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A new approach to graph reconstruction using supercards

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The vertex-deleted subgraph $G - v$, obtained from the graph G by deleting the vertex v and all edges incident to v, is called a *card* of G. The *deck* of G is the multiset of its unlabelled vertex-deleted subgraphs. The *number of common cards* of G and H is the cardinality of a maximum multiset of common cards, i.e., the multiset intersection of the decks of G and H. We introduce a new approach to the study of common cards using supercards, where we define a *supercard* G^+ of G and H to be a graph that has at least one vertex-deleted subgraph isomorphic to G, and at least one isomorphic to H. We show how maximum sets of common cards of G and H correspond to certain sets of permutations of the vertices of a supercard, which we call *maximum saturating sets*. We then show how to construct supercards of various pairs of graphs for which there exists some maximum saturating set X contained in $\mathrm{Aut}(G^+)$. For certain other pairs of graphs, we show that it is possible to construct G^+ and a maximum saturating set X such that the elements of X that are not in $\mathrm{Aut}(G^+)$ are in one-to-one correspondence with a set of automorphisms of a different supercard G^+_λ of G and H. Our constructions cover nearly all of the published families of pairs of graphs that have a large number of common cards.

Keywords: *Graph reconstruction, reconstruction numbers, vertex-deleted subgraphs, supercards, graph automorphisms*

1. Introduction

In this paper all graphs are finite, undirected and contain no loops or multiple edges. Any graph-theoretic terminology and notation not explicitly explained below can be found in Bondy and Murty's text [4]. For more information on the action of a permutation group on the vertices of a graph, we refer the reader to the book by Lauri and Scapellato [14].

Let G be a graph and let $u, v \in V(G)$. We denote the group of all permutations of $V(G)$ by $S_{V(G)}$ and the identity permutation of $S_{V(G)}$ by

$1_{V(G)}$. The *neighbourhood* of v in G is the set $N_G(v)$ consisting of all vertices of G adjacent to v. The cardinality of this set is the *degree* of v in G, i.e., $d_G(v) = |N_G(v)|$. A *leaf* of G is a vertex of degree 1, and an *isolated vertex* of G is a vertex of degree 0. G is *k-regular* if $d_G(v) = k$ for each $v \in V(G)$. The complement of G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We use n to denote $|V(G)|$, the order of G.

Suppose that H is another graph and that γ is a bijection from $V(G)$ to $V(H)$. For any $Z \subseteq V(G)$, we write the image of Z under γ as $\gamma(Z)$. When γ is, moreover, an isomorphism from G to H, i.e., xy is an edge of G if and only if $\gamma(x)\gamma(y)$ is an edge of H, we write $\gamma(G) = H$. We write $G \cong H$ to indicate that G and H are isomorphic. The group of all automorphisms of G, i.e., isomorphisms from G to itself, is denoted by $\mathrm{Aut}(G)$. G is *vertex-transitive* if, for all $u, v \in V(G)$, there exists $\gamma \in \mathrm{Aut}(G)$ such that $\gamma(u) = v$.

Now let $Z \subseteq V(G)$. The Z-deleted subgraph $G - Z$ is obtained from G by deleting all the vertices of Z together with all edges of G incident to a vertex in Z. So $d_{G-Z}(v) = d_G(v) - |N_G(v) \cap Z|$, for all $v \in V(G - Z)$. When $Z = \{v\}$ or $Z = \{u, v\}$, we write $G - Z$ as $G - v$ or $G - u - v$, respectively. The *vertex-deleted subgraph* $G - v$ is also known as a *card* of G, and the multi-set of all n unlabelled cards of G is called the *deck* of G, which we denote by $\mathcal{D}(G)$.

Clearly, if $G \cong H$ then $\mathcal{D}(G) = \mathcal{D}(H)$. The *Reconstruction Conjecture*, first proposed by Kelly and Ulam in 1941 [12, 13, 18], asserts that, when $n > 2$, the converse also holds, i.e., G is isomorphic to H if and only if G has the same collection of n unlabelled cards as H. However, despite the efforts of many graph theorists, the status of the sufficiency of the condition remains unresolved. Surveys on the reconstruction problem can be found in [2] [3] [14].

One approach to tackling this problem has been to consider the *number of common cards* between pairs of graphs in various families (see, for example, [5] or [10]). A *common card of*, or *between*, G and H is any card in the multi-set intersection $\mathcal{D}(G) \cap \mathcal{D}(H)$, and the number of common cards of G and H, denoted by $b(G, H)$, is the cardinality of this multi-set intersection. The Reconstruction Conjecture can then be reformulated as follows: when $n > 2$, $b(G, H) < n$ unless G and H are isomorphic. We note that if $G' \cong G$ and $H' \cong H$ then $b(G', H') = b(G, H)$.

Until recently, there were no known families of pairs of non-isomorphic graphs that had $b(G, H) > \frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n + 9})$. However, Bowler, Brown and Fenner [5] showed that there are, in fact, several infinite families of pairs of non-isomorphic graphs G and H with $b(G, H) = \left\lfloor \frac{2(n-1)}{3} \right\rfloor$. Moreover, they conjectured that $b(G, H)$ is bounded above by $\frac{2(n-1)}{3}$ for large enough n. In a subsequent paper [6], they, together with Myrvold, showed that if G is disconnected and H is connected then $b(G, H) \leq \lfloor \frac{n}{2} \rfloor + 1$. They also characterised all pairs of such graphs that attain this bound (most of these infinite families can also be found in [5]). Results for small graphs, i.e., for $n \leq 11$, have been provided by Baldwin [1], McMullen [15] and Rivshin [17].

In this paper, we introduce a new approach to the study of the maximum number of common cards using supercards, where we define a *supercard* of non-isomorphic graphs G and H to be any graph having at least one vertex-deleted subgraph isomorphic to G, and at least one isomorphic to H. In Section 3, we define such a supercard G^+ and show that there exist subsets of $S_{V(G^+)}$ of cardinality $b(G, H)$, the elements of which correspond to the elements of $\mathcal{D}(G) \cap \mathcal{D}(H)$. We call these subsets *maximum saturating sets.*

It is easy to show that if $\lambda \in \mathrm{Aut}(G^+)$ then λ corresponds to some common card of G and H. Furthermore, it is always possible to find a set of supercards so that every common card corresponds to an automorphism of at least one of these supercards. We shall show that, in all of the published examples we know of, pairs of graphs that have a large number of common cards require automorphisms of at most two supercards to represent all their common cards.

In Section 4, we use vertex-transitive graphs to construct directly supercards G^+, and then define corresponding graphs G and H, where $b(G, H) = \frac{n+1}{2}$. Moreover, we show that there exist corresponding maximum saturating sets that are subsets of $\mathrm{Aut}(G^+)$. We then show how to construct supercards for nearly all of the infinite families of pairs of graphs of odd order that attain the bound $b(G, H) = \frac{n+1}{2}$, when G is disconnected and H is connected (see Theorems 3.4 and 3.6 of [5] and [6], respectively.) We also show that there exist maximum saturating sets that are subsets of the automorphism group of the corresponding supercard.

In Section 5, we show how to construct a second supercard G^+_λ of G and H from G^+, and show how their maximum saturating sets are related. In Section 6, we give examples of supercards and maximum saturating sets such that each element of the set is either an automorphism of G^+ or corresponds to an automorphism of G^+_λ. These examples include a supercard of the infinite family given in Theorem 2.1 of [5] that has the largest value of $b(G, H)$, currently known for large n, i.e., $b(G, H) = \frac{2(n-1)}{3}$; the unique infinite family of even order given in Theorem 3.7 of [6] that attains the bound of $b(G, H) = \frac{n+2}{2}$ when G is disconnected and H is connected; and a generalisation of the infinite family with $b(G, H) = \frac{n+3}{2}$ discovered by Bondy and reported by Myrvold in [16].

2. Preliminary Results

The following gives a simple criterion for two graphs with at least one common card to be isomorphic.

Lemma 2.1. *Let G and H be graphs, and let γ be a bijection from $V(G)$ to $V(H)$. Suppose that there is some vertex v of G such that $\gamma(G-v) = H-\gamma(v)$ and $\gamma(N_G(v)) = N_H(\gamma(v))$. Then $\gamma(G) = H$.*

Proof. γ preserves the adjacencies of the vertices of $G - v$ by the first condition, and those of v by the second. Thus $\gamma(G) = H$. □

Corollary 2.2. *Let G be a graph and let $v \in V(G)$. Suppose that $\gamma \in S_{V(G)}$. Then $\gamma \in \mathrm{Aut}(G)$ if and only if*

$$\gamma(G - v) = G - \gamma(v) \qquad \textit{and} \qquad \gamma(N_G(v)) = N_G(\gamma(v)).$$

We note that the first condition of Corollary 2.2 alone is not sufficient, i.e., we may have $G - v \cong G - u$ even though there is no automorphism of G mapping v to u. This phenomenon is called *pseudosimilarity* and has been explored by Harary and Palmer [11], Lauri and Scapellato [14], and others.

The constructions in Sections 4 and 6 all involve vertex-transitive graphs and their complements. We will make use of the following simple results for regular graphs, which hold, *a fortiori*, for vertex-transitive graphs.

Lemma 2.3. *Let G and H be regular graphs and let $v \in V(G)$. Suppose that there exists a bijection γ from $V(G)$ to $V(H)$ such that $\gamma(G-v) = H-\gamma(v)$. Then $\gamma(G) = H$.*

This result holds, in particular, when $G = H$, in which case γ would be an automorphism of G.

Corollary 2.4. *Let G and H be regular graphs. Then $b(G, H) \neq 0$ if and only $G \cong H$.*

Using Lemma 2.3, is easy to deduce that G is vertex-transitive if and only if every card in $\mathcal{D}(G)$ is isomorphic. It follows that no vertex-transitive graph contains a cut-vertex.

3. Supercards

For the rest of this paper, we assume that G and H are *non-isomorphic* graphs, both of order n. We now show how to use supercards in the study of common cards.

Definition 3.1. A *supercard* of G is any graph of order $n + 1$ whose deck contains a card isomorphic to G.

Definition 3.2. A *common supercard* of G and H is any graph that is a supercard of both G and H, i.e. a graph whose deck contains some card $\widehat{G}$ isomorphic to G and another card $\widehat{H}$ isomorphic to H. For brevity, we refer to such graphs as *supercards of* G *and* H.

Lemma 3.3. *There exists a graph G^+ that is a supercard of G and H if and only if $b(G, H) \geq 1$.*

Proof. Suppose first that G^+ is a supercard of both G and H and let v and w be vertices of G^+ such that $G^+ - w \cong G$ and $G^+ - v \cong H$. Then, since $G^+ - w - v = G^+ - v - w$, it follows that $b(G, H) = b(G^+ - w, G^+ - v) \geq 1$.

Suppose conversely that there exists $s \in V(G)$, $t \in V(H)$, and an isomorphism γ such that $\gamma(G - s) = H - t$. For some $t^* \notin V(G) \cup V(H)$, let G^+ be the graph defined by

$$\begin{aligned} V(G^+) &= V(G) \cup \{t^*\}, \\ E(G^+) &= E(G) \cup \{xt^* \mid x \in V(G-s) \text{ and } \gamma(x)t \in E(H)\}. \end{aligned} \tag{1}$$

Clearly, $G^+ - t^* = G$, so G^+ is a supercard of G. Now let ψ be the bijection from $V(G^+ - s)$ to $V(H)$ defined by $\psi(t^*) = t$ and $\psi(x) = \gamma(x)$ for all $x \in V(G^+ - s - t^*)$. Then $\psi(G^+ - s - t^*) = \gamma(G - s) = H - t$. In addition, $x \in N_{G^+-s}(t^*)$ if and only if $\gamma(x)t \in E(H)$ by (1), and thus $\psi(N_{G^+-s}(t^*)) = N_H(t)$. Hence $\psi(G^+ - s) = H$ by Lemma 2.1, and therefore G^+ is a supercard of H. □

If G^+ is constructed as in (1), then the graph $G^\dagger$ that consists of G^+ with an additional edge st^* is clearly also a supercard of G and H. Moreover, it is easy to see that *any* supercard of G and H can be constructed from (1) in one of these two ways, for some $s \in V(G)$, $t \in V(H)$ and some isomorphism γ.

Definition 3.4. Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$. The set of *active permutations* of G^+ with respect to v and w, denoted by $B_{vw}(G^+)$, is the subset of $S_{V(G^+)}$ defined by

$$\begin{aligned} B_{vw}(G^+) &= \{\lambda \in S_{V(G^+)} \mid \lambda((G^+ - w) - \lambda^{-1}(v)) = (G^+ - v) - \lambda(w)\} \\ &= \{\lambda \in S_{V(G^+)} \mid \lambda(\widehat{G} - \lambda^{-1}(v)) = \widehat{H} - \lambda(w)\}. \end{aligned} \tag{2}$$

We note that $1_{V(G^+)} \in B_{vw}(G^+)$, and that if $\lambda \in B_{vw}(G^+)$ then $\lambda(w) \neq v$, since G and H are not isomorphic.

Definition 3.5. Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$. A *maximum saturating set* of $B_{vw}(G^+)$ is a subset $X \subseteq B_{vw}(G^+)$ that satisfies the following three properties:

(a) $1_{V(G^+)} \in X$;
(b) if λ and π are distinct elements in X then $\lambda^{-1}(v) \neq \pi^{-1}(v)$ and $\lambda(w) \neq \pi(w)$;
(c) there is no σ in $B_{vw}(G^+) \setminus X$ such that $X \cup \{\sigma\}$ satisfies (b).

We note that, for any pair of distinct permutations λ and π in X, (b) guarantees that $G^+ - \lambda^{-1}(v) \neq G^+ - \pi^{-1}(v)$ and $G^+ - \lambda(w) \neq G^+ - \pi(w)$, although either pair of graphs could be isomorphic.

Although condition (c) only ensures that X is *maximal* with respect to (a) and (b), we shall show in Theorem 3.8 that all maximum saturating sets have the same cardinality. This implies that such sets are in fact of *maximum* cardinality with respect to (a) and (b).

In [6], a bipartite graph $B(G, H)$ was introduced to facilitate calculation of $b(G, H)$ when G and H are vertex-disjoint. We generalise that construction here.

Definition 3.6. Let G and H be non-isomorphic graphs of order n, and let V_G and V_H be two disjoint sets of n vertices. Label the elements of V_G and V_H so that the vertex $x_s \in V_G$ corresponds to the vertex s of G, and the vertex $y_t \in V_H$ corresponds to the vertex t of H. We define $B(G, H)$ to be the bipartite graph on $V_G \cup V_H$ such that $x_s y_t \in E(B(G, H))$ if and only if $G - s \cong H - t$.

Since $V_G \cap V_H = \emptyset$, it is easy to see that $b(G, H)$ is the size of any maximum matching in $B(G, H)$, as stated in [6].

We note that any vertex x_s of V_G is adjacent in $B(G, H)$ to *every* vertex y_t of V_H such that $G - s \cong H - t$, and conversely. It follows that every component of $B(G, H)$ must be a complete bipartite graph. Therefore, any *maximal* matching of $B(G, H)$ must be a *maximum* matching, i.e. each maximal matching of $B(G, H)$ has cardinality $b(G, H)$.

Lemma 3.7. *Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\theta(\widehat{G}) = \theta(G^+ - w) = G$ and $\psi(\widehat{H}) = \psi(G^+ - v) = H$, for some isomorphisms θ and ψ. Let $B(G, H)$ be the bipartite graph constructed as in Definition 3.6.*

(a) If $\lambda \in B_{vw}(G^+)$ then $x_{\theta\lambda^{-1}(v)} y_{\psi\lambda(w)}$ is an edge of $B(G, H)$.
(b) Any edge of $B(G, H)$ can be written as $x_{\theta\lambda^{-1}(v)} y_{\psi\lambda(w)}$ for some $\lambda \in B_{vw}(G^+)$.

Proof. (a) Suppose that $\lambda \in B_{vw}(G^+)$, so $\lambda(\widehat{G} - \lambda^{-1}(v)) = \widehat{H} - \lambda(w)$ by (2). Then $\psi(\lambda(\theta^{-1}(G - \theta\lambda^{-1}(v))) = H - \psi\lambda(w)$, hence $x_{\theta\lambda^{-1}(v)}y_{\psi\lambda(w)}$ is an edge of $B(G, H)$.

(b) Suppose now that $x_s y_t$ is an edge of $B(G, H)$, so there exists an isomorphism σ such that $\sigma(G - s) = H - t$. Define λ by $\lambda(\theta^{-1}(s)) = v$, $\lambda(w) = \psi^{-1}(t)$, and $\lambda(u) = \psi^{-1}\sigma\theta(u)$ for all other $u \in V(G^+)$, so $x_s y_t$ is $x_{\theta\lambda^{-1}(v)}y_{\psi\lambda(w)}$. It is straightforward to show that $\lambda \in S_{V(G^+)}$. Moreover, since $\theta(\widehat{G} - \lambda^{-1}(v)) = G - s$, it follows that $\lambda(\widehat{G} - \lambda^{-1}(v)) = \widehat{H} - \lambda(w)$. So $\lambda \in B_{vw}(G^+)$. □

This lemma implies that there is a many-to-one surjection from $B_{vw}(G^+)$ to the edges of $B(G, H)$. Moreover, it follows from the theorem below that the image of a maximum saturating set of $B_{vw}(G^+)$ is a maximum matching of $B(G, H)$.

Theorem 3.8. *Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\theta(\widehat{G}) = \theta(G^+ - w) = G$ and $\psi(\widehat{H}) = \psi(G^+ - v) = H$, for some isomorphisms θ and ψ. Let $Y \subseteq B_{vw}(G^+)$ satisfy properties (a) and (b) of Definition 3.5. Then*

- *(a) $M = \{x_{\theta\pi^{-1}(v)}y_{\psi\pi(w)} \mid \pi \in Y\}$ is a matching in $B(G, H)$, and $|M| = |Y| \leq b(G, H)$.*
- *(b) If Y is not a maximum saturating set of $B_{vw}(G^+)$ then $|Y| < b(G, H)$.*
- *(c) If $|Y| < b(G, H)$ then there is a maximum saturating set X such that $Y \subset X$ (so Y is not a maximum saturating set).*
- *(d) Y is a maximum saturating set of $B_{vw}(G^+)$ if and only if $|Y| = b(G, H)$.*

Proof. (a) By Corollary 3.7(a), M is a set of edges of $B(G, H)$. Moreover, since θ and ψ are isomorphisms, it follows from property (b) of Definition 3.5 that M is a matching and that $|M| = |Y|$. The result follows since $b(G, H)$ is the size of a maximum matching of $B(G, H)$.

(b) Suppose that Y is not a maximum saturating set. Then, by part (c) of Definition 3.5, there exists $\sigma \in B_{vw}(G^+) \setminus Y$ such that $Y \cup \{\sigma\}$ satisfies parts (a) and (b) of that definition. Since $|Y \cup \{\sigma\}| \leq b(G, H)$ by part (a), it follows that $|Y| < b(G, H)$.

(c) Suppose that $|Y| < b(G, H)$ and let M be defined as in (a). Then $|M| < b(G, H)$, so M is not a maximum matching of $B(G, H)$. Now, since each component of $B(G, H)$ is complete, we may construct a maximum matching M' by repeatedly adding non-adjacent edges to M. Furthermore, by Lemma 3.7, each edge of $M' \setminus M$ can be writen as $x_{\theta\lambda^{-1}(v)}y_{\psi\lambda(w)}$ for some $\lambda \in B_{vw}(G^+)$. Hence there exists some non-empty set $Z \subset B_{vw}(G^+)$ such that $M' \setminus M = \{x_{\theta\lambda^{-1}(v)} y_{\psi\lambda(w)} \mid \lambda \in Z\}$ and $|Z| = |M \setminus M'|$. Since M' is a matching and θ and ψ are isomorphisms, $\lambda^{-1}(v) \neq \pi^{-1}(v)$ and $\lambda(w) \neq \pi(w)$ for any distinct λ and π in $Y \cup Z$. Thus the set $X = Y \cup Z$ satisfies properties (a) and (b) of Definition 3.5. Clearly, $|X| = |M'| = b(G, H)$ as M' is a maximum matching of $B(G, H)$. It then follows from part (b) that X is a maximum saturating set.

(d) This follows immediately from (a) to (c). □

We frequently make use of the fact that every maximum saturating set of $B_{vw}(G^+)$ has cardinality $b(G, H)$ without quoting this theorem. We note that it follows from Theorem 3.8(c) that every maximal set satisfying properties (a) and (b) of Definition 3.5 is a maximum saturating set. This justifies our terminology in Definition 3.5.

In Sections 4 and 6, we show how to construct supercards of pairs of graphs with a large number of common cards relative to their order n. The constructions make use of the result of following lemma, that every automorphism of G^+ is an active permutation.

Lemma 3.9. *Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$. Then $\mathrm{Aut}(G^+) \subseteq B_{vw}(G^+)$.*

Proof. Let $\lambda \in \mathrm{Aut}(G^+)$. Now $\lambda(w) \neq v$ as $G \ncong H$, so $\lambda^{-1}(v) \in V(\widehat{G})$ and $v \in V(G^+ - \lambda(w))$. Thus $\lambda(\widehat{G} - \lambda^{-1}(v)) = (G^+ - \lambda(w)) - v = \widehat{H} - \lambda(w)$. Hence $\lambda \in B_{vw}(G^+)$. □

For any maximum saturating set X, we have the following bound on $|X \cap \mathrm{Aut}(G^+)|$.

Lemma 3.10. *Suppose that $b(G, H) \geq 1$. Let G^+ be a supercard of G and H, and let v and w be vertices of G^+ such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$. Let $A = \{\sigma(w) \mid \sigma \in \mathrm{Aut}(G^+)\}$ and let $B = \{\sigma(v) \mid \sigma \in \mathrm{Aut}(G^+)\}$. Then, for any maximum saturating set X,*

$$|X \cap \mathrm{Aut}(G^+)| \leq \min\{|A|, |B|\} \leq \tfrac{n+1}{2}. \tag{3}$$

Proof. Let $\lambda \in X \cap \mathrm{Aut}(G^+)$. Then $\lambda(w)$ and $\lambda^{-1}(v)$ are elements of A and B, respectively, as $\lambda^{-1} \in \mathrm{Aut}(G^+)$. So $|X \cap \mathrm{Aut}(G^+)| \leq \min\{|A|, |B|\}$ by part (b) of Definition 3.5. The second inequality holds since $A \cap B = \emptyset$ as $G \ncong H$. □

Corollary 3.11. *Suppose that the conditions of Lemma 3.10 hold and that there is some maximum saturating set X of $B_{vw}(G^+)$ such that $X \subseteq \mathrm{Aut}(G^+)$. Then*

(a) $b(G,\, H) \leq \min\{|A|,\, |B|\} \leq \frac{n+1}{2}$,
(b) if $b(G,\, H) = \frac{n+1}{2}$ *then* $|A| = |B| = \frac{n+1}{2}$.

4. Supercard Constructions

We now show how to construct directly several families of graphs G^+ such that $\mathcal{D}(G^+)$ contains non-isomorphic cards $G = G^+ - w$ and $H = G^+ - v$ for which there exists a maximum saturating set X of $B_{vw}(G^+)$ such that $X \subseteq \mathrm{Aut}(G^+)$ and $|X| = b(G,\, H) = \frac{n+1}{2}$.

If G_1 and G_2 are disjoint graphs then $G_1 + G_2$ denotes the disjoint union of G_1 and G_2, i.e., the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. If p and q are integers then we write $pG_1 + qG_2$ for a representative of the isomorphism class of the disjoint union that consist of p graphs isomorphic to G_1 and q graphs isomorphic to G_2. We note that G is vertex-transitive if and only if $G \cong kT$ for some connected vertex-transitive graph T.

The *join* of G_1 and G_2, denoted $G_1 \vee G_2$, is the graph $G_1 + G_2$ with additional edges joining every vertex of G_1 with every vertex of G_2. It is easy to see that $\overline{G_1 + G_2} = \overline{G_1} \vee \overline{G_2}$. It is also easy to see that $\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$, from which it follows that G is vertex-transitive if and only if $\overline{G}$ is vertex-transitive.

Lemma 4.1. *Let $G^+ = aT + bS$, where $a \geq 1$ and $b \geq 1$, and S and T are disjoint non-isomorphic connected vertex-transitive graphs, with the proviso that we do not have $S \cong K_{p-1}$ and $T \cong K_p$, for some $p \geq 2$, or vice versa. Let T_1 and S_1 be particular components of G^+ isomorphic to T and S, respectively. Let $w \in V(T_1)$ and $v \in V(S_1)$, and let*

$$G = G^+ - w = (T_1 - w) + (a-1)T + bS \qquad H = G^+ - v = aT + (S_1 - v) + (b-1)S.$$

Then $G \ncong H$ and $B_{vw}(G^+) = \mathrm{Aut}(G^+)$, so $X \subseteq \mathrm{Aut}(G^+)$ for any maximum saturating set X of $B_{vw}(G^+)$. Moreover,

$$b(G,\, H) = \min\{a|V(T)|,\, b|V(S)|\} \leq \tfrac{n+1}{2}.$$

Proof. Clearly, $G \cong H$ since $S \ncong T$. Let A and B be the subsets of $V(G^+)$ defined in Lemma 3.10. Since both T and S are vertex-transitive, clearly $A = V(aT)$ and $B = V(bS)$. In addition, there exists a set of automorphisms $\lambda_1,\, \lambda_2, \ldots,\, \lambda_q$ of G^+, where $q = \min\{a|V(T)|,\, b|V(S)|\}$, such that each

$\lambda_i(w)$ is a distinct element of A and each $\lambda_i^{-1}(v)$ is a distinct element of B, i.e., $\lambda_i(w) \neq \lambda_j(w)$ and $\lambda_i^{-1}(v) \neq \lambda_j^{-1}(v)$, when $i \neq j$. Without loss of generality, we may assume that $\lambda_1 = 1_{V(G^+)}$. So, by Theorem 3.8, there exists a maximum saturating set X of $B_{vw}(G^+)$ that contains each of these q automorphisms λ_i, so $b(G, H) \geq q$.

Without loss of generality, we may assume that $|V(T)| \geq |V(S)|$. Let $\lambda \in B_{vw}(G^+)$, so $\lambda(G - \lambda^{-1}(v)) = H - \lambda(w)$. Clearly, $\lambda(w) \in V(aT)$ as $|V(T)| \geq |V(S)|$. In addition, since T and S are regular, $T - t \ncong S$ for $t \in V(T)$ unless $T \cong K_p$ and $S \cong K_{p-1}$. Since this last case is excluded by hypothesis, it follows that $\lambda^{-1}(v) \in V(bS)$. Let T^* and S^* be the components of G^+ that contain $\lambda(w)$ and $\lambda^{-1}(v)$, respectively. By Corollary 2.4, $b(S, T) = 0$, so $\lambda(T_1 - w) = T^* - \lambda(w)$ and $\lambda(S^* - \lambda^{-1}(v)) = S_1 - v$. Thus $\lambda(T_1) = T^*$ and $\lambda(S^*) = S_1$ by Lemma 2.3. So

$$\lambda(N_G(\lambda^{-1}(v)) = \lambda(N_{S^*}(\lambda^{-1}(v)) = N_{S_1}(v) = N_{G^+-\lambda(w)}(v).$$

Thus $\lambda(G) = G^+ - \lambda(w)$ by Lemma 2.1. Similarly, $\lambda(N_{G^+}(w)) = N_{G^+}(\lambda(w))$, so $\lambda(G^+) = G^+$ by Corollary 2.2. Hence $B_{vw}(G^+) = \mathrm{Aut}(G^+)$. In particular, $X \subseteq \mathrm{Aut}(G^+)$, thus $b(G, H) = q = \min\{a|V(T)|, b|V(S)|\} \leq \frac{n+1}{2}$ by Corollary 3.11(a). $\square$

Corollary 4.2. *Let G^+ be defined as in Lemma 4.1 and let X be a maximum saturating set of $B_{vw}(G^+)$. If $a|V(T)| = b|V(S)|$ then $b(G, H) = |X| = \frac{n+1}{2}$.*

We now show how to extend the construction in Lemma 4.1 to connected supercards. In the following examples, we use a supercard $G^\dagger$ where v and w are adjacent (so that the symmetry of the supercard is easier to see). We recall that if A is vertex-transitive then $A \cong kU$ for some connected vertex-transitive graph U.

Corollary 4.3. *Let $G^\dagger = aT \vee bS$, where $a \geq 1$ and $b \geq 1$, and S and T are disjoint non-isomorphic connected vertex-transitive graphs, with the proviso that $G^\dagger$ is not vertex-transitive and the components of $\overline{G^\dagger}$ are not all isomorphic to either K_p or K_{p-1}, for some $p \geq 2$. Let T_1 and S_1 be particular components of aT and bS isomorphic to T and S, respectively. Let $w \in V(T_1)$ and $v \in V(S_1)$, and let*

$$G = G^\dagger - w = ((T_1 - w) + (a-1)T) \vee bS \quad H = G^\dagger - v = aT \vee ((S_1 - v) + (b-1)S).$$

Then $G \ncong H$, and $B_{vw}(G^\dagger) = \mathrm{Aut}(G^\dagger)$, so $X \subseteq \mathrm{Aut}(G^\dagger)$ for any maximum saturating set X of $B_{vw}(G^\dagger)$. Moreover,

$$b(G, H) = \min\{a|V(T)|, b|V(S)|\} \leq \tfrac{n+1}{2}.$$

Proof. Let $\overline{aT} = \alpha U$ and $\overline{bS} = \beta W$, where U and W are connected vertex-transitive graphs. (Clearly, if $a \geq 2$ then $\alpha = 1$ as in this case $\overline{aT}$ is connected, and similarly for b and β.) Then $\overline{G^\dagger} = \overline{aT} + \overline{bS} = \alpha U + \beta W$, and w is in some component U_1 isomorphic to U and v is in some component W_1 isomorphic to W. So $\overline{G} = (U_1 - w) + (\alpha - 1)U + \beta W$ and $\overline{H} = \alpha U + (W_1 - v) + (\beta - 1)W$. As $G^\dagger$ is not vertex-transitive, neither is $\overline{G^\dagger}$, so $U \not\cong W$. Therefore, by applying Lemma 4.1 to $\overline{G^\dagger}$, $\overline{G}$ and $\overline{H}$, it follows that $\overline{G} \not\cong \overline{H}$, (hence $G \not\cong H$) and $B_{vw}(\overline{G^\dagger}) = \mathrm{Aut}(\overline{G^\dagger})$. It is straightforward to prove that $\lambda(G - \lambda^{-1}(v)) = H - \lambda(w)$ if and only if $\lambda(\overline{G} - \lambda^{-1}(v)) = \overline{H} - \lambda(w)$ for any $\lambda \in S_{V(G^+)}$, so $B_{vw}(G^\dagger) = B_{vw}(\overline{G^\dagger}) = \mathrm{Aut}(\overline{G^\dagger})$. The result then follows as $\mathrm{Aut}(\overline{G^\dagger}) = \mathrm{Aut}(G^\dagger)$. As in Lemma 4.1, it is straightforward to show that $b(G, H) = \min\{a|V(T)|, b|V(S)|\} \leq \frac{n+1}{2}$. □

A more interesting extension is to add a perfect matching to G^+, where each edge of this matching is incident to a vertex of aT and a vertex of bS. We shall make use of the following result that was proved in Lemma 3.3 of [5].

Lemma 4.4. *Let A be a transitive permutation group on the set R, and let t be in R. Then there exists a set of $|R|$ distinct permutations $\{\alpha_u \mid u \in R\} \subseteq A$ such that, for every pair of distinct elements u and v in R,*

(a) $\alpha_u(u) = t$;
(b) $\alpha_u(t) \neq \alpha_v(t)$.

We note that, if we replace α_t by 1_A, (a) and (b) still hold.

Lemma 4.5. *Let T and T^* be disjoint isomorphic connected vertex-transitive graphs of order p, $p \geq 2$, and let ϕ be an isomorphism such that $\phi(T) = T^*$. Suppose that T and $\overline{T}$ have different degrees, and let $G^\dagger$ be the graph constructed from $T + \overline{T^*}$ by adding the perfect matching that joins each vertex u of T to the corresponding vertex $\phi(u)$ of $\overline{T^*}$. Let $w \in V(T)$, $G = G^\dagger - w$, $v = \phi(w) \in V(T^*)$, and $H = G^\dagger - v$. Then $G \not\cong H$, and, moreover, there exists a maximum saturating set X of $B_{vw}(G^\dagger)$ such that $X \subseteq \mathrm{Aut}(G^+)$ and $b(G, H) = |X| = \frac{n+1}{2}$.*

Proof. Clearly, $G \not\cong H$ as T and $\overline{T}$ have different degrees. Now, since T is vertex-transitive, it follows from Lemma 4.4 applied to $\mathrm{Aut}(T)$ and w, that there exists $Y = \{\lambda_i \mid 1 \leq i \leq p\} \subseteq \mathrm{Aut}(T)$ such that $\lambda_1 = 1_{V(T)}$, and $\lambda_i(w) \neq \lambda_j(w)$ and $\lambda_i^{-1}(w) \neq \lambda_j^{-1}(w)$, when $i \neq j$. For each $\lambda_i \in Y$, we define $\widehat{\lambda}_i \in S_{V(G^\dagger)}$ by $\widehat{\lambda}_i(x) = \lambda_i(x)$ for $x \in V(T)$ and $\widehat{\lambda}_i(x) = \phi(\lambda_i(\phi^{-1}(x)))$ for $x \in V(\overline{T^*})$. Clearly, $\widehat{\lambda}_i(w) = \lambda_i(w)$ and

$$\widehat{\lambda}_i^{-1}(v) = \phi(\lambda_i^{-1}(\phi^{-1}(v))) = \phi(\lambda_i^{-1}(w)).$$

In addition, it is easy to show that each $\widehat{\lambda}_i \in \mathrm{Aut}(G^\dagger)$. So, since $\widehat{\lambda}_1 = 1_{V(G^\dagger)}$ and ϕ is an isomorphism, it follows that the set $\widehat{Y} = \{\widehat{\lambda}_i \mid \lambda_i \in Y\}$ is a subset of $\mathrm{Aut}(G^\dagger)$ satisfying properties (a) and (b) of Definition 3.5. Hence, by Theorem 3.8, there exists a maximum saturating set X of $B_{vw}(G^\dagger)$ that contains $\widehat{Y}$. Since $|\widehat{Y}| = p$, the result will then follow if we show that $|X| \leq p = \frac{n+1}{2}$.

Suppose that $d_T(w) > d_{\overline{T^*}}(v)$, and let $d_T(w) = k$, so $d_{\overline{T^*}}(v) = p - k - 1$. Since the case when T is complete is dealt with in Lemma 4.7 below, we may assume that $1 < k < p-1$. Let $\lambda \in X$. Now, H contains $p-1$ vertices of degree $k+1$, whereas G contains only $p-k-1$ such vertices. So, since $\lambda(G - \lambda^{-1}(v)) = H - \lambda(w)$ and $k > 1$, clearly $\lambda(w) \in V(T)$, i.e., there exist at most p distinct choices for $\lambda(w)$. Thus, $|X| \leq |V(T)| \leq p$ by property (b) of Definition 3.5. Therefore, $b(G, H) = |X| = p = \frac{n+1}{2}$ in this case. The case when $d_T(w) < d_{\overline{T^*}}(v)$ can be proved in a similar manner, by showing that $\lambda^{-1}(v) \in V(\overline{T^*})$. $\square$

Example 4.6. Let T and T^* be isomorphic to $K_3 \times K_2$, i.e., the triangular prism on 6 vertices. Then $\overline{T^*} \cong C_6$. We construct $G^\dagger$, G and H as in Lemma 4.5. So there exists a maximum saturating set X such that $X \subseteq \mathrm{Aut}(G^+)$ and $b(G, H) = |X| = 6$.

The perfect matching construction of Lemma 4.5 can also be used in some cases when $T^* \ncong T$; for example, if T^* is isomorphic to either K_p or pK_1. Indeed, as shown in the following lemma, when $T^* \cong K_p$, we obtain the "super-family" of pairs of graphs that attain the bound of $b(G, H) = \frac{n+1}{2}$ when H is connected and G is disconnected [5] [6].

Lemma 4.7. *Let T be a connected vertex transitive graph of order p, $p \geq 2$, and let $G^\dagger$ be the graph constructed from T and p isolated vertices by adding a perfect matching such that each edge of this matching is incident to a vertex of T and an isolated vertex. Let $w \in V(T)$, and let v be the leaf of $G^\dagger$ adjacent to w. Let $G = G^\dagger - w$ and $H = G^\dagger - v$. Then $G \ncong H$ and there exists a maximum saturating set X of $B_{vw}(G^\dagger)$ such that $X \subseteq \mathrm{Aut}(G^\dagger)$ and $b(G, H) = |X| = \frac{n+1}{2}$. Moreover, $B_{vw}(G^\dagger) = \mathrm{Aut}(G^\dagger)$ when $d_T(w) \geq 3$.*

Proof. By construction, G is disconnected and H is connected, so $G \ncong H$. We note that, this is the same pair as in Theorem 3.6 of [6]. It was shown there, using Lemma 4.4, that $b(G, H) = \frac{n+1}{2}$.

Now, when $d_T(w) = 1$, clearly $G^\dagger \cong P_4$, and the two distinct automorphisms of $G^\dagger$ form a maximum saturating set of $B_{vw}(G^\dagger)$. In the case when $d_T(w) = 2$ we have that $T \cong C_p$, and it is easy to see that there is a unique maximum saturating set $X \subseteq \mathrm{Aut}(G^\dagger)$ that is isomorphic to the cyclic group of order p. We now show that $B_{vw}(G^\dagger) = \mathrm{Aut}(G^\dagger)$ when $d_T(w) \geq 3$. We note that, in this case, every vertex of $G^\dagger$ is either a leaf, or is of degree $d_T(w) + 1 \geq 4$ and is adjacent to precisely one leaf.

Let $\lambda \in B_{vw}(G^\dagger)$, so $\lambda(G - \lambda^{-1}(v)) = H - \lambda(w)$. By counting the number of edges in $G - \lambda^{-1}(v)$ and $H - \lambda(w)$, it immediately follows that $d_{G^\dagger}(\lambda(w)) = d_T(w) + 1$ and $d_{G^\dagger}(\lambda^{-1}(v)) = 1$. Let x be the unique vertex of $G^\dagger$ adjacent to $\lambda^{-1}(v)$. It is easy to see that x is w if and only $\lambda(w) = w$. So suppose that x is not w. Now, if w is adjacent to x then $G - \lambda^{-1}(v)$ contains a unique vertex, i.e. x, of degree $d_T(w) - 1$. Otherwise $G - \lambda^{-1}(v)$ does not contain such a vertex, and instead contains a unique vertex, i.e. x, of degree $d_T(w)$ that is not adjacent to a leaf. Similar observations hold for w and $H - \lambda(w)$ depending on whether or not $\lambda(w)$ is adjacent to w. Hence $\lambda(x) = w$ in all cases, and thus $\lambda(N_G(\lambda^{-1}(v)) = N_{G^\dagger - \lambda(w)}(v)$. So $\lambda(G) = G^\dagger - \lambda(w)$ by Lemma 2.1.

Let v^* be the leaf adjacent to $\lambda(w)$. Now, v is the unique isolated vertex of G, and $N_{G^\dagger}(w) \setminus \{v\}$ consists of all the vertices of G of degree $d_T(w)$. Similarly, v^* is the unique isolated vertex of $G^\dagger - \lambda(w)$, and $N_{G^\dagger}(\lambda(w)) \setminus \{v^*\}$ consists of all the vertices of $G^\dagger - \lambda(w)$ of degree $d_T(w)$. It immediately follows that $\lambda(N_{G^\dagger}(w)) = N_{G^\dagger}(\lambda(w))$ as $\lambda(G) = G^\dagger - \lambda(w)$. Therefore, $\lambda(G^\dagger) = G^\dagger$ by Corollary 2.2, so $B_{vw}(G^\dagger) = \mathrm{Aut}(G^\dagger)$. $\square$

The supercard $G^\dagger$ from Lemma 4.7 when T is the Petersen Graph is shown in Figure 1.

We conclude this section with an example from Theorem 3.6 of [5], namely a caterpillar G and a sunshine graph H for which $b(G, H) = \frac{2(n+1)}{5}$. We show that this pair can be constructed by adding a set of $2p$ edges to the graph $C_{3p} + 2pK_1$.

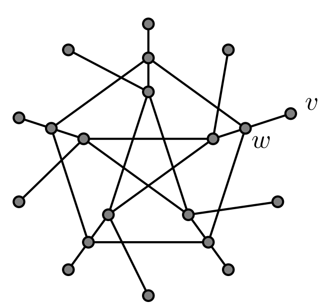

Figure 1: The supercard $G^\dagger$ from Lemma 4.7 when T is the Petersen Graph

Example 4.8. Let $T = C_{3p}$ with vertices $w_0,\, w_1,\, \ldots w_{3p-1}$, and let $v_0,\, v_1,\, \ldots, v_{2p-1}$ be the vertices of $S = 2pK_1$. Let G^+ be the graph constructed from $T+S$ by adding the edge set $\{w_{3i}v_{2i},\, w_{3i}v_{2i+1} \mid 0 \le i \le p-1\}$. Let $w = w_1$, $v = v_1$, and let $G = G^+ - w$ and $H = G^+ - v$. Then G and H are the sunshine-caterpillar pair described in Theorem 3.6 of [5]. We may construct a maximum saturating set X of $B_{vw}(G^+)$ such that $X \subseteq \mathrm{Aut}(G^+)$ and, moreover, the restriction of X to the cycle T is generated by the rotation ϕ defined by $\phi(w_i) = w_{i+3 \pmod 3}$, together with any reflection of the cycle. We note that $X \cong D(2p)$, the Dihedral Group of order $2p$.

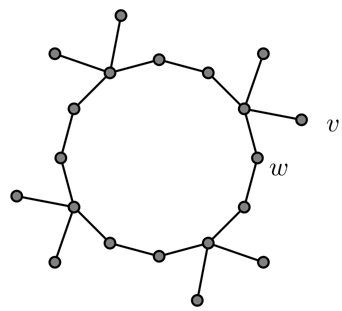

Figure 2: The supercard G^+ from Example 4.8 when $p = 4$

A similar supercard can be constructed from the graph $C_{2p} + pK_1$, for $p \ge 3$, with a matching between alternate vertices on the cycle and the isolated vertices. In this case, there exists a maximum saturating set X such that $|X \setminus \mathrm{Aut}(G^+)| = 2$, where $X \cap \mathrm{Aut}(G^+)$ is isomorphic to the cyclic group $C(p)$. This pair was described by Francalanza in [8] and has $b(G,\, H) = \frac{n+7}{3}$. In a forthcoming paper [6], we shall show that all sunshine-caterpillar pairs that have a large number of common cards can be obtained from this type of construction. Moreover, we can always find a maximum saturating set X such that $X \cap \mathrm{Aut}(G^+)$ is isomorphic to either a cyclic or dihedral group, and $|X \setminus \mathrm{Aut}(G^+)| \le 2$.

5. Two supercards

In Section 4, we presented supercards for which there exist maximum saturating sets X such that $X \subseteq \mathrm{Aut}(G^+)$, and thus $b(G,\, H) \le \frac{n+1}{2}$ by Corollary 3.11. It was shown in [5] that there are infinite families of pairs of graphs for which $b(G,\, H) \approx \frac{2n}{3}$. In this and similar examples, there must exist maximum saturating sets X such that $|X| > \frac{n+1}{2} \ge |X \cap \mathrm{Aut}(G^+)|$. In these cases, we need to consider more than one supercard of G and H.

For the whole of this section, we assume that $b(G, H) \geq 1$, and that G^+ is a supercard of G and H such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$ for distinct vertices v and w of $V(G^+)$.

Definition 5.1. Let $\lambda \in B_{vw}(G^+)$. We define G^+_λ to be the graph $\widehat{G} \cup \lambda^{-1}(\widehat{H})$, i.e., the graph with $V(G^+_\lambda) = V(G^+)$ and

(4) $\quad E(G^+_\lambda) = E(\widehat{G}) \cup \{xw \,:\, x \in V(\widehat{G} - \lambda^{-1}(v)) \text{ and } \lambda(x)\lambda(w) \in E(\widehat{H})\}.$

We further define $\widehat{H}_\lambda = G^+_\lambda - \lambda^{-1}(v)$. Clearly, $G^+_\lambda = G^+$ when $\lambda = 1_{V(G^+)}$.

We note that, since $V(G^+) = V(G^+_\lambda)$, we will consider any element $\pi \in S_{V(G^+)}$ to be a permutation of $V(G^+_\lambda)$ or a bijection between $V(G^+_\lambda)$ and $V(G^+)$, as necessary.

Lemma 5.2. *Let $\lambda \in B_{vw}(G^+)$ and let G^+_λ be the graph defined in Definition 5.1. Then $G^+_\lambda - w = \widehat{G} \cong G$ and $\lambda(G^+_\lambda - \lambda^{-1}(v)) = \lambda(\widehat{H}_\lambda) = \widehat{H} \cong H$, so G^+_λ is a supercard of G and H.*

Proof. $G^+_\lambda - w = \widehat{G}$ by construction. Clearly,

$$\lambda((G^+_\lambda - \lambda^{-1}(v)) - w) = \lambda(\widehat{G} - \lambda^{-1}(v)) = \widehat{H} - \lambda(w),$$

as $\lambda \in B_{vw}(G^+)$. In addition, $\lambda(N_{G^+_\lambda - \lambda^{-1}(v)}(w)) = N_{\widehat{H}}(\lambda(w))$ by (4). Therefore $\lambda(G^+_\lambda - \lambda^{-1}(v)) = \widehat{H}$ by Lemma 2.1, so G^+_λ is a supercard of G and H. □

We now define the set of active permutations of G^+_λ with respect to $\lambda^{-1}(v)$ and w: we replace λ by σ, v by $\lambda^{-1}(v)$, G^+ by G^+_λ and $\widehat{H}$ by $\widehat{H}_\lambda$ in Definition 3.4, to obtain:

$$\begin{aligned} &B_{\lambda^{-1}(v)w}(G^+_\lambda) \\ &\quad= \{\sigma \in S_{V(G^+_\lambda)} \mid \sigma((G^+_\lambda - w) - \sigma^{-1}(\lambda^{-1}(v)) = (G^+_\lambda - \lambda^{-1}(v)) - \sigma(w)\} \\ &\quad= \{\sigma \in S_{V(G^+_\lambda)} \mid \sigma(\widehat{G} - \sigma^{-1}(\lambda^{-1}(v))) = \widehat{H}_\lambda - \sigma(w)\}. \end{aligned}$$

Lemma 5.3. *Let $\lambda \in B_{vw}(G^+)$ and let G^+_λ be the graph defined in Definition 5.1. Then $\lambda^{-1}(B_{vw}(G^+)) = B_{\lambda^{-1}(v)w}(G^+_\lambda)$. Moreover, X is a maximum saturating set of $B_{vw}(G^+)$ that contains λ if and only if $\lambda^{-1}(X)$ is a maximum saturating set of $B_{\lambda^{-1}(v)w}(G^+_\lambda)$ that contains λ^{-1}.*

Proof. By (2), $\pi \in B_{vw}(G^+)$ if and only if $\pi(\widehat{G} - \pi^{-1}(v)) = \widehat{H} - \pi(w)$. This holds if and only if $\lambda^{-1}\pi(\widehat{G} - (\lambda^{-1}\pi)^{-1}\lambda^{-1}(v)) = \widehat{H}_\lambda - \lambda^{-1}\pi(w)$ since $\lambda^{-1}(\widehat{H}) = \widehat{H}_\lambda$, i.e., if and only if $\lambda^{-1}\pi \in B_{\lambda^{-1}(v)w}(G^+_\lambda)$. Therefore $\lambda^{-1}(B_{vw}(G^+)) = B_{\lambda^{-1}(v)w}(G^+_\lambda)$.

Let X be a maximum saturating set of $B_{vw}(G^+)$ that contains λ. Clearly, $\lambda^{-1}(X)$ contains the identity and λ^{-1}. In addition, for each distinct $\pi, \sigma \in X$, it follows from Definition 3.5 that $(\lambda^{-1}\pi)^{-1}\lambda^{-1}(v) \neq (\lambda^{-1}\sigma)^{-1}\lambda^{-1}(v)$ and $\lambda^{-1}\pi(w) \neq \lambda^{-1}\sigma(w)$. So $\lambda^{-1}(X)$ satisfies properties (a) and (b) of Definition 3.5 with respect to $B_{\lambda^{-1}(v)w}(G^+_\lambda)$. So, since $|\lambda^{-1}(X)| = |X| = b(G, H)$, it follows from Theorem 3.8(d) that $\lambda^{-1}(X)$ is a maximum saturating set of $B_{\lambda^{-1}(v)w}(G^+_\lambda)$. The converse implication can be proved in a similar fashion. $\square$

We now show how to find elements of $\lambda^{-1}(X)$ contained in $\mathrm{Aut}(G^+_\lambda)$.

Lemma 5.4. *Let λ and π be distinct permutations in $B_{vw}(G^+)$, and let G^+_λ be the graph defined in Definition 5.1. Suppose that $\pi(N_{\widehat{G}}(\pi^{-1}(v))) = \lambda(N_{\widehat{G}}(\lambda^{-1}(v)))$. Then $\lambda^{-1}\pi(\widehat{G}) = G^+_\lambda - \lambda^{-1}\pi(w)$.*

Proof. Since $\lambda^{-1}\pi \in B_{\lambda^{-1}(v)w}(G^+_\lambda)$ by Lemma 5.3, it follows that $\lambda^{-1}\pi$ is a bijection from $V(\widehat{G})$ to $V(G^+_\lambda - \lambda^{-1}\pi(w))$, such that

$$\lambda^{-1}\pi(\widehat{G} - \pi^{-1}(v)) = (G^+_\lambda - \lambda^{-1}\pi(w)) - \lambda^{-1}(v).$$

Now $\pi(N_{\widehat{G}}(\pi^{-1}(v))) = \lambda(N_{\widehat{G}}(\lambda^{-1}(v)))$, so $\pi(w) \notin \lambda(N_{\widehat{G}}(\lambda^{-1}(v)))$, and therefore $\lambda^{-1}\pi(w) \notin N_{\widehat{G}}(\lambda^{-1}(v))$. So, since by construction $\lambda^{-1}(v)w$ is not an edge of G^+_λ,

$$\begin{aligned} \lambda^{-1}\pi(N_{\widehat{G}}(\pi^{-1}(v))) = N_{\widehat{G}}(\lambda^{-1}(v)) &= N_{\widehat{G}-\lambda^{-1}\pi(w)}(\lambda^{-1}(v)) \\ &= N_{G^+_\lambda-\lambda^{-1}\pi(w)}(\lambda^{-1}(v)). \end{aligned}$$

Thus $\lambda^{-1}\pi(\widehat{G}) = G^+_\lambda - \lambda^{-1}\pi(w)$ by Lemma 2.1. $\square$

Corollary 5.5. *Suppose that X is a maximum saturating set of $B_{vw}(G^+)$ that contains two permutations λ and π satisfying the conditions of Lemma 5.4. If $\lambda^{-1}\pi(N_{G^+_\lambda}(w)) = N_{G^+_\lambda}(\lambda^{-1}\pi(w))$ then $\lambda^{-1}\pi \in \lambda^{-1}(X) \cap \mathrm{Aut}(G^+_\lambda)$.*

Proof. $\lambda^{-1}\pi(\widehat{G}) = G^+_\lambda - \lambda^{-1}(\pi(w))$ by Lemma 5.4. So, if $\lambda^{-1}\pi(N_{G^+_\lambda}(w)) = N_{G^+_\lambda}(\lambda^{-1}\pi(w))$ then $\lambda^{-1}\pi \in \mathrm{Aut}(G^+_\lambda)$ by Corollary 2.2. $\square$

6. Two supercard constructions

In this final section, we construct a number of families of graphs G^+ such that $\mathcal{D}(G^+)$ contains non-isomorphic cards $G = G^+ - w$ and $H = G^+ - v$ for which there is no maximum saturating set of $B_{vw}(G^+)$ contained in $\mathrm{Aut}(G^+)$.

However, there does exist a maximum saturating set X of $B_{vw}(G^+)$ and some permutation $\lambda \in X \setminus \mathrm{Aut}(G^+)$ such that, if G^+_λ is the supercard of G and H given in Definition 5.1 (where $\widehat{G} = G$ and $\widehat{H} = H$) then

$$\lambda^{-1}(X \setminus \mathrm{Aut}(G^+)) = \lambda^{-1}(X) \cap \mathrm{Aut}(G^+_\lambda);$$

i.e., X consists of $|X \cap \mathrm{Aut}(G^+)|$ automorphisms of G^+ and $|X \setminus \mathrm{Aut}(G^+)|$ permutations that each correspond to a different automorphism of G^+_λ.

For convenience, we define $X_{\mathrm{Aut}} = X \cap \mathrm{Aut}(G^+)$, for any maximum saturating set X of $B_{vw}(G^+)$. Since $\lambda^{-1}(X)$ is a maximum saturating set of $B_{\lambda^{-1}(v)w}(G^+_\lambda)$ by Lemma 5.3, we further define

$$\lambda^{-1}(X)_{\mathrm{Aut}} = \lambda^{-1}(X) \cap \mathrm{Aut}(G^+_\lambda).$$

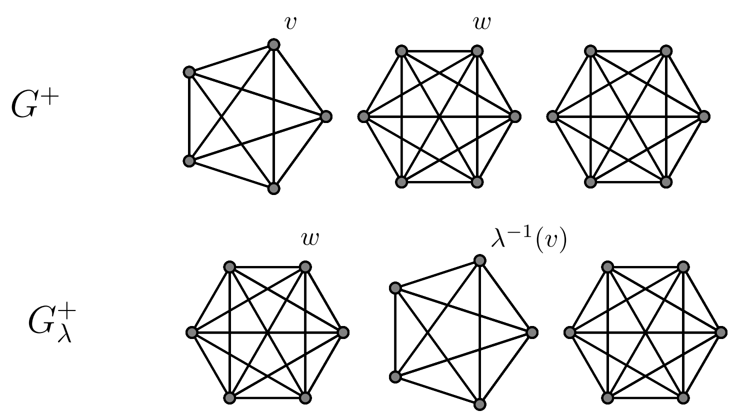

Figure 3: The supercards G^+ and G^+_λ from Lemma 6.1 when $p = 5$

Lemma 6.1. *For $p \geq 3$, let S be a graph isomorphic to K_p, and let T and T^* be graphs isomorphic to K_{p+1}. Let*

$$G^+ = S + T + T^* \cong K_p + 2K_{p+1}.$$

Let $w \in V(T)$, $v \in V(S)$, and let

$$\begin{aligned} G &= G^+ - w = S + (T - w) + T^* \cong 2K_p + K_{p+1} \\ H &= G^+ - v = (S - v) + T + T^* \cong K_{p-1} + 2K_{p+1}. \end{aligned}$$

Then $G \not\cong H$ and there exists a maximum saturating set X of $B_{vw}(G^+)$ and some $\lambda \in X \setminus X_{\mathrm{Aut}}$, where $\lambda^{-1}(v) \in V(T - w)$, such that the supercard G^+_λ

given in Definition 5.1 is

$$G^+_\lambda = (w \vee S) + (T - w) + T^* \cong G^+.$$

In addition, $\lambda^{-1}(X \setminus X_{\mathrm{Aut}}) = \lambda^{-1}(X)_{\mathrm{Aut}}$*,* $|X_{\mathrm{Aut}}| = |\lambda^{-1}(X)_{\mathrm{Aut}}| = p$ *and* $b(G,\, H) = \frac{2(n-1)}{3}$*.*

Proof. Clearly, $G \ncong H$. Let $\lambda \in B_{vw}(G^+)$, so $\lambda(G - \lambda^{-1}(v)) = H - \lambda(w)$. By comparing the components of G and H, it is easy to see that $\lambda(w) \in V(T) \cup V(T^*)$ and $\lambda^{-1}(v) \in V(S) \cup V(T - w)$. Furthermore, since each component of G^+ is complete, there are $2(p+1)$ distinct choices for $\lambda(w)$ and $2p$ distinct choices for $\lambda^{-1}(v)$, and thus $b(G,\, H) = 2p$. Now, if $\lambda \in \mathrm{Aut}(G^+)$ then $\lambda^{-1}(v) \in V(S)$, so $|X_{\mathrm{Aut}}| \le p$ for any maximum saturating set X of $B_{vw}(G^+)$. However, it is easy to see that there exists such a set X for which $|X_{\mathrm{Aut}}| = |X \setminus X_{\mathrm{Aut}}| = p$. Moreover, $\sigma(w) \in V(T)$ for all $\sigma \in X_{\mathrm{Aut}}$, and $\pi(w) \in V(T^*)$ and $\pi^{-1}(v) \in V(T - w)$ for all $\pi \in X \setminus X_{\mathrm{Aut}}$. We note that $\pi(S) = T^* - \pi(w)$, $\pi((T - w) - \pi^{-1}(v)) = S - v$ and $\pi(T^*) = T$, for all such π.

Now let $\lambda \in X \setminus X_{\mathrm{Aut}}$ and let G^+_λ be the supercard given in Definition 5.1. Then, since $\lambda(w)$ is adjacent to every other vertex of T^* in H, and $\lambda^{-1}(v) \in V(T - w)$, it follows that w is adjacent in G^+_λ to every vertex of S. So $G^+_\lambda = (w \vee S) + (T - w) + T^* \cong G^+$. As $|V(T - w)| = p$, it is again easy to show that $|Y_{\mathrm{Aut}}| \le p$ for any maximum saturating set Y of G^+_λ; so $|\lambda^{-1}(X)_{\mathrm{Aut}}| \le p$.

Suppose now that π is a permutation in $X \setminus X_{\mathrm{Aut}}$ distinct from λ. As shown above, $\pi^{-1}(v) \in V(T - w)$ and $\pi((T - w) - \pi^{-1}(v)) = S - v$. Since $T - w$ and $S - v$ are both complete, clearly

$$\pi(N_G(\pi^{-1}(v))) = \lambda(N_G(\lambda^{-1}(v)) = V(S - v).$$

Hence, $\lambda^{-1}\pi(G) = G^+_\lambda - \lambda^{-1}\pi(w)$ by Lemma 5.4. It is easy to see that $\lambda^{-1}\pi(S) = (w \vee S) - \lambda^{-1}\pi(w)$ under this isomorphism. So, since $N_{G^+_\lambda}(w) = V(S)$ and $N_{G^+_\lambda}(\lambda^{-1}\pi(w)) = V((w \vee S) - \lambda^{-1}\pi(w))$, it follows that $\lambda^{-1}\pi(N_{G^+_\lambda}(w)) = N_{G^+_\lambda}(\lambda^{-1}\pi(w))$, and hence $\lambda^{-1}\pi \in \lambda^{-1}(X)_{\mathrm{Aut}}$ by Corollary 5.5. As $|X \setminus X_{\mathrm{Aut}}| = p$ and $|\lambda^{-1}(X)_{\mathrm{Aut}}| \le p$, it immediately follows that $\lambda^{-1}(X \setminus X_{\mathrm{Aut}}) = \lambda^{-1}(X)_{\mathrm{Aut}}$, hence $|\lambda^{-1}(X)_{\mathrm{Aut}}| = p$. □

We note that the infinite family in Lemma 6.1 has the largest value of $b(G,\, H)$ yet published and is conjectured to have the largest possible value of $b(G,\, H)$ for large n [5]. The case when $p = 5$ is illustrated in Figure 3.

In Lemma 6.1, the two supercards are isomorphic. The following lemma gives a construction where this is rarely the case.

Lemma 6.2. *Let S and T be vertex-transitive graphs such that $S \vee T$ is not vertex-transitive. Let S_1 and S_2 be disjoint graphs isomorphic to S, let T_1 and T_2 be disjoint graphs isomorphic to T, and let $s \in V(S_2)$. Now define*

$$G^+ = (S_1 \vee T_1) + ((S_2 - s) \vee T_2) \cong (S \vee T) + ((S - x) \vee T),$$

for any $x \in V(S)$. Let $w \in V(T_1)$ and $v \in V(T_2)$, and let

$$\begin{aligned} G = G^+ - w &= (S_1 \vee (T_1 - w)) + ((S_2 - s) \vee T_2) \\ &\cong (S \vee (T - y)) + ((S - x) \vee T \\ H = G^+ - v &= (S_1 \vee T_1) + ((S_2 - s) \vee (T_2 - v)) \\ &\cong (S \vee T) + ((S - x) \vee (T - y)), \end{aligned}$$

for any $y \in V(T)$. Define S_2^* to be S_2 but with s relabelled as w. Then $G \ncong H$ and there exists a maximum saturating set X of $B_{vw}(G^+)$ and some $\lambda \in X \setminus X_{\mathrm{Aut}}$, where $\lambda^{-1}(v) \in V(S_1)$, such that the supercard G_λ^+ given in Definition 5.1 is

$$G_\lambda^+ = (S_1 \vee (T_1 - w)) + (S_2^* \vee T_2) \cong (S \vee (T - y)) + (S \vee T).$$

In addition, $\lambda^{-1}(X \setminus X_{\mathrm{Aut}}) = \lambda^{-1}(X)_{\mathrm{Aut}}$, $|X_{\mathrm{Aut}}| = |V(T)|$, $|\lambda^{-1}(X)_{\mathrm{Aut}}| = |V(S)|$ and $b(G, H) = \frac{n}{2} + 1$. (We note that G^+ and G_λ^+ are only isomorphic when $(S - x) \vee T \cong S \vee (T - y)$).

Proof. By comparing the components of G and H, clearly $G \ncong H$ and $\lambda(w) \in V(S_1) \cup V(T_1)$ for all $\lambda \in B_{vw}(G^+)$. So $b(G, H) \leq |V(S)| + |V(T)|$, i.e., $b(G, H) \leq \frac{n}{2} + 1$.

Suppose that $\lambda \in \mathrm{Aut}(G^+)$. Then w and $\lambda(w)$ must be in isomorphic components of $\overline{S_1} + \overline{T_1}$ as $\lambda(\overline{S_1} + \overline{T_1}) = \lambda(\overline{S_1 \vee T_1}) = \overline{S_1 \vee T_1} = \overline{S_1} + \overline{T_1}$. Since $S \vee T$ is not vertex-transitive, nor is $\overline{S_1} + \overline{T_1}$, which implies that $\lambda(w) \in V(T_1)$. It therefore follows that $|X_{\mathrm{Aut}}| \leq |V(T)|$ for all maximum saturating sets X of $B_{vw}(G^+)$. Indeed, since both S and T are vertex-transitive, it is easy to show that there exists a maximum saturating set X of $B_{vw}(G^+)$ such that $|X_{\mathrm{Aut}}| = |V(T)|$ and $|X \setminus X_{\mathrm{Aut}}| = |V(S)|$, so $b(G, H) = |V(S)| + |V(T)|$, i.e., $b(G, H) = \frac{n}{2} + 1$. Furthermore, $\sigma^{-1}(v) \in V(T_2)$ for all $\sigma \in X_{\mathrm{Aut}}$, and $\pi(w) \in V(S_1)$ and $\pi^{-1}(v) \in V(S_1)$ for all $\pi \in X \setminus X_{\mathrm{Aut}}$. In addition, for all such π, we have $\pi((S_1 - \pi^{-1}(v)) \vee (T_1 - w)) = (S_2 - s) \vee (T_2 - v)$ and $\pi((S_2 - s) \vee T_2) = (S_1 - \pi(w)) \vee T_1$, where, without loss of generality, we may assume that $\pi(V(T_1 - w)) = V(T_2 - v)$ and $\pi(V(T_2)) = V(T_1)$.

Now let $\lambda \in X \setminus X_{\mathrm{Aut}}$, and let G_λ^+ be the supercard given in Definition 5.1. Then, since $\lambda(w)$ is in $V(S_1)$ and is adjacent to every vertex of T_1

in H, $\lambda(V(S_2 - s)) = V(S_1 - \lambda(w))$ and $\lambda^{-1}(v) \notin V(T_2)$, it follows that w is adjacent in G_λ^+ to every vertex of T_2 and the same vertices of S_2 as s. So $G_\lambda^+ = (S_1 \vee (T_1 - w)) + (S_2^* \vee T_2)$. Since $w \in V(S_2^*)$, it is easy to show in a similar manner to the proof above, that $|Y_{\mathrm{Aut}}| \leq |V(S)|$ for any maximum saturating set Y of G_λ^+; so $|\lambda^{-1}(X)_{\mathrm{Aut}}| \leq |V(S)|$. Furthermore, as in the proof of Lemma 6.1, i.e., using Lemma 5.4 and Corollary 5.5, it is straightfoward to prove that $\lambda^{-1}\pi \in \lambda^{-1}(X)_{\mathrm{Aut}}$ for each $\pi \in X \setminus X_{\mathrm{Aut}}$. Hence $\lambda^{-1}(X \setminus X_{\mathrm{Aut}}) = \lambda^{-1}(X)_{\mathrm{Aut}}$, and therefore $|\lambda^{-1}(X)_{\mathrm{Aut}}| = |V(S)|$. □

The case when $S \vee T$ is vertex-transitive can be easily shown to fit into the two supercard paradigm. An example of this is $K_{p+1} + K_p$. This gives rise to the pair $G \cong K_p + K_p$ and $H = K_{p+1} + K_{p-1}$, for which $b(G, H) = \frac{n}{2} + 1$. This pair was first reported by Harary and Manvel [10].

Corollary 6.3. *Let G^+ be as in Lemma 6.2, and let G^* be the graph obtained by G^+ by adding additional edges between $V(T_1)$ and $V(T_2)$, $V(S_1)$ and $V(S_2)$, or both. The edges added may either be the join between $V(T_1)$ and $V(T_2)$, a matching between $V(T_1)$ and $V(T_2)$ in a similar manner to Lemma 4.5, or the complement of such a matching. Edges between $V(S_1)$ and $V(S_2)$ may be added independently in a similar manner. Corresponding conclusions to those in Lemma 6.2 then hold for G^* and G_λ^*, where G_λ^* is constructed from G_λ^+ by adding the corresponding edges.*

The following example of the construction in Corollary 6.3 is the family of disconnected and connected pairs of graphs of even order that attain the upper bound of $b(G, H) = \frac{n}{2} + 1$ [5] [6].

Example 6.4. Let $S = K_1$, and let T_1 and T_2 both be isomorphic to pK_1 where S, T_1 and T_2 are all disjoint. (Here $S - x$ is the *null* graph, the graph with no vertices.) Let G^+ be the connected graph constructed from the graph $(S \vee T_1) + T_2$ by adding a matching joining each vertex in $V(T_1)$ to a vertex in $V(T_2)$, as illustrated for $p = 5$ in Figure 4. Let $w \in V(T_1)$ and $v \in V(T_2)$ be such that v and w are adjacent in G^+, and let $G = G^+ - w$ and $H = G^+ - v$. Then G and H are the pair of graphs that attain the upper bound of $b(G, H) = \frac{n}{2} + 1$, for even n, when H is connected and G is disconnected, given in Theorem 3.7(c) of [6]. Moreover, there exists a maximum saturating set X of $B_{vw}(G^+)$ that consists of p automorphisms of G^+, and one permutation corresponding to the identity automorphism of G_λ^+, where G_λ^+ is constructed from $\lambda \in X \setminus X_{\mathrm{Aut}}$ as in Lemma 6.2.

Our final example is a generalisation of a pair with $\frac{n+3}{2}$ common cards discovered by Bondy and reported by Myrvold in [16].

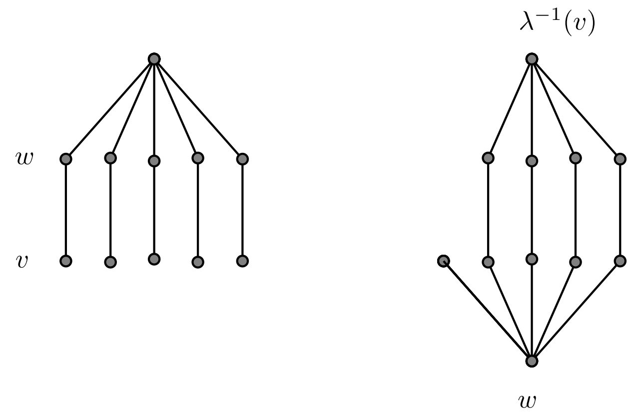

Figure 4: The supercards G^+ and G^+_λ from Example 6.4 when $p = 5$

Lemma 6.5. *Let $T = K_{p,\,p+1}$ and $S = K_{p-1,\,p+1}$, and let $w \in V(T)$ and $v \in V(S)$ be such that $d_T(w) = p$ and $d_S(v) = p-1$. For $a \geq 1$, let A be the graph that consists of $2a-1$ components isomorphic to T, B be the graph that consists of $2a-1$ components isomorphic to S, and C be the graph that consists of $a-1$ components isomorphic to $K_{p,\,p}$. Now let*

$$G^+ = (A+T) + (B+S) + C \cong 2aK_{p,\,p+1} + 2aK_{p-1,\,p+1} + (a-1)K_{p,\,p},$$

and let

$$\begin{aligned} G = G^+ - w &= A + (B+S) + (C + (T-w)) \\ &\cong (2a-1)K_{p,\,p+1} + 2aK_{p-1,\,p+1} + aK_{p,\,p} \\ H = G^+ - v &= (A+T) + B + C + (S-v) \\ &\cong 2aK_{p,\,p+1} + (2a-1)K_{p-1,\,p+1} + (a-1)K_{p,\,p} + K_{p-1,\,p}. \end{aligned}$$

Then $G \ncong H$ and there exists a maximum saturating set X of $B_{vw}(G^+)$ and some $\lambda \in X \setminus X_{\mathrm{Aut}}$, where $\lambda^{-1}(v) \in V(C + (T-w))$, such that the supercard G^+_λ given in Definition 5.1 is

$$\begin{aligned} G^+_\lambda &= (A+T^*) + B + (C + (T-w)) \\ &\cong 2aK_{p,\,p+1} + (2a-1)K_{p-1,\,p+1} + aK_{p,\,p}, \end{aligned} \tag{5}$$

where $T^* \cong T$, $w \in V(T^*)$ and $T^* - w = S$. In addition, $\lambda^{-1}(X \setminus X_{\mathrm{Aut}}) = \lambda^{-1}(X)_{\mathrm{Aut}}$, $|X_{\mathrm{Aut}}| = 2a(p+1)$, $|\lambda^{-1}(X)_{\mathrm{Aut}}| = 2ap$ and

$b(G,\, H) = 2a(2p+1) = \frac{n+1}{2} + a(1-p) + p$. *(We note that G^+ and G^+_λ are not isomorphic.)*

Proof. By comparing the components of G and H, it is easy to see that $G \ncong H$ and $\lambda(w) \in V(A+T)$ for all $\lambda \in B_{vw}(G^+)$. Furthermore, it is straightforward to show that there exists a maximum saturating set X of $B_{vw}(G^+)$ such that, for all $\sigma \in X_{\text{Aut}}$, we have $\sigma^{-1}(v) \in V(B+S)$ and, for all $\pi \in X \setminus X_{\text{Aut}}$, we have $\pi^{-1}(v) \in V(C+(T-w))$. Moreover, $d_{G^+}(\sigma(w)) = p$, $d_{G^+}(\sigma^{-1}(v)) = p-1$ and $d_{G^+}(\pi(w)) = p+1$, for all such σ and π. In addition, if R_π is the component of $C+(T-w)$ that contains $\pi^{-1}(v)$ then $\pi(R_\pi - \pi^{-1}(v)) = S - v$, and if T_π is the component of $A+T$ that contains $\pi(w)$ then, without loss of generality, we may assume that $\pi(S) = T_\pi - \pi(w)$. Clearly, $|X_{\text{Aut}}| = 2a(p+1)$ and $|X \setminus X_{\text{Aut}}| = 2ap$, thus

$$b(G,\, H) = 2a(2p+1) = \frac{n+1}{2} + a(1-p) + p.$$

Now let $\lambda \in X \setminus X_{\text{Aut}}$, and let G^+_λ be the supercard given in Definition 5.1. Then, since $\lambda(w)$ is adjacent to every vertex of T_λ of degree p and $\lambda^{-1}(v) \notin V(S)$, it follows that w is adjacent in G^+_λ to every vertex of S of degree $p-1$. Hence G^+_λ is the graph given in (5). As $\lambda^{-1}(v) \in V(C+(T-w))$, it is easy to see that $|Y_{\text{Aut}}| \leq 2ap$ for any maximum saturating set Y of G^+_λ; so $|\lambda^{-1}(X)_{\text{Aut}}| \leq 2ap$. Furthermore, it is straightfoward to prove, in a similar manner to Lemma 6.1, again using Lemma 5.4 and Corollary 5.5, that $\lambda^{-1}(X \setminus X_{\text{Aut}}) = \lambda^{-1}(X)_{\text{Aut}}$, and hence $|\lambda^{-1}(X)_{\text{Aut}}| = 2ap$. □

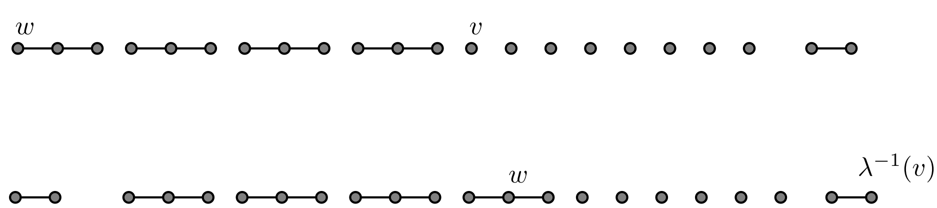

Figure 5: The supercards G^+ and G^+_λ from Lemma 6.5 when $p = 1$ and $a = 2$

Bondy's example corresponds to the case when $p = 1$. Noting that $K_{0,\,2} = 2K_1$, we have $G^+ \cong 2aK_{1,\,2} + 4aK_1 + (a-1)K_2$ and $b(G,\, H) = \frac{n+3}{2}$ in this case (as illustrated in Figure 5 for $a = 2$). The same value for $b(G,\, H)$ is also obtained for the case when $a = 1$.

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