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# Compression and Erdős-Ko-Rado graphs

Fred Holroyd<sup>1</sup>, Claire Spencer<sup>2</sup> and John Talbot<sup>3</sup>

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<sup>1</sup>Department of Pure Mathematics, The Open University,  
Walton Hall, Milton Keynes MK7 6AA, UK  
f.c.holroyd@open.ac.uk

<sup>2</sup>Department of Mathematics, University of Reading,  
Reading, UK  
c.l.spencer@reading.ac.uk

<sup>3</sup>Department of Mathematics, University College London,  
London, WC1E 6BT, UK  
talbot@math.ucl.ac.uk

## Abstract

For a graph  $G$  and integer  $r \geq 1$  we denote the collection of independent  $r$ -sets of  $G$  by  $\mathcal{I}^{(r)}(G)$ . If  $v \in V(G)$  then  $\mathcal{I}_v^{(r)}(G)$  is the collection of all independent  $r$ -sets containing  $v$ . A graph  $G$ , is said to be  $r$ -EKR, for  $r \geq 1$ , iff no intersecting family  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  is larger than  $\max_{v \in V(G)} |\mathcal{I}_v^{(r)}(G)|$ . There are various graphs which are known to have this property: the empty graph of order  $n \geq 2r$  (this is the celebrated Erdős-Ko-Rado theorem), any disjoint union of at least  $r$  copies of  $K_t$  for  $t \geq 2$ , and any cycle. In this paper we show how these results can be extended to other classes of graphs via a compression proof technique.

In particular we extend a theorem of Berge [2], showing that any disjoint union of at least  $r$  complete graphs, each of order at least two, is  $r$ -EKR. We also show that paths are  $r$ -EKR for all  $r \geq 1$ .

# 1 Introduction

An *independent* set in a graph  $G = (V, E)$ , is a subset of the vertices not containing any edges. For an integer  $r \geq 1$  we denote the collection of independent  $r$ -sets of  $G$  by

$$\mathcal{I}^{(r)}(G) = \{A \subset V(G) : |A| = r \text{ and } A \text{ is an independent set}\}.$$

If  $v \in V(G)$  then the collection of independent  $r$ -sets containing  $v$  is

$$\mathcal{I}_v^{(r)}(G) = \{A \in \mathcal{I}^{(r)}(G) : v \in A\}.$$

Such a family is called a *star*.

A graph  $G$  is  *$r$ -EKR* iff no intersecting family of independent  $r$ -sets is larger than the largest star in  $\mathcal{I}^{(r)}(G)$ . If  $G$  is  $r$ -EKR and any intersecting family  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  of maximum size is a star then  $G$  is said to be *strictly  $r$ -EKR*. If  $G$  is  $r$ -EKR and  $\mathcal{I}_v^{(r)}$  is a star of maximum size then we say that  $v$  is a *star centre* for  $G$ .

In this setting the classical Erdős-Ko-Rado theorem can be stated as follows.

**Theorem 1 (Erdős-Ko-Rado [8])** *If  $E_n$  is the empty graph of order  $n$  then  $E_n$  is  $r$ -EKR for  $n \geq 2r$  and strictly so for  $n > 2r$ .*

There exist many results giving Erdős-Ko-Rado theorems for integer sequences (see for example [9] and [7]). Three such results are of direct relevance to the current paper and we restate them below in the language of graphs.

**Theorem 2 (Berge [2], Livingston [14] )** *If  $r \geq 1$ ,  $t \geq 2$  and  $G$  is the disjoint union of  $r$  copies of  $K_t$  then  $G$  is  $r$ -EKR and strictly so unless  $t = 2$ .*

This result was originally proved by Berge, with Livingston providing a characterization of the extremal case. Other proofs of this result were given by Gronau [10] and Moon [16].

In fact Berge proved more than Theorem 2.

**Theorem 3 (Berge [2])** *If  $G$  is the disjoint union of  $r$  complete graphs each of order at least two then  $G$  is  $r$ -EKR.*

A generalization of Theorem 2 was first stated by Meyer [15] and proved by Deza and Frankl [5].

**Theorem 4 (Meyer [15], Deza and Frankl [5])** *If  $r \geq 1$ ,  $t \geq 2$  and  $G$  is the disjoint union of  $n \geq r$  copies of  $K_t$  then  $G$  is  $r$ -EKR and strictly so unless  $t = 2$  and  $r = n$ .*

A new proof of this result using a variant of Katona’s circle method (see [13]) was given by Bollobás and Leader [3].

Another result determining the EKR properties of a graph is the following theorem for cycles and their powers, which was previously a conjecture of Holroyd and Johnson [11]. For  $1 \leq k \leq n$  the  $k^{\text{th}}$  power of the  $n$ -cycle,  $C_n^k$ , is the graph with vertex set  $[n] = \{1, 2, \dots, n\}$  and edges between  $a, b \in [n]$  iff  $1 \leq |a - b \pmod n| \leq k$ .

**Theorem 5 (Talbot[17])** *If  $r, k, n \geq 1$  then  $C_n^k$  is  $r$ -EKR and strictly so unless  $n = 2r + 2$  and  $k = 1$ .*

Erdős-Ko-Rado theorems for other structures have also been widely studied. In particular there are analogues of the Erdős-Ko-Rado theorem for subcubes of cubes [6], Hamming schemes [16], permutations [4] and vector spaces [12].

Many authors also consider generalizations of Erdős-Ko-Rado results to  $t$ -intersecting families of sets (that is to families in which any two members meet in at least  $t$  elements). With such generalizations it is often far more difficult to prove exact (as opposed to asymptotic) results. Indeed the “simplest” generalization of Theorem 1 to  $t$ -intersecting families was a long standing conjecture of Frankl before its proof by Ahlswede and Khachatrian [1]. We are interested in exact results and do not consider any such generalizations.

A useful starting point for the reader interested in the many other known Erdős-Ko-Rado type results is the survey paper of Deza and Frankl [5].

## 2 Results

Although this is not made explicit in [17], the proof of Theorem 5 uses a type of compression that is essentially equivalent to contracting an edge in

the underlying graph. In the present paper we wish to show how this idea can be used to prove that various other graphs are also  $r$ -EKR.

In particular we have the following common generalization of Theorems 3 and 4.

**Theorem 6** *If  $G$  is the disjoint union of  $n \geq r$  complete graphs each of order at least two then  $G$  is  $r$ -EKR.*

Note that Theorem 3 is the case  $n = r$  of Theorem 6, while Theorem 4 is the case of Theorem 6 given by taking all the complete graphs to have the same order.

We remark that our proof of Theorem 6 uses Theorem 4 and so does not yield a new proof of that result.

Our second result is an analogue of the Erdős-Ko-Rado theorem for paths and their powers. For  $1 \leq k \leq n$  the  $k^{\text{th}}$  power of the  $n$ -path,  $P_n^k$ , is the graph with vertex set  $[n] = \{1, 2, \dots, n\}$  and edges between  $a, b \in [n]$  iff  $1 \leq |a - b| \leq k$ .

**Theorem 7** *If  $r, k, n \geq 1$  then  $P_n^k$  is  $r$ -EKR.*

The compression proof technique also extends to other types of graph and our final theorem gives a large class of graphs which are all  $r$ -EKR.

**Theorem 8** *If  $G$  is a disjoint union of  $n \geq 2r$  complete graphs, cycles and paths, including an isolated singleton, then  $G$  is  $r$ -EKR.*

### 3 Proofs

In order to state the key lemmas we require some notation.

If  $e$  is an edge of the graph  $G = (V, E)$  we define  $G/e$  to be the graph obtained from  $G$  by contracting the edge  $e$ . We also define  $G \downarrow e$  to be the graph obtained from  $G$  by removing the vertices in  $e$  as well as their neighbours. As usual we denote the neighbours of a vertex  $v$  by  $\Gamma(v)$ .

The following two technical lemmas relate intersecting families and stars in  $\mathcal{I}^{(r)}(G)$  to intersecting families and stars in  $\mathcal{I}^{(r)}(G/e)$  and  $\mathcal{I}^{(r-1)}(G \downarrow e)$ . These will enable us to prove our main results by induction.

**Lemma 9** Let  $G = (V, E)$  be a graph and  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  be an intersecting family. If  $e = \{v, w\} \in E$  is an edge in  $G$  then there exist families  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  satisfying:

- (i)  $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}|$ .
- (ii)  $\mathcal{B} \subseteq \mathcal{I}^{(r)}(G/e)$  is intersecting.
- (iii)  $\mathcal{C} \subseteq \mathcal{I}^{(r-1)}(G \downarrow e)$  is intersecting.
- (iv)  $\mathcal{D} = \{A \in \mathcal{A} : v \in A \text{ and } \Gamma(w) \cap (A \setminus \{v\}) \neq \emptyset\}$ .
- (v)  $\mathcal{E} = \{A \in \mathcal{A} : w \in A \text{ and } \Gamma(v) \cap (A \setminus \{w\}) \neq \emptyset\}$ .
- (vi) If  $C \in \mathcal{C}$  and  $F \in \mathcal{D} \cup \mathcal{E}$  then  $C \cap F \cap V(G \downarrow e) \neq \emptyset$ .
- (vii) If  $D \in \mathcal{D}$  and  $E \in \mathcal{E}$  then  $D \cap E \cap V(G \downarrow e) \neq \emptyset$ .

**Lemma 10** If  $e = \{v, w\}$  is an edge in the graph  $G = (V, E)$  and  $x \in V(G \downarrow e)$  then

$$|\mathcal{I}_x^{(r)}(G)| = |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{I}_x^{(r-1)}(G \downarrow e)| + |\mathcal{D}_x| + |\mathcal{E}_x|,$$

where

$$\mathcal{D}_x = \{A \in \mathcal{I}_x^{(r)}(G) : v \in A \text{ and } \Gamma(w) \cap (A \setminus \{v\}) \neq \emptyset\}$$

and

$$\mathcal{E}_x = \{A \in \mathcal{I}_x^{(r)}(G) : w \in A \text{ and } \Gamma(v) \cap (A \setminus \{w\}) \neq \emptyset\}.$$

*Proof of Lemma 9:* Let  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  be intersecting. We consider the effect of contracting an edge  $e = \{v, w\}$  in  $G$ . We define a contraction function,  $c : V(G) \rightarrow V(G/e)$  by

$$c(x) = \begin{cases} v, & x = w, \\ x, & x \neq w. \end{cases}$$

Let

$$\begin{aligned} \mathcal{B} &= \{c(A) : A \in \mathcal{A} \text{ and } c(A) \in \mathcal{I}^{(r)}(G/e)\}, \\ \mathcal{C} &= \{A \setminus \{v\} : v \in A \in \mathcal{A} \text{ and } A \setminus \{v\} \cup \{w\} \in \mathcal{A}\}, \\ \mathcal{D} &= \{A \in \mathcal{A} : v \in A \text{ and } \Gamma(w) \cap (A \setminus \{v\}) \neq \emptyset\}, \\ \mathcal{E} &= \{A \in \mathcal{A} : w \in A \text{ and } \Gamma(v) \cap (A \setminus \{w\}) \neq \emptyset\}. \end{aligned}$$

If  $A, B \in \mathcal{A}$  and  $A \neq B$  then  $c(A) = c(B)$  iff  $A\Delta B = \{v, w\}$ . Hence

$$|\{A \in \mathcal{A} : c(A) \in \mathcal{I}^{(r)}(G/e)\}| = |\mathcal{B}| + |\mathcal{C}|.$$

Also if  $A \in \mathcal{A}$  then  $c(A) \notin \mathcal{I}^{(r)}(G/e)$  iff  $A \in \mathcal{D} \cup \mathcal{E}$ . Hence  $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}|$ , which is (i).

The fact that  $\mathcal{B} \subseteq \mathcal{I}^{(r)}(G/e)$  is an intersecting family follows simply because  $\mathcal{A}$  is intersecting, so (ii) holds.

If  $C \in \mathcal{C}$  then  $C \cup \{v\}, C \cup \{w\} \in \mathcal{A}$  hence  $C \in \mathcal{I}^{(r-1)}(G \downarrow e)$ . With a little more thought it is also clear that  $\mathcal{C}$  is an intersecting family. Let  $C, D \in \mathcal{C}$ , if  $C \cap D = \emptyset$  then  $\mathcal{A}$  contains the two disjoint sets  $C \cup \{v\}$  and  $D \cup \{w\}$ . This contradicts the fact that  $\mathcal{A}$  is intersecting. Hence  $\mathcal{C}$  is also intersecting, and so (iii) holds.

The definitions of the families  $\mathcal{D}$  and  $\mathcal{E}$  give (iv) and (v).

To see that (vi) holds let  $C \in \mathcal{C}$ , so  $C \cup \{v\}, C \cup \{w\} \in \mathcal{A}$ . If  $F \in \mathcal{D} \cup \mathcal{E} \subseteq \mathcal{A}$  then  $(C \cup \{w\}) \cap F \neq \emptyset$  and  $(C \cup \{v\}) \cap F \neq \emptyset$  but either  $v \notin F$  or  $w \notin F$ . Hence  $C \cap F \cap V(G \downarrow e) \neq \emptyset$ .

Finally, if  $D \in \mathcal{D}$  and  $E \in \mathcal{E}$  then  $v \in D$  and  $w \in E$  imply that

$$D \cap E \cap (\Gamma(v) \cup \Gamma(w) \cup \{v, w\}) = \emptyset.$$

So  $D \cap E \neq \emptyset$  implies that (vii) must hold.  $\square$

*Proof of Lemma 10:* This follows similarly to the proof of Lemma 9, via contracting the edge  $e = \{v, w\}$ . Let  $c : V(G) \rightarrow V(G/e)$  be as defined in the proof of Lemma 9.

Then  $c$  is a surjection between the families  $\mathcal{I}_x^{(r)}(G) \setminus (\mathcal{D}_x \cup \mathcal{E}_x)$  and  $\mathcal{I}_x^{(r)}(G/e)$ . Moreover if  $A \neq B$  then  $c(A) = c(B)$  iff  $A\Delta B = \{v, w\}$  and the number of sets in  $\mathcal{I}_x^{(r)}(G/e)$  with two preimages under  $c$  is exactly  $|\mathcal{I}_x^{(r-1)}(G \downarrow e)|$ . Hence

$$|\mathcal{I}_x^{(r)}(G)| = |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{I}_x^{(r-1)}(G \downarrow e)| + |\mathcal{D}_x| + |\mathcal{E}_x|.$$

$\square$

*Proof of Theorem 6:* We prove this result by induction on  $r$ . It is clearly true for  $r = 1$  so we may suppose that  $r \geq 2$  and the result holds for smaller values of  $r$ .

We now use induction on the number of vertices in  $G$ . Theorem 4 implies that the result holds when  $G$  consists of  $n \geq r$  copies of  $K_t$ , for  $t \geq 2$ . So let

$$G = K_{t_1} \cup \cdots \cup K_{t_n},$$

with  $2 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ , not all equal. We may suppose that the result holds for all graphs of the correct form with fewer vertices.

Suppose that  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  is intersecting. Let  $v, w \in K_{t_n}$ , we will contract the edge  $e = \{v, w\}$ . Then

$$G/e = K_{t_1} \cup \cdots \cup K_{t_{n-1}},$$

and

$$G \downarrow e = K_{t_1} \cup \cdots \cup K_{t_{n-1}}.$$

Using the notation of Lemma 9 we have  $\mathcal{D} = \mathcal{E} = \emptyset$ . Hence by Lemma 9 (i)

$$|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|. \quad (1)$$

Then for any  $x \in K_{t_1} \subseteq G \downarrow e$  we have, in notation of Lemma 10, that  $\mathcal{D}_x = \mathcal{E}_x = \emptyset$ . So Lemma 10 implies that

$$|\mathcal{I}_x^{(r)}(G)| = |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{I}_x^{(r-1)}(G \downarrow e)|. \quad (2)$$

The observation that  $t_1 \leq t_i$ , for any  $1 \leq i \leq n$ , implies that we also have

$$|\mathcal{I}_x^{(r)}(G/e)| = \max_{v \in V(G/e)} |\mathcal{I}_v^{(r)}(G/e)| \quad (3)$$

and

$$|\mathcal{I}_x^{(r-1)}(G \downarrow e)| = \max_{v \in V(G \downarrow e)} |\mathcal{I}_v^{(r-1)}(G \downarrow e)|. \quad (4)$$

Now  $t_n \geq 3$  so  $G/e$  is a smaller graph of the correct form and hence is  $r$ -EKR. Then Lemma 9 (ii) and (3) imply that

$$|\mathcal{B}| \leq |\mathcal{I}_x^{(r)}(G/e)|. \quad (5)$$

Also  $G \downarrow e$  is  $(r-1)$ -EKR, since the result holds for smaller values of  $r$ . So Lemma 9 (iii) and (4) imply that

$$|\mathcal{C}| \leq |\mathcal{I}_x^{(r-1)}(G \downarrow e)| \quad (6)$$



Hence, using equations (1), (2), (5) and (6), we obtain

$$|\mathcal{A}| \leq |\mathcal{I}_x^{(r)}(G)|.$$

□

We give two proofs of Theorem 7. The first is similar to the other proofs in this paper, using compression. We include this since a similar argument is required for part of the proof of Theorem 8. However we also give a far simpler second proof.

*First Proof of Theorem 7:* We first note that for any  $n$ ,  $r$  and  $k$

$$\max_{x \in V(P_n^k)} |\mathcal{I}_x^{(r)}(P_n^k)|$$

is achieved by taking  $x \in \{1, n\}$ .

Again we prove this result by induction on  $r$ . The result clearly holds for  $r = 1$  so we may assume  $r \geq 2$  and that the result is true for smaller values of  $r$ .

We now prove the result for  $r$  by induction on  $n$ . For  $n < (r - 1)k + r$  there is nothing to prove since  $\mathcal{I}^{(r)}(P_n^k)$  is empty. For  $n = (r - 1)k + r$  the result also holds (since there is only one set in  $\mathcal{I}^{(r)}(P_n^k)$ ). So we may assume that  $n \geq (r - 1)k + r + 1$  and that the result holds for smaller values of  $n$ . In particular  $n \geq k + 3$ .

Let  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(P_n^k)$  be intersecting. Set  $w = n$ ,  $v = n - 1$  and  $e = \{n - 1, n\}$ , and apply Lemma 9. Let  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be the families defined in Lemma 9. In this case  $G/e$  is  $P_{n-1}^k$ , while  $G \downarrow e$  is  $P_{n-k-2}^k$ . Note that  $n \geq k + 3$  implies that  $G \downarrow e$  is non-empty.

We see that in this case  $\mathcal{D}$  is empty and

$$\mathcal{E} = \{A \in \mathcal{A} : n, n - k - 1 \in A\}.$$

Let

$$\mathcal{F} = \{A \setminus \{n\} : A \in \mathcal{E}\}$$

and consider  $\mathcal{C} \cup \mathcal{F}$ . Note that this is a disjoint union since  $n - k - 1$  belongs to every set in  $\mathcal{F}$  but to no set in  $\mathcal{C}$ . Hence

$$|\mathcal{C} \cup \mathcal{F}| = |\mathcal{C}| + |\mathcal{F}| = |\mathcal{C}| + |\mathcal{E}|. \tag{7}$$

Parts (iii) and (vi) of Lemma 9 imply that  $\mathcal{C} \cup \mathcal{F}$  is an intersecting family of independent  $(r-1)$ -sets in the subgraph of  $P_n^k$  induced by  $\{1, 2, \dots, n-k-1\}$ , which is  $P_{n-k-1}^k$ . Our inductive hypothesis for  $r$  then implies that

$$|\mathcal{C} \cup \mathcal{F}| \leq |\mathcal{I}_1^{(r-1)}(P_{n-k-1}^k)|. \quad (8)$$

Now  $G/e$  is  $P_{n-1}^k$ , so part (ii) of Lemma 9 and our inductive hypothesis for  $n$  imply that

$$|\mathcal{B}| \leq |\mathcal{I}_1^{(r)}(P_{n-1}^k)|. \quad (9)$$

Lemma 9 (i), together with equations (7), (8) and (9) imply that

$$|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}| \leq |\mathcal{I}_1^{(r)}(P_{n-1}^k)| + |\mathcal{I}_1^{(r-1)}(P_{n-k-1}^k)|. \quad (10)$$

Applying Lemma 10 we obtain

$$|\mathcal{I}_1^{(r)}(P_n^k)| = |\mathcal{I}_1^{(r)}(P_{n-1}^k)| + |\mathcal{I}_1^{(r-1)}(P_{n-k-2}^k)| + |\mathcal{E}_1|, \quad (11)$$

where

$$\mathcal{E}_1 = \{A \in \mathcal{I}_1^{(r)}(P_n^k) : n-k-1, n \in A\}.$$

Then it is easy to check that

$$|\mathcal{I}_1^{(r-1)}(P_{n-k-1}^k)| = |\mathcal{I}_1^{(r-1)}(P_{n-k-2}^k)| + |\mathcal{E}_1|,$$

and so equations (10) and (11) imply that

$$|\mathcal{A}| \leq |\mathcal{I}_1^{(r)}(P_n^k)|.$$

as required. □

The second proof of Theorem 7 requires the following lemma.

**Lemma 11** *Let  $G$  be an  $r$ -EKR graph with star centre  $v$ . If  $S \subset \Gamma(v)$  then  $G - S$  is also  $r$ -EKR with star centre  $v$ .*

*Proof of Lemma 11:* As  $S \subset \Gamma(v)$ , all the independent  $r$ -sets in  $G$  that contain the star centre  $v$  are independent  $r$ -sets in  $G - S$ . So  $\mathcal{I}_v^{(r)}(G) = \mathcal{I}_v^{(r)}(G - S)$ . Furthermore any independent  $r$ -set in  $G - S$  is also an independent  $r$ -set in  $G$ . So if  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G - S)$  is intersecting then  $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$  and as  $G$  is  $r$ -EKR with star centre  $v$  so

$$|\mathcal{A}| \leq |\mathcal{I}_v^{(r)}(G)| = |\mathcal{I}_v^{(r)}(G - S)|.$$

Hence  $G - S$  is also  $r$ -EKR with star centre  $v$ . □

*Second Proof of Theorem 7:* Using Lemma 11, Theorem 7 is now a simple corollary of Theorem 5.

Theorem 5 implies that  $G = C_{n+k}^k$  is  $r$ -EKR and any vertex  $v$  is a star centre. Taking  $k$  adjacent neighbours of  $v$  for  $S$ , Lemma 11 implies that  $P_n^k = G - S$  is also  $r$ -EKR. □

We turn finally to a proof of Theorem 8. The key ideas have already been presented in Lemmas 9 and 10 as well as in the proofs of Theorems 6 and 7. For this reason our proof is essentially a sketch.

*Proof of Theorem 8:* We will say that a graph  $G$  is  $r$ -mixed, for an integer  $r \geq 1$ , if it satisfies the conditions of Theorem 8. So  $G$  is  $r$ -mixed iff it is the disjoint union of at least  $2r$  complete graphs, paths and cycles, including at least one isolated singleton.

We prove the result by induction on  $r$ . It is clearly true for  $r = 1$  so we may suppose  $r \geq 2$  and that the result holds for smaller values of  $r$ .

We now prove the result for  $r$  by induction on  $|V(G)|$ . If  $G$  is  $r$ -mixed then  $|V(G)| \geq 2r$  with equality iff  $G = E_{2r}$ . So if  $|V(G)| = 2r$  then Theorem 1 implies that  $G$  is  $r$ -EKR. Hence we may suppose that  $|V(G)| > 2r$  and that any  $r$ -mixed graph with fewer vertices is also  $r$ -EKR.

Now either  $G$  is an empty graph of order at least  $2r + 1$ , in which case the result holds by Theorem 1, or  $G$  contains an edge. So we may suppose that  $G$  contains an edge  $e = \{v, w\}$ . We also know that  $G$  contains an isolated singleton  $x$ .

It is easy to check that if  $H$  is any graph with an isolated vertex and  $s \geq 1$  then

$$|\mathcal{I}_x^{(s)}(H)| = \max_{v \in V(H)} |\mathcal{I}_v^{(s)}(H)|. \tag{12}$$

Note also that  $G/e$  is  $r$ -mixed and  $G \downarrow e$  is  $(r-1)$ -mixed. Moreover, both of these graphs have fewer vertices than  $G$  and so by our inductive hypothesis  $G/e$  is  $r$ -EKR and  $G \downarrow e$  is  $(r-1)$ -EKR.

If the edge  $\{v, w\}$  belongs to a complete graph or a path then the proof follows similarly to the proof of Theorem 6 or Theorem 7 respectively so suppose  $\{v, w\}$  is an edge in a cycle  $C_k$ . Note that we may suppose  $k \geq 4$  since otherwise  $C_k$  is a complete graph.

Let  $\mathcal{A} \subseteq I^{(r)}(G)$  be intersecting. We need to show that  $|\mathcal{A}| \leq |\mathcal{I}_x^{(r)}(G)|$ .

We will need to apply compression twice. Let  $a, b$  be the other neighbours of  $v, w$  respectively. So the vertices  $a, v, w, b$  occur in that order on  $C_k$ . Let  $e = \{v, w\}$  and  $f = \{v, b\}$ . We first apply compression to  $e$ .

Using Lemma 9(i) together with our inductive hypothesis we have

$$|\mathcal{A}| \leq |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}|,$$

where  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  are as in the proof of Lemma 9. Now letting

$$\mathcal{D}' = \{D \setminus \{v\} \mid D \in \mathcal{D}\}, \quad \mathcal{E}' = \{E \setminus \{w\} \mid E \in \mathcal{E}\},$$

it is easy to check that

$$\mathcal{F} = \mathcal{C} \cup \mathcal{D}' \cup \mathcal{E}' \subseteq I^{(r-1)}(G/e).$$

Moreover if  $F \in \mathcal{F}$  then

$$F \cap \{a, v, b\} = \begin{cases} \emptyset, & F \in \mathcal{C} \\ \{b\}, & F \in \mathcal{D}' \\ \{a\}, & F \in \mathcal{E}'. \end{cases} \quad (13)$$

One can also check that  $\mathcal{F}$  is intersecting. (This follows from Lemma 9 (iii),(vi),(vii) and (13).)

We now apply compression to  $\mathcal{F}$ , contracting the edge  $f = \{v, b\}$  (this time we map  $v$  to  $b$ ). By (13) this yields an intersecting family  $c(\mathcal{F}) \subseteq \mathcal{I}^{(r-1)}(G/e/f)$ . Note that this is an injective mapping of sets (this is implied by (13)). Hence using our inductive hypothesis once more we obtain

$$|\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}| = |\mathcal{F}| = |c(\mathcal{F})| \leq |\mathcal{I}_x^{(r-1)}(G/e/f)|.$$

Hence

$$|\mathcal{A}| \leq |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{I}_x^{(r-1)}(G/e/f)|. \quad (14)$$

So we need to check that the right hand side of (14) is at most  $|\mathcal{I}_x^{(r)}(G)|$ .

Lemma 10 gives

$$|\mathcal{I}_x^{(r)}(G)| = |\mathcal{I}_x^{(r)}(G/e)| + |\mathcal{I}_x^{(r-1)}(G \downarrow e)| + |\mathcal{D}_x| + |\mathcal{E}_x|. \quad (15)$$

If  $A \in \mathcal{I}_x^{(r-1)}(G/e/f)$  then either  $A \in \mathcal{I}_x^{(r-1)}(G \downarrow e)$  or  $a \in A$  or  $b \in A$ . But

$$\mathcal{D}'_x = \{D \setminus \{v\} \mid D \in \mathcal{D}_x\} = \{A \in \mathcal{I}_x^{(r-1)}(G/e/f) \mid b \in A\}$$

and

$$\mathcal{E}'_x = \{E \setminus \{w\} \mid E \in \mathcal{E}_x\} = \{A \in \mathcal{I}_x^{(r-1)}(G/e/f) \mid a \in A\}.$$

So

$$|\mathcal{I}_x^{(r-1)}(G/e/f)| = |\mathcal{I}_x^{(r-1)}(G \downarrow e)| + |\mathcal{D}'_x| + |\mathcal{E}'_x|.$$

Substituting this in (15) and using (14) gives the desired bound for  $|\mathcal{A}|$ .  $\square$

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