Uncertainty Dynamics in a Model of Economic Inequality

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Abstract

In this article, we consider a stylized dynamic model to describe the economics of a population, expressed by a Langevin-type kinetic equation. The dynamics is defined by a combination of terms, one of which represents monetary exchanges between individuals mutually engaged in trade, while the uncertainty in barter (trade exchange) is modelled through additive and multiplicative stochastic terms which necessarily abide dynamical constraints. The model is studied to estimate three meaningful quantities, the inequality Gini index, the social mobility and the total income of the population. In particular, we investigate the time evolving binary correlations between any two of these quantities.

Keywords: income distribution, economic inequality, social mobility, additive and multiplicative noise

1 Motivation and introductory considerations

The emergence of inequality in income and wealth distribution has attracted considerable interest in recent years. Books by renowned economists addressing this issue (e.g., [1, 2, 3]) are having a widespread diffusion among non specialists too, with frequent tectonic societal impact generating substantial media highlights. Besides, the urge to identify possible mechanisms for the formation of such collective phenomenon prompted the formulation of related models within an enlarged scientific community (see e.g. [4, 5, 6, 7, 8, 9]).

Typically, in these models society is dealt with as a system composed of a large number of individuals who exchange money through binary and other nonlinear interactions. For example, the model proposed in [8] (see also [10]) by two of the present authors is expressed by a system of nonlinear differential equations describing the evolution in time of the income distribution. Specific trading rules which characterise the behavior of individuals of different income classes together with the existence of a taxation and a welfare system are postulated. Despite the stochasticity element due to the presence of transition probabilities for trade and resulting class change of single individuals, the equations governing the variation in time of the fraction of individuals in each income class are deterministic.

An important concept not included in [8] is that of uncertainty. In almost all walks of life the presence of chance is unavoidable, at least to some extent. Incorporating this element to analyse the trade dynamics is then crucial. To keep also uncertainty into consideration, we developed and analysed in [11] and [12] a stochastic model, respectively containing an Ito-type additive and an Ito-type multiplicative noise term [13]. Exploring the resulting dynamics, we were able to recover, at least in some cases, empirically observed patterns. Pondering on all that led us to the conclusion that a more realistic modelling should involve a combination of additive and multiplicative stochastic perturbation. This is what we are doing in this paper.

We stress that, in order to concentrate on the role of noise in the system, here we omit terms describing taxation and redistribution with welfare which were included in [8]: in other words, the "deterministic" component of the present model is simpler (and so was that of the models in [11] and [12]). We limit ourselves to consider the occurrence of a wealth of monetary exchanges between individuals and to this we now add the effect of additive and multiplicative stochastic terms. Another difference with the approach developed in [8, 11, 12] is the following. Whereas the variables in these papers describe the fractions of individuals belonging to different income classes, the variables here are the incomes of the classes. This implies that the random perturbations which we are including here directly affect the income of the classes, sectorially and over the whole economic spectrum.

Our aim is to provide a framework to estimate some important indicators measuring respectively economic inequality and social mobility. In particular, we investigate here the (varying in time) correlations between these two indicators and between each of them and the total income of the population.

The paper is organised as follows. In Section 2 we describe the structure of our model, which is expressed by a Langevin-type kinetic equation and we also propose an algorithm generating a combination of additive and multiplicative noise. The equations are then numerically solved and in Section 3 we report results obtained by taking the average of various quantities over a large number of realisations. Therein, we also compare our results with some real world data [14]. Finally, Section 4 provides some concluding remarks.

2 The equation structure

Consider a population of individuals divided into a finite number n of classes, each one characterized by its average income r_j with $0 < r_1 \leq r_2 \leq \ldots \leq r_n$. Let $x_j(t)$, with $x_j : \mathbf{R} \to [0, +\infty)$ for $1 \leq j \leq n$, denote the fraction at time t of individuals in the j-th class and let $y_j(t) = r_j x_j(t)$, with $y_j : \mathbf{R} \to [0, +\infty)$ be the total income of the class j. In previous work, see e.g. [8], two of us constructed a model for the evolution in time of $x(t) = (x_1(t), ..., x_n(t))$, in correspondence to a whole of economic exchanges taking place between pairs of individuals. From specific behavioral assumptions on individuals of different classes expressed through different transition probabilities, we were led to write down a system of ordinary differential equations with $x(t)$ as unknown function, in fact "deterministic" in the x_j variables. Subsequently, we also included in our models the presence of an additive [11] or a multiplicative [12] noise term. In this paper we aim to discuss a different model, expressed by a Langevin kinetic equation for which both additive and multiplicative noise terms are present and for which the variable is the income vector $y(t) = (y_1(t), ..., y_n(t))$. Specifically, the equations we consider take the form

$$dy_j = \mathcal{A}_j(y)dt + \mathcal{B}_j(y, \eta^A, \eta^M)\sqrt{dt}\,, \qquad 1 \leq j \leq n\,, \tag{1}$$

in which the "deterministic" part

$$\mathcal{A}_j(y)dt = \left(\sum_{h,k=1}^{n} A_{hk}^{j} y_h y_k - y_j \sum_{i=1}^{n} y_i \right) dt \tag{2}$$

is obtained through a reformulation of the r.h.s. of the equations in [8] (we point out however that we disregard here, for sake of simplicity, taxation and redistribution terms) and the stochastic part

$$\mathcal{B}_j(y, \eta^A, \eta^M)\sqrt{dt} = \left(\omega\, \eta_j^A + (1-\omega)\, \eta_j^M\right) \Gamma \sqrt{dt}\,, \tag{3}$$

describes an Ito-type process [13] incorporating additive and multiplicative stochastic noises.

In more detail, the coefficients in the homogeneous quadratic part of $\mathcal{A}_j(y)$ are given by

$$A^i_{hk} = \frac{r_i}{r_h r_k}\, C^i_{hk}\,, \tag{4}$$

with $C^i_{hk} \in [0, 1]$ for $i, h, k \in \{1, ..., n\}$ as in the formula (2) in [8], expressing the probability that an individual of the h-th class will belong to the i-th class after a direct interaction with an individual of the k-th class. In particular, in the expression for $\mathcal{A}_j(y)$ in (2) the identity $\sum_{i=1}^n C^i_{hk} = 1$ for any $h, k \in \{1, ..., n\}$ is used.

As for terms in equation (3), $\eta^A = (\eta_1^A, ..., \eta_n^A)$ and $\eta^M = (\eta_1^M, ..., \eta_n^M)$ respectively denote an additive and a multiplicative noise vector, $\omega \in [0, 1]$ is randomly chosen at each integration step, and Γ denotes the noise amplitude. As in [8], we assume here constant population size during the evolution of the system and normalize it to 1: $\sum_{j=1...n} x_j(t) = 1$ for all $t \geq 0$. For this to occur, the noise vectors have to satisfy a suitable constraint (also the terms in the deterministic part of the equations have been reconstructed so as to satisfy the required condition):

$$\sum_{j=1}^n \frac{\omega\, \eta_j^A + (1-\omega)\, \eta_j^M}{r_j} = 0\,. \tag{5}$$

In view of this, the numerical algorithm reciprocating the dynamics can be represented as follows: at each step one chooses two vectors $\zeta = (\zeta_1, ..., \zeta_n)$ and $\xi = (\xi_1, ..., \xi_n)$ whose components are Gaussian random variables, and then, starting from these, one can define the vectors η^A and η^M by setting

$$\eta_j^A = \zeta_j \;-\; \frac{1}{r_j}\, \frac{\sum_{i=1}^n \frac{\zeta_j}{r_j}}{\sum_{k=1}^n \frac{1}{r_k^2}}\,, \tag{6}$$

and

$$\eta_j^M = y_j \xi_j \;-\; \frac{y_j^2}{r_j}\, \frac{\sum_{i=1}^n \frac{y_j \xi_j}{r_j}}{\sum_{k=1}^n \frac{y_k^2}{r_k^2}}\,. \tag{7}$$

It is easy to verify that the vectors η^A and η^M as in (6) and (7) satisfy $\sum_{j=1}^{n} \frac{\eta_j^A}{r_j} = 0$ and $\sum_{j=1}^{n} \frac{\eta_j^M}{r_j} = 0$. Finally, the parameter ω is randomly chosen in $[0,1]$ and Γ tunes the noise amplitude.

3 Numerical results

Our aim is to evaluate the Gini index G and another indicator, M, which quantifies social mobility during the evolution of equations (1). We are interested as well in getting information on the sign of the correlation of G and M, and on the sign of their correlations with the value μ of the total income of the population.

We recall that the index G was proposed by the Italian statistician Corrado Gini a century ago as a measure of inequality or income or wealth. It is defined as a ratio, whose numerator is given by the area between the Lorenz curve of a distribution and the uniform distribution line, while the denominator is given by the area of the region under the uniform distribution line. It takes values in $[0,1]$.

As for the definition of M, introduced in [15] in a partially different context, we need to recall first a couple of notations it involves. In [8] and [15] (and the same, albeit not explicitly mentioned, holds here in view of the expressions of the coefficients C_{hk}^i which are, as written in Section 2, as in the formula (2) in [8]) we denoted S to be the amount of money exchanged in each trade, and we introduced, in order to bring heterogeneity into the model, suitable parameters $p_{h,k}$ for $h,k = 1,...,n$, with each $p_{h,k}$ measuring the encounter frequency rate of individuals of the h-th and in the k-th class and expressing the probability that in an encounter between an h-individual and a k-individual, the one who pays is the h-individual. Then, M can be defined as

$$M = \frac{1}{\left(1 - \frac{y_1}{r_1} - \frac{y_n}{r_n}\right)} \sum_{i=2}^{n-1} \sum_{k=1}^{n} \frac{S}{(r_{i+1} - r_i)} \, p_{k,i} \, \frac{y_k}{r_k} \, \frac{y_i}{r_i} \,,$$

namely as the collective probability of class advancement of all classes from the 2-th to the $(n-1)$-th one.

For the purpose of computing G and M, we solved numerically the equations (1) and took the average of the quantities of interest out of a large ensamble of stochastic realisations. More specifically, in our simulations, we considered $n = 10$, $r_1 = 10$ and $r_i - r_{i-1} = 10$ for $2 \le i \le n$, Γ equal to 0.001, and, as already pointed out, we chose randomly $\omega \in [0, 1]$ at each integration step. We always considered a stationary solution, reached in the long time, of the equations (1) with $\Gamma = 0$, i.e. in the absence of noise as initial condition. In every simulation, we ensembled averaged over 50 realisations with each run spanning 5000 integration steps.

$\mu(0)$	R_{GM}	$R_{G\mu}$	$R_{M\mu}$
25	- 0.471 $\pm$ 0.058	- 0.125 $\pm$ 0.068	0.901 $\pm$ 0.013
25	- 0.518 $\pm$ 0.045	- 0.136 $\pm$ 0.055	0.883 $\pm$ 0.016
25	- 0.448 $\pm$ 0.053	- 0.110 $\pm$ 0.065	0.906 $\pm$ 0.014
$\mu(0)$	R_{GM}	$R_{G\mu}$	$R_{M\mu}$
30	- 0.617 $\pm$ 0.046	- 0.369 $\pm$ 0.057	0.938 $\pm$ 0.010
30	- 0.641 $\pm$ 0.041	- 0.404 $\pm$ 0.055	0.942 $\pm$ 0.010
30	- 0.642 $\pm$ 0.047	- 0.442 $\pm$ 0.060	0.955 $\pm$ 0.010
$\mu(0)$	R_{GM}	$R_{G\mu}$	$R_{M\mu}$
35	- 0.711 $\pm$ 0.034	- 0.588 $\pm$ 0.047	0.979 $\pm$ 0.003
35	- 0.711 $\pm$ 0.041	- 0.601 $\pm$ 0.050	0.980 $\pm$ 0.004
35	- 0.696 $\pm$ 0.038	- 0.580 $\pm$ 0.050	0.982 $\pm$ 0.003

Table 1: Correlations R_{GM} (Gini and mobility index), $R_{G\mu}$ (Gini index and total income) and $R_{M\mu}$ (mobility index and total income) computed in nine cases in which total income μ is not conserved, with noise amplitude $\Gamma = 0.001$. Averages of 50 realizations, each of 5000 integration steps.

A few samples of the results we obtained are reported in Table 1. Here, nine triplets are displayed with the average values of correlations R_{GM} (Gini and mobility index), $R_{G\mu}$ (Gini index and total income) and $R_{M\mu}$ (mobility index and total income). These values are computed using solutions evolving from three initial conditions for which the initial value of μ is respectively equal to 25, 30 and 35. For any of these initial conditions, three different average results are reported.

As should be evident from the first column of data, the correlation between economic inequality and social mobility represented by R_{GM} is negative for

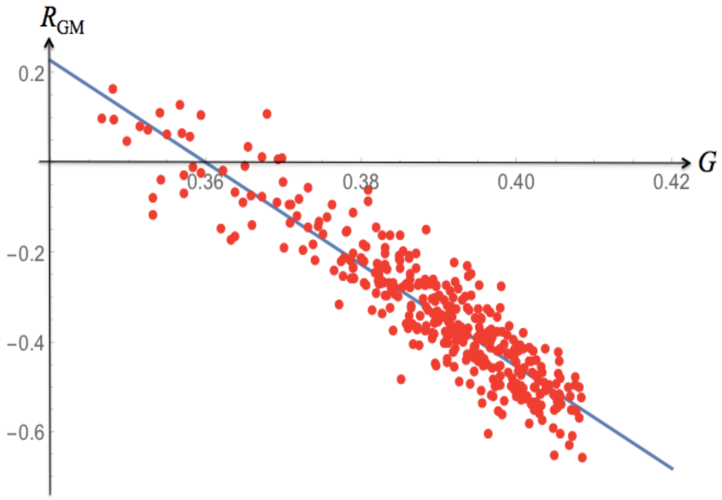

Figure 1: Correlation R_{GM} for different values of G in the range $0.35 < G < 0.41$. Each dot represents the average of 40 realisations of stochastic time-series with 5000 integration steps. The equation of the regression line is $R_{GM} = -11.35\,G + 4.085$.

the three mentioned values of μ. Most remarkably, this negative correlation mimics real world situations [16, 17]. Here, one might wonder what is the meaning of these values of μ. To answer that question, we point out that a relation between the total income μ and the Gini index G can be seen to hold true at equilibrium, approximately given by $G = -0.1594 + 0.03712\mu - 0.0006\mu^2$. Hence, the data in the table can be also thought of as relative to three cases with G close to 0.39, 0.41 and 0.40 respectively. A graphical, quite expressive illustration of the sort of values of R_{GM} we obtained for G (randomly chosen) in the range $0.35 < G < 0.41$ is provided in Figure 1. Each dot therein represents the average over 40 stochastic realisations with 5000 integration steps.

The central data column in Table 1 provide particular values of the negative correlation between G and μ. The corresponding right column shows that there is a strong positive correlation $R_{M\mu}$ between mobility and total income.

Finally, we emphasize that the range of values of G referred to in Figure 1 includes the Gini indices of various countries. These can be found on the web page of the World Bank [14]. For example, the Gini index of the United States whose most recent reported value is relative to the year 2013, is 41.06. The Gini indices (values relative to the year 2012) of the countries where we live, Italy and UK, are 35.16 and 32.57 respectively.

4 Concluding remarks

A stylized dynamic model has been proposed and numerically investigated, which allows to estimate economic inequality and mobility for a population. The model describes a range of economic exchanges between population members driven by a combination of additive and multiplicative stochastic noises, resembling uncertainty in the trade situation. The resulting Langevin-type kinetic equation is represented by a minimalist combination of "deterministic" and stochastic components. The deterministic part assumes phenomenologically supported rules [8] (supposed to be the same for individuals in the same income class) of economic exchanges between pairs of individuals. We emphasise that they do not include effects such as taxation and redistribution, which were studied in other papers [8, 10]. While admittedly our structure does not correspond to the complex nebular real world interactive description, some interesting results are found in connection with the sign of the indicators of inequality and mobility and especially in connection with their correlations.

References

[1] Stiglitz J.E., **The price of inequality: how today's divided society endangers our future**, W.W. Norton & Company, New York (2012).

[2] Atkinson A.B., **Inequality: What can be done?**, Harvard University Press, Cambridge (2015).

[3] Milanovic B., **Global inequality. A new approach for the age of globalization**, Harvard University Press, Cambridge, (2016).

[4] Angle J., *The inequality process as a wealth maximising process* **Physica A**, 367, pp. 388-414, (2006).

[5] Matthes D., and Toscani G., *On steady distributions of kinetic models of conservative economies*, **J. Statist. Phys.**, 130, pp. 1087-1117, (2008).

[6] Chakrabarti A., and Chakraborti, B.K. *Microeconomics of the ideal gas*

like market models **Physica A**, 388, pp. 4151-4158, (2009).

[7] Yakovenko V.M., *Econophysics, Statistical Mechanics Approach to*, **Encyclopedia of Complexity and System Science**, edited by R. A. Meyers, Springer (2009).

[8] Bertotti M.L., and Modanese G., *Microscopic models for welfare measures addressing a reduction of economic inequality*, **Complexity**, vol. 21, 6, pp. 89-98, (2016).

[9] Chattopadhyay A.K., Krishna Kumar T., and Mallick S.K., *Poverty index with time-varying consumption and income distributions*, **Physical Review E**, vol. 95, 032109, (2017)

[10] Bertotti M.L., and Modanese G., *Exchange models for the emergence of income distribution and economic inequality*, **Int. J. of Design & Nature and Ecodynamics**, vol. 11, 4, pp. 620627, (2016).

[11] Bertotti M.L., Chattopadhyay A.K., and Modanese G., *Stochastic effects in a discretized kinetic model of economic exchange*, **Physica A**, vol. 471, 724-732, (2017).

[12] Bertotti M.L., Chattopadhyay A.K., and Modanese G., *Economic inequality and mobility for stochastic models with multiplicative noise*, preprint, arXiv:1702.08391 (2017),

[13] Risken H., **The Fokker-Planck Equation**, Springer Verlag, Berlin (1984).

[14] http://data.worldbank.org/indicator/SI.POV.GINI.

[15] Bertotti M.L., and Modanese G., *Economic inequality and mobility in kinetic models for social sciences*, **Eur. Phys. J Special Topics**, vol. 225, pp. 1945-1958, (2016).

[16] Andrews D., and Leigh. A., *More inequality, less social mobility*, **Appl. Econ. Lett.**, vol.16, pp. 1489-1492, (2009).

[17] Corak M., *Income inequality, equality of opportunity, and intergenerational mobility*, **J. Econ. Perspect.**, vol. 27, pp. 79-102, (2013).