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Dissipative effects in Multilevel Systems

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Abstract. Dissipation is sometimes regarded as an inevitable and regrettable presence in the real evolution of a quantum system. However, the effects may not always be malign, although often non-intuitive and may even be beneficial. In this note we we display some of these effects for N-level systems, where $N = 2, 3, 4$.

We start with an elementary introduction to dissipative effects on the Bloch Sphere, and its interior, the Bloch Ball, for a two-level system. We describe explicitly the hamiltonian evolution as well as the purely dissipative dynamics, in the latter case giving the $t \to \infty$ limits of the motion. This discussion enables us to provide an intuitive feeling for the measures of control-reachable states. For the three-level case we discuss the impossibility of isolating a two-level (qubit) subsystem; this is a Bohm-Aharonov type consequence of dissipation.

We finally exemplify the four-level case by giving constraints on the decay of two-qubit entanglement.

Dedicatory: The seminal work of George Sudarshan and his collaborators some 30 years ago has given us the framework for an analytic discussion of dissipative effects in quantum systems. It gives us great pleasure to dedicate this paper to Professor Sudarshan on his 75th birthday.

1. Introduction

The basic equations which determine the evolution of a hamiltonian quantum system have been well known since the '20s. These may be written in the form of a differential equation for the quantum state, the Schrödinger equation. The standard description of a quantum state suitable for our discussion, which will involve *open* systems, is by means of a density matrix ρ, a positive matrix of trace 1. In this context the state $\rho(t)$ is said to satisfy the quantum Liouville equation, with[1]

Local Form:

$$i\frac{d}{dt}\rho(t) = [H, \rho(t)] \equiv H\rho(t) - \rho(t)H \tag{1.1}$$

[1] we choose units in which $\hbar = 1$.

where H is the total hamiltonian of the system. This may equivalently be written in the *Global Form*:

$$\rho(t) = U(t)\rho_0 U(t)^\dagger, \tag{1.2}$$

where $U(t)$ is the time-evolution operator satisfying the Schrodinger equation

$$i\frac{d}{dt}U(t) = HU(t), \qquad U(0) = I, \tag{1.3}$$

with I the identity operator. For a *hamiltonian* (non-dissipative) system one obtains a unitary evolution of the state.

In the context of *Quantum Control* theory, which we shall briefly explore in Section(3.1), we may assume that $H \equiv H(\mathbf{f})$ depends on a set of control fields f_m which are time-dependent:

$$H(\mathbf{f}) = H_0 + \sum_{m=1}^{M} f_m(t)H_m, \tag{1.4}$$

where H_0 is the internal hamiltonian and H_m is the interaction hamiltonian for the fields f_m for $1 \leq m \leq M$. The advantage of the Liouville equation (1.1) over the unitary evolution equation (1.2) is that it can easily be adapted for dissipative systems by adding a dissipation (super-)operator $L_D[\rho(t)]$:

$$i\dot{\rho}(t) = [H, \rho(t)] + iL_D[\rho(t)]. \tag{1.5}$$

In general, uncontrollable interactions of the system with its environment lead to two types of dissipation: phase decoherence (dephasing) and population relaxation. The former occurs when the interaction with the environment destroys the phase correlations between states, which leads to changes in the off-diagonal elements of the density matrix:

$$\dot{\rho}_{kn}(t) = -i([H, \rho(t)])_{kn} - \Gamma_{kn}\rho_{kn}(t) \tag{1.6}$$

where Γ_{kn} (for $k \neq n$) is the dephasing rate between $|k\rangle$ and $|n\rangle$. The latter happens, for instance, when a quantum particle in state $|n\rangle$ spontaneously emits a photon and moves to another quantum state $|k\rangle$, which changes the populations according to

$$\dot{\rho}_{nn}(t) = -i([H, \rho(t)])_{nn} + \sum_{k\neq n} [\gamma_{nk}\rho_{kk}(t) - \gamma_{kn}\rho_{nn}(t)] \tag{1.7}$$

where $\gamma_{kn}\rho_{nn}$ is the population loss for level $|n\rangle$ due to transitions $|n\rangle \to |k\rangle$, and $\gamma_{nk}\rho_{kk}$ is the population gain caused by transitions $|k\rangle \to |n\rangle$. The population relaxation rate γ_{kn} is determined by the lifetime of the state $|n\rangle$, and for multiple decay pathways, the relative probability for the transition $|n\rangle \to |k\rangle$. Phase decoherence and population relaxation lead to a dissipation superoperator (represented by an $N^2 \times N^2$ matrix) whose non-zero elements are

$$\begin{array}{ll} (L_D)_{[k;n],[k;n]} = -\Gamma_{kn} & k \neq n \\ (L_D)_{[n;n],[k;k]} = +\gamma_{nk} & k \neq n \\ (L_D)_{[n;n],[n;n]} = -\sum_{n\neq k}\gamma_{kn} & \end{array} \tag{1.8}$$

where Γ_{kn} and γ_{kn} are positive numbers, with Γ_{kn} symmetric in its indices. We have here used the convenient notation

$$[m; n] = (m-1)N + n \tag{1.9}$$

The $N^2 \times N^2$ matrix superoperator L_D may be thought of as acting on the N^2-vector $\mathbf{r}$ obtained from ρ by

$$\mathbf{r}_{[m;n]} \equiv \rho_{mn}. \tag{1.10}$$

The resulting vector equation is

$$\dot{\mathbf{r}} = L\mathbf{r} = (L_H + L_D)\mathbf{r} \tag{1.11}$$

where L_H is the anti-hermitian matrix derived from the hamiltonian H.

2. Dissipation theory

The values of the relaxation and dephasing parameters may be determined by experiment, or simply chosen to supply a model for the dissipation phenomenon. But they may not be chosen arbitrarily; the condition of positivity[2] for the state ρ imposes constraints on their values, as does their deduction from rigorous theory.

2.1. Master evolution equations

The standard theory adopted for the discussion of dissipation has the following forms[2]:
Local Form

$$\dot{\rho}(t) = -i[H, \rho(t)] + \frac{1}{2}\sum_k \left([A_k \rho(t), A_k^\dagger] + [A_k, \rho(t) A_k^\dagger] \right) \tag{2.1}$$

where the A_k are (trace-zero) $N \times N$ matrices. It may be shown that the dissipation superoperator L_D arising from Eq. (2.1) has negative eigenvalues. Since the evolution dynamics arises from exponentiation of $L_D t$ it follows that operators $\exp(L_D t)$ in the theory will grow without limit for arbitrary negative t. This means that the set of operator inverses is unbounded and implies a semi-group character to the evolution. This is also implicit in the following global description formalism[3]:
Global Form

$$\rho(t) = \sum_k W_k(t) \rho_0 W_k(t)^\dagger, \tag{2.2}$$

with

$$\sum_k W_k(t)^\dagger W_k(t) = I. \tag{2.3}$$

Equation(2.2) and the condition Eq. (2.3) clearly guarantee both positivity and unit trace for $\rho(t)$.

Further, though less obviously, the global form also implies a *semi-group* description of the evolution. For if we consider the set G whose elements are the *sets* $\{w_k\}$ satisfying Eq. (2.3), then if $g = \{w_k\}$ and $g' = \{w'_k\}$ are two elements of G, then so too is gg', where the product is taken in the sense of set multiplication. Although closed under composition, the only elements of G which possess inverses are the singleton sets $\{U\}$, where U is unitary; thus G is a semi-group.

2.2. Values of the dissipation parameters

To determine the constraints on the dissipation parameters we start with the Kossakowski, Gorini and Sudarshan form of the super-operator

$$L[\rho(t)] = -i[H, \rho(t)] + \frac{1}{2}\sum_{k,k'=1}^{N^2-1} a_{kk'} \left([V_k \rho(t), V_{k'}^\dagger] + [V_k, \rho(t) V_{k'}^\dagger] \right) \tag{2.4}$$

where the V_k, $k = 1, 2, \ldots, N^2 - 1$, are trace-zero, orthonormal operators with respect to the trace inner product $(A, B) = \mathrm{Tr}(A^\dagger B)$ that together with $V_{N^2} = \frac{1}{\sqrt{N}} I$ form a basis for the hermitian operators on the system's Hilbert space. In this form the resulting evolution operator will be *completely positive* if and only if the coefficient matrix $a = (a_{kk'})$ is positive. We note that the Lindblad form (2.1) is essentially a diagonalization of (2.4) for positive coefficient matrices.

A convenient choice for the V_k is to define $N^2 - N$ off-diagonal matrices

$$V_{[m;n]} = \mathbf{e}_{mn}, \qquad m \neq n,\ m, n = 1, 2, \ldots, N. \tag{2.5}$$

[2] In fact, the theory that we adopt here implies the even more stringent constraint of *complete positivity*.

as well as $N-1$ diagonal matrices

$$V_{[m;m]} = \frac{1}{\sqrt{m+m^2}} \left(\sum_{s=1}^{m} \mathbf{e}_{ss} - m\mathbf{e}_{m+1,m+1} \right) \tag{2.6}$$

for $m = 1, 2, \ldots, N-1$, where we use the index notation $[m;n] \equiv (m-1)N + n$ as before, and $\mathbf{e}_{mn}$ is an $N \times N$ matrix whose entries are zero except for a 1 in mth row, nth column position. By comparing the resulting dissipation super-operator with equation (1.8), we can express the relaxation and decoherence parameters in terms of the coefficients $a_{kk'}$ as detailed in [4].

3. Bloch Sphere and Bloch Ball

We now discuss the evolution of a two-level system (qubit). A useful geometric picture of this simple system, for the *pure* state case, is by means of the Bloch sphere and, for a general two-level (mixed) state, the Bloch ball.

3.1. Bloch Sphere and Bloch Ball: N=2

A normalized two-level pure state (qubit) is described by a complex 2-vector $(\alpha, \beta)^T$ where $|\alpha|^2 + |\beta|^2 = 1$. Since for a *single* qubit the overall phase is unimportant, we can set $\alpha = \cos(\theta/2)$, $\beta = e^{-i\phi}\sin(\theta/2)$ where α, β are two (real) angles, and the state may thus be represented by a point

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta \tag{3.1}$$

on the (Bloch) sphere [see Fig 1]. In terms of the density matrix description

$$\rho = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi}\cos(\theta/2)\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2)\sin(\theta/2) & \sin^2(\theta/2) \end{bmatrix} \tag{3.2}$$

and using the (unnormalized) Pauli matrices

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.3}$$

Eq. (3.1) may be succinctly expressed as

$$R_\mu = \mathrm{Tr}(\tau_\mu \rho) \qquad (\mu = 0, 1, 2, 3) \qquad \{R_1, R_2, R_3\} \equiv \{x, y, z\}. \tag{3.4}$$

In this form the Bloch vector may be immediately extended to any 2-level state, pure or mixed, by adopting the same definition of Eq. (3.4) for any state ρ. The length squared of the Bloch Vector, using the definition 3.4, is given by $\mathrm{Tr}(\rho^2) - 2\det\rho$, and thus represents a point in the *Bloch Ball* [Fig. (1)]. The zero vector is attained only for the *maximally random* mixed state $\mathrm{diag}(1/2, 1/2)$.

3.2. Bloch Sphere and Bloch Ball: N > 2.

For the general, N-level, case the concept corresponding to the Bloch vector is the *coherence vector*. We first define a basis for $u(n)$, consisting of N^2 hermitian matrices F_μ analogous to the Pauli matrices Eq. (3.3) above as follows:

$$\{F_\mu : \mu = 1 \ldots N^2\}, \quad \mathrm{Tr}(F_\mu F_\nu) = \delta_{\mu,\nu} \tag{3.5}$$

We choose the F_μ to be trace-zero except $F_{N^2} = \frac{1}{\sqrt{N}} I$ where I is the identity matrix, and define

$$s_\mu = \mathrm{Tr}(\rho F_\mu). \tag{3.6}$$

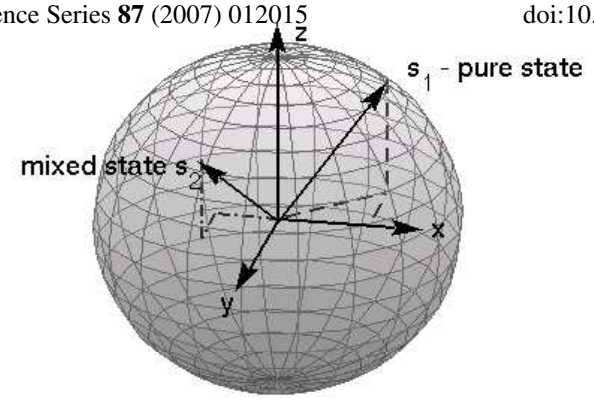

Figure 1. Bloch ball. Pure states are located on the surface of the Bloch ball, mixed states correspond to points in the interior.

Since the N^2-th component $s_{N^2} = \mathrm{Tr}(\rho F_{N^2}) = 1/\sqrt{N}$ is constant, one generally considers the N^2-1-dimensional *coherence* vector $\mathbf{s} = (s_k)_{k=1}^{N^2-1}$, which is the equivalent of the usual 3-dimensional Bloch vector in the $N = 2$ case, apart from the normalisation, which is traditionally chosen to give the unit sphere for $N = 2$.

Since hamiltonian evolution as in Eq. (1.2) leaves $\mathrm{Tr}(\rho^2)$ and $s_{N^2} = (1/N)\,\mathrm{Tr}(\rho^2)$, and consequently the length of the vector $\mathbf{s}$ invariant, it corresponds to a rotation on S^{N^2-2} (or its interior).

3.3. Stable points

For evolution governed by dissipation alone—essentially this corresponds to $L_H = 0$ in Eq. (1.11), but see below—the Bloch vector traces a path in the Bloch Sphere ending on the z-axis. In the case $N = 2$ the dissipation (super-)operator Eq. (1.8) is

$$L_D = \begin{bmatrix} -\gamma_{21} & 0 & 0 & \gamma_{12} \\ 0 & -\Gamma & 0 & 0 \\ 0 & 0 & -\Gamma & 0 \\ \gamma_{21} & 0 & 0 & -\gamma_{12} \end{bmatrix} \tag{3.7}$$

where the non-negative elements $\Gamma, \gamma_{12}, \gamma_{21}$ must satisfy the well-known constraint

$$2\Gamma \geq \gamma_{12} + \gamma_{21}. \tag{3.8}$$

The corresponding evolution equation

$$\dot{\mathbf{r}} = L_D \mathbf{r}. \tag{3.9}$$

has the solution

$$\mathbf{r}(t) = \exp(L_D t)\mathbf{r}(0) \tag{3.10}$$

which corresponds to a value of the state $\rho(t)$

$$\begin{bmatrix} \frac{\rho_{11}(\gamma_{12}+\gamma_{21}E)+\gamma_{12}\rho_{22}(1-E)}{\gamma_{21}+\gamma_{12}} & e^{-t\Gamma}\rho_{12} \\ e^{-t\Gamma}\rho_{21} & \frac{\gamma_{21}\rho_{11}(1-E)+\rho_{22}(\gamma_{21}+\gamma_{12}E)}{\gamma_{21}+\gamma_{12}} \end{bmatrix} \tag{3.11}$$

where $E = e^{-t(\gamma_{21}+\gamma_{12})}$. In the limit $t \to \infty$ we have

$$\rho(\infty) = \begin{bmatrix} \gamma_{12}/(\gamma_{12}+\gamma_{21}) & 0 \\ 0 & \gamma_{21}/(\gamma_{12}+\gamma_{21}) \end{bmatrix} \tag{3.12}$$

which in general is a mixed state, unless one of γ_{12}, γ_{21} is zero. Thus the limit point is $(0, 0, z_\infty)$ on the z-axis of the Bloch ball, where $z_\infty = (\gamma_{12} - \gamma_{21})/(\gamma_{12} + \gamma_{21})$.

Although we have described the purely dissipative dynamics as being defined by $L_H = 0$, in fact it is only the control terms which are absent from Eq. (1.4). The inclusion of H_0 does not alter the above conclusions, since $[L_{H_0}, L_D] = 0$ in our model (see Appendix B); and the off-diagonal terms of Eq. (3.11) are simply multiplied by a phase factor $e^{\pm 2i\epsilon t}$, where ϵ is the energy level scale, which does not change the limit point.

In the general N-level case the matrix L_D has a zero eigenvalue (see Appendix A), whose eigenvector $\mathbf{r}(0)$ corresponds to a *stable point*, as well as a limit point, in the generalized Bloch ball under purely dissipative evolution. This point is also stable under the action of H_0. For example, in the $N = 3$ case, this corresponds to a stable (limit) state $\rho(\infty)$

$$\frac{1}{\mathcal{N}} \begin{bmatrix} \gamma_{13}\gamma_{12} + \gamma_{13}\gamma_{32} + \gamma_{23}\gamma_{12} & 0 & 0 \\ 0 & \gamma_{23}\gamma_{31} + \gamma_{13}\gamma_{21} + \gamma_{23}\gamma_{21} & 0 \\ 0 & 0 & \gamma_{12}\gamma_{31} + \gamma_{32}\gamma_{21} + \gamma_{32}\gamma_{31} \end{bmatrix}$$

with

$$\mathcal{N} = \gamma_{13}\gamma_{12} + \gamma_{13}\gamma_{32} + \gamma_{23}\gamma_{12} + \gamma_{23}\gamma_{31} + \gamma_{13}\gamma_{21} + \gamma_{23}\gamma_{21} + \gamma_{12}\gamma_{31} + \gamma_{32}\gamma_{21} + \gamma_{32}\gamma_{31}$$

These stable or limit states are in general mixed states; however they are pure states for various zero values of the relaxation constants γ_{nm}; for example, in the $N = 3$ case above, when $\gamma_{12} = \gamma_{21} = 0$ and $\gamma_{31} = 0$ or $\gamma_{32} = 0$ and 4 similar patterns.

Renyi entropy

A characterization of the entropy of a N-level state which fits in with the geometric picture outlined above is the so-called *Renyi entropy*, defined for the state ρ by

$$\mathcal{E}_R = -\log_N \mathrm{Tr}(\rho^2), \tag{3.13}$$

which takes the values 0 for a pure state, and 1 for the *maximally random* state $\rho = I/N$, where I is the identity matrix.

For the 2-level dissipative limit of Eq. (3.12) above, we have

$$\mathcal{E}_R = -\log_2({\gamma_{12}}^2 + {\gamma_{21}}^2)/(\gamma_{12} + \gamma_{21})^2 \tag{3.14}$$

which attains its minimal value 0 for $\gamma_{12} = 0$ or $\gamma_{21} = 0$, in which case the stable state is pure, and its maximal value 1 for $\gamma_{12} = \gamma_{21}$ when the stable state is the maximally random state. Similar results hold in the general case.

4. Measures of control-reachable sets

The motivation for the preceding discussion has been to give the reader an intuitive feeling for the behaviour of a qubit under the action of a control hamiltonian in the presence of dissipation. Our objective is to show that a combination of control hamiltonian and dissipative actions may well provide a richer set of targets within the Bloch ball than would be otherwise available to either hamiltonian control or dissipative processes alone. In summary, we observed:

- A purely hamiltonian action behaves as a rotation, and the evolution is restricted to the surface of the Bloch sphere (for a pure state); or to the surface of a spherical shell within the Bloch ball (for a proper mixed state).

- For a purely dissipative process, the evolution traces out a path in the Bloch ball ending on the z-axis.

The set of reachable states in the absence of control is the path traced out by the initial state under the given fixed evolution, which has measure zero in R^{N^2-1}. If the system is subject to controlled evolution governed by a control Hamiltonian of the form (1.4) with controls f_m bounded by $\| f_m \| \leq F$ and $\{\gamma_k\}$ is a fixed set of dissipation parameters, then the set of reachable states in time $t \leq T$ is

$$\{ R(\gamma, f_m, t) : \| f_m \| \leq F, t \leq T \} \tag{4.1}$$

where $R(\gamma, f_m, t)$ is the state we can reach at time t given the controls f_m and dissipation parameters γ. When there is no dissipation, the union of the sets above is a subset of the sphere whose radius is determined by the trace of the initial state, due to the constraint of unitary evolution, and hence, the set of reachable states has Lebesgue measure zero in R^{N^2-1}; in other words, even if the system is *completely controllable*, the set of states that are reachable from any given initial state, has measure zero.

Thus, reachable sets of non-zero measure are only possible as a result of the combination of hamiltonian and dissipative actions. For example, starting with a pure state on the Bloch sphere, the reachable sets are sets of of states consisting of *unions* of portions of spherical shells within the Bloch sphere. In general, these sets have a non-zero measure depending on the strength of the control parameters in relation to the dissipation parameters.

To put these ideas in a slightly more formal context, let us consider the case $N = 2$. If μ is the (Lebesgue) measure in R^3 then taking $\mu(\text{BlochBall}) = 1$ we have

	Action	Dynamical set	Measure
$\mu(0, F, T) = 0$	No dissipation	Bloch sphere	zero
$\mu(\gamma, 0, T) = 0$	No control	Path in Bloch ball	zero
$\mu(\gamma, F, T) > 0$	Finite control and dissipation	subset of Bloch ball	non-zero
$\mu(\gamma, \infty, \infty) = 1$	Infinite control, time	Bloch ball	1

where T is the maximum time to reach the state, F the maximum field strength and γ are the dissipation parameters. So even if we have no control over the dissipative processes, the total set of states we can reach by applying control fields is greater.

This is one case where dissipative effects, if constructively exploited, may result in better outcomes than those due to purely hamiltonian evolution.

5. $N \geq 3$: Bohm-Aharonov type effects

What we mean by *Bohm-Aharonov type effects* in the title of this section, is a certain 'impossibility of isolating' quantum subsystems. We illustrate these effects by considering the dissipative dynamics of a two-level subsystem embedded in a three-level system.

For simplicity, we consider the case of pure dissipation (no control) as discussed in Sec. 3.3. Let us assume that we have a two-level subsystem, embedded in a three-level system, and subject to population relaxation processes from level 2 to 1 and level 1 to 2 at the rates γ_{12} and γ_{21}, respectively. In the absence of other (known) relaxation processes involving the third state, one may naively be tempted to take the evolution of the three-level system to be

$$\rho(t) = \begin{bmatrix} \frac{\rho_{11}(\gamma_{12}+\gamma_{21}E)+\gamma_{12}\rho_{22}(1-E)}{\gamma_{21}+\gamma_{12}} & e^{-t\Gamma}\rho_{12} & \rho_{13} \\ e^{-t\Gamma}\rho_{21} & \frac{\gamma_{21}\rho_{11}(1-E)+\rho_{22}(\gamma_{21}+\gamma_{12}E)}{\gamma_{21}+\gamma_{12}} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} \tag{5.1}$$

where the evolution in the $(1, 2)$ subspace is given by Eq. (3.11), and the populations and coherences involving the third state (which is not directly affected by either control or relaxation processes), remain constant.

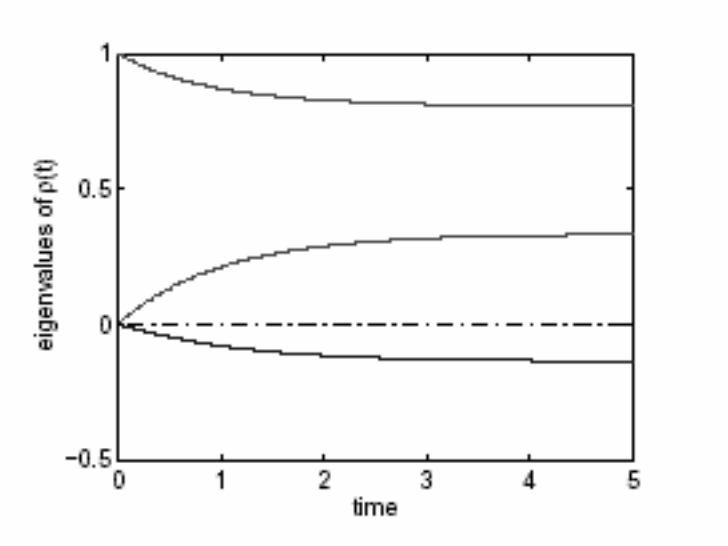

Figure 2. Eigenvalues of Three-Level System assuming 3rd level isolated

However, it is clear by inspection that the above 'solution' leads to a non-physical state for $t \to \infty$ unless $\rho_{13} = \rho_{23} = 0$, i.e., if there is any initial coherence between the $(1,2)$ subspace and the third level then this coherence *must* decay, thus the third state experiences the effect of population relaxation processes entirely restricted to the $(1,2)$ subspace. In fact, one can show that not only must the decoherence rates Γ_{13}, Γ_{23} be non-zero, but $\Gamma_{13} \geq \gamma_{12}/2$ and $\Gamma_{23} \geq \gamma_{12}/2$. Otherwise, the resulting dissipation superoperator governing the dynamics will cease to be completely positive, and if there are initial coherences $\rho_{13}, \rho_{23} \neq 0$, then the state $\rho(t)$ in Eq. (5.1) will become non-physical, not only for $t \to \infty$ but for all $t > 0$. For example, Fig.(2), showing the eigenvalues of (5.1) as a function of time for the pure initial state $(1,1,1)^T/\sqrt{3}$ and $\gamma_{21} = 0$, $\gamma_{12} = \gamma$, $\Gamma = \frac{1}{2}\gamma$, clearly shows that one of the eigenvalues is negative for $t > 0$.

But population relaxation is not the only source of constraints on the decoherence rates for $N > 2$. Even if there is no population relaxation at all, i.e., $\gamma_{kn} = 0$ for all k, n, and the system experiences only pure dephasing, we cannot choose the decoherence rates Γ_{kn} arbitrarily. For example, setting $\Gamma_{12} \neq 0$ and $\Gamma_{23} = \Gamma_{13} = 0$ for our three-level system gives

$$\rho(t) = \begin{bmatrix} \rho_{11} & e^{-\Gamma_{12}t}\rho_{12} & \rho_{13} \\ e^{-\Gamma_{12}t}\rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix}. \tag{5.2}$$

Choosing $\rho(0)$ to be again the (projector onto the) pure state $(1,1,1)^T/\sqrt{3}$, we again obtain a density operator $\rho(t)$ with negative eigenvalues for $t > 0$, as a simple calculation reveals. This shows that additional constraints on the decoherence rates are necessary to ensure that the state of the system remains physical. For $N = 3$ one can show that the pure dephasing rates must satisfy the inequalities [4]

$$(\sqrt{\Gamma_b} - \sqrt{\Gamma_c})^2 \leq \Gamma_a \leq (\sqrt{\Gamma_b} + \sqrt{\Gamma_c})^2 \tag{5.3}$$

where $\{a, b, c\}$ is any permutation of $\{12, 13, 23\}$.

6. $N \geq 4$: Decay of Entanglement

The results of the previous section also have implications for the decay of entanglement. An entangled state of two qubits may be considered in the context of a 4-state system arising from

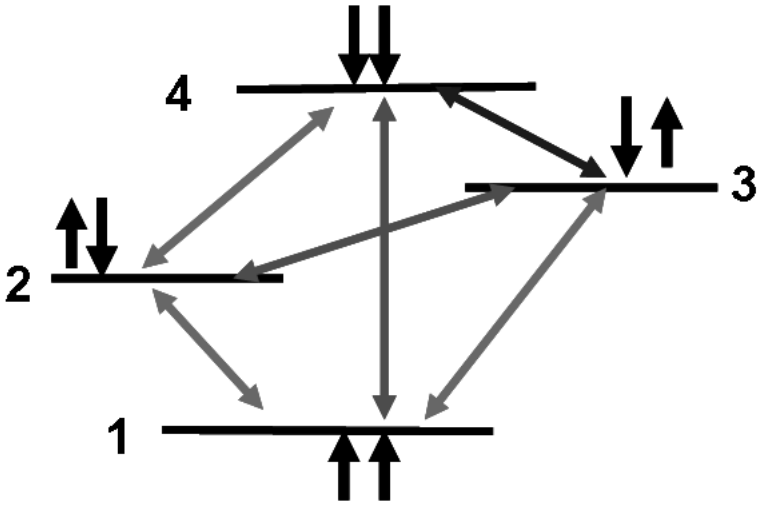

Figure 3. Two-spin entanglement: Four Level System

two qubits as follows: Write the four states as

$$\Psi_1 = |0\rangle|0\rangle, \qquad \Psi_2 = |0\rangle|1\rangle$$
$$\Psi_3 = |1\rangle|0\rangle, \qquad \Psi_4 = |1\rangle|1\rangle$$

where $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$. States that are entangled include for example

$$\begin{aligned} \Psi_1 + \Psi_4 &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \\ \Psi_2 + \Psi_3 &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle) \end{aligned} \tag{6.1}$$

Each of the two states in Eq. (6.1) is *maximally* entangled. For this maximal measure of entanglement to be maintained, the phase relation between the two components of each state must remain unchanged. The entanglement will decay as a result of phase decoherence between the components. Thus the rate of entanglement decay is governed by the phase decoherence parameters $\Gamma_{14}(=\Gamma_{41})$ and $\Gamma_{23}(=\Gamma_{32})$. In the following we show that the rate of entanglement decay is limited by the single qubit dephasing rates regardless of the precise decoherence mechanisms. Our only basic assumption is that the decay processes satisfy the dissipation equations Eq. (2.1) and Eq. (2.2), which form the basis of our discussion here.

For simplicity, we restrict ourselves here to the case where the T_1 relaxation times are much greater than the T_2 decoherence times, and the contribution of population relaxation processes to the decoherence rates is therefore negligible, which is the case for many systems under investigation for quantum information processing applications, although similar results can be obtained for the more general case as well.

Fig. (3) shows that the decay of entanglement is governed by Γ_{14} and Γ_{23}, i.e. giving a total *entanglement* decay rate of $\Gamma_{14} + \Gamma_{23}$, while the other Γs involve *single* spin decay. Constraints imposed by the standard evolution equation Eq. (2.1) impose the following inequalities on the decoherence rates [4]:

$$|\Gamma_{12} + \Gamma_{34} - (\Gamma_{13} + \Gamma_{24})| \leq \Gamma_{14} + \Gamma_{23} \leq \Gamma_{12} + \Gamma_{13} + \Gamma_{24} + \Gamma_{34}. \tag{6.2}$$

Defining the average entanglement decoherence rate Γ^{En} by

$$\Gamma^{En} = \frac{1}{2}(\Gamma_{14} + \Gamma_{23}) \tag{6.3}$$

and the average single-spin decoherence Γ^{Sp} by

$$\Gamma^{Sp} = \frac{1}{4}(\Gamma_{12} + \Gamma_{13} + \Gamma_{24} + \Gamma_{34}) \tag{6.4}$$

we obtain the result that the entanglement cannot decay faster on average than twice the single-qubit decoherence rate.

7. Conclusions

We have looked at the effects of dissipation on an N-level quantum system, for the cases $N = 2, 3, 4$. For the $N = 2$ case, we may avail of the geometric intuition supplied by the Bloch Sphere, and its interior, the Bloch Ball. This allows us to surmise that the reachable states under the joint effects of dissipation and an applied control hamiltonian are of non-zero measure, unlike the case of either of these evolutions being applied separately.

When $N = 3$, specific examples reveal the impossibility of isolation of a two-level subsystem, apart from certain initial states. Thus interaction with a populated third level leads to decoherence effects in the subsystem.

The simplest case of an entangled example occurs for $N = 4$. Although decoherence will cause the entanglement to degrade, complete positivity of the evolution induces universal constraints on the decoherence rates, which lead to *universal upper bounds* on the rate of entanglement decay in terms of the *single* qubit decoherence rates.

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Appendix A

We show that the dissipation super-operator L_D has a zero eigenvalue.

Consider the following partial row sums $R(\alpha)$ of L_D

$$R(\alpha) = \sum_{k=1\ldots N} (L_D)_{[k;k],\alpha} \qquad (\alpha = 1 \ldots N^2).$$

Once again referring to Eq. (1.8) we see that the only non-vanishing contributions are

$$\sum_{k=1\ldots N} \sum_{(n\neq k)} (L_D)_{[k;k],[n;n]} = \sum_{n\neq k} \gamma_{kn}$$

and

$$(L_D)_{[n;n],[n;n]} = -\sum_{n\neq k} \gamma_{kn}$$

whose sum is zero. Thus $R(\alpha)$ gives a vanishing row sum for each column α, and L_D has vanishing determinant, and thus a zero eigenvalue.

Appendix B

We show that the dissipation super-operator L_D commutes with a diagonal free-hamiltonian super-operator L_{H_0}.

We first recall the standard algebraic trick applied in evaluating Liouville equations (see, for example [5]). The correspondence between ρ and $\mathbf{r}$ as given in Eq. (1.10) tells us, after some manipulation of indices, that

$$\rho \to \mathbf{r} \Rightarrow A\rho B \to A \otimes \tilde{B} \ \ \mathbf{r} \tag{7.1}$$

using the Kronecker product. Thus the super-operator corresponding to the hamiltonian term in Eq. (1.5) is

$$H \otimes I - I \otimes \tilde{H} \tag{7.2}$$

and the corresponding super-operator for the diagonal matrix $H_0 = \mathrm{diag}\{h_{(1)}, \ldots h_{(N)}\}$ is

$$(L_{H_0})_{[u;v],[r;s]} = (h_{(u)} - h_{(v)})\delta_{ur}\delta_{vs} \tag{7.3}$$

using the notation of Eq. (1.9) for which

$$(A \otimes B)_{[u;v],[r;s]} = A_{ur}B_{vs}. \tag{7.4}$$

Therefore we have

$$[L_D, L_{H_0}]_{[k;n],[r;s]} = (h_{(k)} - h_{(n)} - h_{(r)} + h_{(s)})(L_D)_{[k;n],[r;s]}. \tag{7.5}$$

Consulting Eq. (1.8) we note that all the non-zero elements of L_D have vanishing coefficients in Eq. (7.5); whence $[L_D, L_{H_0}] = 0$.

The results of Appendix A and Appendix B together tell us that there exists a stable vector under the evolution determined by the joint action of the free hamiltonian $L_{H_0}$and the dissipation super-operator L_D.