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AUTOMORPHISMS OF ALGEBRAS AND ORTHOGONAL POLYNOMIALS

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Abstract. Suitable automorphisms together with a complete classification of representations of some algebras can be used to generate some sets of orthogonal polynomials "at no cost". This is also the case for the nonstandard Klimyk-Gavrilik deformation $U'_q(\mathrm{so}_3)$, which is connected to q-Racah polynomials.

Keywords: orthogonal polynomials, algebra representation, automorphism.

1. Introduction

The connection of orthogonal polynomials with Lie algebras has been known for a long time (see [1–4]. For a nice introduction and a detailed historical survey see [5] and references therein). Having available the classification of irreducible representations and making use of some automorphisms of algebras, one can obtain sets of orthogonal polynomials for free, without the need to proving their properties manually. The same is true for some of their q-analogs. We show this approach on a well-known example of the sl_2 algebra and Krawtchouk polynomials [5], and we then apply the same procedure to the nonstandard Klimyk-Gavrilik deformation $U'_q(\mathrm{so}_3)$, for which the complete classification of its irreducible representations is known (see [6–9]).

2. Lie algebra sl_2

Let us first consider the Lie algebra sl_2 of 2×2 complex matrices with zero trace. It has the standard Chevalley basis

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

These matrices satisfy the commutation relations

$$[h,e]=2e,\quad [h,f]=-2f,\quad [e,f]=h,$$

where $[x,y]=xy-yx$.

Let φ be a finite-dimensional irreducible representation of sl_2 acting on the space V_{N+1} of dimension $N+1$ with some fixed basis via the matrices

$$H=\varphi(h)=\begin{pmatrix}N&0&0&\cdots&0\\0&N-2&0&&0\\0&0&N-4&&0\\\vdots&&&\ddots&\vdots\\0&0&0&\cdots&-N\end{pmatrix},$$

$$\varphi(e)=\begin{pmatrix}0&1&0&\cdots&0\\0&0&2&&0\\0&0&0&\ddots&0\\\vdots&&&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix},$$

$$\varphi(f)=\begin{pmatrix}0&0&0&\cdots&0\\N&0&0&&0\\0&N-1&0&&0\\\vdots&&\ddots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}.$$

Now let us define a matrix S as

$$S=\varphi(e)+\varphi(f)=\begin{pmatrix}0&1&0&\cdots&0\\N&0&2&&0\\0&N-1&0&\ddots&0\\\vdots&&\ddots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}.$$

Making use of an automorphism σ sending

$$h\to e+f,\quad e\to\frac{1}{2}(h-e+f),\quad f\to\frac{1}{2}(h+e-f)$$

and taking into account the classification of irreducible representations of the sl_2 algebra, we see that matrices H and S form a so-called Leonard pair (a Leonard pair is a pair of diagonalizable finite-dimensional linear transformations, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one). Particularly, they have the same eigenvalues.

It follows that there exists a matrix P such that $S=PHP^{-1}$. We can consider the rows of P or the columns of P^{-1} as coordinate vectors of a polynomial. The proposition is that

$$K_j(k;1/2,N)=[P^{-1}]_{kj}=\binom{N}{j}^{-1}P_{jk}\tag{1}$$

where $K_n(x;p,N)$ is n-th Krawtchouk polynomial $n=0,1,\dots,N$ with parameters $p\in(0,1)$ and $N\in\mathbb{N}_0$. The matrix P is defined by a similarity relation up to a multiplicative constant. Equation 1 holds if we choose $P_{00} = 1$.

The Krawtchouk polynomials are defined by means of the hypergeometric function

$$\begin{aligned} K_n(x; p, N) &= {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \,\middle|\, \frac{1}{p}\right) \\ &= \sum_{j=0}^{N} \frac{(-n)_j(-x)_j}{(-N)_j} \frac{1}{p^j j!} \end{aligned}$$

where $(a)_j = \prod_{k=0}^{j-1}(a+k)$ and $(a)_0 = 1$.

To show (1), we can use the similarity relations $P^{-1}S = HP^{-1}$, i.e.,

$$\begin{aligned} [P^{-1}S]_{kj} &= j[P^{-1}]_{k,j-1} + (N-j)[P^{-1}]_{k,j+1}, \\ [HP^{-1}]_{jk} &= (N-2k)[P^{-1}]_{kj}, \end{aligned}$$

construct a recurrence relation

$$\begin{aligned} j[P^{-1}]_{k,j-1} - (N-2k)[P^{-1}]_{k,j} \\ + (N-j)[P^{-1}]_{k,j+1} = 0 \quad (2) \end{aligned}$$

and compare it with general recurrence for Krawtchouk polynomials,

$$\begin{aligned} j(p-1)K_{j-1}(x) - \big(j(p-1) + (j-N)p + x\big)K_j(x) \\ + (j-N)pK_{j+1}(x) = 0 \quad (3) \end{aligned}$$

where $K_n(x) = K_n(x; p, N)$ for fixed p and N. One can show a similar result for the relation $SP = PH$.

This also proves the relation between P^{-1} and P^*, which can be written as $P^{-1} = P^*G^{-1}$, where $G_{jk} = \delta_{jk}\binom{N}{j}$. This means that P is orthogonal with respect to the inner product defined by matrix G. Thus, P^{-1} is orthogonal and so the columns of P^{-1} (Krawtchouk polynomials) are orthogonal with respect to the inner product defining the orthogonality relation

$$\sum_{j=1}^{N} \binom{N}{j} K_m(j)K_n(j) = h_n\delta_{mn}. \quad (4)$$

The inner product is of course determined up to normalization ((4) can be multiplied by the arbitrary sequence a_n). This result corresponds with the general orthogonality relation for Krawtchouk polynomials (see [10])

$$\begin{aligned} \sum_{j=0}^{N} \binom{N}{j} p^j(1-p)^{N-j} K_m(j)K_n(j) \\ = \binom{N}{n}^{-1} \left(\frac{1-p}{p}\right)^n \delta_{mn}. \end{aligned}$$

Relation (4) can be derived, not just proven, using the properties of the Leonard pair. We will make use of the fact that S has $N+1$ distinct eigenvalues and P diagonalizes S, so the columns of P are eigenvectors of S. If we find an inner product such that S is a matrix of a self-adjoint operator with respect to this product, then P will be orthogonal with respect to this product.

We will try to find a diagonal matrix $A = \mathrm{diag}(a_0, \ldots, a_N)$, such that $A^{-1}SA$ is a Hermitian matrix, and so it is a matrix of the self-adjoint operator in the case of a standard inner product. After a change of basis we can see that S is a self-adjoint operator in the case of the inner product defined by the matrix $G = A^*A = \mathrm{diag}(w_0, \ldots, w_n)$. Thus, we require

$$[A^{-1}SA]_{j-1,j} = \overline{[A^{-1}SA]}_{j,j-1},$$

which leads to the condition

$$|a_j|^2 = |a_{j-1}|^2 \frac{\bar{S}_{j,j-1}}{S_{j-1,j}} = |a_{j-1}|^2 \frac{N-j+1}{j}.$$

This request is fulfilled if we choose

$$w_j = |a_j|^2 = \prod_{k=1}^{j} \frac{N-k+1}{k} = \binom{N}{j}.$$

3. Algebra $U_q'(\mathrm{so}_3)$

Now let us consider from the same point of view the algebra $U_q'(\mathrm{so}_3)$, a complex associative algebra generated by three elements I_1, I_2 and I_3 satisfying the relations

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \quad (5)$$

$$q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \quad (6)$$

$$q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2. \quad (7)$$

Let us assume that q is not the root of unity and define matrices $\varphi(I_1)$, $\varphi(I_2)$, $\varphi(I_3)$ by

$$\begin{aligned} [\varphi(I_1)]_{j+1,j} &= \frac{[2M-j]}{q^{-M+j} + q^{M-j}}, \\ [\varphi(I_1)]_{j-1,j} &= -\frac{[j]}{q^{-M+j} + q^{M-j}}, \\ [\varphi(I_1)]_{jk} &= 0 \quad \text{for } k \neq j \pm 1, \\ [\varphi(I_3)]_{jk} &= i[-M+j]\delta_{jk}, \end{aligned}$$

where $M = N/2$ and $[\nu] = (q^\nu - q^{-\nu})/(q - q^{-1})$. (The matrix $\varphi(I_2)$ can be obtained from the third defining relation (7).) Then the triple form is an irreducible so called classical representation of $U_q'(\mathrm{so}_3)$ of dimension $N+1$.

The matrices $\varphi(I_1)$ and $\varphi(I_3)$ have the same eigenvalues, which follows from the classification of all irreducible representations (there is one classical representation per dimension, see [9]) and from the existence of a rotational automorphism which sends

$$I_1 \to I_2, \quad I_2 \to I_3, \quad I_3 \to I_1.$$

Thus, we can construct matrix P such that

$$\varphi(I_1) = P\varphi(I_3)P^{-1}.$$

We will show that matrix P corresponds to q-Racah polynomials. The general q-Racah polynomials are defined by means of hypergeometric series as

$$\begin{aligned}&R_n\big(\mu(x);\alpha,\beta,\gamma,\delta\mid q\big)\\&\quad={}_4\varphi_3\left(\begin{matrix}q^{-n},\alpha\beta q^{n+1},q^{-x},\gamma\delta q^{x+1}\\ \alpha q,\beta\delta q,\gamma q\end{matrix}\;\middle|\;q;q\right)\\&\quad=\sum_{k=0}^{\infty}\frac{[q^{-n}]_k[\alpha\beta q^{n+1}]_k[q^{-x}]_k[\gamma\delta q^{x+1}]_k}{[\alpha q]_k[\beta\delta q]_k[\gamma q]_k}\frac{q^k}{[q]_k},\end{aligned}\tag{8}$$

where $R_n(x;\alpha,\beta,\gamma,\delta\mid q)$ is n-th q-Racah polynomial with parameters α, β, γ, δ, and with $n=0,1,\ldots,N$, where N is a nonnegative integer,

$$\mu(x)=q^{-x}+\gamma\delta q^{x+1},$$
$$[a]_k=\prod_{j=0}^{k-1}(1-aq^j),\quad [a]_0=1.$$

The parameters must satisfy

$$\alpha q=q^{-N}\quad\text{or}\quad\beta\delta q=q^{-N}\quad\text{or}\quad\gamma q=q^{-N}.$$

In the definition of basic hypergeometric orthogonal polynomials it is usually assumed that $q\in(0,1)$. However, in this calculation it is sufficient to assume $q\in\mathbb{R}\setminus\{-1,0,1\}$.

The correspondence has the following form

$$-i^jR_j\big(\mu(k);\alpha,\beta,\gamma,\delta\mid q\big)=[P^{-1}]_{kj}=w_j^{-1}\bar{P}_{jk},\tag{9}$$

where i is an imaginary unit. The weight sequence and parameters are

$$\begin{aligned}w_j&=\frac{[q^{-N}]_j[-q^{-N}]_j}{[q]_j[-q]_j}\frac{1+q^{-N+2j}}{(-q^{-N})^j(1+q^{-N})},\\ \alpha&=\beta=-\gamma=-\delta=iq^{\frac{-N-1}{2}}.\end{aligned}\tag{10}$$

From now, we will again omit the parameters of the polynomials and write only $R_n(\mu(x))$ instead of $R_n(\mu(x);\alpha,\beta,\gamma,\delta\mid q)$.

In order to prove (9), we construct the recurrence relation and compare it to the general form of the recurrence relation for q-Racah polynomials. The equation $P^{-1}\varphi(I_1)=\varphi(I_3)P^{-1}$ gives us the relation

$$\begin{aligned}&-(q^{-N+k}-q^{-k})R_j\big(\mu(k)\big)\\&\quad=\frac{-q^{-N}(1-q^{2n})}{1+q^{-N+2n}}R_{j-1}\big(\mu(k)\big)\\&\qquad+\frac{1-q^{-2N+2n}}{1+q^{-N+2n}}R_{j+1}\big(\mu(k)\big).\end{aligned}\tag{11}$$

We can see that this form corresponds to the general recurrence (see [10])

$$\begin{aligned}&-(1-q^{-x})(1-\gamma\delta q^{x+1})R_n\big(\mu(x)\big)\\&\quad=A_nR_{n+1}\big(\mu(x)\big)-(A_n+C_n)R_n\big(\mu(x)\big)\\&\qquad+C_nR_{n-1}\big(\mu(x)\big),\end{aligned}\tag{12}$$

where

$$\begin{aligned}A_n&=\frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})}{1-\alpha\beta q^{2n+1}}\\&\qquad\frac{(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{1-\alpha\beta q^{2n+2}},\\ C_n&=\frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})},\end{aligned}$$

if the parameters are set up the way as in (10).

The way of deriving the weight sequence is similar to the former case. We again try to find a diagonal matrix A. However, there is no diagonal matrix that transforms $\varphi(I_1)$ to a Hermitian matrix. Nevertheless, we can transform $\varphi(I_1)$ to a symmetric matrix and then show that the transformed matrix is normal. The elements of A have to satisfy

$$\begin{aligned}a_j^2&=a_{j-1}^2\frac{[\varphi(I_1)]_{j,j-1}}{[\varphi(I_1)]_{j-1,j}}\\&=a_{j-1}^2\frac{-(q^{2M-j+1}-q^{-2M+j-1})(q^{-M+j}+q^{M-j})}{(q^j-q^{-j})(q^{-M+j-1}+q^{M-j+1})}.\end{aligned}$$

The elements are determined up to a multiplicative constant as a product

$$\begin{aligned}a_j^2&=\prod_{k=1}^{j}-\frac{(q^{2M-k+1}-q^{-2M+k-1})(q^{-M+k}+q^{M-k})}{(q^k-q^{-k})(q^{-M+k-1}+q^{M-k+1})}\\&=\prod_{k=1}^{j}q^{2M}\frac{(1-q^{-4M+2k-2})(1+q^{-2M+2k})}{(1-q^{2k})(1+q^{-2M+2k-2})}\\&=\prod_{k=1}^{j}\frac{(1-q^{-N+k-1})(1+q^{-N+k-1})(1+q^{-N+2k})}{q^{-N}(1-q^k)(1+q^k)(1+q^{-N+2k-2})}\\&=\frac{[q^{-N}]_j[-q^{-N}]_j}{[q]_j[-q]_j}\frac{1+q^{-N+2j}}{(q^{-N})^j(1+q^{-N})}.\end{aligned}$$

If we assume $q\in(0,1)$ then for all $k\geq 1$ the factor $1-q^{-N+k-1}$ is negative whereas the other factors are positive. Therefore, $|a_j|^2=(-1)^ja_j^2$. It can be easily seen that this holds for all $q\in\mathbb{R}\setminus\{-1,0,1\}$ by similar reasoning. Finally, we have $|a_j|^2=w_j$.

Now we just need to verify that $B:=A^{-1}\varphi(I_1)A$ is normal using the fact that B is symmetric. Thus, we have to verify $\sum B_{jl}\bar{B}_{kl}=\sum\bar{B}_{jl}B_{kl}$. We can just show that for all indices j, k, l we have

$$B_{jl}\bar{B}_{kl}=a_j^{-1}[\varphi(I_1)]_{jl}a_l\bar{a}_k^{-1}\overline{[\varphi(I_1)]}_{kl}\bar{a}_l\in\mathbb{R}.$$

Since $\varphi(I_1)$ is real and $a_l\bar{a}_l=|a_l|^2$, we just have to decide whether a_ja_k is real for indices j, k, whose difference is even (otherwise $[\varphi(I_1)]_{jl}$ or $[\varphi(I_1)]_{kl}$ is zero due to its special form). Considering a_j^2 is real and alternates, we see that a_j and a_k are both real or purely imaginary. Therefore, $a_ja_k\in\mathbb{R}$.

4. Conclusion

On the example of the algebra $U_q'(\mathrm{so}_3)$ we have shown that the existence of a complete classification of representations together with a suitable use of some automorphism can produce as a by-product a set of orthogonal polynomials (see also [11]). Because the algebra $U'_q(\mathrm{so}_3)$ is a special case of Askey-Wilson algebra $AW(3)$, introduced by Zhedanov [12], it would be nice to generalize this approach to this case (see also [13–16]).

References

[1] W. Miller, Jr. Lie theory and difference equations. I. *J Math Anal Appl* **28**:383–399, 1969.

[2] T. H. Koornwinder. Krawtchouk polynomials, a unification of two different group theoretic interpretations. *SIAM J Math Anal* **13**(6):1011–1023, 1982. DOI:10.1137/0513072.

[3] P. Feinsilver. Lie algebras and recurrence relations. I. *Acta Appl Math* **13**(3):291–333, 1988.

[4] Y. I. Granovskiĭ, A. S. Zhedanov. Orthogonal polynomials on Lie algebras. *Izv Vyssh Uchebn Zaved Fiz* **29**(5):60–66, 1986.

[5] K. Nomura, P. Terwilliger. Krawtchouk polynomials, the Lie algebra $\mathfrak{sl}_2$, and Leonard pairs. *Linear Algebra Appl* **437**(1):345–375, 2012. DOI:10.1016/j.laa.2012.02.006.

[6] A. M. Gavrilik, N. Z. Iorgov. q-deformed algebras $U_q(\mathrm{so}_n)$ and their representations. *Methods Funct Anal Topology* **3**(4):51–63, 1997.

[7] A. M. Gavrilik, N. Z. Iorgov, A. U. Klimyk. Nonstandard deformation $U'_q(\mathrm{so}_n)$: the embedding $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$ and representations. In *Symmetries in science, X (Bregenz, 1997)*, pp. 121–133. Plenum, New York, 1998.

[8] A. M. Gavrilik, N. Z. Iorgov. Representations of the nonstandard algebras $U_q(\mathrm{so}(n))$ and $U_q(\mathrm{so}(n-1,1))$ in Gel′ fand-Tsetlin basis. *Ukraïn Fīz Zh* **43**(6-7):791–797, 1998. International Symposium on Mathematical and Theoretical Physics (Kyiv, 1997).

[9] M. Havlíček, S. Pošta. On the classification of irreducible finite-dimensional representations of $U'_q(\mathrm{so}_3)$ algebra. *J Math Phys* **42**(1):472–500, 2001. DOI:10.1063/1.1328078.

[10] R. Koekoek, R. F. Swarttouw. The askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, 1996. `arXiv:math/9602214`.

[11] P. Terwilliger. Leonard pairs and the q-Racah polynomials. *Linear Algebra Appl* **387**:235–276, 2004. DOI:10.1016/j.laa.2004.02.014.

[12] A. S. Zhedanov. "Hidden symmetry" of Askey-Wilson polynomials. *Teoret Mat Fiz* **89**(2):190–204, 1991. DOI:10.1007/BF01015906.

[13] P. Terwilliger. The universal Askey-Wilson algebra. *SIGMA Symmetry Integrability Geom Methods Appl* **7**:Paper 069, 24, 2011. DOI:10.3842/SIGMA.2011.069.

[14] P. Terwilliger. The universal Askey-Wilson algebra and the equitable presentation of $U_q(\mathrm{sl}_2)$. *SIGMA Symmetry Integrability Geom Methods Appl* **7**:Paper 099, 26, 2011.

[15] P. Terwilliger, A. Žitnik. Distance-regular graphs of q-Racah type and the universal Askey–Wilson algebra. *J Combin Theory Ser A* **125**:98–112, 2014. DOI:10.1016/j.jcta.2014.03.001.

[16] P. Terwilliger. The universal Askey-Wilson algebra and DAHA of type $(C_1^\vee, C_1)$. *SIGMA Symmetry Integrability Geom Methods Appl* **9**:Paper 047, 40, 2013.