An exact Turán result for tripartite 3-graphs

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Abstract

Mantel's theorem says that among all triangle-free graphs of a given order the balanced complete bipartite graph is the unique graph of maximum size. We prove an analogue of this result for 3-graphs. Let $K_4^- = \{123, 124, 134\}$, $F_6 = \{123, 124, 345, 156\}$ and $\mathcal{F} = \{K_4^-, F_6\}$: for $n \neq 5$ the unique $\mathcal{F}$-free 3-graph of order n and maximum size is the balanced complete tripartite 3-graph $S_3(n)$ (for $n = 5$ it is $C_5^{(3)} = \{123, 234, 345, 145, 125\}$). This extends an old result of Bollobás that $S_3(n)$ is the unique 3-graph of maximum size with no copy of $K_4^- = \{123, 124, 134\}$ or $F_5 = \{123, 124, 345\}$.

1 Introduction

If $r \geqslant 2$ then an *r-graph* G is a pair $G = (V(G), E(G))$, where $E(G)$ is a collection of r-sets from $V(G)$. The elements of $V(G)$ are called *vertices* and the r-sets in $E(G)$ are called *edges*. The number of vertices is the *order* of G, while the number of edges, denoted by $e(G)$, is the *size* of G.

Given a family of r-graphs $\mathcal{F}$, an r-graph G is *$\mathcal{F}$-free* if it does not contain a subgraph isomorphic to any member of $\mathcal{F}$. For an integer $n \geqslant r$ we define the *Turán number of $\mathcal{F}$* to be

$$\mathrm{ex}(n, \mathcal{F}) = \max\{e(G) : G \text{ an } \mathcal{F}\text{-free } r\text{-graph of order } n\}.$$

The related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [7] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdős–Stone–Simonovits Theorem.

Theorem 1 (Erdős and Stone [5], Erdős and Simonovits [4]) *Let $\mathcal{F}$ be a family of 2-graphs. If* $t = \min\{\chi(F) : F \in \mathcal{F}\} \geqslant 2$, *then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}.$$

It follows that the set of all Turán densities for 2-graphs is $\{0, 1/2, 2/3, 3/4, \ldots\}$.

There is no analogous result for $r \geqslant 3$ and most progress has been made through determining the Turán densities of individual graphs or families of graphs. A central problem, originally posed by Turán, is to determine $\mathrm{ex}(n, K_4^{(3)})$, where $K_4^{(3)} = \{123, 124, 134, 234\}$ is the complete 3-graph of order 4. This is a natural extension of determining the Turán number of the triangle for 2-graphs, a question answered by Mantel's theorem [9]. Turán gave a construction that he conjectured to be optimal that has density 5/9 but this question remains unanswered despite a great deal of work. The current best upper bound for $\pi(K_4^{(3)})$ is 0.561666, given by Razborov [11].

A related problem due to Katona is given by considering cancellative hypergraphs. A hypergraph H is *cancellative* if for any distinct edges $a, b \in H$, there is no edge $c \in H$ such that $a \triangle b \subseteq c$ (where $\triangle$ denotes the symmetric difference). For 2-graphs, this is equivalent to forbidding all triangles. For a 3-graph, it is equivalent to forbidding the two non-isomorphic configurations $K_4^- = \{123, 124, 134\}$ and $F_5 = \{123, 124, 345\}$.

An r-graph G is *k-partite* if there is a partition of its vertices into k classes so that all edges of G contain at most one vertex from each class. It is *complete k-partite* if there is a partition into k classes such that all edges meeting each class at most once are present. If the partition of the vertices of a complete k-partite graph is into classes that are as equal as possible in size then we say that G is *balanced*.

Let $S_3(n)$ be the complete balanced tripartite 3-graph of order n.

Theorem 2 (Bollobás [3]) *For $n \geqslant 3$, $S_3(n)$ is the unique cancellative 3-graph of order n and maximum size.*

This result was refined by Frankl and Füredi [6] and Keevash and Mubayi [8], who proved that $S_3(n)$ is the unique F_5-free 3-graph of order n and maximum size, for n sufficiently large.

The *blow-up* of an r-graph H is the r-graph $H(t)$ obtained from H by replacing each vertex $a \in V(H)$ with a set of t vertices V_a in $H(t)$ and inserting a complete r-partite r-graph between any r vertex classes corresponding to an edge in H. The following result is an invaluable tool in determining the Turán density of an r-graph that is contained in the blow-ups of other r-graphs:

Theorem 3 (Brown and Simonovits [1], [2]) *If F is a k-graph that is contained in a blow-up of every member of a family of k-graphs $\mathcal{G}$, then $\pi(F) = \pi(F \cup \mathcal{G})$.*

Since F_5 is contained in $K_4^-(2)$, Theorems 2 and 3 imply that $\pi(F_5) = 2/9$.

A natural question to ask is which 3-graphs (that are not subgraphs of blow-ups of F_5) also have Turán density 2/9? Baber and Talbot [2] considered the 3-graph $F_6 = \{123, 124, 345, 156\}$, which is not contained in any blow-up of F_5. Using Razborov's flag algebra framework [10], they gave a computational proof that $\pi(F_6) = 2/9$. In this paper, we obtain a new (non-computer) proof of this result. In fact we go further and determine the exact Turán number of $\mathcal{F} = \{F_6, K_4^-\}$.

Theorem 4 *If $n \geqslant 3$ then the unique $\mathcal{F}$-free 3-graph with $ex(n, \mathcal{F})$ edges and n vertices is $S_3(n)$ unless $n = 5$ in which case it is $C_5^{(3)} = \{123, 234, 345, 145, 125\}$.*

As F_6 is contained in $K_4^-(2)$, we have the following corollary to Theorem 4.

Corollary 5 $\pi(F_6) = 2/9$.

2 Turán number

Proof of Theorem 4: The underlying proof structure is the same as that employed by Keevash and Mubayi [8] in their proof of Bollobás's theorem (Theorem 2).

We use induction on n. Note that the result holds trivially for $n = 3, 4$. For $n = 5$ it is straightforward to check that the only $\mathcal{F}$-free 3-graphs with 4 edges are $S_3(5)$, $\{123, 124, 125, 345\}$ and $\{123, 234, 345, 451\}$. Of these the first two are edge maximal while the third can be extended by a single edge to give $C_5^{(3)}$. Thus we may suppose that $n \geqslant 6$ and the theorem is true for $n - 3$.

For $k \geqslant 2$ let $T_k(n)$ be the k-partite Turán graph of order n: this is the complete balanced k-partite graph. We denote the number of edges in $S_3(n)$ and $T_k(n)$ by $s_3(n)$ and $t_k(n)$ respectively. Let G be $\mathcal{F}$-free with $n \geqslant 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges. Since $S_3(n)$ is $\mathcal{F}$-free we have $e(G) \geqslant s_3(n)$.

The inductive step proceeds as follows: select a special edge $abc \in E(G)$ (precisely how we choose this edge will be explained in Lemma 6 below). For $0 \leqslant i \leqslant 3$ let f_i be the number of edges in G meeting abc in exactly i vertices. By our inductive hypothesis we have

$$e(G) = f_0 + f_1 + f_2 + f_3 \leqslant \mathrm{ex}(n - 3, \mathcal{F}) + f_1 + f_2 + 1. \tag{1}$$

Note that unless $n - 3 = 5$ our inductive hypothesis says that $\mathrm{ex}(n - 3, \mathcal{F}) = s_3(n - 3)$ with equality iff $G - \{a, b, c\} = S_3(n - 3)$. For the moment we will assume that $n \neq 8$ and so we have the following bound

$$e(G) \leqslant s_3(n - 3) + f_1 + f_2 + 1, \tag{2}$$

with equality iff $G - \{a, b, c\} = S_3(n - 3)$.

Let $V^- = V(G) - \{a, b, c\}$. For each pair $xy \in \{ab, ac, bc\}$ define $\Gamma_{xy} = \{z \in V^- : xyz \in E(G)\}$ and let $\Gamma_{abc} = \Gamma_{ab} \cup \Gamma_{ac} \cup \Gamma_{bc}$ be the *link-neighbourhood* of abc. Note that since G is K_4^--free this is a disjoint union, so

$$f_2 = |\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}| = |\Gamma_{abc}|.$$

For $x \in \{a, b, c\}$ define $L(x)$ to be the *link-graph of* x, so $V(L(x)) = V^-$ and $E(L(x)) = \{yz \subset V^- : xyz \in E(G)\}$. The *link-graph of the edge abc* is the edge labelled graph L_{abc}

with vertex set V^- and edge set $L(a) \cup L(b) \cup L(c)$. The label of an edge $yz \in E(L_{abc})$ is $l(yz) = \{x \in \{a,b,c\} : xyz \in E(G)\}$. The *weight* of an edge $yz \in L_{abc}$ is $|l(yz)|$ and the weight of L_{abc} is $w(L_{abc}) = \sum_{yz \in L_{abc}} |l(yz)|$. Note that $f_1 = w(L_{abc})$. To ease our presentation we will express the label of an edge as, for example, ab rather than $\{a,b\}$.

By a subgraph of L_{abc} we mean an ordinary subgraph of the underlying graph where the labels of edges are non-empty subsets of the labels of the edges in L_{abc}. For example if $xy \in E(L_{abc})$ has $l(xy) = ab$ then in any subgraph of L_{abc} containing the edge xy it must have label a, b or ab.

A triangle in L_{abc} is said to be *rainbow* iff all its edges have weight one and are labelled a, b, c. Given an edge labelled subgraph H of L_{abc} and an (unlabelled) graph G we say that H is a *rainbow* G if all of the edges in H have weight 1 and all the triangles in H are rainbow.

The following lemma provides our choice of edge abc.

Lemma 6 *If G is an $\mathcal{F}$-free 3-graph with $n \geqslant 6$ vertices and $ex(n,\mathcal{F})$ edges then there is an edge $abc \in E(G)$ such that*

$$f_1 + f_2 = w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(n-3) + n - 3,$$

with equality iff L_{abc} is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$.

Underlying all our analysis are some simple facts regarding $\mathcal{F}$-free 3-graphs that are contained in Lemmas 7 and 8.

Lemma 7 *If G is $\mathcal{F}$-free and $abc \in E(G)$ then the following configurations cannot appear as subgraphs of L_{abc}. Moreover any configuration that can be obtained from one described below by applying a permutation to the labels $\{a,b,c\}$ must also be absent.*

(F_6-1) The triangle xy, xz, yz with $l(xy) = l(xz) = a$ and $l(yz) = b$.

(F_6-2) The pair of edges xy, xz with $l(xy) = ab$ and $l(xz) = c$.

(F_6-3) A vertex $x \in \Gamma_{ab}$ and edges xy, yz with labels $l(xy) = c$ and $l(yz) = a$.

(F_6-4) A vertex $x \in \Gamma_{ab}$ and edges xy, yz, zw with labels $l(xy) = l(zw) = a$ and $l(yz) = b$.

(F_6-5) Vertices $x \in \Gamma_{ac}, y \in \Gamma_{bc}, z \in \Gamma_{ab}$ and the edge xy with label $l(xy) = b$.

(K_4^--1) The triangle xy, xz, yz with $l(xy) = l(xz) = l(yz) = a$.

(K_4^--2) The vertex $x \in \Gamma_{ab}$ and edge xy with label $l(xy) = ab$.

(K_4^--3) The vertices $x, y \in \Gamma_{ab}$ and edge xy with label $l(xy) = a$.

Lemma 8 *If G is $\mathcal{F}$-free and $abc \in E(G)$ then the link-graph and link-neighbourhood satisfy:*

(i) The only triangles in L_{abc} are rainbow.

(ii) *The only K_4s in L_{abc} are rainbow.*

(iii) *L_{abc} is K_5-free.*

(iv) *If $xy \in E(L_{abc})$ has $l(xy) = abc$ then x and y meet no other edges in L_{abc} and $x, y \notin \Gamma_{abc}$.*

(v) *If $V^4_{abc} = \{x \in V^- :$ there is a K_4 containing $x\}$ then $\Gamma_{abc} \cap V^4_{abc} = \varnothing$.*

(vi) *There are no edges in L_{abc} between Γ_{abc} and V^4_{abc}.*

(vii) *If $x \in V^4_{abc}$ then $|l(xy)| \leqslant 1$ for all $y \in V^-$.*

(viii) *If $x \in \Gamma_{ac}$, $y \in \Gamma_{bc}$ and $l(xy) = ab$, then $\Gamma_{bc} = \varnothing$. Moreover, if $xz \in E(L_{abc})$ with $z \neq y$ then $z \notin \Gamma_{abc}$ and $l(xz) = a$, while if $yz \in E(L_{abc})$ with $z \neq x$ then $z \notin \Gamma_{abc}$ and $l(yz) = b$.*

(ix) *If $xy, xz \in E(L_{abc})$, $l(xy) = ab$ and $z \in \Gamma_{abc}$ then $|l(xz)| \leqslant 1$.*

We also require the following identities, that are easy to verify.

Lemma 9 *If $n \geqslant k \geqslant 3$ then*

(i) $s_3(n) = s_3(n-3) + t_3(n-3) + n - 2$.

(ii) $t_3(n) = t_3(n-3) + 2n - 3$.

(iii) $t_3(n) = t_3(n-2) + n - 1 + \lfloor n/3 \rfloor$.

(iv) $t_k(n) = t_k(n-1) + n - \lceil n/k \rceil$.

Let $abc \in E(G)$ be a fixed edge given by Lemma 6.

By assumption $e(G) \geqslant s_3(n)$ so Lemma 9 (i) and Lemma 6 together with the bound on $e(G)$ given by (2) imply that $e(G) = s_3(n)$ and hence $G - \{a, b, c\} = S_3(n-3)$, L_{abc} is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$. To complete the proof we need to show that $G = S_3(n)$. First note that since L_{abc} is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$, Lemma 8 (i) and Lemma 7(F_6-3) imply that no vertex in Γ_{ab} is in an edge with label c and similarly for Γ_{ac}, Γ_{bc}. Hence L_{abc} is the complete tripartite graph with vertex classes Γ_{ab}, Γ_{ac} and Γ_{bc} and the edges between any two parts are labelled with the common label of the parts (e.g. all edges from Γ_{ab} to Γ_{ac} receive label a). So L_{abc} is precisely the link graph of an edge $abc \in S_3(n)$.

In order to deduce that $G = S_3(n)$ we need to show that $G - \{a, b, c\} = S_3(n-3)$ has the same tripartition as L_{abc}. This is straightforward: any edge $xyz \in E(G - \{a, b, c\})$ not respecting the tripartition of L_{abc} meets one of the parts at least twice. But if $x, y, z \in \Gamma_{ab}$ then $|\Gamma_{ac}| \geqslant 2$ so let $u \in \Gamma_{ac}$. Setting $a = 1, b = 2, x = 3, y = 4, z = 5, u = 6$ gives a copy of F_6. If $x, y \in \Gamma_{ab}$ and $z \in \Gamma_{ac}$ then $a = 1, x = 3, y = 4, z = 2$ gives a copy of K_4^-.

Hence $G = S_3(n)$ and the proof is complete in the case $n \neq 8$.

For $n = 8$ we note that if $G - \{a, b, c\}$ is F_5-free then Theorem 2 implies that the result follows as above, so we may assume that $G - \{a, b, c\}$ contains a copy of F_5. In this case it is sufficent to show that $e(G) \leqslant 17 < 18 = s_3(8)$.

If $V(G - \{a, b, c\}) = \{s, t, u, v, w\}$ then we may suppose that $stu, stv, uvw, abc \in G$. Since G is K_4^--free it does not contain suv or tuv. Moreover it contains at most 3 edges from $\{u, v, w\}^{(2)} \times \{a, b, c\}$ and at most 5 edges from $\{s, t, u, v, w\} \times \{a, b, c\}^{(2)}$. Since G is F_6-free it contains no edges from $\{s, t\} \times \{w\} \times \{a, b, c\}$.

The only potential edges we have yet to consider are those in $\{st, su, tu, sv, tv\} \times \{w, a, b, c\}$. Since G is K_4^--free it contains at most 2 edges from std, sud, tud, svd, tvd, for any $d \in \{w, a, b, c\}$. Moreover, since G is F_6-free, if it contains 2 such edges for a fixed d then it can contain at most 3 such edges in total for the other choices of d. Hence at most 5 such edges are present.

Thus in total $e(G) \leqslant 4 + 3 + 5 + 5 = 17$, as required. $\square$

In order to prove Lemma 6 we first need an edge with large link-neighbourhood.

Lemma 10 *If G is K_4^--free 3-graph of order n with $s_3(n)$ edges, then there is an edge $abc \in E(G)$ with $|\Gamma_{abc}| \geqslant n - \lfloor n/3 \rfloor - 3$.*

Proof of Lemma 10: Let G be K_4^--free with n vertices and $s_3(n)$ edges. For $x, y \in V(G)$ let $d_{xy} = |\{x : xyz \in E(G)\}|$. If $uvw \in E(G)$ then $\Gamma_{uvw} = \Gamma_{uv} \cup \Gamma_{uw} \cup \Gamma_{vw}$ is a union of pairwise disjoint sets and $|\Gamma_{uvw}| = d_{uv} + d_{uw} + d_{vw} - 3$. Thus if the lemma fails to hold then for every edge $uvw \in E(G)$ we have $d_{uv} + d_{uw} + d_{vw} \leqslant n - \lfloor n/3 \rfloor - 1$. Note that since $\sum_{xy \in \binom{V}{2}} d_{xy} = 3e(G)$, convexity implies that

$$e(G)(n - \left\lfloor \frac{n}{3} \right\rfloor - 1) \geqslant \sum_{uvw \in E(G)} d_{uv} + d_{uw} + d_{vw} = \sum_{xy \in \binom{V}{2}} d_{xy}^2 \geqslant \frac{9e^2(G)}{\binom{n}{2}}.$$

Thus

$$e(G) \leqslant \frac{1}{18} n(n-1)(n - \lfloor n/3 \rfloor - 1).$$

But it is easy to check that this is less than $s_3(n)$. $\square$

Our next objective is to describe various properties of the link-graph L_{abc} and link-neighbourhood Γ_{abc}.

Lemma 8 (v) allows us to partition the vertices of L_{abc} as $V^- = \Gamma_{abc} \cup V_{abc}^4 \cup R_{abc}$, where $V_{abc}^4 = \{x \in V^- :$ there is a K_4 containing $x\}$ and $R_{abc} = V^- - (\Gamma_{abc} \cup V_{abc}^4)$. To prove Lemma 6 we require the following result to deal with the part of L_{abc} not meeting any copies of K_4.

Lemma 11 *Let H be a subgraph of L_{abc} with $s \geqslant 3$ vertices satisfying $V(H) \cap V_{abc}^4 = \varnothing$. If $H_\Gamma = V(H) \cap \Gamma_{abc}$ and $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$ then*

$$w(H) + |H_\Gamma| \leqslant t_3(s) + s,$$

with equality iff $H_\Gamma = V(H)$ and H is a rainbow $T_3(s)$.

Proof of Lemma 6: Let G be $\mathcal{F}$-free with $n \geqslant 6$ vertices and $\text{ex}(n, \mathcal{F})$ edges. By Lemma 10 we can choose an edge $abc \in E(G)$ such that $|\Gamma_{abc}| \geqslant n - \lfloor n/3 \rfloor - 3$. Let $V^- = \Gamma_{abc} \cup R_{abc} \cup V^4_{abc}$ be the partition of V^- given by Lemma 8 (v). If $s = |V^-|$, $j = |\Gamma_{abc}|$, $k = |R_{abc}|$ and $l = |V^4_{abc}|$ then $n - 3 = s = j + k + l$ and $j \geqslant s - \lfloor s/3 \rfloor - 1 \geqslant j + k - \lfloor (j+k)/3 \rfloor - 1$. We can apply Lemma 11 to $H = L_{abc}[\Gamma_{abc} \cup R_{abc}]$, to deduce that

$$w(L_{abc}[\Gamma_{abc} \cup R_{abc}]) + |\Gamma_{abc}| \leqslant t_3(j+k) + j + k,$$

with equality iff $R_{abc} = \varnothing$ and $L_{abc}[\Gamma_{abc}]$ is a rainbow $T_3(j+k)$. Now if L_{abc} is K_4-free then $V^4_{abc} = \varnothing$ and the proof is complete, so suppose there is a K_4 in L_{abc}. In this case $4 \leqslant |V^4_{abc}| \leqslant n - 3 - |\Gamma_{abc}| \leqslant \lfloor n/3 \rfloor$, so $n \geqslant 12$.

We now need to consider the edges in L_{abc} meeting V^4_{abc}. By Lemma 8 (iii) we know that L_{abc} is K_5-free, while Lemma 8 (vii) says that V^4_{abc} meets no edges of weight 2 or 3, so by Turán's theorem $w(L_{abc}[V^4_{abc}]) \leqslant t_4(l)$.

Lemma 8 (vi) implies that there are no edges from Γ_{abc} to V^4_{abc} so the total weight of edges between $\Gamma_{abc} \cup R_{abc}$ and V^4_{abc} is at most kl. Thus

$$w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(j+k) + j + k + t_4(l) + kl.$$

Finally Lemma 12 with $s = n - 3$ implies that

$$w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(n-3) + n - 3,$$

with equality iff $R_{abc} = V^4_{abc} = \varnothing$ and L_{abc} is a rainbow $T_3(n-3)$ as required. $\square$

Lemma 12 *If $j, k, l \geqslant 0$ are integers satisfying $j + k + l = s \geqslant 5$ and $j \geqslant s - \lfloor s/3 \rfloor - 1$ then*

$$t_3(j+k) + t_4(l) + j + k + kl \leqslant t_3(s) + s, \tag{3}$$

with equality iff $l = 0$.

Proof of Lemma 12: If $l = 0$ then the result clearly holds, so suppose that $l \geqslant 1$, $j + k + l = s \geqslant 5$ and $j \geqslant s - \lfloor s/3 \rfloor - 1$. Let $f(j,k,l)$ be the LHS of (3). We need to check that $\Delta(j,k,l) = f(j,k+1,l-1) - f(j,k,l) > 0$. Since if this holds then we have

$$f(j,k,l) < f(j,k+1,l-1) < \cdots < f(j,k+l,0) = t_3(s) + s.$$

Using Lemma 9 (iv) we have

$$\begin{aligned}\Delta(j,k,l) &= j - \lceil (j+k+1)/3 \rceil + \lceil l/4 \rceil + 1 \\ &= j + \lceil l/4 \rceil - \lfloor (j+k)/3 \rfloor.\end{aligned}$$

So it is sufficient to check that $j + l/4 > (j+k)/3$. This follows easily from $j \geqslant s - \lfloor s/3 \rfloor - 1$, $k \leqslant \lfloor s/3 \rfloor + 1$, $l \geqslant 1$ and $s \geqslant 5$. $\square$

Proof of Lemma 11: We prove this by induction on $s \geqslant 3$. The result holds for $s = 3, 4$ (see the end of this proof for the tedious details) so suppose that $s \geqslant 5$ and the result holds for $s - 2$.

Let H be a subgraph of L_{abc} with $s \geqslant 5$ vertices satisfying $V(H) \cap V^4_{abc} = \varnothing$. Let $H_\Gamma = V(H) \cap \Gamma_{abc}$ and suppose that $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$.

Note that if H contains no edges of weight 2 or 3 then the result follows directly from Turán's theorem and Lemma 8 (i), so we may suppose there are edges of weight 2 or 3. With this assumption it is sufficient to show that

$$w(H) + |H_\Gamma| \leqslant t_3(s) + s - 1.$$

By Lemma 9 (iii) this is equivalent to showing that the following inequality holds:

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + 2s - 2 + \lfloor s/3 \rfloor \tag{4}$$

Case (i): There exists an edge of weight 3, $l(xy) = abc$.

Lemma 8 (iv) implies that $x, y \notin H_\Gamma$ and x, y meet no other edges in H, so we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leqslant w(H') + |H'_\Gamma| + 3 \leqslant t_3(s-2) + s - 2 + 3.$$

Hence (4) holds as required. So we may suppose that H contains no edges of weight 3.

Case (ii): The only edges of weight 2 are contained in H_Γ

Let $xy \in E(H)$ have weight 2, say $l(xy) = ab$. Now Lemma 7 (K_4^--2) implies that $x, y \notin \Gamma_{ab}$, while Lemma 7 (K_4^--3) implies that x, y cannot both belong to Γ_{ac} or Γ_{bc} so we may suppose that $x \in \Gamma_{ac}$ and $y \in \Gamma_{bc}$. Lemma 8 (viii) implies that x, y have no more neighbours in H_Γ. If $H_\Gamma = V(H)$ then we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + 2 + 2,$$

in which case (4) holds, so suppose $V(H) \neq H_\Gamma$.

Let $z \in V(H) - H_\Gamma$ be a neighbour of x in H if one exists otherwise let z be any vertex in $V(H) - H_\Gamma$. By our assumption that all edges of weight 2 are contained in H_Γ, z meets no edges of weight 2. Moreover, by Lemma 8 (viii), all edges containing x (except xy) have label b, so x is not in any triangles in H. Hence x and z have no common neighbours in H and so the total weight of edges meeting $\{x, z\}$ is at most $2 + 1 + s - 3$ (if xz is an edge) and at most $2 + s - 2$ otherwise. Applying our inductive hypothesis to $H' = H - \{x, z\}$ we have

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + 1 + s,$$

and (4) holds.

Case (iii): There is an edge of weight 2 meeting $V(H) - H_\Gamma$.

So suppose that $xy \in E(H)$, $l(xy) = ab$ and $y \notin H_\Gamma$. Lemma 8 (ix) implies that for any $z \in H_\Gamma$ we have $|l(xz)|$, $|l(yz)| \leqslant 1$. Let $\gamma_{xy} = |\{x, y\} \cap H_\Gamma| \leqslant 1$. Thus, since xy is not in any triangles, the total weight of edges meeting $\{x, y\}$ is at most

$$2 + s - 2 + |V(H) - H_\Gamma| - (2 - \gamma_{xy}).$$

Applying the inductive hypothesis to $H' = H - \{x, y\}$ we have

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + s + s - |H_\Gamma| - 2 + 2\gamma_{xy},$$

with equality holding only if $|H'_\Gamma| = s - 2$. Now $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$ implies that

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + 2s - 3 + \lfloor s/3 \rfloor + 2\gamma_{xy}, \tag{5}$$

with equality only if $|H'_\Gamma| = s - 2$ and $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$. If $\gamma_{xy} = 0$ then (4) holds as required, so suppose $\gamma_{xy} = 1$. In this case (4) holds, unless (5) holds with equality. But if (5) is an equality then $|H_\Gamma| = |H'_\Gamma| + 1 = s - 1$, while $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$, which is impossible for $s \geqslant 3$.

We finally need to verify the cases $s = 3, 4$. It is again sufficient to prove that if H contains edges of weight 2 or 3 then $w(H) + |H_\Gamma| \leqslant t_3(s) + s - 1$, thus we need to show that $w(H) + |H_\Gamma|$ is at most 5 if $s = 3$ and at most 8 if $s = 4$.

We note that argument in Case (i) above implies that if H contains an edge of weight 3 then $|H_\Gamma| \leqslant s - 2$ and $w(H) \leqslant 3 + 3\binom{s-2}{2}$, so if $s = 3$ then $w(H) + |H_\Gamma| \leqslant 4$ and if $s = 4$ then $w(H) + |H_\Gamma| \leqslant 8$ so the result holds. So we may suppose there are no edges of weight 3.

Now let xy be an edge of weight 2. Using the fact that xy is not in any triangles and Lemma 8 (viii) and (ix) we find that for $s = 3$ we have $w(H) + |H_\Gamma| \leqslant 2 + 3 - |H_\Gamma|$, while for $s = 4$ we have $w(H) + |H_\Gamma| \leqslant 2 + 6 - |H_\Gamma|$, so the result holds. $\square$

Finally we need to establish our two stuctural lemmas.

Proof of Lemma 7: In each case we describe a labelling of the vertices of the given configuration to show that if it is present then G is not $\mathcal{F}$-free.

(F_6-1) $a = 1$, $b = 5$, $c = 6$, $x = 2$, $y = 3$, $z = 4$.

(F_6-2) $a = 3$, $b = 4$, $c = 5$, $x = 1$, $y = 2$, $z = 6$.

(F_6-3) $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$, $z = 6$.

(F_6-4) $a = 1$, $b = 3$, $x = 2$, $y = 4$, $z = 5$, $w = 6$.

(F_6-5) $a = 5$, $b = 1$, $c = 3$, $x = 4$, $y = 2$, $z = 6$.

(K_4^--1) $a = 1$, $x = 2$, $y = 3$, $z = 4$.

(K_4^--2) $a = 3$, $b = 4$, $x = 1$, $y = 2$.

(K_4^--3) $a = 1$, $b = 2$, $x = 3$, $y = 4$. $\square$

Proof of Lemma 8: We will make repeated use of Lemma 7.

(i) This follows immediately from (F_6-1) and (K_4^--1).

(ii) This follows immediately from (i): if $uvwx$ is a copy of K_4 then we may suppose $l(uv) = a, l(uw) = b, l(vw) = c$, thus $l(ux) = c$ (otherwise (i) would be violated) continuing we see that $uvwx$ must be rainbow.

(iii) This follows immediately from (ii): if $xyzuv$ is a copy of K_5 then by (ii) we may suppose that $l(xy), l(xz), l(xu), l(xv)$ are all distinct single labels from $\{a, b, c\}$ but this is impossible since there are only 3 labels in total.

(iv) This follows immediately from (F_6-2) and (K_4^--2).

(v) If x is in a K_4 then by (ii) it lies in edges with labels a, b, c, so (F_6-3) implies that $x \notin \Gamma_{abc}$.

(vi) If $x \in \Gamma_{abc}$, say $x \in \Gamma_{ab}$, and $y \in V_{abc}^4$ with $xy \in E(L_{abc})$ then (F_6-3) implies that $l(xy) \neq c$, while (F_6-4) implies that $l(xy) \neq a, b$ (since there are t, u, v, w such that $l(yt) = b, l(tu) = a$ and $l(yv) = a, l(vw) = b$).

(vii) This follows immediately from the fact that all $v \in V_{abc}^4$ meet edges with labels a, b, c and (F_6-2).

(viii) (F_6-5) implies that $\Gamma_{bc} = \varnothing$. If $xz \in E(L_{abc})$ then (F_6-3) implies that $l(xz) = a$. Now (K_4-3) implies that $z \notin \Gamma_{ac}$ while (F_6-3) implies that $z \notin \Gamma_{bc}$. Hence $z \notin \Gamma_{abc}$. Similarly if $yz \in E(L_{abc})$ then $l(yz) = b$ and $z \notin \Gamma_{abc}$.

(ix) If $x \in \Gamma_{abc}$ or $y \in \Gamma_{abc}$ then this follows directly from (viii) so suppose that $x, y \notin \Gamma_{abc}$, $l(xy) = ab$ and $|l(xz)| = 2$. In this case, (F_6-2) implies that $l(xz) = ab$ so (K_4-2) implies that $z \in \Gamma_{ac} \cup \Gamma_{bc}$. But then ($F_6$-3) is violated. Hence $|l(xz)| \leqslant 1$. □

3 Conclusion

Many Turán-type results have associated "stability" versions, and we were able to obtain such a result. For reasons of length we state it without proof.

Theorem 13 *For any $\epsilon > 0$ there exist $\delta > 0$ and n_0 such that the following holds: if H is an $\mathcal{F}$-free 3-graph of order $n \geqslant n_0$ with at least $(1 - \delta)\, s_3(n)$ edges, then there is a partition of the vertex set of H as $V(H) = U_1 \cup U_2 \cup U_3$ so that all but at most ϵn^3 edges of H have one vertex in each U_i.*

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