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# The Erdős-Ko-Rado property of various graphs containing singletons

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**Abstract**

Let $G = (V, E)$ be a graph. Let $\mathcal{I}_G$ be the family of all independent sets of $G$. For $r \geq 1$, let $\mathcal{I}_G^{(r)} := \{I \in \mathcal{I}_G : |I| = r\}$. For $v \in V(G)$, let $\mathcal{I}_G^{(r)}(v)$ denote the *star* $\{A \in \mathcal{I}_G^{(r)} : v \in A\}$. $G$ is said to be (*strictly*) *r-EKR* if there exists $v \in V(G)$ such that $(|\mathcal{A}| < |\mathcal{I}_G^{(r)}(v)|)$ $|\mathcal{A}| \leq |\mathcal{I}_G^{(r)}(v)|$ for any non-star family $\mathcal{A}$ of pair-wise intersecting sets in $\mathcal{I}_G^{(r)}$.

Let $\Gamma$ be the family of graphs that are disjoint unions of complete graphs, paths, cycles, including at least one singleton. Holroyd, Spencer and Talbot proved that if $G \in \Gamma$ and $2r$ is no larger than the number of connected components of $G$, then $G$ is $r$-EKR. However, Holroyd and Talbot conjectured that if $G$ is any graph and $2r \leq \mu(G) := \min\{|I| : I \in \mathcal{I}_G, I \text{ maximal}\}$, then $G$ is $r$-EKR, and strictly so if $2r < \mu(G)$. We show that in fact $G$ is $r$-EKR if $2r \leq \alpha(G) := \max\{|I| : I \in \mathcal{I}_G\}$; we do this by proving the result for all graphs that are in a suitable larger set $\Gamma' \supsetneq \Gamma$. We also confirm the conjecture for graphs in an even larger set $\Gamma'' \supsetneq \Gamma'$.

## 1 Introduction

Throughout this paper, we denote the set of natural numbers by $\mathbb{N}$, the set $\{x \in \mathbb{N} : m \leq x \leq n\}$ by $[m, n]$ and $[1, n]$ by $[n]$.

Next, we give some terminology and notation relating to graph theory.

A graph $G = (V, E) = (V(G), E(G))$ is assumed to be finite, simple and undirected unless specified otherwise. (An infinite graph is temporarily introduced in Definition 1.6 and a directed graph in Definition 1.9, but these are the only such graphs to appear.) We denote a typical edge of $G$ by $vw$ where $v, w \in V(G)$. For any $v \in V(G)$, the set of *neighbours* of $v$ (that is, vertices adjacent to $v$) will be denoted by $N_G(v)$, and

$N_G(v) \cup \{v\}$ will be denoted by $\hat{N}_G(v)$. An *independent set* of vertices of $G$ is a set of pairwise non-adjacent vertices.

We denote the complete graph, the path, and the cycle on $n$ vertices by $K_n$, $C_n$ and $P_n$, respectively. The *length* of $P_n$ is $n-1$. A *singleton* is a vertex of $G$ that is adjacent to no other vertex, and the *empty* graph on $E_n$ is the graph consisting of $n$ singletons and no edges.

Let $G$ be any graph; then the *distance* $d(v,w)$ between vertices $v$ and $w$ in the same connected component of $G$ is the length of the shortest path between $v$ and $w$. For $k \in \mathbb{N}$ the $k^{\text{th}}$ *power* of $G$, denoted by $G^k$, is the graph with vertex set $V(G)$ where $vw \in E(G^k)$ iff $d(v,w) \leq k$. Note that $P_n^k = K_n$ for $k \geq n-1$, while $C_n^k = K_n$ for $k \geq n/2$.

If $G$ is a graph and $S \subseteq V(G)$, then the subgraph $H$ of $G$ *induced* by $S$ has $V(H) = S$, two vertices of $H$ being adjacent in $H$ iff they are adjacent in $G$.

Finally, the *Cartesian product* $G \times H$ of two graphs has $V(G \times H) = V(G) \times V(H)$, two vertices $(v,w)$ and $(x,y)$ being adjacent in $G \times H$ iff either $v = x$ and $wy \in E(H)$ or $vx \in E(G)$ and $w = y$.

Next, we introduce notation for certain families of sets of vertices of a graph.

We denote the family of all independent sets of vertices of $G$ by $\mathcal{I}_G$. Then $\alpha(G)$ and $\mu(G)$ denote, respectively, the maximum and minimum sizes of a maximal member of $\mathcal{I}_G$ under set-inclusion.

For $r \geq 1$, let $\mathcal{I}_G^{(r)} := \{I \in \mathcal{I}_G \colon |I| = r\}$. For $v \in V(G)$, let $\mathcal{I}_G^{(r)}(v)$ denote the *star of* $\mathcal{I}_G^{(r)}$, that is, $\{A \in \mathcal{I}_G^{(r)} \colon v \in A\}$ .

More generally, for any family $\mathcal{A}$ of sets, the *stars* of $\mathcal{A}$ are the subfamilies $\mathcal{A}(x) := \{A \in \mathcal{A} \colon x \in A\}$ (where we assume $x \in \bigcup_{A \in \mathcal{A}} A$). The family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect.

In [24], Holroyd and Talbot introduced the following definition that is inspired by the classical Erdős-Ko-Rado (EKR) Theorem [15]: $G$ is said to be *$r$-EKR* if no intersecting family $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}$ is larger than the largest star of $\mathcal{I}_G^{(r)}$, and to be *strictly $r$-EKR* if no non-star intersecting family $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}$ is as large as the largest star of $\mathcal{I}_G^{(r)}$.

It is interesting that many EKR-type results can be expressed in terms of the $r$-EKR or strict $r$-EKR property of some graph $G$ and $r \in X \subseteq [\alpha(G)]$. This observation was made in [24] and inspired a number of other results about the EKR properties of certain graphs. Before coming to the crux of this paper, we give a brief review of such results, recalling certain well-known classes of graphs and also defining new ones.

The EKR Theorem [15] and the Hilton-Milner Theorem [21] may be expressed in terms of empty graphs as follows.

**Theorem 1.1 (Erdős, Ko, Rado [15]; Hilton, Milner [21])** *Let $r \leq n/2$. Then $E_n$ is $r$-EKR, and strictly so if $r < n/2$.*

The work of Cameron and Ku [8] (inspired by the work in [12]) on intersecting permutations and the works of Ku and Leader [29] and Li and Wang [31] on intersecting partial permutations can be summed up and phrased as follows.

**Theorem 1.2 (Cameron, Ku [8]; Ku, Leader [29]; Li, Wang [31])** *Let $G = K_n \times K_n$. Then $G$ is strictly $r$-EKR for all $r \in [n]$.*

Suppose $G$ is a graph whose vertex set has a partition $V(G) = V_1 \cup ... \cup V_k$ into *partite sets* such that any two vertices are adjacent iff they belong to distinct partite sets. Such a graph is said to be a *complete multipartite graph*, or more particularly a *complete $k$-partite graph*. (Thus if $|V_1| = ... = |V_k| = 1$, then $G = K_k$.)

A well-known intersection theorem that was first stated by Meyer [33] and proved by Deza and Frankl [12] and Bollobás and Leader [4] can be phrased as follows.

**Theorem 1.3 (Meyer [33]; Deza, Frankl [12]; Bollobás, Leader [4])** *Let $r \leq n$ and $k \geq 2$. Let $G$ be the disjoint union of $n$ copies of $K_k$. Then $G$ is $r$-EKR, and strictly so unless $r = n$ and $k = 2$.*

Other proofs were obtained by Engel [13] and Erdős et al. [14]. Holroyd, Spencer and Talbot [23] extended non-strict part of Theorem 1.3 by showing that if $G$ is the disjoint union of $n$ complete graphs each of order at least 2 then $G$ is $r$-EKR for all $r \leq n$.

Holroyd and Talbot [24] considered the problem for complete multipartite graphs.

**Theorem 1.4 (Holroyd, Talbot [24])** *Let $G$ be the disjoint union of two complete multipartite graphs. Let $r \leq \mu(G)/2$. Then $G$ is $r$-EKR, and strictly so if $r < \mu(G)/2$.*

This result follows immediately from the case $k = 1$ of the next result (see [24]).

**Theorem 1.5 (Borg, Holroyd [7])** *Let $G$ be the disjoint union of $k$ complete multipartite graphs and a non-empty set $V_0$ of singletons. Let $1 \leq r \leq \mu(G)/2$. Then:*
*(i) $G$ is $r$-EKR;*
*(ii) $G$ fails to be strictly $r$-EKR iff $2r = \mu(G) = \alpha(G)$, $3 \leq |V_0| \leq r$, $k = 1$.*

The following is the first of two definitions that are needed to state the new results presented in this paper (Theorems 1.14 and 1.15).

**Definition 1.6 (Borg [6])** *For a monotonic non-decreasing (mnd) sequence $\mathbf{d} = \{d_i\}_{i \in \mathbb{N}}$ of non-negative integers, let $M := M(\mathbf{d})$ be the graph such that $V(M) = \{x_i \colon i \in \mathbb{N}\}$ and, for $x_a, x_b \in V(M)$ with $a < b$, $x_a x_b \in E(M)$ iff $b \leq a + d_a$. Let $M_n := M_n(\mathbf{d})$ be the sub-graph of $M$ induced by the subset $\{x_i \colon i \in [n]\}$ of $V(M)$. We call $M_n$ an mnd graph.*

In the case $d_i = d$ ($i \in \mathbb{N}$), the graph $M_n(\mathbf{d})$ is just the $d^{\text{th}}$ power $P_n^d$.

**Theorem 1.7 (Holroyd, Spencer, Talbot [23])** *If $d \geq 1$ and $G$ is a $d^{\text{th}}$ power of a path, then $G$ is $r$-EKR for all $r \geq 1$.*

In [6], the $r$-EKR and strict $r$-EKR problems are solved for any mnd graph $M_n$ and any integer $r$ except for $r > \alpha(M_n)/2$ when $d_1 = 0$, and $\mathcal{I}_{M_n}^{(r)}$ is labeled *type I* iff the integers $n$, $r$ and $d_i$ ($i \in \mathbb{N}$) satisfy certain conditions (one of which is $d_1 = d_3 = 1$).

**Theorem 1.8 (Borg [6])** *Let* $\mathbf{d} = \{d_i\}_{i\in\mathbb{N}}$ *be an mnd sequence, and let* $M_n := M_n(\mathbf{d})$.
*(i) If* $d_1 > 0$ *and* $r \le \alpha(M_n)$*, then* $M_n$ *is* $r$*-EKR, and strictly so unless* $\mathcal{I}^{(r)}_{M_n}$ *is type I.*
*(ii) If* $d_1 = 0$ *and* $r \le \alpha(M_n)/2$*, then* $M_n$ *is* $r$*-EKR, and strictly so if* $r < \alpha(M_n)/2$.

We now come to our second important definition. We shall represent the vertices of $C_n$ by $v_1, ..., v_n$ and take $E(C_n)$ to be in the natural way, i.e. $E(C_n) = \{v_1v_2, ..., v_{n-1}v_n, v_nv_1\}$.

**Definition 1.9** *For* $n > 2$, $1 \le k < n-1$, $0 \le q < n$*, let* ${}_qC_n^{k,k+1}$ *be the graph with vertex set* $\{v_i \colon i \in [n]\}$ *and edge set* $E(C_n^k) \cup \{v_iv_{i+k+1 \text{ modulo } n} \colon 1 \le i \le q\}$.

If $q > 0$, then we call ${}_qC_n^{k,k+1}$ a *modified* $k^{\text{th}}$ power of a cycle; essentially it is a $k^{\text{th}}$ power for some of the cycle and a $(k+1)^{\text{th}}$ power for the remainder of the cycle.

A nice EKR-type result of Talbot [35] for *separated sets* can be stated as follows.

**Theorem 1.10 (Talbot [35])** *Let* $r \le \alpha(C_n^k)$*. Then* $C_n^k$ *is* $r$*-EKR, and strictly so unless* $k = 1$ *and* $n = 2r + 2$.

The *clique number* $\text{cl}(G)$ of a graph $G$ is the size of a largest complete sub-graph of $G$. Hilton and Spencer proved the following.

**Theorem 1.11 (Hilton and Spencer [22])** *Let* $G$ *be the disjoint union of graphs* $G_0, G_1, ..., G_n$ *such that* $\text{cl}(G_0) \le \min\{\text{cl}(G_i) \colon i \in [n]\}$*, where* $G_0$ *is a power of a path and* $G_i$ $(i \in [n])$ *is a power of a cycle. Then* $\mathcal{I}^{(r)}_G$ *is EKR for all* $r \le \alpha(G)$.

As we explain later, the work in this paper is inspired by the following result.

**Theorem 1.12 (Holroyd, Spencer, Talbot [23])** *Let* $G$ *be the disjoint union of* $n$ *connected components, each a complete graph, path, cycle or singleton, including at least one singleton. Then* $G$ *is* $r$*-EKR for all* $r \le n/2$.

Unlike all the preceding theorems, this result does not live up to Conjecture 1.13 (below), because for an arbitrary graph $G$, $\mu(G)$ is at least as large as the number of connected components of $G$ and may be much larger.

As we hinted earlier, the idea of the graph-theoretical formulation we have been discussing emerged in [24], in which Holroyd and Talbot initiated the study of the general EKR problem for independent sets of graphs and made the following conjecture.

**Conjecture 1.13 (Holroyd, Talbot [24])** *Let* $G$ *be any graph, and let* $r \le \mu(G)/2$*. Then* $G$ *is* $r$*-EKR, and strictly so if* $r < \mu(G)/2$.

By proving Theorem 1.4, they provided an example of a graph $G$ such that $G$ obeys the conjecture and, as we demonstrate in a stronger fashion below, $G$ may not be $r$-EKR if $\mu(G)/2 < r < \alpha(G)$ (it is easy to see that for such a graph $G$, $G$ is $r$-EKR for $r = \alpha(G)$). They gave various other examples of graphs $H$ and values $r > \mu(H)/2$ for which $H$ is *not* $r$-EKR, and one particularly interesting example of this kind has $r = \alpha(H)$. The idea behind Conjecture 1.13 is that if $I$ is any maximal independent set of a graph $G$ with $\mu(G) \geq 2r$, then, since $|I| \geq \mu(G)$, it holds by the EKR Theorem that $(I, \emptyset)$ (i.e. the empty graph with vertex set $I$) is $r$-EKR, and strictly so if $\mu(G) > 2r$.

We now show that there are graphs $G$ such that $\mu(G) < \alpha(G)$ and $G$ is not $r$-EKR for all $\mu(G)/2 < r < \alpha(G)$. Indeed, let $G$ be the graph consisting of a 3-set $V_0$ of singletons and a complete bipartite graph with partite sets $V_1$ and $V_2$ of sizes 5 and 4 respectively. So $7 = \mu(G) < \alpha(G) = 8$. For $r \in [\alpha(G)]$, let $\mathcal{J}_r$ be a star of $\mathcal{I}_G^{(r)}$ with centre $x \in V_0$, and let $\mathcal{A}_r := \{A \in \mathcal{I}_G^{(r)} : |A \cap V_0| \geq 2\}$. Clearly $\mathcal{J}_r$ is a star of $\mathcal{I}_G^{(r)}$ of largest size. However, for $\mu(G)/2 < r < \alpha(G)$, we have $|\mathcal{A}_r| > |\mathcal{J}_r|$. This proves what we set out to show.

Conjecture 1.13 seems very hard to prove or disprove. However, restricting the problem to some classes of graphs with singletons makes it tractable. Theorem 1.1 and the example that we gave above demonstrate the fact that when an arbitrary number of singletons are allowed in a graph $G$, $G$ may not be $r$-EKR for $r > \mu(G)/2$.

We now come to the objective of this paper, which is to provide an improvement of the techniques in [23] that enables us to confirm the conjecture for the class of graphs in Theorem 1.12 and even larger classes. The key idea that leads us to this improvement is to consider a suitable larger class of graphs, namely to allow copies of mnd graphs and modified powers of cycles in the disjoint union specified in Theorem 1.12. Since the proof goes by induction, we will need to perform certain deletions on the original graph. When a deletion is performed on a power of a cycle, which is the most difficult component to treat, we obtain a modified power of a cycle (mpc) or a power of a path, and if a deletion is performed on an mpc then we obtain an mnd graph or an mpc or a power of a cycle. So the idea is that every time a deletion is performed, the resulting graph is in the admissible class. Although not necessary for our main aim, we show that our method allows us to include *trees* (connected cycle-free graphs) as components; the scope is to illustrate the fact that the method we employ works for many classes of graphs.

**Theorem 1.14** *Conjecture 1.13 is true if G is a disjoint union of complete multipartite graphs, copies of mnd graphs, powers of cycles, modified powers of cycles, trees, and at least one singleton.*

Our method also allows to improve Theorem 1.12 beyond Conjecture 1.13.

**Theorem 1.15** *Let $G$ be a disjoint union of complete graphs, copies of mnd graphs, powers of cycles, modified powers of cycles, and at least one singleton. Let $r \leq \alpha(G)/2$. Then $G$ is r-EKR, and strictly so if $r < \alpha(G)/2$.*

Note that in this result we cannot include components like complete multipartite graphs or trees, because otherwise, as we have shown above, $G$ may not be $r$-EKR for $\mu(G)/2 < r \leq \alpha(G)/2$.

## 2 The compression operation

In the context of set combinatorics, a *compression operation* (or simply a *compression*) is a function that maps a family of sets to another family while retaining its size and (usually) some other important properties. Loosely speaking, a compression replaces a particular element of the ground set by another particular element whenever possible.

In the graph-theoretic context the ground set is $V(G)$ and we are interested in independent subsets of $V(G)$. The shift operation $\delta_{u,v}$ is defined on any such set as follows:

$$\delta_{u,v}(F) := \begin{cases} (F\backslash\{v\}) \cup \{u\} & \text{if } u \notin F, v \in F \text{ and } (F\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G; \\ F & \text{otherwise} \end{cases}$$

Then compression $\Delta_{u,v}$ acts on subfamilies of $\mathcal{I}_G$, as follows. Let $\mathcal{F}$ be a subfamily of $\mathcal{I}_G$. Then for each $A \in \mathcal{F}$, define

$$\Delta_{u,v}(\mathcal{F}) := \{\delta_{u,v}(A) \colon A \in \mathcal{F}\} \cup \{A \in \mathcal{F} \colon \delta_{u,v}(A) \in \mathcal{F}\}.$$

It should be clear that $\delta_{u,v}$ preserves the sizes of sets while $\Delta_{u,v}$ preserves the sizes of families of sets.

Let $G$ be a graph, $v \in V(G)$. We use $G - v$ to denote the graph obtained from $G$ by deleting $v \in V(G)$ (and hence edges incident to $v$), and $G \downarrow v$ to denote the graph obtained by deleting also all vertices in $N_G(v)$ (and incident edges). Next, for any $\mathcal{F} \subseteq \mathcal{I}_G$, we define the following subfamilies of $\mathcal{F}$:

$$\mathcal{F}\langle v\rangle := \{A\backslash\{v\} \colon A \in \mathcal{F}(v)\} \subseteq \mathcal{I}_{G\downarrow v}, \quad \mathcal{F}\overline{(v)} := \{A \in \mathcal{F} \colon v \notin A\} \subseteq \mathcal{I}_{G-v}.$$

**Lemma 2.1** *Let $uv \in E(G)$. Let $\mathcal{F} \subset \mathcal{I}_G^{(r)}$ be an intersecting family, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. Then:*
*(i) $\mathcal{A}\overline{(v)}$ is intersecting;*
*(ii) if $|N_G(u)\backslash\hat{N}_G(v)| \leq 1$ then $\mathcal{A}\langle v\rangle$ is intersecting;*
*(iii) if $N_G(u)\backslash\hat{N}_G(v) = \emptyset$, then $\mathcal{A}$ and $\mathcal{A}\overline{(v)} \cup \mathcal{A}\langle v\rangle$ are intersecting.*

**Proof.** We begin with the observation that since $uv \in E(G)$, the 2-set $\{u, v\}$ is not contained in any set of $\mathcal{I}_G$, and hence $\mathcal{F}$ may be partitioned as $\bigcup_{i=1}^{5} \mathcal{F}_i$ where

$\mathcal{F}_1 := \{F \in \mathcal{F} \colon u \in F, v \notin F\}$,
$\mathcal{F}_2 := \{F \in \mathcal{F} \colon \{u, v\} \cap F = \emptyset\}$,
$\mathcal{F}_3 := \{F \in \mathcal{F} \colon v \in F, u \notin F$ and $(F\backslash\{v\}) \cup \{u\} \in \mathcal{F}_1\}$,
$\mathcal{F}_4 := \{F \in \mathcal{F} \colon v \in F, u \notin F$ and $(F\backslash\{v\}) \cup \{u\} \notin \mathcal{I}_G\}$,
$\mathcal{F}_5 := \{F \in \mathcal{F} \colon v \in F, u \notin F$ and $(F\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G\backslash\mathcal{F}_1\}$.

Moreover, $\mathcal{A} = \bigcup_{i=1}^{4} \mathcal{F}_i \cup \mathcal{A}_5$ where $\mathcal{A}_5 := \{(F\backslash\{v\}) \cup \{u\} \colon F \in \mathcal{F}_5\}$.

Note that $\mathcal{A}\overline{(v)} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{A}_5$. Since $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{A}_5$ are each intersecting, to prove (i) we need merely verify that if $A \in \mathcal{F}_1 \cup \mathcal{F}_2, B \in \mathcal{A}_5$, then $A \cap B \neq \emptyset$. Now consider the set $C \in \mathcal{F}_5$ such that $(C\backslash\{v\}) \cup \{u\} = B$. Since $\mathcal{F}$ is intersecting, there exists $x \in V(G)\backslash\{v\}$ such that $x \in A \cap C$. So $x \in A \cap B$. Hence (i).

We next prove (ii). So suppose $|N_G(u)\backslash\hat{N}_G(v)| \leq 1$. Clearly $\mathcal{A}\langle v\rangle = (\mathcal{F}_3 \cup \mathcal{F}_4)\langle v\rangle$. If $A \in \mathcal{F}_3$, then the set $A' := A\backslash\{v\} \cup \{u\}$ is in $\mathcal{F}_1$, and hence, for any $F \in \mathcal{F}_3 \cup \mathcal{F}_4$, $(A \cap F)\backslash\{v\} = (A' \cap F)\backslash\{v\} \neq \emptyset$ (as $u \notin F$ and $\mathcal{F}$ is intersecting). Thus we need merely show that $\mathcal{F}_4\langle v\rangle$ is intersecting. If $N_G(u)\backslash\hat{N}_G(v) = \emptyset$, then $\mathcal{F}_4 = \emptyset$, as $(A\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G$

whenever $A \in \mathcal{I}_G$ and $v \in A$. If $N_G(u) \backslash \hat{N}_G(v) = \{x\}$ for some $x \in V(G)$, then $x \neq v$ and every set $F \in \mathcal{F}_4$ must own $x$; thus $\mathcal{F}_4\langle v \rangle$ is indeed intersecting.

We finally prove (iii). So suppose $N_G(u) \backslash \hat{N}_G(v) = \emptyset$. Thus $\mathcal{F}_4 = \emptyset$. Clearly, $\bigcup_{i=1}^3 \mathcal{F}_i$ and $\mathcal{A}_5$ are intersecting. Thus, to show that $\mathcal{A}$ is intersecting, we must show that if $A \in \bigcup_{i=1}^3 \mathcal{F}_i, B \in \mathcal{A}_5$, then $A \cap B \neq \emptyset$. The set $C := (B \backslash \{u\}) \cup \{v\}$ is in $\mathcal{F}$ and so $A \cap C \neq \emptyset$. Suppose $A \cap C = \{v\}$. Then $A \in \mathcal{F}_3$; but then $D := (A \backslash \{v\}) \cup \{u\}$ is in $\mathcal{F}_1$ and $D \cap C = \emptyset$, a contradiction. So $(A \cap C) \backslash \{v\} \neq \emptyset$ and hence $A \cap B \neq \emptyset$. Therefore $\mathcal{A}$ is intersecting. By (ii), it follows that $\mathcal{A}\overline{(v)} \cup \mathcal{A}\langle v \rangle$ is intersecting. □

# 3 Vertex deletion lemmas

It frequently happens that a vertex of a graph may be deleted without decreasing $\mu$ or $\alpha$. This is important to our improvement of Theorem 1.12; in this section we develop several vertex deletion lemmas that will be employed in the proofs of Theorems 1.14 and 1.15.

**Lemma 3.1** *Let $G$ be a graph, and let $v \in V(G)$. Then*

$$\min\{\mu(G \downarrow v), \mu(G - v)\} \geq \mu(G) - 1.$$

**Proof:** Let $Z$ be a maximal independent set of $G \downarrow v$ of minimum size; then $Z \cup \{v\}$ is a maximal independent set of $G$, hence $\mu(G \downarrow v) \geq \mu(G) - 1$. Now let $Z$ be a maximal independent set of $G - v$. If $Z$ is not maximal in $G$, then $Z \cup \{v\}$ is. Thus $\mu(G - v) \geq \mu(G) - 1$. □

**Corollary 3.2** *Let $r \leq \frac{1}{2}\mu(G)$, and let $v, w \in V(G)$. Then:*
*(i) $r - 1 < \frac{1}{2}\mu(G \downarrow v)$;*
*(ii) $r - 1 \leq \frac{1}{2}\mu((G - v) \downarrow w)$.*

**Proof.** Lemma 3.1 implies:
(i) $r - 1 < \frac{1}{2}(\mu(G) - 1) \leq \frac{1}{2}\mu(G \downarrow v)$;
(ii) $r - 1 \leq \frac{1}{2}(\mu(G) - 2) \leq \frac{1}{2}(\mu(G - v) - 1) \leq \frac{1}{2}\mu((G - v) \downarrow w)$. □

The next lemma relies on a well-known property of trees: any tree other than a singleton has a vertex with only one neighbour.

**Lemma 3.3** *Let $T$ be a tree with $|V(T)| \geq 2$, and let $w \in V(T)$ such that $N_T(w)$ consists only of one vertex $v$. Then*

$$\mu(T - v) \geq \mu(T).$$

**Proof.** Let $Z$ be a maximal independent set of $T - v$. Since $w$ is a singleton of $T - v$, we must have $w \in Z$. So $Z$ is also a maximal independent set of $T$ because $vw \in E(T)$. Thus $\mu(T - v) \geq \mu(T)$. □

**Lemma 3.4** *Let $M_n(\mathbf{d})$ be as in Definition 1.6, and let $M_n := M_n(\mathbf{d})$. Let $d_1 > 0$. Then*
*(i)* $\mu(M_n - x_2) \geq \mu(M_n)$;
*(ii)* $\alpha(M_n - x_2) \geq \alpha(M_n)$;
*(iii)* $\alpha(M_n \downarrow x_2) \geq \alpha(M_n) - 2$.

**Proof.** Let $Z$ be a maximal independent set of $M_n - x_2$. Then $x_1 \in Z$ or $x_1x_z \in E(M_n - x_2)$ for some $x_z \in Z$. Suppose $x_1 \in Z$. Since $d_1 > 0$, we have $x_1x_2 \in E(M_n)$, and hence $Z$ is a maximal independent set of $M_n$. Now suppose $x_1x_z \in E(M_n - x_2)$ for some $x_z \in Z$. Then, by definition of $M_n$, $z \leq 1 + d_1 < 2 + d_2$, and hence $x_2x_z \in E(M_n)$. Thus, $Z$ is again a maximal independent set of $M_n$. Hence (i).

Now let $I$ be an arbitrary independent set of $M_n$. If $x_2 \notin I$ then $I$ is an independent set of $M_n - x_2$. Suppose $x_2 \in I$ instead. Since $d_1 > 0$, $x_1 \notin I$. It is therefore easy to see that $\{x_{j-1} \colon j \in [n], x_j \in I\}$ is an independent set of $M_n - x_2$ of size $|I|$. Hence (ii).

Clearly $I$ can contain at most 2 vertices in $V(M_n)\backslash V(M_n \downarrow x_2)$. Hence (iii). □

**Lemma 3.5** *Let ${}_qC_n^{k,k+1}$ be as in Definition 1.9, and let $q > 0$. Then:*
*(i)* $\mu({}_qC_n^{k,k+1} - v_{k+2}) \geq \mu({}_qC_n^{k,k+1})$;
*(ii)* $\alpha({}_qC_n^{k,k+1} - v_{k+2}) \geq \alpha({}_qC_n^{k,k+1})$;
*(iii)* $\alpha({}_qC_n^{k,k+1} \downarrow v_{k+2}) \geq \alpha({}_qC_n^{k,k+1}) - 2$.

**Proof.** Let $C := {}_qC_n^{k,k+1}$ and $V := V(C)$. If $N_C(v_1) = V\backslash\{v_1\}$ then trivially $\mu(C - v_{k+2}) = \mu({}_qC_n^{k,k+1}) = 1$. So suppose $N_C(v_1) \neq V\backslash\{v_1\}$. Let $Z$ be a maximal independent set of $C - v_{k+2}$, and let $s := \min\{i \colon v_i \in Z\}$, $t := \max\{i \colon v_i \in Z\}$. If $s \leq k+1$ then $v_sv_{k+2} \in E(C)$, and hence $Z$ is also maximal in $C$. Suppose $s \geq k+3$. Suppose also that $v_{k+2}v_s \notin E(C)$. Then $v_{k+1}v_s \notin E(C - v_{k+2})$ and, since $q < n$ (by definition of $C$) and $s \leq t \leq n$, $v_tv_{k+1} \notin E(C - v_{k+2})$. So $Z \cup \{v_{k+1}\} \in \mathcal{I}_{C - v_{k+2}}$, but this contradicts the maximality of $Z$. So $v_{k+2}v_s \in E(C)$, and hence $Z$ is also maximal in $C$. Hence (i).

Now let $I$ be an arbitrary independent set of $C$. If $v_{k+2} \notin I$ then $I$ is an independent set of $C - v_{k+2}$. Suppose $v_{k+2} \in I$ instead. Note that $v_1 \notin I$ as $v_1v_{k+2} \in E(C)$. By construction of $C$, $\{v_{j-1} \colon j \in [n], v_j \in I\}$ is an independent set of $C - v_{k+2}$ of size $|I|$. Hence (ii).

Clearly $I$ can contain at most 2 vertices in $V(C)\backslash V(C \downarrow v_{k+2})$. Hence (iii). □

**Lemma 3.6** *Let $n \geq 2k+2$. Then:*
*(i)* $\mu(C_n^k - v_{k+1} - v_{2k+2}) \geq \mu(C_n^k)$;
*(ii)* $\alpha(C_n^k - v_{k+1} - v_{2k+2}) \geq \alpha(C_n^k)$;
*(iii)* $\alpha(C_n^k \downarrow v_{k+1}) \geq \alpha(C_n^k) - 2$.

**Proof.** Let $Z$ be a maximal independent set of $C_n^k - v_{k+1} - v_{2k+2}$. If $Z$ contains $z \in \{v_{k+2}, ..., v_{2k+1}\}$ then $zv_{k+1}, zv_{2k+2} \in E(C_n^k)$, and hence $Z$ is also maximal in $C_n^k$. Now consider $Z \cap \{v_{k+2}, ..., v_{2k+1}\} = \emptyset$. Thus, if $zv_{k+1}, zv_{2k+2} \notin E(C_n^k)$ for all $z \in Z$ then $Z \cup \{v\}$ is an independent set of $C - v_{k+1} - v_{2k+2}$ for all $v \in \{v_{k+2}, ..., v_{2k+1}\}$, but this is a contradiction. We therefore have $zw \in E(C_n^k)$ for some $z \in Z$ and $w \in \{v_{k+1}, v_{2k+1}\}$. Suppose $w = v_{k+1}$ and $Z \cup \{v_{2k+2}\}$ is an independent set of $C_n^k$. Then $zv_{2k+1} \notin E(C_n^k -$

$v_{k+1} - v_{2k+2}$), and hence $Z \cup \{v_{2k+1}\}$ is an independent set of $C_n^k - v_{k+1} - v_{2k+2}$, a contradiction. By symmetry, we can neither have both $w = v_{2k+2}$ and $Z \cup \{v_{k+1}\}$ an independent set of $C_n^k$. Therefore there exist $z_1, z_2 \in Z$ such that $z_1v_{k+1}, z_2v_{2k+2} \in E(C_n^k)$, and hence $Z$ is maximal in $C_n^k$. Hence (i).

(ii) and (iii) follow by the same arguments for the corresponding parts in Lemma 3.5. □

# 4 Proof of Theorem 1.14

We shall now use the lower bounds obtained in Lemmas 3.4, 3.5 and 3.6 to prove Theorem 1.14. Before proceeding to the main proof, we need two straightforward lemmas concerning stars.

We remark that whenever we use a notation of the kind $\mathcal{F}(x)(y)$ we mean the family $(\mathcal{F}(x))(y)$, which, according to the notation we set up earlier, is the family $\{A \in \mathcal{F}(x) \colon y \in A\}$ $(= \{A \in \mathcal{F} \colon x, y \in A\})$. The same applies for notation like $\mathcal{F}(x)\overline{(y)}$, $\mathcal{F}(x)\langle y \rangle$, etc.

**Lemma 4.1** *Let $G$ be a graph containing an edge $vw$ and a singleton $x$. Suppose $2 \leq r \leq \alpha(G)$. Then $|\mathcal{I}_G^{(r)}(v)| \leq |\mathcal{I}_G^{(r)}(x)|$, and the inequality is strict if $r \leq \mu(G)$.*

**Proof.** Since $x$ is a singleton, $A \backslash \{y\} \cup \{x\} \in \mathcal{I}_G^{(r)}$ for any $A \in \mathcal{I}_G^{(r)}\overline{(x)}$ and $y \in A$. Setting $\mathcal{J} := \{A \backslash \{v\} \cup \{x\} \colon A \in \mathcal{I}_G^{(r)}(v)\overline{(x)}\}$, it follows that $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)}$. Given that $vw \in E(G)$, we have $\mathcal{I}_G(v)(w) = \emptyset$, and hence actually $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)} \backslash \mathcal{I}_G^{(r)}(x)(w)$; also, $\mathcal{I}_G^{(r)}(x)(w) \subseteq \mathcal{I}_G^{(r)}(x)\overline{(v)}$, and hence $|\mathcal{J}| \leq |\mathcal{I}_G^{(r)}(x)\overline{(v)}| - |\mathcal{I}_G^{(r)}(x)(w)|$. We therefore have

$$\begin{aligned} |\mathcal{I}_G^{(r)}(v)| &= |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{I}_G^{(r)}(v)\overline{(x)}| = |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{J}| \\ &\leq |\mathcal{I}_G^{(r)}(x)(v)| + |\mathcal{I}_G^{(r)}(x)\overline{(v)}| - |\mathcal{I}_G^{(r)}(x)(w)| \\ &= |\mathcal{I}_G^{(r)}(x)| - |\mathcal{I}_G^{(r)}(x)(w)|. \end{aligned}$$

Now suppose $r \leq \mu(G)$. Since $\{x, w\} \in \mathcal{I}_G^{(2)}$, there exists $I \in \mathcal{I}_G^{(r)}$ such that $\{x, w\} \subset I$, i.e. $\mathcal{I}_G^{(r)}(x)(w) \neq \emptyset$. Thus $|\mathcal{I}_G^{(r)}(v)| < |\mathcal{I}_G^{(r)}(x)|$. □

**Lemma 4.2** *Let $G$ be a graph with $\mu(G) \geq 2r$. Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{I}_G^{(r)}$ such that $\mathcal{A}\langle v \rangle = \mathcal{I}_{G \downarrow v}^{(r-1)}(y) \neq \emptyset$ for some $y \in V(G \downarrow v)$. Then $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$.*

**Proof.** Suppose there exists $A \in \mathcal{A}\overline{(v)}$ such that $y \notin A$. We are given that $\mathcal{I}_{G \downarrow v}^{(r-1)}(y) \neq \emptyset$, and so $\mathcal{I}_G^{(r)}(v)(y) \neq \emptyset$. Therefore there exists a maximal independent set $Y$ of $G$ such that $v, y \in Y$. Given that $2r \leq \mu(G)$, we have $2r \leq |Y|$. Since $y, v \in Y \backslash A$, it follows that there exists an $r$-subset $A'$ of $Y \backslash A$ containing $\{y, v\}$. So $A' \backslash \{v\} \in \mathcal{I}_{G \downarrow v}^{(r-1)}(y)$, and hence $A' \in \mathcal{A}(v)$. But $A \cap A' = \emptyset$, which contradicts $\mathcal{A}$ intersecting. Hence result. □

**Proof of Theorem 1.14.** The result is trivial for $r = 1$, so we assume $r \geq 2$ and use induction on $|E(G)|$. If $|E(G)| = 0$ then the result is given by Theorem 1.1, so we assume that $|E(G)| > 0$. This means that $G$ contains a non-singleton component. If $G$ consists solely of complete multipartite graphs and singletons then the result is given by Theorem 1.5. We now consider the case when $G$ contains a connected component $G_1$ that is neither a singleton nor a complete mulitpartite graph.

Let $G_2$ be the graph obtained by removing $G_1$ from $G$. Note that

$$\mu(G) = \mu(G_1) + \mu(G_2).$$

Since $G_1$ contains no singletons and $G$ contains at least one singleton, $G_2$ contains some singleton $x$.

Let $r \leq \mu(G)/2$, and let $\mathcal{F}$ be an extremal intersecting sub-family of $\mathcal{I}_G^{(r)}$. Let $\mathcal{J} := \mathcal{I}_G^{(r)}(x)$. So $|\mathcal{J}| \leq |\mathcal{F}|$. Lemma 4.1 tells us that $\mathcal{J}$ is a largest star of $\mathcal{I}_G^{(r)}$ and that, for any $v \in V(G_1)$, $\mathcal{J}\langle v\rangle$ and $\mathcal{J}\overline{(v)}$ are largest stars of $\mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{I}_{G-v}^{(r)}$ respectively.

Now $G_1$ is one of the following: a tree, a copy of an mnd graph, a modified power of a cycle, a power of a cycle. We consider each of these four possibilities separately and in the order we have listed them. We will actually show that in each of the first three cases, $G$ is in fact strictly $r$-EKR even if $r = \mu(G)/2$.

*Case I: $G_1$ is a tree $T$, $|V(T)| \geq 2$.* So there exists $u \in V(G_1)$ such that $N_{G_1}(u)$ consists solely of one vertex $v$ (see the preceding section). Let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. Since $N_G(u) = N_{G_1}(u) = \{v\}$, it follows by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

Since $G_1$ contains no cycles, $G_1 - v$ and $G_1 \downarrow v$ contain no cycles, and hence $G_1 - v$ and $G_1 \downarrow v$ are disjoint unions of trees and singletons. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

By Corollary 3.2(i), $r - 1 < \mu(G \downarrow v)/2$. By Lemma 3.3, $\mu(G_1 - v) \geq \mu(G_1)$; so $\mu(G - v) = \mu(G_1 - v) + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{A}\langle v\rangle \subset \mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{A}\overline{(v)} \subset \mathcal{I}_{G-v}^{(r)}$, the inductive hypothesis gives us $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$. So $|\mathcal{A}| \leq |\mathcal{J}|$. Since $|\mathcal{F}| = |\mathcal{A}|$ and $\mathcal{F}$ is extremal, $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}\overline{(v)}| = |\mathcal{J}\overline{(v)}|$. Since $r - 1 < \mu(G \downarrow v)/2$, it follows by the inductive hypothesis that $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}^{(r-1)}(y)$ for some $y \in V(G \downarrow v)$. Thus, by Lemma 4.2, $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$. If $y$ is not a singleton of $G$ then Lemma 4.1 gives us $|\mathcal{I}_G^{(r)}(y)| < |\mathcal{J}|$, but this leads to the contradiction that $|\mathcal{F}| < |\mathcal{J}|$. So $y$ is a singleton of $G$, and hence $\mathcal{F} \subseteq \mathcal{I}_G^{(r)}(y)$ (as $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$). Therefore $G$ is strictly $r$-EKR.

*Case II: $G_1$ is an mnd graph $M_n := M_n(\mathbf{d})$.* Since $G_1$ contains no singletons, $n \geq 2$ and $d_1 \geq 1$. Let $v := x_2$ and $u := x_1$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. By definition of $M_n$ and $d_1 \geq 1$, $N_{G_1}(u) \subset \hat{N}_{G_1}(v)$. Since $N_G(u) = N_{G_1}(v)$, it follows by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

Clearly, $G_1 - v$ is a copy of $M_{n-1}(\{d_i'\}_{i\in\mathbb{N}})$, where $d_1' = d_1 - 1$ and $d_i' = d_{i+1}$ for all $i \geq 2$. Also, if $n \leq 2 + d_2$ then $G_1 \downarrow v = (\emptyset, \emptyset)$, and if $n > 2 + d_2$ then $G_1 \downarrow v$ is a copy of $M_{n-2-d_2}(\{d_i''\}_{i\in\mathbb{N}})$ where $d_i'' = d_{i+2+d_2}$ for all $i \geq 1$. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in the preceding case, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 3.4(i).

*Case III: $G_1$ is a modified k'th power of a cycle, i.e. $G_1 = {}_qC_n^{k,k+1}$ for some $q > 0$.* We set $u := v_{k+1}$ and $v := v_{k+2}$, and we note that the condition $q < n$ in the definition of ${}_qC_n^{k,k+1}$ implies $N_{G_1}(u) \subseteq \hat{N}_{G_1}(v)$ and hence $N_G(u) \subseteq \hat{N}_G(v)$. Thus, for $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, we know by Lemma 2.1(iii) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting.

If $n = k+2$ then $G_1 = K_n$, which is a special complete multipartite graph; contradiction. So $n \geq k+3$.

Suppose $v_{k+3}v_1 \in E(G_1)$. It is easy to see that we then have $\hat{N}_{G_1}(v) = V(G_1) = \hat{N}_{G_1}(v_1)$, which gives $\mu(G_1 - v) = \mu(G_1) = 1$ and $G_1 \downarrow v = (\emptyset, \emptyset)$. Thus, by the same line of argument for the preceding cases, we conclude that $G$ is strictly $r$-EKR.

So suppose $v_{k+3}v_1 \notin E(G_1)$. Then $V(G_1 \downarrow v) = \{v_m, ..., v_n\}$ where

$$m = \begin{cases} 2k+3 & \text{if } q < k+2; \\ 2k+4 & \text{if } q \geq k+2. \end{cases}$$

Let $n' := n - m + 1$. By considering the bijection $\beta\colon V(G_1 \downarrow v) \to \{x_j : j \in [n']\}$ defined by $\beta(v_l) = x_{n-l+1}$ ($l \in [m, n]$), one can see that $G_1 \downarrow v$ is a copy of $M_{n'}(\{d_i\}_{i\in\mathbb{N}})$ where

$$d_i = \begin{cases} k & \text{if } i \leq n - (q+k+1); \\ k+1 & \text{if } i > n - (q+k+1). \end{cases}$$

It is also not difficult to check that

$$G_1 - v \text{ is a copy of } \begin{cases} {}_{n+q-k-2}C_{n-1}^{k-1,k} & \text{if } q < k+1; \\ C_{n-1}^{k} & \text{if } q = k+1; \\ {}_{q-k-2}C_{n-1}^{k,k+1} & \text{if } q > k+1. \end{cases}$$

So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in Case I, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 3.5(i).

*Case IV: $G_1$ is a $k^{\text{th}}$ power of a cycle $C_n$, i.e. $G_1 = C_n^k$.* Let $u := v_k$ and $v := v_{k+1}$. If $n < 2k+2$ then $G_1 = K_n$, which is a special complete multipartite graph; contradiction. So $n \geq 2k+2$. Let $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$. Since $N_G(u) \backslash \hat{N}_G(v) = \{v_n\}$, Lemma 2.1(ii) tells us that $\mathcal{A}\langle v\rangle$ and $\mathcal{A}\overline{(v)}$ are intersecting.

Clearly, $G_1 \downarrow v$ is a power of a path. As in Case I, it follows that $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$.

Now $G_1 - v$ is a path (if $k = 1$) or a copy of ${}_{n-k-1}C_{n-1}^{k-1,k}$ (if $k > 1$); however, we are not guaranteed that $\mu(G_1 - v) \geq \mu(G_1)$ (this is the case if, for example, $G_1 = C_4^1$). Let $\mathcal{G} := \mathcal{A}\overline{(v)}$. Let $u' := v_{2k+1}$ and $v' := v_{2k+2}$, and let $\mathcal{B} := \Delta_{u',v'}(\mathcal{G})$. Clearly, $N_{G-v}(u') = N_{G_1-v}(u') \subset \hat{N}_{G_1}(v')$. Thus, by Lemma 2.1(ii), $\mathcal{B}\langle v'\rangle \cup \mathcal{B}\overline{(v')}$ is intersecting.

If $k = 1$ then $G_1 - v - v'$ is a disjoint union of a path and a singleton, and if $k > 1$ then $G_1 - v - v'$ is a copy of ${}_{n-2k-2}C_{n-2}^{k-1,k}$. It is easy to see that $G_1 - v \downarrow v'$ is a power of a path. So $G - v - v'$ and $G - v \downarrow v'$ belong to the class of graphs specified in the theorem.

By Corollary 3.2(ii), $r - 1 \leq \mu(G - v \downarrow v')/2$. By Lemma 3.3, $\mu(G_1 - v - v') \geq \mu(G_1)$; so $\mu(G - v - v') = \mu(G_1 - v - v') + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{B}\langle v'\rangle \subset \mathcal{I}^{(r-1)}_{G-v\downarrow v'}$ and $\mathcal{B}\overline{(v')} \subset \mathcal{I}^{(r)}_{G-v-v'}$, the inductive hypothesis gives us $|\mathcal{B}\langle v'\rangle| \leq |\mathcal{J}\overline{(v)}\langle v'\rangle|$ and $|\mathcal{B}\overline{(v')}| \leq |\mathcal{J}\overline{(v)}\overline{(v')}|$. So $|\mathcal{G}| = |\mathcal{B}| \leq |\mathcal{J}\overline{(v)}|$. Since $\mathcal{F} = |\mathcal{A}| = |\mathcal{A}\langle v\rangle| + |\mathcal{G}| \leq |\mathcal{J}\langle v\rangle| + |\mathcal{J}\overline{(v)}|$, we have $|\mathcal{F}| \leq |\mathcal{J}|$, and hence $G$ is $r$-EKR.

Now suppose $r < \mu(G)/2$. Since $|\mathcal{F}| = |\mathcal{A}|$ and $\mathcal{F}$ is extremal, we must have $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{G}| = |\mathcal{J}\overline{(v)}|$. By Corollary 3.2(i), we have $r - 1 < \mu(G \downarrow v)/2$, and hence, by the inductive hypothesis, $\mathcal{A}\langle v\rangle = \mathcal{I}^{(r-1)}_{G\downarrow v}(y_1)$ for some $y_1 \in V(G \downarrow v) \subset V(G)\backslash\{u, v\}$. Since $|\mathcal{G}| = |\mathcal{J}\overline{(v)}|$, we have $|\mathcal{B}\langle v'\rangle| = |\mathcal{J}\overline{(v)}\langle v'\rangle|$ and $|\mathcal{B}\overline{(v')}| = |\mathcal{J}\overline{(v)}\overline{(v')}|$. Given that $r < \mu(G)/2$, we have $r - 1 < (\mu(G) - 2)/2 \leq \mu(G - v \downarrow v')/2$ by Lemma 3.1. Thus, by the inductive hypothesis, $\mathcal{B}\langle v'\rangle = \mathcal{I}^{(r-1)}_{G-v\downarrow v}(y_2)$ for some $y_2 \in V(G - v \downarrow v')$. By Lemma 4.2, $\mathcal{B} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$. We next show that $y_1 = y_2$.

If $y_2$ is not a singleton of $G - v$ then Lemma 4.1 gives us $|\mathcal{I}^{(r)}_{G-v}(y_2)| < |\mathcal{J}\overline{(v)}|$, but this leads to the contradiction that $|\mathcal{G}| < |\mathcal{J}\overline{(v)}|$. So $y_2$ is a singleton of $G - v$, and hence, since $G_1 - v$ contains no singletons, $y_2 \in V(G)\backslash V(G_1) \subset V(G)\backslash\{u, v, u', v'\}$. Note that, by definition of $\mathcal{B}$, $\mathcal{B}(v') \subseteq \mathcal{G}$. Thus, since $\mathcal{B}\langle v'\rangle = \mathcal{I}^{(r-1)}_{G-v\downarrow v'}(y_2)$, we have $\mathcal{V} := \mathcal{I}^{(r)}_{G-v}(y_2)(v') \subseteq \mathcal{G}$. Suppose $y_1 \neq y_2$. Let $A_1 \in \{I \in \mathcal{V} \colon u, y_1 \notin I\}$ (note that $A_1$ exists since $y_2$ is a singleton of $G - v$ and, by Lemma 3.1, $\mu(G - v) \geq \mu(G) - 1 \geq 2r - 1$). So $A_1 \in \mathcal{G}$, $\{u, v\} \cap A_1 = \emptyset$, and hence $A_1 \in \mathcal{F}$. Recall that $y_1 \in V(G \downarrow v)$, which means that $y_1 v \notin E(G)$; let $Y$ be a maximal independent set of $G$ containing $y_1$ and $v$. Since $2r \leq \mu(G) \leq |Y|$ and $\{y_1, v\} \cap A_1 = \emptyset$, the family $\mathcal{Y} := \{A \in \binom{Y\backslash A_1}{r} \colon y_1, v \in A\}$ is non-empty. Let $A_2 \in \mathcal{Y}$; note that $A_2 \in \mathcal{I}^{(r)}_G(y_1)(v)$. Since $\mathcal{A}\langle v\rangle = \mathcal{I}^{(r-1)}_{G\downarrow v}(y_1)$, we have $\mathcal{A}(v) = \mathcal{I}^{(r)}_G(y_1)(v)$ and hence $A_2 \in \mathcal{A}(v)$. Now, by definition of $\mathcal{A}$, $\mathcal{A}(v) \subseteq \mathcal{F}$. Hence $A_2 \in \mathcal{F}$. But $A_1 \cap A_2 = \emptyset$, which contradicts $\mathcal{F}$ intersecting. So $y_1 = y_2$ indeed.

Since $y_2 \notin \{u', v'\}$ and $\mathcal{B} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$, we clearly have $\mathcal{G} \subseteq \mathcal{I}^{(r)}_{G-v}(y_2)$. So we have $\mathcal{F} = \mathcal{A}(v) \cup \mathcal{G} \subseteq \mathcal{I}^{(r)}_G(y_2)$. This proves that $G$ is strictly $r$-EKR. $\square$

# 5 Proof of Theorem 1.15

Theorem 1.15 is trivial for $r = 1$, so we assume $r \geq 2$ and prove the result by induction on $|E(G)|$. If $|E(G)| = 0$ then the result is given by Theorem 1.1, so we assume that $|E(G)| > 0$. This means that $G$ contains a non-singleton component $G_1$. Let $G_2$ be the graph obtained by removing $G_1$ from $G$. Note that

$$\alpha(G) = \alpha(G_1) + \alpha(G_2).$$

Since $G_1$ contains no singletons and $G$ contains at least one singleton, $G_2$ contains some singleton $x$.

Let $r \leq \alpha(G)/2$, and let $\mathcal{F}$ be an extremal intersecting sub-family of $\mathcal{I}^{(r)}_G$. Let $\mathcal{J} := \mathcal{I}^{(r)}_G(x)$. So $|\mathcal{J}| \leq |\mathcal{F}|$. By Lemma 4.1, $\mathcal{J}$ is a largest star of $\mathcal{I}^{(r)}_G$, and, for any $v \in V(G_1)$, $\mathcal{J}\langle v\rangle$ and $\mathcal{J}\overline{(v)}$ are largest stars of $\mathcal{I}^{(r-1)}_{G\downarrow v}$ and $\mathcal{I}^{(r)}_{G-v}$ respectively.

Note that a complete graph is an mnd graph, so we need to consider the following possible cases for $G_1$.

*Case I: $G_1$ is an mnd graph $M_n := M_n(\mathbf{d})$.* As in Case II of the Proof of Theorem 1.15, we take $v := x_2$, $u := x_1$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

By (ii) and (iii) of Lemma 3.4, we have $\alpha(G_1 - v) \geq \alpha(G_1)$ and $\alpha(G_1 \downarrow v) \geq \alpha(G_1) - 2$; so $\alpha(G - v) = \alpha(G_1 - v) + \alpha(G_2) \geq \alpha(G_1) + \alpha(G_2) = \alpha(G) \geq 2r$ and $\alpha(G \downarrow v) = \alpha(G_1 \downarrow v) + \alpha(G_2) \geq \alpha(G_1) - 2 + \alpha(G_2) = \alpha(G) - 2 \geq 2r - 2 = 2(r-1)$. Therefore, since $\mathcal{A}\langle v\rangle \subset \mathcal{I}^{(r-1)}_{G\downarrow v}$ and $\mathcal{A}\overline{(v)} \subset \mathcal{I}^{(r)}_{G-v}$, the inductive hypothesis gives us $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$. So $|\mathcal{F}| = |\mathcal{A}| \leq |\mathcal{J}|$, and hence $G$ is $r$-EKR.

*Case II: $G_1$ is a modified k'th power of a cycle, i.e. $G_1 = {}_qC_n^{k,k+1}$ for some $q > 0$.* As in Case III of the Proof of Theorem 1.15, we take $u := v_{k+1}$, $v := v_{k+2}$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v\rangle \cup \mathcal{A}\overline{(v)}$ is intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem. The rest follows as in Case I, except that we use Lemma 3.5 instead of Lemma 3.4.

*Case III: $G_1$ is a $k^{\text{th}}$ power of a cycle $C_n$, i.e. $G_1 = C_n^k$.* As in Case IV of the Proof of Theorem 1.15, we take $u := v_k$, $v := v_{k+1}$ and $\mathcal{A} := \Delta_{u,v}(\mathcal{F})$, and we obtain that $\mathcal{A}\langle v\rangle$ and $\mathcal{A}\overline{(v)}$ are intersecting and that $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem. As in Case I, we get $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$, $|\mathcal{A}\overline{(v)}| \leq |\mathcal{J}\overline{(v)}|$ and hence $|\mathcal{F}| = |\mathcal{A}| \leq |\mathcal{J}|$; the only difference is that we use Lemma 3.6 instead of Lemma 3.4. So $G$ is $r$-EKR. □

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