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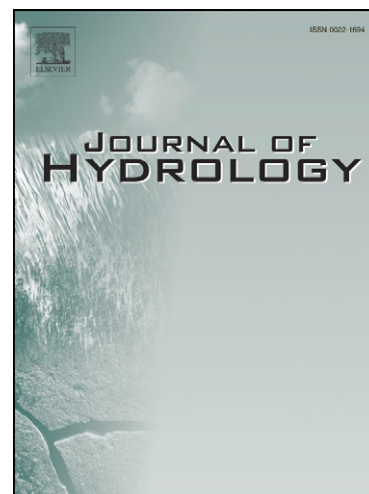
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4 **STEADY-STATE DITCH-DRAINAGE OF TWO-LAYERED SOIL REGIONS**5 **OVERLYING AN INVERTED V-SHAPED IMPERMEABLE BED WITH**6 **EXAMPLES OF THE DRAINAGE OF BALLAST BENEATH RAILWAY TRACKS**

7

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9

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12

13 **Abstract**

14 Water-table heights due to steady surface accretion in drained two-layered soil regions
15 overlying an inverted V -shaped impermeable bed are obtained using both the Dupuit-
16 Forchheimer approximate analysis with flow assumed parallel to the bed and also from
17 numerical solutions of Laplace's equation for the head distribution. For illustration, water-
18 table profiles obtained by the two procedures are compared for surface accretion draining to
19 ditches in a typical two-layered ballast foundation for a railway track where a very permeable
20 ballast material overlies a less permeable sub-grade on top of an inverted V-shaped
21 impermeable bed that slopes away both sides from a central line to drainage ditches. These
22 results are found to be in good agreement except very near the drainage ditches where the
23 Laplace numerical solution takes into consideration a surface of seepage that is ignored in the
24 Dupuit-Forchheimer analysis. The Dupuit-Forchheimer analysis is also in good agreement

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25 with results of a laboratory model experiment. It is concluded that the approximate Dupuit-
26 Forchheimer analysis can be used with confidence in these situations. It is used to investigate
27 the effect on the water-table elevation caused by the reduction of hydraulic conductivity of
28 the porous materials due to clogging.

29

30 *Keywords:* Drainage, Layered soils, Sloping bed, Dupuit-Forchheimer analysis, Laplace
31 numerical solutions, Railway ballast beds

32

33 **1. Introduction**

34

35 Water flow due to surface accretion to drains in two-layered porous bodies overlying an
36 undulating impermeable base is a problem occurring both in agricultural lands and
37 engineering structures. In ridge and furrow drained lands a permeable structured surface soil
38 overlies less permeable soil that lies above an impermeable base that rises and falls, with the
39 furrows acting as drainage ditches for rainfall infiltrating through the soils. Similarly, ballast
40 beds, that provide a foundation for railway tracks, often consist of a layer of very permeable
41 material overlying a layer of finer less permeable material laid on top of an impermeable sub-
42 grade whose surface slopes away from a peak midway between drainage channels.

43 The Dupuit-Forchheimer approximation is conventionally used in investigations of the
44 two-dimensional groundwater problem presented by flow to transverse drains due to steady
45 accretion on the surface of lands overlying a moderately sloping impermeable bed, either
46 assuming horizontal flow (Werner, 1957; Schmid and Luthin, 1964; Yates et al., 1985) or
47 more realistically assuming flow parallel to the sloping bed (Wooding and Chapman, 1966;
48 Childs, 1971; Towner, 1975; Lesaffre, 1987; Chapman, 1980). Towner (1975) showed that
49 the Hele-Shaw viscous flow analogue results of Guitjens and Luthin (1965) agreed with

50 Dupuit-Forchheimer calculations for even large slopes when the flow was assumed parallel to
51 the bed although the agreement was poor when the flow was assumed horizontal.

52 These studies all considered transverse drains on a continually rising sloping bed. For
53 steady-state drainage of lands overlying an impermeable bed that rises to a peak midway
54 between uniformly spaced parallel drainage ditches, the water-table height midway between
55 drains is a maximum. The Dupuit-Forchheimer analysis is then simpler than that for the
56 problem of interception of rainfall over sloping lands by parallel ditch drains along the
57 contours where the location of the maximum water-table height is part of the solution. As
58 shown by Towner (1975) for drains along contours and by Youngs and Rushton (2009) for
59 the present drainage situation for a uniform soil, there is little difference in the results
60 assuming horizontal flow and those assuming flow parallel to the sloping bed when the slope
61 is less than 10%. However, the difference becomes significant for larger slopes when the
62 Dupuit-Forchheimer analysis assuming horizontal flow gives poor agreement with numerical
63 computations of the water-table profile solutions based on Laplace's equation describing the
64 head, while the analysis assuming flow parallel to the slope gives good agreement.

65 In all these studies the soil over the sloping bed was assumed to be uniform. For soils
66 whose hydraulic conductivity varies with height, Guirinsky's (1946) extension of the Dupuit-
67 Forchheimer analysis can be used for soils overlying a horizontal impermeable base, while
68 Youngs' (1965, 1966) seepage analysis, founded on Charny's (1951) work on flow through
69 earth dams, gives an exact formulation of the problem of groundwater flow in layered soils,
70 leading to estimates of bounds for the water-table profiles. This exact analysis was extended
71 to groundwater flow in layered sloping lands (Youngs, 1971), but does not provide solutions
72 for the water-table profiles.

73 In this paper the drainage of a two-layered soil region overlying an inverted V-shaped
74 impermeable bed has been addressed analytically by assuming Dupuit-Forchheimer
75 conditions with flow parallel to the bed and with the water table drawn down to the water
76 level in ditch drains. We assume the impermeable bed slopes away at a uniform angle from a

77 peak midway between the ditches. We compare and discuss our results with those obtained
78 from numerical solutions of Laplace's equation for the head assuming boundary conditions
79 that include the existence of a seepage face at the drainage outfall. Comparisons are also
80 made between the Dupuit-Forchheimer calculations of the water-table profile and the
81 laboratory model results for steady surface accretion on drained railway ballast foundations
82 published by the Department of Civil and Environmental Engineering, University of
83 Massachusetts, Amherst, USA (Heyns, 2000). We also demonstrate the use of the Dupuit-
84 Forchheimer analysis by examining the effect on the water-table elevation due to clogging of
85 the ballast.

86

87 2. The physical problem

88

89 We consider the two-dimensional flow region through a cross section of the two-layered
90 soil that is sketched in Fig.1. The soil overlies an impermeable bed that slopes downwards at
91 an angle α to the horizontal from a central plane at $x = 0$. The soil consists of a lower layer of
92 depth t with hydraulic conductivity K_0 overlain by a more permeable layer of conductivity K_1 .
93 There is uniform steady accretion q over the surface which drains at $x = \pm D$ to a head H_D .
94 The accretion maintains the water table at a height $H(x)$ above the floor at position x over the
95 area. Above the outfall water level H_D at $x = D$ a seepage surface exists to a height h_f .

96 The location of the water table in the two layers varies depending on the slope of the bed,
97 the accretion rate, the ratio of the hydraulic conductivities and the outfall head. When the
98 outfall head is below the boundary of the two soil regions, for small values of q , $q < q_1$, the
99 water table is wholly contained in the lower soil layer of conductivity K_0 . This situation is
100 shown in Fig. 2a for a sufficiently small value of $q < q_1$ (Case 1) that does not allow the
101 water table to rise above the impermeable sloping base at the centre $x = 0$. If $q_1 < q < q_1$
102 (Case 2) the water table is above the base at $x = 0$ and the situation becomes that shown in

103 Fig. 2b. For a range of values of q , $q_1 < q < q_2$ (Case 3), the water table is in the more
104 permeable soil region with conductivity K_1 over a section $x_1 < x < x_2$ of the region but is in the
105 lower soil region near the centre and near the drainage ditch (Fig.2c). Again, the water table
106 might drop to the impermeable base at $x = 0$ when the water table configuration in the lower
107 layer in the central region becomes similar to that of Fig. 2a. For large values of q , $q > q_2$
108 (Case 4a), the water table can be in the lower layer in a region near the outfall but in the
109 upper layer over the rest of the region near the centre. This is shown in Fig. 2d. However, the
110 height of the seepage surface h_f can be above the boundary between the two layers (Case 4b).
111 In this case the water table lies wholly in the upper soil layer as shown in Fig.2e. When the
112 outfall ditch level is above the boundary between the two layers (Case 5), for smaller values
113 of q and larger slopes, the water table could drop into the lower ballast layer as shown in Fig.
114 2f, but for large values of q and smaller slopes the water table is wholly in the upper layer.
115 Conditions giving rise to these situations can occur with ridge and furrow lands and also with
116 railway ballast foundations with a very permeable ballast overlying a less permeable sub-
117 ballast. When significant rainfall occurs, the water table rises progressively through the
118 lower layer into the very permeable layer above as shown in Cases 1, 2, 3 and possibly 4(a)
119 and/or 4(b); when the rainfall stops the water table falls progressively through the situations
120 described by these conditions. Case 5 occurs when there is drainage surcharge and the water
121 head builds up in the drainage channel.

122 The flow in each soil layer can be obtained by solving Laplace's equation $\nabla^2 h = 0$ for
123 the hydraulic head h at (x,z) in the groundwater region. The boundary conditions of the
124 problem are shown in Fig.3a. These are that there is no flow through the base of the lower
125 layer and through the plane of symmetry at $x = 0$ and there is continuity of flow and
126 hydraulic head between layers with the vertical flux, assumed equal to the accretion rate q ,
127 through the water table where $h = z = H$. Thus we assume that flow is vertical in the
128 unsaturated soil above the water table where it is refracted on entry. It was argued by Childs
129 (1945) that this was a reasonable assumption for uniform soils in considering water-table

130 heights in drained lands, but Kacimov (2003) has demonstrated that with soils overlying a V-
131 shaped impermeable bed, the flow diverges from the vertical in the unsaturated region,
132 leading to non-uniform flux through the water table. However, for small angles of slope such
133 divergence may be assumed to be small. At $x = D$ water drains out of the soil into the ditch
134 where there is a surface of seepage, so that $h = H_D$, $0 < z < H_D$, and $h = z$, $H_D < z < h_f$, where
135 H_D is the ditch-water level and h_f is the height of the top of the seepage face. Analytical
136 solutions of Laplace's equation with these boundary conditions have not been possible so that
137 numerical methods of solution are needed.

138

139 3. Dupuit-Forchheimer solutions

140

141 The boundary conditions to be applied with the approximate Dupuit-Forchheimer analysis
142 to obtain the water-table profiles in the two-layered drainage situation shown in Fig.1 when
143 flow is assumed parallel to the impermeable base, are shown in Fig.3b. The water-table
144 height is a maximum at the centre of the soil region at $x = 0$. With a uniform accretion rate q ,
145 assumed to be the vertical flux through the water table, the flow per unit width down the
146 slope is qx , discharging qD into the ditch. However, as discussed by Youngs and Rushton
147 (2009) the assumption of flow parallel to the slope requires the ditch face to be normal to the
148 sloping bed and the inclusion of fictitious flow regions upslope from the central plane and
149 another overhanging the ditch as shown in Fig.3b. When α is small, as in the examples given
150 later in this paper, the overhang becomes unimportant, increasing the total inflow by less than
151 0.5 % for a 5% slope.

152

153 3.1: Water table in upper layer

154

155 When the water table is in the upper soil layer, the flow per unit width assumed parallel
 156 to the sloping bed is

157

$$158 \quad qx = -\{K_0 t + K_1 [H - t - (D - x) \tan(\alpha)]\} \cos(\alpha) \frac{dH}{ds} \quad (1)$$

159

160 where s is the coordinate measured down-slope with $s = 0$ corresponding to the water-table
 161 height at $x = 0$ and $s = s_D$ (a function of the slope of the bed, accretion rate and hydraulic
 162 conductivities of the layers) at $x = D$ (see Fig.3b), so that

163

$$164 \quad s_D - s = \frac{D - x}{\cos(\alpha)} + [H - (D - x) \tan(\alpha)] \sin(\alpha)$$

165

166 giving

167

$$168 \quad \frac{ds}{dx} = \cos(\alpha) - \frac{dH}{dx} \sin(\alpha) \quad (2)$$

169

170 as deduced by Childs (1971) and Youngs and Rushton (2009).

171

172 Thus, in terms of the horizontal coordinate x

173

$$174 \quad qx = -\{K_0 t + K_1 [H - t - (D - x) \tan(\alpha)] - qx \tan(\alpha)\} \frac{dH}{dx} \quad (3)$$

175

176 An analytical solution of eq.(3) is obtained using the substitution

177

$$178 \quad w = [H - D \tan(\alpha) - (1 - K_0/K_1)t + (1 - q/K_1)x \tan(\alpha)]/x \quad (4)$$

179

180 Equation (3) then becomes

181

$$182 \quad \frac{dx}{dw} = \frac{-xw}{w^2 - (1 - q/K_1) \tan(\alpha)w + q/K_1} \quad (5)$$

183

184 so that after integration, x as a function of w is given by

185

$$186 \quad x(w) = x(w_i) \exp \left[- \int_{w_i}^w \frac{wdw}{w^2 - (1 - q/K_1) \tan(\alpha)w + q/K_1} \right] \quad (6)$$

187

188 where the lower integration limit of w is w_i at $x = x(w_i)$. Noting that

189

$$190 \quad \int \frac{vdv}{av^2 + bv + c} = \frac{1}{2} \ln(av^2 + bv + c) - \frac{b}{a\sqrt{4ac - b^2}} \tan^{-1} \left(\frac{2av + b}{\sqrt{4ac - b^2}} \right) \\ = f(v) \text{ say} \quad (7)$$

191

192 we can write the solution of eq.(6) in the form

193

$$194 \quad x(w) = x(w_i) \exp[- \{f(w) - f(w_i)\}] \quad (8)$$

195

196 with $a = 1$, $b = -(1 - q/K_1) \tan(\alpha)$ and $c = q/K_1$ in $f(v)$ of eq.(7). In eq.(8) w_i is obtained from197 eq.(4) for the given value of H at $x(w_i)$. With eq.(8) giving the coordinate x as a function of198 w , the water-table height H at a given x is obtained from eq.(4) so that

199

$$200 \quad H = wx(w) + D \tan(\alpha) + (1 - K_0/K_1)t - (1 - q/K_1)x(w) \tan(\alpha) \quad (9)$$

201

202 3.2: Water table in lower layer

203

204 When the water table is located in the lower layer, the flow q is given by (Youngs and
205 Rushton, 2009)

206

$$207 \quad qx = -\{K_0[H - (D - x) \tan(\alpha)] - qx \tan(\alpha)\} \frac{dH}{dx} \quad (10)$$

208

209 With u defined by

210

$$211 \quad u = [H - D \tan(\alpha) + (1 - q/K_0)x \tan(\alpha)]/x \quad (11)$$

212

$$213 \quad \frac{dx}{du} = \frac{-xu}{u^2 - (1 - q/K_0) \tan(\alpha)u + q/K_0} \quad (12)$$

214

215 so that

216

$$217 \quad \begin{aligned} x(u) &= x(u_i) \exp \left[- \int_{u_i}^u \frac{u \, du}{u^2 - (1 - q/K_0) \tan(\alpha)u + q/K_0} \right] \\ &= x(u_i) \exp[-\{f(u) - f(u_i)\}] \end{aligned} \quad (13)$$

218

219 with $a=1$, $b = -(1 - q/K_0)\tan(\alpha)$ and $c = q/K_0$ in $f(v)$ of eq.(7). The lower limit of integration

220 u_i in eq.(13) is obtained from eq.(11) for the known value of H at $x(u_i)$. With eq.(13) giving

221 x as a function of u , the water-table height H at a given x is then found from eq.(11) as

222

$$223 \quad H = ux(u) + D \tan(\alpha) - (1 - q/K_0)x(u) \tan(\alpha) \quad (14)$$

224

225 The Dupuit-Forchheimer analysis of the drainage problem assumes the water-table height
 226 is drawn down to the ditch-water level. Thus at the ditch from eqs.(11) and (4)

227

$$228 \quad u = \frac{H_D - D \tan(\alpha)}{D + H_D \tan(\alpha)} + \left(1 - \frac{q}{K_0}\right) \tan(\alpha), \quad x = D + H_D \tan(\alpha), \quad H_D < t \quad (15)$$

229

$$230 \quad w = \frac{H_D - t(1 - K_0/K_1) - D \tan(\alpha)}{D + H_D \tan(\alpha)} + \left(1 - \frac{q}{K_1}\right) \tan(\alpha), \quad x = D + H_D \tan(\alpha), \quad H_D > t \quad (16)$$

231

232 At the positions x_1 and x_2 where the water table crosses over from one layer to the other, $u =$
 233 $t/x - q/K_0 \tan(\alpha)$ and $w = K_0 t/K_1 x - q/K_1 \tan(\alpha)$. The water-table profile is obtained by finding x
 234 as a function of u from eq.(13) when the water table is in the lower layer or as a function of w
 235 from eq.(8) when it is in the upper layer; H at a given x is obtained from eq.(14) or eq.(9),
 236 using the appropriate value of u_i or w_i in eq.(13) or (8). Values of u and w where the water
 237 table crosses the interface between layers are found by trial and error, hence determining x_1
 238 and x_2 . The calculation of the water-table profile in a practical example is given in Table 1.

239

240 3.3: Application to individual cases

241

242 *Case 1: Water table wholly in lower layer, $H_D < t$ at $x = D$, $H = D \tan(\alpha)$ at $x = 0$.* Fig.2a
 243 sketches this situation. The water table lies wholly in the lower layer with the accretion rate q
 244 $< q_1'$ insufficient to raise the water table to meet the boundary between the two layers at any
 245 distance from the outfall. The situation is thus that discussed by Youngs and Rushton (2009).
 246 Also the accretion rate in this case is insufficient to raise the water table above the
 247 impermeable base at the centre. $x(u)$ is calculated from eq.(13) with u_i given by eq.(15),
 248 and the water-table height H found from eq.(14). In this case the parameter u is finite at $x = 0$
 249 and is found by trial and error. The limiting value of q/K_0 below which the water table meets

250 the impermeable bed at $x = 0$ is $(\tan \alpha)^2/4$ (Youngs and Rushton, 2009). It is to be noted that
 251 the water table meeting the impermeable bed at $x = 0$ results from the assumption of the
 252 uniform surface accretion travelling to meet the water table without diverging from the
 253 vertical. If the divergence (as would be the case at large slope angles) were taken into
 254 consideration, then the water table would meet the impermeable bed at some distance down
 255 slope.

256

257 *Case 2: Water table wholly in lower layer, $H_D < t$ at $x = D$, $H > D \tan(\alpha)$ at $x = 0$.* Fig.2b
 258 illustrates this case when the accretion rate $q_1' < q < q_1$ is sufficient to raise the water table
 259 above the impermeable floor at the centre but insufficient for the water table to penetrate into
 260 the upper layer. Again this corresponds to the situation considered in Youngs and Rushton
 261 (2009). The calculations proceed in the same way as for Case 1, but in this case $q/K_0 >$
 262 $(\tan \alpha)^2/4$ at $x = 0$ and $u \rightarrow 0$ as $x \rightarrow 0$.

263

264 *Case 3: Water table in lower layer $x < x_1$ and $x > x_2$, in upper layer $x_1 < x < x_2$, $H_D < t$ at*
 265 $x = D$. This is the situation sketched in Fig.2c. The accretion rate $q_1 < q < q_2$ is such that the
 266 water table is in the lower layer in the vicinity of the central area but crosses into the upper
 267 layer at $x = x_1$ before descending into the lower layer at $x = x_2$ to drain at $x = D$ to an outfall
 268 at height $H_D < t$. The range of values of accretion rate $q_1 < q < q_2$ when this occurs is
 269 determined by the hydraulic conductivities of the two layers, the thickness t of the lower layer
 270 and the slope of the bed. In the region $x_2 < x < D$, H is obtained from eq.(14) as for Case 1
 271 with u_i given by eq.(15) and $x(u)$ calculated from eq.(13) with the calculation proceeding until
 272 $u = t/x_2 - q/K_0 \tan(\alpha)$. For $x_2 > x > x_1$ the water table height H is given by eq.(9) with $w_i =$
 273 $K_0 t/K_1 x_2 - q/K_1 \tan(\alpha)$ and $x(w)$ given by eq.(8), the calculation proceeding until $w = K_0 t/K_1 x_1$
 274 $- q/K_1 \tan(\alpha)$. Between $x_1 > x > 0$ the water table is obtained from eqs.(13) and (14) with $u_i =$

275 $t/x_1 - q/K_0 \tan(\alpha)$. q_1 , the lower value of q , is when $H - D \tan(\alpha) \rightarrow t$ as $x_2 \rightarrow 0$. q_2 , the upper
276 value, is when $H - D \tan(\alpha) \rightarrow t$ as $x_1 \rightarrow 0$.

277

278 *Case 4: Water table in lower layer $x > x_2$, in upper layer $x < x_2$, $H_D < t$ at $x = D$.* This
279 situation, sketched in Fig.2d for the case of a seepage surface with $h_f < t$ and in Fig.2e for a
280 seepage surface with $h_f > t$, occurs when the accretion rate is increased beyond q_2 . The
281 procedure for the Dupuit-Forchheimer calculation of the water table is the same as for Case 3
282 except only the first two calculations are performed. Since the seepage surface at the outfall is
283 neglected in the Dupuit-Forchheimer calculations and the water table in the soil is assumed to
284 be drawn down to the ditch water level, Case 4b shown in Fig.2e would give the same result
285 as for Case 4a shown in Fig.2d with the water table wrongly calculated to cross into the lower
286 layer.

287

288 *Case 5: Outfall above layer boundary.* With the water level at the outfall above the
289 boundary between the two layers, the water table can penetrate into the lower layer for small
290 accretion rates and large slopes, as illustrated in Fig.2f. The calculation of the water-table
291 profile then follows the last two parts of the procedure given for Case 3 with w_i given by
292 eq.(16). For large accretion rates and small slopes the water table is located always in the
293 upper layer when the outfall is above the boundary between layers and is calculated from
294 eq.(9).

295

296 4. Numerical solutions of Laplace's equation

297

298 The reliability of the application of the Dupuit-Forchheimer analysis can be checked by
299 obtaining numerical solutions of Laplace's equation with the physical boundary conditions
300 given in Fig.3a. Both the water-table profile and the height of the seepage surface are
301 unknown and emerge as part of the solution in the numerical investigation.

302 Numerical solutions for specific problems can be obtained using the finite difference
303 approximation method (Rushton and Redshaw, 1979). Due to the unknown location of the
304 water table and hence the top of the seepage face, an iterative technique is required with a
305 series of trial solutions for the water-table location. When the water table is entirely in the
306 lower layer, Cases 1 and 2 (Figs.2a and 2b), or entirely in the upper layer, Case 4b (Fig.2e), a
307 systematic series of trial solutions leads to the water-table elevations and the height of the top
308 of the seepage face, h_f . However, when the water table crosses the interface between the two
309 regions, many trials are required before a satisfactory approximation is obtained for the
310 crossover points and the water-table elevation. For most problems, a regular rectangular
311 finite difference mesh is used. On the other hand, for Case 4(a) the steep fall in the water
312 table from the interface towards the downstream boundary, requires closer vertical grid lines
313 towards the downstream face. Due to the sudden change in the hydraulic gradients at the
314 interface between the two layers, solutions were obtained for a ratio of the hydraulic
315 conductivities in the upper and lower layers of $K_1/K_0 = 10$. For larger values of K_1/K_0 , it is
316 difficult to obtain reliable finite difference solutions. For all the numerical solutions, results
317 are presented for the water table profile, the equipotentials and the height of the seepage face
318 h_f .

319

320 5. Dupuit-Forchheimer calculations and Laplace numerical results

321

322 The Dupuit-Forchheimer calculations of the water-table profiles for different cases that can
323 arise for drainage of a two-layered permeable region overlying an inverted V-shaped
324 impermeable bed are compared with Laplace numerical solutions for the particular example
325 presented by the drainage of ballast foundations beneath railway tracks. The results are given
326 in Figs 4-9. In these we considered a typical railway ballast geometry with a half-width $D =$
327 2.25 m of depth $t = 0.125$ m; the slope of the base above the horizontal is $= \tan^{-1} 0.05$. The

328 upper layer is assumed to have a hydraulic conductivity $K_1 = 10 K_0$. (Note that in these
329 figures the vertical coordinate is five times that of the horizontal.) In most cases, there is
330 good agreement between water-table elevations deduced from the two approaches except in
331 the vicinity of the drainage ditch where the Dupuit-Forchheimer analysis ignores the
332 existence of a seepage face which is included in the Laplace solution. For Case 4(b) shown
333 in Fig.8 the downstream water level is at the base of the aquifer so that the water table,
334 according to the Dupuit-Forchheimer approximation, falls to this level. However, in the
335 Laplace solution the seepage face is found to be above the interface. Consequently there is a
336 significant difference between the water-table elevations for large x near the drainage ditch.
337 Due to the discrete mesh used in the numerical solution of the Laplace equation, the accuracy
338 of the water-table elevation is about 0.005 m. When the outfall is above the boundary of the
339 two layers, the seepage surface in the more permeable layer included in the Laplace solution
340 is a small distance above the interface; when the outfall level is below the interface it is a
341 prominent feature. The equipotentials obtained are very nearly normal to the impermeable
342 base except near the outfall ditch and midway between ditches.

343 The area over which the upper layer contains the water table when the accretion rate
344 becomes large depends on the slope of the bed, the hydraulic conductivities of the two layers,
345 the elevation of the downstream water level and the accretion rate. Fig.10 plots the values of
346 x_1 and x_2 , the distances at which the water table crosses the boundary between layers (see Fig.
347 1), against the ratio of the accretion rate to the hydraulic conductivity of the lower layer when
348 the ditch-water level is zero for the examples given in Figs 4 to 8. It is seen that x_1 , the cross-
349 over distance nearest to the watershed, occurs over a limited range $0.0069 < q/K_0 < 0.016$. The
350 horizontal line on Fig. 10 refers to $q/K_0 = 0.01$ for which $x_1 = 0.317$ m and $x_2 = 1.75$ m as in
351 the example shown in Fig. 6. For Cases 1 and 2, $q/K_0 < 0.0069$; consequently the water table
352 is always below the interface. For Case 4(a) shown in Fig. 7 with $q/K_0 = 0.02$, $x_2 = 2.05$ m,

353 but there is no value for x_1 because the water table does not fall below the interface towards
354 the watershed.

355

356 6. Dupuit-Forchheimer results and laboratory experiments

357

358 Heyns (2000) reported an extensive study of the drainage of railway ballast using a tilting
359 tank containing a sub-ballast overlain by a ballast; water was sprayed from nozzles to
360 simulate rainfall. His main interest was in non-steady state conditions, especially the
361 recession of the water table after the rainfall ceased. However, the simulated rainfall
362 continued for a sufficient time for a steady-state to be reached.

363 In Fig.11 we compare one of his steady-state results with our Dupuit-Forchheimer
364 calculations; water-table heights deduced from five piezometers are shown in the figure by
365 the symbol +. Due to the experimental technique, in which the piezometers are connected to
366 the base of the tank, the accuracy of the estimates of water table elevation is unlikely to be
367 better than 0.005 m. A two-layered railway ballast bed was modelled with a lower less
368 permeable layer of porous material having a hydraulic conductivity equal to 65 md^{-1} (K_0) to a
369 depth $t = 0.14 \text{ m}$ overlain by a very permeable material of conductivity 3250 md^{-1} (K_1). In
370 the experiments water was sprayed on to the surface for three hours at a rate 2.7 md^{-1} .
371 Outflows occurred from both the upper and lower layers at a distance 1.88 m from the mid-
372 plane; this is assumed to be the value of D used in the calculations. Two Dupuit-Forchheimer
373 calculations are included on the figure. The full line corresponds to $H_D = 0.0$ with the water
374 table in the lower layer in the region midway between drains, crossing into the upper layer
375 before entering again the lower layer (Case 3 of Section 3); for the broken line $H_D = 0.14 \text{ m}$
376 where the water table does not enter again the lower layer (Case 5), chosen to represent
377 conditions actually observed in the experiment of outflow both from the upper and lower
378 layers at this downstream boundary. It is seen that good agreement is obtained between the
379 experimental and Dupuit-Forchheimer estimates.

380 Table 1 contains some results of calculations that give the Dupuit-Forchheimer plots in
381 Fig.11, following the procedure given for Case 3 in Section 3. These calculations were
382 performed using computer algebra with the mathematical package Mathcad (Mathsoft Inc.,
383 201 Broadway, Cambridge, Mass., U.S.A.)

384 Parsons (1990) discusses the reduction of hydraulic conductivity of the ballast [upper
385 layer] due to clogging of pores. From experiments using a falling head permeameter, Parsons
386 found that the hydraulic conductivity for moderately clean ballast is typically one tenth of the
387 value for clean ballast; for moderately fouled ballast the hydraulic conductivity is about one-
388 twenty fifth of the value for clean ballast. In Fig.12 we also show the calculated water-table
389 profiles when K_0 and K_1 are reduced to 0.1 and 0.04 of their original values to represent
390 clogging. With the hydraulic conductivities at 0.1 of the original values, the maximum
391 water- table elevation is 0.24 m above the base (0.1 m above the interface). For moderately
392 fouled ballast, with hydraulic conductivities set at 0.04 of the original values, the maximum
393 water-table elevation is 0.34 m above the base (0.2 m above the interface). This means that
394 the water table approaches the bottom of the track sub-structure a condition that needs to be
395 avoided. For hydraulic conductivities 1% of their original values, the maximum water table
396 elevation is 0.61 m above the impermeable base.

397

398 7. Discussion and conclusion

399

400 This paper has particular relevance to the problem of drainage of the ballast beneath
401 railway tracks that, if not attended to, risks the water table reaching the level of the sleepers
402 when severe operational speed restrictions have to be imposed due to reduced strength of the
403 track foundations. The problem considered here also occurs in agricultural ridge and furrow
404 drainage where the water table is controlled to provide suitable conditions for root
405 development and livestock grazing. In these situations there is an underlying impermeable
406 undulating base, overlain by a permeable surface layer on top of a less permeable layer. We

407 have used the Dupuit-Forchheimer analysis to consider the steady-state drainage of two-
408 layered soil regions overlying an inverted V-shaped impermeable bed that approximates the
409 situation. Our results have been compared with numerical solutions of Laplace's equation for
410 the head distribution and also the results of laboratory model experiments of the drainage of a
411 railway ballast foundation.

412 These steady-state results provide a theoretical background for more general time-variant
413 studies of the problem (Rushton and Ghataora, 2009). Cases 1 to 4 discussed here can occur
414 with the drainage of the railway ballast where a very permeable ballast overlies sub-ballast
415 with a hydraulic conductivity at least an order of magnitude less. Case 5 is relevant to the
416 situation when there is surcharge in the drain and a low accretion rate.

417 The upper layer clearly plays a major role in preventing the water table rising to the surface
418 when the accretion rate is large. Our results show that the high conductivity of the surface
419 layer insures the water table to follow close to the boundary between the layers when the
420 accretion rate is sufficient for the water table to rise above the less permeable layer.

421 The agreement obtained between the Dupuit-Forchheimer results and the Laplace
422 numerical calculations of the water-table profile in drained two-layered soils overlying a
423 sloping bed gives confidence in using the approximate analysis in these situations. This is
424 important since two-dimensional numerical computations involving a water table crossing
425 between layers are time consuming and particularly difficult when there is a large difference
426 in conductivity in the two layers, while Dupuit-Forchheimer calculations are easily performed
427 with computer algebra. Further confirmation of the efficacy of the application of the Dupuit-
428 Forchheimer is given with the agreement between the calculations of the water-table profile
429 and the experimental results of Heyns (2000) laboratory experiment of a two-layered railway
430 track ballast foundation. Application of the Dupuit-Forchheimer analysis shows the effect on
431 the water-table heights in such situations due to the fouling of the ballast and sub-ballast with
432 the consequent reduction in hydraulic conductivities.

433

434 **Notation**

435

436 D = drainage ditch half-spacing (L);437 H = water-table elevation (L);438 H_0 = water-table elevation at watershed (L);439 H_D = ditch-water level (L);440 h = hydraulic head (potential) (L);441 K_0 = hydraulic conductivity of lower layer (LT^{-1});442 K_1 = hydraulic conductivity of upper layer (LT^{-1});443 q = accretion rate (LT^{-1});444 q_1 = limiting accretion rate for the water table to be wholly in the lower layer (LT^{-1});445 q_1' = limiting accretion rate for the water table to meet the impermeable bed at the watershed446 and be wholly contained in the lower layer (LT^{-1});447 q_2 = limiting accretion rate for the water table to be in the upper layer over a section of the448 region but in the lower layer near the watershed (LT^{-1});449 s = coordinate along sloping bed (L)450 s_D = value of s at $x = D$ (L);451 t = thickness of lower layer (L);452 u, w = parameters used in calculating the water-table profile when the water table is located

453 below and above the boundary between layers, respectively;

454 u_i, w_i = lower integration limits of u, w ;455 x = horizontal coordinate (L);456 x_1, x_2 = coordinates where the water table crosses the boundary between the layers (L);457 z = vertical coordinate (L);

458 = slope angle.

459

460

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505

506

507 **Legend to Figures**

508

509 Fig. 1. Drainage to ditches in two-layered soil regions overlying an inverted V-shaped

510 impermeable base.

511

512 Fig. 2. The water-table profiles in a drained two-layered soil overlying an inverted V-shaped

513 impermeable base for the different cases discussed in the text.

514

515 Fig.3. (a) The boundary conditions of the two-dimensional physical problem; (b) the flow

516 conditions assumed in the Dupuit-Forchheimer analysis.

517

518 Fig. 4. Calculated water-table profiles for Case 1 sketched in Fig.2a.

519

520 Fig. 5. Calculated water-table profiles for Case 2 sketched in Fig.2b.

521

522 Fig. 6. Calculated water-table profiles for Case 3 sketched in Fig.2c.

523

524 Fig. 7. Calculated water-table profiles for Case 4a sketched in Fig.2d.

525

526 Fig. 8. Calculated water-table profiles for Case 4b sketched in Fig.2e.

527

528 Fig. 9. Calculated water-table profiles for Case 5 sketched in Fig.2f.

529

530 Fig. 10. Locations x_1 and x_2 where the water table crosses the interface as a function of the
531 accretion rate q .

532

533 Fig. 11. Steady-state water-table profiles calculated by the Dupuit-Forchheimer analysis
534 compared with Heyns' (2000) laboratory experiment: ($K_1 = 3250 \text{ md}^{-1}$; $K_0 = 65 \text{ md}^{-1}$; $q = 2.7$
535 md^{-1}).

536

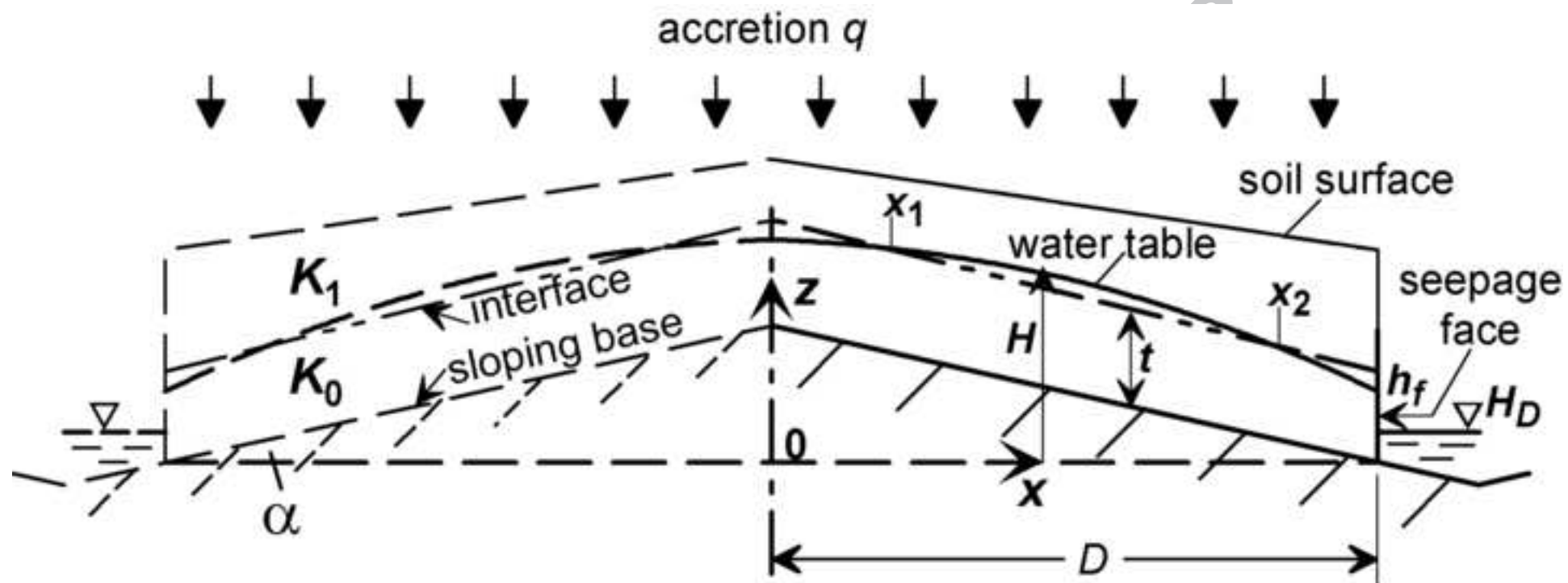
537 Fig. 12. Effect of clogging of ballast on water-table elevations.

538

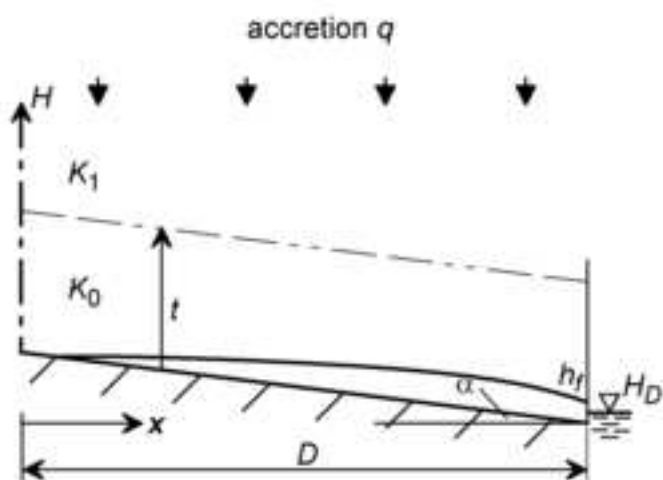
539 **Table**

540

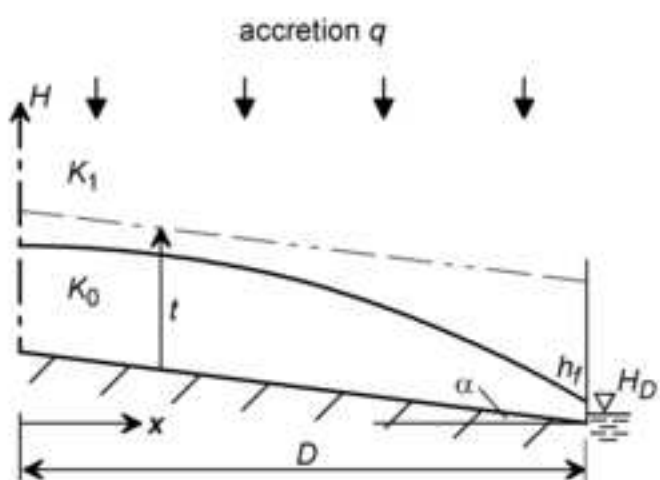
541 Table 1. Calculation of the steady-state water-table profile shown in Fig.11 for zero ditch-
542 water level and for a ditch-water level at the top of the sub-ballast.



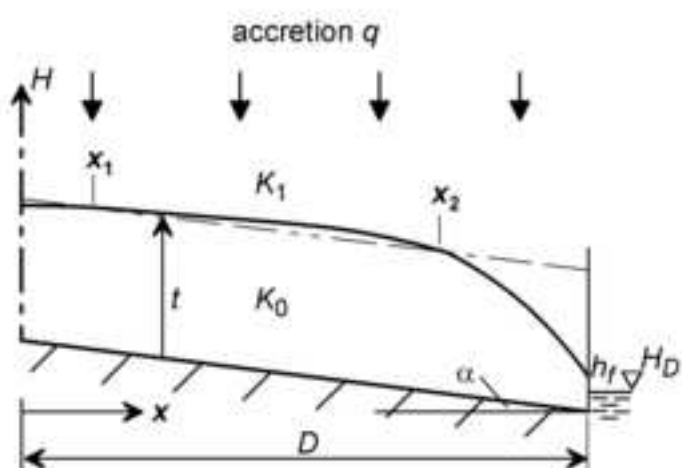
(a) Case 1



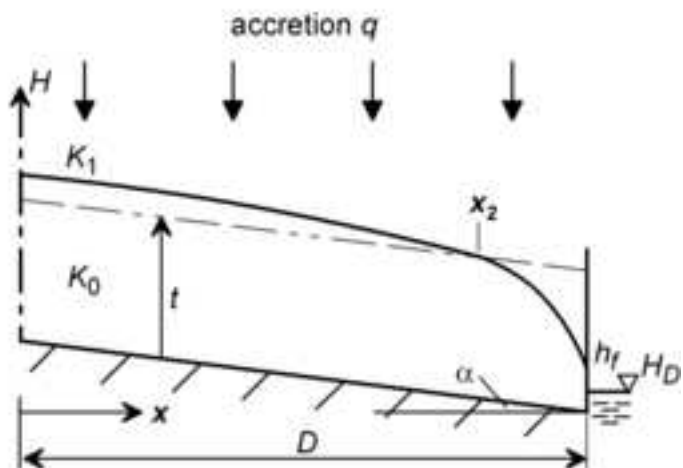
(b) Case 2



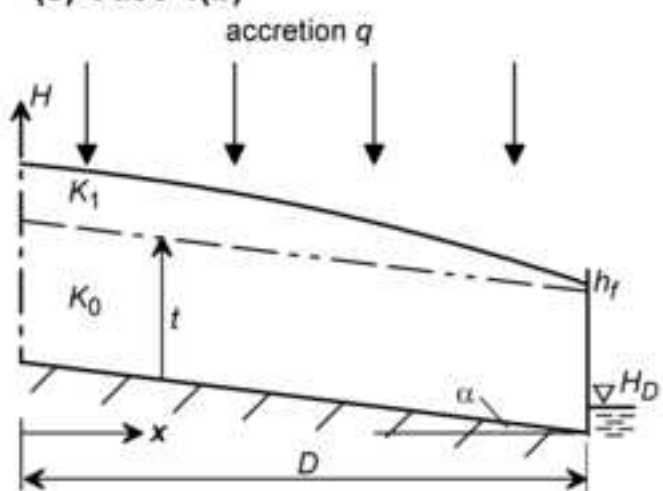
(c) Case 3



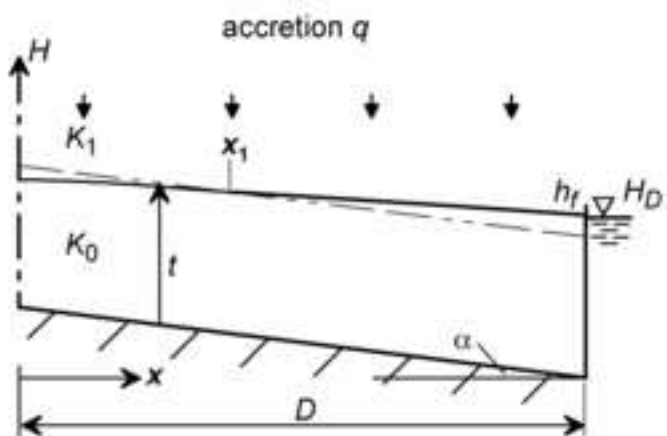
(d) Case 4(a)

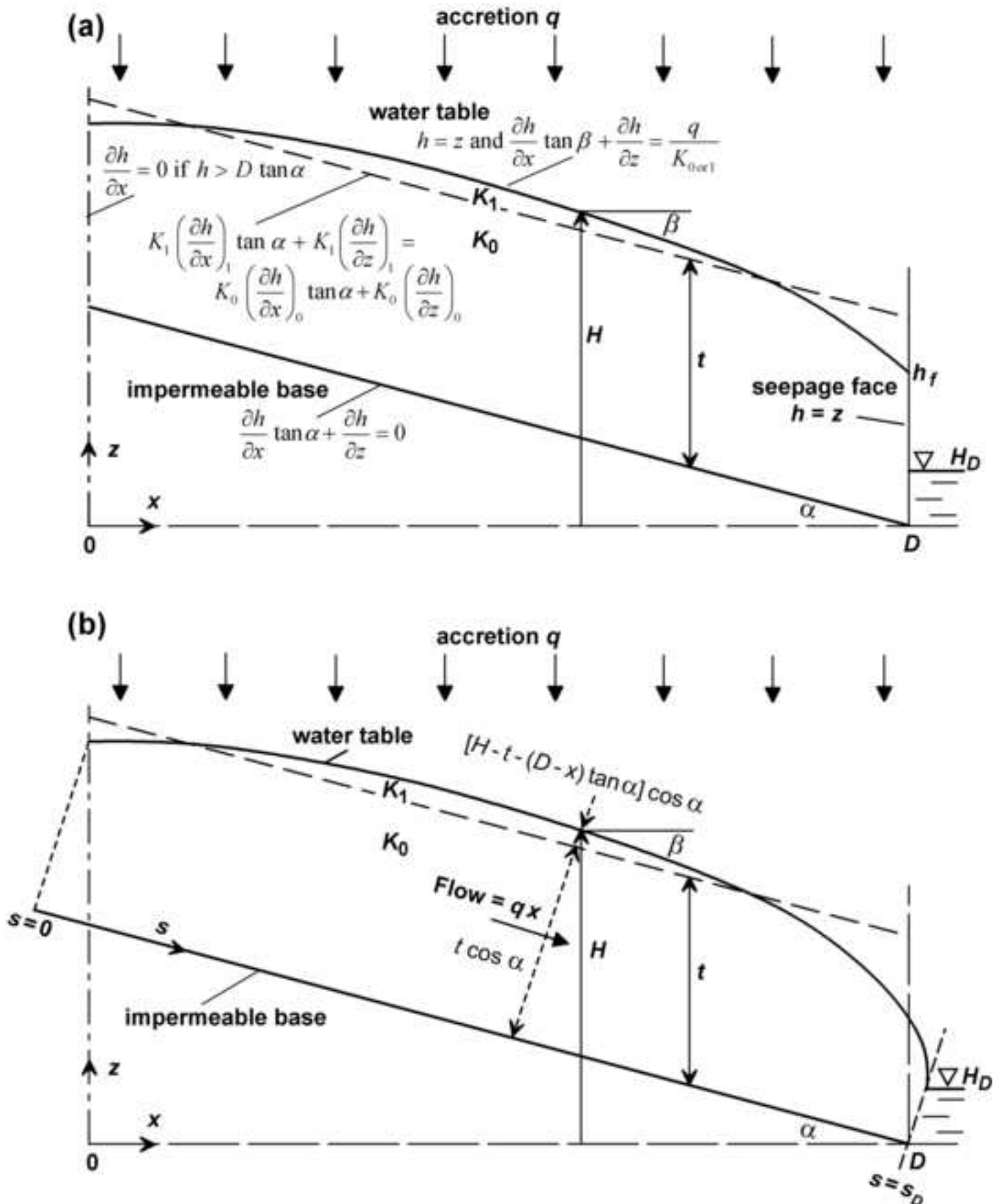


(e) Case 4(b)



(f) Case 5





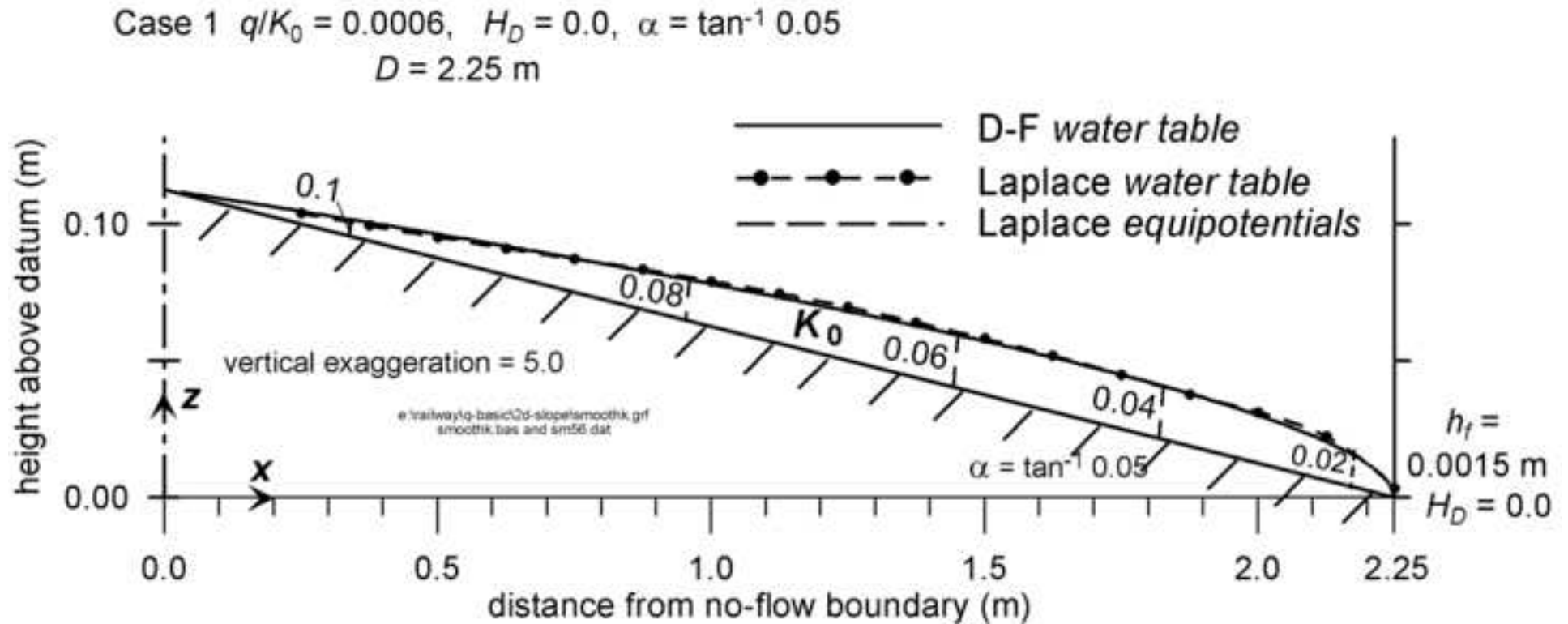
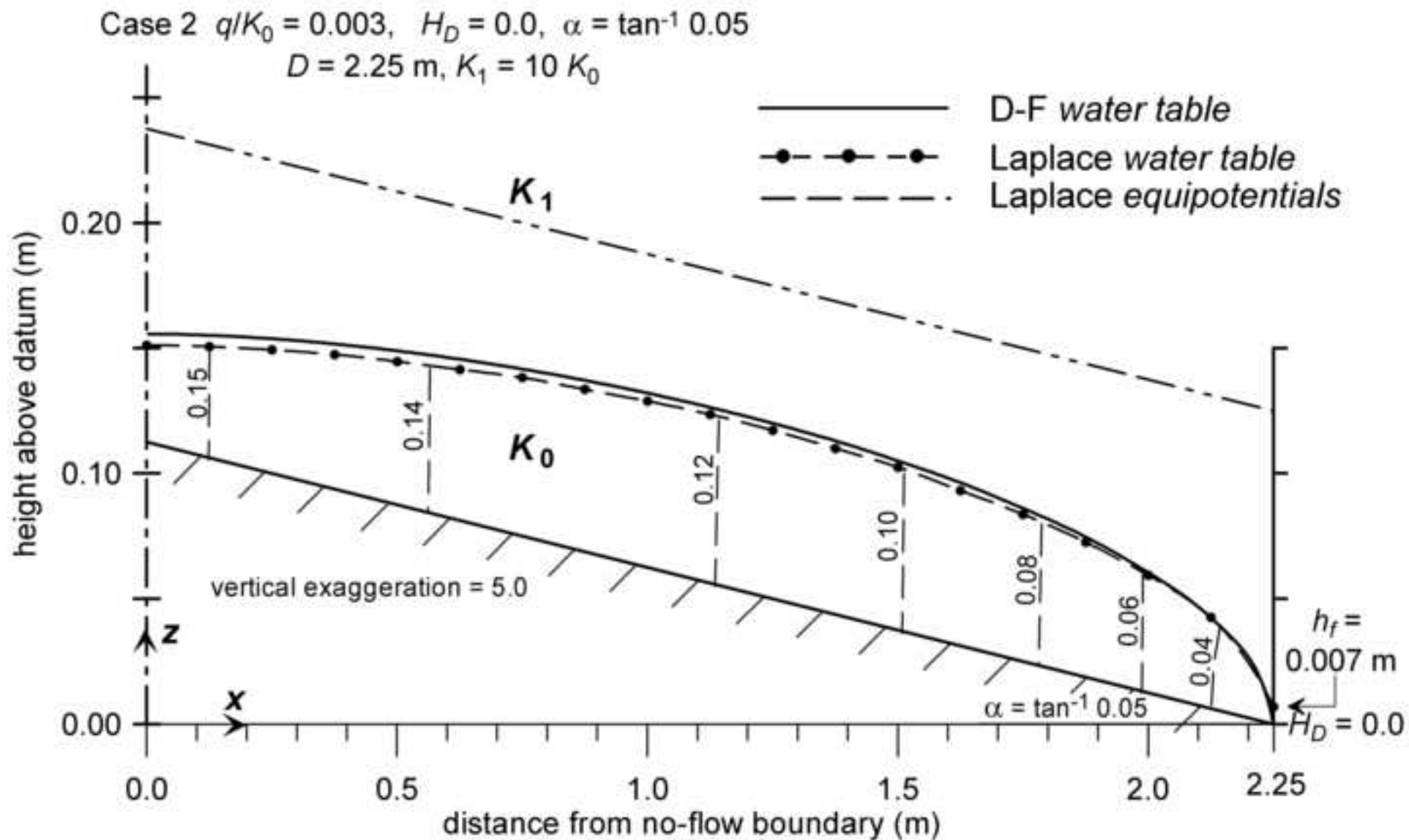
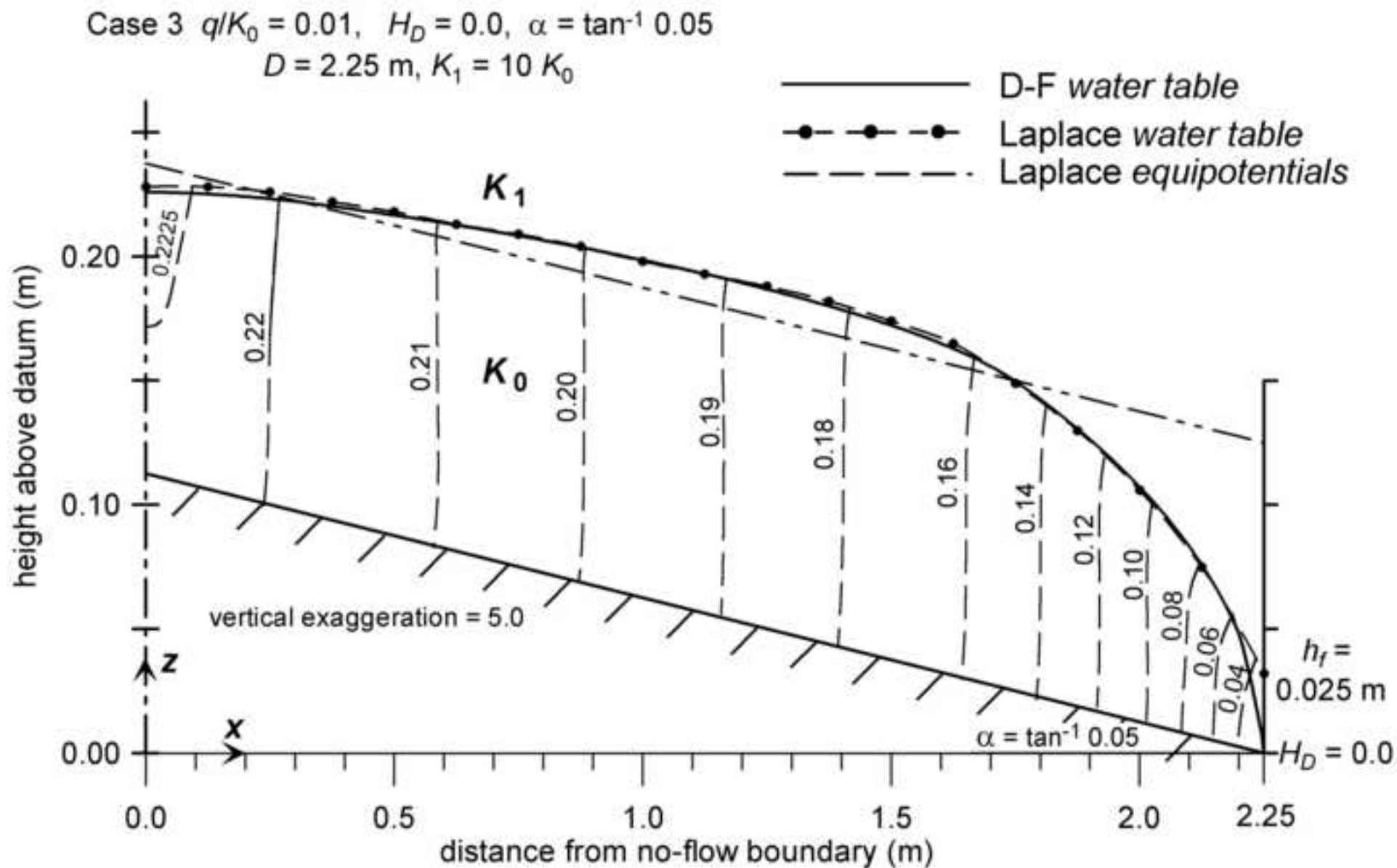
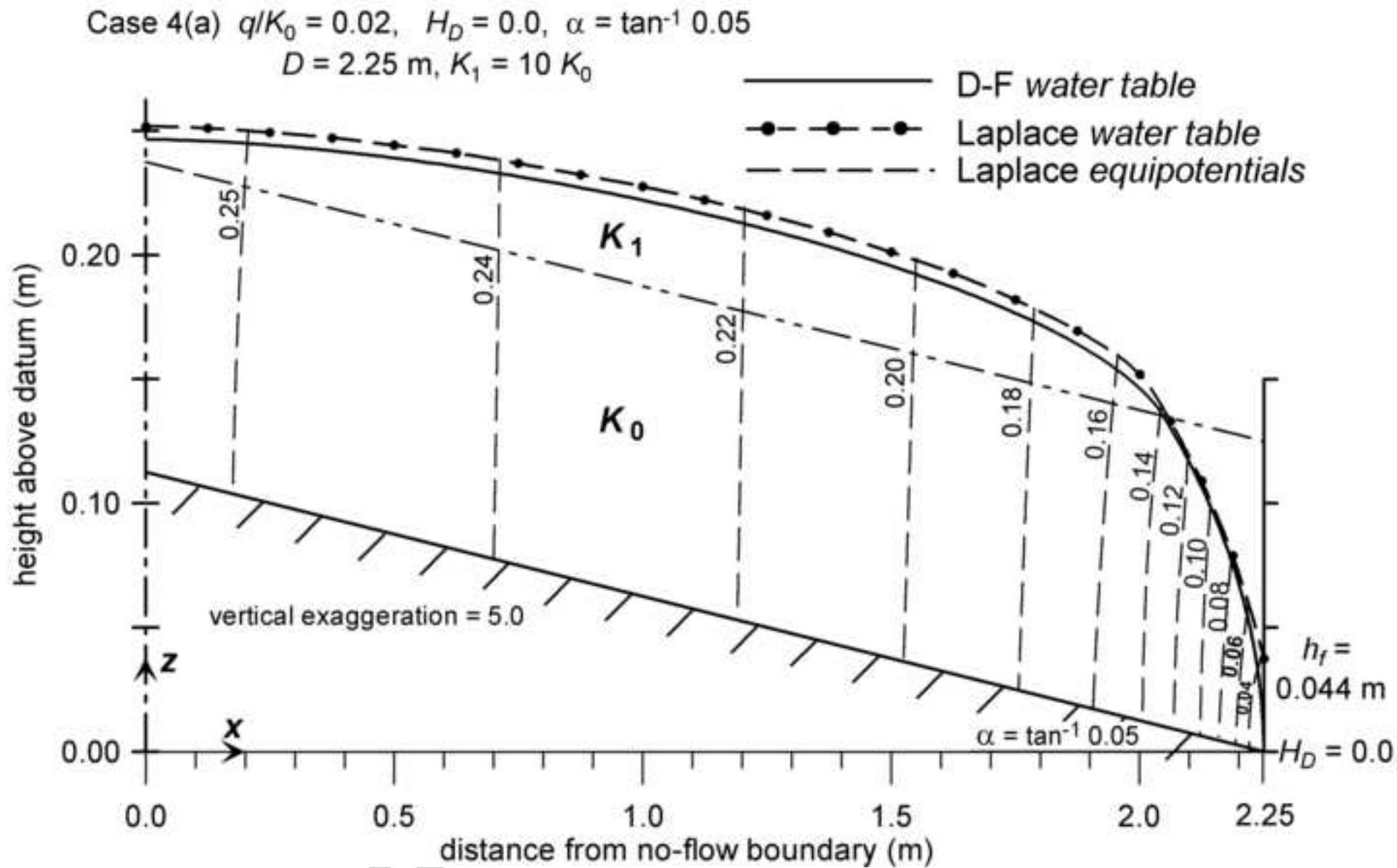


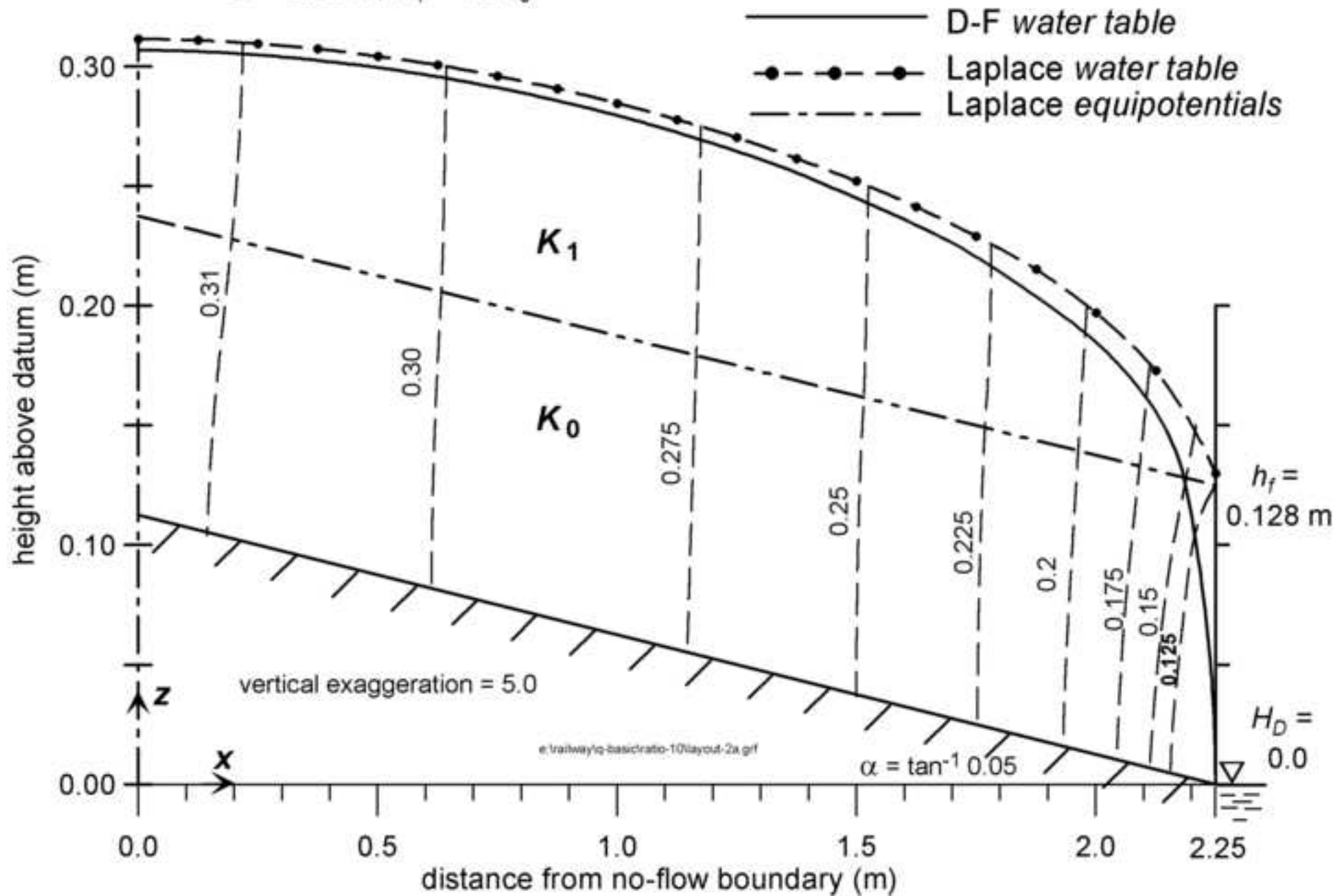
Figure 5

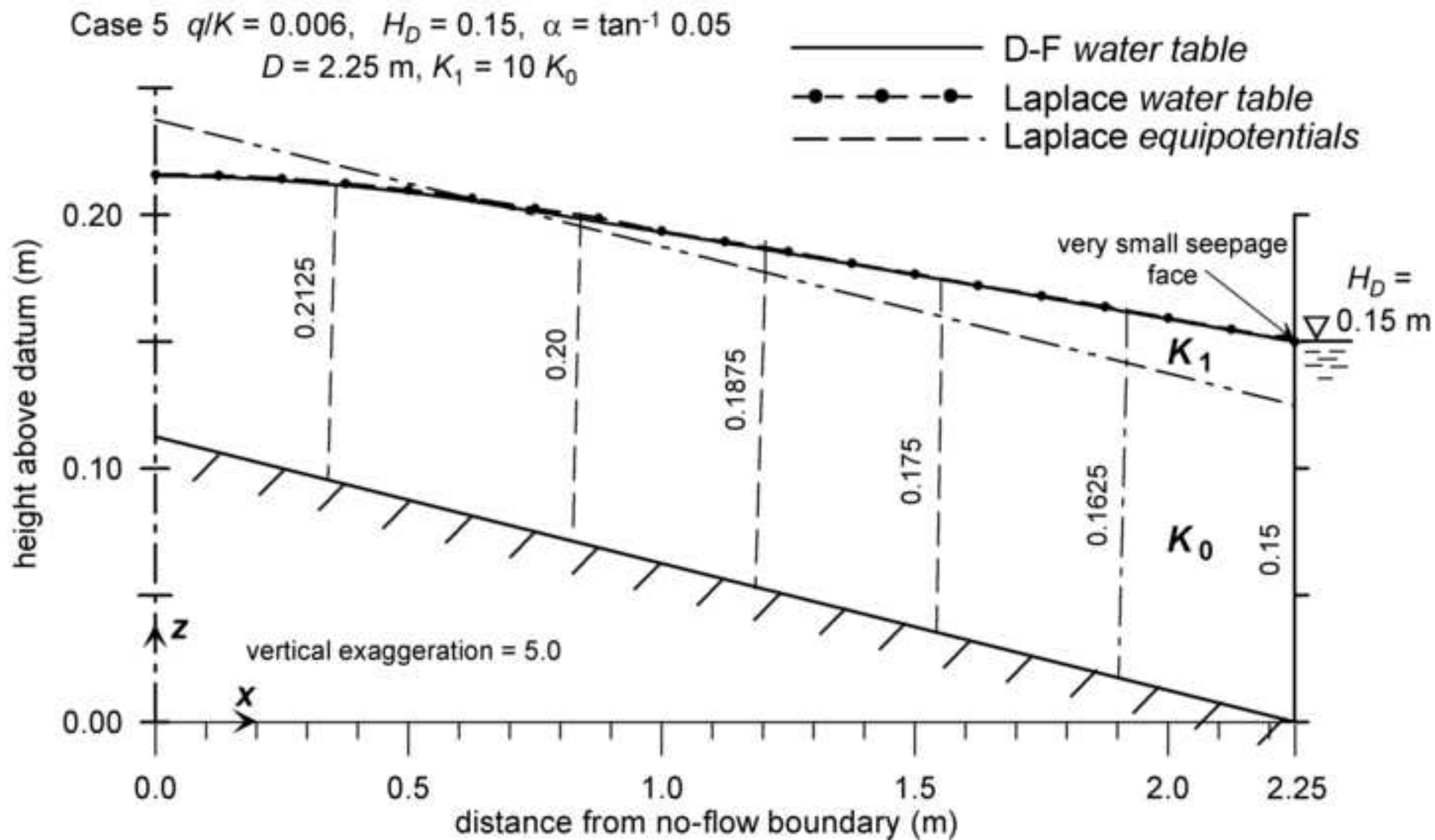


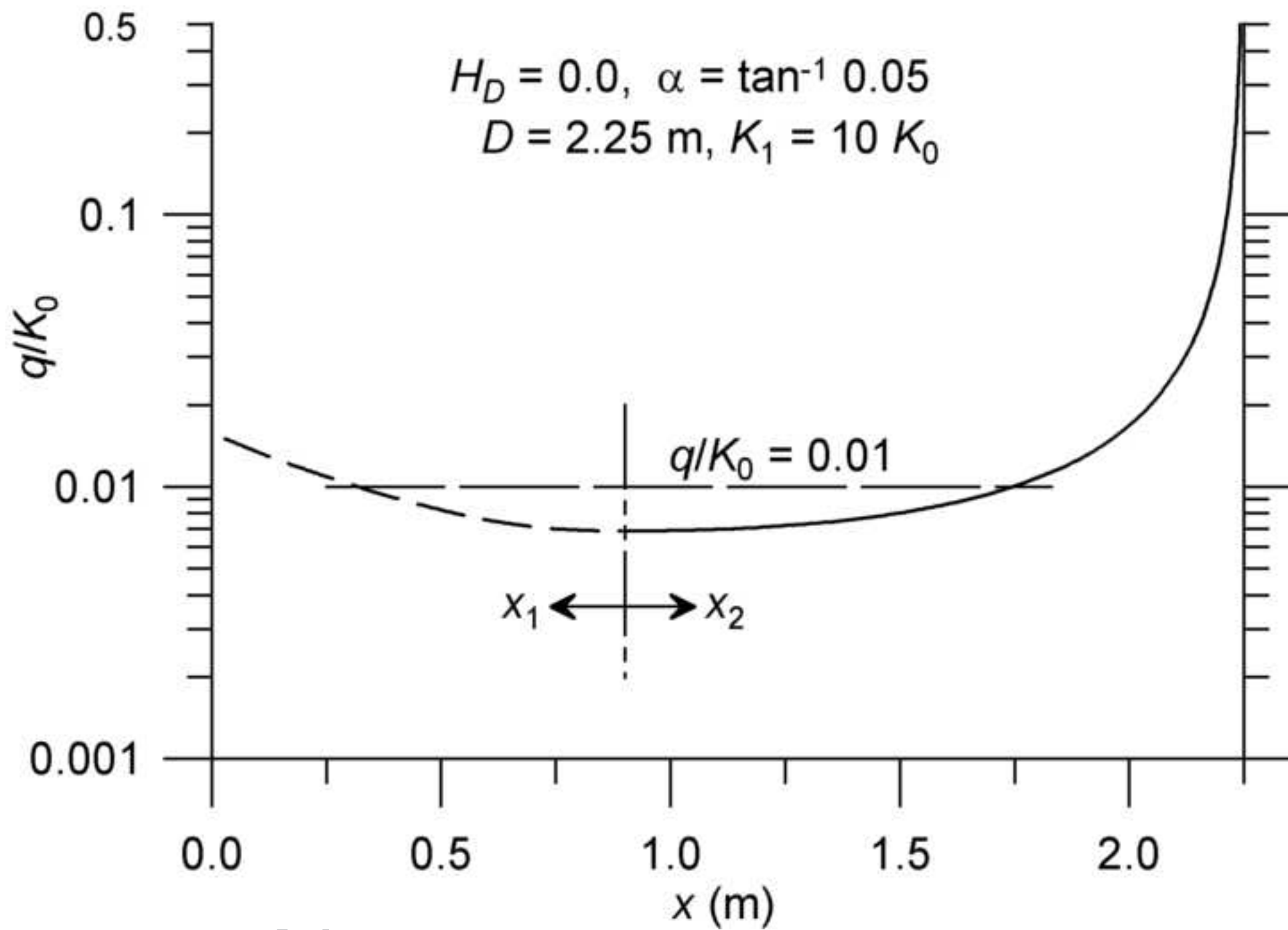


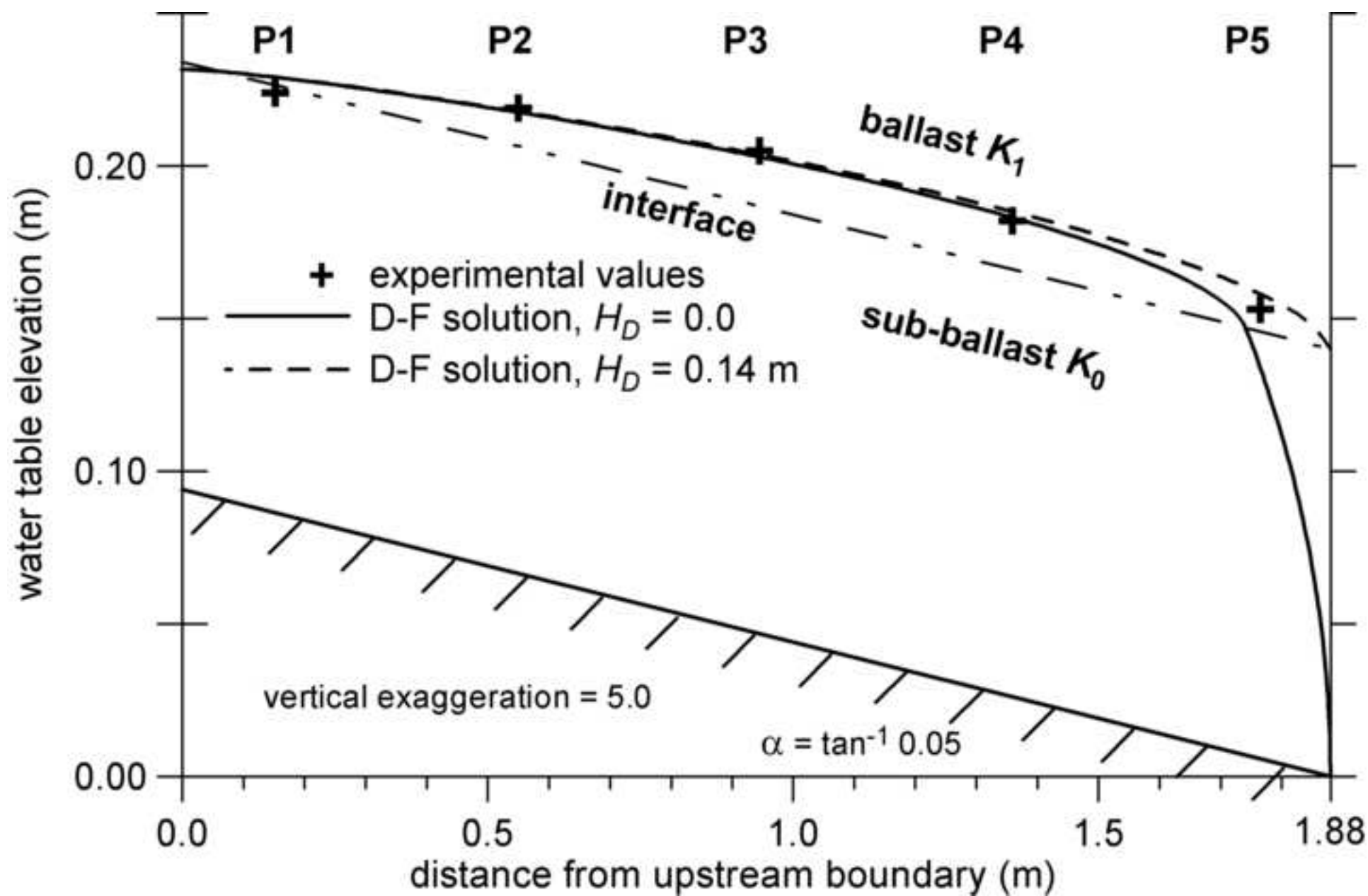


Case 4(b) $q/K_0 = 0.055$, $H_D = 0.0$ m, $\alpha = \tan^{-1} 0.05$
 $D = 2.25$ m, $K_1 = 10 K_0$









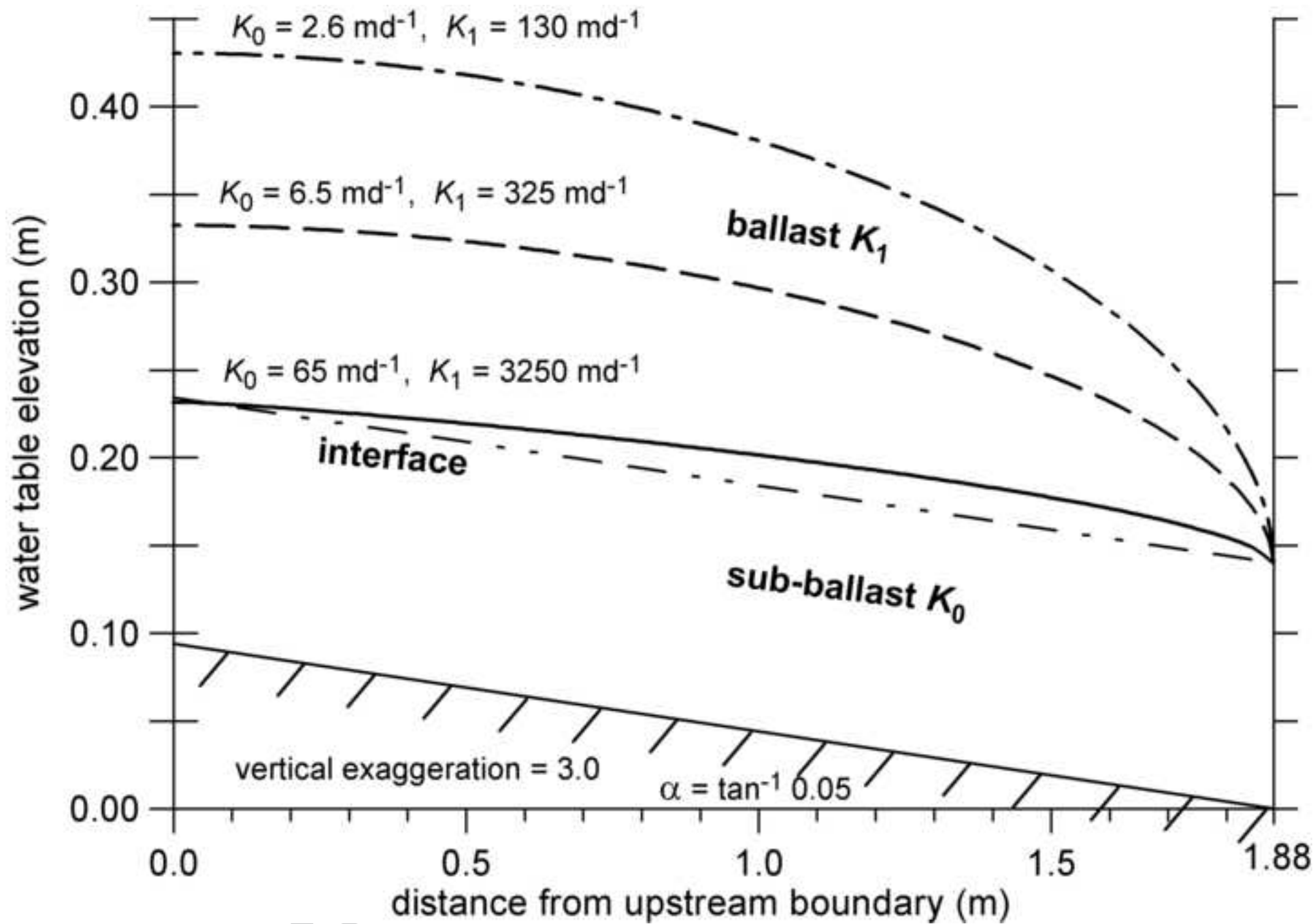


Table 1. Calculation of water-table profiles shown in Fig. 11.

u	w	x (m) eq.(8) or (13)]	H (m) eq.(9) or (14)
$H_D = 0$			
- 0.00208		1.880 (D)	0
0		1.880	0.0039
0.002		1.880	0.0077
0.005		1.879	0.0133
0.01		1.878	0.0228
0.02		1.871	0.0418
0.04		1.844	0.0794
0.06		1.800	0.1157
0.07797	0.001559	1.749 (x_2)	0.1466
	0.005	1.734	0.1532
	0.01	1.645	0.1655
	0.015	1.440	0.1809
	0.02	1.111	0.1979
	0.025	0.7328	0.2130
	0.03	0.4308	0.2227
	0.035	0.2509	0.2275
	0.04	0.1592	0.2297
	0.05	0.07615	0.2313
2.99780	0.059956	0.04669 (x_1)	0.2317
5		0.02784	0.2319
10		0.01386	0.2320
100		0.001380	0.2321
1000		0.0001380	0.2321
$H_D = 0.14$ m			
	0.001442	1.880 (x_2, D)	0.14
	0.01	1.776	0.1603
	0.015	1.555	0.1768
	0.02	1.200	0.1953
	0.025	0.7909	0.2115
	0.03	0.4650	0.2220
	0.035	0.2708	0.2272
	0.04	0.1683	0.2296
	0.06	0.05028	0.2318
3.16237	0.063250	0.04424 (x_1)	0.2319
5		0.02786	0.2320
10		0.01387	0.2321
100		0.001381	0.2322
1000		0.0001381	0.2322