PERIODIC POINTS OF LATITUDINAL MAPS OF THE m-DIMENSIONAL SPHERE

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Abstract. Let f be a smooth self-map of the m-dimensional sphere S^m. Under the assumption that f preserves latitudinal foliations with the fibres S^1, we estimate from below the number of fixed points of the iterations of f. The paper generalizes the results obtained by Pugh and Shub and by Misiurewicz.

1. Introduction

Estimating of the growth rate of the number of periodic points for smooth self-maps of compact manifolds is a challenging problem. There are two issues in this problem: an upper bound and a lower one. It is known from the late 90's (Kaloshin [5]) that for a typical diffeomorphism such growth can be arbitrary fast, so there is no upper bound.

The aim of this paper is to find a lower bound in case of self-maps of the m-dimensional sphere S^m preserving latitudinal foliations with the fibres S^1. Our work is strictly related to well-known Shub Conjecture formulated in 1974 and being still an open problem.

Shub and Sullivan [12] considered maps with isolated fixed points of f^n for all n and showed that unboundedness of the sequence of Lefschetz numbers $\{L(f^n)\}_{n=1}^{\infty}$ implies that $\{N(f^n)\}_{n=1}^{\infty}$ is also unbounded, where $N(f^n) = |\operatorname{Fix}(f^n)|$ is the number of fixed points of f^n. The growth of unbounded Lefschetz numbers is exponential, thus Shub conjectured in 1974 that for smooth maps the growth rate of the number of periodic points is also at least (asymptotically) exponential (Problem 4 in [11]):

$$\limsup_{n\to\infty} \frac{\log N(f^n)}{n} \geq \limsup_{n\to\infty} \frac{\log |L(f^n)|}{n}. \tag{1.1}$$

In particular, if the considered manifold is a sphere, then the conjecture takes the form:

$$\limsup_{n\to\infty} \frac{\log N(f^n)}{n} \geq \log |\operatorname{Deg}(f)|, \tag{1.2}$$

where $\operatorname{Deg}(f)$ denotes the degree of the self-map of the sphere.

There were no progress in solving Shub conjecture in the whole generality within the last decades, and the same questions were repeated by Shub as open problems during International Congress of Mathematicians in Madrid in 2006 [10] (Problem 2 and Problem 3).

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Up to now, Shub gave the positive answer for rational maps [9]. Babenko and Bogatyi proved that the growth rate of periodic points of smooth maps should be at least linear [1]. Graff and Jezierski showed that the linear growth can be realized on simply-connected manifolds up to any prescribed period [2].

Hernández-Corbato and Ruiz del Portal proved the conjecture (1.2) for S^2 but replacing the smoothness condition by the assumption that all periodic orbits of a considered map are isolated as invariant sets and the map has no sources of degree r with $|r| > 1$ (where the degree of a source p is its fixed point index $\mathrm{ind}(f, p)$ and the degree of a periodic source of period s is $\mathrm{ind}(f^s, p)$) [3].

Recently, Pugh and Shub [8] and Misiurewicz [7] considered smooth self-maps of S^2 with additional assumption that f preserves latitudinal foliations with the fibres S^1 and found the estimate from below for the number of fixed points of f^n as well as proved Shub Conjecture (1.2) in that case.

Our paper deal with the case of smooth (i.e. C^1) latitudinal self-maps of S^m, where $m \geq 2$. By application of the methods of topological degree in the base (which is $(m-1)$-dimensional now), we extend the results of [7] and [8] to higher dimensions. We obtain the lower bound for $N(f^n)$ in terms of so-called drops (Theorem 5.3) and get asymptotic exponential growth of $N(f^n)$ in many cases (Corollary 5.4).

2. Latitudinal self-maps of a sphere

Consider the m-dimensional sphere $S^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^m : x_1^2 + \dots + x_{m+1}^2 = 1\}$. All maps considered in that paper are assumed to be continuous.

Define the set $J \subset \mathbb{R}^{m-1}$:

$$J = \{(x_3, \dots, x_{m+1}) : x_3^2 + \dots + x_{m+1}^2 \leq 1\},$$

which is a closed $(m-1)$-dimensional ball D^{m-1}.

Now we introduce the class of maps that will be investigated in the paper. Let us define a map $l : S^m \to J$ by the following formula:

$$l(x) = l(x_1, \dots, x_{m+1}) = (x_3, \dots, x_{m+1}). \tag{2.1}$$

Definition 2.1. A map $f : S^m \to S^m$ will be called *latitudinal* if $l(x) = l(y)$ implies $l(f(x) = l(f(y))$.

The next lemma is a straightforward consequence of this definition.

Lemma 2.2. *For a given latitudinal map f there is a uniquely determined map $\varphi : J \to J$ such that*

$$\varphi \circ l = l \circ f, \tag{2.2}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} S^m & \xrightarrow{f} & S^m \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} \\ J & \xrightarrow{\varphi} & J \end{array}$$

Remark 2.3. Each fixed point x of φ in $\mathrm{Int}\, J$ is associated with an invariant circle of f. Indeed, let $x = (x_3, \dots, x_{m+1}) \in J$. Then we define $S_x \subset S^m$ as $S_x =$

$\{(x_1, x_2, x_3, \dots, x_{m+1}) : x_1^2 + x_2^2 = 1 - \|x\|\}$, and we have $l^{-1}(x) = S_x$. By (2.2), the condition $\varphi(x) = x$ implies

$$l(f(l^{-1}(x))) = \varphi(l(l^{-1}(x))) = x,$$

and thus $f(l^{-1}(x)) \subset l^{-1}(x)$, which is equivalent to $f(S_x) \subset S_x$.

However, if $x \in \partial J$ i.e. $x_3^2 + \dots + x_{m+1}^2 = 1$ then $l^{-1}(x) \in S^m$ is a singleton. Thus each point $x \in \partial J$ may be identified with the point $(0, 0, x_3, \dots, x_{m+1})$ in S^m with two first coordinates equal to zero.

For the rest of the paper we will assume that $f : S^m \to S^m$ is a latitudinal map of class C^1.

Now we want to find sufficient conditions for φ to have fixed points in certain subsets of the interior of J. In the case $m = 2$ this was not difficult, because those subsets were intervals mapped onto the interior of J and it was easy to rule out the possibility that the fixed point is an endpoint of J. Here the situation is much more complicated and this part of the paper is its core.

Definition 2.4. Let us consider a subset $U \subset J$. We will call such a subset *regular* if the following conditions are satisfied:

(i) U is an open subset of $\operatorname{Int} J$,
(ii) $\varphi(U) = \operatorname{Int} J$,
(iii) $\varphi(\partial U) \subset \partial J$,
(iv) the degree of the circle maps $f|_{l^{-1}(x)}$ does not depend on $x \in U$.

Notice that the condition (iv) is satisfied if U is a connected set, as then all maps $f|_{l^{-1}(x)}$ are homotopic to each other.

If $x \in \operatorname{Int} J$, then $\varphi^{-1}(x)$ is a closed subset of $\overline{U}$. However, $\varphi^{-1}(x) \subset U$, so $\varphi^{-1}(x)$ is a compact subset of U. As a result, $\deg(\varphi, U, x)$ is well-defined for every $x \in \operatorname{Int} J$ (we use the classical definition of degree, cf. [6], that is, the degree is defined as a sum of signs of the Jacobian of φ at $\varphi^{-1}(x)$ for x being a regular value). Notice that $\deg(\varphi, U, x)$ is independent of the choice of $x \in \operatorname{Int} J$, because $\operatorname{Int} J$ is path-connected (cf. [6]). We will denote this value of $\deg(\varphi, U, x)$ by $\deg(\varphi, U)$.

We will also define another degree for a regular set U. Namely, the common value of degree of the circle maps $f|_{l^{-1}(x)}$ for all $x \in U$ will be called the *latitudinal degree* of U and will be denoted by $\deg(U)$.

3. Periodic points in regular sets

In this section we prove the following result.

Theorem 3.1. *Let U be a regular subset of J and assume that either $|\deg(U)| \neq 1$ or there are no fixed points of φ in ∂U. Then $\deg(\varphi, U) \neq 0$ implies the existence of a fixed point of φ in U.*

We will prove this theorem is steps. We begin by defining a map $\Psi : J \times J \setminus \Delta \to \partial J$, where Δ is the diagonal in $J \times J$. For distinct points $x, y \in J$ we take the ray starting at x and passing through y; then the point of intersection of this ray with ∂J is $\Psi(x, y)$. Then we define a map $\psi : \overline{U} \to \partial J$ by

$$\psi(x) = \begin{cases} \Psi(x, \varphi(x)) & \text{if } x \notin \operatorname{Fix}(\varphi), \\ x & \text{if } x \in \operatorname{Fix}(\varphi), \end{cases} \tag{3.1}$$

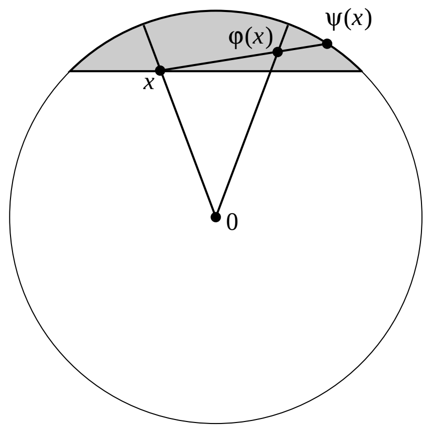

FIGURE 1. The situation from the proof of Lemma 3.2.

where $\mathrm{Fix}(\varphi)$ is the set of fixed points of φ.

The map Ψ is continuous (in fact, even real analytic), so the only discontinuities of ψ can occur at the fixed points of φ. We will be showing that if $|\deg(U)| \neq 1$ then at every fixed point of φ in ∂U the map ψ is continuous.

Lemma 3.2. *Let U be a regular subset of J and assume that φ has no fixed points in U. Assume additionally that every fixed point of φ in ∂U has a neighborhood such that for every x in this neighborhood*

$$\|\varphi(x)\| \geq \|x\|. \tag{3.2}$$

Then the map ψ (defined by the formula (3.1)) is continuous.

Proof. Let x_0 be a fixed point of φ and let $x \in \overline{U}$ be a point close to x_0. Intersect J by the plane through the origin and the points $x, \varphi(x)$. Then in this plane the picture is as in Figure 1. The horizontal line in this figure is perpendicular to the bisector of the angle formed by the rays from the origin through x and $\varphi(x)$. The point $\psi(x)$ lies in the shaded "cap," so its distance from x is not larger than the diameter of the cap. As x approaches x_0, the points x and $\varphi(x)$ get closer and closer to each other and closer and closer to ∂J, so the diameter of the cap goes to zero. Thus, $\psi(x)$ approaches x, which approaches x_0, so $\psi(x)$ approaches x_0. This proves the continuity of ψ at the fixed points of φ, and therefore everywhere. □

If $z = (x_1, x_2, x_3, \ldots, x_{m+1}) \in S^m$ then we will write

$$\|z\|_2 = \sqrt{x_1^2 + x_2^2}.$$

Lemma 3.3. *If $z \in S^m$ and $x = l(z)$ then inequality (3.2) is equivalent to*

$$\|f(z)\|_2 \leq \|z\|_2. \tag{3.3}$$

Proof. We have

$$\|z\|_2^2 = 1 - \|l(z)\|^2 = 1 - \|x\|^2$$

and

$$\|f(z)\|_2^2 = 1 - \|l(f(z))\|^2 = 1 - \|\varphi(l(z))\|^2 = 1 - \|\varphi(x)\|^2,$$

and the equivalence follows. ∎

Now we assume that $p \in \partial J$ is a fixed point of φ. By the second part of Remark 2.3 the points of ∂J are in $1-1$ correspondence with the points $l^{-1}(\partial J)$ in S^m. We will consider the point $l^{-1}(p)$ which has the form $l^{-1}(p) = (0, 0, \cdot, \dots, \cdot)$. For the sake of convenience (as the sphere is homogeneous) we take as $l^{-1}(p)$ the point $z_0 = (0, 0, 0, \dots, 0, 1)$.

In the proof of Lemma 3.5 below, sometimes we will be working in the local coordinates in a neighborhood of z_0. Namely, we choose on $S^m_+ = \{(x_1, \dots, x_{m+1}) : x_1^2 + \dots + x_{m+1}^2 = 1, x_{m+1} > 0\}$ an m-dimensional local coordinate system, by taking first m coordinates of the points $z = (x_1, \dots, x_m, x_{m+1})$ (then $x_{m+1} = \sqrt{1 - x_1^2 - \dots - x_m^2}$). Abusing notation we will use the same letters f and l for the maps in this coordinate system.

In the proof of the first part of Lemma 3.5, we will use the following fact proved in [7].

Lemma 3.4. *A planar C^1 map with the fixed point at $(0,0)$ that maps circles centered at $(0,0)$ to circles centered at $(0,0)$, such that the degrees of the map restricted to these circles are different from ± 1, has the derivative at $(0,0)$ equal to zero.*

Lemma 3.5. *Let U be a regular subset of J and let $p \in \partial U$ be a fixed point of φ. Assume that $|\deg(U)| \neq 1$. Then for all z from a small neighborhood of $l^{-1}(p)$ in $l^{-1}(\overline{U})$ the inequality (3.3) holds.*

Proof. We will use the notation introduced earlier; in particular, $l^{-1}(p) = z_0 = (0, 0, 0, \dots, 0, 1)$. Let

$$Df_{z_0} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.4}$$

where A is a 2×2 matrix.

We will show first that

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5}$$

Let us consider the map $\tilde{f} : D^2((0,0); 1) \to \mathbb{R}^2$, defined by the formula

$$\tilde{f}(x_1, x_2) = P\left(f\left(x_1, x_2, 0, \dots, 0, \sqrt{1 - x_1^2 - x_2^2}\right)\right),$$

where $D^2((0,0); 1)$ denotes the two-dimensional disc centered at $(0,0)$ with radius 1, and P is the projection to the first two coordinates: $P(y_1, \dots, y_{m+1}) = (y_1, y_2)$. As f is a latitudinal map, $\tilde{f}$ maps circles centered at $(0,0)$ to circles centered at $(0,0)$. Indeed, let us consider a circle $S = \{(x_1, x_2) : x_1^2 + x_2^2 = r^2\}$ for some $r > 0$. Then, if

$$f\left(x_1, x_2, 0, \dots, 0, \sqrt{1 - x_1^2 - x_2^2}\right) = (y_1, y_2, y_3, \dots, y_{m+1}),$$

we have

$$
\begin{aligned}
\|\tilde{f}(x_1,x_2)\|^2 &= y_1^2+y_2^2 = 1-y_3^2-\ldots-y_{m+1}^2 \\
&= 1-\left\| l\left(f\left(x_1,x_2,0\ldots,0,\sqrt{1-x_1^2-x_2^2}\right)\right)\right\|^2 \\
&= 1-\left\|\varphi\left(0,\ldots,0,\sqrt{1-r^2}\right)\right\|^2 = const.
\end{aligned} \tag{3.6}
$$

Furthermore, notice that by the definition of $\tilde{f}$ its derivative at $(0,0)$ is equal to

$$D_{(0,0)}\tilde{f} = A.$$

Summarizing, by our assumption $|\deg(U)| \neq 1$ and by (3.6) $\tilde{f}$ maps circles centered at $(0,0)$ to circles centered at $(0,0)$. They map $\tilde{f}$ on those circles is conjugate to the map used to define $\deg(U)$. Thus, by Lemma 3.4, A is the zero matrix.

At this moment we start using the local coordinate system. By the smoothness of f there exists an open ball $V = \{(x_1,\ldots,x_m) : x_1^2+\ldots+x_m^2 < \delta\}$, centered at z_0, such that

$$f(V) \subset S^m_+ \tag{3.7}$$

and

$$\forall_{z\in V}\ \|Df_z - Df_{z_0}\| \leq 1. \tag{3.8}$$

Now, for a given $z \in l^{-1}(\overline{U}) \cap V$, we will find a point z_1 of some special form in a neighborhood of z_0.

Let $z = (x_1,\ldots,x_m) \in l^{-1}(\overline{U}) \cap V$. We move with the point w along the segment $\{(\lambda x_1, \lambda x_2, x_3,\ldots,x_m) : 0 \leq \lambda \leq 1\}$. As λ decreases (notice that we remain in V), w is approaching $l^{-1}(\partial J)$ and thus it has to cross $l^{-1}(\partial U)$. In this way we find a point

$$z_1 = (\lambda x_1, \lambda x_2, x_3, \ldots, x_m) \in l^{-1}(\partial U) \cap V,$$

such that $l(f(z_1)) = \varphi(l(z_1)) \in \partial J$ and thus (cf. the second part of Remark 2.3)

$$f(z_1) = (0,0,\cdot,\ldots,\cdot). \tag{3.9}$$

Set $g(w) = f(w) - Df_{z_0}(w)$. Then $Dg = Df - Df_{z_0}$. By the Mean Value Theorem applied for the map g on the segment $[z_1, z]$ we get

$$\|f(z)-f(z_1)-Df_{z_0}(z-z_1)\|_2 \leq \sup_{\theta\in[0,1]} \|Dg_\xi\|\cdot\|z-z_1\|, \tag{3.10}$$

where $\xi = z_1 + \theta(z - z_1)$.

As $\|Dg_\xi\| = \|Df_\xi - Df_{z_0}\|$ and for each point ξ in the segment $[z_1, z]$ we have $\|Df_\xi - Df_{z_0}\| \leq 1$ by (3.8), we get

$$\|f(z)-f(z_1)-Df_{z_0}(z-z_1)\|_2 \leq \|z-z_1\|. \tag{3.11}$$

Notice that by the special form of $f(z_1)$ given in (3.9) we have

$$\|f(z)\|_2 = \|f(z)-f(z_1)\|_2.$$

Now we prove our main inequality (3.3). Using (3.11) we get

$$
\begin{aligned}
\|f(z)\|_2 &= \|f(z)-f(z_1)\|_2 \\
&\leq \|f(z)-f(z_1)-Df_{z_0}(z-z_1)\|_2 + \|Df_{z_0}(z-z_1)\|_2.
\end{aligned} \tag{3.12}
$$

On the other hand, by the equality (3.5) which states that A is the zero matrix, and the equality

$$z - z_1 = ((1-\lambda)x_1, (1-\lambda)x_2, 0, \dots, 0), \tag{3.13}$$

we get $Df_{z_0}(z - z_1) = (0, 0, \cdot, \dots, \cdot)$, so $\|Df_{z_0}(z - z_1)\|_2 = 0$.

Thus the inequality (3.12) reduces to

$$\|f(z)\|_2 \le \|f(z) - f(z_1) - Df_{z_0}(z - z_1)\|_2. \tag{3.14}$$

Finally, taking into account the inequality (3.11) and (3.13) and the fact that $0 \le \lambda \le 1$, we obtain

$$\|f(z)\|_2 \le \|z - z_1\| \le \sqrt{x_1^2 + x_2^2} \le \|z\|_2. \tag{3.15}$$

This completes the proof. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that φ has no fixed points in U. If φ has no fixed points in ∂U, then clearly the map $\psi : \overline{U} \to \partial J$, given by (3.1), is continuous. If φ has a fixed point in ∂U, but $|\deg(U)| \neq 1$, then by Lemmas 3.2, 3.3 and 3.5, also ψ is continuous.

Define a homotopy $\Phi : [0,1] \times \overline{U} \to J$ between φ and ψ by

$$\Phi(t, x) = (1-t)\varphi(x) + t\psi(x). \tag{3.16}$$

Clearly, Φ is continuous. Since $\varphi(\partial U) \subset \partial J$, the maps φ and ψ coincide on ∂U, and therefore for each t the map $\Phi(t, \cdot)$ coincides with φ on ∂U. In particular, it maps ∂U to ∂J.

As a result, by the homotopy invariance of the degree we get that $\deg(\varphi, U) = \deg(\psi, U)$. On the other hand, $\deg(\psi, U) = 0$ because $\psi(U) \subset \partial J$. This is a contradiction. □

4. Periodic points of φ

In order to estimate the number of fixed points of iterates of f, we decompose $\varphi^{-1}(\operatorname{Int} J)$ into its connectivity components. We will be interested in those components to which we can apply Theorem 3.1.

Definition 4.1. Connectivity components U of $\varphi^{-1}(\operatorname{Int} J)$ that are contained in $\operatorname{Int} J$ and for which $\deg(\varphi, U) \neq 0$, will be called *drops*.

Lemma 4.2. *Each drop is a regular set.*

Proof. Condition (i) of Definition 2.4 follows from the fact that U is an open subset of J in the topology of J and is a subset of the interior of J. The fact that $\deg(\varphi, U) \neq 0$ implies (ii). Condition (iii) is satisfied because U is a connectivity component of $\varphi^{-1}(\operatorname{Int} J)$ and is contained in $\operatorname{Int} J$. Connectivity of U implies (iv). □

Lemma 4.3. *The number of drops is finite.*

Proof. Take an arbitrary point $p \in \operatorname{Int} J$ and consider $\varphi^{-1}(p)$, which is a compact set. The family of all components of $\varphi^{-1}(\operatorname{Int} J)$ is an open cover of $\varphi^{-1}(p)$. We choose its finite subcover. Then among the elements of this subcover there are all drops, because they are pairwise disjoint and have non-empty intersection with $\varphi^{-1}(p)$. □

In Theorem 3.1 we have an additional assumption that either $|\deg(U)| \neq 1$ or there are no fixed points of φ in ∂U. Since we will be using it for iterates of φ, we have to change fixed points to periodic points. This justifies the following definition.

Definition 4.4. A drop U having a periodic point of φ on the boundary and such that $|\deg(U)| = 1$, will be called *tricky*.

Given drops $A_0, A_1, \dots, A_{n-1}$, we consider the set

$$U = U(A_0, \dots, A_{n-1}) = \bigcap_{i=0}^{n-1} \varphi^{-i}(A_i).$$

Lemma 4.5. *Assume that $U = U(A_0, \dots, A_{j-1})$ is regular for φ^j and set $V = U(A_0, \dots, A_{j-1}, A_j)$. Then $\varphi^j(\partial V) \subset \partial A_j$.*

Proof. We have $V = U \cap \varphi^{-j}(A_j)$. Take $x \in \partial V$. If $x \in \partial U$, then, since U is regular for φ^j, we get $\varphi^j(x) \in \partial J$, so $\varphi^j(x) \in \partial A_j$. If $x \in \operatorname{Int} U$, then also $\varphi^j(x) \in \partial A_j$, because otherwise some neighborhood of x would be contained in $\operatorname{Int} U$ and in $\operatorname{Int} \varphi^{-j}(A_j)$, and this contradicts the assumption $x \in \partial V$. Thus, $\varphi^j(\partial V) \subset \partial A_j$. □

Lemma 4.6. *The set $U(A_0, \dots, A_{n-1})$ is regular for φ^n and*

$$\deg(\varphi^n, U(A_0, \dots, A_{n-1})) = \prod_{i=0}^{n-1} \deg(\varphi, A_i). \tag{4.1}$$

Proof. We proceed by induction. For $n = 1$ we have $U = A_0$, so it is a drop, and therefore regular by Lemma 4.2. Clearly, (4.1) holds.

Assume that $U = U(A_0, \dots, A_{n-1})$ is regular for φ^n and (4.1) holds, and prove that that $V = U(A_0, \dots, A_{n-1}, A_n)$ is regular for φ^{n+1} and (4.1) holds with n replaced by $n + 1$.

By Lemma 4.5 (for $j = n$), we have $\varphi^n(\partial V) \subset \partial A_n$. Since A_n is regular, we get $\varphi(\partial A_n) \subset \partial J$. Therefore $\varphi^{n+1}(\partial V) \subset \partial J$, so V is regular for φ^{n+1}.

By multiplicativity of the degree we have

$$\deg(\varphi^{n+1}, V) = \deg(\varphi^n, V) \deg(\varphi, A_n).$$

By the excision property of the degree, $\deg(\varphi^n, V) = \deg(\varphi^n, U)$, because for every $x \in A_n$ we have $\varphi^{-n}(x) \cap U \subset V$. This proves (4.1) with n replaced by $n + 1$. □

Now we are ready to prove the main theorem of this section.

Theorem 4.7. *For every choice of drops $A_0, \dots, A_{n-1}$ such that not all of them are tricky, there exists a fixed point of φ^n in $U = U(A_0, \dots, A_{n-1})$.*

Proof. Assume that there is i such that the drop A_i is not tricky. Then by Lemma 4.6 the set U is regular for φ^n and either its latitudinal degree for φ^n satisfies $|\deg(U)| \neq 1$ or there is no periodic point of φ in ∂A_i. Moreover, $\deg(\varphi^n, U) \neq 0$.

We want to use Theorem 3.1 for φ^n instead of φ. We can do it, because $\varphi^n \circ l = l \circ f^n$, that is, φ^n plays the same role for the latitudinal map f^n as φ plays for f. By what we wrote in the preceding paragraph, to show that the assumptions of Theorem 3.1 are satisfied it remains to prove that if there is a fixed point of φ^n in ∂U then there is a periodic point of φ in ∂A_i.

To do it, assume that $x \in \partial U$ is a fixed point of φ^n. Since U is regular for φ^n, we have $\varphi^n(\partial U) \subset \partial J$. Therefore $x \in \partial J$. Thus, for every $j \geq 1$ we have $x \in \partial U(A_0, \dots, A_j)$ and by Lemma 4.6, the set $U(A_0, \dots, A_{j-1})$ is regular for φ^j. Therefore, by Lemma 4.5, $\varphi^j(x) \in \partial A_j$ (since $x \in \partial J$, we have also $\varphi^0(x) \in \partial A_0$). However, $\varphi^n(\varphi^j(x)) = \varphi^j(x)$, so $\varphi^j(x)$ is a periodic point for φ. $\square$

As a consequence, we get immediately the following estimate.

Corollary 4.8. *Let g denote the number of non-tricky drops. Then*

$$|Fix(\varphi^n)| \geq g^n.$$

5. Estimate for the number of periodic points of f

In this section we will prove the main result of the paper, which gives the estimate for the number of periodic points for smooth latitudinal self-maps of the m-dimensional sphere, much more refined than Corollary 4.8. This theorem is a generalization of the results obtained for S^2 in [7] and [8].

We start with recalling a well-known fact related to the estimate for the number of periodic points of continuous self-maps of the circle, cf. [4].

Lemma 5.1. *Let $h : S^1 \to S^1$ be a continuous map of degree D. Then $|Fix(h)| \geq |D - 1|$.*

We introduce the following notation.

Definition 5.2. For a latitudinal map $f : S^m \to S^m$, we denote by:

k - the total number of drops,
k_+ - the number of drops with positive latitudinal degree,
k_- - the number of drops with negative latitudinal degree,
p - the number of tricky drops,
p_+ - the number of tricky drops with latitudinal degree $+1$,
p_- - the number of tricky drops with latitudinal degree -1.
u - the sum of absolute values of latitudinal degrees over all drops.

Theorem 5.3. *Let $f : S^m \to S^m$ be a C^1 latitudinal map. Then*

$$|\operatorname{Fix}(f^n)| \geq k^n - (k_+ + k_-)^n + u^n - (k_+ - k_-)^n - p^n + (p_+ - p_-)^n. \tag{5.1}$$

Proof. First, we find an estimate without taking into account the existence of tricky drops. By Remark 2.3, each fixed point of φ^n generates an invariant circle of f^n. By Lemma 5.1, each such invariant circle produces $|D-1|$ fixed points of f^n, where D is a degree of f^n on this circle.

We interpret the set of all drops as an alphabet $\mathcal{A}$. By Theorem 4.7, at least some fixed points of φ^n may be identified with the words of length n (maybe there are more fixed points of φ^n, and this is one of the reasons why we get only an inequality). We will denote by d_a the latitudinal degree of the drop identified with the letter a. Then the degree d_w of f^n on the invariant circle determined by the word $w = w_1 \dots w_n$ is equal to the product of latitudinal degrees of all drops that are the letters of the word, that is, $d_w = d_{w_1} \cdot \dots \cdot d_{w_n}$.

We start our calculations by finding the number of fixed points of f^n obtained from words of degree 0. We get it by subtracting from the number of all words the number of the words of non-zero degree. In such a way, we get $k^n - (k_+ + k_-)^n$ fixed points.

Now we count the number of fixed points of f^n obtained from the words with non-zero degree (we denote their set by W_n). Each such word gives the contribution to $|\operatorname{Fix}(f^n)|$ in dependence on the value of its degree d_w. We will write w_i for the ith letter of a word w. This contribution is

$$
\begin{aligned}
(5.2)\qquad \sum_{w\in W_n} \begin{cases} |d_w|-1 \text{ if } d_w>0 \\ |d_w|+1 \text{ if } d_w<0 \end{cases} &= \sum_{w\in W_n} |d_w| - \sum_{w\in W_n} \operatorname{sign}(d_w) \\
&= \sum_{w\in W_n}\prod_{i=1}^{n} |d_{w_i}| - \sum_{w\in W_n}\prod_{i=1}^{n} \operatorname{sign}(d_{w_i}) \\
&= \left(\sum_{a\in\mathcal{A}} |d_a|\right)^n - \left(\sum_{a\in\mathcal{A}} \operatorname{sign}(d_a)\right)^n = u^n - (k_+ - k_-)^n,
\end{aligned}
$$

where in the third equality we change the sum of products into the product of sums. We are taking into account that instead of summing over the words of degree non-zero we can sum over all words, since if $d_w = 0$ then $|d_w| - \operatorname{sign}(d_w) = 0$.

Summing up this part of the proof, we obtained the following estimate:

$$
\operatorname{Fix}(f^n) \geq k^n - (k_+ + k_-)^n + u^n - (k_+ - k_-)^n. \tag{5.3}
$$

However, we have to introduce corrections to this estimate, because some drops may be tricky. Namely, according to Theorem 4.7, we have to subtract the number of words of length n with all letters tricky. Denote the set of all such words Z_n. For such a word w we have $|d_w| = 1$, so its contribution to the right-hand side of (5.3) was computed as $1 - d_w$. This means that we have to subtract from the right-hand side of (5.3) the following expression.

$$
\begin{aligned}
(5.4)\qquad \sum_{w\in Z_n} (1-d_w) &= \sum_{w\in Z_n} 1 - \sum_{w\in Z_n}\prod_{i=1}^{n} d_{w_i} \\
&= p^n - \left(\sum_{a \text{ is tricky}} d_a\right)^n = p^n - (p_+ - p_-)^n.
\end{aligned}
$$

After subtracting the correction (5.4) from (5.3), we get (5.1). ☐

From Theorem 5.3 we can obtain estimates for the asymptotic exponential growth of the number of periodic points of f^n in many situations.

Corollary 5.4. *Let $N(f^n) = |Fix(f^n)|$. We will consider the following cases.*

(1) *There exists a drop of latitudinal degree 0. Then $k > k_+ + k_- \geq p \geq |p_+ - p_-|$ and $u \geq |k_+ - k_-|$, and thus*

$$
\liminf_{n\to\infty} \frac{1}{n} \log N(f^n) \geq \log k.
$$

(2) *Either there exists a drop U with $|\deg(U)| \geq 2$, or both k_+ and k_- are non-zero. Then $u > k_+ + k_- \geq |k_+ - k_-|$ and $u > k_+ + k_- \geq p \geq |p_+ - p_-|$, and thus*

$$
\liminf_{n\to\infty} \frac{1}{n} \log N(f^n) \geq \log u.
$$

(3) *All drops have latitudinal degree* -1*, and there exists a non-tricky drop. Then* $u = k = k_- > p \geq |p_+ - p_-|$*, and thus*

$$\limsup_{n\to\infty} \frac{1}{n} \log N(f^n) \geq \log k = \log u.$$

The above corollary is related to the Shub Conjecture, as it allows one to obtain exponential asymptotic growth of the number of periodic points in terms of drops in many cases. It is interesting that for the 2-dimensional sphere the conjecture is true for all smooth latitudinal maps (see [7], Theorem 3.5). On the other hand, in the cases not listed in Corollary 5.4, i.e., when Theorem 5.3 is too weak to detect exponential growth of periodic points, it still may happen that the Shub Conjecture is true, as it is illustrated by the following example.

Example 5.5. Let $\mathbb{R}^4 = \{(x_1, x_2, z) : x_1, x_2 \in \mathbb{R} \text{ and } z \in \mathbb{C}\}$. We consider a map $f : S^3 \to S^3$, given by the formula $f(x_1, x_2, z) = (O_\alpha(x_1, x_2), z^n)$, where $O_\alpha(x_1, x_2)$ is an irrational rotation and $n \geq 2$. Then $J = D^2$ and $\partial J = S^1$. The map $\varphi : J \to J$ is given by $\varphi(z) = z^n$. Thus, there is only one proper drop U and $\deg(U) = 1$. In $\operatorname{Int} J$ the map φ has only one fixed point $(0, 0)$ and f has no periodic points on the invariant circle $l^{-1}(0, 0)$. However, the growth rate of the number of periodic points of f on $l^{-1}(\partial J)$ is exponential.

The above example motivates us to formulate a weak version of Shub Conjecture (i.e. for latitudinal maps) related to higher dimensions.

Conjecture 5.6. *Let* f *be a latitudinal* C^1 *self-maps of* S^m*,* $m > 2$*. Then Shub Conjecture* (1.2) *holds.*

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