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Singularities of meromorphic functions with Baker domains

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Abstract

We show that if f is a transcendental meromorphic function with a finite number of poles and f has a cycle of Baker domains of period p, then there exist $C > 1$ and $r_0 > 0$ such that

$$\left\{z : \frac{1}{C}r < |z| < Cr\right\} \cap \operatorname{sing}(f^{-p}) \neq \varnothing, \quad \text{for } r \geqslant r_0.$$

We also give examples to show that this result fails for transcendental meromorphic functions with infinitely many poles.

1. *Introduction*

Let f be a meromorphic function which is not rational of degree one and denote by $f^n, n \in \mathbb{N}$, the nth iterate of f. The Fatou set, $F(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that $(f^n)_{n\in\mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z. The complement, $J(f)$, of $F(f)$ is called the Julia set of f. An introduction to the properties of these sets can be found in, for example, [**6**] for rational functions and in [**7**] for transcendental meromorphic functions.

The set $F(f)$ is completely invariant so for any component U of $F(f)$ there exists, for each $n \in \mathbb{N}$, a component U_n of $F(f)$ such that $f^n(U) \subset U_n$. If $U_p = U$, for some minimal $p \in \mathbb{N}$, then we say that U is a periodic component of *period* p, and there are then five possible types of periodic components; see [**7**, theorem 6]. In particular, U is called a *Baker domain* if there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ as $n \to \infty$, for $z \in U$, but $f^p(z_0)$ is not defined. In this case, there is at least one component U_k, $1 \leqslant k \leqslant p$, with the property that $f^{np}(z) \to \infty$ as $n \to \infty$ for $z \in U_k$. If U is a Baker domain of a transcendental *entire* function f, then $f^n(z) \to \infty$ as $n \to \infty$ for $z \in U$ and, moreover, U is simply connected [**2**, theorem 3·1]. However, a transcendental meromorphic function (even one with only finitely many poles) can have a multiply connected Baker domain; see [**8**, example 1], for example.

For $p \in \mathbb{N}$, we denote by $\operatorname{sing}(f^{-p})$ the set of finite singularities of f^{-p}; that is, the set of points $w \in \mathbb{C}$ such that some branch of f^{-p} cannot be analytically continued through w. The set $\operatorname{sing}(f^{-1})$ consists of the critical values and finite asymptotic values of f, and for $p \in \mathbb{N}$ we have

$$f^{p-1}(\operatorname{sing}(f^{-1}) \setminus A_{p-1}) \subseteq \operatorname{sing}(f^{-p}) \subseteq \bigcup_{j=0}^{p-1} f^j(\operatorname{sing}(f^{-1}) \setminus A_j),$$

where

$$A_j = \{z : f^j \text{ is not analytic at } z\};$$

see [**10**, theorem 7·1·2], and also [**1**, lemma 2] for the case of a transcendental entire function.

It follows from the proof of [**14**, theorem A] that if f is a transcendental meromorphic function and f has a cycle of Baker domains of period p, then the set $\operatorname{sing}(f^{-p})$ is unbounded. Here we show that if f has only finitely many poles, then this result can be strengthened to deduce that the set $\operatorname{sing}(f^{-p})$ is not too sparse.

Theorem 1·1. *Let f be a transcendental meromorphic function with a finite number of poles and with a cycle of Baker domains of period p. Then there exist constants $C > 1$ and $r_0 > 0$ such that*

$$\left\{z : \frac{1}{C}r < |z| < Cr\right\} \cap \mathit{sing}(f^{-p}) \neq \varnothing, \quad \textit{for } r \geqslant r_0.$$

Theorem 1·1 was first proved by Bargmann [**4**] for the case of a transcendental entire function. The proof in the case of a transcendental meromorphic function f with a finite number of poles has to overcome the difficulty that the Baker domains of such an f need not be simply connected.

In Section 3, we give examples to show that the result of Theorem 1.1 does not hold for transcendental meromorphic functions with infinitely many poles. These examples are of the form

$$f(z) = cz + \sum_{n=1}^{\infty}\left(\frac{1}{a_n - z} - \frac{1}{a_n + z}\right) = cz + \sum_{n=1}^{\infty}\frac{2z}{a_n^2 - z^2}, \tag{1·1}$$

where $c \geqslant 1$, $a_{n+1} > a_n > 0$, for $n \in \mathbb{N}$, and $a_{n+1}/a_n \to \infty$ as $n \to \infty$.

It follows from [**3**, section 4] that, if f is of the form (1·1), then $J(f) \subseteq \mathbb{R}$, because the upper and lower half-planes are invariant under f. Also, since $c \geqslant 1$, we have $f^n(z) \to \infty$ as $n \to \infty$ for z in these half-planes. Thus $F(f)$ consists of one or two invariant Baker domains, depending on whether $J(f)$ is a proper subset of $\mathbb{R}$ or $J(f) = \mathbb{R}$. In fact it turns out that if $c > 1$ then both of these possibilities can occur whereas if $c = 1$ then the Julia set is always equal to $\mathbb{R}$.

We show that if f is of the form (1·1) then the positions of the singularities of f^{-1} depend on the values of c and a_n. More precisely, we prove the following result. Here and later we use the notation $B(z, r) = \{w : |w - z| < r\}$, where $z \in \mathbb{C}, r > 0$.

Theorem 1·2. *Let f be of the form* (1.1). *Then there exist $R > 0$ and $N_1 \in \mathbb{N}$ such that*

$$\mathit{sing}(f^{-1}) \subset B(0, R) \cup \bigcup_{n=N_1}^{\infty} B(\pm c a_n, 5\sqrt{c}).$$

Since $a_{n+1}/a_n \to \infty$ as $n \to \infty$, this result is sufficient to show that the conclusion of Theorem 1·1 does not hold for functions of the form (1·1).

2. *Proof of Theorem* 1·1

We first recall the following fundamental result from [**11**, theorem 1]. Here we use the notation $\mathbb{H} = \{z : \Re(z) > 0\}$.

Lemma 2·1. *Let f be a transcendental meromorphic function with a finite number of poles, and with a Baker domain U which is periodic with period p. Then there exist a simply connected domain V in U, an analytic function ϕ defined in U and a Möbius transformation T such that*:

(a) *V is absorbing for f^p, that is, $f^p(V) \subset V$ and for each compact $K \subset U$ there exists N such that $f^{Np}(K) \subset V$;*
(b) *$\phi : U \to \Omega \in \{\mathbb{H}, \mathbb{C}\}$ and ϕ is univalent in V;*
(c) *$T : \Omega \to \Omega$ is a bijection, and $\phi(V)$ is absorbing for T;*
(d) *$\phi(f^p(z)) = T(\phi(z))$, for $z \in U$.*

The triple (V, ϕ, T) is called a *conformal conjugacy*, or *eventual conjugacy*, of f^p in U, and V is called a *fundamental set* for f^p in U. We note that properties (b) and (d) imply that f^p is univalent in V, a key fact in the proof of Theorem 1·1.

We require two further results from earlier work.

Lemma 2·2. *Let f be a transcendental meromorphic function with a Baker domain U which is periodic with period p, such that $f^{np}(z) \to \infty$ as $n \to \infty$ for $z \in U$. Then, for any $a \in U$ and any path $\Gamma = \bigcup_{n=0}^{\infty} f^{np}(\Gamma_0)$, where Γ_0 joins a to $f^p(a)$ in U and $0 \notin \Gamma$, there is a positive constant C_0 such that*

$$\frac{1}{C_0}|z| \leqslant |f^p(z)| \leqslant C_0|z|, \quad \text{for } z \in \Gamma.$$

This result is a special case of [**13**, theorem 1]. For the case of a transcendental meromorphic function with a finite number of poles it can alternatively be deduced from [**7**, lemma 7] by using the fact that the complement of any periodic component of the Fatou set must contain an unbounded closed connected set; see [**15**, theorems 1, 2, and 4].

We also need a result about the size of the image, under a transcendental meromorphic function with a finite number of poles, of a large Jordan curve surrounding the origin; see [**8**, proof of theorem F] or [**15**, lemma 4].

Lemma 2·3. *Let f be a transcendental meromorphic function with a finite number of poles. Then there exist $\rho > R > 0$ such that if γ is any Jordan curve surrounding $\{z : |z| = \rho\}$ and $B(\gamma)$ is the doubly connected domain bounded by $\{z : |z| = R\}$ and γ, then the outer boundary component of $f(B(\gamma))$ is a subset of $f(\gamma)$ and lies in $\{z : |z| > 2\rho\}$.*

The proof of Theorem 1·1 is carried out in two main steps.

Lemma 2·4. *Let f be a transcendental meromorphic function with a finite number of poles and with a Baker domain U which is periodic with period p, such that $f^{np}(z) \to \infty$ as $n \to \infty$ for $z \in U$. Then there is a path Γ defined by a continuous map $\gamma : [0, \infty) \to U$ such that*:

(a) *$\gamma(t) \to \infty$ as $t \to \infty$;*
(b) *$f^p(\gamma(t)) \to \infty$ as $t \to \infty$;*
(c) *$f^p(\Gamma) \subset \Gamma$;*

(d) *there exist* $c_1, c_2 > 0$ *such that*

$$c_1|z| \leqslant |f^p(z)| \leqslant c_2|z|, \quad \textit{for } z \in \Gamma;$$

(e) *there exist* $c_3, c_4 > 0$ *such that*

$$c_3 \leqslant |(f^p)'(z)| \leqslant c_4, \quad \textit{for } z \in \Gamma.$$

Proof. Our proof of Lemma 2·4 follows that of [**4**, lemma 3] with some simplifications; we give the details for completeness. By Lemma 2·1, there is a conformal conjugacy (V, ϕ, T) of f^p in U. Let $a \in V$ and let $\gamma : [0, \infty) \to V$ be a continuous map such that $\gamma(0) = a, \gamma(1) = f^p(a)$, and

$$\gamma(t + n) = f^{np}(\gamma(t)), \quad \text{for } t \in [0, 1],\ n \in \mathbb{N}.$$

Then γ has properties (a), (b) and (c). Also property (d) holds by Lemma 2·2, since we may assume without loss of generality that $\Gamma = \gamma([0, \infty)) \subset \mathbb{C} \setminus \{0\}$.

For $z \in V$ and $n \in \mathbb{N}$, we define

$$\phi_n(z) = \frac{f^{np}(z) - f^{np}(a)}{(f^{np})'(a)}.$$

Note that the functions f^{np} and hence ϕ_n are univalent in V by Lemma 2·1, and

$$\phi_n(a) = 0, \quad \phi_n'(a) = 1, \quad \text{for } n \in \mathbb{N}. \tag{2·1}$$

We now make several uses of the well-known fact that if $\mathcal{F}$ is a family of functions analytic in a domain G, which is uniformly bounded at some point of G, then $\mathcal{F}$ is a normal family in G if and only if $\mathcal{F}$ is locally uniformly bounded in G.

First, we deduce that the sequence $\phi_n, n \in \mathbb{N}$, forms a normal family in V. Indeed, the family of univalent functions g in the unit disc D satisfying $g(0) = 0$ and $g'(0) = 1$ is locally uniformly bounded in D by the Koebe distortion theorem, so the normality of $\phi_n, n \in \mathbb{N}$, in V follows by use of the Riemann mapping theorem.

Therefore, the functions $\Phi_n = \phi_n \circ f^p, n \in \mathbb{N}$, also form a normal family in V, as do both $\phi_n', n \in \mathbb{N}$, and $\Phi_n', n \in \mathbb{N}$. Moreover, the functions ϕ_n' and Φ_n' are zero-free in V, by univalence, so $1/\phi_n', n \in \mathbb{N}$, and $1/\Phi_n', n \in \mathbb{N}$, also form normal families in V, by Hurwitz's theorem. Hence there exist constants $c_3, c_4 > 0$ such that

$$c_3 \leqslant \left|\frac{\Phi_n'(z)}{\phi_n'(z)}\right| \leqslant c_4, \quad \text{for } z \in \gamma([0, 1]), n \in \mathbb{N}. \tag{2·2}$$

Now for $n \in \mathbb{N}$ and $t \in [0, 1]$, we have

$$(f^p)'(\gamma(t + n)) = (f^p)'(f^{np}(\gamma(t))) = \frac{(f^{(n+1)p})'(\gamma(t))}{(f^{np})'(\gamma(t))} = \frac{\Phi_n'(\gamma(t))}{\phi_n'(\gamma(t))},$$

so part (e) holds by (2·2).

For the second step in our proof of Theorem 1·1 we use the log transform technique of Eremenko and Lyubich [**9**]. Bargmann's proof of this step in [**4**] used his result [**4**, theorem 1] on normal families of meromorphic covering maps. However, he also mentioned the approach we use below, and this works well for meromorphic functions with a finite number of poles.

Lemma 2·5. *Let f be a transcendental meromorphic function with a finite number of poles and with a Baker domain U which is periodic with period p, such that $f^{np}(z) \to \infty$*

as $n \to \infty$ for $z \in U$. If there are positive sequences r_n and R_n such that $R_n > r_n$, $r_n \to \infty$ as $n \to \infty$,

$$\frac{R_n}{r_n} \longrightarrow \infty \quad as\ n \longrightarrow \infty \tag{2·3}$$

and

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap sing(f^{-p}) = \varnothing, \tag{2·4}$$

where $A_n = \{z : r_n < |z| < R_n\}$, then for any sequence z_n such that $|f^p(z_n)| = \sqrt{r_n R_n}$ and $z_n \to \infty$ as $n \to \infty$, we have

$$\left|\frac{z_n (f^p)'(z_n)}{f^p(z_n)}\right| \longrightarrow \infty \quad as\ n \longrightarrow \infty. \tag{2·5}$$

Proof. Let r_n, R_n and z_n satisfy the hypotheses of the lemma. First we choose a periodic point c of f. We may assume that the annuli $A_n, n \in \mathbb{N}$, surround the corresponding periodic cycle and also the poles of f.

Let $w_n = f^p(z_n)$, $n = 1, 2, \ldots$, so $|w_n| = \sqrt{r_n R_n}$, and put

$$S_n = \{t : \ln r_n < \Re(t) < \ln R_n\}, \quad n = 1, 2, \ldots.$$

Also, choose $t_n \in S_n$ such that $e^{t_n} = w_n$ and then let h_n denote the branch of $f^{-p} \circ \exp$ that maps t_n to z_n. Since $A_n \cap \operatorname{sing}(f^{-p}) = \varnothing$, the branch h_n can be analytically continued along all paths from t_n in S_n to give a single-valued analytic function in S_n, by the monodromy theorem.

Two cases can then arise (see [**12**, page 283] or [**14**]), as follows.

(a) The function h_n is univalent in S_n.

(b) The function h_n is periodic in S_n with period $2\pi i m_n$, for some minimal $m_n \in \mathbb{N}$. In this case, we have

$$h_n(t) = \phi_n(\exp(t/m_n)), \quad \text{for } t \in S_n,$$

where ϕ_n is univalent in the annulus $\{s : r_n^{1/m_n} < |s| < R_n^{1/m_n}\}$.

In case (a), $h_n(S_n)$ is a simply connected domain and $c \notin h_n(S_n)$. Thus, for an appropriate branch of the logarithm function,

$$F_n(t) = \log(h_n(t) - c), \quad t \in S_n,\ n \in \mathbb{N}, \tag{2·6}$$

is a single-valued analytic (even univalent) map of S_n onto a domain that contains no vertical line segment of length greater than 2π and hence no disc of radius greater than π. By applying Bloch's theorem (or Koebe's theorem) in the disc $\{t : |t - t_n| < \frac{1}{2}\ln(R_n/r_n)\}$, we deduce that

$$|F_n'(t_n)| \leqslant \frac{C_1}{\ln(R_n/r_n)}, \quad \text{for } n \in \mathbb{N},$$

where C_1 is a positive absolute constant. On expressing this inequality in terms of f, we obtain

$$\left|\frac{(z_n - c)(f^p)'(z_n)}{f^p(z_n)}\right| \geqslant \frac{1}{C_1}\ln(R_n/r_n), \quad \text{for } n \in \mathbb{N}. \tag{2·7}$$

In case (b), $h_n(S_n)$ is a doubly connected domain, with $c \notin h_n(S_n)$, and

$$f^p(z) = (\phi_n^{-1}(z))^{m_n}, \quad \text{for } z \in h_n(S_n),$$

is an m_n to 1 mapping of $h_n(S_n)$ onto A_n. First consider case (b)(i), where c lies in the unbounded complementary component of $h_n(S_n)$. Then we can define $F_n(t)$ as in (2.6) to be a single-valued (but not necessarily univalent) analytic map of S_n onto a domain that contains no disc of radius greater than π. Hence (2.7) holds once again.

Next consider case (b)(ii), where c lies in the bounded complementary component of $h_n(S_n)$. Here we can use the monodromy theorem again to define

$$F_n(t) = \log(h_n(t) - c), \quad t \in S_n, \; n \in \mathbb{N},$$

as a single-valued analytic map of S_n onto a simply connected domain bounded by two curves, which is invariant under translation by $2\pi i$. Let

$$Q_n = \{t : \ln r_n < \Re(t) < \ln R_n, |\Im(t - t_n)| < \pi m_n\}, \quad n \in \mathbb{N}.$$

Then Q_n is a rectangle contained in S_n, with centre t_n, and $h_n(t) = \phi_n(\exp(t/m_n))$ maps Q_n (univalently) onto $h_n(S_n) \setminus \alpha_n$, where α_n is a cross-cut of $h_n(S_n)$ joining the two boundary components. Thus $F_n(Q_n)$ is a quadrilateral containing no vertical line segment of length greater than 2π, and hence no disc of radius greater than π. By applying Bloch's theorem to F_n in the disc $\{t : |t - t_n| < \min\{\pi m_n, \frac{1}{2}\ln(R_n/r_n)\}\}$, we deduce that

$$\left|\frac{(z_n - c)(f^p)'(z_n)}{f^p(z_n)}\right| \geqslant \frac{1}{C_1}\min\{\pi m_n, \frac{1}{2}\ln(R_n/r_n)\}, \quad \text{for } n \in \mathbb{N}. \tag{2·8}$$

Now note that if the lemma is false, then we may assume, by taking a subsequence if necessary, that the sequence

$$\left|\frac{z_n(f^p)'(z_n)}{f^p(z_n)}\right|, \quad n \in \mathbb{N}, \tag{2·9}$$

is bounded. Since $R_n/r_n \to \infty$ as $n \to \infty$, it follows from (2·7) that case (b)(ii) must occur for all sufficiently large n. Thus we may assume that case (b)(ii) occurs for all n and deduce from (2·8) that the corresponding sequence m_n, $n \in \mathbb{N}$, is bounded.

Let γ_n be the image under h_n of $\{t : \Re(t) = \frac{1}{2}\ln(r_n R_n)\}$. Then γ_n is a simple closed curve with c inside γ_n, and f^p is an m_n to 1 mapping of γ_n onto $\beta_n = \{w : |w| = \sqrt{r_n R_n}\}$. Since $z_n \in \gamma_n$ and $z_n \to \infty$ as $n \to \infty$ (by hypothesis), we deduce that $\text{dist}(\gamma_n, c) \to \infty$ as $n \to \infty$. For if not, there exist $R_0 > 0$ and a subsequence γ_{n_k}, $k = 1, 2, \ldots$, such that $\gamma_{n_k} \cap \{z : |z - c| = R_0\} \neq \varnothing$, for $k = 1, 2, \ldots$. Then each circle $\{z : |z - c| = R\}, R > R_0$, meets all but a finite number of the γ_{n_k} and so contains a point where f^p is not analytic (since, on r_n, $|f^p| = \sqrt{r_n R_n \to \infty}$). This contradicts the fact that there is a closed countable set E such that f^p is analytic in $\mathbb{C} \setminus E$.

By adjusting the radius of the circle β_n slightly, we can arrange that f^p remains an m_n to 1 mapping of a simple closed curve γ_n onto β_n, but there are no critical values on β_n of $f, f^2, \ldots, f^{p-1}$. Then the closed curves

$$f(\gamma_n), f^2(\gamma_n), \ldots, f^{p-1}(\gamma_n),$$

contain no critical points of $f^{p-1}, \ldots, f^2, f$, respectively. Hence each $f^j(\gamma_n)$, $j =$

$1, 2, \ldots, p-1$, is a simple closed curve. Since f has only finitely many poles and $\mathrm{dist}(\gamma_n, c) \to \infty$ as $n \to \infty$, it follows from Lemma 2·3 that, for n large enough, each $f^j(\gamma_n)$, $j = 1, 2, \ldots, p-1$, winds positively round 0, at most m_n times.

Therefore, by the argument principle, we have

$$m_n \geqslant \mathrm{wnd}(f(\gamma_n), 0) = \sum_k \mathrm{wnd}(\gamma_n, a_k) - \sum_k \mathrm{wnd}(\gamma_n, b_k),$$

where a_k are the zeros of f inside γ_n and b_k are the poles of f inside γ_n. Since m_n is bounded and f has only finitely many poles, we deduce that the number of zeros of f inside γ_n is uniformly bounded. Because $\mathrm{dist}(\gamma_n, c) \to \infty$ as $n \to \infty$ and c lies inside γ_n, it follows that f has a finite number of zeros in $\mathbb{C}$. This argument applies similarily to the zeros of $f - a$, for any $a \in \mathbb{C}$, so we obtain a contradiction to the fact that f has a transcendental singularity at ∞. Thus the sequence (2·9) cannot be bounded, so the proof of Lemma 2·5 is complete.

We now prove Theorem 1·1. We can assume that f satisfies the hypotheses of Lemma 2·4. If the theorem is false, then there exist positive sequences r_n and R_n such that $R_n > r_n$, $r_n \to \infty$ as $n \to \infty$, $R_n/r_n \to \infty$ as $n \to \infty$ and

$$\{z : r_n < |z| < R_n\} \bigcap \mathrm{sing}(f^{-p}) = \varnothing, \quad \text{for } n \in \mathbb{N}.$$

Hence Lemma 2·5 applies. Now let Γ be the path given by Lemma 2·4, and for n sufficiently large let w_n be the point where Γ meets $\{w : |w| = \sqrt{r_n R_n}\}$. Then $w_n = f^p(z_n)$ for some $z_n \in \Gamma$, for n sufficiently large. We have $z_n \to \infty$ as $n \to \infty$, by Lemma 2·2. The conclusions of Lemmas 2·4 and 2·5 about $|z_n (f^p)'(z_n)/f^p(z_n)|$ are now contradictory, so the proof of Theorem 1·1 is complete.

3. *Examples of Baker domains*

Here we give examples to show that the result of Theorem 1·1 does not hold for transcendental meromorphic functions with infinitely many poles. Throughout this section f is a function of the form (1·1); that is,

$$f(z) = cz + \sum_{n=1}^{\infty} \left(\frac{1}{a_n - z} - \frac{1}{a_n + z} \right) = cz + \sum_{n=1}^{\infty} \frac{2z}{a_n^2 - z^2},$$

where $c \geqslant 1$, $a_{n+1} > a_n > 0$, for $n \in \mathbb{N}$, and $a_{n+1}/a_n \to \infty$ as $n \to \infty$.

We saw in the Introduction that $J(f) \subseteq \mathbb{R}$ and that $F(f)$ consists of one or two invariant Baker domains, depending on whether $J(f)$ is a proper subset of $\mathbb{R}$ or $J(f) = \mathbb{R}$. Later in this section we show that if $c = 1$, then only the second of these two cases can occur whereas both cases can occur if $c = 2$. Also note that $J(f)$ is symmetric in the imaginary axis since f is odd.

Theorem 1·2 shows that the result of Theorem 1·1 does not hold for such functions f. To prove Theorem 1·2, we need the following lemma which also plays a key role later in this section. Here and later the constants $N_0, N_1, \ldots$, depend on the particular function f.

Lemma 3·1. *There exists $N_0 \in \mathbb{N}$ such that, if $N \geqslant N_0$, $a_N \leqslant |z| < a_{N+1}$ and $z \neq \pm a_N$, then*

$$\left| f(z) - cz - \frac{2z}{a_N^2 - z^2} - \frac{2z}{a_{N+1}^2 - z^2} \right| \leqslant \frac{5N}{|z|}$$

and

$$\left| f'(z) - c - \frac{2(a_N^2 + z^2)}{(a_N^2 - z^2)^2} - \frac{2(a_{N+1}^2 + z^2)}{(a_{N+1}^2 - z^2)^2} \right| \leqslant \frac{5N}{|z|^2}.$$

Proof. Suppose that $a_N \leqslant |z| < a_{N+1}$ and $z \neq \pm a_N$. Since $a_{n+1}/a_n \to \infty$ as $n \to \infty$ we deduce that, for N sufficiently large,

$$\begin{aligned}
\left| f(z) - cz - \frac{2z}{a_N^2 - z^2} - \frac{2z}{a_{N+1}^2 - z^2} \right| &= \left| \sum_{n=1}^{N-1} \frac{2z}{a_n^2 - z^2} + \sum_{n=N+2}^{\infty} \frac{2z}{a_n^2 - z^2} \right| \\
&\leqslant \sum_{n=1}^{N-1} \frac{4|z|}{|z|^2} + \sum_{n=N+2}^{\infty} \frac{4|z|}{a_n^2} \\
&\leqslant \frac{4(N-1)}{|z|} + \frac{4}{|z|} \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&\leqslant \frac{5N}{|z|},
\end{aligned}$$

as required. Similarly, if N is sufficiently large, then we have

$$\begin{aligned}
\left| f'(z) - c - \frac{2(a_N^2 + z^2)}{(a_N^2 - z^2)^2} - \frac{2(a_{N+1}^2 + z^2)}{(a_{N+1}^2 - z^2)^2} \right| &= \left| \sum_{n=1}^{N-1} \frac{2(a_n^2 + z^2)}{(a_n^2 - z^2)^2} + \sum_{n=N+2}^{\infty} \frac{2(a_n^2 + z^2)}{(a_n^2 - z^2)^2} \right| \\
&\leqslant \sum_{n=1}^{N-1} \frac{4}{|z|^2} + \sum_{n=N+2}^{\infty} \frac{4}{a_n^2} \\
&\leqslant \frac{4(N-1)}{|z|^2} + \frac{4}{|z|^2} \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&\leqslant \frac{5N}{|z|^2},
\end{aligned}$$

as required.

We are now in a position to prove Theorem 1·2. First note that f has no finite asymptotic values by Lemma 3·1. Now consider the critical values of f. Using Lemma 3·1 and the identities

$$\frac{2z}{a_n^2 - z^2} = \frac{1}{a_n - z} - \frac{1}{a_n + z} \quad \text{and} \quad \frac{2(a_n^2 + z^2)}{(a_n^2 - z^2)^2} = \frac{1}{(a_n - z)^2} + \frac{1}{(a_n + z)^2},$$

we can deduce that there exists $N_1 \geqslant N_0$ such that the critical points of f lie in

$$B(0, a_{N_1}) \cup \bigcup_{n=N_1}^{\infty} \left\{ z : \frac{1}{2\sqrt{c}} \leqslant |z \pm a_n| \leqslant \frac{2}{\sqrt{c}} \right\},$$

and also

$$f\left(\bigcup_{n=N_1}^{\infty} \left\{ z : \frac{1}{2\sqrt{c}} \leqslant |z \pm a_n| \leqslant \frac{2}{\sqrt{c}} \right\} \right) \subset \bigcup_{n=N_1}^{\infty} B(\pm c a_n, 5\sqrt{c}).$$

Since the poles of both f and f' are at the points $\pm a_n$, $n \in \mathbb{N}$, it follows that there exists $R > 0$ such that any critical points within $B(0, a_{N_1})$ map to points within $B(0, R)$. This completes the proof of Theorem 1·2.

We now show that when $c = 1$ the Julia set is always equal to the real line, so $F(f)$ consists of two invariant Baker domains. We remark that this fact follows from a general result of Bargmann [**5**, theorem 2·24] on the Julia sets of inner functions, but here we give a direct proof based on Lemma 2·2.

THEOREM 3·1. *If $c = 1$, then $J(f) = \mathbb{R}$.*

Proof. We use proof by contradiction. Assume that there exists $x_0 \in \mathbb{R} \setminus J(f)$. Then $F(f)$ is a single invariant Baker domain and so $f^n(x_0) \to \infty$ as $n \to \infty$. Also, it follows from Lemma 2.2 that there exists $C_0 > 1$ such that

$$\frac{1}{C_0}|f^n(x_0)| \leqslant |f^{n+1}(x_0)| \leqslant C_0|f^n(x_0)|, \quad \text{for } n \in \mathbb{N}. \tag{3·1}$$

We now obtain a contradiction to the fact that $f^n(x_0) \to \infty$ as $n \to \infty$ by showing that if N is sufficiently large and $|f^n(x_0)| \leqslant 2C_0 a_N$ for some $n \in \mathbb{N}$, then $|f^{n+1}(x_0)| \leqslant 2C_0 a_N$. We begin by noting that if $|f^n(x_0)| \leqslant 2a_N$, for some n, $N \in \mathbb{N}$, then it follows from (3·1) that $|f^{n+1}(x_0)| \leqslant 2C_0 a_N$. So, it is sufficient to show that there exists $N_2 \in \mathbb{N}$ such that

$$2a_N \leqslant |x| \leqslant 2C_0 a_N \;\Rightarrow\; |f(x)| \leqslant |x|, \quad \text{for } N \geqslant N_2,\ x \in \mathbb{R}. \tag{3·2}$$

Since $f(z) = z + \sum_{n=1}^{\infty} 2z/(a_n^2 - z^2)$, the implication (3·2) will be proved if we can show that there exists $N_2 \in \mathbb{N}$ such that

$$2a_N \leqslant |x| \leqslant 2C_0 a_N \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{a_n^2 - x^2} < 0, \quad \text{for } N \geqslant N_2,\ x \in \mathbb{R}, \tag{3·3}$$

and

$$2a_N \leqslant |x| \leqslant 2C_0 a_N \;\Rightarrow\; \sum_{n=1}^{\infty} \left|\frac{2x}{a_n^2 - x^2}\right| \leqslant |2x|, \quad \text{for } N \geqslant N_2,\ x \in \mathbb{R}. \tag{3·4}$$

It follows from Lemma 3·1 that (3·4) is true and so it remains to prove (3·3). If $2a_N \leqslant |x| \leqslant 2C_0 a_N$ and N is sufficiently large, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{a_n^2 - x^2} &= \frac{1}{a_N^2 - x^2} + \sum_{r=1}^{N-1}\left(\frac{1}{a_{N-r}^2 - x^2} + \frac{1}{a_{N+r}^2 - x^2}\right) + \sum_{n=2N}^{\infty} \frac{1}{a_n^2 - x^2} \\
&< \frac{1}{a_N^2 - 4C_0^2 a_N^2} + \sum_{r=1}^{N-1}\left(\frac{1}{-x^2} + \frac{2}{a_{N+r}^2}\right) + 2\sum_{n=2N}^{\infty} \frac{1}{a_n^2} \\
&< \frac{1}{-4C_0^2 a_N^2} + \frac{2}{a_{2N}^2}\sum_{n=0}^{\infty} \frac{1}{2^n} \\
&< 0.
\end{aligned}$$

This proves (3·3) and hence Theorem 3·1.

We now show that if $c > 1$ then it is possible for $F(f)$ to consist of either one or two invariant Baker domains, depending on the values of the constants a_n. In each case, we use the following corollary of Lemma 3·1.

Corollary 3·1. *There exists $N_3 \in \mathbb{N}$ such that, if $|z| \geqslant a_{N_3}$ and $|z \pm a_n| \geqslant 3$, for each $n \in \mathbb{N}$, then*

$$|f(z) - cz| < 2.$$

First we give an example with $c = 2$ and $J(f) \neq \mathbb{R}$, so that $F(f)$ consists of one multiply-connected Baker domain.

Theorem 3·2. *Let $c = 2$ and $a_n = 2^{p_n}$, for each $n \in \mathbb{N}$, where $p_{n+1} - p_n \to \infty$ as $n \to \infty$. Then $a_{n+1}/a_n \to \infty$ as $n \to \infty$ and $J(f) \neq \mathbb{R}$.*

Proof. Let N_3 be the integer given by Corollary 3·1 for this sequence a_n. Then take $N \in \mathbb{N}$ such that

$$2^{N+1} - 3 > 2^N + 3 > a_{N_3} \tag{3·5}$$

and put

$$A_n = \{z : 2^n + 3 \leqslant |z| \leqslant 2^{n+1} - 3\}, \quad \text{for } n \geqslant N.$$

It follows from Corollary 3·1 and (3·5) that, for $n \geqslant N$ and $z \in A_n$, we have

$$2^{n+1} + 3 < 2(2^n + 3) - 2 \leqslant |f(z)| \leqslant 2(2^{n+1} - 3) + 2 < 2^{n+2} - 3,$$

so $f(A_n) \subset A_{n+1}$, for $n \geqslant N$. Thus $\bigcup_{n \geqslant N} A_n \subset F(f)$, by Montel's theorem. This completes the proof of Theorem 3·2.

Finally, we give an example with $c = 2$ where the constants a_n are chosen so that $J(f) = \mathbb{R}$.

Theorem 3·3. *Let $c = 2$ and $a_n = 2^{n^2} n$, for each $n \in \mathbb{N}$. Then $a_{n+1}/a_n \to \infty$ as $n \to \infty$ and $J(f) = \mathbb{R}$.*

Proof. We begin by noting that, since f is increasing on each component of the set $\mathbb{R}\backslash\{\pm a_n : n \in \mathbb{N}\}$, it follows from Corollary 3·1 that if $x > 0$ is sufficiently large and $|x - a_n| \geqslant 6$, for each $n \in \mathbb{N}$, then

$$f((x-3, x+3)) \supset (2(x-3)+2, 2(x+3)-2) \supset (2x-3, 2x+3).$$

Thus, if n is sufficiently large and

$$\left|2^m n - 2^{p^2} p\right| \geqslant 6, \quad \text{for } 0 \leqslant m < n^2,\ p \in \mathbb{N}, \tag{3·6}$$

then

$$f^{n^2}((n-3, n+3)) \supset \left(2^{n^2} n - 3, 2^{n^2} n + 3\right) = \left(a_n - 3, a_n + 3\right),$$

and hence

$$(n-3, n+3) \cap J(f) \neq \varnothing. \tag{3·7}$$

We now determine which values of n fail to satisfy (3·6). We begin by noting that, if there exists $0 \leqslant m < n^2$ such that $|2^m n - 2^{p^2} p| < 6$ for some $p \in \mathbb{N}$, then p is large (since n is large), $p < n$ (since $m < n^2$), and so $p^2 > m$.

If $m = 0$, then

$$\begin{aligned}|2^m n - 2^{p^2} p| < 6 &\iff |n - 2^{p^2} p| < 6\\ &\iff n = 2^{p^2} p \pm 0, 1, \ldots, 5.\end{aligned}$$

If $m = 1$, then

$$|2^m n - 2^{p^2} p| < 6 \iff |2n - 2^{p^2} p| < 6$$
$$\iff |n - 2^{p^2-1} p| < 3$$
$$\iff n = 2^{p^2-1} p \pm 0, 1, 2.$$

If $m = 2$, then

$$|2^m n - 2^{p^2} p| < 6 \iff |4n - 2^{p^2} p| < 6$$
$$\iff |n - 2^{p^2-2} p| < 3/2$$
$$\iff n = 2^{p^2-2} p \pm 0, 1.$$

If $m \geqslant 3$, then

$$|2^m n - 2^{p^2} p| < 6 \iff |2^{m-3} n - 2^{p^2-3} p| < 6/8$$
$$\iff n = 2^{p^2-m} p.$$

In particular, if (3·6) fails to be satisfied for some $m \geqslant 3$, $p \in \mathbb{N}$, then n must be even because $m < p^2$.

If n and hence p is sufficiently large, then we have

$$2^{p^2-2} p + 10 < 2^{p^2-1} p, \quad 2^{p^2-1} p + 10 < 2^{p^2} p \quad \text{and} \quad 2^{p^2} p + 10 < 2^{(p+1)^2-2}(p+1).$$

Therefore, if $a > 0$ is sufficiently large, then the interval $[a, a+14]$ contains an integer n for which (3·6) is satisfied. Hence, by (3·7) and the symmetry of $J(f)$, there exists $a_0 > 0$ such that

$$[a-3, a+17] \cap J(f) \neq \varnothing, \quad \text{for } |a| \geqslant a_0. \tag{3·8}$$

Now suppose that there is an interval $I \subset F(f) \cap \mathbb{R}$ and that I has length ε. We have $f'(x) \geqslant 2$ on $\mathbb{R} \setminus \{\pm a_n : n \in \mathbb{N}\}$, and so $f^n(I)$ contains an interval I_n of length $2^n \varepsilon$. We know that $f^n(z) \to \infty$ as $n \to \infty$ if $z \in F(f)$ and so, for sufficiently large n, it follows from (3·8) that $I_n \cap J(f) \neq \varnothing$. This is a contradiction, and so we must have $J(f) = \mathbb{R}$ as claimed.

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