatrix} = 0 \tag{7.2.10a}$$

which in a sense are dual to those of (7.2.2a). From (7.2.10a) we have that

$$z(s)^t = \underline{v}^t(s)C(sI - A)^{-1} \tag{7.2.10b}$$

If we now the coprime factorisation of $C(sI - A)^{-1}$, i.e.

$$C(sI - A)^{-1} = \tilde{D}(s)^{-1}\tilde{N}(s)$$

(7.2.11a)

then

$$\underline{z}(s)^t = \underline{v}(s)^t \tilde{D}(s)^{-1}\tilde{N}(s) \tag{7.2.10c}$$

and by defining $\underline{f}(s)^t = \underline{v}(s)^t \tilde{D}(s)^{-1}$, this leads to

$$\underline{v}(s)^t = \underline{f}(s)^t \tilde{D}(s)$$
$$\underline{z}(s)^t = \underline{f}(s)^t \tilde{N}(s) \tag{7.2.11b}$$

or

$$\left[\underline{z}(s)^t, \quad \underline{v}(s)^t \right] = \underline{f}(s)^t \left[\tilde{N}(s), \quad \tilde{D}(s) \right] . \tag{7.2.11c}$$

From the above, the left coprime MFDs of the transfer function can be obtained as shown below

$$G(s) = C(sI - A)^{-1}B + E = \tilde{D}(s)^{-1}\tilde{N}(s)B + E$$
$$= \tilde{D}(s)^{-1}\left\{ \tilde{N}(s)B + \tilde{D}(s)E \right\} = D'(s)^{-1}N'(s) \tag{7.2.12a}$$

Proposition 7.3: If the system is minimal and $\tilde{D}(s)$, $\tilde{N}(s)$ are left coprime MFDs of $C(sI - A)^{-1}$, then $D'(s)$, $N'(s)$ where

$$D'(s) = \tilde{D}(s), \ N'(s) = \tilde{N}(s)B + \tilde{D}(s)E \tag{7.2.12b}$$

are left coprime MFDs of $G(s)$.

Proof:

$\left(\tilde{D}(s), \ \tilde{N}(s) \right)$ is the *state-output generator pair,* and the generation of the left coprime MFDs is described by

$$\begin{bmatrix} \tilde{N}(s), & \tilde{D}(s), & \tilde{N}(s) \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \hline 0 & I & E \\ \hline 0 & 0 & B \end{bmatrix} = \begin{bmatrix} \tilde{N}(s), & D'(s), & N'(s) \end{bmatrix} \tag{7.2.13}$$

The above is the dual of the relationship of (7.2.9a). It should be noted that

$$G(s) = D'(s)^{-1} N'(s) = N(s)D(s)^{-1} \tag{7.2.14a}$$

and thus some interesting relationships between the input-state and state-output generator pairs are derived below. In fact (7.2.14a) implies that

$$N'(s)D(s) - D'(s)N(s) = 0 \tag{7.2.14b}$$

and from (7.2.6) and (7.2.12) we have that

$$\{\tilde{N}(s)B + \tilde{D}(s)E\}\overline{D}(s) - \tilde{D}(s)\{C\overline{N}(s) + E\overline{D}(s)\} = 0 \tag{7.2.15a}$$

or

$$\tilde{N}(s)B\overline{D}(s) - \tilde{D}(s)C\overline{N}(s) = 0 \tag{7.2.15b}$$

∎

The above leads to the following result:

Proposition 7.4: If $\left(\overline{N}(s),\ \overline{D}(s)\right)$ is an input-state and $\left(\tilde{N}(s),\ \tilde{D}(s)\right)$ a state-output generator pair, then the following relationship holds true

$$\begin{bmatrix} \tilde{N}(s), & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} \overline{D}(s) \\ \overline{N}(s) \end{bmatrix} = 0 \tag{7.2.15c}$$

Proof:

It should be noted that $\left(\tilde{N}(s),\ \tilde{D}(s)\right)$ contain information on observability indices and $\left(\overline{D}(s),\ \overline{N}(s)\right)$ on controllability indices. Condition (7.2.15c) thus expresses constraints on their values. The computation of state output generator pairs is based on the fact that (7.2.11) implies

$$\begin{bmatrix} \tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} sI - A \\ -C \end{bmatrix} = 0 \tag{7.2.16a}$$

and if M and $C^\dagger$ are right annihilators and inverses of the full rank output matrix C, then by multiplying on the right by the full rank matrix $\begin{bmatrix} M & \vdots & C^\dagger \end{bmatrix}$, the following result is obtained

$$\begin{bmatrix} \tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} sI - A \\ -C \end{bmatrix} \begin{bmatrix} M & \vdots & C^\dagger \end{bmatrix} = 0$$

or

$$\begin{bmatrix} \tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} sM - AM & sC^\dagger - AC^\dagger \\ 0 & -I_m \end{bmatrix} = 0 \qquad\blacksquare$$

Thus we are led to the following result:

Proposition 7.5: The left numerator $\tilde{N}(s)$ is constructed as a minimal basis of the left kernel of $sM - AM$ i.e.

$$\tilde{N}(s)(sM - AM) = 0 \quad . \tag{7.2.16b}$$

This leads to a left denominator with the pair $\left(\tilde{N}(s), \tilde{D}(s) \right)$ coprime, computed as

$$\tilde{D}(s) = \tilde{N}(s)(sC^\dagger - AC^\dagger) \tag{7.2.16c}$$

$$\blacksquare$$

The above expressions together with (7.2.15c) may be used to work out more detailed relationships between the controllability and observability indices of the system. Starting from (7.2.15c) and using (7.2.16c) and (7.2.16b), it is readily shown that:

Remark 7.3: [7] The numerators $\tilde{N}(s)$ and $\overline{N}(s)$ of the output-state and input-state generator pairs are related as

$$\tilde{N}(s) \left\{ s \left(BB^\dagger - C^\dagger C \right) - \left(BB^\dagger A - AC^\dagger C \right) \right\} \overline{N}(s) = 0 \tag{7.2.17}$$

where $\tilde{N}(s)$ is a minimal basis of $\mathcal{N}_l\{sM - AM\}$ and $\overline{N}(s)$ is a minimal basis of

$\mathcal{N}_r \{sN - NA\}$. ∎

7.2.2. Computation of Input - State generator pairs

We consider now the computation of the input-state and state output pairs which are crucial for our current study. By definition

$$(sI - A)^{-1} B = \overline{N}(s)\overline{D}(s)^{-1} \qquad (7.2.18)$$

and this implies that

$$B = (sI - A)\overline{N}(s)\overline{D}(s)^{-1} \Leftrightarrow B\overline{D}(s) = (sI - A)\overline{N}(s) \Leftrightarrow$$
$$\Leftrightarrow [sI - A, \quad -B]\begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \end{bmatrix} = 0 \qquad (7.2.19)$$

Remark 7.4: The computation of a pair $\left(\overline{D}(s), \ \overline{N}(s)\right)$ is equivalent to computing a minimal basis for the right kernel of $[sI - A, \quad -B]$. ∎

Reduced complexity computations may be achieved by using the pair $(N, B^\dagger)$ for the B matrix where N is a left annihilator and $B^\dagger$ a left inverse of B, i.e.

$$\rho(B) = p, NB = 0, B \in \mathbb{R}^{(n-p)\times n}, \rho(N) = n - p, B^\dagger B = I .$$

Using $(N, B^\dagger)$, (7.2.19) is equivalent to the following set of conditions

$$(sN - NA)\overline{N}(s) = 0, \ \overline{D}(s) = B^\dagger (sI - A)\overline{N}(s) \qquad (7.2.20)$$

Remark 7.5: The results developed later in this chapter on minimal bases of matrix pencils are used for computing $\overline{N}(s)$. Then $\overline{D}(s)$ is defined by (7.2.20) and the numerator and denominator of the MFD of $G(s)$ by

$$N(s) = C\overline{N}(s) + E\overline{D}(s), \ D(s) = \overline{D}(s) \qquad (7.2.21)$$

∎

The above results form the basis for a numerical method for computing MFDs [4]. The current treatment is algebraic in nature and it provides links with fundamental aspects of the underlying system structure.

7.2.3. Closed -loop Eigenvectors and frequency transmission

The algebraic analysis given before is now used to characterise the structure of closed loop eigenvectors and to produce a new characterisation of them. The solution to the frequency transmission problem [35] is defined by

$$
\begin{bmatrix} \lambda_n I - A & -B & 0 \\ -C & -E & -I \end{bmatrix} \begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = 0 \tag{7.2.22a}
$$

and thus from Proposition 7.2 and condition (7.2.18), it can be shown that:

Proposition 7.6: The solution of the input, state and output rectilinear motion problem is given by

$$
\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \underline{x}(\lambda_i) \\ \underline{u}(\lambda_i) \\ \underline{y}(\lambda_i) \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ E\overline{D}(\lambda_i) + C\overline{N}(\lambda_i) \end{bmatrix} \underline{h}_i = Q_r(\lambda_i)\underline{h}_i \tag{7.2.22b}
$$

∎

The above generates all solutions of the frequency transmission problem in parametric form. In fact, $Q_r(\lambda_i)$ is a basis for the total composite transmission space [35]. This framework will be subsequently used to derive an eigenstructure assignment method.

Remark 7.6: [35] The solutions of the frequency transmission problem are given by the fact that the generation of any general frequency requires that the state $\underline{x}(t)$ be restricted in an (A, B)-invariant subspace. This condition may be ensured by selecting an appropriate release condition x_0 that lies in this particular subspace and some appropriate rectilinear input trajectory with the same frequency. The resulting output trajectory will then be the sum of the rectilinear motions whose frequency components are defined by the same exponential. ∎

7.2.4. Pole assignment by output feedback and Closed-Loop Eigenvectors

Kimura [70] in his paper about pole assignment by output feedback deals with the problem of pole assignment with incomplete state observation. It is shown that if the system is controllable and observable and if $n \leq r+m-1$, an almost arbitrary set of distinct closed-loop poles is assignable by gain output feedback, where n, r and m are the numbers of state variables, inputs and outputs, respectively. This result improves considerably the ones obtained so far about this problem.

Different from the conventional approach using the characteristic equation, an approach based on the properties of the eigenspaces of the closed-loop dynamics is used, which gives a new light on the various problems in the linear system theory. It is also shown, as a direct consequence of this result, that the minimum order of the dynamic compensator required for almost arbitrary pole assignment of overall closed-loop system is not greater than $n-m-r+1$.

For the sake of simplicity the strictly proper case is first considered, i.e. when $E = 0$. If K_0 is the output feedback matrix, then the closed loop eigenvectors and eigenvalues are defined by

$$\left(\lambda_i I - A - BK_O C\right)\underline{x}_i = 0 \tag{7.2.23a}$$

where $\{\lambda_i \in \mathbb{C}\}$ is a complex conjugate set and the set of corresponding eigenvectors $\{x_i, i \in n\}$ is linearly independent. For this case, (7.2.22b) takes the form

$$\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ C\overline{N}(\lambda_i) \end{bmatrix} \underline{h}_i \tag{7.2.23b}$$

and

$$\underline{y}_i = C\underline{x}_i, \quad \underline{u}_i = K_O \underline{y}_i, \quad \forall i \in n. \tag{7.2.23c}$$

From (7.2.22b)

$$\underline{u}_i = \underline{u}(\lambda_i) = \overline{D}(\lambda_i)\underline{h}_i, \quad \underline{y}_i = \underline{y}(\lambda_i) = C\overline{N}(\lambda_i)\underline{h}_i$$

and thus (7.2.22c) leads to

$$\left[\overline{D}(\lambda_i) - K_O C\overline{N}(\lambda_i)\right]\underline{h}_i = 0 \tag{7.2.24}$$

Remark 7.7: $D_K(s) = \overline{D}(s) - K_O C\overline{N}(s)$ is the denominator of the closed loop transfer function under output feedback and thus $\underline{h}_i$ are the vectors associated with the loss of rank of the $D_K(s)$ denominator (closed loop poles). The selection of $\underline{h}_i$ has to be such that the eigenvectors of (7.2.23b) defined by

$$\underline{x}_i = \underline{x}(\lambda_i) = \overline{N}(\lambda_i)\underline{h}_i, \quad \forall i \in n$$

(7.2.23d)

have to be linearly independent. If $\underline{h}_i$ are treated as free parameters then a design problem may be posed as that aiming at maximising the orthogonality of the $\left[N(\lambda_1)\underline{h}_1, \quad \dots, \quad N(\lambda_n)\underline{h}_n\right]$ frame. ∎

Remark 7.8: Given that $(sI - A)\overline{N}(s) = B\overline{D}(s)$, then for an eigenvalue $\lambda \in \sigma(A)$, $\overline{D}(\lambda)$ is rank deficient and $\exists\, \underline{h}_\lambda(\lambda)$, then

$$\overline{D}(\lambda)\underline{h}_\lambda = 0, \quad \underline{x}_\lambda = \overline{N}(\lambda)\underline{h}_\lambda$$

where $\underline{x}_\lambda$ is the λ-closed loop eigenvector since $(\lambda I - A)\overline{N}(\lambda)\underline{h}_\lambda = 0$. ∎

The selection of parameter $\underline{h}_i$ is dependent on the input vector $\underline{u}_i$ and the denominator of the input-state transfer function $\overline{D}(s)$ defined by equation (7.2.4). It is also dependent on the eigenvectors determined by (7.2.23a) and the corresponding condition (7.2.23d).

The resulting selection problem is thus a crucial one because issues such as linear independence and orthogonality are involved. This approach is independent of the

feedback used and can be employed for procedures that lead to eigenstructure assignment.

Since the fundamental result was presented by Wonham [9] the problem of pole assignment has received much attention and has been expected to bridge the gap between classical and modern control theory.

The result of Wonham states that, if the system is controllable, it is pole assignable, that is, the eigenvalues of the closed-loop system can be assigned arbitrarily by selecting an appropriate state feedback. Since the complete state observation which was assumed is unlikely to most practical situations, it has been desirable to find the condition under which the system is pole-assignable with incomplete state observation.

If some dynamic elements are allowed in the feedback loop, an elegant result of Brasch and Pearson [70] gave an answer to this question. Another approach has been the one using the Luenberger observer [70]. In case only a gain feedback is allowed, however, the problem still remains open in spite of several authors' efforts.

The first result obtained along this line was that of Davison [70] who showed that if the system is both controllable and observable, m poles of closed-loop system are assignable almost arbitrarily by gain output feedback, where m is the number of independent outputs. This result was extended by Davison and Chatterjee [70] and by Sridhar and Lindorff [70] who showed that under the same conditions, $\max(r, m)$ eigenvalues are assignable almost arbitrarily by gain output feedback, where r is the number of independent inputs.

The results are not practical because nothing was said about the remaining $n - \max(r, m)$ poles, where n is the number of state variables. Kimura's paper derived a more workable criterion for the pole assignability by gain output feedback.

Different from other papers, the eigenvectors play a fundamental role rather the characteristic polynomial. The main result is that if $n \leq r + m - 1$ the system is always pole-assignable by gain output feedback provided that a slight modification of the poles to be assigned is tolerable.

7.2.5. Poles and Zeros

The simplest case of an autonomous system is one that has no physical inputs, i.e. $\underline{u}(t) = 0$. This reduces the state space description to merely $\underline{\dot{x}} = A\underline{x}$ and $\underline{y} = C\underline{x}$. For a forced system (i.e. a system with physical inputs) a frequency s_0 is said to be transmitted through the system when the application of a signal with this same frequency is applied to the inputs. The system then yields an output response of the same frequency.

However, when $\underline{u}(t) = 0$, a frequency cannot be transmitted in this fashion. This does not imply that the system itself, which is free responding under zero input conditions, is not capable of exciting a response of an exponential type. The notion of the zeros of a system is strongly related to the physical situation whereby the system has an identically zero output whilst the states and inputs are not themselves identically zero.

It has been shown [34], [105] that given a transfer function matrix $G(s)$ there are certain specific values of the complex frequency s associated with certain specific non-zero input transform vectors $\underline{u}(s)$ in the input space that transform the output vector $\underline{y}(s)$ to zero. The matrix $G(s)$ corresponds to an external description of the system behaviour in terms of how sets of exponential signals are propagated through it.

In the following part of the analysis, the case of selecting the parameter $\underline{h}_i$ for both cases of input and output zeroing will be examined. The behaviour form provides an ideal characterisation of the poles and zeros and corresponding directions, because from equation (6.3.7). In fact, starting from the behavioural description

$$\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} \underline{h}(s) = Q_r(s)\underline{h}(s) \qquad (7.2.25)$$

the following results can be readily deduced:

Corollary 7.1: (*Characterisation of Poles*) Consider the zero input problem with $\underline{u}(t) = 0$. Then $\forall \lambda : |D(\lambda)| = 0 \ \exists \ \underline{h}_p$ such that

$$D(\lambda)\underline{h}_p = 0 \text{ and } \underline{u}_p = 0$$

(7.2.26a)

then λ_i is a pole of the system with corresponding eigenvectors and output pole directions defined by

$$\underline{x}_p = \overline{N}(\lambda)\underline{h}_p, \quad \underline{y}_p = C\overline{N}(\lambda)\underline{h}_p \qquad (7.2.26b)$$

∎

Corollary 7.2: (*Characterisation of Zeros*) Consider the output zeroing problem i.e. $y(t) = 0$. Then $\forall z : \mathcal{N}_r\{C\overline{N}(z) + E\overline{D}(z)\} \neq \{0\} \ \exists \ \underline{h}_z$ such that

$$\left[C\overline{N}(z) + E\overline{D}(z)\right]\underline{h}_z = 0 = \underline{y}_z \qquad (7.2.27a)$$

then z is a zero with corresponding state and input zero directions of the system defined by

$$\underline{x}_z = \overline{N}(z)\underline{h}_z, \quad \underline{u}_z = \overline{D}(z)\underline{h}_z \qquad (7.2.27b)$$

∎

From the behaviour viewpoint, poles and zeros are distinct frequency solutions of zero input and zero output problems. Thus

$$\underline{y}(s) = 0 \iff \exists \underline{h}_z(s) : \left[C\overline{N}(s) + E\overline{D}(s)\right]\underline{h}_z = 0 \ . \qquad (7.2.28a)$$

The polynomial solution $\underline{h}_z(s)$ then defines the vectors

$$\underline{x}_z(s) = \overline{N}(s)\underline{h}_z(s), \quad \underline{u}_z(s) = \overline{D}(s)\underline{h}_z(s) \tag{7.2.28b}$$

which in turn defines the output nulling controllability spaces for the system [9].

Remark 7.9: The zeros are those frequencies associated with the further expansion of the kernel of $\left[C\overline{N}(s) + E\overline{D}(s) \right]$ and the corresponding $\underline{x}_z$ are independent from those of colsp.$\{\underline{x}_z\}$. ∎

7.2.6. Design of state feedback controllers using Eigenvector parametrisation

The general analysis on the solution of the system equations in an algebraic-behavioural sense leads to a parameterisation of closed loop eigenvectors and an explicit design of state feedback that assigns the eigenstructure, and is presented here. The problem of state feedback is defined as stated below:

Problem 7.1: Given a complex symmetric set $\Lambda = \{\lambda_i, \ i \in n\}$, find an independent set of closed loop eigenvectors $\{\underline{x}(\lambda_i) = \underline{x}_i, \ i \in n\}$ with corresponding input directions $\{\underline{u}(\lambda_i) = \underline{u}_i, \ i \in n\}$ such that

$$K_S \underline{x}_i = \underline{u}_i, \ \forall i \in n \tag{7.2.29a}$$

or equivalently

$$K_S \left[\underline{x}_1, \ \underline{x}_2, \ ..., \ \underline{x}_n \right] = \left[\underline{u}_1, \ \underline{u}_2, \ ..., \ \underline{u}_n \right] \ \Rightarrow K_S X(\Lambda) = U(\Lambda) \tag{7.2.29b}$$

∎

The above problem can be solved if the frame $X(\Lambda)$ has full rank. Furthermore, it is necessary for the frame $X(\Lambda)$ to be as close to orthogonality as possible, [65], since this is related to robustness. Clearly, if for the given Λ a frame $X(\Lambda)$ which has full rank may be found, the solution of (7.2.29b) is not unique and for the selected frame $X(\Lambda)$ it is shown that

$$K_S = U(\Lambda)X(\Lambda)^{-1}. \tag{7.2.30}$$

Some important issues that emerge here are:

(i) Selection of an independent set of eigenvectors for any given Λ.

(ii) Given a set Λ select the most orthogonal frame, if a procedure for selection of independent vectors is found.

(iii) Selection of an appropriate stable spectrum Λ that may achieve the best orthogonality

Considering the first of the interrelated problems, condition (7.2.23b) is used to characterise the solution of the rectilinear motion problem, i.e.

$$
\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ N(\lambda) \end{bmatrix} \underline{h}_i , \ i = 1, \ 2, \ \dots, \ n \tag{7.2.31}
$$

Critical for the above characterisation is the computation of minimal bases, as well as their parametrisation. In fact, let us assume that in the factorisation

$$
(sI - A)^{-1} B = \overline{N}(s)\overline{D}(s)^{-1} \tag{7.2.32a}
$$

$\overline{N}(s)$ is an ordered minimal basis [96] which is expressed as

$$
\overline{N}(s) = \begin{bmatrix} \underline{\bar{n}}_1(s), & \underline{\bar{n}}_2(s), & \dots, & \underline{\bar{n}}_p(s) \end{bmatrix} \tag{7.2.33b}
$$

where $\delta\left[\underline{\bar{n}}_i(s)\right] = \varepsilon_i$, $i \in p$ and $\varepsilon_1 \le \varepsilon_2 \le \dots \le \varepsilon_p$, and

$$
\underline{\bar{n}}_i(s) = \overline{N}_{\varepsilon_i} \underline{e}_{\varepsilon_i}(s), \ \underline{e}_{\varepsilon_i}(s) = \begin{bmatrix} 1, & s, & \dots, & s^{\varepsilon_i} \end{bmatrix} \tag{7.2.33c}
$$

where $\overline{N}_{\varepsilon_i} \in \mathbb{R}^{n \times (\varepsilon_i + 1)}$. From the properties of minimal bases of matrix pencils, $rank\left(\overline{N}_{\varepsilon_i}\right) = \varepsilon_i + 1$ and the space $\mathcal{R}_i = sp\left\{\overline{N}_{\varepsilon_i}\right\}$ is an $\varepsilon_i + 1$ dimensional controllability subspace. The use of minimal bases suggests a simple procedure for selection of an independent eigenframe.

This in turn, splits the selection of spectrum from the selection of the most orthogonal frame and reveals the critical role of the parameterization of minimal bases in the overall problem. In the next section we consider the selection of the full rank eigenframe and its relation to the definition of state feedback and then we consider the problem of minimal basis parameterization and its role to the shaping of frames.

7.2.7 Selection of an independent Eigenframe and resulting state feedback

The selection of an eigenframe that corresponds to a given closed loop spectrum is based on the following steps.

STEP (1): For every $\Lambda = \{\lambda_i, i \in n\}$ symmetric, it is possible to partition it into the following subsets $\Lambda_{\varepsilon_1} = \{\lambda_1^{\varepsilon_1}, \lambda_{21}^{\varepsilon_1}, \ldots, \lambda_{\varepsilon_1+1}^{\varepsilon_1}\}$, ..., $\Lambda_{\varepsilon_{p1}} = \{\lambda_1^{\varepsilon_p}, \lambda_{21}^{\varepsilon_p}, \ldots, \lambda_{\varepsilon_1+1}^{\varepsilon_p}\}$. It is assumed that each of the Λ_{ε_i} subsets with $\varepsilon_i + 1 = \sigma_i$ eigenvalues is also symmetric. The partitioning corresponds to the dimensions of the controllability subspaces defined by the $\varepsilon_i + 1$ indices. Clearly

$$\Lambda = \{\lambda_i, i \in n\} = \Lambda_{\varepsilon_1} \cup \Lambda_{\varepsilon_2} \cup \ldots \cup \Lambda_{\varepsilon_p} \tag{7.2.34}$$

Definition 7.1: For a given set of Λ and a system with controllability indices $\{\sigma_i = \varepsilon_i + 1, i \in p\}$, the ability to split Λ into symmetric subsets Λ_{ε_i} such that (7.2.34) holds true characterises a property referred to as *compatibility* of the Λ, with respect to the $\{\sigma_i, i \in p\}$ sets. ∎

In the following we will assume compatibility of the Λ, $\{\sigma_i, i \in p\}$ sets.

Remark 7.10: Compatibility of the Λ, $\{\sigma_i, i \in p\}$ sets implies that the minimal decomposition of the state space implied by the minimal basis can lead to a real state

feedback matrix. If compatibility is not valid, nonminimal decompositions will have to be dealt with, i.e. controllability subspaces of higher dimensions. This may be readily overcome but requires additional work going through the results characterising the possible dimensions of controllability subspaces [71], [8]. ∎

STEP (2): Having assumed compatibility, the free parameters in the selection of eigenvectors are defined as

$$h_1 = \ldots = h_{\sigma_1} = \begin{bmatrix} 1, & 0, & \ldots, & 0 \end{bmatrix}^t = \underline{e}_1$$
$$h_{\sigma_1+1} = \ldots = h_{\sigma_1+\sigma_2} = \begin{bmatrix} 0, & 1, & 0, & \ldots, & 0 \end{bmatrix}^t = \underline{e}_2$$
$$\vdots$$
$$h_{\sigma_1+\ldots+\sigma_{p-1}+1} = \ldots = h_n = \begin{bmatrix} 0, & \ldots, & 0, & 1 \end{bmatrix}^t = \underline{e}_p$$

STEP (3): For every ε_i, the $\sigma_i = \varepsilon_i + 1$ vectors are defined based on the common

$$h = \begin{bmatrix} 0, & \ldots, & 0, & 1, & 0, & \ldots, & 0 \end{bmatrix}^t$$
$$\leftarrow \quad i \quad \rightarrow$$

and the selected spectrum Λ_{ε_i} as

$$\underline{x}_j^{\varepsilon_i} = \overline{N}_{\varepsilon_i} \begin{bmatrix} 1 \\ \lambda_j^i \\ \vdots \\ \lambda_j^{i\varepsilon_i} \end{bmatrix}, \quad j = 1, \ldots, \varepsilon_i + 1 \tag{7.2.35a}$$

and thus a set of vectors

$$\begin{bmatrix} \underline{x}_1^{\varepsilon_1}, & \ldots, & \underline{x}_{\sigma_i}^{\varepsilon_i} \end{bmatrix} = \overline{N}_{\varepsilon_i} \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1^i & & \lambda_{\sigma_1}^i \\ \vdots & & \vdots \\ \lambda_1^{i\varepsilon_i} & \cdots & \lambda_{\sigma_i}^{i\varepsilon_i} \end{bmatrix} = \overline{N}_{\varepsilon_i} V\left(\Lambda_{\varepsilon_i}\right) \tag{7.2.35b}$$

can be derived, where $\overline{N}_{\varepsilon_i} \in \mathbb{R}^{n \times \sigma_i}$ and $V(\Lambda_{\varepsilon_i}) \in \mathbb{R}^{\sigma_i \times \sigma_i}$, since it is assumed that the eigenvalues are distinct (for the sake of simplicity), the Vandermonde matrix has full

rank. For the case of repeated eigenvalues, corresponding Jordan vectors can be defined by using derivatives of the $\underline{e}_{\varepsilon_i}(s) = \begin{bmatrix} 1, & s, & \ldots, & s^{\varepsilon_i} \end{bmatrix}$ vector evaluated at λ_i.

Proposition 7.7: For any given symmetric set Λ_{ε_i}, the set of vectors

$$X\left(\Lambda_{\varepsilon_i}\right) = \overline{N}_{\varepsilon_i} V\left(\Lambda_{\varepsilon_i}\right) = \overline{N}_{\varepsilon_i} V_{\varepsilon_i} \tag{7.2.36a}$$

is linearly independent. Furthermore, if the original set is a compatibly partitioning set as in (7.2.33), then the set of vector

$$X\left(\Lambda\right) = \begin{bmatrix} X\left(\Lambda_{\varepsilon_1}\right), & \ldots, & X\left(\Lambda_{\varepsilon_p}\right) \end{bmatrix} = \begin{bmatrix} \overline{N}_{\varepsilon_1}, & \overline{N}_{\varepsilon_1}, & \ldots, & \overline{N}_{\varepsilon_p} \end{bmatrix} \begin{bmatrix} V_{\varepsilon_1} & & & 0 \\ & V_{\varepsilon_2} & & \\ & & \ddots & \\ 0 & & & V_{\varepsilon_p} \end{bmatrix} \tag{7.2.36b}$$

is symmetric (pairwise complex conjugate) within each of the σ_i subsets and it is linearly independent. ∎

STEP (4): For every Λ_{ε_i} set and with $\underline{h} = \underline{e}_i$ vector, it is possible to define the input vectors $\underline{u}_j^{\varepsilon_i}$, $j = 1, \ldots, \varepsilon_1 + 1$ in a systematic way. Firstly, we express $\overline{D}(s)$ as

$$\overline{D}(s) = \begin{bmatrix} \underline{d}_1(s), & \underline{d}_2(s), & \ldots, & \underline{d}_p(s) \end{bmatrix} \tag{7.2.37a}$$

where $\partial\left[\underline{d}_i(s)\right] = \varepsilon_i + 1 = \sigma_i$, $i \in p$. Then

$$\underline{u}_j^{\varepsilon_i} = \underline{d}_i\left(\lambda_j^i\right), j = 1, 2, \ldots, \sigma_i \tag{7.2.37b}$$

and for the set Λ_{ε_i}, a new set is then defined by

$$U\left(\Lambda_{\varepsilon_i}\right) = \begin{bmatrix} \underline{u}_1^{\varepsilon_i}, & \ldots, & \underline{u}_{\sigma_i}^{\varepsilon_1} \end{bmatrix} = \begin{bmatrix} \underline{d}_i(\lambda_1^i), & \ldots, & \underline{d}_i\left(\lambda_{\sigma_i}^i\right) \end{bmatrix} \tag{7.2.37c}$$

STEP (5): The state feedback matrix that assigns Λ as closed loop eigenvalues with $X(\Lambda)$ as the corresponding closed loop eigenvectors is then defined by

$$K_S = \left[U\left(\Lambda_{\varepsilon_1}\right), \quad \ldots, \quad U\left(\Lambda_{\varepsilon_p}\right)\right]\left[X\left(\Lambda_{\varepsilon_1}\right), \quad \ldots, \quad X\left(\Lambda_{\varepsilon_p}\right)\right]^{-1} = U\left(\Lambda\right)X\left(\Lambda\right)^{-1} \qquad (7.2.35)$$

Remark 7.11: The construction of the frame $X(\Lambda)$ is based on the properties of minimal bases of matrix pencils [38] and thus this theory is instrumental in defining all such families of eigenframes. ∎

The advantage of this construction is that it leads to maximal rank feedback and provides constructive means for shaping the properties of the eigenframe $X(\Lambda)$. Furthermore the selection of the $\underline{h}_i$ vectors for each of the subspaces of the decomposition is arbitrary and this expresses the p degrees of freedom in the eigenstructure assignment, which may be further explored to achieve additional properties of the eigenframe beyond the linear independence.

7.2.8 Parameterisation of ordered minimal bases and the selection of Eigenframes

In the previous section we have developed an expression for the closed loop eigenframe (see condition (7.2.36b)) that clearly indicates that the eigenframe $X(\Lambda)$ may be factorised into the matrices

$$\overline{N}_{\varepsilon_1,\ldots,\varepsilon_p} = \left[\overline{N}_{\varepsilon_1}, \quad \overline{N}_{\varepsilon_2}, \quad .., \quad \overline{N}_{\varepsilon_p}\right] \qquad (7.2.36a)$$

$$\overline{V}_{\varepsilon_1,\ldots,\varepsilon_p} = \text{block diag}\,\{\overline{N}_{\varepsilon_1}, \quad \overline{N}_{\varepsilon_1}, \quad .., \quad \overline{N}_{\varepsilon_p}\} \qquad (7.2.36b)$$

where the first is a basis for the state space, with the matrices $\overline{N}_{\varepsilon_i}$ defining basis matrices for the $\mathcal{R}_i = sp\{\overline{N}_{\varepsilon_i}\}$ minimal $(\varepsilon_i + 1 = \sigma_i)$-dimension controllability

subspace and the matrix $\overline{V}_{\varepsilon_1,\ldots,\varepsilon_p}$ is a block diagonal Vandermode matrix, which has full rank when the spectrum has distinct values. As long as we select a set of minimal dimension controllability subspaces and distinct closed-loop spectra, the matrix

$$X(\Lambda) = \left[X(\Lambda_{\varepsilon_1}), X(\Lambda_{\varepsilon_2}), \cdots, X(\Lambda_{\varepsilon_p}) \right] = \left[\overline{N}_{\varepsilon_1}, \overline{N}_{\varepsilon_2}, \cdots, \overline{N}_{\varepsilon_p} \right] \begin{bmatrix} V_{\varepsilon_1} & & & 0 \\ & V_{\varepsilon_2} & & \\ & & \ddots & \\ 0 & & & V_{\varepsilon_p} \end{bmatrix}$$

$$= \overline{N}_{\varepsilon_1,\ldots,\varepsilon_p} \cdot \overline{V}_{\varepsilon_1,\ldots,\varepsilon_p} \tag{7.2.37}$$

has always full rank. This clearly demonstrates that independence can always be guaranteed and that the selection of the "best eigenframe" has as two components the selection of the spectrum that leads to the most orthogonal $\overline{V}_{\varepsilon_1,\ldots,\varepsilon_p}$ and the selection of the "most orthogonal" controllability subspace decomposition of the state space. This may be referred to as the "separation principle" for eigen-structure shaping.

The latter problem is linked to the parameterization of controllability subspaces, which is equivalent to the problem of parameterization of ordered minimal bases of the $\mathcal{N}_r = \mathcal{N}_r\{sN - NA\}$ rational vector space. The fundamentals of this parameterization are summarized below [96]:

Let us denote by $\mathcal{Z} = \mathcal{N}_r\{sN - NA\}$ the state-space rational vector space, $\dim \mathcal{Z} = p$. The space $\mathcal{Z}$ has always minimal bases which may be ordered by the ordered set of minimal indices denoted by

$$\mathcal{I}_z = \left\{ \left(\varepsilon_i, r_i \right), i \in \underset{\sim}{n}, 0 \le \varepsilon_1 < \ldots < \varepsilon_n \right\} \tag{7.2.38}$$

where ε_i are the distinct values of column minimal indices of $sN - NA$ and r_i denotes their corresponding multiplicity. Note that $\varepsilon_1 \ge 0, \sum_{i=1}^{n} r_i = p$, where p is the

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number of inputs and the set $\mathcal{I}_z$ is referred to as the set of restricted controllability indices.

For every ε_i, we define as $\sigma_i = \varepsilon_i + 1$ the corresponding controllability index of the system $\mathcal{S}(A,B)$. Using the ordered representation, the set of controllability indices, defined also as the set of column minimal indices of $[sI - A, -B]$ is given by

$$\mathcal{I}_c = \left\{ (\sigma_i, r_i), i \in \underline{n}, 1 \leq \sigma_1 < \dots < \sigma_n \right\} \tag{7.2.39}$$

Remark 7.12: If $rank(B) = p$, then all controllability indices of $\mathcal{S}(A,B)$ are positive

∎

Remark 7.13: The set of controllability indices satisfy

$$\sum_{i=1}^{n} \sigma_i r_i \leq n \tag{7.2.40}$$

with equality holding if and only if the system $\mathcal{S}(A,B)$ is controllable. ∎

Consider a minimal basis for $\mathcal{Z}$, which may be represented by a basis matrix $Z(s) \in \mathbb{R}^{n \times p}[s]$. We may always use the set $\mathcal{I}_z$ to order the columns of the basis matrix and this representation is denoted by, [96],

$$Z(s) = \left[Z_1(s); Z_2(s); \dots; Z_n(s) \right] \in \mathbb{R}^{n \times p}[s] \tag{7.2.41}$$

where $Z_i(s) \in \mathbb{R}^{n \times r_i}[s], Z_i(s) = \left[z_{i1}(s); \dots; z_{ir_i}(s) \right], \partial \left[\underline{z}_{ik}(s) \right] = \varepsilon_i$. Such minimal bases will be referred to as $\mathcal{I}_z$ – ordered minimal bases. The relationship between any two $\mathcal{I}_z$ – ordered minimal bases is defined by the following result.

Lemma 7.1: Let $Z^1(s), Z^2(s)$ be two ordered minimal bases of rational vector space $\mathcal{Z}$ characterised by the ordered set of minimal indices $\mathcal{I}_z = \left\{ (\varepsilon_i, r_i), i \in \underline{n}, 0 \leq \varepsilon_1 < \dots < \varepsilon_n \right\}$. There always exists an $\mathbb{R}[s]$ – unimodular matrix $W(s) \in \mathbb{R}^{n \times n}[s]$ such that

$$Z^2(s) = Z^1(s)W(s) \tag{7.2.42}$$

where $W(s)$ has the following structure

$$
\begin{array}{cccccc}
r_1 & r_2 & r_3 & \cdots & r_n & \\
\begin{bmatrix}
W_1 & W_{12}(s) & W_{13}(s) & \cdots & W_{1n}(s) \\
0 & W_2 & W_{23}(s) & \cdots & W_{2n}(s) \\
\vdots & 0 & \ddots & \ddots & \vdots \\
0 & \vdots & \cdots & W_{n-1} & W_{n-1,n}(s) \\
0 & 0 & \cdots & \cdots & W_n
\end{bmatrix}
&
\begin{matrix}
r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n
\end{matrix}
\end{array}
\tag{7.2.43}
$$

where $W_i \in \mathbb{R}^{r_i \times r_i}, i \in \underset{\sim}{n}, |W_i| \neq 0, W_{i,j}(s) = s^{p_{i,j}}\hat{W}_{i,j} + \ldots + \hat{W}_{i,j} \in \mathbb{R}^{r_i \times r_j}[s]$ with $p_{i,j} \leq \varepsilon_j - \varepsilon_i$, but otherwise arbitrary. ∎

The above result introduces a parameterisation of all ordered-minimal bases and it is central to our parameterisation of problem of all closed loop eigenstructures. The above result may expressed in a Toeplitz matrix equivalent set up, which is more appropriate for discussing properties of associated controllability subspaces with the ordered minimal basis.

Using the ordered representation of the minimal basis $Z(s) = [Z_1(s); \ldots; Z_i(s); \ldots; Z_n(s)]$, the general polynomial vector of the *i-th* block $\underline{z}_{i,j}(s)$ has degree ε_i and may be expressed as

$$\underline{z}_{i,j}(s) = \underline{z}_{i,j}^0 + s\underline{z}_{i,j}^1 + \ldots + s^{\varepsilon_i}\underline{z}_{i,j}^{\varepsilon_i}, i \in \underset{\sim}{n}, j \in \underset{\sim}{r_i} \tag{7.2.44}$$

or as

$$\underline{z}_{i,j}(s) = \begin{bmatrix} \underline{z}_{i,j}^0, \underline{z}_{i,j}^1, \ldots, \underline{z}_{i,j}^{\varepsilon_i} \end{bmatrix} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\varepsilon_i} \end{bmatrix} = \tilde{Z}_{i,j}\underline{e}_{\varepsilon_i}(s) \tag{7.2.45}$$

where $\tilde{Z}_{i,j} \in \mathbb{R}^{n \times \sigma_i}, \sigma_i = \varepsilon_i + 1$ and $rank\{\tilde{Z}_{i,j}\} = \sigma_i$. Clearly, $sp\{\tilde{Z}_{i,j}\} = \mathbb{R}_{i,j}$ is a minimal dimension controllability subspace with $\dim \mathbb{R}_{i,j} = \sigma_i$.

Note that condition (7.2.45) allows a clear characterization of the closed loop eigenvectors associated with $\mathbb{R}_{i,j}$. In fact under the spectrum compatibility conditions previously described we have that for the $\underline{z}_{i,j}(s)$ polynomial vector we can define the set of σ_i independent closed loop eigenvectors associated with the distinct frequencies $\Lambda_{i,j} = \{\lambda_{i,j}^1, ..., \lambda_{i,j}^{\sigma_i}\}, i = 1, ..., n, j = 1, ..., r_i$. These vectors are expressed as

$$\underline{z}_{i,j}(\lambda) = \tilde{Z}_{i,j}\underline{e}_{\varepsilon_i}(\lambda), \lambda \in \Lambda_{i,j} \tag{7.2.46}$$

and the set of all such vectors is represented in a matrix form as

$$\begin{aligned} Z_{i,j}(\Lambda_{i,j}) &= \left[\underline{z}_{i,j}(\lambda_{i,j}^1); ....; \underline{z}_{i,j}(\lambda_{i,j}^{\sigma_i})\right] = \\ &= \tilde{Z}_{i,j}\left[\underline{e}_{\varepsilon_i}(\lambda_{i,j}^1); ....; \underline{e}_{\varepsilon_i}(\lambda_{i,j}^{\sigma_i})\right] = \\ &= \tilde{Z}_{i,j}V_{\sigma_i}^j \end{aligned} \tag{7.2.47}$$

where $V_{\sigma_i}^j = \left[\underline{e}_{\varepsilon_i}(\lambda_{i,j}^1); ....; \underline{e}_{\varepsilon_i}(\lambda_{i,j}^{\sigma_i})\right]$ is defined by the distinct spectrum $\Lambda_{i,j}$ and thus $V_{\sigma_i}^j \in \mathbb{C}^{\sigma_i \times \sigma_i}$ is a Vandermonde matrix and has full rank due to the distinct spectrum assumption.

Assuming that for each $\mathbb{R}_{i,j}$ we assign a distinct spectrum $\Lambda_{i,j}$ (spectrum of distinct spaces may have overlapping), the overall closed loop eigenframe associated with the spectrum

$$\Lambda = \bigcup_{i=1}^{n}\bigcup_{j=1}^{r_i}\Lambda_{i,j} \tag{7.2.48}$$

may be expressed as

$$Z(\Lambda) = \left[....; \tilde{Z}_{i1} V_{\sigma_i}^1; ....; \tilde{Z}_{ir_i} V_{\sigma_i}^{r_i}; ....\right] diag\left\{....; V_{\sigma_i}^1; ....; V_{\sigma_i}^{r_i}; ....\right\} =$$
$$= \left[....; \tilde{Z}_i; ....\right] diag\left\{....; V_{\sigma_i}; ....\right\} = \qquad (7.2.49)$$
$$\tilde{Z}\tilde{V}(\Lambda)$$

where

$$\tilde{Z}_i = \left[\tilde{Z}_{i1}; ....; \tilde{Z}_{ir_i}\right] \in \mathbb{R}^{n \times \sigma_i r_i} \qquad (7.2.50)$$

$$\tilde{V}_{\sigma_i} = diag\left[\tilde{V}_{\sigma_i}^1; ....; \tilde{V}_{\sigma_i}^{r_i}\right] \in \mathbb{C}^{\sigma_i r_i \times \sigma_i r_i} \qquad (7.2.51)$$

and $\tilde{Z} \in \mathbb{R}^{n \times n}, \tilde{V}(\Lambda) \in \mathbb{C}^{n \times n}$ are full rank matrices, $\tilde{Z}$ corresponds to the given minimal basis, and $\tilde{V}(\Lambda)$ defined by the given spectrum Λ. The analysis so far leads to the following result:

Proposition 7.8: For any ordered minimal basis $Z(s)$ of $\mathcal{Z}$ as defined by (7.2.41) and the set of indices $\mathcal{I}_z = \left\{(\varepsilon_i, r_i), i \in \underline{n}, 0 \leq \varepsilon_1 < .... < \varepsilon_n\right\}$, and for any spectrum Λ expressed as by (7.2.48), a closed loop eigenstructure $Z(\Lambda)$ is expressed in a factorised form as

$$Z(\Lambda) = \tilde{Z}\tilde{V}(\Lambda) \qquad (7.2.52)$$

where $\tilde{V}(\Lambda)$ is a Vandermonde matrix and $\tilde{Z}$ is the ordered basis matrix of the minimal controllability subspaces decomposition of the state space, induced by the minimal basis. ∎

For a given spectrum Λ, the problem of defining all possible closed loop eigenstructures is thus equivalent to defining the structure of all possible $\tilde{Z}$ matrices. This problem clearly relates to the parameterisation of all possible minimal bases of $\mathcal{Z}$. To establish this, we have to translate the algebraic parameterisation introduced by *lemma 7.1.* to space set up and this is considered next.

We first note that

$$Z_i(s) = \left[\underline{z}_{i1}(s), \ldots, \underline{z}_{ir_i}(s) \right] = \left[\tilde{Z}_{i1}\underline{e}_{\varepsilon_i}(s), \ldots, \tilde{Z}_{ir_i}\underline{e}_{\varepsilon_i}(s) \right] =$$

$$= \left[\tilde{Z}_{i1}; \ldots; \tilde{Z}_{ir_i} \right] \begin{bmatrix} \underline{e}_{\varepsilon_i}(s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \underline{e}_{\varepsilon_i}(s) \end{bmatrix} = \tag{7.2.53}$$

$$= \tilde{Z}_i E_{\varepsilon_i r_i}(s)$$

where $\tilde{Z}_i \in \mathbb{R}^{n \times r_i \sigma_i}, E_{\varepsilon_i, r_i}(s) \in \mathbb{R}^{r_i \sigma_i \times r_i}[s]$. Furthermore we can also express $Z_i(s)$ as

$$Z_i(s) = s^{\varepsilon_i} Z_{\varepsilon_i}^i + \ldots + s Z_{\varepsilon_i}^1 + Z_{\varepsilon_i}^0 \tag{7.2.54}$$

which may be expressed as

$$Z_i(s) = \left[Z_{\varepsilon_i}^i + \ldots + Z_{\varepsilon_i}^1 + Z_{\varepsilon_i}^0 \right] \begin{bmatrix} s^{\varepsilon_i} I_{r_i} \\ \hline \vdots \\ \hline sI_{r_i} \\ \hline I_{r_i} \end{bmatrix} = \hat{Z}_i D_{\varepsilon_i, r_i}(s) \tag{7.2.55}$$

where $\tilde{Z}_i \in \mathbb{R}^{n \times r_i \sigma_i}, D_{\varepsilon_i, r_i}(s) \in \mathbb{R}^r$. Clearly, $\tilde{Z}_i, \hat{Z}_i$ are related by column permutations and thus we may summarise as:

Remark 7.14: There always exists a permutation matrix $U_i \in \mathbb{R}^{r_i \sigma_i \times r_i \sigma_i}$ such that

$$\tilde{Z}_i = \hat{Z}_i U_i \tag{7.2.56}$$

∎

We shall refer to U_i as the ε_i −normal permutation. Using the above remark we may also state:

Remark 7.15: Let $Z(\Lambda)$ be the closed loop eigenframe for the spectrum Λ, compatible with $\mathscr{I}_z$, and expressed by (7.2.52) as

$$Z(\Lambda) = \tilde{Z}\tilde{V}(\Lambda)$$
$$= \left[\ldots; \tilde{Z}_i; \ldots \right] diag\{\ldots; \tilde{V}_{\sigma_i}; \ldots\} \tag{7.2.57}$$

If U_i are the ε_i-normal permutation matrices, and

$$U = diag\{....;U_i;....\}$$

(7.2.58)

then $Z(\Lambda)$ may be expressed as

$$Z(\Lambda) = \left[....;\tilde{Z}_i;....\right] diag\left\{....;U_i\tilde{V}_{\sigma_i};....\right\}$$

(7.2.59)

∎

Expression (7.2.59) is more appropriate for the study of the parameterisation of frames since the submatrices $\hat{Z}_i$ are closer to the Toeplitz representation of the minimal bases which is considered next.

Definition 7.2: [96] Consider the $\mathcal{I}_z$-structured ordered minimal basis of $\mathcal{Z}$ expressed as:

$$Z(s) = \left[Z_1(s);....;Z_i(s);....;Z_n(s)\right] \in \mathbb{R}^{n \times p}[s]$$

(7.2.60)

where $Z_i(s) \in \mathbb{R}^{n \times r_i}[s]$, degree of $Z_i(s)$ being ε_i and assume to be expressed as

$$Z_i(s) = s^{\varepsilon_i} Z_{\varepsilon_i}^i +....+ s Z_1^i + Z_0^i$$

(7.2.61)

(i) We define as its *k-th* order Toeplitz representation with $k \geq \varepsilon_i$ the Toeplitz matrix

$$T_{k-\varepsilon_i+1}^{\varepsilon_i}(Z_i) = \begin{bmatrix} z_{\varepsilon_i}^i & 0 & \cdots & 0 \\ \vdots & z_{\varepsilon_i}^i & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ z_1^i & \vdots & & z_{\varepsilon_i}^i \\ z_0^i & z_1^i & & \vdots \\ 0 & z_0^i & & \vdots \\ \vdots & \ddots & \ddots & z_1^i \\ 0 & \cdots & 0 & z_0^i \end{bmatrix} \begin{array}{l} \left. \vphantom{\begin{matrix} z \\ \vdots \\ \vdots \end{matrix}} \right\} (k-\varepsilon_i)-blocks \end{array} \in \mathbb{R}^{(k+1)n \times (k+1-\varepsilon_i)r_i}$$

(7.2.62)

$$\underbrace{\qquad\qquad}_{(k-\varepsilon_i)-blocks}$$

for all $k : \varepsilon_i \leq k$.

(ii) Using the above notation we may define the *canonical Toeplitz matrix* of the minimal basis $Z(s)$ as

$$T_{\varepsilon_n}(Z) = \left[T_{\varepsilon_n - \varepsilon_1 + 1}^{\varepsilon_1}(Z_1); T_{\varepsilon_n - \varepsilon_2 + 1}^{\varepsilon_2}(Z_2); \dots; T_{\varepsilon_n - \varepsilon_n + 1}^{\varepsilon_n}(Z_n) \right] \tag{7.2.63}$$

which is a $(\varepsilon_n + 1)n \times \nu, \nu = \sum_{j=1}^{n}(\varepsilon_n - \varepsilon_j + 1)r_j$ matrix. ∎

Using this Toeplitz representation of the minimal basis $Z(s)$ we can express the parameterization of minimal bases result, i.e. *lemma 7.1* as indicated below:

Proposition 7.9: **[96]** Let $Z(s)$ and $Z'(s)$ be two $\{(\varepsilon_i, r_i), i \in \underline{n}\}$−ordered minimal bases matrices of $\mathcal{Z}$. There always exists a matrix $Q_n(W) \in \mathbb{R}^{\nu \times \nu}, |Q_n(W)| \neq 0$ Toeplitz type matrix, $\nu = \sum_{j=1}^{n}(\varepsilon_n - \varepsilon_j + 1)r_j$ such that

$$T_{\varepsilon_n}(Z') = T_{\varepsilon_n}(Z) Q_{\varepsilon_n}(W) \tag{7.2.64}$$

where

$$Q_{\varepsilon_n}(W) = \begin{bmatrix} T_{\varepsilon_n - \varepsilon_1 + 1}^{\varepsilon_1 - \varepsilon_1}(W_1) & T_{\varepsilon_n - \varepsilon_2 + 1}^{\varepsilon_2 - \varepsilon_1}(W_1) & \cdots & T_{\varepsilon_n - \varepsilon_n + 1}^{\varepsilon_n - \varepsilon_1}(W_{1n}) \\ 0 & T_{\varepsilon_n - \varepsilon_2 + 1}^{\varepsilon_2 - \varepsilon_1}(W_2) & \cdots & T_{\varepsilon_n - \varepsilon_n + 1}^{\varepsilon_n - \varepsilon_2}(W_{2n}) \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & T_{\varepsilon_n - \varepsilon_n + 1}^{\varepsilon_n - \varepsilon_n}(W_n) \end{bmatrix} \tag{7.2.65}$$

with $T_{k - \delta + 1}^{\delta}(W)$ the Toeplitz matrices defined on arbitrary polynomial matrices of degree δ. Conversely if for any two minimal bases $Z(s)$ and $Z'(s)$ structured by the same $\mathcal{J}$ set, their canonical Toeplitz matrices $T_{\varepsilon_n}(Z'), T_{\varepsilon_n}(Z)$ are minimal bases for the same system. ∎

To illustrate the above we consider the following example

Example 7.1: Let $\mathcal{I}_Z = \{(\varepsilon_1, r_1), (\varepsilon_2, r_2), (\varepsilon_3, r_3), \text{ with } \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 4\}$. A minimal basis for $\mathcal{Z}$ structured by $\mathcal{I}_z$ may be denoted as

$$Z(s) = [Z_1(s); Z_2(s); Z_3(s)]$$

where

$$Z_1(s) = sZ_1^1 + Z_0^1$$
$$Z_2(s) = s^2 Z_2^2 + sZ_1^2 + Z_0^2$$
$$Z_3(s) = s^4 Z_4^3 + s^3 Z_3^3 + s^2 Z_2^3 + sZ_1^3 Z_0^3$$

The canonical Toeplitz representation of $Z(s)$ is given by

$$T_4(Z) = \begin{bmatrix} Z_1^1 & 0 & 0 & 0 & Z_2^2 & 0 & 0 & Z_4^3 \\ Z_0^1 & Z_1^1 & 0 & 0 & Z_1^2 & Z_2^2 & 0 & Z_3^3 \\ 0 & Z_0^1 & Z_1^1 & 0 & Z_0^2 & Z_1^2 & Z_2^2 & Z_2^3 \\ 0 & 0 & Z_0^1 & Z_1^1 & 0 & Z_0^2 & Z_1^2 & Z_1^3 \\ 0 & 0 & 0 & Z_0^1 & 0 & 0 & Z_0^2 & Z_0^3 \end{bmatrix} \tag{7.2.66}$$

$$= [T_4^1(Z_1); T_3^2(Z_2); T_1^4(Z_3)]$$

Any other order minimal basis $Z'(s)$ has a Toeplitz representation $T_4(Z')$ and this is related to $T_4(Z)$ as

$$T_4(Z') = T_4(Z) Q_4(W) \tag{7.2.67}$$

where $Q_4(W)$ is defined by

$$Q_4(W) = \begin{bmatrix} W_1 & 0 & 0 & 0 & W_1^{12} & 0 & 0 & W_3^{13} \\ 0 & W_1 & 0 & 0 & W_0^{12} & W_1^{12} & 0 & W_2^{13} \\ 0 & 0 & W_1 & 0 & 0 & W_0^{12} & W_1^{12} & W_1^{13} \\ 0 & 0 & 0 & W_1 & 0 & 0 & W_0^{12} & W_0^{13} \\ \hline 0 & 0 & 0 & 0 & W_2 & 0 & 0 & W_2^{13} \\ 0 & 0 & 0 & 0 & 0 & W_2 & 0 & W_1^{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & W_2 & W_0^{13} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_3 \end{bmatrix} = \begin{bmatrix} T_4^0(W_1) & T_3^1(W_{12}) & T_1^3(W_{13}) \\ 0 & T_3^0(W_2) & T_1^2(W_{23}) \\ 0 & 0 & T_1^0(W_3) \end{bmatrix}$$

$$(7.2.68)$$

where $W_i \in \mathbb{R}^{r_i \times r_i}$, $|W_i| \neq 0$ and W_k^{ij} are arbitrary matrices of appropriate dimensions. For the two minimal bases $Z(s)$, $Z'(s)$ the block-column representations $\hat{Z}$ and $\hat{Z}'$ are given by

$$\hat{Z} = \left[Z_1^1, Z_0^1; Z_2^2, Z_1^2, Z_0^2; Z_4^3, Z_3^3, Z_2^3, Z_1^3, Z_0^3 \right]$$
$$\hat{Z}' = \left[Z_1^{1'}, Z_0^{1'}; Z_2^{2'}, Z_1^{2'}, Z_0^{2'}; Z_4^{3'}, Z_3^{3'}, Z_2^{3'}, Z_1^{3'}, Z_0^{3'} \right]$$

and thus (7.2.67) implies the following conditions

$$Z_1^{1'} = Z_1^1 W_1, Z_0^{1'} = Z_0^1 W_1,$$
$$Z_2^{2'} = Z_1^1 W_1^{12} + Z_2^2 W_2, Z_1^{2'} = Z_0^1 W_1^{12} + Z_1^1 W_0^{12} + Z_1^2 W_2,$$
$$Z_0^{2'} = Z_0^1 W_0^{12} + Z_0^2 W_2, Z_4^{4'} = Z_1^1 W_3^{13} + Z_2^2 W_2^3 + Z_4^3 W_3,$$
$$Z_3^{3'} = Z_0^1 W_3^{13} + Z_1^1 W_2^{13} + Z_1^2 W_2^{23} + Z_2^2 W_1^{23} + Z_3^3 W_3,$$
$$Z_2^{3'} = Z_0^1 W_2^{13} + Z_1^1 W_1^{13} + Z_0^2 W_2^{23} + Z_1^2 W_1^{23} + Z_2^2 W_0^{23} + Z_2^3 W_3,$$
$$Z_1^{3'} = Z_0^1 W_1^{13} + Z_1^1 W_0^{13} + Z_0^2 W_1^{23} + Z_1^2 W_0^{23} + Z_1^3 W_3$$
$$Z_0^{3'} = Z_0^1 W_0^{13} + Z_0^2 W_0^{23} + Z_0^3 W_3$$

The above relationships imply that the block column representations $\hat{Z}$ and $\hat{Z}'$ are related as

$$\hat{Z}' = \hat{Z} R(W) \qquad (7.2.69)$$

where $R(W)$ is defined as

$$\begin{bmatrix} W_1 & 0 & W_1^{12} & W_0^{12} & 0 & W_3^{13} & W_2^{13} & W_1^{13} & W_0^{13} & 0 \\ & W_1 & 0 & W_1^{12} & W_0^{12} & 0 & W_3^{13} & W_2^{13} & W_1^{13} & W_0^{13} \\ & & W_2 & 0 & 0 & W_2^{23} & W_1^{23} & W_0^{23} & 0 & 0 \\ & & & W_2 & 0 & 0 & W_2^{23} & W_1^{23} & W_0^{23} & 0 \\ & & & & W_2 & & 0 & W_2^{23} & W_1^{23} & W_0^{23} \\ & & \mathbf{0} & & & W_3 & 0 & 0 & 0 & 0 \\ & & & & & & W_3 & 0 & 0 & 0 \\ & & & & & & & W_3 & 0 & 0 \\ & & & & & & & & W_3 & 0 \\ & & & & & & & & & W_3 \end{bmatrix}$$

The above provides an alternative expression of the parameterisation of minimal bases and thus of the structure of eigenframes.

7.3. Conclusion

This chapter has introduced a behavioural framework for discussing system properties such as poles, zeros, as well introducing a new parameterization of possible closed loop eigenstructures. The approach makes use of minimal bases theory of matrix pencils and provides new ways for computing a pole assigning state feedback matrix K_S that explores the algebraic properties of minimal bases and associated controllability subspaces.

The methodology starts off by deriving the total behaviour under zero initial conditions of a minimal system. The problem of optimal distribution of eigenvalues to guarantee stability and maximal orthogonality of the eigenframe may be now addressed using Grammian based criteria and exploiting the separation of the spectrum selection and the selection of the most appropriate controllability subspaces decomposition of the state space.

The properties of characteristic bases of controllability subspaces may be explored and this has the advantage that expresses the desirable closed loop eigenframe in terms of differences of the open and closed loop spectra. The latter may allow the linking of robustness criteria (orthogonality of the frame) to pole mobility.

The general behaviour framework introduced here provides the means to also examine problems of the creation of Forney dynamical indices from controllability indices, or observability indices, in terms of problems of general model projection involving selection of the matrix B or C respectively. Such problems are generalisations of the squaring down problem and are issues for further research.

CHAPTER 8

MINIMISATION OF THE ANGLE BETWEEN CLOSED-LOOP EIGENVECTORS VIA NON-SMOOTH OPTIMISATION ALGORITHM

8.1. Introduction

The aim of this chapter is to provide the required new concept of the relative positioning between subspaces that can be used in:

- Sensitivity of closed-loop eigenvectors
- Relative measures of controllability

In fact, the main subject of this chapter is to develop an optimization algorithm that will allow us to design and build of the solutions of minimizing the condition number of the gram matrix to assign and to reduce the Sensitivity of closed-loop eigenvetors for any controllable system(s), obtaining a system which is robust to the perturbations injected to the eigenvalues or their relative eigenvectors, i.e. the error is minimized.

So far the parameterisation of the closed-loop eigenvectors based on two different methods have been studied; the first method using the connection of open-loop /closed-loop spectra (Chapter Five) and the second method, based on the algebraic interpretation of controllability subspaces via using ordered minimal bases of matrix pencil. This also led to the parameterisation of minimal bases itself by constructing the relative Toeplitz matrices (Chapter Seven).

In this chapter, our main focus is on the optimization problem related to the parameterisations of eigenframes obtained from the first method, i.e. by using optimisation over condition number of relative Gramian matrix. This optimisation problem has been formulated in chapter Five and presented in equation (5.6.4) as follows:

- Suppose each entry of $V(\underline{x})$ is a continuously differentiable function of $\underline{x} \in \mathbb{R}^m$, then each entry of $A(\underline{x}) = U_\mu^t(\underline{x}).U_\mu(\underline{x})$ is also a continuously differentiable function of x. We consider the following minimization problem:

$$\text{minimize} \qquad \mathcal{K}(A(\underline{x}))$$

$$\text{subject to} \qquad \underline{x} \in \mathcal{X} \qquad\qquad\qquad (8.1.1)$$

where $\mathcal{X}$ is a convex set in $\mathbb{R}^m$ and $U_\mu(\underline{x})$ the closed-loop eigenvector matrix corresponding to closed-loop eigenvalue(s) μ.

In this case, the aim is to find the non- zero elements of any vector $\underline{m}$ in equation (5.5.3) (Refer to Chapter Five), such that the overall condition number of the closed-loop eigenvactor $U_\mu(\underline{x})$ will be minimized (the angle between eigenvectors be maximized).

Clearly one may continue with the optimisation regarding to the parameterisation of minimal bases by constructing the relative Toeplitz matrices (Chapter Seven) as we will not concentrate on this part and so this optimisation will be considered to be an open issue. It is worth mentioning that the target for second optimisation problem will be to find the best possible sets of minimal bases such that the resulting controllability subspaces contain maximal angle as possible.

Regarding to the optimisation over the first method and in the event of having repeated eigenvalues, we may need to use a non-smooth optimisation method, i.e. a non-smooth (or at some cases semi-smooth) optimisation problem will be developed, depending on the values of the open-loop or closed-loop eigenvalues and relative eigenvectors.

So for that reason, in this chapter, we will first review the fundamental elements of non-smooth optimisation [11] by employing Clarke's generalized gradient of the function $\kappa(A(x))$ in equation (8.1.1). Most optimization methods are only efficient for convex and smooth problems. To develop efficient algorithms to solve (8.1.1) [11], we adopt the Clarke generalized gradient and the exponential smoothing function. Then the Smoothing Conjugate Gradient optimisation algorithm [79] will be presented in order to be used for solving such non-smooth problems. At each iteration, we use the function value of the smoothing approximation of the objective function in (8.1.1) and update the smoothing parameter.

8.2. Non-Smooth optimisation problem definitions

As it has been studied in pervious chapters, the condition number and/or the determinant of a Gram matrix, are often used to measure the sensitivity of the polynomial approximation. Given a polynomial basis, consider the problem of finding a set of points and/or weights which minimizes the condition number of the Gram matrix or equivalently maximize the determinant of the Gram matrix.

Throughout this chapter, we denote by the $\mathbb{S}_n$ space of symmetric $n \times n$ matrices with the standard inner products:

$$< A, B >= \sum_{i,j=1}^{n} a_{ij}b_{ij}, \qquad \forall A = (a_{ij}), B = (b_{ij}) \in \mathbb{S}_n.$$

We denote by $\mathbb{S}_n^+$ and $\mathbb{S}_n^{++}$, the cone of symmetric positive semi definite $n \times n$ matrices and the cone of symmetric positive definite $n \times n$ matrices, respectively.

For $A \in \mathbb{S}_n$, we denote by $\lambda(A) \in \mathbb{R}^n$, the vector of its eigenvalues ordered in a decreasing order:

$$\lambda_1(A) \geq \ldots \geq \lambda_n(A).$$

The Euclidean condition number of a nonzero matrix $A \in \mathbb{S}_n$ is defined by [12]

$$k(A) = \begin{cases} \dfrac{\lambda_1(A)}{\lambda_n(A)} & \text{If A is nonsingular} \\ \infty & \text{If A is singular} \end{cases}$$

Optimizing eigenvalue functions have been studied extensively in [80], [81], [82],[83], [84], [85] Recently there some publications including the work done by P. Marechal and J. J. Ye, [86], studying the following optimization problem

$$\text{minimize } k(A)$$

$$\text{subject to } A \in \Omega \tag{8.2.1}$$

where Ω is a compact convex subset of $\mathbb{S}_n^+$. From the definition, it is clear that if $\Omega \cap \mathbb{S}_n^{++}$ is not empty, then a minimizer for (8.2.1) must belong to $\mathbb{S}_n^{++}$. However, if $\Omega \cap \mathbb{S}_n^{++}$ is empty, then (8.2.1) has no optimal solution [11].

Here we are interested in the minimal condition number for matrices in the form $A = V^T V$, where $V \in \mathbb{R}^{l \times n}$ with $\ell \geq n$, and $rank(V) = n$. Clearly $A \in \mathbb{S}_n^{++}$.

Let $\|.\|$ denote the Euclidean vector norm and matrix norm. The Euclidean condition number of V is defined by [87]

$$\mathcal{K}(V) = \max_{y \neq 0} \frac{\|y\|}{\|V_y\|} \max_{z \neq 0} \frac{\|V_z\|}{\|\underline{z}\|} = \|V\| \|V^\dagger\| = \sqrt{\mathcal{K}(A)} = \frac{\sqrt{\lambda_1(A)}}{\sqrt{\lambda_n(A)}}$$

where $V^\dagger = (V^T V)^{-1} V^T$ is the Moore-Penrose generalized inverse of V.

The quantity $\kappa(V)$ has been widely used in the sensitivity analysis of interpolation and approximation; however, there is little work on efficient optimization methods to find optimal weights and nodes which minimize $\kappa(V)$ with a fixed n.

As it mentioned before, suppose each entry of $V(\underline{x})$ is a continuously differentiable function of $\underline{x} \in \mathbb{R}^m$ [11], then each entry of $A(\underline{x}) = V(\underline{x})^t V(\underline{x})$ is also a continuously differentiable function of $\underline{x}$.

8.3. Generalized gradient of $\mathcal{K}(A(\underline{x}))$

In this section, we present an expression of the Clarke generalized gradient of $\mathcal{K}(A(\underline{x}))$. The presented formation in this section is based on the work done by Chen, R. S. Womersley and J. Ye [11] on the area of minimization the condition number of a gram matrix. In order to explain the results clearly, we divide this section into different subsections. First we recall for $\partial\mathcal{K}(A(\underline{x}))$ and then we give an expression of the generalized gradient for $\mathcal{K}(A(\underline{x}))$ with $A(\underline{x}) = V(\underline{x})^t V(\underline{x})$.

8.3.1. $\kappa(A)$

For, $A \in \mathbb{S}_n$ the notation $diag(\lambda(A)) \in \mathbb{S}_n$ is used for the diagonal matrix with the vector $\lambda(A) \in \mathbb{R}^n$ on the main diagonal [11]. It is known that any $A \in \mathbb{S}_n^+$ admits an eigenvalue decomposition:

$$A = U(A).diag(\lambda(A)).U(A)^T$$

with a square orthogonal matrix $U(A)$, $U(A)^T U(A) = I_n$, whose columns are eigenvectors of A. Let $u_i(A)$ be the ith column of matrix $U(A)$.

***Proposition 8.1*: The Clarke generalized gradient [80], [85], [88]**

Let $A \in \mathbb{S}_n$ .The Clarke subdifferential of $\lambda_1(A)$ is given by:

$$\partial\lambda_1(A) = \{G = \sum_{i=1}^{d(A)} \tau_i u_i(A)u_i(A)^T : \tau \geq 0, i = 1.......d(A), \sum_{i=1}^{d(A)} \tau_i = 1\}$$

where $d(A)$ is the multiplicity of the largest eigenvalue of the matrix A. The Clarke generalized gradient of $\lambda_n(A)$ is given by

$$\partial\lambda_n(A) = \{H = \sum_{i=1}^{b(A)} \gamma_i u_{n-i+1}(A)u_{n-i+1}(A)^T : \gamma_i \geq 0, i = 1.......b(A), \sum_{i=1}^{b(A)} \tau_i = 1\}$$

where $b(A)$ is the multiplicity of the smallest eigenvalue of the matrix A. ∎

For the Clarke generalized gradient of quotients, we have the following proposition:

Proposition 8.2: [86, Proposition 4.2.] Assume that $A \in \mathbb{S}_n^{++}$, then κ is Clarke regular at A and its Clarke generalized gradient at A is given by

$$\partial \kappa(A) = \lambda_n(A)^{-1}(\partial \lambda_1(A) - \kappa(A)\partial \lambda_n(A)) .$$

The following two submatrices of $U(A)$

$U_\alpha(A) = \{u_1(A),......,u_d(A)\}$, and $U_\beta(A) = \{u_{n-b(A)+1}(A),......,u_n(A)\}$

are formed by the orthonormal bases for the eigenspaces corresponding to the largest eigenvalue and the smallest eigenvalue of A. ∎

Applying the two above propositions, we have the following formula for $\partial k(A)$:

Proposition 8.3: [11]For $A \in \mathbb{S}_n^{++}$, let $d(A)$ be the multiplicity of the largest eigenvalue of matrix A, and $b(A)$ be the multiplicity of the smallest eigenvalue of matrix A. Then

$$\partial \kappa(A) = \lambda_n(A)^{-1}(\partial \lambda_1(A) - \kappa(A)\partial \lambda_n(A))$$

$$= \{Y \in R^{n \times n} : Y_{pq} = \frac{1}{\lambda_n(A)} < U_\alpha^T(A)e_p e_q^T U_\alpha(A), p_\alpha > - \frac{k(A)}{\lambda_n(A)} < U_\beta^T(A)e_p e_q^T U_\beta(A), P_\beta > ,$$

$p = 1,...,n$, $q = 1,...,n$ where $P_\alpha \in D_{d(A)}^+, tr(P_\alpha) = 1, P_\beta \in D_{b(A)}^+, tr(P_\beta) = 1\}$.

Proof:

By proposition 8.1, for any $G \in \partial \lambda_1(A)$, there is a $P\alpha \in D_{d(A)}^+$, with $tr(P\alpha) = 1$ such that each element G_{pq} of G can be written as

$$G_{pq} = \langle (e_p^T U_\alpha(A))^T e_q^T U_\alpha(A), P_\alpha \rangle = \langle U_\alpha^T(A)e_p e_q^T U_\alpha(A), P_\alpha \rangle.$$

Similarly, for any $H \in \partial \lambda_n(A)$, there is $P_\beta \in D_{d(A)}^+$ with $tr(P_\beta) = 1$ such that each element H_{pq} of H can be written as

$$H_{pq} = \langle U_\beta^T(A)e_p e_q^T U_\beta(A), P_\beta \rangle.$$

The desired formula follows from Proposition 8.2. ∎

Remark 8.1: In the case where $\lambda_1(A)=\lambda_n(A)$, we have $U=U_\alpha=U_\beta$ and

$$\partial k(A)= \{Y\in\mathbb{R}^{n\times n}:Y_{Pq}=\frac{1}{\lambda_n(A)}\langle U(A)e_p e_q^T U(A),P_\alpha\rangle-\frac{1}{\lambda_n(A)}\langle U(A)e_p e_q^T U(A),P_\beta\rangle,$$

$p=1,...,n$, $q=1,...,n$ where $P_\alpha\in D_{d(A)}^+,tr(P_\alpha)=1,P_\beta\in D_{b(A)}^+,tr(P_\beta)=1\}$

$$=\{Y\in R^{n\times n}:Y_{Pq}=\frac{1}{\lambda_n(A)}\max_{1\le i\le n}|U(A)e_p e_q^T U(A)|_{ii}\,[-1,1],p=1,...,n,q=1,...,n\}.$$

Such a matrix A would have the global minimal condition number 1 and it is clear that $0\in\partial\kappa(A)$. ∎

8.3.2. $\mathcal{K}(A(\underline{x}))$ with $A(\underline{x})=V(\underline{x})^t V(\underline{x})$

Let $V(\underline{x})$ be an $l\times n$ matrix [11] with each entry being a continuously differentiable function of $\underline{x}\in\mathbb{R}^m$. The differentiability of V implies that each entry of $A(\underline{x})=V(\underline{x})^t V(\underline{x})\in\mathbb{R}^{n\times n}$ is a continuously differentiable function of $\underline{x}$.

Let $\mathcal{X}\subset\mathbb{R}^m$ be a nonempty, compact and convex set. It is convenient to define a function $f:\mathcal{X}\to\mathbb{R}$ by

$$f(\underline{x})= \mathcal{K}(A(\underline{x})) \tag{8.3.1}$$

We assume that for any $\underline{x}\in\mathcal{X},rank(V(\underline{x}))=n$. We consider (8.2.1) in the following version.

$$\text{Minimize } f(\underline{x})$$

$$\text{subject to } \underline{x}\in\mathcal{X}. \tag{8.3.2}$$

Since $\lambda_1(A)$ is a convex function of A and $\lambda_n(A)$ is a concave function of A then $\lambda_1(A)$ and $\lambda_n(A)$ are Lipschitz continuous functions of A.

Definition 8.1: [89] For a general function $f:I\to Q$ where I is a set of rational numbers (typically I may be an interval of rational numbers: $\{\underline{x}\in Q:a\le\underline{x}\le b\}$ for some rational numbers a and b, if $\underline{x}_1$ and $\underline{x}_2$ are two numbers in I, then $|\underline{x}_2-\underline{x}_1|$ is the

change in the input and $\left| f(\underline{x}_2) - f(\underline{x}_1) \right|$ is the corresponding change in the output. We say that f is Lipschitz continuous with Lipschitz constant L_f on I, if there is a (necessarily nonnegative) constant L_f such that

$$\left| f(\underline{x}_1) - f(\underline{x}_2) \right| \leq L_f \left| \underline{x}_1 - \underline{x}_2 \right| \quad \text{for all } \underline{x}_1, \underline{x}_2 \in I \tag{8.3.3}$$

By the continuous differentiability of $A(\underline{x})$, $\lambda_1(A(\underline{x}))$ and $\lambda_n(A(\underline{x}))$ are Lipschitz continuous functions on $\mathcal{X}$. Moreover, there are positive constants $\underline{\lambda}_n$ and $\overline{\lambda}_1$, such that

$$\underline{\lambda}_n \leq \lambda_n(A(\underline{x})) \quad \text{and} \quad \lambda_1(A(\underline{x})) \leq \overline{\lambda}_1 \quad \forall \underline{x} \in \mathcal{X}.$$

Hence f is Lipschitz continuous and satisfies

$$1 \leq f(\underline{x}) \leq \frac{\overline{\lambda}_1}{\underline{\lambda}_n} \quad \forall \underline{x} \in \mathcal{X} \tag{8.3.4}$$

This, together with the continuity of f on $\mathcal{X}$, ensures the existence of a solution of (8.3.2). Denote [11]

$$A_k(\underline{x}) = \frac{\partial A(\underline{x})}{\partial \underline{x}_k} \quad k = 1, \cdots, m.$$

By the definition of A_{pq}, V and A, we have

$$A_k(x) = \sum_{p=1}^{l}\sum_{q=1}^{n} \frac{\partial A(V)}{\partial V_{pq}} \frac{\partial V_{pq}}{\partial x_k} = \sum_{p=1}^{l}\sum_{q=1}^{n} A_{pq}(V) \frac{\partial V_{pq}}{\partial x_k} = \sum_{p=1}^{l}\sum_{q=1}^{n} (e_q v_p^T + v_p e_q^T) \frac{\partial V_{pq}}{\partial x_k} \in \mathbb{S}_n.$$

Let $d(x)$ be the multiplicity of the largest eigenvalues of $A(\underline{x})$, and $b(\underline{x})$ be the multiplicity of the smallest eigenvalue of $A(\underline{x})$. Let $A(\underline{x})$ admit an eigenvalue decomposition

$$A(\underline{x}) = U(\underline{x}) diag(\lambda(A(\underline{x})) U(\underline{x})^t \quad \text{with} \quad U(\underline{x})^t U(\underline{x}) = I_n.$$

Let

$$U_\alpha = (u_1(\underline{x}), \cdots, u_{d(\underline{x})}(\underline{x})) \quad \text{and} \quad U_\beta = (u_{n-b(\underline{x})}(\underline{x}), \cdots, u_n(\underline{x})). \qquad \blacksquare$$

Proposition 8.4: [11] Suppose that $rank(V(\underline{x})) = n$, then f is Clarke regular at $\underline{x}$ and the Clarke generalized gradient of f is

$$\partial f(\underline{x}) = \left\{ g \in \mathbb{R}^n : g_k = \frac{1}{\lambda_n(A(\underline{x}))} < U_\alpha^t A_k(\underline{x}) U_\alpha, p_\alpha > -\frac{\mathcal{K}(A(\underline{x}))}{\lambda_n(A(\underline{x}))} < U_\beta^t A_k(\underline{x}) U_\beta, p_\beta >, \right.$$
$$\left. p = 1,\cdots,n, \; q = 1,\cdots,n \text{ where } P_\alpha \in D_{d(A)}^+, tr(P_\alpha) = 1, P_\beta \in D_{b(A)}^+, tr(P_\beta) = 1 \right\}$$

Proof : The proof is similar to the one related to Propos. 8.3. ∎

8.4. Smoothing approximation

In this part, we will show the smoothed approximation of the function of condition number $\mathcal{K}(A(\underline{x}))$ such that $A(\underline{x}) = V(\underline{x})^t V(\underline{x})$.

The exponential smoothing function [11] has been used for continuous min-max problem [90] and for minimizing the largest eigenvalue of a symmetric matrix [86],[91,26]. Applying the exponential smoothing function for the largest and the smallest eigenvalue functions, we introduce the smoothing function of the condition number as follows [11]:

$$\tilde{f}(x,\mu) = -\frac{\ln(\sum_{i=1}^{n} e^{\lambda_i(A(x))/\mu})}{\ln(\sum_{i=1}^{n} e^{-\lambda_i(A(x))/\mu})}. \tag{8.4.1}$$

In numerical computations, we use an equivalent formula

$$\tilde{f}(x,\mu) = \frac{\lambda_1(A(x)) + \mu \ln(\sum_{i=1}^{n} e^{\lambda_i(A(x))-\lambda_1(A)))/\mu})}{\lambda_n(A(x)) - \mu \ln(\sum_{i=1}^{n} e^{(\lambda_n(A(x))-\lambda_i(A(x)))/\mu})} \tag{8.4.2}$$

which is more stable numerically compared to equation in (8.4.1).

In this section we will show that this smoothing function has various nice properties including the gradient consistency property. These properties ensure that any accumulation point of the sequence generated by some smoothing methods is a Clarke stationary point. For example, the smoothing

projected gradient (SPG) method [92] or smoothing conjugate gradient method [79] can be used to solve (8.4.2).

Theorem 8.1: **[11]** Let f and $\tilde{f}(.,\mu)$ be defined by (8.3.1) and (8.4.1) respectively. Then

(*i*) $\tilde{f}(.,\mu)$ is continuously differentiable for any fixed $\mu > 0$ with gradient

$$\frac{\partial \tilde{f}_{(x,\mu)}}{\delta x_k} = \frac{-1}{\mu \ln(\sum\limits_{i=1}^{n} e^{-\lambda_i(A(x))/\mu})} \left[\frac{1}{\sum\limits_{i=1}^{n} e^{\lambda_i(A(x))/\mu}} \sum\limits_{i=1}^{n} e^{\lambda_i(A(x))/\mu} u_i(x)^T A_k(x) u_i(x) - \frac{\tilde{f}_{(x,y)}}{\sum\limits_{i=1}^{n} e^{-\lambda_i(A(x))/\mu}} \sum\limits_{i=1}^{n} e^{-\lambda_i(A(x))/\mu} u_i(x)^T A_k(x) u_i(x) \right].$$

(*ii*) There exists a constant $c > 0$, such that for any $\underline{x} \in \mathcal{X}$ and $\mu \le \dfrac{\lambda_n}{2 \ln n}$

$$0 \le \tilde{f}(\underline{x},\mu) - f(\underline{x}) \le c\mu. \tag{8.4.3}$$

(*iii*) For any $\overline{x} \in \mathcal{X}$ (where $\mathcal{X}$ is a local Lipschitz continuous function), $\left\{ \lim\limits_{\underline{x} \to \underline{x}, \mu \downarrow 0} \Delta_{\underline{x}} \tilde{f}(\underline{x},\mu) \right\}$ is nonempty and bounded.

Moreover, $\tilde{f}(.,\mu)$ satisfies the gradient consistent property, that is,

$$\left\{ \lim\limits_{\underline{x} \to \underline{x}, \mu \downarrow 0} \Delta_{\underline{x}} \tilde{f}(\underline{x},\mu) \right\} \subset \partial f(\overline{x})$$

(*iv*) For any fixed $\mu > 0$, the gradient of $\tilde{f}(\underline{x},\mu)$ is Lipschitz continuous, that is, for any $\underline{x}, \underline{y} \in \mathcal{X}$, there exists a constant $L\mu$ such that

$$\left\{ \nabla \tilde{f}(\underline{x},\mu) - \nabla \tilde{f}(\underline{y},\mu) \right\} \le L_\mu \|\underline{x} - \underline{y}\| \tag{8.4.4}$$

Proof: See [11] for the proof. ∎

According to Theorem (8.1), we can construct globally convergent smoothing methods for solving (8.3.2). In the smoothing methods, we can update the iterates $\underline{x}^k$ and smoothing parameter μ_k in an appropriate way which depends on the method used for the smoothing problems. Alternatively one can apply

smoothing steepest descent method via the same method used in [93] for the smoothing problem. We have the following global convergence theorem.

Theorem 8.2 **[11]:** From any starting point $\underline{x}_0 \in \mathcal{X}$, the sequence $\left\{\underline{x}^k\right\}$ generated by the smoothing optimisation method [92] is contained in $\mathcal{X}$ and any accumulation point $\overline{x}$ of $\left\{\underline{x}^k\right\}$ is a Clarke stationary point, that is, there is $g \in \partial f(\overline{x})$ such that

$$< g, \underline{x} - \overline{x} > \geq 0, \forall \underline{x} \in \mathcal{X}.$$

Proof. From Theorem 8.1., we know that Assumption 2.1 in [92] holds, and

$$\left\{\lim_{\underline{x}^k \to \underline{x}, \mu \downarrow 0} \Delta_{\underline{x}} \tilde{f}(\underline{x}^k, \mu)\right\} \subset \partial f(\overline{x}).$$

By Theorem 2.1 in [92], we have the conclusion of this theorem. ∎

By virtue of [86, Proposition 5.1], Theorem 8.2, has the following immediate consequences.

Corollary 8*.1* **[11]:** Under the assumptions of Theorem 8.2. if the function f is pseudo-convex in a neighbourhood $B(\overline{x}) \subset \mathcal{X}$, then the accumulation point is a local optimal solution and if the function f is pseudo-convex on $\mathcal{X}$, then the accumulation point is a global optimal solution. ∎

Although we have generated a *MATLAB* programme for a smooth case based on steepest Descent method (Appendix_1), however, in the next section, we will introduce also a non-smooth optimization algorithm for solving the optimization problem defined in equation (8.3.2) in the case of having a non-smooth problem.

8.5. Non-smooth Optimisation Algorithm

In this section, we will describe a non-smooth optimisation algorithm using Smoothing Conjugate Gradient Method introduced in [79]. During this part, we consider the general iterative scheme for solving (8.3.2) as

$$\underline{x}_{k+1} = \underline{x}_k + \alpha_k d_k, k = 0, 1, \cdots, \tag{8.5.1}$$

where α_k is a positive scalar and d_k is a search direction given by some formula. Before stating the non-smooth algorithm, it is necessary to recall the following definition:

Definition 8.2: [79] Let $\tilde{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be a locally Lipschitz continuous function. We call $\tilde{f} = \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ a smoothing function of f, $\tilde{f}(.,\mu)$ is continuously differentiable in R^n for any fixed $\mu \in R_{++}$, and $\lim_{\mu \downarrow 0} \tilde{f}(\underline{x},\mu) = f(\underline{x})$ for any fixed $\underline{x} \in \mathbb{R}^n$.

By denoting $\nabla \tilde{f}(\underline{x},\mu) = \nabla_{\underline{x}} \tilde{f}(\underline{x},\mu), \tilde{g}_k = \nabla \tilde{f}(\underline{x},\mu_k)$, the following smoothing conjugate gradient method for non-smooth and non-convex optimization is presented [79]:

Algorithm 8.1: Smoothing Conjugate Gradient Method

Step 1: Choose constants $\varepsilon_0 > 0, r \geq 0$. Choose $\delta \in (0,1), \rho, \gamma_1 \in (0,1), \mu_0 > 0, \gamma > 0$ and initial point $\underline{x}_0 \in \mathbb{R}^n$. Let $d_0 = -\tilde{g}_0$. Set $k := 0$.

Step 2: Compute the stepsize α_k by the *Armijo line search*, that is for

$$\alpha_k = \max\{\rho^0, \rho^1, \cdots\}, \text{ satisfying } \tilde{f}(\underline{x}_k + \rho^m d_k, \mu_k) \leq \tilde{f}(\underline{x}, \mu_k) + \partial \rho^m \tilde{g}_k^t d_k$$

Set $\underline{x}_{k+1} = \underline{x}_k + \alpha_k d_k$.

Note: Armijo line search can be described simply be the following:

- Given $\alpha_{init} > 0$ (e.g. $\alpha_{init} = 1$), let $\alpha^{(0)} = \alpha_{init}$ and $l = 0$.

- Until $f(\underline{x}^k + \alpha^{(l)} \rho^k) \leq f(\underline{x}^k) + \alpha^{(l)} \beta. \left[g^k \right]^t p^k$

    (i) Set $\alpha^{(l+1)} = \tau \alpha^{(l)}$, where $\tau \in (0,1)$ is fixed (e.g. $\tau = \frac{1}{2}$),

    (ii) Increment l by 1.

- $\alpha^k = \alpha^{(l)}$.

Step 3: If $\left\| \nabla \tilde{f}(\underline{x}_{k+1}, \mu_k) \right\| \geq \gamma \mu_k$, then set $\mu_{k+1} = \mu_k$; otherwise, choose $\mu_{k+1} = \gamma_1 \mu_k$.

Step 4: Compute d_{k+1} by the following formula

$$d_{k+1} = -\tilde{g}_{k+1} + \left(\frac{\tilde{g}_{k+1}^T \tilde{z}_k}{d_k^T \tilde{z}_k} - \frac{2\|\tilde{z}_k\|^2 \; \tilde{g}_{k+1}^T d_k}{(d_k^T \tilde{z}_k)^2} \right) d_k + \frac{\tilde{g}_{k+1}^T d_k}{d_k^T \tilde{z}_k} \tilde{z}_k,$$

where $\tilde{z}_k = \tilde{y}_k + \left(\varepsilon_0 \|\tilde{g}_{k+1}\|^r + \max\left\{ 0, -\frac{s_k^T \tilde{y}_k}{s_k^T s_k} \right\} \right) s_k$, $\tilde{y}_k = \tilde{g}_{k+1} - \tilde{g}_k$ and $s_k = x_{k+1} - x_k$.

Step 5: $k := k+1$. Go to Step 2.

End.

Theorem 8.3: [11] Suppose $\tilde{f}(.,\mu)$ is a smoothing function of f. If for every fixed $\mu > 0$, $\tilde{f}(.,\mu)$ satisfies the following assumptions:

(i)   For any $\hat{x} \in \mathbb{R}^n$, the level set $\mathcal{S}(\hat{x}) = \{x \in \mathbb{R}^n \,|\, f(x) \le f(\hat{x})\}$ is bounded.

(ii) f is continuously differentiable and there exists a constant $L > 0$ such that for any $\hat{x} \in \mathbb{R}^n$, the gradient of f satisfies

$$\|g(x) - g(y)\| \le L\|x - y\|, \quad x, y \in \mathcal{S}(\hat{x})$$

then a sequence $\{x^k\}$ generated by Algorithm 5.1, satisfies

$$\lim_{k\to\infty} \mu_k = 0 \quad \text{and} \quad \liminf_{k\to\infty} \left\| \nabla \tilde{f}(x_k, \mu_{k-1}) \right\| = 0.$$

Proof: [11, Theorem 2.6] Denote $K = \{k \,|\, \mu_{k+1} = \gamma_1 \mu_k\}$. If K is infinite then there exists an integer $\bar{k}$ such that for all $k > \bar{k}$

$$\left\| \nabla \tilde{f}(x_k, \mu_{k-1}) \right\| \ge \eta \mu_{k-1} \tag{8.5.2}$$

and $\mu_k = \mu_{\bar{k}} =: \bar{\mu}$ in step 3 of the Algorithm 8.1. Since $\tilde{f}(.,\mu)$ is a smooth function, the non-smooth algorithm 5.1, can be reduced to a smooth algorithm for solving $\min_{x \in \mathbb{R}^n} \tilde{f}(x, \bar{\mu})$.

Hence, by the given two assumptions on $\tilde{f}(.,\mu)$, we have (from [11,Theorem 2.4]) that $\lim_{k\to\infty}\left\|\nabla\tilde{f}(\underline{x}_k,\overline{\mu})\right\|=0$, which contradicts with (8.2.12). This shows that K must be infinite and $\lim_{k\to\infty}\mu_k=0$.

Since K is infinite, we can assume that $K=\{k_0,k_1,...\}$ with $k_0<k_1<....$ Then we have:

$$\lim_{i\to\infty}\left\|\nabla\tilde{f}(x_{k_i+1},\mu_{k_i})\right\|\le\gamma\lim_{i\to\infty}\mu_{k_i}=0.\qquad\blacksquare$$

8.6. Numerical examples

Example 8.1: Consider the system $\dot{\underline{x}}(t)=A\underline{x}(t)+B\underline{u}(t)$, where

$$A=\begin{pmatrix}0&1&0\\-2&3&0\\5&1&3\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&3\\0&1\end{pmatrix}$$

The eigenvalues of A can be calculated as: $\sigma(A)=\{\lambda_1,\lambda_2,\lambda_3\}$ where $\lambda_1=1,\lambda_2=2,\lambda_3=3$. It is required to move these by using state feedback L to locations $\sigma(A-BL)=\{\mu_1,\mu_2,\mu_3\}$, where $\mu_1=-1,\mu_2=-2,\mu_3=-3$ and to use the remaining degrees of freedom which are available to minimize the condition number of the closed-loop matrix $A-BL$. Let:

$$\underline{b}=x\underline{b}_1+y\underline{b}_2=B\underline{m},\underline{m}^t=(x,y)$$

where $\underline{b}_1,\underline{b}_2$ denote the first and second column of B, respectively. Consider the controllability properties of the system $(A,\underline{b})$ as a function of parameters x and y. The controllability matrix is:

$$\Gamma_c(A,\underline{b})=\begin{pmatrix}\underline{b}&A\underline{b}&A^2\underline{b}\end{pmatrix}=\begin{pmatrix}0&x+3y&3x+9y\\x+3y&3x+9y&7x+21y\\y&x+6y&11x+42y\end{pmatrix}=x\begin{pmatrix}0&1&3\\1&3&7\\0&1&11\end{pmatrix}+y\begin{pmatrix}0&3&9\\3&9&21\\1&6&42\end{pmatrix}$$

The determinant of $\Gamma_c(A,\underline{b})$ can factored as:

$$\det\Gamma_c(A,\underline{b})=-2(4x+13y)(x+3y)^2.$$

Thus, the system is controllable unless

$$y = -\frac{4}{13}x \quad \text{or} \quad y = -\frac{1}{3}x$$

hence it is possible to assign the closed-loop eigenvalues at the required locations via state feedback, unless the design parameters (x, y) are constrained by either one of the two equations above. The eigenvector matrix of the corresponding closed-loop matrix $A - BL$ with $\sigma(A - BL) = \{-1, -2, -3\}$ is parametrised as:

$$U_\mu(x, y) = \begin{pmatrix} \dfrac{x}{6} + \dfrac{y}{2} & \dfrac{x}{12} + \dfrac{y}{4} & \dfrac{x}{20} + \dfrac{3y}{20} \\[2mm] -\dfrac{x}{6} - \dfrac{y}{2} & -\dfrac{x}{6} - \dfrac{y}{2} & -\dfrac{3x}{20} - \dfrac{9y}{20} \\[2mm] -\dfrac{x}{6} - \dfrac{3y}{4} & -\dfrac{x}{20} - \dfrac{7y}{20} & -\dfrac{x}{60} - \dfrac{13y}{60} \end{pmatrix}$$

Note that, as expected, all elements of this matrix are linear functions of the two parameters. The determinant of $U_\mu(x, y)$ can be calculated as:

$$\det U_\mu(x, y) = \frac{(4x + 13y)(x + 3y)^2}{43200}$$

As expected, loss of controllability is associated with singularity of $U_\mu(x, y)$ or, equivalently, $\kappa(U_\mu(x, y)) = \infty$ when $\det \Gamma_c(A, \underline{b}) = 0$. The corresponding parametrisation of the closed-loop matrix $A - BL$ with the required spectrum is:

$$A - BL = \begin{pmatrix} 0 & 1 & 0 \\[2mm] -\dfrac{15y}{4x + 13y} - 23 & \dfrac{60y}{4x + 13y} - 9 & \dfrac{15y}{4x + 13y} - 15 \\[2mm] 5 - \dfrac{288y^2 + 84xy}{(4x + 13y)(x + 3y)} & 1 - \dfrac{96y^2 + 48xy}{(4x + 13y)(x + 3y)} & 3 - \dfrac{60y}{4x + 13y} \end{pmatrix}$$

It can be verified that the eigenvalues of $A - BL$ are at $\sigma(A - BL) = \{-1, -2, -3\}$. The problem now reduces to the minimization of $\kappa(U_\mu(x, y))$ (or its Gram matrix) over the variables (x, y). Since this problem is clearly scale-invariant, we introduce polar

coordinates for the two variables, i.e. $x = r\cos\theta$ and $y = r\sin\theta$, and set without loss of generality $r = 1$. If we define,

$$\hat{U}_\mu(\theta) := U_\mu(r\cos\theta, r\sin\theta))|_{r=1}$$

the equivalent optimization of $\kappa(\hat{U}_\mu(\theta)) : [-\pi, \pi) \to R_+$ can be carried out over a single parameter θ. We now make two additional observations:

(a) Let

$$\theta_1 = -\tan^{-1}\left(\frac{4}{13}\right) \cong -0.2985 \quad \text{and} \quad \theta_2 = -\tan^{-1}\left(\frac{1}{3}\right) \cong -0.3248$$

Then in the interval $[-\pi, \pi)$ the graph of $\kappa(\hat{U}_\mu(\theta))$ has vertical asymptotes at $\theta = \theta_2$, $\theta = \theta_1$, $\theta = \pi + \theta_2 \cong 2.8198$ and $\theta = \pi + \theta_1 \cong 2.8431$. These correspond to the two loss-of-controllability conditions derived earlier. The minimum of the function is located in the interval $(\theta_1, \pi + \theta_2) \cong (-0.2985, 2.8198)$.

(b) In the interval $(\theta_1, \pi + \theta_2)$ the function $\kappa(\hat{U}_\mu(\theta))$ is not only continuous (as expected) but also differentiable since the eigenvalues of $\hat{U}_\mu(\theta)$ are distinct for every $\theta \in (\theta_1, \pi + \theta_2)$. The graph of the function in this interval is shown in Figure 8.1 and appears to be quasi-convex.

The minimization of the condition number is straightforward in this case (since the problem is one-dimensional and the function is differential) and MATLAB's function fminbnd.m (Appendix 1.) was used to perform the optimization between bounds $-0.2 \leq \theta \leq 2.5$ and error tolerance 10^{-8}. The optimum was identified as $\theta^* = -0.142223$ corresponding to a minimum condition number $\kappa^* = 144.267$. The intermediate results of the algorithm are summarised in the table 8.1:
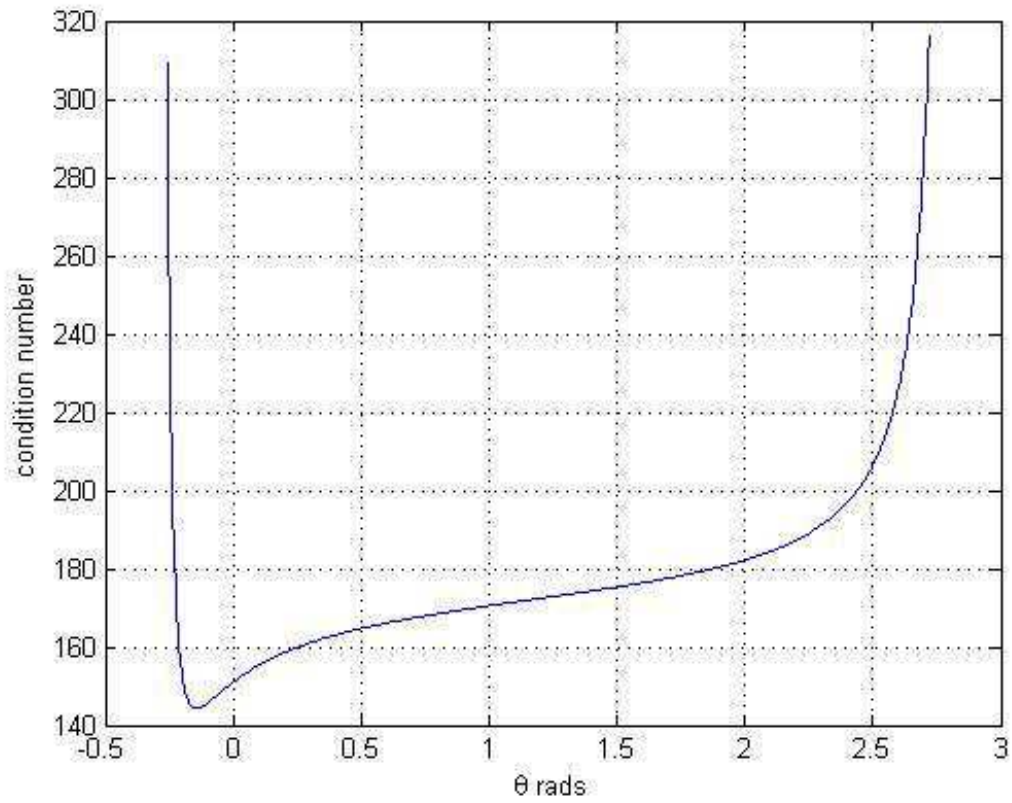
Figure 8.1: Graph of condition number $\kappa(\hat{U}_\mu(\theta))$

Iteration	θ	$\kappa(\hat{U}_\mu(\theta))$	Procedure
1	0.831308	168.809	Initial
2	1.46869	174.982	Golden
3	0.437384	163.67	Golden
4	0.19392	158.338	Golden
5	0.0434588	152.943	Golden
6	-0.049534	148.26	Golden
7	-0.107007	145.223	Golden

8	-0.142527	144.267	Golden
9	-0.173044	145.905	Parabolic
10	-0.135794	144.311	Parabolic
11	-0.143699	144.27	Parabolic
12	-0.142155	144.267	Parabolic
13	-0.142225	144.267	Parabolic
14	-0.142223	144.267	Parabolic
15	-0.142223	144.267	Parabolic
16	-0.142223	144.267	Parabolic
17	-0.142224	144.267	Golden
18	-0.142223	144.267	Golden
19	-0.142223	144.267	Golden
20	-0.142223	144.267	Parabolic
21	-0.142223	144.267	Parabolic
22	-0.142223	144.267	Parabolic

Table 8.1: Condition number minimisation

The optimising values of the original variables can now be obtained as $x^* = \cos(\theta^*) = 0.9899$ and $y^* = \sin(\theta^*) = -0.1417$. Although not particularly useful here, a family of smooth functions $g(\theta, \nu)$ were also generated for the three smoothing parameter values $\nu_1 = 7 \cdot 10^{-5}$, $\nu_2 = 6 \cdot 10^{-5}$, $\nu_3 = 5 \cdot 10^{-5}$. These are shown in Figure 8.2 along with the graph of the function $\kappa(\hat{U}_\mu(\theta))$ for $-0.25 \leq \theta \leq -0.23$ (for larger values of θ the graphs are almost identical).
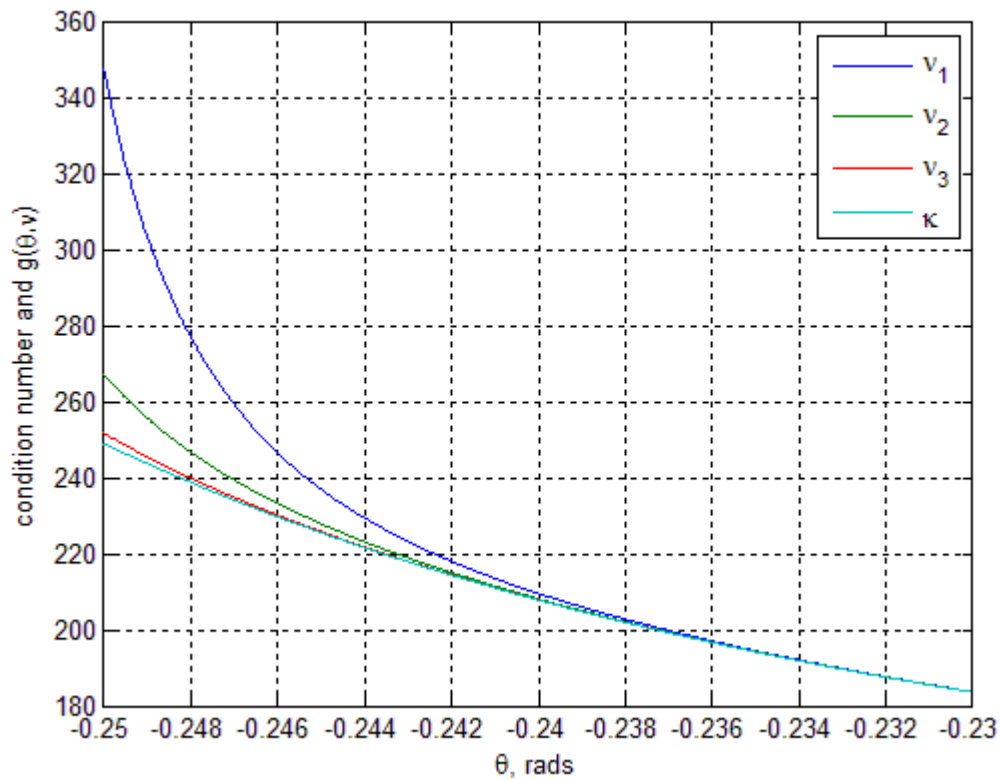
Figure 8.2: Condition number and smooth-parameter approximations

Example 8.2:

In this example, we try our Algorithm on a 3 by 3 Vandermonde-like matrix. In general, Vandermonde –like matrices are assumed to be ill-conditioned. However, by choosing a decent starting point and using our presented MATLAB programme (Appendix_1), this time, we aim to minimize the condition number of the Gram matrix of this given matrix:

$$A = \begin{bmatrix} 1 & x & x^2 \\ 1 & -0.2 & 0.04 \\ 1 & 2 & 4 \end{bmatrix}$$

choosing smoothing factor to be $v = 0.2$ and initial point of $x=1$, the following results are achieved:

Iteration	x_data	Cond. Number
1	1	109.1902
2	-1.0275	48.7713
3	-2.1691	42.8793
4	-1.6791	31.71
5	-1.594	31.3693
6	-1.5809	31.3611
7	-1.5793	31.3609
8	-1.58	31.3609

Table 8.2: Condition number minimisation $\nu = 0.2$

By changing ν from 0.2 to 0.4 we will get:

Iteration	x_data	Cond. Number
01	1	120.75
2	-2.187	54.8796
3	-0.9732	54.4117
4	-1.6988	31.8574
5	-1.6478	31.5274
6	-1.6189	31.4179
7	-1.6023	31.3807

8	-1.5928	31.3681
9	-1.5876	31.3638
10	-1.5848	31.3623
11	-1.5836	31.3619
12	-1.58359	31.3619

Table 8.3: Condition number minimisation ν =0.4

Observing the results in Table 8.2 and 8.3, it can be seem that the value of smoothing factor has an slight effect on the slops of the graph. In fact by having smaller smoothing factor, the graph will be smoother and the estimation will be more accurate. We should note that just the real numbers have been used in example 8. 2, so clearly, if one can apply the complex values, it might smaller values for the condition number of Gram matrix related to the given Vandermonde matrix.

8.7. Conclusion

In this chapter, a new non-smooth algorithm was introduced in order for the condition number of a closed-loop eigenvector matrix (or its Grammian) to be minimised. This ensures that the angle between closed-loop eigenframes obtained by equation (5.5.3) within Chapter Five has been maximised.

As noted earlier, the main objective of this chapter has been to provide the required new concept of the relative positioning between subspaces that can be used in sensitivity of closed-loop eigenvectors and relative measures of controllability.

So the development of the desired optimization algorithm has been done such that by obtaining a closed-loop eigenvetors with a least possible sensitivity for any controllable system(s), the system will be robust to the perturbations injected to the eigenvalues or their relative eigenvectors, i.e. the error is minimized. The performance and numerical aspects of the algorithm is a topic for further research.

CHAPTER 9

CONCLUSION AND FUTURE RESEARCH

9.1. Achievements, Results

This thesis has studied the problem of eigenstructure assignment as part of the family of Control system design problems. Parameter variations or perturbations can usually be found in many practical problems in the area of control system design. The presence of uncertainty in the system usually have a major negative impact on the performance and stability of a closed-loop system, assumed to be designed based on the nominal model of the system. So clearly, by reducing (or in fact minimizing) the sensitivity of eigenvalues to perturbations and parameter variations, the possibility of instability of the closed-loop system will be reduced in the case of applying the designed controller to the real system.

In general, for a multivariable system, where a set of desired closed-loop eigenvalues is given, introducing a feedback gain matrix, is not unique. So variety of different methods has been introduced on the best choice of feedback matrix, such that a robust closed-loop system is produced. These methods include Kautsky, et al., 1985, Owens and O'Reilly, 1989, Duan, 1992, etc. In order to achieve such robustness, several measures have been introduced where the first was introduced by Kautsky, et al., 1985; this measure was based on the condition number of the eigenvector matrix of the closed-loop system and many of the robust eigenstructure assignment methods try to obtain a minimised sensitivity via these measures. The approach was motivated by the work of Wilkinson [1], which linked insensitivity of eigenvalues to parameter uncertainty to orthogonality of eigenframes. This work has been also here the main motivation of the techniques developed.

In this thesis, the main objective has been the development of a new approach for robust closed-loop eigenframe by combining the measure of orthogonality with results from Geometric Control Theory and in particular the parametrisation of controllability subspaces (cs) ([8], [36], [71], [96]) which leads to a family of direct sum decompositions of the state space. Our work mainly was based on the parameterization of closed-loop eigenframes based on the open and closed loop spectra and also the algebraic characterization and parameterization of controllability subspaces developed in Karcanias [8], [96] on his algebraic characterisation of Geometric Theory concepts based on matrix pencil theory.

The basis of the introduced approach is based on a new parameterisation of the set of closed-loop eigenvectors as vectors corresponding to certain desirable closed loop frequencies within the controllability subspaces of a given system. Given the spectrum the problem that then arises is selecting the most orthogonal decomposition of the state space in terms of cs and then reducing the selection of the set of closed-loop eigenvectors by developing a non-smooth optimisation algorithm.

The development of this new approach required the development of a measure for the degree of orthogonality, or measure of "skewness" between subspaces of the state space, in a direct sum decomposition and thus developed a concept of angle between these sets of subspaces. These measures were developed and were based on the Condition Number, Determinant of Gram Matrix and Spread of Singular Values (a deviation measure of the singular values). Using these measurement tools, we then developed some important results on the conditions on which the skewness of these subspaces is minimized.

We have applied these results on the task of defining the least skewness closed loop eigenframe by using the parameterisation of eigenframes based on the "mobility of open loop to closed-loop spectra" method introduced by N. Karcanias (Presented in Chapter Five) and the alternative parameterisation based on the cs. This has led to an optimally-conditioned closed-loop eigenvector matrix via optimization techniques using condition number of closed-loop eigenvector matrix and guarantees minimization of sensitivity for a defined closed-loop system where the state

feedback matrix is computed using the "mobility of open loop to closed-loop spectra" parameterisation.

The assumption during our work for the choice of both open and closed-loop eigenvalues was to assume distinct eigenvalues (real, or complex). However the algorithms could be easily extended to the case of repeated eigenvalues. The achievements and results of the thesis are reported in a number of Chapters dealing with the different aspects of the overall study are presented.

Chapter Two has reviewed some important mathematical and control topics, required in the reminder of this thesis. Within mathematical section of this chapter, attention has been given to those definitions, theorems and proofs (whenever required) which have been widely used across the report. Elements including the three measurement tools: Condition Number, Gram Matrix and Singular Value Decomposition which have been applied in order to measure the skewness of subspaces of a state space in direct sum decomposition (Refer to Chapter Five). The same revision has gone through some major control-related fundamentals which have been strongly linked to objectives of our research. Among these, are Matrix Fraction Description (MFD) and Minimal Bases of Matrix Pencil which are of the required elements when it comes to the study of Controllability Subspaces (cs) decomposition of the state space.

Chapter Three has reviewed the basic concepts of Eigenstructre assignment along with some famous background results. Central to these, has been the notion of Robust eigenstructure assignment which is the notion underpinning the task of developing the most orthogonal closed-loop eigenframes for a control system. In this chapter, we have reviewed the Rectilinear Motion and its relative implications for the characterisation of closed loop eigenvectors. We then have been looking at the connection between forced rectilinear motions and closed-loop eigenstructure followed by the difference between the frequency and vector correspondence for the two cases of A and (A, B)-invariance. We have examined the basis of eigenstructure assignment via state and output feedbacks and provided the characterisation of the state and output feedbacks required for closed loop assignment of eigenstructures.

We have then focused on the Purturbation of eigenvalues, introduced by Wilkinson (1965), and its links to the corresponding eigenvectors. We then explained and its relative robust eihenstructure assignment. This followed by the revision of the work done by Kautsky and et al. (1985) on robust eigenstructure assignment. We have then examined some other approaches to eigenstructure assignments and have identified some open areas linked to the objectives of thesis.

Chapter Four has provided a geometric review of eigenstructure assignment via one-dimensional *A*-Invariant and (*A*,*B*)-Invariant subspaces. The main focus of the this chapter has been given to the properties of (*A*,*B*) – invariant subspaces, since special subfamilies of them have the controllability property which is represented by the family of controllability subspaces. This provides the required background for defining the controllability subspaces decomposition of the state space required for the subsequent chapters.

Chapter Five has dealt with the problem of parametrisation of closed-loop eigenframes and introduced two new parametrerisations: first the parametrisation of eigenframes based on the open and closed loop spectra mobility and second the characterisation based on the properties and parametrisation of controllability subspaces (cs) [4]. We have presented the algebraic characterization of controllability subspaces and minimal dimension controllability subspaces as the main elements underpinning the results on eigenstructure assignment of this chapter. Using this information, the assignability of the relative spectrum to a controllability subspace has been derived and so the eigenvalue placement algorithm introduced in [4] based on open-loop/ closed-loop spectra mobility has been presented. This has led to the introduction of parametrization of the closed-loop eigenvectors resulting from this method which was also further discussed together with the optimization problem considered in **chapter Eight**.

Chapter Six has dealt with the development of some measures of skewness for a set of subspaces defining a direct sum decomposition of a state space. This chapter provides an important bases not only for the study of selection of the closed-loop eigenframes, but also helps finding the optimal angle between ordered minimal

bases of controllability subspaces where the overall controllability space is created from all the subspaces in a direct sum decomposition. Three diagnostic measurement tools have been defined: The Condition number, the Gram Matrix and the Singular Values Decomposition. Then using each tool, we have investigated the conditions under which the angle between subspaces in a direct sum decomposition is maximized. This was then followed by numerical tests. In order to achive the targeted outcome, we have derived the necessary conditions for the Gramian determinant to be maximized, the Condition number to be minimized and the Spread of singular values or a deviation measure of the singular values to be minimized. This has provided alternative tools for the angle between the subspaces in a direct sum decomposition to be maximized or in another word, tools for minimization of sensitivity in robust design to be minimized.

Chapter Seven has presented an algebraic description of the total system behaviour which allows the study of closed loop eigenvectors in a systematic way by providing a new parameterisations. This will then leads to an algebraic characterisation of the input, state and output behaviour in an implicit formulation and it is given based on properties of MFD descriptions which will remain open for future studies. Within this chapter, a behavioural framework for discussing system properties such as poles, zeros, as well as introducing a new parameterization of possible closed loop eigenstructures has been introduced. The approach makes use of minimal bases theory of matrix pencils and provides new ways for computing a pole assigning state feedback matrix K_S that explores the algebraic properties of minimal bases and associated controllability subspaces. The methodology starts off by deriving the total behaviour under zero initial conditions of a minimal system. In fact, each full rank closed-loop eigenframe can be written as the product of a matrix of ordered minimal bases (of matrix pencil) and a matrix containing all the existing poles of the system in the form of Vandermonde matrix. In this chapter, the parametrisation of these minimal bases via Toeplitz form matrices is presented. This is followed by computing a general formula for any other minimal bases with the same degree using a Toeplitz matrix construction such that the controllability subspaces separated by the largest possible angles can be identified.

Finally **Chapter Eight** has provided a new method based on non-smooth optimisation algorithm in order to maximise the angle between closed-loop eigenvectors introduced in the previous chapters. This algorithm has provided a guaranteed convergence to an optimal solution if a descent starting point is chosen. For the development of the algorithm , we have initially discussed the concept of non-sooth optimisation and its relative theorems and results. We have reviewed some useful formulas applied to optimize the condition number of a Gram matrix and then we have applied these results to a general Gram matrix. Finally, by using these results, we have developed a *MATLAB* programme to implement the developed algorithm. We have also included some numerical examples in order to examine the accuracy of our developed programme. This programme may need to be adjusted/modified to be used if one wishes to try with repeated and/or complex conjugate poles.

9.2. Future Research

The thesis has considered a number of issues related to the representation, parameterisation and selection of closed-loop eigenframes, related to the problem of robust eigenstructure assignment. There are still many issues that remain open related to the main problem of eigenstructure assignment. Some of the problems that require further research are listed below:

(i) The basis of our robustness result is the Wilkinson [1] result that has been stated for distinct eigenvalues. Extension of the result to matrices of a *non-simple strucrure* (repeated eigenvalues), as well as use of other robustness of eigenvalues results are still open issues.

(ii) Investigation of the links between degree of skewness of eigenstructure and the degree of presence of system properties such as controllability, observability, and stability. Such investigations may provide links with the bounded gain and bounded state and output feedback design, as well as similar problems for observer design.

(iii) The selection of the most orthogonal controllability subspaces decomposion of the state space has been defined, but the general solution has to be worked out using an appropriate optimisation process. The applications of this decomposition related to problems such as decoupling and disturbance rejection need to be investigated.

(iv) Assuming that the spectrum is not specified, but it is only required to be stable. Use the controllability subspace methodology for eigenstructrure assignment to select stable spectra that lead to the most orthogonal eigenframes. This is a prelude to studying robust stabilisation based on eigenstructure properties.

(v) There are strong indications that the skewness of the eigenframes are linked to properties such as finite settling time stabilisation (FSTS) such as results presented in [94]. A general solution to this problem is still open and in particular linking the measure of skewness to the FSTS property.

Such properties are very important and have not been paid the appropriate attention in the study of eigenstructure assignment problems. It is essential that the solutions obtained are such that the sensitivity of the assigned eigenvalues to system modelling discrepancies and external disturbances is minimised. Apart from positoning the closed-loop eigenframes, there are some areas which have not been considered during this present research and will remain open for future study such as:

(vi) The selection of eigenframes is linked to the degree of resulting controllability and observability and thus it can also be used for the selection of input and output matrices of a system, that is the problem of *System Instrumentation* [95]. The development of such methodology still remains an open issue.

(vii) The methodology developed here is based on non-smooth optimisation. This is a general challenging problem and methodologies suited to the special nature of the problem have to be developed.

(viii) The new approach based on the parameterisation of controllability subspaces, the selection of the most orthogonal decomposition and then selection of eigenvectors within each of the controllability saubspaces allows the study of

stabilisation instead of assignment. The principle is the selection of arbitrary but stable spectra that leads to the most orthogonal set of eigenvectors in each subspace.

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APPENDIX_1

```matlab
% smooth_fun (cond_fun)
%
a=[ 0 1 0;
    -2 3 0 ;
    5 1 3];
%
b = [ 0 0;
    1 3;
    0 1];
 %
%  syms x y real
%  %
%  bb=x*b(:,1)+y*b(:,2);
%  Gamma = [bb a*bb a*a*bb];
%  d=det(Gamma);
%  d1=factor(d);
 %
 u1=[-1 -1 3]';
 u2=[1 2 -7]';
 u3=[0 0 1]';
 %
 lam=[1 2 3]; % eigvalues of A
 mu=[-1 -2 -3]; % required CL eigenvalues
 %
 u=[u1 u2 u3]; % right e-vactor matrix
 v=inv(u);
 v1=(v(1,:))'; %left e-vectors
 v2=(v(2,:))';
 v3=(v(3,:))';
 %
%  alpha1s=v1'*b*[x;y];
%  alpha2s=v2'*b*[x;y];
%  alpha3s=v3'*b*[x;y];
%  alphas=[alpha1s alpha2s alpha3s];
%  %
%  u_mu1s=(alphas(1)/(mu(1)-lam(1)))*u1+(alphas(2)/(mu(1)-
lam(2)))*u2+(alphas(3)/(mu(1)-lam(3)))*u3;
%  u_mu2s=(alphas(1)/(mu(2)-lam(1)))*u1+(alphas(2)/(mu(2)-
lam(2)))*u2+(alphas(3)/(mu(2)-lam(3)))*u3;
%  u_mu3s=(alphas(1)/(mu(3)-lam(1)))*u1+(alphas(2)/(mu(3)-
lam(2)))*u2+(alphas(3)/(mu(3)-lam(3)))*u3;
%  %
%  u_mus=[u_mu1s u_mu2s u_mu3s]; % CL right e-vector matrix symbolic form
%  ls=-[x x x;y y y]*inv(u_mus);
%  acls=a-b*ls;
%  simplify(acls);
%
%  d2=det(u_mus);
%  d3=factor(d2);

%  syms psi real
%  u_mus_psi=subs(u_mus,{x,y},[cos(psi),sin(psi)]);
%  u_mus_psi=simplify(u_mus_psi);
%  lam1=eig(u_mus_psi'*u_mus_psi);
```

```matlab
%return

np=1000;
%theta=linspace(theta1,pi+theta2,np);
%theta=linspace(theta2+0.001,theta1-0.001,np);
theta=linspace(-0.25,1,np);
%theta=linspace(theta1,theta2,np);
sm_fun=zeros(3,np);
cond_array1=zeros(1,np);
sp=[7e-5 6e-5 5e-5]; % smoothing parameter
%

for i=1:np

    i
    m=[cos(theta(i)) sin(theta(i))]';
    alpha1=v1'*b*m;
    alpha2=v2'*b*m;
    alpha3=v3'*b*m;
    alpha=[alpha1 alpha2 alpha3];
    u_mu1=(alpha(1)/(mu(1)-lam(1)))*u1+(alpha(2)/(mu(1)-
lam(2)))*u2+(alpha(3)/(mu(1)-lam(3)))*u3;
    u_mu2=(alpha(1)/(mu(2)-lam(1)))*u1+(alpha(2)/(mu(2)-
lam(2)))*u2+(alpha(3)/(mu(2)-lam(3)))*u3;
    u_mu3=(alpha(1)/(mu(3)-lam(1)))*u1+(alpha(2)/(mu(3)-
lam(2)))*u2+(alpha(3)/(mu(3)-lam(3)))*u3;
    u_mu=[u_mu1 u_mu2 u_mu3];
    l=-[m m m]*inv(u_mu);
    acl=a-b*l;
    cond_array1(i)=cond(u_mu);
    aa=u_mu'*u_mu; % A(x)
    %
    eig1=eig(aa); % eigenvalues
    eig1=sort(eig1); % sort
    eig1=eig1([3:-1:1]'); % largest first
    %
    for isp=1:3
        tmp1=eig1(1)+sp(isp)*log(sum(exp((eig1-eig1(1))./sp(isp))));
        tmp2=eig1(3)-sp(isp)*log(sum(exp((-eig1+eig1(3))./sp(isp))));
        sm_fun(isp,i)=sqrt(tmp1/tmp2);
        %keyboard
    end
end
%
figure(1)
plot(theta,cond_array1);
%
figure(2)
plot(theta,sm_fun(1,:),theta,sm_fun(2,:),theta,sm_fun(3,:),theta,cond_array
1);

%-------------------------------------------------------------------

return

for i=1:np
    for j=1:np
        i
    %m=[0 1]';
```

```matlab
   %m=[cos(theta(i)) sin(theta(j))]';
   m=[mval1(i) mval2(j)]';
   alpha1=v1'*b*m;
   alpha2=v2'*b*m;
   alpha3=v3'*b*m;

   alpha=[alpha1 alpha2 alpha3];
   %alpha=[1 1 4];

   %b*m-alpha(1)*u1-alpha(2)*u2-alpha(3)*u3
   %

   u_mu1=(alpha(1)/(mu(1)-lam(1)))*u1+(alpha(2)/(mu(1)-
lam(2)))*u2+(alpha(3)/(mu(1)-lam(3)))*u3;
   u_mu2=(alpha(1)/(mu(2)-lam(1)))*u1+(alpha(2)/(mu(2)-
lam(2)))*u2+(alpha(3)/(mu(2)-lam(3)))*u3;
   u_mu3=(alpha(1)/(mu(3)-lam(1)))*u1+(alpha(2)/(mu(3)-
lam(2)))*u2+(alpha(3)/(mu(3)-lam(3)))*u3;
   %
   u_mu1-(1/6)*[1 -1 -1]';
   u_mu2-(1/60)*[5 -10 -3]';
   u_mu3-(1/60)*[3 -9 -1]';
   %
   u_mu=[u_mu1 u_mu2 u_mu3];
   l=-[m m m]*inv(u_mu);
   acl=a-b*l;
%[v,d]=eig(acl);
%       %
%acl*u_mu1-mu(1)*u_mu1;
%acl*u_mu2-mu(2)*u_mu2;
%acl*u_mu3-mu(3)*u_mu3;
   %
   if ~any(isnan(acl))
       cond_array(i,j)=cond(acl);
   end

   if mval1(i)<0
      if mval2(j) <= (-1/3)*mval1(i)
         cond_array(i,j)=0;
      end
   elseif mval1(i) > 0
       if mval2(j) <= (-4/13)*mval1(i)
         cond_array(i,j)=0;
      end
   end

  end
 end

 %plot(theta,cond_array);
 contour(cond_array, [183 183.5 184 184.5 185 185.5 186]);
%-------------------- end of function cond_fun ------------------------
------------------
```