Universal singular sets in the calculus of variations

ARMA manuscript No.
(will be inserted by the editor)

Universal singular sets in the calculus of variations

MARIANNA CSÖRNYEI, BERND KIRCHHEIM, TOBY C. O'NEIL, DAVID PREISS, STEFFEN WINTER

Abstract

For regular one-dimensional variational problems, Ball and Nadirashvilli introduced the notion of the universal singular set of a Lagrangian L and established its topological negligibility. This set is defined to be the set of all points in the plane through which the graph of some absolutely continuous L-minimizer passes with infinite derivative.

Motivated by Tonelli's partial regularity results, the question of the size of the universal singular set in measure naturally arises. Here we show that universal singular sets are characterized by being essentially purely unrectifiable — that is, they intersect most Lipschitz curves in sets of zero length and that any compact purely unrectifiable set is contained within the universal singular set of some smooth Lagrangian with given superlinear growth. This gives examples of universal singular sets of Hausdorff dimension two, filling the gap between previously known one-dimensional examples and Sychev's result that universal singular sets are Lebesgue null.

We show that some smoothness of the Lagrangian is necessary for the topological size estimate, and investigate the relationship between growth of the Lagrangian and the existence of (pathological) rectifiable pieces in the universal singular set.

We also show that Tonelli's partial regularity result is stable in that the energy of a 'near' minimizer u over the set where it has large derivative is controlled by how far u is from being a minimizer.

Contents

Correspondence to: Toby C. O'Neil

1. Introduction

In his paper [15] of 1915, Tonelli gave a rigorous treatment of the variational problem

$$\mathscr{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \to \min$$

over the class of absolutely continuous u subject to a Dirichlet boundary condition. It is now well known that the crucial assumptions for attainment of the minimum are superlinear growth and convexity of the Lagrangian $L(x,u,p)$ in p. We will always consider this the natural setting unless otherwise stated. In addition, some smoothness or, at the least, some kind of continuity with respect to (x,u) is required, see [13]. The vectorial situation is also an active area of investigation, see for example [10]. In Section 2.1 we present corresponding results that are natural in our framework but do not lie at the core of our later arguments.

Equally important, but perhaps more surprising, Tonelli also obtains in [15] the first partial regularity results for the minimizer of such Lagrangians. He proves that any such minimizer has a continuous derivative provided one allows values in the extended real line: $u' \in C([a,b], \mathbb{R} \cup \{-\infty, \infty\})$. Hence the singular set of such a minimizer, that is the set where it has infinite derivative, is closed, and since u is absolutely continuous, it has measure zero.

In the converse direction, sharper examples are given in [1] that are not restricted to the case of superlinear growth and Davie shows in [5] that, for any compact null set $E \subset \mathbb{R}$, there is a smooth, convex and superlinear Lagrangian and an appropriate choice of boundary conditions such that any minimizer has infinite derivatives precisely on E. Superlinear Lagrangians clearly prefer bounded derivatives of u' for minimizing $\mathscr{L}(u)$. Thus the existence of singularities must be enforced by very steep "wells" in the (x,u,p)-energy landscape of L — the natural question arises as to how these wells can be distributed.

Motivated by this, Ball and Nadirashvili introduce in [2] the notion of the *universal singular set* of a Lagrangian L: a point (x,y) is in the universal singular set of L if there is a choice of boundary conditions so that there is a corresponding minimizer u for which $u(x) = y$ and $|u'(x)| = +\infty$. They show that for Lagrangians of class C^3 the universal singular set is a countable union of nowhere dense closed sets and thus of the first Baire category. In [14], Sychëv lowers the smoothness assumption to $L \in C^1$ and, more importantly, shows that the universal singular set is of zero (2-dimensional) Lebesgue measure.

In light of these results, the question about the "true" size of universal singular sets naturally arises: for example, one can ask whether a universal singular set can have positive length or even Hausdorff dimension larger than one. Here, we show that the key to investigating universal singular sets is to understand their geometric structure rather than just making size estimates. We show in Section 4 that for any given compact purely unrectifiable set E and any prescribed superlinear growth, there is a smooth Lagrangian whose universal singular set contains E. In particular, there are smooth Lagrangians whose universal singular sets have Hausdorff dimension two and contain non-trivial continua. The converse is also true in the sense that, if a given compact set E is such that for any superlinear growth there is a corresponding Lagrangian whose universal set contains E, then E is purely unrectifiable. On the other hand, we show that for a prescribed superlinear growth, there is always a Lagrangian whose universal singular set contains rectifiable pieces, although the universal singular set is always 'almost' purely unrectifiable in the sense that it can only intersect members of a 'small' class of rectifiable curves in a set of positive length. Together these results (Theorems 7–9) imply the results of Sychëv [14], and Ball and Nadirashvili [2] on the size of universal singular sets.

In fact, the result concerning the almost pure unrectifiability of the universal singular set still holds even when we consider Lagrangians that satisfy much weaker hypotheses than those given in Tonelli's paper, in particular, convexity in p can be dropped. In order to have a satisfactory existence theorem for this more general setting, we use the notion of a generalized minimizer — an absolutely continuous function satisfying the given boundary conditions that is an appropriate limit of a sequence of almost minimizers. This notion is related to, but differs slightly from, the usual minimizers of the relaxed problem, as discussed in Section 2.3. The difference can be made explicit by using some instances of the so-called Lavrentiev phenomenon, where approximation of a minimizer u on $\{x : |u'(x)| = +\infty\}$ by a smooth function fails. The first example of this type was given by Lavrentiev in [8], and then simplified by [9] (see also [1]) — a survey of recent progress in understanding this behaviour is given in [11]. The corresponding notion of the universal singular set for generalized minimizers of a Lagrangian is still 'almost' purely unrectifiable, and so Sychëv's result that universal singular sets have area zero holds in this broader context. However, it is possible in this setting to construct a continuous Lagrangian whose associated universal singular set is residual in the plane (see Theorem 12 of Section 4), and so the result of Ball and Nadirashvili requires at least some smoothness.

Finally, let us briefly describe the structure of the paper. In Section 2, apart from presenting some lower semicontinuity results for later use, we mainly introduce the concepts necessary to handle superlinear but non-convex Lagrangians. Under the weak smoothness conditions that we assume later, our generalized minimizers introduced in Section 2.2 do not fit completely into the usual regularization scheme — the details are given in Section 2.3. In Section 2.4 we study Tonelli's regularity results for our setting (see also [4]) and show that for Lagrangians that are locally uniformly Lipschitz in y, regularity of minimizers is stable in the sense that the energy of 'near' minimizers on the set where they have large derivative is controlled by how far the 'near' minimizer is from being a minimizer — see Corollary 5. Note that Tonelli's original approach of building auxiliary Lagrangians allows us to lower the smoothness assumptions on the Lagrangian whereas the approach given in [1] applies only to Lagrangians that are C^3 and strictly convex in p (but not necessarily of superlinear growth). As superlinear growth is essential for our estimates on the universal singular set, Tonelli's approach seems more natural.

In Sections 3 and 4 we investigate the structure of universal singular sets — a reader only interested in the case of convex Lagrangians could bypass Section 2 since the results we use from this section are well known for this situation. In Section 4, after establishing the general scheme of construction, we derive the basic examples of "large" universal singular sets. In the rest of the section we complete our studies of the interplay between the growth of L and possible "tangential" behaviour of the universal singular set and the link between smoothness of L and its topological size.

Basic notions and notation

The set $\mathrm{AC}[a,b]$ denotes the collection of all absolutely continuous real-valued functions on the closed interval $[a,b]$, and for $u \in AC[a,b]$, u' denotes the derivative of u.

$\|x\|$ denotes the usual Euclidean norm of a point x, and $\|x\|_\infty$ denotes the sup norm of x. We occasionally also use $\|f\|$ to denote the sup-norm of a function f, provided f is bounded. For $r > 0$ and $S \subset \mathbb{R}^2$, $B(S,r)$ denotes the open r-neighbourhood of S.

A set $E \subset \mathbb{R}^2$ is purely unrectifiable if it meets each Lipschitz curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ in a set of zero length: $\mathscr{H}^1(E \cap \{\gamma(t) : t \in \mathbb{R}\}) = 0$. A set $E \subset \mathbb{R}^2$ is rectifiable if there is a countable collection of Lipschitz curves $\gamma_i \colon \mathbb{R} \to \mathbb{R}^2$ for which $\mathscr{H}^1\,(E \setminus \bigcup_{i=1}^\infty \gamma_i(\mathbb{R})) = 0$.

We say that a function $L \colon \mathbb{R}^3 \to \mathbb{R}$ is a *Lagrangian* if:

- L is bounded from below and locally bounded from above;
- L is Borel measurable;
- there is a superlinear function $\omega \colon \mathbb{R} \to \mathbb{R}$ such that $L(x,y,p) \geq \omega(p)$ for all $(x,y,p) \in \mathbb{R}^3$.

Recall that superlinearity of ω means that $\lim_{|p|\to\infty} \omega(p)/|p| = \infty$.

Let $a < b$ be real numbers. For $u \in \mathrm{AC}[a,b]$ we let

$$\mathscr{L}(u) = \mathscr{L}(u;a,b) := \int_a^b L(x,u(x),u'(x))\,dx;$$

note that the integral exists thanks to the lower boundedness and Borel measurability of L. Recall that $u \in \mathrm{AC}[a,b]$ is a *minimizer* for the Lagrangian L on $[a,b]$ if

$$\mathscr{L}(u) = \mathscr{L}(a,u(a);b,u(b)),$$

where

$$\mathscr{L}(a,A;b,B) = \inf\{\mathscr{L}(v) : v \in \mathrm{AC}[a,b], v(a) = A,\, v(b) = B\}.$$

Notice that $\mathscr{L}(a,A;b,B)$ is finite; to see this it suffices to consider the affine function that joins (a,A) to (b,B). The same argument shows that for any constant C, $\mathscr{L}(a,A;b,B)$ is bounded on every bounded set of $(a,A;b,B)$ for which $a<b$ and $|B-A|\le C|b-a|$.

To show that 'almost' minimizers satisfy an approximate version of Tonelli's partial regularity theorem, we will need to measure how far a given function is from being a minimizer. A convenient such measure, the *excess of* $u\in \mathrm{AC}[a,b]$ *over the interval* $[a,b]$ is defined by

$$\mathscr{E}(u;a,b) := \mathscr{L}(u;a,b) - \mathscr{L}(a,u(a);b,u(b)).$$

Of course, $\mathscr{E}(u;a,b)=0$ if and only if u is a minimizer. The fact that a restriction of a minimizer to a subinterval is a minimizer has a simple quantitative version:

Lemma 1. *If* $a\le\alpha<\beta\le b$ *then* $\mathscr{E}(u;\alpha,\beta)\le\mathscr{E}(u;a,b)$.

Proof. Assuming, as we may, that $\mathscr{E}(u;a,b)<\infty$ and so also $\mathscr{L}(u;a,b)<\infty$, we extend any $v\in \mathrm{AC}[\alpha,\beta]$ with $v(\alpha)=u(\alpha)$ and $v(\beta)=u(\beta)$ to $\tilde v\in \mathrm{AC}[a,b]$ by $\tilde v = u$ on $[\alpha,\beta]\setminus[a,b]$. Then

$$\begin{aligned}\mathscr{L}(u;a,\alpha)+\mathscr{L}(v;\alpha,\beta)+\mathscr{L}(u;\beta,b)&\\ &=\mathscr{L}(\tilde v;a,b)\\ &\ge \mathscr{L}(u;a,b)-\mathscr{E}(u;a,b)\\ &=\mathscr{L}(u;a,\alpha)+\mathscr{L}(u;\alpha,\beta)+\mathscr{L}(u;\beta,b)-\mathscr{E}(u;a,b),\end{aligned}$$

giving the statement.

Given a Lagrangian $L\colon \mathbb{R}^3\to\mathbb{R}$, the *universal singular set of* L is defined to be all points $(x_0,y_0)\in\mathbb{R}^2$ for which there are $a\le x_0\le b$ with $a<b$ and a minimizer u for L on $[a,b]$ such that $u(x_0)=y_0$ and u has infinite derivative at x_0.

Of course, under the above assumptions, there may well be no minimizers for L and so the notion of the universal singular set may make little sense. In Section 2.2 we define a weaker notion of minimizers for which the existence result is nearly trivial and for which the corresponding universal singular sets enjoy the same smallness properties as those for ordinary minimizers.

Those of our results that do not include study of the behaviour of minimizers when the derivative is infinite (so, in particular, our results of Section 2.4 on Tonelli's partial regularity) are readily transferable to the vector-valued case, with essentially identical arguments. It is not clear, however, how the universal singular set should be defined in this situation. A natural definition is to say that (x_0,y_0) belongs to the universal singular set if there is a minimizer u such that $u(x_0)=y_0$ and $\lim_{x\to x_0}\|u(x)-u(x_0)\|/|x-x_0|=\infty$, which is equivalent to the standard definition in the scalar case. In this case only one of our arguments for the scalar case extends, showing that the graphs of absolutely continuous functions meet the universal singular set in a set of measure zero.

2. Generalized minimizers and regularity

It is obvious that under our general assumption on the Lagrangian no existence or regularity results for minimizers can hold. One of our goals here is to give a notion of generalized minimizers for which the existence results hold (and are, in fact, nearly trivial), but for which we will see later that the universal singular set is as small as in the standard situation. We start by revising some classical results on compactness and lower semicontinuity, then discuss two ways of generalizing the notion of a minimizer, and finally show that Tonelli's partial regularity results hold, under only a mild assumption on the Lagrangian, even for these generalized minimizers.

2.1. Compactness and lower semicontinuity

Mainly for the sake of future reference, we record here standard arguments showing that under our assumptions on the Lagrangians, the sets of functions with uniformly bounded Lagrangian enjoy weak compactness properties, even if we allow changing boundary data. Recall that we only consider Lagrangians that are Borel measurable, lower bounded, locally bounded from above and superlinear.

Proposition 1. *For every real K the set of functions*

$$\{u \in \mathrm{AC}[a,b] : -K \le a < b \le K,\ \mathscr{L}(u;a,b) \le K\}$$

is uniformly equicontinuous and the set of their derivatives

$$\{u' : u \in \mathrm{AC}[a,b] : -K \le a < b \le K,\ \mathscr{L}(u;a,b) \le K\}$$

is equiintegrable.

Proof. Suppose that $L(x,y,p) \ge \omega(p)$ where ω is superlinear and bounded from below. Given any $\varepsilon > 0$, let $\tau = \varepsilon/2K$ and use that ω is superlinear and bounded from below to find C so that $|p| \le C + \tau\omega(p)$ for all $p \in \mathbb{R}$. If $u \in \mathrm{AC}[a,b]$, $\mathscr{L}(u;a,b) \le K, M \subset [a,b]$ and $|M| < \varepsilon/2C$ then $\int_M |u'(x)|\,dx \le \int_M (C + \tau\omega(u'(x)))\,dx \le C|M| + \tau\mathscr{L}(u;a,b) < \varepsilon$. This establishes equiintegrability. Equicontinuity follows from equiintegrability and from $|u(\beta) - u(\alpha)| \le \int_{[\alpha,\beta]} |u'(x)|\,dx$.

It is rather natural to complement this statement by a lower semicontinuity result, for which we however need more stringent assumptions. Recall that $u_n \in \mathrm{AC}[a,b]$ converge to $u \in \mathrm{AC}[a,b]$ weakly if and only if they converge to u pointwise and the set of u_n' is equiintegrable; of course, in this situation, pointwise convergence of the u_ns is equivalent to their uniform convergence.

In the following theorem, our extra assumptions are that the Lagrangian is convex in p and lower semicontinuous in y for each fixed (x,p); this should be contrasted with our results concerning the regularity of 'near' minimizers where we only require the Lagrangian to be (locally) Lipschitz in y, see Section 2.4.

Theorem 1. *Suppose that $L(x,y,p)$ is a Lagrangian that is convex in p for each fixed (x,y), and lower semicontinuous in y for each fixed (x,p). Then the map $u \in \mathrm{AC}[a,b] \mapsto \mathscr{L}(u;a,b)$ is weakly sequentially lower semicontinuous.*

Proof. The theorem follows from Theorem 3.6 of [3] (see also [7]) provided that we show that L is lower semicontinuous as a function of (y,p) for fixed x. This follows easily from the fact L is convex in p, lower semicontinuous in y and locally bounded: for fix $x \in \mathbb{R}$ and suppose that $y_n \to y_0$ and $p_n \to p_0$. By the convexity of $L(x,y_n,p)$ in p, $L(x,y_n,p) \geq f_n(p)$ where

$$f_n(p) = \begin{cases} (p_0-p)L(x,y_n,p_0-1)+(1+(p-p_0))L(x,y_n,p_0) & \text{if } p \geq p_0 \\ (p-p_0)L(x,y_n,p_0+1)+(1+(p_0-p))L(x,y_n,p_0) & \text{if } p \leq p_0 \end{cases}$$

Use local upper boundedness of L to find a constant C so that $L(x,y_n,p_0 \pm 1) \leq C$ for all n; hence $L(x,y_n,p) \geq f_n(p) \geq g_n(p)$ where

$$g_n(p) = \begin{cases} C(p_0-p)+(1+(p-p_0))L(x,y_n,p_0) & \text{if } p \geq p_0 \\ C(p-p_0)+(1+(p_0-p))L(x,y_n,p_0) & \text{if } p \leq p_0 \end{cases}$$

Assuming, as we may, that the sequence $L(x,y_n,p_0)$ has a (necessarily finite) limit, which is at least $L(x,y_0,p_0)$ by lower semicontinuity for fixed (x,p_0), we see that $g_n(p)$ converge uniformly on bounded sets to a continuous function $g(p)$ such that $g(p_0) \geq L(x,y_0,p_0)$. Hence $\liminf_{n\to\infty} L(x,y_n,p_n) \geq \liminf_{n\to\infty} g_n(p_n) \geq g(p_0) \geq L(x,y_0,p_0)$, which shows the claim.

An existence result is an immediate corollary.

Theorem 2. *Suppose that $L(x,y,p)$ is a Lagrangian that is convex in p for each fixed (x,y) and lower semicontinuous in y for each fixed (x,p). Then for any $a<b$ and $A,B \in \mathbb{R}$ there is a minimizer $u \in \mathrm{AC}[a,b]$ for which $u(a)=A$ and $u(b)=B$.*

Proof. Choose $u_n \in \mathrm{AC}[a,b]$ so that $u_n(a)=A$, $u_n(b)=B$ and

$$\lim_{n\to\infty} \mathscr{L}(u_n;a,b) = \mathscr{L}(a,A;b,B) < \infty.$$

By Proposition 1 the sequence u_n has a subsequence weakly converging to some $u \in \mathrm{AC}[a,b]$ which still satisfies the same boundary conditions, and by Theorem 1, $\mathscr{L}(u;a,b) \leq \lim_{n\to\infty} \mathscr{L}(u_n;a,b) = \mathscr{L}(a,A;b,B)$, hence u is a minimizer.

2.2. Generalized minimizers

Here we briefly discuss several possible ways of extending the notion of minimizers. Initially we consider generalized and constrained minimizers. Later we give yet another, $\tilde{\mathscr{L}}$-minimizers, used for technical purposes only and which, in fact, turn out to be equivalent to the notion of constrained minimizers. In Section 2.3, we describe the relationship between constrained and relaxed minimizers.

Our original reason for introducing them was to enlarge the universal singular set to a 'generalized universal singular set', for which our size estimates could still be valid. Although this is true, we show in Proposition 3 that for Lagrangians satisfying the classical assumptions 'generalized universal singular sets' are in fact universal singular sets. Even for Lagrangians that are only continuous in (y, p) (and not necessarily convex in p), we will see in Section 2.3 that their 'generalized universal singular sets' are in fact universal singular sets for the Lagrangian convexified in p. Therefore, since it can hardly lead to any confusion, we skip the 'generalized' and, after proving the necessary results, use the term 'universal singular sets' even for sets defined via generalized minimizers.

We also use generalized minimizers in Section 2.4 to give non-technical formulations of variants of Tonelli's regularity theorem, which shows that Tonelli's results have remarkable stability.

We say that $u \in C[a,b]$ is a *generalized minimizer* for the Lagrangian L on $[a,b]$ if its restriction to (a,b) is a locally uniform limit of a sequence $u_n \in \mathrm{AC}[a_n,b_n]$ such that $\mathscr{E}(u_n;a_n,b_n) \to 0$.

We say that $u \in C[a,b]$ is a *constrained minimizer* for the Lagrangian L on $[a,b]$ if it is a uniform limit of a sequence $u_n \in \mathrm{AC}[a,b]$ such that $u_n(a) = u(a)$, $u_n(b) = u(b)$ and $\mathscr{E}(u_n;a,b) \to 0$.

Some remarks may be in order. First, we should explain that in the definition of generalized minimizers, the convergence of u_n to u on (a,b) requires that $\limsup_{n\to\infty} a_n \leq a$ and $b \leq \liminf_{n\to\infty} b_n$. However Lemma 1 shows that, equivalently, we could have required that $a_n \to a$ and $b_n \to b$. More generally, if $\varepsilon_n \to 0$ and $a_n, b_n \in (a,b)$ with $a_n \to a$, $b_n \to b$, then for every generalized minimizer u for L on $[a,b]$ there are $u_n \in \mathrm{AC}[a_n,b_n]$ such that $|u_n - u| < \varepsilon_n$ on $[a_n,b_n]$ and $\mathscr{E}(u_n;a_n,b_n) \to 0$: it suffices to replace the u_n from the definition by u_{k_n} such that $|u_{k_n} - u| < \varepsilon_n$ on $[a_n,b_n]$ and use Proposition 1. With such a_n, b_n and u_n, we use that

$$\mathscr{L}(u_n;a_n,b_n) \leq \mathscr{L}(v_n;a_n,b_n) + \mathscr{E}(u_n;a_n,b_n),$$

where v_n is an affine function joining $(a_n, u_n(a_n))$ and $(b_n, u_n(b_n))$, together with the fact that the sequence $\mathscr{L}(v_n;a_n,b_n)$ is bounded, to infer from the equiintegrability bound of Proposition 1 that the u_n are equiabsolutely continuous. Hence generalized minimizers are in fact absolutely continuous. (In what follows, we therefore consider only functions $u \in \mathrm{AC}[a,b]$.)

The above argument also shows that the same notion of minimizers would be obtained had we required just pointwise convergence of u_n to u on (a,b).

It is obvious that constrained minimizers are generalized minimizers. And the notion of minimizers has been weakened so much that an existence result is essentially trivial: By taking, for any $a < b$ and $A, B \in \mathbb{R}$, a sequence $u_n \in \mathrm{AC}[a,b]$ such that $u_n(a) = A$, $u_n(b) = B$ and $\mathscr{E}(u_n;a,b) \to 0$, and using Proposition 1 together with the Arzela-Ascoli Theorem, we see that constrained minimizers exist. We record some of these facts in the following theorem.

Theorem 3. *Let L be a Lagrangian. Suppose that $a < b$ and $A, B \in \mathbb{R}$. Then there is a constrained minimizer such that $u(a) = A$ and $u(b) = B$. Moreover, every minimizer is a constrained minimizer, every constrained minimizer is a generalized minimizer and every generalized minimizer is absolutely continuous.*

Notice also that Lemma 1 implies that the restriction of a generalized minimizer to a subinterval is a generalized minimizer; the same statement for constrained minimizers is not so obvious (but true; see Corollary 2). As with ordinary minimizers, the maximum and minimum of two constrained minimizers with the same boundary conditions is a constrained minimizer.

Proposition 2. *Let u,v be constrained minimizers for L on $[a,b]$ such that $u(a) \le v(a)$ and $u(b) \le v(b)$. Then $\max(u,v)$ and $\min(u,v)$ are constrained minimizers for L on $[a,b]$.*

Proof. Let $u_n, v_n \in \mathrm{AC}[a,b]$ be such that $u_n(a) = u(a)$, $v_n(a) = v(a)$, $u_n(b) = u(b)$, $v_n(b) = v(b)$, $\mathscr{E}(u_n;a,b) \to 0$ and $\mathscr{E}(v_n;a,b) \to 0$. Then $\mathscr{E}(\max(u_n,v_n);a,b) + \mathscr{E}(\min(u_n,v_n);a,b) = \mathscr{E}(u_n;a,b) + \mathscr{E}(u_n;a,b) \to 0$, hence $\mathscr{E}(\max(u_n,v_n);a,b) \to 0$ and $\mathscr{E}(\min(u_n,v_n);a,b) \to 0$.

For generalized minimizers this argument fails, and indeed, the maximum of two generalized minimizers with the same boundary conditions need not be a generalized minimizer. A variant of the argument shows that the analogue of the above statement holds for generalized minimizers u and v provided we assume that $u(a) < v(a)$ and $u(b) < v(b)$. (These remarks are not used in this paper.)

Easy examples show that none of the inclusions from the last sentence of Theorem 3 may be reversed. Generalized minimizers may fail to be constrained even for Lagrangians satisfying the classical assumptions (under which, of course, constrained minimizers coincide with minimizers); for example for

$$L(x,y,p) = (x^3 - y^5)^2 p^{20} + \varepsilon_0 p^2,$$

where $\varepsilon_0 > 0$ is a small enough constant. Indeed, let $u_n \in \mathrm{AC}[0,1]$ be a minimizer for $u_n(0) = -1/n$, $u_n(1) = 1$. The usual estimates proving the Lavrentiev phenomenon (see either [6] or [3,8,9]) show that there is $\delta > 0$ such that $u_n(x) \le \frac{1}{2}x^{3/5}$ for all n and all $0 \le x \le \delta$. Hence the limit u of (a subsequence of) the u_n's stays below $\frac{1}{2}x^{3/5}$ on $[0,\delta]$. But, by the Lavrentiev phenomenon estimates mentioned above, any such u is of energy larger than that of $\tilde{u}(x) = x^{3/5}$ and so is only a generalized minimizer, not a minimizer.

In this example, we see that a generalized minimizer fails to be constrained because it is on a higher energy level. We now show that this is the only way this phenomenon can occur. This will then lead us to recognition that our two notions of generalized minimizers give rise to the same universal singular set. For this, it is convenient to introduce some notation for the 'generalized energy'. For $u \in \mathrm{AC}[a,b]$ denote by $\tilde{\mathscr{L}}(u;a,b)$ the infimum of all possible $\liminf_{n\to\infty} \mathscr{L}(u_n;a_n,b_n)$ where $a_n \to a$, $b_n \to b$ and $u_n \in \mathrm{AC}[a_n,b_n]$ converge to u on (a,b). We also let

$$\tilde{\mathscr{L}}(a,A;b,B) = \inf\left\{\tilde{\mathscr{L}}(u;a,b) : u \in \mathrm{AC}[a,b],\, u(a) = A,\, u(b) = B\right\}.$$

and say that u is an *$\tilde{\mathscr{L}}$-minimizer* (for L on $[a,b]$) if

$$\tilde{\mathscr{L}}(u;a,b) = \tilde{\mathscr{L}}(a,u(a);b,u(b)).$$

Lemma 2. *For any $a < b$ and $A,B \in \mathbb{R}$, $\tilde{\mathscr{L}}$-minimizers exist.*

Proof. By Proposition 1 and the Arzela-Ascoli Theorem, it suffices to find $u_n \in \mathrm{AC}[a_n,b_n]$ such that $(a_n,u_n(a_n)) \to (a,A)$, $(b_n,u_n(b_n)) \to (b,B)$ and

$$\limsup_{n\to\infty} \mathscr{L}(u_n;a_n,b_n) \le \tilde{\mathscr{L}}(a,A;b,B).$$

To find the u_n we start by choosing $v_n \in \mathrm{AC}[a,b]$ such that

$$v_n(a) = A,\ v_n(b) = B \text{ and } \tilde{\mathscr{L}}(v_n;a,b) < \tilde{\mathscr{L}}(a,A;b,B) + 1/n.$$

We then choose $a < a_n < a + 1/n$ and $b - 1/n < b_n < b$ such that $|v_n(a_n) - v_n(a)| < 1/n$ and $|v_n(b_n) - v_n(b)| < 1/n$ and finish by using the definition of $\tilde{\mathscr{L}}(v_n;a,b)$ to find $\alpha_n \in (a-1/n,a_n)$, $\beta_n \in (b_n,b+1/n)$, and $u_n \in \mathrm{AC}[\alpha_n,\beta_n]$ such that $|u_n(a_n) - v_n(a_n)| < 1/n$, $|u_n(b_n) - v_n(b_n)| < 1/n$ and $\mathscr{L}(u_n;\alpha_n,\beta_n) < \tilde{\mathscr{L}}(u_n;a,b) + 1/n$. Then $\limsup_{n\to\infty} \mathscr{L}(u_n;a_n,b_n) \le \limsup_{n\to\infty} \mathscr{L}(u_n;\alpha_n,\beta_n) \le \lim_{n\to\infty} \tilde{\mathscr{L}}(u_n;a,b) = \tilde{\mathscr{L}}(a,A;b,B)$.

Lemma 3. *Every $\tilde{\mathscr{L}}$-minimizer $u \in \mathrm{AC}[a,b]$ is a (uniform) limit of a sequence $u_n \in \mathrm{AC}[a,b]$ such that $u_n(a) = u(a)$, $u_n(b) = u(b)$ and $\mathscr{L}(u_n;a,b) \to \tilde{\mathscr{L}}(u;a,b)$.*

Proof. Let $\varepsilon > 0$; we have to find a $v \in \mathrm{AC}[a,b]$ such that $v(a) = u(a)$, $v(b) = u(b)$, $|v(x) - u(x)| < \varepsilon$ on $[a,b]$ and $\mathscr{L}(v;a,b) < \tilde{\mathscr{L}}(u;a,b) + \varepsilon$.

Find $v_n \in \mathrm{AC}[a_n,b_n]$ converging to v on (a,b) such that $\mathscr{L}(v_n;a_n,b_n) < \tilde{\mathscr{L}}(u;a,b) + 2^{-n-4}\varepsilon$. For $a < s < t < b$ denote $\varphi(s,t) = \limsup_{n\to\infty} \mathscr{L}(v_n;s,t)$. We shall assume that for rational s,t this lim sup is actually a limit; if necessary, this can be achieved by passing to a subsequence of v_n.

We claim that for every $\eta > 0$, there is $\delta > 0$ such that $\varphi(s,t) < \eta$, whenever $a < s < t < a + \delta$. To see this, note that $\sigma := \sup_{a<s<t<b} \varphi(s,t) \le \tilde{\mathscr{L}}(u;a,b) < \infty$ and find $a < s_0 < t_0 < b$ such that $\varphi(s_0,t_0) > \sigma - \eta$. If $\varphi(s,t) \ge \eta$ for some $a < s < t < s_0$, we would pick rational $\tilde{s} \in (a,s)$ and $\tilde{t} \in (t,t_0)$, and use that $\varphi(\tilde{s},\tilde{t}) = \lim_{n\to\infty} \mathscr{L}(v_n;\tilde{s},\tilde{t})$ to obtain a contradiction by estimating $\varphi(\tilde{s},t_0) \ge \varphi(\tilde{s},\tilde{t}) + \varphi(s_0,t_0) > \sigma$.

Similarly we show that for every $\eta > 0$ there is $\delta > 0$ such that $\varphi(s,t) < \eta$ whenever $b - \delta < s < t < b$.

Hence, for each $j = 1,2,\dots$ we may find $\delta_j > 0$ such that $\varphi(s,t) < 2^{-j-4}\varepsilon$ provided that either $a < s < t < a + \delta_j$ or $b - \delta_j < s < t < b$.

Choose $\alpha_j \searrow a$ and $\beta_j \nearrow b$ so that $\alpha_j < a + \delta_j$, $\beta_j > b - \delta_j$, $\alpha_1 < \beta_1$ and u has a finite derivative at each of the points α_j, β_j. Choose also $0 < \tau_j < \min(\alpha_j - \alpha_{j+1}, \beta_{j+1} - \beta_j)$ and $C_j < \infty$ so that $|u(x) - u(\alpha_j)| \le C_j|x - \alpha_j|$ whenever $|x - \alpha_j| \le \tau_j$ and $|u(x) - u(\beta_j)| \le C_j|x - \beta_j|$ whenever $|x - \beta_j| \le \tau_j$.

Let $R_j := (C_j+2)(b-a) + \max_{t\in[a,b]} |u(t)|$ and find $M_j < \infty$ such that $L(x,y,p) \le M_j$ for $x \in [a,b]$, $|y| \le R_j$ and $|p| \le C_j + 2$. Also find $0 < \eta_j < \tau_j$ so that $\eta_j M_j < 2^{-j-2}$.

For each j choose $n_j \ge j$ so large that $a_{n_j} < \alpha_j$, $b_{n_j} > \beta_j$, $|u_{n_j}(x) - u(x)| < 2^{-j-4}\eta_j$ on $[\alpha_j,\beta_j]$, and, if $j \ge 2$, $\mathscr{L}(v_{n_j};\alpha_j,\alpha_{j-1}) < 2^{-j-3}\varepsilon$ and $\mathscr{L}(v_{n_j};\beta_{j-1},\beta_j) < 2^{-j-3}\varepsilon$. (To get the last requirements we have used that $\varphi(\alpha_j,\alpha_{j-1}) < 2^{-j-3}\varepsilon$ and $\varphi(\beta_{j-1},\beta_j) < 2^{-j-3}\varepsilon$.)

Define $v\colon [a,b] \to \mathbb{R}$ by

$$v(x) = \begin{cases} u(x) & \text{for } x = a \text{ and } x = b \\ v_{n_1} & \text{on } [\alpha_1, \beta_1] \\ v_{n_{j+1}}(x) & \text{on } [\alpha_{j+1}, \alpha_j - \eta_j] \cup [\beta_j + \eta_j, \beta_{j+1}] \\ \text{affine} & \text{on } [\alpha_j - \eta_j, \alpha_j] \text{ and on } [\beta_j, \beta_j + \eta_j]. \end{cases}$$

Clearly, v is locally absolutely continuous on (a,b) and, since $|v(x) - u(x)| < \eta_j$ for $x \in (a, \alpha_j) \cup (\beta_j, b)$, it is also continuous on $[a,b]$.

Since on $(\alpha_j - \eta_j, \alpha_j)$, $|v| \le \max(v_{n_j}(\alpha_j), v_{n_{j+1}}(\alpha_j - \eta_j)|) + (b-a) \le R_j$ and $|v'| = |v_{n_j}(\alpha_j) - v_{n_{j+1}}(\alpha_j - \eta_j)|/\eta_j \le |u(\alpha_j) - u(\alpha_j - \eta_j)|/\eta_j + 2 \le C_j + 2$, we have $\mathscr{L}(v; \alpha_j - \eta_j, \alpha_j) \le M_j \eta_j < 2^{-j-4}\varepsilon$. A similar argument gives that $\mathscr{L}(v; \beta_j, \beta_j + \eta_j) \le M_j \eta_j < 2^{-j-4}\varepsilon$. Also, recall that n_{j+1} has been chosen so that $\mathscr{L}(v_{n_{j+1}}; \alpha_{j+1}, \alpha_j - \eta_j) < 2^{-j-4}\varepsilon$ and $\mathscr{L}(v_{n_{j+1}}; \beta_j + \eta_j, \beta_{j+1}) < 2^{-j-4}\varepsilon$, and infer from $L \ge 0$ that $\mathscr{L}(v_{j_1}, \alpha_1, \beta_1) \le \mathscr{L}(v_{j_1}, a_1, b_1) < \tilde{\mathscr{L}}(u; a, b) + \varepsilon/2$.

Adding all these estimates of $\mathscr{L}$ together gives $\int L(x, v(x), v'(x))\,dx < \tilde{\mathscr{L}}(u; a, b) + \varepsilon$. Since this also implies that v' is integrable over $[a,b]$, we see that v, being an indefinite Lebesgue integral of v' on $[a,b]$, is absolutely continuous on $[a,b]$. Hence v has all the required properties and we are done.

Corollary 1. *$\tilde{\mathscr{L}}(a,A;b,B) = \mathscr{L}(a,A;b,B)$ for every a,A,b,B, and the notions of constrained and $\tilde{\mathscr{L}}$-minimizers coincide.*

Proof. Clearly, $\tilde{\mathscr{L}}(a,A;b,B) \le \mathscr{L}(a,A;b,B)$. Hence, if u is any $\tilde{\mathscr{L}}$-minimizer, Lemma 3 shows that it is also a constrained minimizer and that $\mathscr{L}(a,A;b,B) \le \tilde{\mathscr{L}}(a,A;b,B)$; since $\tilde{\mathscr{L}}$ minimizers exist, this also shows that $\tilde{\mathscr{L}}(a,A;b,B) = \mathscr{L}(a,A;b,B)$. Having this equality, it is clear that every constrained minimizer is an $\tilde{\mathscr{L}}$-minimizer.

Various properties of constrained minimizers follow from these facts. We just record the following

Corollary 2. *The restriction of a constrained minimizer to a subinterval is a constrained minimizer.*

Proof. Let u be a constrained minimizer on $[a,b]$, with the corresponding $u_n \in \mathrm{AC}[a,b]$ and let $c \in (a,b)$. Denoting by v_n and w_n the restrictions of u_n to $[a,c]$ and $[c,b]$, respectively, and using Corollary 1, we find

$$\begin{aligned} \mathscr{L}(a,u(a);b,u(b)) &\le \mathscr{L}(a,u(a);c,u(c)) + \mathscr{L}(c,u(c);b,u(b)) \\ &= \tilde{\mathscr{L}}(a,u(a);c,u(c)) + \tilde{\mathscr{L}}(c,u(c);b,u(b)) \\ &\le \tilde{\mathscr{L}}(u;a,c) + \tilde{\mathscr{L}}(u;c,b) \\ &\le \liminf_{n\to\infty} (\mathscr{L}(v_n;a,c) + \mathscr{L}(w_n;c,b)) \\ &= \mathscr{L}(a,u(a);b,u(b)). \end{aligned}$$

Since $\tilde{\mathscr{L}}(u;a,c) \ge \tilde{\mathscr{L}}(a,u(a);c,u(c))$ and $\tilde{\mathscr{L}}(u;c,b) \ge \tilde{\mathscr{L}}(c,u(c);b,u(b))$, we have

$$\tilde{\mathscr{L}}(u;a,c) = \tilde{\mathscr{L}}(a,u(a);c,u(c)) \text{ and } \tilde{\mathscr{L}}(u;c,b) = \tilde{\mathscr{L}}(c,u(c);b,u(b)).$$

Hence the restrictions of u to $[a,c]$ and $[c,b]$ are $\tilde{\mathscr{L}}$-minimizers, and thus constrained minimizers by Corollary 1.

Lemma 4. *Suppose that $u \in \mathrm{AC}[a,b]$ is a generalized minimizer for L such that $u'_+(a) = \infty$. Then for every $B > u(b)$ there is a constrained minimizer v on $[a,b]$ such that $v(a) = u(a)$, $v(b) = B$, and v has infinite right derivative at a.*

Proof. Let $w \in \mathrm{AC}[a,b]$ be a constrained minimizer such that $w(a) = u(a)$ and $w(b) = B$. If the right derivative of w at a is infinite, we are done. So assume that this is not the case, hence $\liminf_{x\searrow a} \frac{|w(x)-w(a)|}{|x-a|} < \infty$. Since $u'_+(a) = +\infty$, this implies, in particular, that there is $c \in (a,b)$ such that $w(c) < u(c)$. Also use that $w(b) > u(b)$ to find $c < d < b$ so that $w(d) > u(d)$.

Let $c > \beta_n \searrow a$ and $M < \infty$ be such that $|w(\beta_n) - w(a)| < M(\beta_n - a)$ for each n, and find $a < \alpha_n < \beta_n$ so that $|w(\beta_n) - w(\alpha_n)| < M(\beta_n - \alpha_n)$. Denoting $\delta_n = \frac{1}{2}\min(\beta_n - \alpha_n, b-d, u(c)-w(c), B-u(b))$, we use that w is a constrained minimizer to find $w_n \in \mathrm{AC}[a,b]$ such that $w_n(a) = w(a)$, $w_n(b) = w(b)$, $|w_n(x) - w(x)| < \delta_n$ on $[a,b]$, and $\mathscr{L}(w_n;a,b) < \tilde{\mathscr{L}}(w;a,b) + 1/n$. Also, we use that u is a generalized minimizer to find $a_n < \alpha_n < b - \delta_n < b_n$ and $u_n \in \mathrm{AC}[a_n,b_n]$ such that $|u_n(x) - u(x)| < \delta_n$ on $[\alpha_n, b-\delta_n]$ and $\mathscr{E}(u_n;a_n,b_n) < 1/n$.

Since $w_n(c) < u_n(c)$ and $w_n(d) > u_n(d)$, there are $c_n \in (c,d)$ such that $u_n(c_n) = w_n(c_n)$. By considering the function defined as u_n on $[a_n, a+\delta_n] \cup [c_n,b_n]$, as w_n on $[\alpha_n, c_n]$ and affine on $[a+\delta_n, \alpha_n]$, we see that $\mathscr{L}(u_n;a_n,b_n) \le \mathscr{L}(u_n;a_n,a+\delta_n) + \mathscr{L}(u_n;c_n,b_n) + \mathscr{L}(w_n;\alpha_n,c_n) + M(\beta_n-\alpha_n) + \mathscr{E}(u_n;a_n,b_n)$, hence $\mathscr{L}(u_n;\alpha_n,c_n) \le \mathscr{L}(w_n;\alpha_n,c_n) + 2/n$.

Defining v_n as u_n on $[\alpha_n,c_n]$ and as w_n on $[c_n,b_n]$, we therefore have $\mathscr{L}(v_n;\alpha_n,b) = \mathscr{L}(u_n;\alpha_n,c_n) + \mathscr{L}(w_n;c_n,b) \le \mathscr{L}(w_n;\alpha_n,b) + 2/n \le \mathscr{L}(w_n;a,b) + 2/n \to \tilde{\mathscr{L}}(w;a,b)$. Hence, choosing a subsequence along which c_n converges to, say, $\tilde{c} \in [c,d]$, and defining v as u on $[a,\tilde{c}]$ and as w on $[\tilde{c},b]$, we see that $\tilde{\mathscr{L}}(v;a,b) \le \tilde{\mathscr{L}}(w;a,b) = \tilde{\mathscr{L}}(a,v(a);b,v(b))$. So v is an $\tilde{\mathscr{L}}$-minimizer, and so, by Corollary 1, it is a constrained minimizer. Finally, $v'_+(a) = \infty$ since $v \ge u$ on $[a,c]$.

Proposition 3. *The following statements about a point $(x_0,y_0) \in \mathbb{R}^2$ are equivalent.*

1. *There are $a \le x_0 \le b$ with $a < b$ and a generalized minimizer u for L on $[a,b]$ such that $u(x_0) = y_0$ and u has infinite derivative at x_0.*
2. *There are $a \le x_0 \le b$ with $a < b$ and a constrained minimizer u for L on $[a,b]$ such that $u(x_0) = y_0$ and u has infinite derivative at x_0.*

Moreover, if $L(x,y,p)$ is continuous in (y,p) and convex in p, these statements hold if and only if

3. *There are $a \le x_0 \le b$ with $a < b$ and a minimizer u for L on $[a,b]$ such that $u(x_0) = y_0$ and u has infinite derivative at x_0.*

Proof. By Lemma 4, 2 holds provided that there is $b > x_0$ and a generalized minimizer u for L on $[a,b]$ such that $u(x_0) = y_0$ and $u'_+(x_0) = \infty$. Symmetric arguments shows that 2 also holds if $u'_-(x_0) = -\infty$ or if there is $a < x_0$ and a generalized minimizer u for L on $[a,b]$ such that $u(x_0) = y_0$ and u has infinite left derivative at x_0. Hence 1⇒2.

The implication 2⇒1 is obvious since every constrained minimizer is a generalized one.

Under the additional assumptions on the Lagrangian, 2 is equivalent to 3, since then the notions of constrained minimizers and minimizers coincide.

This statement allows us to define the *universal singular set of a Lagrangian* as all points $(x_0,y_0)\in\mathbb{R}^2$ for which the statement 1, or equivalently 2, of Proposition 3 holds.

We now return to our example illustrating the difference between general and constrained minimizers and observe that not only were the two minimizers on different energy levels, but behind the whole discrepancy was the fact that an endpoint of the minimizers belonged to the universal singular set:

Proposition 4. *Suppose that $u\in \mathrm{AC}[a,b]$ is such that*

$$\liminf_{x\searrow a}\frac{|u(x)-u(a)|}{|x-a|}<\infty \textit{ and } \liminf_{x\nearrow b}\frac{|u(x)-u(b)|}{|x-b|}<\infty$$

and that v_n is a sequence of absolutely continuous functions on $[a_n,b_n]$ such that $(a_n,v_n(a_n))\to(a,u(a))$, $(b_n,v_n(b_n))\to(b,u(b))$ and $\mathscr{E}(v_n;a_n,b_n)\to 0$ (but possibly converging to a limit different from u). Then

$$\limsup_{n\to\infty}\mathscr{L}(v_n;a_n,b_n)\le\tilde{\mathscr{L}}(u;a,b).$$

Proof. We may assume that $\tilde{\mathscr{L}}(u;a,b)<\infty$ and $\lim_{n\to\infty}\mathscr{L}(v_n;a_n,b_n)$ exists; this will allow us to pass to a subsequence of v_n. Denote by M the maximum of the two limits inferior in the assumption and choose $\alpha_n\searrow a$ and $\beta_n\nearrow b$ such that $\alpha_n<\beta_n$, $|u(\alpha_n)-u(a)|\le(M+1)|\alpha_n-a|$ and $|u(\beta_n)-u(a)|\le(M+1)|\beta_n-b|$. Let $\delta_n=\min(\alpha_n-a,b-\beta_n)/2$. Passing to a subsequence of v_n if necessary, we also assume $\|(a_n,v_n(a_n))-(a,u(a))\|<\delta_n$ and $\|(b_n,v_n(b_n))-(b,u(b))\|<\delta_n$.

Find $u_n\in\mathrm{AC}[\alpha_n,\beta_n]$ such that $|u_n(x)-u(x)|<\delta_n$ on $[\alpha_n,\beta_n]$ and $\tilde{\mathscr{L}}(u;a,b)=\lim_{n\to\infty}\mathscr{L}(u_n;\alpha_n,\beta_n)$.

Since $a_n<\alpha_n<\beta_n<b_n$, we may define w_n on $[a_n,b_n]$ which agrees with v_n at the points a_n,b_n, with u_n on $[\alpha_n,\beta_n]$ and is affine on $[a_n,\alpha_n]$ and $[\beta_n,b_n]$.

Choose $R>0$ so large that the graphs of v_n, u_n, and so also of w_n are contained in $[-R,R]^2$. Let C be an upper bound for $L(x,y,p)$ for $|x|,|y|\le R$ and $|p|\le M+4$.

Observing that $\alpha_n-a_n\ge\delta_n$, we estimate that $|w_n(a_n)-w_n(\alpha_n)|\le|u(a)-u(\alpha_n)|+2\delta_n\le(M+4)|a-\alpha_n|$ and, similarly, that $|w_n(b_n)-w_n(\beta_n)|\le(M+4)|b_n-\beta_n|$. Hence, letting $l_n=|\alpha_n-a|+|b-\beta_n|$,

$$\begin{aligned}\mathscr{L}(v_n;a_n,b_n)&\le\mathscr{L}(w_n;a_n,b_n)+\mathscr{E}(v_n;a_n,b_n)\\&\le\mathscr{L}(w_n;\alpha_n,\beta_n)+Cl_n+\mathscr{E}(u_n;a_n,b_n)\\&\to\tilde{\mathscr{L}}(u;a,b),\end{aligned}$$

and it suffices to take a limit.

Using the definition of $\tilde{\mathscr{L}}(v;a,b)$ and Proposition 1, we have the following result.

Corollary 3. *If u satisfies the assumptions of Proposition 4 and v has the same boundary values then* $\tilde{\mathscr{L}}(v;a,b) \leq \tilde{\mathscr{L}}(u;a,b)$.

If u is a constrained minimizer satisfying the assumptions of Proposition 4, then every generalized minimizer having the same boundary values is constrained.

2.3. Approximation and Relaxation

In this section, we show how the usual relaxation (i.e. convexification) procedure applies to the study of universal singular sets for general Lagrangians. A special case of Theorem 5 says that, under the additional assumption that the Lagrangian L is also continuous in (y,p), generalized minimizers of L starting and ending at points outside the universal singular set of the convexified Lagrangian L^c are necessarily minimizers of L^c. As a corollary, in Theorem 4 we show that under the same assumptions on L, the generalized universal singular set of L coincides with the universal singular set of its convexification L^c.

All Lagrangians $L(x,y,p)$ in this section are assumed to be continuous in (y,p) for each fixed x.

Before introducing relaxation, we show that all $u \in AC[a,b]$ can be approximated (in the space $W^{1,1}(a,b)$, and whilst preserving the boundary conditions) by a function whose energy is not much greater than that of u and which is C^1 on a dense open subset of (a,b) of full measure. This would be easy if u were Lipschitz on a dense open subset of (a,b) of full measure; however, under our assumptions, this need not be the case.

Proposition 5. *For every* $u \in \mathrm{AC}[a,b]$ *with* $\mathscr{L}(u;a,b) < \infty$ *and every* $\varepsilon > 0$, *there is* $v \in \mathrm{AC}[a,b]$ *such that*

$$v(a) = u(a),\, v(b) = u(b),\, \int_a^b |v' - u'|\, dx < \varepsilon,\, \mathscr{L}(v;a,b) \leq \mathscr{L}(u;a,b) + \varepsilon,$$

and almost every point of $[a,b]$ *has a neighbourhood on which v is affine.*

Proof. For $a \leq \alpha < \beta \leq b$ let $u_{\alpha,\beta}$ denote the affine function that joins $(\alpha, u(\alpha))$ and $(\beta, u(\beta))$. Let $\eta = \varepsilon/(b-a)$.

Our proof depends upon showing that for almost every $\alpha \in (a,b)$,

$$\mathscr{L}(u_{\alpha,\beta};\alpha,\beta) < \mathscr{L}(u;\alpha,\beta) + \eta(\beta - \alpha), \tag{1}$$

whenever $\beta > \alpha$ is close enough to α.

We start by showing how (1) gives the Proposition: Let $\mathscr{J}$ be the family of intervals $J = [\alpha,\beta] \subset (a,b)$ such that

$$\int_\alpha^\beta |u'_{\alpha,\beta}(x) - u'(x)|\, dx < \eta |J| \text{ and } \mathscr{L}(u_{\alpha,\beta};\alpha,\beta) < \mathscr{L}(u;\alpha,\beta) + \eta |J|$$

Since almost every α is a Lebesgue point of u' for which (1) holds, almost every α has the property that $[\alpha,\beta] \in \mathscr{J}$ whenever $\beta > \alpha$ is close enough to α. Hence $\mathscr{J}$ covers (a,b) in the sense of Vitali, and so the Vitali covering theorem provides us with disjoint $J_j = [\alpha_j, \beta_j] \in \mathscr{J}$ covering almost all of (a,b).

Define $v\colon [a,b] \to \mathbb{R}$ by

$$v(x) = \begin{cases} u(x), & \text{for } x \in [a,b] \setminus \bigcup_j J_j, \\ u_{\alpha_j,\beta_j}(x), & \text{on } J_j. \end{cases}$$

Since $\int_{J_j} |v' - u'|\, dx < \eta |J_j|$ and v agrees with u at the endpoints of the intervals J_j and outside their union, we see that v' is integrable, $\int_a^b |v' - u'|\, dx < \eta(b-a) = \varepsilon$ and v is an indefinite integral of v'. Consequently, $v \in \mathrm{AC}[a,b]$ and, since $\{J_j\}$ covers almost all of (a,b), v is affine in a neighbourhood of almost every point and

$$\mathscr{L}(v;a,b) \le \sum_j \mathscr{L}(v;\alpha_j,\beta_j) \le \sum_j (\mathscr{L}(u;\alpha_j,\beta_j) + \eta |J_j|) < \mathscr{L}(u;a,b) + \varepsilon.$$

It remains to prove that (1) holds for almost every $\alpha \in (a,b)$: For those $x \in (a,b)$ at which u is differentiable, define $g(x)$ to be the largest number for which

$$|L(x,y,p) - L(x,u(x),u'(x))| < \tfrac{1}{2}\eta, \text{ whenever } \|(y,p) - (u(x),u'(x))\| < g(x).$$

Since $L(x,y,p)$ is a continuous function in (y,p) for every x, g is a strictly positive measurable function. (Measurability follows upon observing that $\{x : g(x) < c\}$ is the union of the sets $\{x : |L(x,y,p) - L(x,u(x),u'(x))| > \frac{1}{2}\eta\}$ over rational y,p for which $\|(y,p) - (u(x),u'(x))\| < c$.)

Let $\alpha \in (a,b)$ be a point at which u is differentiable and at which both g and u' are approximately continuous. Since both the values and the slopes of $u_{\alpha,\beta}$ have a bound independent of $\beta \in (\alpha,b)$, there is $C < \infty$ such that $L(x,u_{\alpha,\beta}(x),u'_{\alpha,\beta}(x)) \le C$ for every $\beta > \alpha$ and $x \in (\alpha,\beta)$. If $\beta > \alpha$ is close enough to α, then $|u_{\alpha,\beta}(x) - u(x)| < \frac{1}{2}g(\alpha)$ for every $x \in (\alpha,\beta)$ and the set

$$T = \{x \in (\alpha,\beta) : |u'_{\alpha,\beta}(x) - u'(x)| > \tfrac{1}{2}g(\alpha) \text{ or } |g(x) - g(\alpha)| > \tfrac{1}{2}g(\alpha)\}$$

has measure less that $\eta(\beta-\alpha)/(2C)$. Letting $S = (\alpha,\beta) \setminus T$ and observing that for almost all $x \in S$, $\|(u_{\alpha,\beta}(x),u'_{\alpha,\beta}(x)) - (u(x),u'(x))\| < g(\alpha)$ and so $|L(x,u_{\alpha,\beta}(x),u'_{\alpha,\beta}(x)) - L(x,u(x),u'(x))| < \frac{1}{2}\eta$, we find

$$\begin{aligned} \mathscr{L}(u_{\alpha,\beta};\alpha,\beta) &= \int_S L(x,u_{\alpha,\beta}(x),u'_{\alpha,\beta}(x))\, dx + \int_T L(x,u_{\alpha,\beta}(x),u'_{\alpha,\beta}(x))\, dx \\ &\le \int_S L(x,u(x),u'(x))\, dx + \tfrac{1}{2}\eta |S| + C|T| \\ &< \mathscr{L}(u;\alpha,\beta) + \eta(\beta-\alpha), \end{aligned}$$

which finishes the proof.

We now come to the main part of this section. Since we will work with two Lagrangians simultaneously, we write $\mathscr{L}_L(u;a,b)$ instead of $\mathscr{L}(u;a,b)$ for $\int_a^b L(x,u(x),u'(x)\, dx$; we use $\mathscr{L}_L(a,A;b,B)$ and $\mathscr{E}_L(u;a,b)$ similarly. We define relaxed minimizers as follows. Given a Lagrangian L, we denote by L^c the convexification of L with respect to the third variable; thus $L^c(x,y,p)$ is equal to

$$\inf\{\lambda_1 L(x,y,p_1) + \lambda_2 L(x,y,p_2) : \lambda_i \ge 0,\ \lambda_1 + \lambda_2 = 1 \text{ and } p = \lambda_1 p_1 + \lambda_2 p_2\}.$$

We say that $u \in AC[a,b]$ is a *relaxed minimizer for L* if it minimizes the L^c energy with respect to its boundary data in a and b, that is,

$$\mathscr{L}_{L^c}(u;a,b) = \mathscr{L}_{L^c}(a,u(a);b,u(b)).$$

Provided that the (superlinear) lower bound ω for L is convex, it is clear that L^c has the same lower bound.

Lemma 5. *If L be a Lagrangian that is continuous in (y,p) for each fixed x, then L^c is continuous in (y,p) for each fixed x.*

Proof. Let x be fixed, and suppose $(y^n,p^n) \to (y^0,p^0)$ and $\varepsilon > 0$. If $\lambda_i \geq 0$ are such that

$$\lambda_1 + \lambda_2 = 1,\ p^0 = \lambda_1 p_1 + \lambda_2 p_2,$$

and

$$\lambda_1 L(x,y^0,p_1) + \lambda_2 L(x,y^0,p_2) < L^c(x,y^0,p^0) + \varepsilon,$$

then we may change the p_i and λ_i slightly, relabelling them if necessary, to get $p_1 < p^0 < p_2$. Then for large n, there are $\lambda_i^n \geq 0$, with $\lambda_1^n + \lambda_2^n = 1$ for which $p^n = \lambda_1^n p_1 + \lambda_2^n p_2$; moreover $\lambda_i^n \to \lambda_i$. Hence

$$\begin{aligned} L^c(x,y^n,p^n) &\leq \lambda_1^n L(x,y^n,p_1) + \lambda_2^n L(x,y^n,p_2) \\ &\to \lambda_1 L(x,y^0,p_1) + \lambda_2 L(x,y^0,p_2) \\ &< L^c(x,y^0,p^0) + \varepsilon, \end{aligned}$$

and we see that $\limsup_{n\to\infty} L^c(x,y^n,p^n) \leq L^c(x,y^0,p^0)$.

For the opposite direction, it suffices to show that there are bounded p_i^n and $\lambda_i^n \geq 0$ for which

$$\lambda_1^n + \lambda_2^n = 1,\ p^n = \lambda_1^n p_1^n + \lambda_2^n p_2^n$$

and

$$\lambda_1^n L(x,y^n,p_1^n) + \lambda_2^n L(x,y^n,p_2^n) \leq L^c(x,y^n,p^n) + \varepsilon;$$

the inequality $\liminf_{n\to\infty} L^c(x,y^n,p^n) \geq L^c(x,y^0,p^0)$ then follows by taking limits over subsequences and arbitrary $\varepsilon > 0$. If $L^c(x,y^0,p^0) > L^c(x,y,p)$, then the superlinearity of L implies that any p_i^n satisfying the above conditions are bounded. If $L^c(x,y^0,p^0) = L^c(x,y,p)$, we take $p_1^n = p_1 < p^0 < p_2 = p_2^n$, sufficiently close to p^0; the desired inequality follows by continuity.

Lemma 6. *For every $u \in \mathrm{AC}[a,b]$ and $\varepsilon > 0$, there is $v \in \mathrm{AC}[a,b]$ such that*

$$v(a) = u(a),\ v(b) = u(b),\ |v(x) - u(x)| < \varepsilon \text{ for every } x \in [a,b],$$

and

$$\mathscr{L}_L(v;a,b) \leq \mathscr{L}_{L^c}(u;a,b) + \varepsilon.$$

Proof. Since the statement is obvious when $\mathscr{L}_{L^c}(u;a,b) = \infty$ and since adding a constant to L does not change the inequalities to be proved, we assume that $\mathscr{L}_{L^c}(u;a,b) < \infty$ and $L(x,y,p) \geq |p|$. By Proposition 5 used for the Lagrangian L^c we may also assume that almost every point of $[a,b]$ has a neighbourhood on which u is affine.

Our proof depends upon showing that, given any $\eta > 0$, almost every $\alpha \in (a,b)$ has the property that for every $\beta \in (\alpha,\beta)$ that is close enough to α, there are functions $u_{\alpha,\beta} \in \mathrm{AC}[\alpha,\beta]$, for which $u_{\alpha,\beta}(\alpha) = u(\alpha)$, $u_{\alpha,\beta}(\beta) = u(\beta)$ and

$$\mathscr{L}_L(u_{\alpha,\beta};\alpha,\beta) < \mathscr{L}_{L^c}(u;\alpha,\beta) + \eta(\beta-\alpha). \tag{2}$$

We first show how this gives the Lemma. Let $\eta = \varepsilon/(b-a)$ and let $\mathscr{J}$ be the family of those intervals $J = [\alpha,\beta] \subset (a,b)$ for which $\eta|J| < \varepsilon/3$, $\mathscr{L}_{L^c}(u;\alpha,\beta)) < \varepsilon/6$, and there is $v_J \in \mathrm{AC}[\alpha,\beta]$ with $v_J(\alpha) = u(\alpha)$, $v_J(\beta) = u(\beta)$ and $\mathscr{L}_L(v_J;\alpha,\beta) < \mathscr{L}_{L^c}(u;\alpha,\beta) + \eta|J|$.

Since (2) implies that $\mathscr{J}$ covers (a,b) in the sense of Vitali, we may use the Vitali covering theorem to find disjoint $J_j = [\alpha_j,\beta_j] \in \mathscr{J}$ whose union covers almost all of (a,b). Let $v_j = v_{J_j}$.

Define $v\colon [a,b] \to \mathbb{R}$ by

$$v(x) = \begin{cases} u(x), & \text{for } x \in [a,b] \setminus \bigcup_j J_j, \\ v_j(x), & \text{for } x \in J_j. \end{cases}$$

Then

$$\begin{aligned} \sum_j \mathscr{L}_L(v_j;\alpha_j,\beta_j) &< \mathscr{L}_{L^c}(u;a,b) + \sum_j \eta|J_j| \\ &\leq \mathscr{L}_{L^c}(u;a,b) + \varepsilon. \end{aligned}$$

Hence, since $\int_{\alpha_j}^{\beta_j} |v'| \leq \mathscr{L}_L(v_j;\alpha_j,\beta_j)$ and since v agrees with u at the endpoints of the intervals J_j and outside their union, we see that v' is integrable and v is an indefinite integral of v'. Consequently, $v \in \mathrm{AC}[a,b]$ and $\mathscr{L}_L(v;a,b) < \mathscr{L}_{L^c}(u;a,b) + \varepsilon$.

If $x \in [\alpha_j,\beta_j]$, then

$$\begin{aligned} |v(x) - u(x)| &\leq |v(x) - v(\alpha_j)| + |u(x) - u(\alpha_j)| \\ &\leq \mathscr{L}_L(v;\alpha_j,\beta_j) + \mathscr{L}_{L^c}(u;\alpha_j,\beta_j) \\ &< 2\mathscr{L}_{L^c}(u;\alpha_j,\beta_j) + \eta|J_j| \leq \varepsilon. \end{aligned}$$

So, on recalling that $v(x) = u(x)$ for $x \notin \bigcup_j [\alpha_j,\beta_j]$, we see that the inequality $|v(x) - u(x)| < \varepsilon$ holds for all $x \in [a,b]$, and the Lemma is proved.

It only remains to prove (2). It is enough for us to show that the statement holds for those $\alpha \in (a,b)$ for which:

- there is $\tau > 0$ such that u is affine on $[\alpha-\tau,\alpha+\tau]$;
- α is a Lebesgue point of $x \mapsto L(x,u(x),p)$ for every rational p;
- α is a Lebesgue point of $x \mapsto L^c(x,u(x),u'(x))$.

Let $\eta > 0$. Choose $p_1, p_2 \in \mathbb{R}$ and $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$, $u'(\alpha) = \lambda_1 p_1 + \lambda_2 p_2$ and $\lambda_1 L(\alpha, u(\alpha), p_1) + \lambda_2 L(\alpha, u(\alpha), p_2) < L^c(\alpha, u(\alpha), u'(\alpha)) + \eta$. Since L is continuous in p, we can change p_i and λ_i slightly so that p_1 and p_2 are rational. Let $R = \max(|p_1|, |p_2|)$ and let $C < \infty$ be an upper bound of $L(x,y,p)$ for $x \in [a,b]$, $|y - u(x)| \leq 1$ and $|p| \leq R$.

Choose $\beta \in (\alpha, \alpha + \tau)$ close enough to α so that

$$\int_\alpha^\beta |L^c(x, u(x), u'(x)) - L^c(\alpha, u(\alpha), u'(\alpha))| \, dx < \eta(\beta - \alpha)$$

and

$$\int_\alpha^\beta |L(x, u(x), p_i) - L(\alpha, u(\alpha), p_i)| \, dx < \eta(\beta - \alpha).$$

Since $L(x,y,p)$ is continuous in y, there is $0 < \delta < 1$ so that the sets

$$S_i = \{x \in (\alpha, \beta) : |L(x, y, p_i) - L(x, u(x), p_i)| < \eta \text{ for } |y - u(x)| \leq \delta\} \ (i = 1, 2),$$

each have measure at least $(1 - \eta/C)(\beta - \alpha)$. Choose $m \in \mathbb{N}$ so that

$$(|p_1| + |p_2| + |u'(\alpha)|)(b - a)/m < \delta.$$

Set $\alpha_j = \alpha + j(\beta - \alpha)/m$, $\beta_j = \alpha_j + \lambda_1(\beta - \alpha)/m$, $P_1 = \bigcup_{j=0}^{m-1}(\alpha_j, \beta_j)$, $P_2 = \bigcup_{j=0}^{m-1}(\beta_j, \alpha_{j+1})$ and $Q_i = P_i \cap S_i$, and observe that

$$|Q_i| \leq |P_i| = \lambda_i(\beta - \alpha) \text{ and } |P_i \setminus Q_i| \leq \eta(\beta - \alpha)/C.$$

Define $w \in \mathrm{AC}[\alpha, \beta]$ by $w(\alpha) = u(\alpha)$, $w' = p_1$ on P_1, and $w'(x) = p_2$ on P_2. Then $w(\alpha_j) = u(\alpha_j)$ for $j = 1, 2, \ldots, m-1$, and $|w(x) - u(x)| < \delta$ for $x \in [\alpha, \beta]$. Thus $|L(x, w(x), p_i) - L(x, u(x), p_i)| < \eta$ for $x \in Q_i$, and so

$$\begin{aligned}
&\left| \int_{Q_i} L(x, u(x), p_i) \, dx - |Q_i| L(\alpha, u(\alpha), p_i) \right| \\
&\leq \int_{Q_i} |L(x, u(x), p_i) - L(\alpha, u(\alpha), p_i)| \, dx \leq \eta(\beta - \alpha).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{P_i} L(x, w(x), p_i) \, dx &\leq \int_{Q_i} L(x, w(x), p_i) \, dx + C|P_i \setminus Q_i| \\
&\leq \int_{Q_i} L(x, u(x), p_i) \, dx + \eta|Q_i| + \eta(\beta - \alpha) \\
&\leq |Q_i| L(\alpha, u(\alpha), p_i) + \eta(\beta - \alpha) + 2\eta(\beta - \alpha) \\
&\leq \lambda_i L(\alpha, u(\alpha), p_i)(\beta - \alpha) + 3\eta(\beta - \alpha).
\end{aligned}$$

Since $\mathscr{L}_L(w; \alpha, \beta)$ is the sum of these integrals over $i = 1, 2$, we get

$$\begin{aligned}
\mathscr{L}_L(w; \alpha, \beta) &\leq (\lambda_1 L(\alpha, u(\alpha), p_1) + \lambda_2 L(\alpha, u(\alpha), p_2))(\beta - \alpha) + 6\eta(\beta - \alpha) \\
&\leq L^c(\alpha, u(\alpha), u'(\alpha))(\beta - \alpha) + 7\eta(\beta - \alpha) \\
&\leq \mathscr{L}_{L^c}(u; \alpha, \beta) + 8\eta(\beta - \alpha),
\end{aligned}$$

and, noting that $w(\beta) = u(\beta)$, we see that the required statement holds with $u_{\alpha,\beta} = w$, provided that the above construction was started with $\eta/9$.

This Lemma clearly implies that $\mathscr{L}_{L^c}(a,A;b,B) = \mathscr{L}_L(a,A;b,B)$ for any $a < b$ and $A,B \in \mathbb{R}$, and that every relaxed minimizer is a constrained minimizer. This also implies that every constrained minimizer is a relaxed one: if $u \in \mathrm{AC}[a,b]$, $u(a) = A$, $u(b) = B$ is a constrained minimizer, then $\mathscr{L}_{L^c}(a,A;b,B) \le \mathscr{L}_{L^c}(u;a,b) \le \mathscr{L}_L(u;a,b) = \mathscr{L}_L(a,A;b,B) = \mathscr{L}_{L^c}(a,A;b,B)$. We record this in

Theorem 4. *If the Lagrangian L is continuous in (y,p) for each $x \in \mathbb{R}$, then a function $u \in \mathrm{AC}[a,b]$ is a constrained minimizer for L if and only if it is a minimizer for L^c.*

An immediate consequence of this result and Proposition 4 is the following theorem.

Theorem 5. *Let L be a Lagrangian that is continuous in (y,p) for each $x \in \mathbb{R}$. Let $a < b$ and A,B be given and suppose that there is a relaxed minimizer $\hat{u}$ such that $\hat{u}(a) = A$, $\hat{u}(b) = B$,*

$$\liminf_{x \to a} \frac{|\hat{u}(x) - A|}{|x-a|} < \infty \text{ and } \liminf_{x \to b_-} \frac{|\hat{u}(x) - B|}{|x-b|} < \infty.$$

Then every generalized minimizer with $u(a) = A$ and $u(b) = B$ is also a relaxed one.

The description of the universal singular set of a Lagrangian as the universal singular set of the convexified Lagrangian also follows from Theorem 4.

Corollary 4. *If L is a Lagrangian that is continuous in (y,p) for each $x \in \mathbb{R}$, then the universal singular set of L coincides with the universal singular set of L^c.*

2.4. Tonelli regularity

In this section, we prove a version of Tonelli's partial regularity theorem that is valid for functions that are close to minimizers. The main point of Tonelli's theorem is that, when the slope of a minimizer u has, between two points α, β of its domain, a certain bound and α, β are close enough (depending on the bound of the slope), then the minimizer is Lipschitz between the points, and even $|u'| \le C$ on $[\alpha, \beta]$ where C depends only on the bound and the points. Our idea is that, when the slope of a given function u, between two points α, β of its domain, has a certain bound and α, β are close enough (depending on the bound for the slope), then the measure of the set $\{x \in [\alpha,\beta] : |u'(x)| > C\}$ (where C depends only on the bound and the points) should be controlled by the excess.

In the rest of this section, in addition to our usual assumptions on the Lagrangian L (Borel, bounded from below, locally bounded from above, and superlinear), we assume that L is locally Lipschitz in y uniformly for (x,p) in any compact set, that is:

(L) For every $R > 0$, there is $C \ge 0$ such that $|L(x,y_1,p) - L(x,y_2,p)| \le C|y_1 - y_2|$ whenever $|x|, |y|, |p| \le R$.

This is weaker than the assumptions in the papers [15] and [2].

Lemma 7. *Let L be a Lagrangian satisfying the Lipschitz condition* (L). *For each $R > 0$, there are $M, \delta > 0$ such that if*

1. $[\alpha,\beta] \subset [a,b] \subset [-R,R]$ *with* $|b-a| < \delta$*, and*
2. $u \in \mathrm{AC}[a,b]$ *satisfies* $|u(x)| \le R$ *for* $x \in [a,b]$ *and*

$$\mathscr{E}(u;\alpha,\beta) + |u(\beta) - u(\alpha)| + \left| \int_{\{x\in[a,b]:|u'(x)|>M\}} u'(x)\,dx \right| \le R(\beta - \alpha),$$

then

$$\int_{\{x\in[a,b]:|u'(x)|>M\}} L(x,u(x),u'(x))\,dx \le 2\mathscr{E}(u;a,b).$$

Proof. Since adding a positive constant to L does not change the validity of the Lemma's hypotheses and only strengthens the conclusion, we may assume without any loss of generality that $L(x,y,p) \ge |p|$ for all x,y,p.

Let $R > 0$ be given.

Fix $C \ge 1$ for which $L(x,y,p) \le C$ whenever $|x|,|y|,|p| \le R$. Choose $N \ge R+1$ so that

$$\omega(p) \ge 2(R+C)|p| \text{ for } |p| \ge N,$$

and pick $D \ge CR$ for which $L(x,y,p) \le D$ whenever $|x| \le R$, $|y| \le N$ and $|p| \le N+2R$.

We can now define the required constants M and δ: first, choose $M \ge N+2R$ so that $\omega(p) \ge 10\left(1+\frac{D}{R}\right)|p|$ for $|p| \ge M$ and then choose $\delta > 0$ so that

$$|L(x,y_1,p) - L(x,y_2,p)| \le |y_1 - y_2|/\delta$$

whenever $|x| \le R$, $|y_1|,|y_2| \le 5R+4RM$ and $|p| \le M$.

Now suppose that intervals $[\alpha,\beta] \subset [a,b]$ and a function $u \in \mathrm{AC}[a,b]$ satisfy the assumptions of the lemma. Denoting by $\hat{u}$ the affine function on $[\alpha,\beta]$ for which $\hat{u}(\alpha) = u(\alpha)$ and $\hat{u}(\beta) = u(\beta)$, we infer that

$$\mathscr{L}(u;\alpha,\beta) \le \mathscr{E}(u;\alpha,\beta) + \mathscr{L}(\hat{u};\alpha,\beta) \le (R+C)(\beta-\alpha). \tag{3}$$

Let $Y = \{x \in [\alpha,\beta] : |u'(x)| > N\}$ and observe that, by the definition by N,

$$2(R+C)\int_Y |u'(x)|\,dx \le \int_\alpha^\beta L(x,u(x),u'(x))\,dx \le (R+C)(\beta-\alpha).$$

Hence $|Y| \le (\beta-\alpha)/(2N) \le \frac{1}{2}(\beta-\alpha)$ and

$$|\{x \in [\alpha,\beta] : |u'(x)| \le N\}| = |[\alpha,\beta] \setminus Y| \ge \tfrac{1}{2}(\beta-\alpha). \tag{4}$$

Let $Z = \{x \in [a,b] : |u'(x)| > M\}$. For future reference, we note that

$$10\left(1+\frac{D}{R}\right)\int_Z |u'(x)|\,dx \le \int_Z \omega(u'(x))\,dx \le \int_Z L(x,u(x),u'(x))\,dx, \tag{5}$$

which implies, in particular, that

$$C|Z| \le \frac{D}{R}\int_Z |u'(x)|\,dx \le \frac{1}{10}\int_Z L(x,u(x),u'(x))\,dx. \tag{6}$$

We also notice that the assumption $|u| \le R$ implies that for all $x \in [a,b]$,

$$\begin{aligned}\left|\int_{\{t\in[a,x]:|u'(t)|>M\}} u'(t)\,dt\right| &= \left|\int_a^x u'(t)\,dt - \int_{\{t\in[a,x]:|u'(t)|\le M\}} u'(t)\,dt\right| \\ &\le |u(x)-u(a)| + M(x-a) \le 2R(1+M).\end{aligned} \tag{7}$$

We now define a function $v \in \mathrm{AC}[a,b]$ to provide an estimate of the excess $\mathscr{E}(u;a,b)$. Combining our hypothesis that $|\int_Z u'(x)\,dx| \le R(\beta-\alpha)$ with (4) gives

$$\left|\int_Z u'(x)\,dx\right| \le 2R|\{x\in[\alpha,\beta] : |u'(x)| \le N\}|.$$

Hence we may find a measurable set $X \subset \{x \in [\alpha,\beta] : |u'(x)| \le N\}$ with

$$|X| = \frac{1}{2R}\left|\int_Z u'(x)\,dx\right|.$$

Denoting the sign of $\int_Z u'(x)\,dx$ by σ, we define $\varphi\colon [a,b] \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} u'(x), & \text{if } x \in Z, \\ -2\sigma R, & \text{if } x \in X, \\ 0, & \text{if } x \in [a,b]\setminus(X\cup Z). \end{cases}$$

Define $v \in \mathrm{AC}[a,b]$ by

$$v(x) = u(x) - \int_a^x \varphi(t)\,dt.$$

The estimates (8)–(11) that follow compare various energy integrals of u and v.

We use two estimates of $|u(x)-v(x)|$ for $x \in [a,b]$: the obvious one that $|u(x)-v(x)| \le 2\int_Z |u'(t)|\,dt$ and, a consequence of (7), that $|u(x)-v(x)| \le 4R(M+1)$, which implies $|v(x)| \le 5R+4RM$, since $|u(x)| \le R$.

Our choice of δ together with the observations that $|v'(x)| \le \max(0, N+2R, M) = M$ and $|u(x)-v(x)| \le 2\int_Z |u'(t)|\,dt$ for $x \in [a,b]$, and (5) give

$$\begin{aligned}&\int_a^b (L(x,v(x),v'(x)) - L(x,u(x),v'(x)))\,dx \\ &\qquad \le \frac{1}{\delta}\int_a^b |u(x)-v(x)|\,dx \le \frac{2(b-a)}{\delta}\int_Z |u'(t)|\,dt \\ &\qquad \le 2\int_Z |u'(t)|\,dt \le \frac{1}{5}\int_Z L(x,u(x),u'(x))\,dx.\end{aligned} \tag{8}$$

For $x \in X$, we have $|u(x)| \le R$, $|u'(x)| \le N$ and $|v'(x)| \le N+2R$, so the definitions of the constant D and the set X together with (5) give

$$\begin{aligned}&\int_X (L(x,u(x),v'(x)) - L(x,u(x),u'(x)))\,dx \\ &\qquad \le 2D|X| \le \frac{D}{R}\int_Z |u'(t)|\,dt \le \frac{1}{5}\int_Z L(x,u(x),u'(x))\,dx.\end{aligned} \tag{9}$$

For $x \in Z$, we have $|u(x)| \le R$ and $v'(x) = 0$, so $L(x,u(x),v'(x)) = L(x,u(x),0) \le C$. Hence, by (6),

$$\int_Z (L(x,u(x),v'(x)) - L(x,u(x),u'(x)))\,dx \le C|Z| - \int_Z L(x,u(x),u'(x))\,dx \le -\frac{9}{10}\int_Z L(x,u(x),u'(x))\,dx. \quad (10)$$

Finally, $u'(x) = v'(x)$ for $x \in [a,b] \setminus (X \cup Z)$, and so

$$\int_{[a,b]\setminus(X\cup Z)} (L(x,u(x),v'(x)) - L(x,u(x),u'(x)))\,dx = 0. \quad (11)$$

Adding the inequalities (8)–(11), we find

$$\int_a^b (L(x,v(x),v'(x)) - L(x,u(x),u'(x)))\,dx \le -\frac{1}{2}\int_Z L(x,u(x),u'(x))\,dx.$$

Since $v(a) = u(a)$ and $v(b) = u(b)$, we conclude

$$\int_Z L(x,u(x),u'(x))\,dx \le 2\int_a^b (L(x,u(x),u'(x)) - L(x,v(x),v'(x)))\,dx \le 2\mathscr{E}(u;a,b),$$

as required.

Corollary 5. *Let L be a Lagrangian satisfying the Lipschitz condition* (L). *For each $R > 0$, there are $M, \delta > 0$ such that if*

1. *$[\alpha,\beta] \subset [a,b] \subset [-R,R]$ with $|b-a| < \delta$, and*
2. *$u \in \mathrm{AC}[a,b]$ satisfies $|u(x)| \le R$ for $x \in [a,b]$ and*

$$\mathscr{E}(u;a,b) + |u(\beta) - u(\alpha)| \le R(\beta - \alpha),$$

then

$$\int_{\{x\in[a,b]:|u'(x)|>M\}} L(x,u(x),u'(x))\,dx \le 2\mathscr{E}(u;a,b).$$

Proof. Again, we may assume without any loss of generality that $L(x,y,p) \ge |p|$ for all x,y,p.

Let $C > 0$ be such that $L(x,y,p) \le C$ whenever $|x|,|y|,|p| \le R$. We show that the statement holds for M, δ obtained from Lemma 7 used with R replaced by $R_0 := 4(R+C)$. For this, assume that $[\alpha,\beta]$ and $u \in \mathrm{AC}[a,b]$ are as in the assumptions of the Corollary.

Call an interval $[c,d]$ *good* if $[a,b] \supset [c,d] \supset [\alpha,\beta]$ and

$$\int_{\{x\in[c,d]:|u'(x)|>M\}} L(x,u(x),u'(x))\,dx \le 3R(\beta - \alpha).$$

Notice that, by Lemma 7 (used with $R = R_0$ and $[a,b] = [c,d]$), if $[c,d]$ is a good interval, then

$$\int_{\{x\in[c,d]:|u'(x)|>M\}} L(x,u(x),u'(x))\,dx \le 2\mathscr{E}(u;c,d).$$

In particular, it suffices to show that $[a,b]$ is good. Since $2\mathscr{E}(u;c,d) < 3R(\beta-\alpha)$, continuity of the integral implies that every good interval distinct from $[a,b]$ is contained in a larger good interval. Hence $[a,b]$ is good, provided we show that at least one good interval exists. Thus it is enough to show that $[\alpha,\beta]$ is good.

Let $\hat{u}$ denote the affine function on $[\alpha,\beta]$ for which $\hat{u}(\alpha) = u(\alpha)$ and $\hat{u}(\beta) = u(\beta)$. Then

$$\int_\alpha^\beta |u'(x)|\,dx \le \mathscr{L}(u;\alpha,\beta) \le \mathscr{E}(u;\alpha,\beta) + \mathscr{L}(\hat{u};\alpha,\beta) \le (R+C)(\beta-\alpha).$$

Hence Lemma 7 gives

$$\int_{\{x\in[\alpha,\beta]:|u'(x)|>M\}} L(x,u(x),u'(x))\,dx \le 2\mathscr{E}(u;\alpha,\beta) \le 3R(\beta-\alpha),$$

as required.

Corollary 6. *Let L be a Lagrangian satisfying the Lipschitz condition* (L). *For every $R,N>0$, there are $M,\delta>0$ such that if $[a,b]\subset[-R,R]$ with $|b-a|<\delta$, and if $u\in \mathrm{AC}[a,b]$ is a generalized minimizer for L with $|u|<R$ on $[a,b]$, then either*

- $|u(\beta)-u(\alpha)| \le M(\beta-\alpha)$ *whenever $a\le\alpha\le\beta\le b$, or*
- $|u(\beta)-u(\alpha)| \ge N|\beta-\alpha|$ *whenever $a\le\alpha\le\beta\le b$.*

Proof. We again may assume that $L(x,y,p)\ge|p|$ for all x,y,p. We show that the conclusion holds for M and δ obtained from Corollary 5 when R is replaced by $R_0 := R+N$.

Let $[a,b]$ and u be as in the assumptions and find sequences $u_k\in \mathrm{AC}[a_k,b_k]$ so that: $u_k\to u$, $\mathscr{E}(u_k;a_k,b_k)\to 0$, and $[a_k,b_k]\nearrow[a,b]$.

If the second alternative of the corollary does not hold, then there are $a\le\alpha<\beta\le b$ for which $|u(\beta)-u(\alpha)|<N(\beta-\alpha)$. Without loss of generality we can assume that $a<\alpha<\beta<b$. Then, for sufficiently large k, $[a_k,b_k]\supset[\alpha,\beta]$, $|u_k|\le R$, and

$$\mathscr{E}(u_k;a_k,b_k) + |u_k(\beta)-u_k(\alpha)| \le R_0(\beta-\alpha).$$

Hence, by Corollary 5,

$$\begin{aligned}\int_{\{x\in[a_k,b_k]:|u_k'(x)|>M\}} |u_k'(x)|\,dx &\le \int_{\{x\in[a_k,b_k]:|u_k'(x)|>M\}} L(x,u_k(x),u_k'(x))\,dx \\ &\le 2\mathscr{E}(u_k;a_k,b_k)\to 0,\end{aligned}$$

showing that the first alternative of the Corollary holds.

Remark 1. Lemma 7 and Corollaries 5 and 6 also hold without change of proof in the vector-valued case, extending the work of [4]. In the real-valued case more precise information is apparently available (see Proposition 6 following); however it follows immediately from Corollary 6.

Proposition 6. *Let L be a Lagrangian satisfying the Lipschitz condition* (L). *Then for every $R,N>0$, there are $M,\delta>0$ such that if $[a,b]\subset[-R,R]$ with $|b-a|<\delta$, and if $u\in \mathrm{AC}[a,b]$ is a generalized minimizer for L with $|u|<R$ on $[a,b]$, then either*

- $|u(\beta)-u(\alpha)|\le M(\beta-\alpha)$ *whenever* $a\le\alpha<\beta\le b$, *or*
- $u(\beta)-u(\alpha)\ge N(\beta-\alpha)$ *whenever* $a\le\alpha<\beta\le b$, *or*
- $u(\beta)-u(\alpha)\le -N(\beta-\alpha)$ *whenever* $a\le\alpha<\beta\le b$.

Proof. Let M and δ be as given by Corollary 6 for R. Corollary 6 implies that if the first alternative does not hold, then $|u(\beta)-u(\alpha)|\ge N(\beta-\alpha)$ whenever $a\le\alpha<\beta\le b$. In this case it is enough to show that, for $a\le\alpha<\beta\le b$, $u(\beta)-u(\alpha)$ is either always positive or always negative. But this is obvious, since otherwise there are $a\le\alpha<\beta\le b$ such that $u(\beta)-u(\alpha)=0$ in which case Corollary 6 implies that the first alternative occurs.

Another way of presenting our results is closer to the usual formulation of Tonelli's partial regularity theorem. It follows directly from the fact that u has a finite derivative almost everywhere and the previous Proposition.

Proposition 7. *Let L be a Lagrangian satisfying the Lipschitz condition* (L). *Then for every generalized minimizer u on $[a,b]$, there are disjoint closed Lebesgue null sets $E_+,E_-\subset[a,b]$ such that*

- *u is locally Lipschitz on* $[a,b]\setminus(E_+\cup E_-)$;
- $\lim_{s\neq t,\,\max(d(s,E_+),d(t,E_+),|t-s|)\to 0}(u(t)-u(s))/(t-s)=\infty$;
- $\lim_{s\neq t,\,\max(d(s,E_-),d(t,E_-),|t-s|)\to 0}(u(t)-u(s))/(t-s)=-\infty$.

Many standard variants of Tonelli's regularity results may be obtained, under appropriate smoothness and strict convexity assumptions, by deducing the Euler-Lagrange equation on the intervals where the minimizer is Lipschitz. Since this is a straightforward use of known methods, we do not do this here.

3. The structure of universal singular sets

This section is devoted to the study of the size of the intersection of universal singular sets with rectifiable curves. Our aim is to show that universal singular sets intersect many curves in a set of zero length.

Recall that the universal singular set of L is defined to be all points $(x_0,y_0)\in\mathbb{R}^2$ for which there are $a\le x_0\le b$ with $a<b$ and a $u\in\mathrm{AC}[a,b]$ that is a generalized minimizer for L with $u(x_0)=y_0$ and $|u'(x_0)|=\infty$. However, for continuous Lagrangians that are convex in p our argument does not use generalized minimizers.

It turns out that, although the universal singular set of any Lagrangian meets the graph of every absolutely continuous function in a set of linear measure zero, the situation is considerably more delicate when it comes to curves that may have vertical tangents: vertical lines also meet the universal singular set in a null set, but, as shown in section 4, some rectifiable curves may actually meet it in a set of positive linear measure.

In this section, we consider the following problems:

1. Which curves have the property that they meet the universal singular set of each Lagrangian with a given superlinear growth in a null set?
2. Which curves have the property that they meet the universal singular set of every Lagrangian in a null set?
3. Which curves have the property that they meet a set contained in the universal singular set of Lagrangians with arbitrary superlinear growth in a null set?
4. For which Lagrangians is the universal singular set of the first category?

We now describe our results. The examples of section 4 show that these results are close to a complete picture.

Our answer to problem (1) is given by the following theorem.

Theorem 6. *Let $\omega\colon \mathbb{R} \to \mathbb{R}$ be even, convex and superlinear. Suppose that an absolutely continuous curve $\gamma(t) = (x(t), y(t))\colon [a,b] \to \mathbb{R}^2$ is such that for almost all $t \in [a,b]$, either*

$$\limsup_{s\to t} \left| \frac{y(s)-y(t)}{x(s)-x(t)} \right| < \infty \tag{12}$$

or

$$\liminf_{s\to t} |x(s)-x(t)|\ \omega\left(\frac{y(s)-y(t)}{x(s)-x(t)}\right) > 0. \tag{13}$$

Then $\{\gamma(t) : t \in [a,b]\}$ meets the universal singular set of any Lagrangian L for which $L(x,y,p) \geq \omega(p)$ in a set of linear measure zero. (When $x(s)-x(t) = 0$, we take

$$\left| \frac{y(s)-y(t)}{x(s)-x(t)} \right| \text{ to be zero and } |x(s)-x(t)|\omega\left(\frac{y(s)-y(t)}{x(s)-x(t)}\right) \text{ to be } \infty.)$$

Before proving this Theorem, we show how it is used to answer questions (2)–(4). Since absolutely continuous functions (considered as curves $x \mapsto (x, f(x))$) satisfy (12), and vertical lines satisfy (13), we have an answer to question (2).

Theorem 7. *Graphs of absolutely continuous functions and vertical lines meet the universal singular set of any Lagrangian in a set of linear measure zero.*

The answers to problems (3) and (4) also follow from Theorem 6, but a little more work is needed.

Theorem 8. *Suppose that $E \subset \mathbb{R}^2$ is such that for any superlinear ω there is a Lagrangian $L(x,y,p) \geq \omega(p)$ whose universal singular set contains E. Then E is purely unrectifiable.*

Proof. It suffices to show that $|\{t \in (a,b) : \gamma(t) \in E\}| = 0$ for any injective C^1 curve $\gamma(t) = (x(t), y(t))\colon [a,b] \to \mathbb{R}^2$ for which $\gamma'(t) \neq 0$ for all $t \in [a,b]$. This follows from Theorem 6 provided we find a superlinear function ω for which

$$\liminf_{s\to t} |x(s)-x(t)|\ \omega\left(\frac{|y(s)-y(t)|}{|x(s)-x(t)|}\right) > 0 \text{ whenever } x'(t) = 0. \tag{14}$$

Let $\delta_n \searrow 0$ be a strictly decreasing sequence such that for $s,t \in [a,b]$ with $s \neq t$,

$$|y(s)-y(t)| > (n+1)|x(s)-x(t)| \text{ whenever } x'(t) = 0 \text{ and } |x(s)-x(t)| < \delta_n.$$

Such a sequence exists, since otherwise we could find, for some n, sequences $s_k, t_k \in [a,b]$ with

$$x'(t_k) = 0,\ |y(s_k) - y(t_k)| \le (n+1)|x(s_k) - x(t_k)| \text{ and } |x(s_k) - x(t_k)| \to 0.$$

But then, by passing to a subsequence, we could assume that $s_k \to s$ and $t_k \to t$ with

$$x'(t) = 0,\ |y(s) - y(t)| \le (n+1)|x(s) - x(t)| \text{ and } x(s) = x(t).$$

But this means $y(s) = y(t)$, and so, since γ is injective, $s = t$. But then $|y'(t)| > 0 = x'(t)$, and so $|y(s_k) - y(t_k)| > (n+1)|x(s_k) - x(t_k)|$ for k large enough — a contradiction.

Let $\kappa\colon (0,\infty) \to (0,\infty)$ be a continuous decreasing function for which $\kappa(\delta_n) = n$. Let κ^{-1} be the inverse of κ and let ω be any convex superlinear function satisfying $\omega(p) = \omega(|p|) \ge 1/\kappa^{-1}(p)$. If $s,t \in (a,b)$ and $\delta_{n+1} \le |x(s) - x(t)| < \delta_n$, then

$$\frac{|y(s) - y(t)|}{|x(s) - x(t)|} \ge n+1 = \kappa(\delta_{n+1}) \ge \kappa(|x(s) - x(t)|).$$

Hence

$$|x(s) - x(t)|\,\omega\left(\frac{|y(s) - y(t)|}{|x(s) - x(t)|}\right) \ge |x(s) - x(t)|\,\omega(\kappa(|x(s) - x(t)|)) \ge 1,$$

for every $s \in [a,b]$ for which $|x(s) - x(t)| < \delta_1$, and so, in particular, for $|s - t|$ small enough. It follows that (14) holds and we are done.

Our answer to problem (4) is given in the following Theorem where we show that, assuming the Lagrangian satisfies the Lipschitz condition (L) of subsection 2.4, the universal singular set is of the first Baire category. Our proof is based on the regularity results of Section 2.4 and the 'almost' pure unrectifiability of universal singular sets described in Theorem 6. We show in Section 4 that some additional assumption on the Lagrangian is necessary.

Theorem 9. *If L is a Lagrangian that satisfies the Lipschitz condition* (L)*, then the universal singular set of L is a countable union of closed sets. In particular, it is a first category set.*

Proof. Let L be a Lagrangian that satisfies the Lipschitz condition (L). Let S_k be the set of all $P = (a,A) \in \mathbb{R}^2$ for which there is a generalized minimizer u^P on $[a, a+\frac{1}{k}]$ with $|u^P| \le k$ and $u'(a) = +\infty$. By Proposition 6, for every $N > 0$, there is $0 < \delta_N < \frac{1}{k}$ such that

$$u^P(y) - u^P(x) \ge N(y - x) \text{ whenever } a \le x < y \le a + \delta_N.$$

So if $P_l \in S_k$ converge to some $P \in \mathbb{R}^2$, then the corresponding functions u^{P_l} converge (up to a subsequence) to some generalized minimizer u on $[a, a+1/k]$ for which $|u| \le k$, and

$$u(y) - u(x) \ge N(y - x) \text{ whenever } a \le x < y \le a + \delta_N.$$

Hence $u'(a) = +\infty$, and so $P \in S_k$ implying that S_k is a closed set.

The universal singular set of L is a union of $\bigcup_{k=1}^{\infty} S_k$ together with three other sets obtained by symmetrical constructions. Since each of these sets is a countable unions of closed sets, the universal singular set is a countable union of closed sets.

Moreover, Theorem 6 implies that the interior of the universal singular set of L is empty. Hence the universal singular set of L is a first category set, since a countable union of closed sets in the plane that is not of the first category has nonempty interior.

The key to the proof of Theorem 6, and hence these results, is in understanding that, for fixed $(a,A) \in \mathbb{R}^2$, the functional $(x,X) \mapsto \mathscr{L}(a,A;x,X)$ increases (or decreases) steeply in many directions from (b,B) whenever (b,B) is in the universal singular set. This result is of independent interest.

Lemma 8. *Let $\omega\colon \mathbb{R} \to \mathbb{R}$ be convex and superlinear, and suppose that L is a Lagrangian satisfying $L(x,y,p) \geq \omega(p)$ for all x,y,p.*

If $a,A,b,B \in \mathbb{R}$ with $a < b$, and $u \in \mathrm{AC}[a,b]$ satisfies $u(a) = A$, $u(b) = B$ and has infinite left-derivative at b, $u'_-(b) = \infty$, then for all $C,D > 0$

$$\lim_{T \ni (x,X) \to (b,B)} \frac{\mathscr{L}(a,A;x,X) - \mathscr{L}(u;a,b)}{\|(x,X) - (b,B)\|} = -\infty,$$

where

$$T := \left\{(x,X) : x < b, X < B \text{ and } (b-x)\omega\left(\frac{B-X}{b-x}\right) > D\right\} \cup \{(x,X) : x \geq b \text{ and } X - B < C(x-b)\}.$$

Proof. Since L is bounded from below, $\mathscr{L}(u;a,b) > -\infty$ and so, the case $\mathscr{L}(u;a,b) = \infty$ being trivial, we may assume that $\mathscr{L}(u;a,b)$ is finite.

Fix $0 < K < \infty$; it is enough to show that for any $(x,X) \in T$ that is sufficiently close to (b,B), $\mathscr{L}(a,A;x,X) < \mathscr{L}(u;a,b) - K\|(x-b,X-B)\|$.

Let $M = \sup_{x \in [a,b+1], |y| \leq |B|+1} |L(x,y,2C)|$. Choose $p_0 > 2C+1$ so that ω is increasing and $\omega(p) > (K + 2M(C+1))(1+1/C)p$ on $[p_0,\infty)$. Choose $\alpha \in (\max(a,b-1),b)$ sufficiently close to b so that

1. $(b-\alpha)\omega(p) < D$ for $0 \leq p \leq p_0$;
2. $\mathscr{L}(u;s,t) < D$ whenever $\alpha \leq s < t \leq b$;
3. $|u(x) - B| < 1$ for $\alpha \leq x \leq b$;
4. $u(x) < B + p_0(x-b)$ for $\alpha \leq x < b$.

Since 4 with $x = \alpha$ implies $B + 2C(\alpha - b) > u(\alpha)$, there is $0 < \delta < b - \alpha$ so that

$$X_0 + 2C(\alpha - x_0) > u(\alpha) \text{ whenever } \|(x_0,X_0) - (b,B)\| < \delta.$$

Fix $(x_0,X_0) \in T$ for which $\|(x_0 - b, X_0 - B)\| < \delta$. Note that

$$B - X_0 > C(b - x_0);$$

for, if $x_0 < b$, then $(b-\alpha)\omega((B-X_0)/(b-x_0)) > D$, so 1 implies that $B-X_0 > p_0(b-x_0)$, and if $x_0 \geq b$, then the definition of T gives directly that $B-X_0 > C(b-x_0)$.

Let $\beta = \min(x_0, b)$. We show that

$$X_0 + 2C(\beta - x_0) < u(\beta). \tag{15}$$

If $x_0 \geq b$, then $\beta = b$, $u(\beta) = B$, and (15) follows from $X_0 - B < C(x_0 - b) < 2C(x_0 - b)$.

If $x_0 < b$, inequality (15) is just $X_0 < u(x_0)$. Assuming, for a contradiction, that $X_0 \geq u(x_0)$, we use $(B-u(x_0))/(b-x_0) > p_0$, the monotonicity of ω on $[p_0, \infty)$ and that $(x_0, X_0) \in T$ to infer

$$(b-x_0)\omega((B-u(x_0))/(b-x_0)) \geq (b-x_0)\omega((B-X_0)/(b-x_0)) > D.$$

Jensen's inequality then gives

$$\begin{aligned} D > \mathscr{L}(u; x_0, b) &\geq \int_{x_0}^{b} \omega(u'(x))\,dx \\ &\geq (b-x_0)\omega\left(\frac{u(b)-u(x_0)}{b-x_0}\right) \\ &\to (b-x_0)\omega\left(\frac{B-u(x_0)}{b-x_0}\right) > D \text{ — a contradiction.} \end{aligned}$$

Since $X_0 + 2C(\alpha - x_0) > u(\alpha)$ and $X_0 + 2C(\beta - x_0) < u(\beta)$, there is $\tau \in (\alpha, \beta) \subset (\alpha, b)$ such that $X_0 + 2C(\tau - x_0) = u(\tau)$.

We use two estimates of $\|(x_0 - b, X_0 - B)\|$. First, by rearranging

$$X_0 - 2C(b-\tau) - 2C(x_0 - b) = u(\tau) < B - (2C+1)(b-\tau),$$

we find $b - \tau < B - X_0 + 2C(b - x_0) \leq (2C+1)\|(x_0 - b, X_0 - B)\|$. Hence

$$x_0 - \tau \leq (x_0 - b) + (b - \tau) < 2(C+1)\|(x_0 - b, X_0 - B)\|. \tag{16}$$

For the second estimate, we recall that $\|(x_0 - b, X_0 - B)\| < \delta$ and $(x_0, X_0) \in T$, and so $X_0 - B < C(x_0 - b)$. We also notice that

$$B - u(\tau) = B - X_0 + 2C(x_0 - \tau) \geq B - X_0.$$

Thus if $x_0 \leq b$, then $0 \leq C(b - x_0) < B - X_0$, and so

$$\|(x_0 - b, X_0 - B)\| \leq (b - x_0) + (B - X_0) < (1 + 1/C)(B - u(\tau)).$$

Whereas if $x_0 > b$, we use

$$B - u(\tau) = B - X_0 + 2C(x_0 - \tau) \geq B - X_0 + 2C(x_0 - b) > C(x_0 - b)$$

to infer that $|X_0 - B| \leq \max(B - u(\tau), C(x_0 - b)) = B - u(\tau)$ and so, in either case,

$$\|(x_0 - b, X_0 - B)\| \leq (x_0 - b) + |X_0 - B| < (1 + 1/C)(u(b) - u(\tau)). \tag{17}$$

Let

$$v(x) = \begin{cases} X_0 + 2C(x - x_0) & \text{on } [\tau, x_0], \\ u(x) & \text{on } [a, \tau]. \end{cases}$$

Noting that $[\tau, x_0] \subset (b-1, b+1)$ and $|v(x)| \le |B| + 1$ on $[\tau, x_0]$, we infer from (16) that

$$\mathscr{L}(v; \tau, x_0) \le M(x_0 - \tau) < 2M(C+1)\|(x_0 - b, X_0 - B)\|.$$

However Jensen's inequality and (17) imply

$$\begin{aligned} \mathscr{L}(u; \tau, b) \ge \int_\tau^b \omega(u'(x))\,dx \ge (b-\tau)\omega\left(\frac{u(b)-u(\tau)}{b-\tau}\right) \\ > (K + 2M(C+1))\|(x_0 - b, X_0 - B)\|. \end{aligned}$$

Hence

$$\begin{aligned} \mathscr{L}(v; a, x_0) &\le \mathscr{L}(u; a, b) - \mathscr{L}(u; \tau, b) + \mathscr{L}(v; \tau, x_0) \\ &< \mathscr{L}(u; a, b) - K\|(x_0 - b, X_0 - B)\|, \end{aligned}$$

giving $\mathscr{L}(a, A; x_0, X_0) < \mathscr{L}(u; a, b) - K\|(x_0 - b, X_0 - B)\|$, as required.

In order to prove Theorem 6, we use the following well-known result showing that certain sets of reals always have linear measure zero,. (See [12] for many ramifications.)

Lemma 9. *The set of points at which a function $f\colon [a,b] \to \mathbb{R}$ has an infinite one-sided derivative has linear measure zero.*

Proof. It suffices to consider the set $E = \{x \in (a,b) : f'_+(x) = +\infty\}$. Let $E_n = \{x \in E : f(x) \le f(y) \text{ for } x < y < x + \frac{1}{n}(b-a)\}$ and $E_{n,k} = E_n \cap [a + (k-1)(b-a)/n, a + k(b-a)/n)$. Then $E = \bigcup_{n=1}^\infty \bigcup_{k=1}^n E_{n,k}$, and so it is enough to show that each $E_{n,k}$ has measure zero.

Fix $n, k \in \mathbb{N}$ and observe that $f(x) \le f(y)$ whenever $x, y \in E_{n,k}$ and $x < y$. Hence there is a nondecreasing function g that agrees with f on $E_{n,k}$. If $x \in E_{n,k}$ is not a right-isolated point of $E_{n,k}$ (that is, $(x, x+\delta) \cap E_{n,k} \ne \emptyset$ for every $\delta > 0$), then g does not have a finite derivative at x. But g, being monotonic, has a finite derivative almost everywhere, and so $E_{n,k}$ has measure zero since it can have at most countably many right-isolated points.

Proof of Theorem 6. Given $(a, A) \in \mathbb{R}^2$, denote by $S_{a,A}$ the set of $(b, B) \in \mathbb{R}^2$ for which $a < b$ and there is $u \in \mathrm{AC}[a,b]$ that is a constrained minimizer for L with $u(a) = a$, $u(b) = B$ and $u'_-(b) = +\infty$. It is enough to show that the union of the sets, $S := \bigcup_{a,A} S_{a,A}$, is null on the curves described in the Theorem; the whole universal singular set is a union of this set and three other sets that are given by similar definitions, and for which symmetrical arguments apply.

In fact we can assume that the union defining S is only taken over rational a and A. For given $(b, B) \in S_{a,A}$ and a corresponding constrained minimizer $u \in \mathrm{AC}[a,b]$, we pick a rational $\hat{a} \in (a,b)$ and a rational $\hat{A} < u(\hat{a})$, use Corollary 2 to infer that u is a constrained minimizer on $[\hat{a}, b]$, use Theorem 3 to find a constrained minimizer

v on $[\hat{a}, b]$ with $v(\hat{a}) = \hat{A})$ and $v(b) = B$, and use Proposition 2 to infer that $\min(u, v)$ is a constrained minimizer, and conclude that $(b, B) \in S_{\hat{a}, \hat{A}}$.

Hence it is enough to show that $S_{a,A}$ is a null set for all $a, A \in \mathbb{R}$.

Let $\gamma(t) = (x(t), y(t))$ be an absolutely continuous curve defined on an interval $[a, b]$ and having the properties (12) and (13) from the Theorem. Since γ maps both the set of points at which it is non-differentiable and the set of points at which it has derivative zero to a set of linear measure zero, we restrict our attention to the set E of those $t \in (\alpha, \beta)$ at which γ is differentiable, has nonzero derivative and has one of the properties from the Theorem. Then subsets of E have Lebesgue measure zero if and only if their image under γ has linear measure zero; so it is enough to show that $F := \{t \in E : \gamma(t) \in S_{a,A}\}$ has Lebesgue measure zero. We do this by showing that a one-sided derivative of the function $f(t) := \mathscr{L}(a, A; x(t), y(t))$ is infinite at every point of F.

So let $t \in F$. If (12) holds, choose C so that

$$\limsup_{s \to t} |(y(s) - y(t))/(x(s) - x(t))| < C < \infty$$

and $D > 0$ arbitrarily. If (13) holds, choose D so that

$$\liminf_{s \to t} |x(s) - x(t)| \omega((y(s) - y(t))/(x(s) - x(t))) > D > 0$$

and $C < \infty$ arbitrarily. Let $(b, B) = (x(t), y(t))$ and

$$\begin{aligned} T := \{(x, X) : x < b, X < B, (b - x)\omega((B - X)/(b - x)) > D\} \\ \cup \{(x, X) : x \geq b, X - B < C(x - b)\}. \end{aligned}$$

Then Lemma 8 applied with any constrained minimizer u witnessing that $(b, B) \in S_{a,A}$, and the fact that $\mathscr{L}(u; a, b) = \mathscr{L}(a, A; b, B)$, gives

$$\lim_{T \ni (x,X) \to (b,B)} \frac{\mathscr{L}(a, A; x, X) - \mathscr{L}(a, A; b, B)}{\|(x, X) - (b, B)\|} = -\infty. \tag{18}$$

Since $\gamma'(t) \neq 0$, we have $0 < \lim_{s \to t} \|((x(s), y(s)) - (b, B)\| / |s - t| < \infty$ and one of the following cases must occur:

- If $x'(t) > 0$, then (12) holds. Hence for $s > t$ sufficiently close to t we have $x(s) - B < C(y(s) - b)$, and so $(x(s), y(s)) \in T$. Thus (18) implies that $f'_+(t) = -\infty$.
- If $x'(t) < 0$, then (12) holds. Hence $(x(s), y(s)) \in T$ for $s < t$ sufficiently close to t. Thus (18) implies that $f'_-(t) = \infty$.
- If $x'(t) = 0$ and $y'(t) > 0$, then (13) holds, and we see that $(x(s), y(s)) \in T$ for $s > t$ sufficiently close to t. Hence (18) implies that $f'_+(t) = -\infty$.
- If $x'(t) = 0$ and $y'(t) < 0$, then (13) holds, and we see that $(x(s), y(s)) \in T$ for $s > t$ sufficiently close to t. Hence (18) implies that $f'_-(t) = \infty$.

We see that in each case, f has an infinite one-sided derivative at t. Hence Lemma 9 shows that F has measure zero, and the Theorem is proved.

4. Lagrangians with large universal singular sets

In this section we show that the results of Theorems 6–9 are close to being optimal.

For Theorems 6–8, we give examples of smooth Lagrangians satisfying classical conditions (including convexity in p) for which the universal singular set is as large as possible. In fact, the Lagrangians in Theorems 10 and 11 have the following special form: we assume that we are given a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(0) = 0$, and we construct Lagrangians L for which

($\star$) $L(x,y,p) = \omega(p) + F(x,y,p)$ where F satisfies:
- ($\star_1$) $F \in C^\infty(\mathbb{R}^3)$;
- ($\star_2$) $F \geq 0$ and for all $x, y \in \mathbb{R}$, $F(x,y,0) = 0$;
- ($\star_3$) $p \mapsto F(x,y,p)$ is convex for each fixed (x,y).

Notice that for such Lagrangians, the classical existence theorems hold, and the universal singular set corresponds with that defined by Ball and Nadirashvili.

The main result of this section is given in the following theorem.

Theorem 10. *Fix a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(p) \geq \omega(0) = 0$, and let $S \subset \mathbb{R}^2$ be a purely unrectifiable compact set. Then there is a Lagrangian satisfying ($\star$) whose universal singular set contains S.*

Recall that purely unrectifiable compact subsets of $\mathbb{R}^2$ may have Hausdorff dimension two and may contain non-trivial continua; so, in spite of Theorem 7, universal singular sets may be rather large.

We complement this result by a more particular example showing that, even when one restricts to compact sets, Theorem 10 does not provide a complete answer.

Theorem 11. *Fix a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(p) \geq \omega(0) = 0$. Then there is a rectifiable compact set $S \subset \mathbb{R}^2$ of positive linear measure that is contained in the universal singular set of some Lagrangian satisfying ($\star$).*

Unlike the measure zero result of Sychëv, our generalisation of the first category result of Ball and Nadirashvili (Theorem 9) is shown only under (mild) additional smoothness assumptions on the Lagrangian. The following result shows that this is necessary.

Theorem 12. *Fix a superlinear function $\omega\colon \mathbb{R} \to [0,\infty)$ for which $\omega(0) = 0$. Then there is a continuous Lagrangian L with $L(x,y,p) \geq \omega(p)$ for $(x,y,p) \in \mathbb{R}^3$ and whose universal singular set is residual in $\mathbb{R}^2$.*

We start by describing the general ideas behind our constructions. To construct Lagrangians with a large singular set S, and with a given superlinearity ω, we employ the idea of calibrations. Basically, we prescribe a field of minimizers (better: functions that should be minimizers in the future) that have infinite derivative when passing through the points of S. These minimizers are given by the equation

$u' = \psi(x,u)$ for a suitable function ψ that we expect to have singular behaviour whenever $(x,u) \in S$. We also choose (at this stage completely independently) the potential of the energy of our field of minimizers; that is, a function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$. Since we want each of our (future) minimizers u to satisfy $\int_a^b L(x,u,u')\,dx = \Phi(b,u(b)) - \Phi(a,u(a))$, and so L is a null Lagrangian for the minimizers u, we have to define $L(x,y,p) = \Phi_x(x,y) + p\Phi_y(x,y)$ for $p = \psi(x,y)$. The superlinearity condition means that we require

$$\Phi_x(x,y) + \psi(x,y)\Phi_y(x,y) \geq \omega(\psi). \tag{19}$$

The calibration argument, which is formally given by

$$\begin{aligned}\int_a^b L(x,u,u')\,dx &\geq \int_a^b \Phi_x(x,u(x)) + \Phi_y(x,u(x))u'(x)\,dx\\ &= \Phi(b,u(b)) - \Phi(a,u(a)),\end{aligned}$$

leads to the second requirement, namely

$$L(x,y,p) \geq \Phi_x(x,y) + p\Phi_y(x,y) \text{ whenever } (x,y) \notin S. \tag{20}$$

We manage to avoid the set S here because the pure unrectifiability of S should imply that, along trajectories, the inequality (20) holds almost everywhere. However the fact that Φ cannot behave regularly at the points of S returns to haunt us. The easier difficulty, that (20) may have no continuous solution L, is avoided rather simply by requiring that close to S, Φ_x is negative and $|\Phi_y|$ is much smaller that $|\Phi_x|$; hence the function on the right hand side of (20) is locally bounded from above. The harder problem is that, for some absolutely continuous $u(x)$, the function $\Phi(x,u(x))$ may fail to be absolutely continuous. To handle this, we use the pure unrectifiability requirement on S to construct Φ in such a way that, at least for increasing $u(x)$ (to which the problem may be reduced by a simple trick), the composition $\Phi(x,u(x))$ maps null sets to null sets. Other properties of $\Phi(x,u(x))$ and classical real analysis then imply that it is in fact absolutely continuous, justifying the use of the formal calibration argument indicated above.

Before embarking on the technical details which, unfortunately, involve a little more than the above basic description, we should comment on (19). Since our main discussion did not involve any condition on ψ, we are free to choose it subject only to (19). It is therefore more natural to begin by defining Φ satisfying all the requirements alluded to above and then choose a suitable ψ. Noting that the first part of our argument leads to $\Phi_x < 0$, we impose $\Phi_y > 0$ (the signs are, of course, arbitrary; these come from the requirement that the relevant minimizers be increasing) and observe that the increase of energy proved in Section 3 shows that $\psi > -\Phi_x/\Phi_y$ (at least close to the points of the universal singular set, and it should be substantially bigger there). Hence we decide to take $\psi = -2\Phi_x/\Phi_y$. This transforms (19) into $-\Phi_x \geq \omega(-2\Phi_x/\Phi_y)$, which is easy to achieve. In reality, technical points force us to require similar but much stronger inequalities. We therefore begin our argument by giving the details of the requirements on Φ leading to the construction of the required Lagrangian in Lemma 11, and only then describe the particular constructions of Φ giving the proofs of Theorems 10 and 11, respectively.

Before giving the proofs of Theorems 10 and 11, we record a couple of results that are used in the constructions of the required Lagrangians.

The following simple lemma tells us how to smooth the corners of a particular piecewise-affine function.

Lemma 10. *There is a C^∞ function $\gamma\colon \{(p,a,b) \in \mathbb{R}^3 : b > 0\} \to \mathbb{R}$ such that:*

10.a. $p \mapsto \gamma(p,a,b)$ *is convex;*
10.b. $\gamma(p,a,b) = 0$ *for* $p \le a-1$*;*
10.c. $\gamma(p,a,b) = b(p-a)$ *for* $p \ge a+1$*;*
10.d. $\gamma(p,a,b) \ge \max(0, b(p-a))$.

Proof. Let $\eta : \mathbb{R} \to [0,1]$ be a non-decreasing C^∞ function such that $\eta(p) = 0$ for $p \le -1$, $\eta(p) = 1$ for $p \ge 1$, and $\int_{-1}^{1} \eta(t)dt = 1$. Define $\gamma(p,a,b) = b\int_{-\infty}^{p-a} \eta(t)dt$. Then 10.a holds since $\partial\gamma/\partial p$ is non-decreasing, 10.b is obvious since for $p \le a-1$ the integrand vanishes, for 10.c we have $b\int_{-\infty}^{p-a} \eta(t)dt = b\int_{-1}^{1} \eta(t)dt + b\int_{1}^{p-a} \eta(t)dt = b(p-a)$, and 10.d follows from the previous statements.

The following lemma gives sufficient conditions to assert the existence of a Lagrangian satisfying ($\star$) with a given compact set inside its universal singular set, and plays a central role in the proofs of Theorems 10 and 11.

To simplify notation in what follows, u always denotes a real-valued function defined on an interval and U denotes the corresponding function from the same interval to the plane given by $U(x) = (x, u(x))$.

Lemma 11. *Fix a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(p) \ge \omega(0) = 0$. Let $S \subset \mathbb{R}^2$ be a compact set and $\Phi \in C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus S)$. Suppose that:*

11.α. *Φ is decreasing in x and increasing in y;*
11.β. $-\Phi_x \ge 4\Phi_y > 0$ *on* $\mathbb{R}^2 \setminus S$*;*
11.γ. $\Phi_y > 4\omega'(-2\Phi_x/\Phi_y)$ *on* $\mathbb{R}^2 \setminus S$*;*
11.δ. $\lim_{0<\mathrm{dist}((x,y),S)\to 0} \Phi_x/\Phi_y = -\infty$*;*
11.ε. *for all $a \le b$ and each non-decreasing $u \in \mathrm{AC}[a,b]$ the sets*

$$\{x : U(x) \in S\} \text{ and } \{\Phi(U(x)) : U(x) \in S\}$$

are Lebesgue null.

Then there is a Lagrangian L satisfying ($\star$) that has the following properties:

11.a. *If $u \in \mathrm{AC}[a,b]$ for some $a \le b \in \mathbb{R}$, then*

$$\int_a^b L(x,u,u')\,dx \ge \Phi(U(b)) - \Phi(U(a)).$$

11.b. *Equality holds in 11.a if and only if*

$$2\Phi_x(x,u(x)) + \Phi_y(x,u(x))u'(x) = 0 \text{ for almost every } x \in [a,b]. \tag{21}$$

11.c. *Every $u \in \mathrm{AC}[a,b]$ satisfying (21) is a minimizer for L on $[a,b]$.*

11.d. If through each $(x_0, y_0) \in S$ there passes a locally absolutely continuous solution $u\colon \mathbb{R} \to \mathbb{R}$ of (21)*, then S is contained in the universal singular set of L.*

Proof. Define auxiliary functions $\psi, \theta, \xi \in C^\infty(\mathbb{R}^2 \setminus S)$ by

$$\psi = -2\Phi_x/\Phi_y,\ \theta = \Phi_y - \omega'(\psi) \text{ and } \xi = (-\Phi_x + \omega(\psi) - \omega'(\psi)\psi)/\theta;$$

so that

$$\omega(\psi) + (p-\psi)\omega'(\psi) + (p-\xi)\theta = \Phi_x + p\Phi_y.$$

Note that $11.\beta$, $11.\gamma$ and the properties of ω guarantee that

$$\theta > 3\omega'(\psi) \geq 3\omega'(8) > 0 \text{ on } \mathbb{R}^2 \setminus S,$$

and so ξ is well-defined.

More precisely, we can use $11.\beta$ and $11.\gamma$ to find that

$$\Phi_y \geq \theta \geq (1 - \tfrac{1}{4})\Phi_y$$

and, since $\omega(p) - \omega'(p)p \leq \omega(0) = 0$ for $p \geq 0$,

$$\begin{aligned} -\Phi_x \geq -\Phi_x + \omega(\psi) - \omega'(\psi)\psi = \xi\theta &\geq -\Phi_x - \omega'(\psi)\psi \\ &= (\tfrac{1}{2}\Phi_y - \omega'(\psi))\psi \\ &\geq (\tfrac{1}{2} - \tfrac{1}{4})\psi\Phi_y = -\tfrac{1}{2}\Phi_x. \end{aligned}$$

Hence

$$\xi \geq -\frac{1}{2}\frac{\Phi_x}{\Phi_y} \geq 2 \geq 1 \text{ and } \psi \geq \tfrac{3}{2}\xi \geq \xi + \tfrac{1}{2}\xi \geq \xi + 1 \text{ on } \mathbb{R}^2 \setminus S \tag{22}$$

and so, by $11.\delta$,

$$\lim_{0<\operatorname{dist}((x,y),S)\to 0} \xi(x,y) = \infty. \tag{23}$$

Let γ be the function given by Lemma 10, and define $F\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x,y,p) = \begin{cases} \gamma(p, \xi(x,y), \theta(x,y)) & \text{for } (x,y) \in \mathbb{R}^2 \setminus S, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $F \in C^\infty(\mathbb{R}^3 \setminus (S \times \mathbb{R}))$. By (23), for each $p_0 \in \mathbb{R}$ there is an open set $\Omega \supset S$ so that $\xi \geq p_0 + 1$ on Ω; hence $F = 0$ on $\Omega \times (-\infty, p_0)$ and we see that $F \in C^\infty(\mathbb{R}^3)$. Defining $L(x,y,p) = \omega(p) + F(x,y,p)$, it is easy to check that L satisfies $(\star)$.

We first note a few basic properties of L. Since $\xi \geq 1$, $\omega(p) \geq \omega(0)$ and ω is strictly convex, it follows that $L(x,y,p) \geq L(x,y,0) = \omega(0)$ and the inequality is strict when $p \neq 0$. If $(x,y) \notin S$, we use 10.d, strict convexity of ω and the definitions of θ and ξ, to infer that

$$L(x,y,p) \geq \omega(\psi) + \omega'(\psi)(p-\psi) + \theta(p-\xi) = \Phi_x + p\Phi_y$$

with equality holding if and only if $p = \psi$.

We also note some simple consequences of the assumption 11.ε. If $u \in \mathrm{AC}[a,b]$ is non-decreasing, then $U(x) \notin S$ for almost every x, and so $\Phi \circ U$ is differentiable for almost every $x \in [a,b]$ and $(\Phi \circ U)' = \Phi_x + \Phi_y u'$. Combining this with the properties of L shown above, we see that for almost every x, $L(x,u,u') \geq (\Phi \circ U)'$ with equality holding if and only if $u' = \psi(x,u)$. We also note that $\Phi \circ U$ has the Lusin property — it maps null sets to null sets: subsets of $U^{-1}(S)$ are mapped to null sets because of 11.ε and null subsets of its complement are mapped to null sets because on this set $\Phi \circ U$ is locally absolutely continuous.

We now show that L satisfies 11.a–11.d.

Condition 11.α implies that for $u \in \mathrm{AC}[a,b]$, if $\Phi(U(a)) \leq \Phi(U(b))$, then $u(a) < u(b)$. Hence if such a u is also not non-decreasing, then there is a non-decreasing $v \in \mathrm{AC}[a,b]$ for which $v(a) = u(a)$, $v(b) = u(b)$ and, for almost every x, either $v'(x) = 0$, or $v(x) = u(x)$ and $v'(x) = u'(x)$. Since $\{x : 0 = v'(x) \neq u'(x)\}$ must have positive measure, we see that

$$\int_a^b L(x,u,u')\,dx > \int_a^b L(x,v,v')\,dx.$$

Since $L \geq 0$, it follows that, to prove 11.a and 11.b, we may restrict ourselves to non-decreasing $u \in \mathrm{AC}[a,b]$ satisfying $\Phi(U(a)) \leq \Phi(U(b))$. For such a u, let $G = (a,b) \setminus U^{-1}(S)$ and let $(a_j,b_j) \subset [a,b]$ be a sequence of those components of G for which $\Phi(U(a_j)) < \Phi(U(b_j))$. Since 11.$\varepsilon$ implies that $(\Phi \circ U)(U^{-1}(S))$ is a null set,

$$\begin{aligned}
\int_a^b L(x,u,u')\,dx &\geq \sum_j \int_{a_j}^{b_j} L(x,u,u')\,dx \\
&\geq \sum_j \int_{a_j}^{b_j} \max(0,(\Phi \circ U)')\,dx \\
&\geq \sum_j \Phi(U(b_j)) - \Phi(U(a_j)) \\
&\geq \Phi(U(b)) - \Phi(U(a)).
\end{aligned}$$

The first inequality is an equality only if $G = \bigcup_j (a_j,b_j)$ and the second only if $L(x,u,u') = (\Phi \circ U)'$, and we have already shown that this can happen only when $u' = \psi(x,u)$ almost everywhere.

Conversely, if $u'(x) = \psi(x,u)$ almost everywhere, then u is increasing and $(\Phi \circ U)' \geq 0$ almost everywhere. Together with the fact that $\Phi \circ U$ has the Lusin property, this implies that $\Phi \circ U$ is absolutely continuous, see [12, Chapter IX,§7.7]. Moreover, $L(x,u,u') = (\Phi \circ U)'$ almost everywhere, and so we find $\int_a^b L(x,u,u')\,dx = \Phi(U(b)) - \Phi(U(a))$.

Statements 11.c and 11.d follow directly from 11.a and 11.b.

Remark 2. A slightly more general version of this lemma is obtained by introducing a different function $\psi \in C^\infty(\mathbb{R}^2 \setminus S)$ satisfying $\psi \geq -\Phi_x/\Phi_y$, $\Phi_y > 4\omega'(\psi)$ and such that (22) and (23) hold. Then there is a Lagrangian satisfying $(\star)$ for which 11.a–11.d hold with (21) replaced by $u'(x) = \psi(x,u(x))$. This can be used to give examples with highly non-unique minimizers.

4.1. Proof of Theorem 10: constructing a singular set containing a given unrectifiable set

Recall that $\|\cdot\|_\infty$ denotes the sup-norm on $\mathbb{R}^2$ and $\|f\|$ denotes the sup-norm of f, provided f is bounded. In this section λ denotes Lebesgue measure on $\mathbb{R}$, and for $a, b \in \mathbb{R}^2$, $[a,b]$ denotes the closed line segment joining the points.

Lemma 12. *Let $S \subset \mathbb{R}^2$ be a compact purely unrectifiable set, $e \in \mathbb{R}^2$ and $\tau > 0$. Then there is $f \in C^\infty(\mathbb{R}^2)$ for which:*

- $0 \le f(x) \le \tau$ *for all* $x \in \mathbb{R}^2$*;*
- $\operatorname{dist}(\nabla f(x), [0,e]) < \tau$ *for all* $x \in \mathbb{R}^2$*;*
- $\sup_{x\in S} \|\nabla f(x) - e\|_\infty < \tau$.

Proof. The case $e = 0$ is trivial. We can assume without loss of generality that e is the unit vector in the positive direction of the x axis. Let $0 < \varepsilon < \tau$. It is enough to show that there is a Lipschitz function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that

- $0 \le g \le \varepsilon$ on $\mathbb{R}^2$;
- $g_x \in [0,1]$ and $g_y \in [-\varepsilon, \varepsilon]$ at almost every point of $\mathbb{R}^2$;
- $g_x = 1$ in a neighbourhood of S.

For then a suitable mollification of g gives f.

Let $C = 1/\varepsilon$, and let Ω be a nonempty open set containing S such that if $\gamma\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant at most C, then the length of $\{s : (s,\gamma(s)) \in \Omega\}$ is at most ε. (To see that such an open set exists, suppose instead that for each $n \in \mathbb{N}$, there is a Lipschitz function $\gamma_n\colon \mathbb{R} \to \mathbb{R}$ for which the length of $P_n = \{x \in \mathbb{R} : (x, \gamma_n(x)) \in B(S, \frac{1}{n})\}$ is at least ε. On choosing a pointwise convergent subsequence of (γ_n), converging to γ, say, we find that for each $s \in \limsup_n P_n$, $(s,\gamma(s)) \in K$, and $\lambda(\limsup_n P_n) > 0$, contradicting the unrectifiability of S, since γ is Lipschitz.)

Define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \sup\left\{x - b + \int_{(s,\gamma(s))\in\Omega} (1 - \tfrac{1}{C}|\gamma'(s)|)\, ds\right\},$$

where the supremum is taken over all $b \in \mathbb{R}$ and $\gamma\colon (-\infty, b] \to \mathbb{R}$, with $b \ge x$, $\operatorname{Lip}(\gamma) \le C$ and $\gamma(b) = y$.

The choice $b = x$ and $\gamma(s) = y$ for $s \le b$ shows that $g(x,y) \ge 0$, and it is also clear that $g(x,y) \le \sup_\gamma \lambda\{s : (s,\gamma(s)) \in \Omega\} \le \varepsilon$. The definition of g implies that for arbitrary (x,y) and $t > 0$,

$$g(x+t,y) \le g(x,y) + t,$$

with equality when the horizontal line segment joining (x,y) and $(x+t,y)$ lies in Ω.

If $(x,y) \in \mathbb{R}^2$, $b \ge x$, $t_1 \ge 0$ and $|t_2| \le Ct_1$, then the extension of the curve $\gamma\colon (-\infty, b] \to \mathbb{R}$ to the interval $[b, b+t_1]$ by a linear segment for which $\gamma(b+t_1) =$

$\gamma(b)+t_2$ shows that $g(x+t_1,y+t_2) \geq g(x,y)$. In particular, for every (x,y) and $t>0$, $g(x,y) \leq g(x+t,y)$ and

$$g(x,y\pm t) \in [g(x-\tfrac{1}{C}t,y), g(x+\tfrac{1}{C}t,y)] \subset [g(x,y)-\tfrac{1}{C}t, g(x,y)+\tfrac{1}{C}t].$$

That is, $g(x,y\pm t)-g(x,y) \in [-\varepsilon t, \varepsilon t]$, and so g has the required properties.

Lemma 13. *Let $S \subset \mathbb{R}^2$ be a compact purely unrectifiable set, Ω an open set that contains S, $h^0 \in C^\infty(\mathbb{R}^2)$, $e^0, e^1 \in \mathbb{R}^2$ and $\varepsilon > 0$. Then there is $h^1 \in C^\infty(\mathbb{R}^2)$ so that:*

1. $\|h^1 - h^0\| < \varepsilon$;
2. $h^1 = h^0$ *outside* Ω;
3. $\operatorname{dist}(\nabla h^1(x), [e^0,e^1]) < \varepsilon + \|\nabla h^0(x) - e^0\|_\infty$ *for* $x \in \mathbb{R}^2$; *and*
4. $\|\nabla h^1(x) - e^1\|_\infty < \varepsilon + \|\nabla h^0(x) - e^0\|_\infty$ *for* $x \in S$.

Proof. Choose $\delta > 0$ so that $B(S,2\delta) \subset \Omega$. Let $\tau \in (0, \varepsilon/(1+\delta^{-1}))$ and take f to be the function given by Lemma 12 for $e = e^1 - e^0$. Choose $g \in C^\infty(\mathbb{R}^2)$ so that

$$0 \leq g \leq 1,\ g = 1 \text{ on } S,\ g = 0 \text{ outside } \Omega \text{ and } \|\nabla g(x)\|_\infty \leq 1/\delta \text{ for } x \in \mathbb{R}^2.$$

Set $h^1 = h^0 + fg$. Clearly $h^1 = h^0$ outside Ω and

$$\sup_x \|h^1(x) - h^0(x)\|_\infty \leq \sup_x \|f(x)\|_\infty \leq \tau < \varepsilon.$$

Moreover for $x \in \mathbb{R}^2$,

$$\begin{aligned}
\operatorname{dist}(\nabla h^1(x), [e^0,e^1]) &= \operatorname{dist}((\nabla h^0(x) - e^0) + (g\nabla f)(x) + (f\nabla g)(x), [0,e]) \\
&\leq \|\nabla h^0(x) - e^0\|_\infty + \operatorname{dist}((g\nabla f)(x), [0,e]) + \|(f\nabla g)(x)\|_\infty \\
&< \|\nabla h^0(x) - e^0\|_\infty + \tau + \tau/\delta. \\
&< \|\nabla h^0(x) - e^0\|_\infty + \varepsilon.
\end{aligned}$$

If $x \in S$, we replace $[e^0,e^1]$ by e^1 and $[0,e]$ by e in these inequalities and use that $g = 1$ on S to obtain

$$\|\nabla h^1(x) - e^1\|_\infty < \|\nabla h^0(x) - e^0\|_\infty + \tau(1+1/\delta).$$

Proof of Theorem 10. For $k \geq 0$, we let

$$B_k = 4 + 4\omega'(5\cdot 2^{k+4}),\ A_k = 3\cdot 2^{k+2}B_k \text{ and } \eta_k = 1 - 2^{-k-1},$$

and use Lemma 13 recursively with vectors $e_k = (-A_k, B_k)$ to define functions $\Phi^k \in C^\infty(\mathbb{R}^2)$, open sets $\Omega_k \subset \mathbb{R}^2$ and numbers $\varepsilon_k > 0$ such that:

$$\Phi^0(x,y) = -A_0x + B_0y,\ \Omega_0 = \mathbb{R}^2 \text{ and } \varepsilon_0 = 1/4; \tag{24}$$

$$\|\nabla\Phi^k(x) - e_k\|_\infty < \eta_k \text{ for } x \in \overline{\Omega_k}; \tag{25}$$

if $a < b$, $u \in C[a,b]$ is non-decreasing and $\Phi \in C(\mathbb{R}^2)$ satisfies $\|\Phi - \Phi^k\| < 2\varepsilon_k$, then

$$\lambda\left(\{\Phi(x,u(x)) : (x,u(x)) \in \Omega_k\}\right) \leq 1/k; \tag{26}$$

and, for $k \geq 1$,

$$\|\Phi^k - \Phi^{k-1}\| < \varepsilon_{k-1}; \tag{27}$$

$$\Phi^k = \Phi^{k-1} \text{ outside } \Omega_{k-1}; \tag{28}$$

$$\operatorname{dist}(\nabla\Phi^k(x), [e_{k-1}, e_k]) < \eta_k \text{ for } x \in \overline{\Omega_{k-1}}; \tag{29}$$

$$S \subset \Omega_k,\ \overline{\Omega_k} \subset B(S, 2^{-k}) \cap \Omega_{k-1} \text{ and } \varepsilon_k < \varepsilon_{k-1}/2. \tag{30}$$

(We interpret $1/0$ as ∞ in (26).)

To see that this possible, we use (24) to define Φ^0, Ω_0 and ε_0, and observe that (25) and (26) trivially hold for $k = 0$. We also observe that $\nabla\Phi^0 = e_0$, which can be considered as the appropriate version of (29) for this case. For $k \geq 1$, define Φ^k as the function obtained by using Lemma 13 with $\Omega = \Omega_{k-1}$, $h^0 = \Phi^{k-1}$, $e^0 = e_{k-1}$, $e^1 = e_k$ and $\varepsilon = \varepsilon_{k-1}$. Then (27) and (28) are just properties of Φ^k given by the lemma.

Inequality (29) follows by induction, using the properties of Φ^{k-1} and (25) for $k-1$,

$$\begin{aligned}\operatorname{dist}(\nabla\Phi^k(x), [e_{k-1}, e_k]) &< \varepsilon_{k-1} + \|\nabla\Phi^{k-1}(x) - e_{k-1}\|_\infty \\ &< \varepsilon_{k-1} + \eta_{k-1} \leq \eta_k \text{ for } x \in \overline{\Omega_{k-1}}.\end{aligned}$$

For $x \in S$, we have

$$\begin{aligned}\|\nabla\Phi^k(x) - e_k\| &< \varepsilon_{k-1} + \|\nabla\Phi^{k-1}(x) - e_{k-1}\| \\ &< \varepsilon_{k-1} + \eta_{k-1} \leq \eta_k;\end{aligned}$$

by continuity these inequalities hold for some open set $\Omega \supset S$ and so the only requirement on Ω_k to ensure the validity of (25) is to choose Ω_k so that $\overline{\Omega_k} \subset \Omega$.

Since S is purely unrectifiable, we may find $\delta > 0$ sufficiently small so that both

$$B(S, 3\delta) \subset \Omega \cap B(S, 2^{-k}) \cap \Omega_{k-1}$$

and for each non-decreasing $u \in C(\mathbb{R})$, the linear measure of $U(\mathbb{R}) \cap B(S, 3\delta)$ does not exceed $1/(k(A_k + B_k + 6))$. In particular, for $a < b$ and any non-decreasing $u \in C[a, b]$, there are

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_q < b_q \leq b$$

so that the length of each $U([a_j, b_j])$ is δ and,

$$U([a,b]) \cap B(S,\delta) \subset \bigcup_{j=1}^{q} U([a_j, b_j]) \subset U([a,b]) \cap B(S, 3\delta).$$

The second inclusion implies $q\delta \leq ((A_k + B_k + 6)k)^{-1}$. In particular, if we set $\Omega_k = B(S, \delta)$, then for non-decreasing $u \in C(\mathbb{R})$,

$$\lambda\left(\{(x, u(x)) : x \in \mathbb{R}\} \cap \Omega_k\right) \leq \frac{1}{k(A_k + B_k + 6)}. \tag{31}$$

Let $\varepsilon_k = \min(\frac{1}{2}\varepsilon_{k-1}, \delta)$. If Φ is as in (26), we use the fact that the estimates from (25) are valid on $B(S, 3\delta)$ to estimate

$$\begin{aligned}\lambda\left(\{\Phi(x,u(x)) : (x,u(x)) \in \Omega_k\}\right) &= \lambda\left(\Phi(U([a,b]) \cap B(S,\delta))\right)\\ &\le \sum_{j=1}^{q} \lambda(\Phi(U(a_j,b_j)))\\ &\le q(4\varepsilon_k + (A_k + B_k + 2)\delta)\\ &\le q(A_k + B_k + 6)\delta \le 1/k.\end{aligned}$$

Hence (26) and (27)–(30) hold.

By (27), the sequence Φ^k converges uniformly to some $\Phi \in C(\mathbb{R}^2)$. We now show that Φ satisfies the hypotheses of Lemma 11. From (30) and (28), we infer that $\Phi \in C^\infty(\mathbb{R}^2 \setminus S)$. Suppose $(x,y) \in \mathbb{R}^2 \setminus S$, then there is $k \in \mathbb{N}$ such that $(x,y) \in \overline{\Omega_{k-1}} \setminus \overline{\Omega_k}$. By (28),

$$\Phi_x(x,y) = \Phi_x^k(x,y) \text{ and } \Phi_y(x,y) = \Phi_y^k(x,y).$$

Hence (29) implies that

$$\Phi_y(x,y) \ge B_{k-1} - 1 \ge B_0 - 1 \ge 3 \text{ and } \Phi_x(x,y) \le -A_{k-1} + 1 \le -47;$$

in particular $11.\alpha$ and the second inequality of $11.\beta$, namely $\Phi_y > 0$, hold. By (29) there is $0 \le s \le 1$ so that

$$|\nabla\Phi(x,y) - (se_{k-1} + (1-s)e_k)| < 1.$$

Hence

$$\begin{aligned}-\Phi_x(x,y) &\le sA_{k-1} + (1-s)A_k + 1 \le 3 \cdot 2^{k+2}(sB_{k-1} + (1-s)B_k) + 1\\ &\le 3 \cdot 2^{k+2}(\Phi_y(x,y) + 1) + 1 \le 5 \cdot 2^{k+2}\Phi_y(x,y)\end{aligned}$$

and so $-2\Phi_x(x,y)/\Phi_y(x,y) \le 5 \cdot 2^{k+3}$ on $\overline{\Omega_{k-1}} \setminus \overline{\Omega_k}$. Similarly,

$$\begin{aligned}-\Phi_x(x,y) &\ge sA_{k-1} + (1-s)A_k - 1 \ge 3 \cdot 2^{k+1}(sB_{k-1} + (1-s)B_k) - 1\\ &\ge 3 \cdot 2^{k+1}(\Phi_y(x,y) - 1) - 1 \ge 2^{k+1}\Phi_y(x,y).\end{aligned}$$

This gives $11.\delta$ and the first inequality of $11.\beta$. It also shows that

$$\Phi_y(x,y) \ge B_{k-1} - 1 > 4\omega'(5 \cdot 2^{k+3}) \ge 4\omega'(-2\Phi_x/\Phi_y),$$

and so $11.\gamma$ holds.

It remains to verify $11.\varepsilon$. So suppose $a < b$ and let $u \in \mathrm{AC}[a,b]$ be non-decreasing. The set $\{x : (x,u(x)) \in S\}$ is null since S is purely unrectifiable. Moreover, (27) and $\varepsilon_k \le \varepsilon_{k-1}/2$ imply that for each $k \in \mathbb{N}$, $|\Phi - \Phi^k| < 2\varepsilon_k$, hence (26) implies that the measure of $\{\Phi(x,u(x)) : (x,u(x)) \in S\}$ is no more than $1/k$ for all $k \in \mathbb{N}$. Hence $\{\Phi(x,u(x)) : (x,u(x)) \in S\}$ is also Lebesgue null and we deduce that $11.\varepsilon$ holds. Hence, by Lemma 11, there is a Lagrangian L satisfying $(\star)$ and 11.a–11.d.

In order to show that S is contained within the universal singular set of L, it only remains to show that the assumption of 11.d holds. Let $\psi_k = -2\Phi_x^k/\Phi_y^k$ and $\psi = -2\Phi_x/\Phi_y$. Let $(x_0, y_0) \in S$ be given. Since ψ_k is Lipschitz for each k, we can find $u_k \in C^1(\mathbb{R})$ so that $u_k(x_0) = y_0$ and $u_k'(x) = \psi_k(x, u_k)$. We show that $\{u_k\}$ is an equicontinuous family. For given $\tau > 0$ choose $q \in \mathbb{N}$ so that $q > 2/\tau$.

Since u_k is non-decreasing, (31) implies

$$\lambda\left(\{(x, u_k(x)) : (x, u_k(x)) \in \Omega_q\}\right) < 1/q < \tau/2.$$

Let $\sigma = \frac{1}{5}2^{-(q+4)}\tau$ and consider any $0 < t - s < \sigma$. If

$$\{(x, u_k(x)) : s < x < t\} \cap \Omega_q = \emptyset,$$

then, since $0 < \psi_k \leq 5 \cdot 2^{q+3}$ on $\mathbb{R}^2 \setminus \Omega_q$,

$$0 \leq u_k(t) - u_k(s) \leq 5 \cdot 2^{q+3}(t - s).$$

Hence, in the general case, $0 \leq u_k(t) - u_k(s) < 5 \cdot 2^{q+3}(t - s) + 1/q < \tau$. We infer that a subsequence of u_k converges locally uniformly to a non-decreasing continuous function u that satisfies $u(x_0) = y_0$ and for which $u' = \psi(x, u)$ whenever $(x, u(x)) \notin S$. Since S is purely unrectifiable, this implies that u is locally absolutely continuous, and so the hypothesis of 11.d holds. Thus S is contained in the universal singular set of L.

4.2. Proof of Theorem 11: a singular set meeting a rectifiable curve in positive length

Let $A_k = 4^{k+5}$, $B_k = 2^{k+4}$ and recursively choose C_k to be very large; the particular inequalities we need will follow, for example, by setting $C_0 = 0$ and picking

$$C_k > 8\left(\omega'(6A_{k+2}) + 1 + \sum_{j=0}^{k-1} C_j(1 + A_j) + (1 + \sum_{j=0}^{k-1} C_j)A_k\right). \tag{32}$$

Define sets $T_k \subset \mathbb{R}$ and positive constants ℓ_k and ε_k recursively as follows. Let

$$T_0 = [0, 1],\ \ell_0 = \infty \text{ and } \varepsilon_0 = \varepsilon_{-1} = 1.$$

For $k \geq 1$, in the k-th step write T_{k-1} as a finite union of non-overlapping closed intervals J each of length less than ε_{k-1}/A_k and define $T_k \subset T_{k-1}$ so that for each of these intervals, $T_k \cap J$ is a closed interval concentric with J of length $\lambda(J)(A_{k-1} - B_k)/A_k$. Then let

$$\ell_k = \tfrac{1}{2}\min(\mathrm{dist}(T_k, \mathbb{R} \setminus T_{k-1}), \varepsilon_{k-1}) \text{ and } \varepsilon_k = 2^{-k-4}\ell_k/C_{k+1}.$$

Define $\chi_k \in C^\infty(\mathbb{R})$ recursively by setting $\chi_0(x) = A_0 x$, and for $k \geq 1$ by defining

$$\chi_k = \chi_{k-1} \text{ outside } T_{k-1} \text{ and at the endpoints of the intervals } J,$$

and by requiring

$$\chi_k' = A_k \text{ on } T_k \text{ and } B_k \le \chi_k' \le A_k \text{ on } T_{k-1};$$

the existence of χ_k is guaranteed by $0 < B_k < A_{k-1} < A_k$. Notice that $\|\chi_k - \chi_{k-1}\| < \varepsilon_{k-1}$.

Let $S_k = \{(x, \chi_k(x)) : x \in T_k\}$, the graph of χ_k over T_k, and let $\Omega_0 = \mathbb{R}^2$, and $\Omega_k = B(S_k, \ell_k)$, an open neighbourhood around S_k.

Choose functions $\beta_k \in C^\infty(\mathbb{R})$ so that $\beta_0 \equiv 0$ and for $k \ge 1$,

$$\beta_k(0) = 0, \beta_k' \in [0, C_k] \text{ everywhere and } \beta_k' = \begin{cases} C_k & \text{on } (-\varepsilon_{k-1}, \varepsilon_{k-1}), \\ 0 & \text{outside } (-2\varepsilon_{k-1}, 2\varepsilon_{k-1}). \end{cases}$$

Choose functions $\alpha^k \in C^\infty(\mathbb{R}^2)$ so that $\alpha^0 \equiv \alpha^1 \equiv 1$ and for $k \ge 2$

$$0 \le \alpha^k \le 1,\ \|\nabla \alpha^k\| \le 2/\ell_{k-1} \text{ and } \alpha^k = \begin{cases} 1 & \text{on } \Omega_k, \\ 0 & \text{outside } B(\Omega_k, \ell_{k-1}). \end{cases}$$

Define $\zeta_0(x) = \chi_0(x) = A_0 x$, $\Phi^0(x,y) = 0$, and for $k \ge 1$:

$$\zeta_k(x) = \chi_k(x) + \Phi^{k-1}(x, \chi_k(x))/C_k$$

and

$$\Phi^k(x,y) = \textstyle\sum_{j=0}^k \alpha^j(x,y)\beta_j(y - \zeta_j(x)).$$

An easy calculation shows that

$$\Phi_x^k(x,y) = \sum_{j=0}^k \alpha_x^j(x,y)\beta_j(y - \zeta_j(x)) - \sum_{j=0}^k \alpha^j(x,y)\beta_j'(y - \zeta_j(x))\zeta_j'(x)$$

and

$$\Phi_y^k(x,y) = \sum_{j=0}^k \alpha_y^j(x,y)\beta_j(y - \zeta_j(x)) + \sum_{j=0}^k \alpha^j(x,y)\beta_j'(y - \zeta_j(x)).$$

Hence, since $\|\nabla\alpha^0\| = \|\nabla\alpha^1\| = 0$, $\|\nabla\alpha^j\| \le 2/\ell_{j-1}$ for $j \ge 2$, and $|\beta_j| \le 2C_j\varepsilon_{j-1}$, we estimate $\sum_{j=0}^\infty \|\nabla\alpha^j\||\beta_j| \le \sum_{j=2}^\infty 4C_j\varepsilon_{j-1}/\ell_{j-1} \le 1$, and so

$$|\Phi_x^k(x,y) + \sum_{j=0}^k \alpha^j(x,y)\beta_j'(y - \zeta_j(x))\zeta_j'(x)| \le 1, \text{ and} \tag{33}$$

$$|\Phi_y^k(x,y) - \sum_{j=0}^k \alpha^j(x,y)\beta_j'(y - \zeta_j(x))| \le 1. \tag{34}$$

This implies

$$\|\zeta_k' - \chi_k'\| \le 1; \tag{35}$$

indeed, (35) certainly holds for $k=0$ and hence for $k \geq 1$, we use (33), (34), induction and (32) to find that for $x \in \mathbb{R}$,

$$\begin{aligned}
|\zeta_k'(x)-\chi_k'(x)| &= |(\Phi^{k-1}(x,\chi_k(x)))'|/C_k \\
&\leq (|\Phi_x^{k-1}(x,\chi_k(x))|+|\Phi_y^{k-1}(x,\chi_k(x))||\chi_k'(x)|)/C_k \\
&\leq (1+\textstyle\sum_{j=0}^{k-1}\|\alpha^j\|\|\beta_j'\|\|1+\chi_j'\|+(1+\sum_{j=0}^{k-1}\|\alpha^j\|\|\beta_j'\|)A_k)/C_k \\
&\leq (1+\textstyle\sum_{j=0}^{k-1}C_j(1+A_j)+(1+\sum_{j=0}^{k-1}C_j)A_k)/C_k \leq 1.
\end{aligned}$$

We also observe that

$$|\zeta_k(x)-\chi_k(x)| \leq \tfrac{1}{2}\varepsilon_{k-1} \text{ for } x \in T_{k-1}, \text{ and} \tag{36}$$

$$\Phi^k(x,\chi_k(x)) = 0 \text{ for } x \in T_k; \tag{37}$$

these inequalities certainly hold for $k=0$. If (37) holds for $k-1$, then for $x \in T_{k-1}$, by estimating Φ_y^{k-1} from (34) and using (32), we find

$$\begin{aligned}
|\zeta_k(x)-\chi_k(x)| &= |\Phi^{k-1}(x,\chi_k(x))-\Phi^{k-1}(x,\chi_{k-1}(x))|/C_k \\
&\leq \left(1+\sum_{j=0}^{k-1}C_j\right)|\chi_k(x)-\chi_{k-1}(x)|/C_k \\
&\leq \tfrac{1}{8}C_k\varepsilon_{k-1}/C_k \leq \tfrac{1}{2}\varepsilon_{k-1},
\end{aligned}$$

which is (36). Hence, since for $x \in T_k$, $\alpha^k(x,\chi_k(x)) = 1$ and $\beta_k(\chi_k(x)-\zeta_k(x)) = C_k(\chi_k(x)-\zeta_k(x))$, we have

$$\begin{aligned}
\Phi^k(x,\chi_k(x)) &= \Phi^{k-1}(x,\chi_k(x))+\alpha^k(x,\chi_k(x))\beta_k(\chi_k(x)-\zeta_k(x)) \\
&= C_k(\zeta_k(x)-\chi_k(x))+C_k(\chi_k(x)-\zeta_k(x)) = 0,
\end{aligned}$$

which is (37).

Next we note that, since $\varepsilon_k+\ell_{k+1} \leq \ell_k$ and $\varepsilon_{k-1}+\varepsilon_k+\ell_{k+1}+\ell_k \leq \ell_{k-1}$,

$$\Omega_{k+1} \subset \Omega_k \text{ and } B(\Omega_{k+1},\ell_k) \subset \Omega_{k-1}. \tag{38}$$

and, since $\ell_k+\frac{1}{2}\varepsilon_{k-1} \leq \varepsilon_{k-1}$, that

$$|y-\zeta_k(x)| \leq \varepsilon_{k-1} \text{ for } (x,y) \in \Omega_k. \tag{39}$$

To deal with $(x,y) \in \Omega_q \setminus \Omega_{q+1}$ we need finer estimates of the partial derivatives of Φ^k. Let $r := \min(q,k)$ and $s := \min(q+2,k)$ and observe that (38) implies $\alpha^j(x,y) = 0$ for $j > q+2$, so the sums in equations (33) and (34) finish with $j = s$. Also,

$$\left|\sum_{j=0}^{r-1}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))\right| \leq \sum_{j=0}^{r-1}\|\beta_j'\| \leq \sum_{j=0}^{r-1}C_j < -1+\tfrac{1}{8}C_r$$

and, using (35),

$$\left|\sum_{j=0}^{r-1}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))\zeta_j'(x)\right| \le \sum_{j=0}^{r-1}\|\beta_j'\|(1+\|\chi_j'\|)$$
$$\le \sum_{j=0}^{r-1}(1+A_j)C_j < -1+\tfrac{1}{8}C_r.$$

Hence

$$|\Phi_x^k(x,y)+\textstyle\sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))\zeta_j'(x)| \le \tfrac{1}{8}C_r, \text{ and} \tag{40}$$
$$|\Phi_y^k(x,y)-\textstyle\sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))| \le \tfrac{1}{8}C_r. \tag{41}$$

Further, using that $\alpha^r(x,y)=1$ and deducing from (39) that $\beta_j'(y-\zeta_j(x))=C_j$ and from (35) that $\zeta_j'(x)\ge 0$ for all j and $B_r/2\le\zeta_j'(x)\le 2A_{r+2}$ for $r\le j\le s$, we get from (40) and (41) that

$$7A_{r+2}C_{r+2} \ge -\Phi_x^k(x,y) \ge \tfrac{1}{4}B_rC_r \quad \text{and} \quad 4C_{r+2} \ge \Phi_y^k(x,y) \ge \tfrac{7}{8}C_r. \tag{42}$$

Using (41) and (42) we see that

$$\Phi_y^k(x,y) \ge 7|\Phi_y^k(x,y)-\textstyle\sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))|,$$

and so

$$\tfrac{6}{7}\Phi_y^k(x,y) \le \sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x)) \le \tfrac{8}{7}\Phi_y^k(x,y).$$

Thus

$$\tfrac{3}{7}B_r\Phi_y^k(x,y) \le \sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))\zeta_j'(x) \le \tfrac{16}{7}A_{r+2}\Phi_y^k(x,y).$$

However (40) and (42) imply that

$$|-\Phi_x^k(x,y)-\sum_{j=r}^{s}\alpha^j(x,y)\beta_j'(y-\zeta_j(x))\zeta_j'(x)| \le \tfrac{1}{8}C_r \le \tfrac{1}{7}\Phi_y^k(x,y),$$

and so

$$\tfrac{1}{7}\left(-1+3B_r\right)\Phi_y^k(x,y) \le -\Phi_x^k(x,y) \le \tfrac{1}{7}\left(1+16A_{r+2}\right)\Phi_y^k(x,y),$$

giving

$$\tfrac{2}{7}B_r\Phi_y^k(x,y) \le -\Phi_x^k(x,y) \le 3A_{r+2}\Phi_y^k(x,y). \tag{43}$$

We are now in a position to use Lemma 11. Since

$$\sum_{k=1}^{\infty}\|\alpha^k\|\|\beta_k\| \le 2\sum_{k=1}^{\infty}C_k\varepsilon_{k-1} < \infty,$$

the function

$$\Phi(x,y) := \lim_{k\to\infty}\Phi^k(x,y) = \sum_{k=0}^{\infty}\alpha^k(x)\beta_k(y-\zeta_k(x))$$

is continuous on $\mathbb{R}^2$. Since $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, $\chi := \lim_{k\to\infty} \chi_k$ is also a continuous function on $\mathbb{R}$. Moreover χ is uniformly continuous on $\mathbb{R}$, since $\chi(x) - A_0(x)$ is zero outside $[0,1]$, and χ is increasing, since for all $k \geq 1$ and $y > x$

$$\chi_k(y) - \chi_k(x) \geq \min(A_0, B_1)(y-x) = B_1(y-x).$$

Let $T = \bigcap_{k=0}^{\infty} T_k$ and $S = \{(x, \chi(x)) : x \in T\}$. Then $\lambda(T) = 0$, since $\lambda(T_k) = (A_{k-1} - B_k)\lambda(T_{k-1})/A_k$ and $\prod_k (A_{k-1} - B_k)/A_k = 0$. Similarly, we observe that

$$\lambda(\chi_k(T_k)) = A_k \lambda(T_k) = (A_{k-1} - B_k)\lambda(T_{k-1}) = (A_{k-1} - B_k)\lambda(\chi_{k-1}(T))/A_{k-1},$$

$\prod_k (A_{k-1} - B_k)/A_{k-1} > 0$, and, since $\chi_{k+1}(T_{k+1}) \subset \chi_k(T_k)$, $\chi(T) = \cap_{k=1}^{\infty} \chi_k(T_k)$. Hence $\lambda(\chi(T)) > 0$, and it follows that S is a rectifiable set of positive linear measure. Note also that (38) implies that $S = \bigcap_{k=0}^{\infty} \Omega_k$. In particular, the sum defining Φ is locally finite on $\mathbb{R}^2 \setminus S$. This means that $\Phi \in C^{\infty}(\mathbb{R}^2 \setminus S)$ and that hypotheses $11.\alpha$, $11.\beta$, $11.\gamma$ and $11.\delta$ of Lemma 11 follow immediately from (32), (42) and (43). The condition $11.\varepsilon$ holds because for any $u\colon \mathbb{R} \to \mathbb{R}$, $\{x : (x, u(x)) \in S\} \subset T$ is a null set and $\{\Phi(x, u(x)) : (x, u(x)) \in S\} \subset \{0\}$, since (37) implies that $\Phi(x,y) = 0$ on S.

Finally, to verify that the assumptions of 11.d hold, for $k \geq 1$ let

$$\psi_k = -2\Phi_x^k/\Phi_y^k \text{ and } \psi = -2\Phi_x/\Phi_y.$$

Fix $(x_0, y_0) \in S$.

Since ψ_k is Lipschitz for each k, we can find $u_k \in C^1(\mathbb{R})$ so that $u_k(x_0) = y_0$ and $u_k'(x) = \psi_k(x, u_k)$. We show that $\{u_k\}$ is an equicontinuous family. For given $\tau > 0$ choose q so that

$$2\ell_q + 2\sum_{j\geq q} \varepsilon_j < \tau/6$$

and use the uniform continuity of χ to choose $\sigma \leq \frac{1}{18}\tau/A_{q+2}$ so that for $0 \leq t - s < \sigma$,

$$0 \leq \chi(t) - \chi(s) < \tau/6.$$

Consider $0 < t - s < \sigma$. If

$$\{(x, u_k(x)) : s < x < t\} \cap \Omega_q = \emptyset,$$

we have, since ψ_k is at most $6A_{q+2}$,

$$0 \leq u_k(t) - u_k(s) \leq 6A_{q+2}(t-s) < \tau/3.$$

If $(s, u_k(s)) \in \overline{\Omega_q}$ and $(t, u_k(t)) \in \overline{\Omega_q}$, we have

$$0 \leq u_k(t) - u_k(s) \leq \chi_q(t) - \chi_q(s) + 2\ell_q \leq \chi(t) - \chi(s) + 2\ell_q + 2\sum_{j\geq q} \varepsilon_j < \tau/3.$$

In the general case, the interval $[t,s]$ can be written as the union of three non-overlapping intervals each of which satisfies one of these conditions, so $0 \leq u_k(t) - u_k(s) < \tau$. We infer that a subsequence of u_k converges locally uniformly to a non-decreasing continuous function u that satisfies $u(x_0) = y_0$ and for which $u' = \psi(x,u)$ whenever $(x,u(x)) \notin S$. Note that

$$\begin{aligned}\frac{d}{dx}\Phi^k(x,u_k) &= \Phi^k_x(x,u_k(x)) + \Phi^k_y(x,u_k(x))u_k'(x)\\ &= \Phi^k_x(x,u_k) + \Phi^k_y(x,u_k) \times \frac{-2\Phi^k_x(x,u_k)}{\Phi^k_y(x,u_k)}\\ &= -\Phi^k_x(x,u_k(x)) \geq 1.\end{aligned}$$

Hence $\Phi^k(t,u_k(t)) - \Phi^k(s,u_k(s)) \geq t-s$, and so $\Phi(t,u(t)) - \Phi(s,u(s)) \geq t-s$ for $t > s$. Since $\Phi(x,y) = 0$ on S, the graph of u meets S only at the point (x_0,y_0). Hence $u' = \psi(x,u)$ for $x \in \mathbb{R} \setminus \{x_0\}$ and the monotonicity of u implies that it is locally absolutely continuous. Lemma 11 now guarantees the existence of a Lagrangian with the required properties.

4.3. Proof of Theorem 12: a continuous Lagrangian with residual universal singular set

This Lagrangian is constructed by a variant of the general construction of Lagrangians with large universal singular sets that we described earlier; the technicalities are simpler in one sense, since we do not have to care about the smoothness of the Lagrangian, however complications are caused by the fact that we have to work with derivatives of non-smooth functions.

Lemma 14. *Suppose that $z_0 \in \mathbb{R}^2$, $\varepsilon > 0$ and $0 \neq P \in \mathbb{R}^2$. Then there are an open set $G \subset \mathbb{R}^2$, a Lipschitz function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ and a continuous function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that*

14.1. $z_0 \in \overline{G}$, $\operatorname{diam}(G) < \varepsilon$ and ∂G is Lebesgue null;
14.2. $|\Phi| \leq \varepsilon$ on $\mathbb{R}^2$, $\|\varphi\| \leq \varepsilon$, $\nabla\Phi = \varphi$ on $\mathbb{R}^2 \setminus \overline{G}$ and $\nabla\Phi = P$ on G;
14.3. if $\|z - z_0\| \geq \varepsilon$, then $|\Phi(w) - \Phi(z)| \leq \varepsilon\|w - z\|$ for all $w \in \mathbb{R}^2$;
14.4. if $\gamma\colon [a,b] \to \mathbb{R}^2$ is absolutely continuous, then
  (a) $\frac{d}{dx}\Phi(\gamma(x)) = P \cdot \gamma'(x)$ for a.e. x for which $\gamma(x) \in \overline{G}$, and
  (b) $\frac{d}{dx}\Phi(\gamma(x)) = \varphi(\gamma(x)) \cdot \gamma'(x)$ for a.e. x for which $\gamma(x) \notin G$.

Proof. Since the statement does not depend on the choice of origin and coordinates, we may assume that $z_0 = (0,0)$ and $P = (0,c)$, where $c > 0$.

Let $0 < \delta < 1$ be small enough; the exact conditions are determined below. Choose $f \in C^1(\mathbb{R})$ so that

$$f(x) = 0 \text{ for } |x| \geq \delta,\ 0 < f(x) < \delta \text{ for } x \in (-\delta,\delta) \text{ and } |f'(x)| < \delta \text{ for all } x.$$

Let $G = \{(x,y) : 0 < y < f(x)\}$, and define

$$\Phi(x,y) = \begin{cases} 0 & \text{if } y \leq 0,\\ cy & \text{if } 0 < y \leq f(x),\\ cf(x) & \text{if } y > f(x),\end{cases}$$

and, for $(x,y) \in \mathbb{R}^2 \setminus G$, let

$$\varphi(x,y) = \begin{cases} (0,0) & \text{if } y \leq 0, \\ (cf'(x),0) & \text{if } y \geq f(x). \end{cases}$$

Then 14.1 holds, provided $3\delta < \varepsilon$. To ensure $|\Phi| \leq \varepsilon$ on $\mathbb{R}^2$, we require $\delta c \leq \varepsilon$. The same assumption on δ also guarantees $\|\varphi\| \leq \varepsilon$ on $\mathbb{R}^2 \setminus G$. Since φ is (well-defined and) continuous on $\mathbb{R}^2 \setminus G$, we can then extend it to a continuous function on $\mathbb{R}^2$ with $\|\varphi\| \leq \varepsilon$. The remainder of 14.2 is obvious.

If $[z,w]$ does not meet $\overline{G}$, 14.3 holds, since $\|\nabla\Phi(x)\| \leq \varepsilon$ for every $x \in [z,w]$. If $[z,w]$ meets $\overline{G}$, then the intersection has length at most 3δ and $\|z-w\| \geq \varepsilon - 2\delta$. Since $\|\nabla\Phi(x)\| \leq c$ for $x \in [z,w] \cap \overline{G}$ and $\|\nabla\Phi(x)\| \leq \delta c$ for $x \notin [z,w] \cap \overline{G}$, we use that $\|w-z\| \geq \|z-z_0\| - \operatorname{diam}(G) \geq \varepsilon - 2\delta$ to estimate that

$$|\Phi(w) - \Phi(z)| \leq 3\delta c + \delta c\|w-z\| \leq \varepsilon\|w-z\|,$$

provided that $3\delta c \leq \frac{1}{2}\varepsilon(\varepsilon - 2\delta)$ and $\delta c \leq \frac{1}{2}\varepsilon$.

Finally, if $\gamma = (\gamma_1, \gamma_2)$ is as in 14.4, then, since $\Phi(\gamma(x)) = P \cdot \gamma(x)$ whenever $\gamma(x) \in \overline{G}$, we conclude that $\frac{d}{dx}\Phi(\gamma(x)) = P \cdot \gamma'(x)$ whenever $\gamma(x) \in \overline{G}$, x is not an isolated point of $\gamma^{-1}(\overline{G})$, and γ and $\Phi \circ \gamma$ are differentiable at x. Similarly, we infer from $\Phi(\gamma(x)) = cf(\gamma_1(x))$ when $\gamma_2(x) \geq f(\gamma_1(x))$, that $\frac{d}{dx}\Phi(\gamma(x)) = cf'(\gamma_1(x))\gamma_1'(x) = \varphi(\gamma(x)) \cdot \gamma'(x)$ for almost all x for which $\gamma_2(x) \geq f(\gamma_1(x))$, and, from $\Phi(\gamma(x)) = 0$ when $\gamma_2(x) \leq 0$, that $\frac{d}{dx}\Phi(\gamma(x)) = 0 = \varphi(\gamma(x)) \cdot \gamma'(x)$ for almost all x for which $\gamma_2(x) \leq 0$.

In the following, $\mathscr{H}^1_\infty(A)$ denotes the one-dimensional Hausdorff capacity of a set $A \subset \mathbb{R}^2$; that is,

$$\mathscr{H}^1_\infty(A) = \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j\right\}.$$

Recall that for $u\colon A \subset \mathbb{R} \to \mathbb{R}$, U denotes the function $x \mapsto (x, u(x))$.

Lemma 15. *Let $\emptyset \neq H \subset \mathbb{R}^2$ be open, $\varepsilon > 0$ and $P = (-A, B) \in \mathbb{R}^2$ with $A, B > 0$. Then there is an open set $G \subset H$ that is dense in H, a Lipschitz function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$, and a continuous function $L\colon \mathbb{R}^3 \to \mathbb{R}$ for which*

15.1. $\mathscr{H}^1_\infty(G) \leq \varepsilon$;
15.2. $|\Phi| \leq \varepsilon$ on $\mathbb{R}^2$;
15.3. $|L(x,y,p) - \max(0, -A + Bp)| \leq \varepsilon(1 + |p|)$;
15.4. $\int_a^b L(x,u,u')\,dx \geq \Phi(U(b)) - \Phi(U(a))$ for every $u \in \mathrm{AC}[a,b]$;
15.5. if $u \in \mathrm{AC}[a,b]$ satisfies $u'(x) \geq A/B + 1$ for a.e. x for which $U(x) \in G$, and there is an absolutely equicontinuous sequence $u_j \in \mathrm{AC}[a,b]$ converging to u such that $|u_j'(x)| \leq A/B - 1$ for a.e. x for which $U_j(x) \notin G$, then $\int_a^b L(x,u,u')\,dx = \Phi(U(b)) - \Phi(U(a))$.

Proof. We may assume that $2\varepsilon(2+A/B) \le \min(A,B)$. Let $z_1, z_2, \ldots \in H$ be a sequence that is dense in H.

We use Lemma 14 recursively to define open sets $G^k \subset \mathbb{R}^2$ that have pairwise disjoint closures, Lipschitz functions $\Phi^k \colon \mathbb{R}^2 \to \mathbb{R}$ and continuous functions $\varphi^k \colon \mathbb{R}^2 \to \mathbb{R}^2$. In addition, we also define real numbers $\varepsilon_k > 0$ and continuous functions $\zeta^k \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\sum_{j=1}^{k} \nabla\Phi^j = \sum_{j=1}^{k} \zeta^j \text{ on } \bigcup_{j=1}^{k} G^j \text{ for each } k, \tag{44}$$

and

$$\|\zeta^k(x)\| \le \varepsilon_k \text{ for } x \in \mathbb{R}^2 \text{ and } k \ge 2.$$

To start the recursion, we choose $0 < \varepsilon_1 < \frac{1}{2}\varepsilon$ so that $\{z : \|z - z_1\| \le \varepsilon_1\} \subset H$, let $P_1 = (-A_1, B_1) = (-A, B)$, let $\zeta^1(z) = (-A, B)$, and use Lemma 14 with $z_0 = z_1$, $\varepsilon = \varepsilon_1$ and $P = P_1$ to define G^1, Φ^1 and φ^1.

Assume now that $k \ge 2$ and that all objects are defined for $1 \le j < k$. Let i_k be the first index for which $z_{i_k} \notin \bigcup_{j=1}^{k-1} \overline{G^j}$, and choose $0 < \varepsilon_k < 2^{-k}\varepsilon$ so that

$$\{z : \|z - z_{i_k}\| \le \varepsilon_k\} \subset H \setminus \bigcup_{j<k} \overline{G^j}.$$

Let

$$P_k = (-A_k, B_k) = \sum_{j=1}^{k-1} (\zeta^j(z_{i_k}) - \nabla\Phi^j(z_{i_k})).$$

(Notice that $\|P_k - P\| \le 2\varepsilon$, and so $A_k, B_k > 0$.) Since $z \mapsto \sum_{j=1}^{k-1}(\zeta^j(z) - \nabla\Phi^j(z))$ is continuous at z_{i_k}, there is $0 < \varepsilon_k' \le \varepsilon_k$ so that

$$\left\| P_k - \sum_{j=1}^{k-1} (\zeta^j(z) - \nabla\Phi^j(z)) \right\| \le \varepsilon_k \text{ on } B(z_{i_k}, \varepsilon_k').$$

Now use Lemma 14 with $z_0 = z_{i_k}$, $\varepsilon = \varepsilon_k'$ and $P = P_k$ to define G^k, Φ^k and φ^k. Let

$$\zeta^k(z) = \begin{cases} \nabla\Phi^k(z) & \text{on } \bigcup_{j=1}^{k-1} \overline{G^j}, \text{ and} \\ P_k - \sum_{j=1}^{k-1}(\zeta^j(z) - \nabla\Phi^j(z)) & \text{for } z \in \overline{G^k}. \end{cases}$$

Then $\zeta^k \colon \bigcup_{j=1}^{k} \overline{G^j} \to \mathbb{R}^2$ is continuous and $\|\zeta^k\| \le \varepsilon_k$. Hence we may extend ζ^k to a continuous function $\zeta^k \colon \mathbb{R}^2 \to \mathbb{R}^2$ for which $\|\zeta^k\| \le \varepsilon_k$. It remains to verify (44): on $\bigcup_{j=1}^{k-1} G^j$ we have

$$\sum_{j=1}^{k} \zeta^j = \zeta^k + \sum_{j=1}^{k-1} \nabla\Phi^j = \sum_{j=1}^{k} \nabla\Phi^j,$$

and on G_k,

$$\sum_{j=1}^{k} \zeta^j = P_k + \sum_{j=1}^{k-1} \nabla\Phi^j = \sum_{j=1}^{k} \nabla\Phi^j.$$

Since $|\Phi^k| \le \varepsilon_k$ on $\mathbb{R}^2$, the series $\Phi := \sum_{j=1}^{\infty} \Phi^j$ defines a continuous function on $\mathbb{R}^2$. Moreover, $|\Phi| \le \varepsilon$ (so 15.2 holds) and Φ is Lipschitz, since the sets G^j are disjoint, and so each partial sum $\sum_{j=1}^{k} \Phi^j$ is Lipschitz. Also observe that $\left\| \nabla \left(\sum_{j=1}^{k} \Phi^j \right)(x) \right\| \le A + B + 2\varepsilon$ for almost every x, since $\|\nabla \Phi^j\| \le \varepsilon_j$ outside $\overline{G^j}$, $\nabla \Phi^j = P_j$ on G^j, $\|P_j - P\| \le 2\varepsilon$ and $\|P\| \le A + B$.

Clearly the set $G = \bigcup_{j=1}^{\infty} G^j$ is an open dense subset of H, and $\mathscr{H}^1_\infty(G) \le \sum_{j=1}^{\infty} \varepsilon_k \le \varepsilon$ so 15.1 holds.

Write $\varphi^k = (\varphi_1^k, \varphi_2^k)$ and $\zeta^k = (\zeta_1^k, \zeta_2^k)$, and define functions $L^{\text{out}}, L^{\text{in}} : \mathbb{R}^3 \to \mathbb{R}$ by

$$L^{\text{out}}(x,y,p) = \sum_{j=1}^{\infty} \left(\varphi_1^j(x,y) + p\varphi_2^j(x,y) \right)$$

and

$$L^{\text{in}}(x,y,p) = \sum_{j=1}^{\infty} \left(\zeta_1^j(x,y) + p\zeta_2^j(x,y) \right).$$

Recalling that $\|\varphi^k\| \le \varepsilon_k$ for all k, $\|\zeta^k\| \le \varepsilon_k$ for $k \ge 2$ and $\zeta^1 = (-A, B)$, we see that $L^{\text{out}}, L^{\text{in}}$ are continuous and satisfy $|L^{\text{out}}| \le \varepsilon(1 + |p|)$ and $|L^{\text{in}} - (-A + Bp)| \le \varepsilon(1 + |p|)$ on $\mathbb{R}^3$. It follows that the function

$$L(x,y,p) := \max\left(L^{\text{in}}(x,y,p), L^{\text{out}}(x,y,p) \right)$$

satisfies 15.3. To prove the remaining statements, we first show that

$$L(x,y,p) = L^{\text{out}}(x,y,p), \text{ when } |p| \le A/B - 1 \tag{45}$$

and

$$L(x,y,p) = L^{\text{in}}(x,y,p), \text{ when } (x,y) \in G \text{ and } p \ge A/B + 1. \tag{46}$$

Indeed, if $|p| \le A/B - 1$, then, since $2\varepsilon \le B^2/A$,

$$L^{\text{in}}(x,y,p) \le -A + Bp + \varepsilon(1 + |p|) \le -\varepsilon(1 + |p|) \le L^{\text{out}}(x,y,p),$$

and if $p \ge A/B + 1$, then, since $2\varepsilon(2 + A/B) \le B$,

$$L^{\text{in}}(x,y,p) \ge -A + Bp - \varepsilon(1 + |p|) \ge \varepsilon(1 + |p|) \ge L^{\text{out}}(x,y,p).$$

Let $Z_k = \bigcup_{j=k}^{\infty} \{z \in \mathbb{R}^2 : \|z - z_{i_j}\| < \varepsilon_j\}$, $Z = \bigcap_{k=1}^{\infty} Z_k$ and observe that if $z \notin Z_k$ then $|\Phi^j(w) - \Phi^j(z)| \le \varepsilon_j \|w - z\|$ for all $w \in \mathbb{R}^2$ and $j \ge k$. Since Φ^k is differentiable on $\mathbb{R}^2 \setminus \partial G^k$, we infer that

$$\nabla \Phi(z) = \sum_{j=1}^{\infty} \nabla \Phi^j(z), \text{ when } z \notin Z \cup \bigcup_{j=1}^{\infty} \partial G^j. \tag{47}$$

The same argument, together with the fact that $\partial G^k \cap Z_{k+1} = \emptyset$, shows that

$$\nabla\left(\sum_{j\neq k} \Phi^j\right)(z) = \sum_{j\neq k} \nabla\Phi^j(z), \text{ when } z \in \partial G^k. \tag{48}$$

If $u \in \mathrm{AC}[a,b]$, then the function $\Phi \circ U$ is absolutely continuous, since Φ is Lipschitz. Noting that $\mathscr{H}^1_\infty(Z) = 0$, and so $U(x) \notin Z$ for almost every x, we see that its derivative is described by one of the following three cases for almost every x:

1. If $U(x) \in G$, then (47) and (44) give $\frac{d}{dx}\Phi(U(x)) = L^{\mathrm{in}}(x,u,u')$.
2. If $U(x) \notin Z \cup \bigcup_{j=1}^\infty \overline{G^j}$, then (47) gives that $\frac{d}{dx}\Phi(U(x)) = L^{\mathrm{out}}(x,u,u')$.
3. If $U(x) \in \partial\left(\bigcup_{j=1}^\infty G^j\right)$, then (48) and 14.4 of Lemma 14 give that $\frac{d}{dx}\Phi(U(x)) = L^{\mathrm{in}}(x,u,u') = L^{\mathrm{out}}(x,u,u')$.

Statement 15.4 of the lemma follows immediately from this and the definition of L. To deduce 15.5, let u satisfy its assumptions. Since $u' \geq A/B+1$ for a.e. x for which $U(x) \in G$, case 1 and (46) imply that $\frac{d}{dx}\Phi(x,u) = L(x,u,u')$ for such x. By case 3, the same expression for the derivative of $\Phi(x,u)$ holds for a.e. x for which $U(x) \in \partial G_k$ for some k. Hence it is enough to show that it holds for a.e. x for which $U(x) \notin \bigcup_{j=1}^\infty \overline{G^j}$, since then it will hold almost everywhere implying that $\mathscr{L}(u;a,b) = \Phi(b,u(b)) - \Phi(a,u(a))$.

Let W_k be the x-projection of $\bigcup_{j=k+1}^\infty \overline{G^j}$. If K is a compact subset of $\{x : U(x) \notin \bigcup_{j=1}^\infty \overline{G^j}\}$ and k is fixed, then for all sufficiently large l, $U_l(x) \notin \bigcup_{j=1}^k \overline{G^j}$ for all $x \in K$. Hence $u_l'(x) \leq A/B-1$ for almost all $x \in K \setminus W_k$. Since u_l' converges to u' weakly in $L^1[a,b]$, we infer that $u'(x) \leq A/B-1$ for almost all $x \in K\setminus W_k$. Since k is arbitrary and the measure of W_k tends to zero, we have that $u'(x) \leq A/B-1$ for almost all $x \in K$. Finally, we use that K is arbitrary to deduce that $u'(x) \leq A/B-1$ for almost all x for which $U(x) \notin \bigcup_{j=1}^\infty \overline{G^j}$ and conclude from case 2 and (45) that $\frac{d}{dx}\Phi(x,u(x)) = L(x,u,u')$ for almost all x from this set, as required.

Proof of Theorem 12. Let $\omega \geq 0$ be convex and superlinear. We let

$$A_k = 1+\omega(2^{2k+2}),\ B_k = 2^{-2k+1}A_k \text{ and } \varepsilon_k = 2^{-k}(1+2^{2k+2})^{-1}.$$

To start the recursive construction, let $G_0 = \mathbb{R}^2$, $\Phi_0(x,y) = -A_0x + B_0y$ and $L_0(x,y,p) = -A_0 + B_0p$. (For future use, notice that $\int_a^b L_0(x,u,u')\,dx = \Phi_0(U(b)) - \Phi_0(U(a))$ for every $u \in \mathrm{AC}[a,b]$.) Using Lemma 15 recursively with $H = G_{k-1}$, $\varepsilon = \varepsilon_k$ and $P = (-A_k, B_k)$, we define open sets $G_k \subset G_{k-1}$ that are dense in G_{k-1}, Lipschitz functions $\Phi_k\colon \mathbb{R}^2 \to \mathbb{R}$, and continuous functions $L_k\colon \mathbb{R}^3 \to \mathbb{R}$ with the properties described there.

By 15.2, $\Phi(x,y) := \sum_{k=0}^\infty \Phi_k(x,y)$ is continuous on $\mathbb{R}^2$. If $|p| < 2^{2j-1}$, then 15.3 implies that $|L_k(x,y,p)| \leq \varepsilon_k(1+2^{2j-1})$ for $k \geq j$. Hence both

$$\tilde{L}(x,y,p) := \sum_{k=0}^\infty \max(-2^{-k}, L_k(x,y,p))$$

and

$$L(x,y,p) := \max(\omega(p), \tilde{L}(x,y,p))$$

are continuous functions on $\mathbb{R}^3$.

Clearly L is superlinear. Noting that

$$\max(0, -A_j + B_j p) = \begin{cases} 0 & \text{when } 0 \le p \le A_j/B_j, \\ -A_j + B_j p & \text{when } p \ge A_j/B_j, \end{cases}$$

we deduce from 15.3 that for $0 \le p \le A_j/B_j$,

$$L_j(x,y,p) \ge -\varepsilon_j(1 + A_j/B_j)$$

and for $p > A_j/B_j$,

$$\begin{aligned} L_j(x,y,p) &\ge B_j p - A_j - \varepsilon_j(1+p) \\ &= (B_j - \varepsilon_j)(p - A_j/B_j) - \varepsilon_j(1 + A_j/B_j) \quad \ge -\varepsilon_j(1 + A_j/B_j). \end{aligned}$$

Hence for any $p \ge 0$,

$$L_j(x,y,p) \ge -\varepsilon_j(1 + A_j/B_j) \ge -2^{-j}, \tag{49}$$

and so for $p \in [2^{2k}, 2^{2k+2}]$, we use 15.3 for L_k, the fact that $L_0(x,y,p) \ge 0$ and (49) to estimate

$$\begin{aligned} \tilde{L}(x,y,p) &= \sum_{j=0}^{\infty} L_j(x,y,p) \\ &= L_k(x,y,p) + \sum_{j=1, j\neq k}^{\infty} L_j(x,y,p) \\ &\ge -A_k + pB_k - \sum_{j=1}^{\infty} \varepsilon_j(1 + 2^{2j+2}) \\ &\ge A_k - 1 = \omega(2^{2k+2}) \ge \omega(p). \end{aligned}$$

Hence for $p \ge 1$,

$$L(x,y,p) = \sum_{j=0}^{\infty} L_j(x,y,p) \text{ and } L_j(x,y,p) \ge -2^{-j}. \tag{50}$$

If $u \in \mathrm{AC}[a,b]$, we have $\int_a^b L_j(x,u,u')\,dx \ge \Phi_j(U(b)) - \Phi_j(U(a))$ by 15.4, and so

$$\begin{aligned} \mathscr{L}(u;a,b) &\ge \int_a^b \sum_{j=0}^{\infty} \max(-2^{-j}, L_j(x,u,u'))\,dx \\ &= \sum_{j=0}^{\infty} \int_a^b \max(-2^{-j}, L_j(x,u,u'))\,dx \\ &\ge \sum_{j=0}^{\infty} (\Phi_j(U(b)) - \Phi_j(U(a))) \\ &= \Phi(U(b)) - \Phi(U(a)) \end{aligned} \tag{51}$$

Let $G = \bigcap_{k=0}^{\infty} G_k$. Since $G_0 = \mathbb{R}^2$ and G_k is dense in G_{k-1}, G is a residual subset of $\mathbb{R}^2$. We show that G is contained in the universal singular set of L. For this, assume that $(x_0, y_0) \in G$ and define $\psi\colon \mathbb{R}^2 \to [1, \infty]$ by $\psi(z) = A_k/B_k - 1 = 2^{2k-1} - 1$ for $z \in G_{k-1} \setminus G_k$ and $\psi(z) = \infty$ for $z \in G$. Then ψ is lower semicontinuous.

Let $\psi_k\colon \mathbb{R} \to [1, \infty)$ be bounded continuous functions such that $\psi_k \nearrow \psi$ as $k \to \infty$. Then the equation $u' = \psi_k(x, u)$ has a global C^1 solution u_k such that $u_k(x_0) = y_0$.

Denote by $f(x)$ the supremum of those values A_k/B_k for which $(x, y) \in G_{k-1}$ for some y. Then

$$\int_a^b f(x)\,dx \le (b-a)A_1/B_1 + \sum_{k=2}^{\infty} \mathscr{H}^1_\infty(G_{k-1}) A_k/B_k < \infty$$

(the convergence of the series follows from 15.1), so f is locally integrable. By definition, $\psi(x, y) \le f(x)$ for every $(x, y) \in \mathbb{R}^2$. Since $\psi_k \le \psi$, we infer that $0 \le u_k' \le f$, and conclude that the sequence u_k is locally absolutely equicontinuous. Hence it has a subsequence converging (locally uniformly) to a locally absolutely continuous function $u\colon \mathbb{R} \to \mathbb{R}$ such that $u(x_0) = y_0$. Since $u_l' = \psi_l(x, u_l) \ge \psi_k(x, u_l)$ for $l \ge k$ and since $\psi_k(x, u_l)$ converges locally uniformly to $\psi_k(x, u)$ as $l \to \infty$, we have $u' \ge \psi_k(x, u)$ for all k and so $u' \ge \psi(x, u)$.

Since $u'(x) = \psi(x, u) \ge A_k/B_k + 1$ when $(x, u) \in G_k$ and

$$0 \le u_l'(x) \le \psi_l(x, u_l) \le \psi(x, u_l) \le A_k/B_k - 1 \text{ when } (x, u) \notin G_k,$$

u satisfies the assumptions of 15.5 for each interval $[a, b]$ and each k. Hence $\int_a^b L_k(x, u, u')\,dx = \Phi_k(b, u(b)) - \Phi_k(a, u(a))$. Since $u' \ge \psi(x, u) \ge 1$, (50) implies that $L(x, u, u') = \sum_{j=0}^{\infty} L_j(x, u, u')$ and $L_j(x, u, u') \ge -2^{-j}$. This justifies exchange of integration and summation, allowing us to conclude that $\int_a^b L(x, u, u')\,dx = \sum_{j=0}^{\infty} \int_a^b L_j(x, u, u')\,dx = \Phi(U(b)) - \Phi(U(a))$ for any $a < b$ and so, because of (51), u is a minimizer on any interval $[a, b]$. Finally, since

$$\lim_{x \to x_0} u'(x) \ge \lim_{(x,y) \to (x_0, y_0)} \psi(x, y) = \infty,$$

we have $u'(x_0) = \infty$ and we are done.

References

1. J. M. Ball and V. J. Mizel: One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. *Arch. Rational Mech. Anal.* **90**, 325–388 (1985)
2. J. M. Ball and N. Nadirashvili: Universal singular sets for one-dimensional variational problems. *Calc. Var. Partial Differential Equations* **1**, 429–438 (1993)
3. G. Buttazzo, M. Giaquinta and S. Hildebrandt: *One-dimensional variational problems*. Oxford University Press, 1998
4. F. H. Clarke and R. B. Vinter: Regularity properties of solutions to the basic problem in the calculus of variations. *Trans. Amer. Math. Soc.* **289**, 73–98 (1985)
5. A. M. Davie: Singular minimisers in the calculus of variations in one dimension. *Arch. Rational Mech. Anal.* **101**, 161–177 (1988)

6. A. FERRIERO: *The Lavrentiev phenomenon in the Calculus of Variations*, PhD Thesis. Università degli Studi di Milano-Bicocca, 2004
7. A. D. IOFFE: On lower semicontinuity of integral functionals. I. *SIAM J. Control Optimization* **15**, 521–538 (1977)
8. M. LAVRENTIEV: Sur quelques problemes du calcul des variations. *Ann. Matem. Pura Appl.* **4**, 7–28 (1926)
9. B. MANIÀ: Sopra un esempio di Lavrentiev. *Boll. Un. Matem. Ital.* **13**, 147–153 (1934)
10. R. ČERNÝ AND J. MALÝ: Another counterexample to lower semicontinuity in calculus of variations. *J. Convex Anal.* **9**, 295–299 (2002)
11. V. J. MIZEL: Recent progress on the Lavrentiev phenomenon with applications. *Lecture Notes in Pure and Appl. Math.* **225**, 257–261 (2002)
12. S. SAKS: *Theory of the Integral*. Hafner Publishing Company, New York, 1937
13. J. SERRIN: On the definition and properties of certain variational integrals. *Trans. Amer. Math. Soc.* **101**, 139–167 (1961)
14. M. A. SYCHËV: The Lebesgue measure of a universal singular set in the simplest problems of the calculus of variations. *Siberian Math. J.* **35**, 1220–1233 (1994)
15. L. TONELLI: Sur un méthode directe du calcul des variations. *Rend. Circ. Mat. Palermo* **39** (1915)

Department of Mathematics,University College London, Gower Street, London, WC1E 6BT, UK
email:mari@math.ucl.ac.uk

and

Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford, OX1 3LB, UK
email:kirchhei@maths.ox.ac.uk

and

Department of Mathematics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, UK
email:t.c.oneil@open.ac.uk

and

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
email:d.preiss@warwick.ac.uk

and

Friedrich-Schiller-Universität Jena, Mathematisches Institut,D-07737 Jena, Germany
email:winter@minet.uni-jena.de