Bethe subalgebras in affine Birman–Murakami–Wenzl algebras and flat connections for q-KZ equations.6

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Dedicated to Professor Rodney Baxter on the occasion of his 75th Birthday.

Abstract. Commutative sets of Jucys–Murphy elements for affine braid groups of $A^{(1)}, B^{(1)}, C^{(1)}, D^{(1)}$ types were defined. Construction of R-matrix representations of the affine braid group of type $C^{(1)}$ and its distinguish commutative subgroup generated by the $C^{(1)}$-type Jucys–Murphy elements are given. We describe a general method to produce flat connections for the two-boundary quantum Knizhnik- Zamolodchikov equations as necessary conditions for Sklyanin's type transfer matrix associated with the two-boundary multicomponent Zamolodchikov algebra to be invariant under the action of the $C^{(1)}$-type Jucys–Murphy elements. We specify our general construction to the case of the Birman–Murakami–Wenzl algebras (BMW algebras for short). As an application we suggest a baxterization of the Dunkl–Cherednik elements Y's in the double affine Hecke algebra of type A.

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Key words. $C^{(1)}$-type affine braid group, Jucys–Murphy subgroup, Yang Baxter equations of types A and C, Baxterization, affine Hecke and Birman–Murakami-Wenzl algebras, Bethe subalgebras, Gaudin models. Flat connections and two-boundary Knizhnik–Zamolodchikov equations.

1 Introduction

The quantum Knizhnik–Zamolodchikov equation (q-KZ equation for shot) is a system of difference equations which has been introduced by F.Smirnov [42], [43], during the study of form factors of integrable models, and independently, by I. Frenkel and N.Reshetikhin, [11] during the study of the representation theory of quantum affine algebras. Since that time the literature that enter into the treatment of qKZ equations, their generalizations and applications, are enormous. We mention here only a few:

- [22], which is concerned to the study of correlation functions of integrable systems;
- [4], which is devoted to applications to the representation theory of affine Hecke algebras;
- [27], [46], [38], which are concerned to the study of variety applications to Algebraic Combinatorics and Algebraic Geometry of certain class of solutions to (boundary) q-KZ equations.
- [39], devoted to the study of Jackson integral solutions of the boundary quantum Knizhnik-Zamolodchikov equation(s) with applications to the representation theory of quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$.

In the present paper we describe a general method for construction of *two-boundary* quantum KZ equations associated with affine Birman–Murakami–Wenzel algebras (BMW algebras) [1], [30], [45], [8], and give several examples to illustrate our method. The underlying idea of our construction is to describe relations/equations among the generators of the multicomponent two-boundary Zamolodchikov algebras [13] which imply that the natural action of the distinguish commutative subgroup of the affine braid group $B_n(C^{(1)})$ of type $C^{(1)}$ generated by the Jucys–Murphy elements $\{JM_i\}$, $i = 1, 2, \ldots, n$, preserves the "monodromy matrix" associated with the Zamolodchikov algebras in question, see Sections 2, 3 and 4 for details. For example, in Section 2 we describe *distinguish* commutative subgroups in the (non-twisted) affine braid groups of classical types. The generators of these distinguish subgroups will be called *universal Jucys–Murphy elements*, or JM-elements for short. Note that the well-known JM-elements in the group ring of the symmetric group [25], or (affine) Hecke [12], [9], Birman–Murakami-Wenzl [21] and cyclotomic Hecke (and cyclotomic BMW) algebras, are images of the universal JM-elements. The main objective of our paper is to construct *Baxterization* of the JM-elements in the affine Birman–Murakami–Wenzl algebras of type $C^{(1)}$, i.e. to construct mutually commuting family of elements $JM_i(x) \in BMW(C^{(1)}) \otimes \mathbb{Q}(x)$ depending on spectral parameter x, such that $JM_i(0) = JM_i$, $\forall i$.

Now let us say few words about the content of our paper.
As it was mentioned, in Section 2 we recall definitions of *distinguish* commutative subgroups in the affine braid groups of classical types. Since the generators of these commutative subgroups are the major origin of the Jucys–Murphy elements in a big variety of algebras, we include the definitions and proofs of universal JM-elements basic properties.

We want to stress that in all known cases, such as the group ring of the symmetric groups, (affine, cyclotomic) Hecke, Brauer, BMW algebras, the corresponding JM-elements come from the distinguish commutative subgroup in the corresponding (affine) braid group of classical type. In fact, birational representations of affine braid group associated with semisimple Lie algebras, give rise to the well-known and widely used integrable systems such as Heisenberg chains and Gaudin models, [15], [14], Painlevé equations, [35] and the literature quoted therein.

In Section 4 we describe a way how to construct R-matrix representations of the affine braid group $B_n(C^{(1)})$ of type $C^{(1)}$, and use these constructions to define the corresponding *quantum* qKZ *equations* and two sets of *flat connections* associated with the former.

Section 5 contains one of our main results concerning of construction of *flat connections* based on the study of two-boundary (multi-component) Zamolodchikov algebras. Namely, qKZ equations are making their appearance to ensure that the two boundary Zamolodchikov algebra in question is invariant under the action of the distinguish commutative subgroup in the corresponding affine braid group. In Section 5.2 we present our main construction, namely that of *flat connections* for quantum Knizhnik–Zamolodchikov equations derived from the study of two boundary Zamolodchikov algebra and the $B_n(C^{(1)})$ universal Jucys–Murphy elements.

In Section 6 we specify our general constructions presented in Section 5 to the case of affine BMW algebras, and construct flat connections for the algebra $BMW(C^{(1)})$. To pass from general construction to the case of the affine Birman–Murakami–Wenzl algebras of type $C^{(1)}$, we rely on the use of embedding the braid group $B_n(C^{(1)})$ into the algebra $BMW(C^{(1)})$.

In Section 7 we construct *baxterized Jucys–Murphy elements* in the affine BMW algebras. Our approach is based on Sklyanin's transfer matrix method [1], [40],[41]. The key to apply the Sklyanin transfer matrix method to construction of *baxterized JM-elements* $\bar{y}_n(x;\vec{z}_{(n)})$, see (7.4), lies in the fact that the family of algebras $\{BMW_n(C)\}_{n\geq 1}$ can be provided with the *Markov trace*, namely, there exists a unique homomorphism

$$Tr_{n+1} : BMW_{n+1}(C) \longrightarrow BMW_n(C), \quad \forall n \geq 1$$

which satisfies a set of "good" properties, stated in Proposition 7.2 (cf [23], [24], [15], [7]). Let's point out here on another important fact that the Jucys–Murphy element $y_n(x)$ satisfies the reflection equation (7.5). We also introduce a family of mutually commuting elements $\tau_n(x;;\vec{z}_(n)) \in BMW_n(C)$, the so-called *dressing JM-operators* which are an analogue of the Sklyanin transfer matrices [40], and the coefficients in the expansion of $\tau_n(x;;\vec{z}_{(n)})$ over the variable x (for the homogeneous case $z_i = 1, \forall$) are the Hamiltonians for the open Birman–Murakami–Wenzl chain models with nontrivial boundary conditions, see e.g. [15], and example at the end of Section 7.1. Section 7.2 is devoted to construction of the Bethe subalgebras in the affine $BMW_n(C)$ algebras and a factorizibility property of the corresponding qKZ connections. We will show that the flat connections $\mathsf{A}'_i(z)$, see (7.26), are images under the map (7.28) of certain elements $\mathsf{J}_i \in B_n(C)$ which under the special limit (6.31) one can deduce the BMW analog (7.29) of the Cherednik's connections have been introduced in [4] for Hecke algebras. As an application, in Section 7 we construct a *baxterization* of the type A Dunkl–Cherednik elements $Y_i \in DAHA$, which have been in-depth studied in [4].

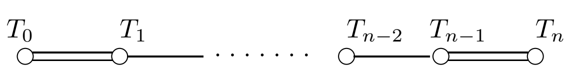
2 Affine braid groups of type $A^{(1)}, B^{(1)}, C^{(1)}, D^{(1)}$ and Jucys–Murphy elements

First consider affine braid group $B_n(C^{(1)})$ with generators $\{T_0, \ldots, T_n\}$ subject to defining relations

$$T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1} \ , \quad i = 1, \ldots, n-2 \ , \tag{2.1}$$

$$\begin{aligned} T_1\, T_0\, T_1\, T_0 &= T_0\, T_1\, T_0\, T_1 \ , \\ T_{n-1}\, T_n\, T_{n-1}\, T_n &= T_n\, T_{n-1}\, T_n\, T_{n-1} \ , \end{aligned} \tag{2.2}$$

where T_0, T_n — two affine generators. Let $||m_{ij}||$ be symmetric matrix with integer coefficients $m_{ij} \geq 2$. The structure relations (2.1), (2.2) of the group $B_n(C^{(1)})$ can be written as $\underbrace{T_i\, T_j\, T_i \cdots}_{m_{ij}} = \underbrace{T_j\, T_i\, T_j \cdots}_{m_{ji}}$ and correspond to the Coxeter graph of the type $C^{(1)}$



(2.3)

where the number of lines between nodes i and j is equal to $(m_{ij} - 2)$. Note that for the group $B_n(C^{(1)})$ defined by (2.1), (2.2) we have two automorphisms ρ_1 and ρ_2:

$$\rho_1(T_i) = T_i^{-1} \ , \quad \rho_2(T_i) = T_{n-i} \ . \tag{2.4}$$

The well known statement is:

[1]Naive replacement of generators T_i in (2.5) by its baxterization $T_i(u/v)$ defined in (6.15), leads to the set of elements in the BMW algebra, which do not commute in general

Proposition 2.1. *The affine braid group $B_n(C^{(1)})$ contains the commutative subgroups which are generated by the following sets of elements*

$$
\begin{array}{l}
\bullet \;\; J_i = \Big(T_{i-1}^{-1}\cdots T_1^{-1}\Big)\,\Big(T_0\cdots T_n\Big)\,\Big(T_{n-1}\cdots T_i\Big)\,, \quad i=1,\dots,n\,, \\
\bullet \;\; \overline{J}_i = \Big(T_{i-1}\cdots T_1\Big)\,\Big(T_0\cdots T_n\Big)\,\Big(T_{n-1}^{-1}\cdots T_i^{-1}\Big)\,, \quad i=1,\dots,n\,, \\
\bullet \;\; (\mathit{Jucys-Murphy\ elements}) \quad a_i := \Big(T_{i-1}\cdots T_1\Big)\,T_0\,\Big(T_1\cdots T_{i-1}\Big)\,, \quad i=1,\dots,n\,, \\
\bullet \;\; (\mathit{Jucys-Murphy\ elements}) \quad b_i := \Big(T_i\cdots T_{n-1}\Big)\,T_n\,\Big(T_{n-1}\cdots T_i\Big) \quad i=1,\dots,n\,.
\end{array}
\tag{2.5}
$$

Proof. The proof of commutativity of the elements a_i is straightforward and follows from the fact that $[a_i,\, T_j]=0$ for $i>j$. The commutativity of the elements b_i follows from the commutativity of elements a_i since we have $b_{n-i+1}=\rho_2(a_i)$, where the automorphism ρ_2 is defined in (2.4).

Now we prove the commutativity of the elements J_i (it will be important for our consideration below). We introduce the element

$$
X = \prod_{k=0}^{n} T_k = T_0\cdots T_n\,. \tag{2.6}
$$

For this element we have the following identities

$$
\begin{array}{c}
X\,T_i = T_{i+1}\,X\,, \quad (i=1,\dots,n-2)\,, \\
T_1\cdot X^2 = T_1\cdot T_0\,T_1\,T_0\,(T_2\,T_1)\,(T_3\,T_2)\cdots(T_{n-1}\,T_{n-2})T_n\,T_{n-1}\,T_n = X^2\,T_{n-1}\,,
\end{array}
\tag{2.7}
$$

where in the proof of these identities we have used (2.1), (2.2). With the help of the operator X (2.6) the element $\overline{J}_k$ (2.5) can be written as

$$
\overline{J}_k = T_{k-1}\cdots T_1\cdot X\cdot T_{n-1}^{-1}\cdots T_k^{-1} = T_{k-1}\cdots T_2\cdot X\cdot T_n^{-1}T_{n-1}^{-1}\cdots T_k^{-1}\,.
$$

Let $k>r$. Then by using (2.1), (2.2) and (2.7) we have

$$
\begin{array}{c}
\overline{J}_k\,\overline{J}_r = (T_{k-1}\cdots T_1\cdot X\cdot T_{n-1}^{-1}\cdots T_k^{-1})\cdot(T_{r-1}\cdots T_1\cdot X\cdot T_{n-1}^{-1}\cdots T_r^{-1}) = \\
= (T_{k-1}\cdots T_1)\cdot X\cdot(T_{r-1}\cdots T_1)\cdot(T_{n-1}^{-1}\cdots T_k^{-1})\cdot X\cdot(T_{n-1}^{-1}\cdots T_r^{-1}) = \\
= (T_{k-1}\cdots T_1)\cdot(T_r\cdots T_2)\cdot X\cdot X\cdot(T_{n-2}^{-1}\cdots T_{k-1}^{-1})\cdot(T_{n-1}^{-1}\cdots T_r^{-1}) = \\
= (T_{r-1}\cdots T_1)\cdot(T_{k-1}\cdots T_1)\cdot X^2\cdot(T_{n-1}^{-1}\cdots T_r^{-1})\cdot(T_{n-1}^{-1}\cdots T_k^{-1}) = \\
= (T_{r-1}\cdots T_1)\cdot(T_{k-1}\cdots T_2)\cdot X^2\cdot T_{n-1}\cdot(T_{n-1}^{-1}\cdots T_r^{-1})\cdot(T_{n-1}^{-1}\cdots T_k^{-1}) = \\
= (T_{r-1}\cdots T_1)\cdot X\cdot(T_{k-2}\cdots T_1)\cdot(T_{n-1}^{-1}\cdots T_{r+1}^{-1})\cdot X\cdot(T_{n-1}^{-1}\cdots T_k^{-1}) = \\
= (T_{r-1}\cdots T_1)\cdot X\cdot(T_{n-1}^{-1}\cdots T_r^{-1})\cdot(T_{k-1}\cdots T_1)\cdot X\cdot(T_{n-1}^{-1}\cdots T_k^{-1}) = \overline{J}_r\,\overline{J}_k\,,
\end{array}
$$

where to obtain the last line we use the identity ($k>r$)

$$
\begin{array}{c}
(T_{k-2}\cdots T_1)\cdot(T_{n-1}^{-1}\cdots T_{r+1}^{-1}) = (T_{k-2}\cdots T_r)\cdot(T_{r-1}\cdots T_1)\cdot(T_{n-1}^{-1}\cdots T_k^{-1})(T_{k-1}^{-1}\cdots T_{r+1}^{-1}) = \\
= (T_{n-1}^{-1}\cdots T_k^{-1})(T_{k-2}\cdots T_r)\cdot(T_{k-1}^{-1}\cdots T_{r+1}^{-1})(T_{r-1}\cdots T_1) = \\
= (T_{n-1}^{-1}\cdots T_k^{-1}T_{k-1}^{-1})(T_{k-1}T_{k-2}\cdots T_r)\cdot(T_{k-1}^{-1}\cdots T_{r+1}^{-1})(T_{r-1}\cdots T_1) = \\
= (T_{n-1}^{-1}\cdots T_{k-1}^{-1})(T_{k-2}^{-1}\cdots T_r^{-1})\cdot(T_{k-1}\cdots T_r)(T_{r-1}\cdots T_1) = (T_{n-1}^{-1}\cdots T_r^{-1})\cdot(T_{k-1}\cdots T_1)\,.
\end{array}
$$

The commutativity of the elements J_i follows from the commutativity of the elements $\overline{J}_i$ since we have $\rho_1(\rho_2(\overline{J}_{n-i+1}))=J_i^{-1}$, where automorphisms ρ_1 and ρ_2 are defined in (2.4). ■

The quotient of the group $B_n(C^{(1)})$ by additional relations $T_i^2=1$ $(\forall i)$ is called Coxeter group of the type $C^{(1)}$. This group is denoted as $W_n(C^{(1)})$. At the end of this Section we present the explicit realization of $W_n(C^{(1)})$ which we use below. Introduce the set of spectral parameters $(z_1, \dots, z_n)$, $z_i \in \mathbb{C}$. Now we define a representation $\mathsf{s}: \; T_i \to s_i$ of B_n:

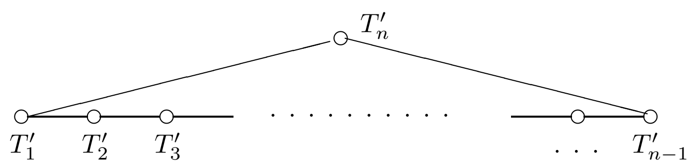
$$
\begin{array}{rcll}
s_i & : & (z_1, \dots, z_i, z_{i+1}, \dots, z_n) \;\; \to \;\; (z_1, \dots, z_{i+1}, z_i, \dots, z_n) & (i = 1, \dots, n-1) \; , \\
s_0 & : & (z_1, z_2, \dots, z_n) \;\; \to \;\; (\sigma(z_1), z_2, \dots, z_n) \; , & \\
s_n & : & (z_1, \dots, z_{n-1}, z_n) \;\; \to \;\; (z_1, \dots, z_{n-1}, \bar{\sigma}(z_n)) \; , &
\end{array}
\tag{2.8}
$$

where σ, $\bar{\sigma}$ are two involutive mappings $\mathbb{C} \to \mathbb{C}$ such that $(\sigma)^2 = 1$, $(\bar{\sigma})^2 = 1$. We specify these involutions in next Sections. From (2.8) one can check that operators s_0, s_i, s_n satisfy (2.1), (2.2) and moreover we have $s_0^2 = s_n^2 = s_i^2 = 1$. Thus, equations (2.8) give the representation of the Coxeter group $W_n(C^{(1)})$. For special choices of σ and $\bar{\sigma}$, namely $\sigma(z) = 1 - z$ and $\bar{\sigma}(z) = -z$, the representation (2.8) have been used in [4],[44].

Remark 1. Denote by $B_n(C)$ the subgroup of the affine braid group $B_n(C^{(1)})$ generated by elements T_i $(i = 0, \dots, n-1)$ with defining relations given in (2.1) and in first line of (2.2). The group $B_n(C)$ is associated to the Coxeter graph of C-type

Consider the homomorphism (projection) ρ: $B_n(C^{(1)}) \to B_n(C)$ such that $\rho(T_i) = T_i$ $(i = 0, \dots, n-1)$ and $\rho(T_n) = 1$. It is clear that under this projection we have $a_i = \rho(\overline{J}_i)$ and it means that the commutativity of a_i follows from the commutativity of $\overline{J}_i$. The elements a_i given in (2.5) generate the commutative set in the subgroup $B_n(C) \subset B_n(C^{(1)})$.

Remark 2. Denote by $B_n(A^{(1)})$ the affine braid group which corresponds to the affine A-type Coxeter graph



We call group $B_n(A^{(1)})$ $(n > 2)$ a periodic A-type braid group. This group is generated by invertible elements T'_i $(i = 1, \dots, n)$ and according to its Coxeter graph we have the defining relations

$$
T'_i \, T'_{i+1} \, T'_i = T'_{i+1} \, T'_i \, T'_{i+1} \; , \quad i = 1, \dots, n \; , \tag{2.9}
$$

where we impose the periodic conditions $T'_{i+n} = T'_i$.

Note that the group $B_n(A^{(1)})$ possesses automorphisms

$$
\rho_3(T'_i) = T'_{i+1} \; , \quad \rho_4(T'_i) = T'_{n-i+1} \; , \quad \rho_5(T'_i) = T'^{-1}_i \; . \tag{2.10}
$$

Define the extension $\bar{B}_n(A^{(1)})$ of the group $B_n(A^{(1)})$ by adding an additional generator $\bar{X}$ with defining relations (cf. (2.7))

$$
\bar{X} \, T'_i = T'_{i+1} \, \bar{X} \quad (i = 1, \dots, n) \quad \Rightarrow \quad T'_1 \cdot \bar{X}^2 = \bar{X}^2 \cdot T'_{n-1} \; . \tag{2.11}
$$

Namely, we add operator $\bar{X}$ which serves the automorphism ρ_3: $\rho_3(T'_i) = \bar{X} \, T'_i \bar{X}^{-1}$ in (2.10). Then for the group $\bar{B}_n(A^{(1)})$ one can construct the following commuting sets of elements

$$
\begin{array}{c}
J'_k = T'^{-1}_{k-1} \cdots T'^{-1}_1 \cdot \bar{X} \cdot T'_{n-1} \cdots T'_k \quad (k = 1, \dots, n) \; , \\
\bar{J}'_k = \rho_5(J'_k) = T'_{k-1} \cdots T'_1 \cdot \bar{X} \cdot T'^{-1}_{n-1} \cdots T'^{-1}_k \quad (k = 1, \dots, n) \; ,
\end{array}
\tag{2.12}
$$

where we have defined $\rho_5(\bar{X}) = \bar{X}$ (this is compatible with (2.11)).

Now we introduce the element $\bar{T}_n$ in $B_n(C^{(1)})$ as following

$$\bar{T}_n := X^{-1}\, T_1 \cdot X = X\, T_{n-1}\, X^{-1} \;\; \in \;\; B_n(C^{(1)}) \; , \tag{2.13}$$

where X is given in (2.6). The element (2.13) satisfies periodic braid relations

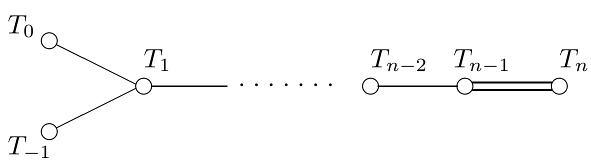
$$\bar{T}_n\, T_{n-1}\, \bar{T}_n = T_{n-1}\, \bar{T}_n\, T_{n-1} \; , \quad \bar{T}_n\, T_1\, \bar{T}_n = T_1\, \bar{T}_n\, T_1 \; ,$$

where we have used (2.7). Thus, we have the homomorphic maps (embeddings) ρ': $B_n(A^{(1)}) \to B_n(C^{(1)})$ and ρ'': $\bar{B}_n(A^{(1)}) \to B_n(C^{(1)})$ such that

$$\begin{gathered} \rho'(T'_i) = T_i \quad (i = 1, \dots, n-1) \; , \qquad \rho'(T'_n) = \bar{T}_n \; , \\ \rho''(T'_i) = T_i \quad (i = 1, \dots, n-1) \; , \qquad \rho''(T'_n) = \bar{T}_n \; , \quad \rho''(\bar{X}) = X \; . \end{gathered}$$

It means that $B_n(A^{(1)})$ and $\bar{B}_n(A^{(1)})$ are subgroups in $B_n(C^{(1)})$ with generators $(T_1, \dots, T_{n-1}, \bar{T}_n)$ and $(T_1, \dots, T_{n-1}, \bar{T}_n, X)$, respectively.

Remark 3. Consider the braid group $B_{n+1}(B^{(1)})$ which is associated to the graph



The defining relations for this group are

$$\begin{gathered} T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1} \; , \quad i = 0, 1, \dots, n-1 \; , \\ T_{-1}\, T_1\, T_{-1} = T_1\, T_{-1}\, T_1 \;\; , \qquad T_{-1}\, T_0 = T_0\, T_{-1} \; , \\ T_{n-1}\, T_n\, T_{n-1}\, T_n = T_n\, T_{n-1}\, T_n\, T_{n-1} \; . \end{gathered} \tag{2.14}$$

Introduce the element

$$\tilde{T}_0 = T_{-1}\, T_0 \; , \tag{2.15}$$

which in view of (2.14) satisfies relation

$$\tilde{T}_0\, T_1\, \tilde{T}_0\, T_1 = T_1\, \tilde{T}_0\, T_1\, \tilde{T}_0 \tag{2.16}$$

So, $B_n(C^{(1)})$ is a subgroup in $B_{n+1}(B^{(1)})$ and we have the homomorphism (embedding) $\tilde{\rho}$: $B_n(C^{(1)}) \to B_{n+1}(B^{(1)})$ which is defined by the map

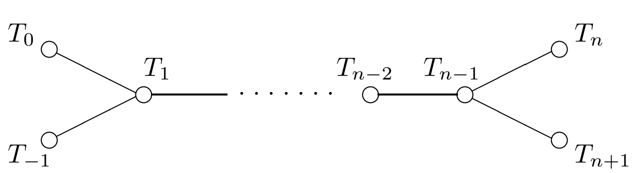
$$\tilde{\rho} \;:\; T_0 \;\to\; \tilde{T}_0 \; , \quad T_i \;\to\; T_i \qquad (i = 1, \dots, n) \; . \tag{2.17}$$

Thus, according to the Proposition 2.1 we have the following commuting sets for the group $B_{n+1}(B^{(1)})$

$$\begin{aligned} \tilde{J}_i &= \left(\prod_{k=i-1}^{1} T_k^{-1} \right) \tilde{X} \left(\prod_{k=n-1}^{i} T_k \right) \quad (i = 1, \dots, n) \; , \\ \overline{\tilde{J}}_i &= \left(\prod_{k=i-1}^{1} T_k \right) \tilde{X} \left(\prod_{k=n-1}^{i} T_k^{-1} \right) \quad (i = 1, \dots, n) \; , \end{aligned} \tag{2.18}$$

where $\tilde{X} = \tilde{T}_0\, T_1 \cdots T_n$ is the image of the element $X \in B_{n+1}(C^{(1)})$ presented in (2.6).

Remark 4. The braid group $B_{n+2}(D^{(1)})$ which is associated with the graph



has defining relations

$$
\begin{aligned}
T_i\,T_{i+1}\,T_i &= T_{i+1}\,T_i\,T_{i+1}\ , \qquad i=0,1,\dots,n\ ,\\
T_{-1}\,T_1\,T_{-1} &= T_1\,T_{-1}\,T_1\ \ , \qquad T_{-1}\,T_0 = T_0\,T_{-1}\ ,\\
T_{n-1}\,T_{n+1}\,T_{n-1} &= T_{n+1}\,T_{n-1}\,T_{n+1}\ \ , \qquad T_n\,T_{n+1} = T_{n+1}\,T_n\ .
\end{aligned}
\tag{2.19}
$$

Note that the element $\tilde{T}_n = T_n\,T_{n+1}$ obeys relations

$$
\tilde{T}_n\,T_{n-1}\,\tilde{T}_n\,T_{n-1} = T_{n-1}\,\tilde{T}_n\,T_{n-1}\,\tilde{T}_n\ .
$$

Thus the elements $(T_{-1},T_0,T_1,\dots,T_{n-1},\tilde{T}_n)$ generate the subgroup $B_{n+1}(B^{(1)})$ in $B_{n+2}(D^{(1)})$ and we have the homomorphism (embedding) $\rho_0 : B_{n+1}(B^{(1)}) \to B_{n+2}(D^{(1)})$ such that

$$
\rho_0 : T_i \ \ \to \ \ T_i \quad (i=-1,0,1,\dots,n-1)\ , \quad \rho_0 : T_n \ \ \to \ \ \tilde{T}_n\ .
\tag{2.20}
$$

Define the element (cf. (2.6))

$$
X'' = \tilde{T}_0\,T_1\cdots T_{n-1}\,\tilde{T}_n\ ,
$$

where $\tilde{T}_0$ is defined as in (2.15). Then we again have two sets of commuting elements (cf. (2.5), (2.18))

$$
\begin{aligned}
J''_i &= \left(\prod_{k=i-1}^{1} T_k^{-1}\right) X'' \left(\prod_{k=n-1}^{i} T_k\right) \quad (i=1,\dots,n)\ ,\\
\overline{J}''_i &= \left(\prod_{k=i-1}^{1} T_k\right) X'' \left(\prod_{k=n-1}^{i} T_k^{-1}\right) \quad (i=1,\dots,n)\ .
\end{aligned}
\tag{2.21}
$$

Finally we stress that the quotient of the group $B_{n+2}(D^{(1)})$ with respect to the relations $T_0 = T_{-1}$ (or $T_n = T_{n+1}$) is isomorphic to the braid group $B_{n+2}(D)$ associated to the Coxeter graph of classical D-type. The commutative elements in this case are given by the same formulas as in (2.18), where instead of $\tilde{X}$ we have to substitute element $X(D) = T_0^2 T_1\cdots T_{n-1}\tilde{T}_n$ (or $X(D) = \tilde{T}_0\,T_1\cdots T_{n-1}T_n^2$).

3 General picture

1. Affine root systems and affine Weyl groups (see [5, Section 1]).

Let R_n be a root system of type $A_n, B_n, \dots, F_n, G_n$. We will write R also for the type of the root system. Let $\alpha_1,\dots,\alpha_n \in R_n$ be simple roots, $\omega_1^\vee,\dots,\omega_n^\vee$ — fundamental coweights, $(\omega_i^\vee,\alpha_i) = \delta_i^j$, θ — the maximal root. The Dynkin diagram of the affine root system $R_n^{(1)}$ is obtained by adding the root $-\theta$ to the simple roots $\alpha_1,\dots,\alpha_n$. The affine simple root is $\alpha_0 = [-\theta, 1]$ in the notation of [5].

For $\alpha \in R_n$, denote $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. Let $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ be the coroot lattice, $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}\,\omega_i^\vee$ be the coweight lattice, and $P_+^\vee = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i^\vee$

Let s_α be the reflection corresponding to a root $\alpha \in R_n^{(1)}$, and $s_i = s_{\alpha_i}$. The Weyl group W of type R_n is generated by the reflections $s_1,\dots,s_n$.

The affine Weyl group $W^{(a)}$ of type $R_n^{(1)}$ is generated by the reflections $s_0, s_1,\dots,s_n$ and is isomorphic to the semidirect product $W \ltimes Q^\vee$, with $s_0 = \theta^\vee s_\theta$. Here we identify W and $Q^\vee$ with the respective subgroups of W^a.

The extended affine Weyl group $W^{(b)}$ of type $R_n^{(1)}$ is the semidirect product $\widetilde{W} = W \ltimes P^\vee$. It is also isomorphic to the semidirect product $\Pi \ltimes W^a$, where $\Pi = P^\vee/Q^\vee$. The elements of the subgroup $\Pi \subset \widetilde{W}$ "permute" the reflections $s_0,\dots,s_n$ — for any i and $\pi \in \Pi$, $\pi s_i \pi^{-1} = s_j$ for some $j = \pi[i]$.

Define the length on $\widetilde{W}$ by $\ell(s_i) = 1$ and $\ell(\pi) = 0$ for $\pi \in \Pi$. Then for $b, b' \in P_+^\vee \subset \widetilde{W}$,

$$
\ell(b+b') \ = \ \ell(b) + \ell(b')\,,
\tag{3.1}
$$

see [5, Proposition 1.4].

The affine braid group $B(R_n^{(1)})$ is generated by the elements $S_0,\dots,S_n$ subject to the same braid relations as $s_0,\dots,s_n$ (we use S_i to keep distinction from the generators T_i in Section 2.) The extended affine braid group $\widetilde{B}(R_n^{(1)})$ is the semidirect product $\Pi\ltimes B(R_n^{(1)})$ — for any i and $\pi\in\Pi$, $\pi S_i\pi^{-1}=S_{\pi[i]}$, (cf. with relations (i), (ii) in [5, Definition 3.1]).

For $\widetilde{w}\in\widetilde{W}$ with a reduced decomposition $\widetilde{w}=\pi s_{i_1}\dots s_{i_k}$, $\pi\in\Pi$, $k=\ell(\widetilde{w})$, the element $S_{\widetilde{w}}=\pi S_{i_1}\dots S_{i_k}\in\widetilde{W}$ does not depend on the reduced decomposition, and $S_{\widetilde{w}\widetilde{w}'}=S_{\widetilde{w}}S_{\widetilde{w}'}$ provided $\ell(\widetilde{w}\widetilde{w}')=\ell(\widetilde{w})+\ell(\widetilde{w}')$, $\widetilde{w},\widetilde{w}'\in\widetilde{W}$. Hence, the elements S_b, $b\in P_+^\vee\subset\widetilde{W}$ generate a commutative subgroup of $\widetilde{W}$ because $S_bS_{b'}=S_{b+b'}=S_{b'}S_b$ for any $b,b'\in P_+^\vee\subset\widetilde{W}$, see (3.1). (Cf. with [5, formula (3.8)].)

For fundamental coweights $\omega_1^\vee,\dots,\omega_n^\vee$, set

$$Y_i\,=\,S_{\omega_i^\vee},\qquad i=1,\dots,n\,. \tag{3.2}$$

The elements $Y_1,\dots,Y_n\in\widetilde{W}$ pairwise commute.

2. Groups $\widehat{B}(C_n^{(1)})$ and $\widetilde{B}(C_n^{(1)})$.

The group $\widehat{B}(C_n^{(1)})$ is generated by the elements $T_0,\dots,T_n\in B(C_n^{(1)})$, see (2.1), (2.2) and by the element U with relations

$$UT_iU^{-1}\,=\,T_{n-i}\,,\quad i=0,\dots,n\,. \tag{3.3}$$

In other words, $UGU^{-1}=\rho_2(G)$ for any $G\in B(C_n^{(1)})$, where ρ_2 is given by formula (2.4). The element U^2 is central.

Set $I_i\,=\,J_1\dots J_i$, $i=1,\dots,n$, where $J_1,\dots,J_n$ are given by (2.5). Also,

$$I_i\,=\,(XT_{n-1}\dots T_i)^{\,i},\qquad i=1,\dots,n\,, \tag{3.4}$$

where $X=T_0\dots T_n$, see (2.6). Let

$$Z\,=\,T_0\dots T_{n-1}\,T_0\dots T_{n-2}\,\dots\,T_0\,T_1\,T_0\,U\,. \tag{3.5}$$

The element Z commutes with $T_1,\dots,T_{n-1}$ and X, and hence by (3.4), commutes with $I_1,\dots,I_n$. Moreover, $Z^2=I_nU^2$. One more nice formula

$$I_i\,=\,X^i\,T_{n-i}\dots T_1\,T_{n-i+1}\dots T_2\,\dots\,T_{n-1}\dots T_i\,. \tag{3.6}$$

The group $\widetilde{B}(C_n^{(1)})$ is the quotient of $\widehat{B}(C_n^{(1)})$ by relation $U^2=1$. The identification is $S_i=T_i$, $i=0,\dots,n$, and $\Pi=\{1,U\}$. Also, $Y_i=I_i$, $i=1,\dots,n-1$, and $Y_n=Z$.

3. Groups $B(B_n^{(1)})$ and $\widetilde{B}(B_n^{(1)})$.

The group $\widetilde{B}(B_n^{(1)})$ is the quotient of $B(C_n^{(1)})$ by relation $T_0^2=1$. The identification is $S_i=T_i$, $i=1,\dots,n$, $S_0=T_0T_1T_0$, and $\Pi=\{1,T_0\}$. Thus $S_0S_1=S_1S_0$ and $S_0S_2S_0=S_2S_0S_2$. Also $Y_i=I_i$, $i=1,\dots,n$. The commutative subgroup in $B(B_n^{(1)})$ is generated by the products $J_1J_i=I_1\,I_i\,I_{i-1}^{-1}$, $i=1,\dots,n$. Here $I_0=1$.

The relation with elements (2.18), (2.21) is explained farther.

4. Groups $B(D_n^{(1)})$ and $\widetilde{B}(D_n^{(1)})$.

The groups $B(D_n^{(1)})$ and $\widetilde{B}(D_n^{(1)})$ are subquotients of $\widetilde{B}(C_n^{(1)})$. Let $\widetilde{B}'(C_n^{(1)})$ be the quotient of $\widetilde{B}(C_n^{(1)})$ by relations $T_0^2=1$, $T_n^2=1$. (Recall that $U^2=1$ in $\widetilde{B}(C_n^{(1)})$.) The subgroup $B(D_n^{(1)})\subset\widetilde{B}'(C_n^{(1)})$ is generated by $S_0=T_0T_1T_0$, $S_n=T_nT_{n-1}T_n$, and $S_i=T_i$, $i=1,\dots,n-1$.

Let $\Pi_n=\{1,T_0U,(T_0U)^2,(T_0U)^3\}=\{1,T_0U,T_0T_n,T_nU\}$ if n is odd, and $\Pi_n=\{1,T_0T_n,U,T_0T_nU\}$ if n is even. The subgroup $\widetilde{B}(D_n^{(1)})\subset\widetilde{B}'(C_n^{(1)})$ is generated by $B(D_n^{(1)})$ and $\Pi=\Pi_n$.

Also $Y_i = I_i,\ i = 1, \ldots, n-2,\ Y_{n-1} = I_{n-1}Z^{-1}$ and $Y_n = Z$. The subgroup Π_n can be recovered from the requirement that I_1 and Z belong to the subgroup generated by $B(D_n^{(1)})$ and $\Pi = \Pi_n$.

The commutative subgroup in $B(D_n^{(1)})$ is generated by the products $J_1 J_i = I_1\, I_i\, I_{i-1}^{-1},\ i = 1, \ldots, n$. Here $I_0 = 1$.

5. Recursive definition of $I_1, \ldots, I_n$.

Let $T_0', \ldots, T_{n-1}'$ denote the generators of $B(C_{n-1}^{(1)})$, and similarly for $I_1', \ldots, I_{n-1}'$. There is an embedding

$$\mu : B(C_{n-1}^{(1)}) \,\to\, B(C_n^{(1)})\,, \tag{3.7}$$

$$\mu(T_i') \,=\, T_i\,, \quad i = 0, \ldots, n-2\,, \qquad \mu(T_{n-1}') \,=\, T_{n-1}T_nT_{n-1}\,.$$

Then

$$\mu(I_i') \,=\, I_i\,, \quad i = 1, \ldots, n-1\,. \tag{3.8}$$

This suggests a proof that the elements $I_1, \ldots, I_{n-1}, Z$ pairwise commute. Since $Z^2 = I_n U^2$, by induction it suffices to prove only that Z commutes with $I_1, \ldots, I_{n-1}$. This follows from the fact that Z commutes with $T_1, \ldots, T_{n-1}$ and X, and formula (3.4).

The recursive definition of $I_1, \ldots, I_n$ is reminiscent of the construction of Gelfand-Zetlin subalgebras, though I_n is not central.

6. Relation with elements (2.18), (2.21).

To get elements (2.18) compose the embedding μ with automorphisms (2.4) for $B(C_{n-1}^{(1)})$ and $B(C_n^{(1)})$: $\lambda = \rho_2 \circ \mu \circ \rho_2'$,

$$\lambda(T_i') \,=\, T_{i+1}\,, \quad i = 1, \ldots, n-1\,, \qquad \lambda(T_0') \,=\, T_1T_0T_1\,. \tag{3.9}$$

The elements of the image $\lambda\big(B(C_{n-1}^{(1)})\big)$ commute with T_0. The next formulae define one more embedding $\tilde\lambda : B(C_{n-1}^{(1)}) \to B(C_n^{(1)})$:

$$\tilde\lambda(T_i') \,=\, T_{i+1}\,, \quad i = 1, \ldots, n-1\,, \qquad \tilde\lambda(T_0') \,=\, T_0T_1T_0T_1\,. \tag{3.10}$$

Taking the quotient by the relation $T_0^2 = 1$ projects $B(C_n^{(1)})$ into $\widetilde{B}(B_n^{(1)})$ and formulae (3.10) into

$$\tilde\lambda(T_i') \,=\, S_{i+1}\,, \quad i = 1, \ldots, n-1\,, \qquad \tilde\lambda(T_0') \,=\, S_0S_1\,. \tag{3.11}$$

(Recall that $S_0 = T_0T_1T_0$ and $S_i = T_i,\ i = 1, \ldots, n$.) Formulae (3.11) coincide with the embedding $\tilde\rho$ in (2.17) up to relabeling of generators.

To get elements (2.21), the game is similar. First take an embedding $B(C_{n-2}^{(1)}) \to B(C_n^{(1)})$,

$$T_i'' \mapsto T_{i+1}\,, \quad i = 1, \ldots, n-3\,, \qquad T_0'' \mapsto T_0T_1T_0T_1\,, \quad T_{n-2}'' \mapsto T_{n-1}T_nT_{n-1}T_n\,, \tag{3.12}$$

where $T_0'', \ldots, T_{n-2}''$ are the generators of $B(C_{n-2}^{(1)})$, and then the quotient by the relations $T_0^2 = 1$, $T_n^2 = 1$. Then formulae (3.12) induce an embedding $B(C_{n-2}^{(1)}) \to \widetilde{B}(B_n^{(1)})$,

$$T_i'' \mapsto S_{i+1}\,, \quad i = 1, \ldots, n-3\,, \qquad T_0'' \mapsto S_0S_1\,, \quad T_{n-2}'' \mapsto S_{n-1}S_n\,. \tag{3.13}$$

Recall that $S_0 = T_0T_1T_0,\ S_n = T_nT_{n-1}T_n$, and $S_i = T_i,\ i = 1, \ldots, n-1$. Formulae (3.13) coincide with the embedding ρ_0 in (2.20) up to relabeling of generators.

7. One more automorphism of $B(C_n^{(1)})$.

Consider the element Z, see (3.5). In addition to commutativity

$$ZT_i = T_i Z\,, i = 1, \dots, n-1\,, \qquad ZX \,=\, XZ\,, \tag{3.14}$$

we also have

$$ZT_1 \dots T_n = T_0 \dots T_{n-1} Z\,, \qquad ZT_1 \dots T_{n-1} = T_1 \dots T_{n-1} Z\,, \tag{3.15}$$

that is, $ZT_0^{-1}X = XT_n^{-1}Z$ and $ZT_0^{-1}XT_n^{-1} = T_0^{-1}XT_n^{-1}Z$. Consider an automorphism

$$\varphi : \widehat{B}(C_n^{(1)}) \to \widehat{B}(C_n^{(1)})\,, \qquad \varphi(G) \,=\, ZGZ^{-1}\,. \tag{3.16}$$

Then $\varphi(T_i) \,=\, T_i,\ i = 1, \dots, n-1$,

$$\varphi(T_0) \,=\, T_0\, T_1 \dots T_{n-1}\, T_n\, T_{n-1}^{-1} \dots T_0^{-1} \,=\, XT_nX^{-1}. \tag{3.17}$$

$$\varphi(T_n) \,=\, T_{n-1}^{-1} \dots T_1^{-1} T_0\, T_1 \dots T_{n-1} \,=\, T_nX^{-1}T_0XT_n^{-1} \,=\, J_nT_n^{-1}, \tag{3.18}$$

Notice that Z commutes with $I_1, \dots, I_n$ given by (3.4), that is, $\varphi(I_i) = I_i,\ i = 1, \dots, n$.

The subgroup $B(C_n^{(1)})$ is invariant under the automorphism φ.

4 R-matrix representation of $B_n(C^{(1)})$.

Define an R-operator acting in the tensor product $V \otimes V$ of two N-dimensional vector spaces V

$$R(x,y) \cdot (\vec{e}_{k_1} \otimes \vec{e}_{k_2}) = (\vec{e}_{i_1} \otimes \vec{e}_{i_2})\; R^{i_1 i_2}_{k_1 k_2}(x,y)\;. \tag{4.1}$$

Here vectors $\{\vec{e}_1, \dots, \vec{e}_N\}$ form a basis in V and components $R^{i_1 i_2}_{k_1 k_2}(x,y)$ are functions of two spectral parameters x and y. Let operator R satisfies Yang-Baxter equation:

$$R_{12}(x,y)\; R_{13}(x,z)\; R_{23}(y,z) = R_{23}(y,z)\; R_{13}(x,z)\; R_{12}(x,y) \;\;\in\;\; \mathrm{End}(V \otimes V \otimes V)\;, \tag{4.2}$$

where we have used the standard matrix notations [10]. Now we introduce two matrices $||K^i_j|| \in \mathrm{Mat}(V)$ and $||\overline{K}^i_j|| \in \mathrm{Mat}(V)$ with elements which are operators acting in the spaces $\widetilde{V}$ and $\widetilde{V}'$, respectively. In other words we have two operators $K \in \mathrm{End}(V \otimes \widetilde{V})$ and $\overline{K} \in \mathrm{End}(V \otimes \widetilde{V}')$. Let these operators be solutions of the equation

$$R_{12}(x,y)\; K_1(x)\; R_{21}(y,\bar{x})\; K_2(y) = K_2(y)\; R_{12}(x,\bar{y})\; K_1(x)\; R_{21}(\bar{y},\bar{x}) \;\in\; \mathrm{End}(\tilde{V} \otimes V \otimes V)\;, \tag{4.3}$$

which is called reflection equation and equation (cf. (4.3))

$$R_{12}(x,y)\; \overline{K}_2(y)\; R_{21}(\tilde{y},x)\; \overline{K}_1(x) = \overline{K}_1(x)\; R_{12}(\tilde{x},y)\; K_2(y)\; R_{21}(\tilde{y},\tilde{x}) \;\in\; \mathrm{End}(V \otimes V \otimes \tilde{V}')\;, \tag{4.4}$$

which is called dual reflection equation. We explain this terminology and the meaning of the equations (4.2), (4.3) and (4.4) in the next Section. In equations (4.3) and (4.4) we have used notations

$$\bar{x} = \sigma(x)\;, \quad \tilde{x} = \bar{\sigma}(x)\;, \tag{4.5}$$

where σ and $\bar{\sigma}$ are the same involutive mappings $\mathbb{C} \to \mathbb{C}$ which were introduced in (2.8).

Using operator $R(x,y)$, which is defined in (4.1) and (4.2), we introduce the set of R-operators $R_{k,k+1}(x,y)$ $(k = 1, \dots, n-1)$ which act in the space $V^{\otimes n}$

$$R_{k,k+1}(x,y) = I^{\otimes(k-1)} \otimes R(x,y)\; \otimes I^{\otimes(n-k-1)}\;. \tag{4.6}$$

For us it will be also convenient to introduce operators

$$\hat{R}_k(x,y) \equiv \hat{R}_{k,k+1}(x,y) = I^{\otimes(k-1)} \otimes P \cdot R(x)\; \otimes I^{\otimes(n-k-1)} \qquad (k = 1, \dots, n-1)\;, \tag{4.7}$$

$$R_{k,r}(x,y) = P_{r,k+1} \cdot (I^{\otimes(k-1)} \otimes R(x) \otimes I^{\otimes(n-k-1)}) \cdot P_{r,k+1} \; , \tag{4.8}$$

where P is a permutation operator in $V \otimes V$

$$P \cdot (v_1 \otimes v_2) = (v_2 \otimes v_1) \qquad \forall v_1, v_2 \in V \; ,$$

and $P_{r,k} = P_{k,r}$ is the permutation operator in $V^{\otimes n}$ such that

$$P_{r,k}(v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_r \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_r \otimes \cdots \otimes v_k \otimes \cdots \otimes v_n) \; .$$

In terms of operators (4.7) equations (4.2), (4.3) and (4.4) can be written in the form

$$\hat{R}_k(x,y)\; \hat{R}_{k+1}(x,z)\; \hat{R}_k(y,z) = \hat{R}_{k+1}(y,z)\; \hat{R}_k(x,z)\; \hat{R}_{k+1}(x,y) \; , \tag{4.9}$$

$$\hat{R}_{12}(x,y)\; K_1(x)\; \hat{R}_{12}(y,\bar{x})\; K_1(y) = K_1(y)\; \hat{R}_{12}(x,\bar{y})\; K_1(x)\; \hat{R}_{12}(\bar{y},\bar{x}) \; , \tag{4.10}$$

$$\hat{R}_{12}(x,y)\; \overline{K}_2(y)\; \hat{R}_{12}(\tilde{y},x)\; \overline{K}_2(x) = \overline{K}_2(x)\; \hat{R}_{12}(\tilde{x},y)\; \overline{K}_2(y)\; \hat{R}_{12}(\tilde{y},\tilde{x}) \; , \tag{4.11}$$

where

$$K_k(x) = I^{\otimes(k-1)} \otimes K(x) \otimes I^{\otimes n-k-1} \; , \quad \overline{K}_k(x) = I^{\otimes(k-1)} \otimes \overline{K}(x) \otimes I^{\otimes n-k-1} \qquad (k = 1, \dots, n-1) \; .$$

Introduce the set of spectral parameters $\{z_1, \dots, z_n\}$. By using the group of the elements s_i (see (2.8)) and matrices $\hat{R}_k(z_k, z_{k+1})$, $K_1(z_1)$, $\overline{K}_n(z_n)$ we construct the representation ρ of the affine group $B_n(C^{(1)})$ in $\tilde{V} \otimes V^{\otimes n} \otimes \tilde{V}'$

$$\rho(T_i) \;=\; s_i\, \hat{R}_i(z_i, z_{i+1}) \quad (i = 1, \dots, n-1) \; , \quad \rho(T_0) \;=\; K_1(z_1)\, s_0 \; , \quad \rho(T_n) \;=\; \overline{K}_n(z_n)\, s_n \; . \tag{4.12}$$

One can directly check that $\rho(T_i)$ $(i = 0, \dots, n)$ satisfy defining relations in (2.1), (2.2) if $\hat{R}_k(z_k, z_{k+1})$ and $K_1(z_1)$, $\overline{K}_n(z_n)$ satisfy relations (4.9), (4.10), (4.11).

Further we will use the operator D_{z_k} such that for any wave function $\Psi(z_1, \dots, z_n)$ and any operator $f(z_1, \dots, z_n)$ we have

$$\begin{gathered} D_{z_k} \cdot f(z_1, \dots, z_k, \dots, z_n) = f(z_1, \dots, \tilde{\bar{z}}_k, \dots, z_n) \cdot D_{z_k} \; , \\ D_{z_k} \cdot \Psi(z_1, \dots, z_k, \dots, z_n) = \Psi(z_1, \dots, \tilde{\bar{z}}_k, \dots, z_n) \; , \end{gathered} \tag{4.13}$$

where $\tilde{\bar{z}}_k = \bar{\sigma}(\sigma(z_k))$. We note that the operator D_{z_k} in (4.13) can be written in the representation (2.8) as

$$D_{z_k} = (s_{k-1} \cdots s_1)(s_0 \cdots s_n)(s_{n-1} \cdots s_k) = \mathsf{s}(J_k) \; , \tag{4.14}$$

where elements J_k were introduced in (2.5).

Theorem 4.1. *The images of the commutative elements (2.5) are operators in* $\tilde{V} \otimes V^{\otimes n} \otimes \tilde{V}'$

$$\begin{aligned} \rho(J_i) = \mathsf{A}_i &= \hat{R}^{-1}_{k-1}(z_{k-1}, z_k) \cdots \hat{R}^{-1}_1(z_1, z_k)\; K_1(z_k)\; \hat{R}_1(z_1, \bar{z}_k) \cdots \hat{R}_{k-1}(z_{k-1}, \bar{z}_k) \cdot \\ &\hat{R}_k(z_{k+1}, \bar{z}_k) \cdots \hat{R}_{n-1}(z_n, \bar{z}_k)\; \overline{K}_n(\bar{z}_k)\; \cdot D_{z_k}\; \cdot \hat{R}_{n-1}(z_k, z_n) \cdots \hat{R}_k(z_k, z_{k+1}) \; , \\ \rho(\overline{J}_i) = \overline{\mathsf{A}}_i &= \hat{R}_{k-1}(z_k, z_{k-1}) \cdots \hat{R}_1(z_k, z_1)\; K_1(z_k)\; \hat{R}_1(z_1, \bar{z}_k) \cdots \hat{R}_{k-1}(z_{k-1}, \bar{z}_k) \cdot \\ &\hat{R}_k(z_{k+1}, \bar{z}_k) \cdots \hat{R}_{n-1}(z_n, \bar{z}_k)\; \overline{K}_n(\bar{z}_k)\; \cdot D_{z_k}\; \cdot \hat{R}^{-1}_{n-1}(z_n, z_k) \cdots \hat{R}^{-1}_k(z_{k+1}, z_k) \; , \end{aligned} \tag{4.15}$$

form two sets of flat connections for quantum Knizhnik-Zamolodchikov equations

$$\begin{gathered} \mathsf{A}_k(z_1, \dots, z_k, \dots, z_n)\;\; \Psi(z_1, \dots, z_k, \dots, z_n) = \Psi(z_1, \dots, z_k, \dots, z_n) \; , \\ \overline{\mathsf{A}}_k(z_1, \dots, z_k, \dots, z_n)\;\; \overline{\Psi}(z_1, \dots, z_k, \dots, z_n) = \overline{\Psi}(z_1, \dots, z_k, \dots, z_n) \; , \end{gathered} \tag{4.16}$$

where functions $\Psi, \overline{\Psi} \in \tilde{V} \otimes V^{\otimes n} \otimes \tilde{V}'$.

Proof. Formulas (4.15) are obtained by direct calculations. The flatness of the connections (4.16)

$$[\mathsf{A}_k,\, \mathsf{A}_j] = 0 = [\overline{\mathsf{A}}_k,\, \overline{\mathsf{A}}_j] \; ,$$

follows from the Proposition 2.1. ∎

5 Flat connections for quantum Knizhnik-Zamolodchikov equations. Approach with Zamolodchikov algebra.

5.1 Zamolodchikov algebra.

Introduce a set of operators $A^i(z)$ $(i = 1, 2, \dots, N)$ which act in the complex vector space $\mathcal{H}$. Each operator $A^i(z)$ is a function of the spectral parameter z. The operators $A^i(z)$ are generators of the algebra $\mathcal{Z}$ with quadratic defining relations (see e.g. [13] and references therein)

$$A^{i_1}(x)\; A^{i_2}(y) = R^{i_1 i_2}_{k_1 k_2}(x, y)\; A^{k_2}(y)\; A^{k_1}(x)\;. \tag{5.1}$$

where $R^{i_1 i_2}_{k_1 k_2}(x, y) \in \mathbb{C}$ are functions of the spectral parameters x and y and also are components of an R-operator acting in the tensor product $V \otimes V$ of two N-dimensional vector spaces V (see (4.1)). The algebra $\mathcal{Z}$ is called Zamolodchikov algebra. Relations (5.1) can be written in concise matrix notations [10] as following

$$A^{1\rangle}(x)\; A^{2\rangle}(y) = R_{12}(x, y)\; A^{2\rangle}(y)\; A^{1\rangle}(x)\;. \tag{5.2}$$

Consider the product $A^{i_1}(x)A^{i_2}(y)A^{i_3}(z)$ of three operators and reorder it with the help of (5.1) as following

$$A^{i_1}(x)A^{i_2}(y)A^{i_3}(z) \;\;\to\;\; A^{k_3}(z)A^{k_2}(y)A^{k_1}(x)\;,$$

in two different ways in accordance with the arrangement of brackets

$$(A^{i_1}(x)A^{i_2}(y))A^{i_3}(z) = A^{i_1}(x)(A^{i_2}(y)A^{i_3}(z))\;. \tag{5.3}$$

As a result we obtain the self-consistence condition for the matrix $R(x, y)$ in the form of the Yang-Baxter equation (4.2). The solutions $R(x, y)$ of the equation (4.2) define Zamolodchikov algebra (5.1).

Now we extend (see [13]) the algebra $\mathcal{Z}$ by adding new "boundary" operators B^α $(\alpha = 1, 2, \dots, M)$ which act in $\mathcal{H}$ and obey relations

$$\begin{gathered} A^i(x)\; B^\alpha = K^{i\alpha}_{k\beta}(x)\; A^k(\bar{x})\; B^\beta \quad \Rightarrow \quad A^{1\rangle}(x)\; B = K_1(x)\; A^{1\rangle}(\bar{x})\; B\;, \\ \bar{x} = \sigma(x) \in \mathbb{C}\;, \end{gathered} \tag{5.4}$$

where $\bar{x}$ is a reflected spectral parameter and σ − involutive operation $\mathbb{C} \to \mathbb{C}$ such that $\sigma^2 = 1$. E.g., for rational and trigonometric R-matrices (pay attention to the special dependence of spectral parameters)

$$R(x, y) = R(x - y)\;, \quad R(x, y) = R(x/y)\;, \tag{5.5}$$

one can take $\sigma = \sigma_a$ and $\sigma = \sigma^{tri}_b$, respectively, where

$$\sigma_a(x) = a - x\;, \quad \sigma^{tri}_b(x) = b/x\;, \tag{5.6}$$

and $a, b \in \mathbb{C}$ are parameters which specify involutions σ, σ^{tri}. Matrix K with components $K^{i\alpha}_{k\beta}(x)$ acts in the space $V \otimes \widetilde{V}$, where $\widetilde{V}$ is M-dimensional vector space. This matrix is called reflection matrix and describes a reflection of particles from right boundary [13]. For simplicity, in the second formula in (5.4) and below, we omit indices $\alpha, \beta, \dots$ related to the space $\widetilde{V}$.

In the same way as in (5.3), one can consider two different ways for the reordering of special product of 3 generators (including B^α):

$$[A^{i_1}(x)\; A^{i_2}(y)]\; B = A^{i_1}(x)\; [A^{i_2}(y)\; B] \quad \longrightarrow \quad A^{k_1}(\bar{x})\; A^{k_2}(\bar{y})\; B\;.$$

As a result, in addition to the Yang-Baxter equation (4.2), we obtain new consistence condition for the reflection matrix K in the form of the reflection equation (4.3).

Now besides the "right" boundary operators B^α with relations (5.4), we also introduce the "left" boundary operators $\overline{B}_{\alpha'}$ $(\alpha' = 1, \dots, M')$ with relations

$$\begin{gathered} \overline{B}_{\beta'}\; A^i(x) = \overline{K}^{i\alpha'}_{k\beta'}(x)\; \overline{B}_{\alpha'}\; A^k(\tilde{x}) \quad \Rightarrow \quad \overline{B}\; A^{1\rangle}(x) = \overline{K}_1(x)\; \overline{B}\; A^{1\rangle}(\tilde{x})\;, \\ \tilde{x} = \bar{\sigma}(x) \in \mathbb{C}\;, \end{gathered} \tag{5.7}$$

where $\bar{\sigma}$ is another involutive operation in $\mathbb{C}$: $\bar{\sigma}^2 = 1$ (e.g., one can define $\bar{\sigma}$ as in (5.6) but with another parameters a, b). Equations (5.7) and operator $\overline{K}(x) \in \mathrm{End}(V \otimes \tilde{V}')$, where $\widetilde{V}'$ is M'-dimensional vector space, describe the reflection of particles from the left boundary. Two different ways for the reordering of the product of 3 generators (including $\overline{B}$):

$$\overline{B}\; A^{i_1}(x)\; A^{i_2}(y) \quad \longrightarrow \quad \overline{B}\; A^{k_1}(\tilde{x})\; A^{k_2}(\tilde{y}) \; ,$$

give additional consistence condition in the form of the dual reflection equation (4.4).

Note that applying defining relations (5.2), (5.4) and (5.7) twice, we deduce three unitary relations for matrices R, $\overline{K}$ and K

$$R_{12}(x,y)\; R_{21}(y,x) = I \otimes I \; , \quad K_1(x)\; K_1(\bar{x}) = I \otimes \widetilde{I} \; , \quad \overline{K}_1(x)\; \overline{K}_1(\tilde{x}) = I \otimes \widetilde{I}' \; , \tag{5.8}$$

where I, $\widetilde{I}$ and $\widetilde{I}'$ – unite operators in V, $\widetilde{V}$ and $\widetilde{V}'$, correspondingly.

In physics the matrices R and K, $\overline{K}$ which satisfy equations (4.2), (4.3), (4.4) and (5.8) describe the factorizable scattering on a half line [13], [3], or define the integrable spin chains with nontrivial boundary conditions [40].

Note that if matrices R, K and $\overline{K}$ satisfy unitarity conditions (5.8), then for the representation (4.12) we have $(\rho(T_i))^2 = I$, where I is the unit operator in $\tilde{V} \otimes V^{\otimes n} \otimes \tilde{V}'$. Thus, in this case the equations (4.12) define the representation of the Coxeter group $W_n(C^{(1)})$.

5.2 Flat connections for quantum Knizhnik-Zamolodchikov equations.

Consider the boundary Zamolodchikov algebra $\mathcal{Z}_{LR}$ with generators $\{A^i(x),\; B^\alpha,\; \overline{B}_{\beta'}\}$. Namely, the algebra $\mathcal{Z}_{LR}$ includes the generators $A^i(x)$ of the Zamolodchikov algebra $\mathcal{Z}$ and both left and right boundary operators B^α and $\overline{B}_{\beta'}$. Consider the special element in $\mathcal{Z}_{LR}$:

$$[\Psi^\alpha_{\beta'}]^{i_n \dots i_1}(z_n, \dots, z_k, \dots, z_1) = \overline{B}_{\beta'}\; A^{i_n}(z_n) \cdots A^{i_k}(z_k) \cdots A^{i_1}(z_1)\; B^\alpha \; , \tag{5.9}$$

and push the k-th operator $A^{i_k}(z_k)$, in the ordered product $\big(A^{i_n}(z_n) \cdots A^{i_1}(z_1)\big)$ in the right hand side of (5.9), with the help of equations (5.1) to the right. Then we reflect this operator from the right boundary operator B^α with the help of (5.4), and push the reflected operator $A_{(k)}(\bar{z}_k)$ backward to the left with the help of (5.1) up to the left boundary operator $\overline{B}_{\beta'}$. Then we reflect the operator $A_{(k)}(\bar{z}_k)$ from this boundary operator and finally place the operator $A_{(k)}(\tilde{\bar{z}}_k)$ on its initial k-th position in the ordered product $A_{(n)}(z_n) \cdots A_{(2)}(z_2)\; A_{(1)}(z_1)$. As a result we obtain the equation

$$\begin{gathered}(\Psi^\alpha_{\beta'})^{i_1 \dots i_n}(z_n, \dots, z_k, \dots, z_1) = [\mathcal{A}_k(z_1, \dots, z_k, \dots, z_n)]^{i_1 \dots i_n;\, \alpha\gamma'}_{j_1 \dots j_n;\, \beta'\delta}\; (\Psi^\delta_{\gamma'})^{j_1 \dots j_n}(z_n, \dots, \tilde{\bar{z}}_k, \dots, z_1) \; , \\ \tilde{\bar{z}}_k = \bar{\sigma}(\sigma(z_k)) \; ,\end{gathered} \tag{5.10}$$

where involutions σ and $\bar{\sigma}$ were introduced in (5.4) and (5.7) while the matrix

$$\begin{gathered}[\mathcal{A}_k(z_1, \dots, z_k, \dots, z_n)]_{12 \dots n} = K_k(z_k;\; \vec{z}_{(1,k-1)}) \cdot \overline{K}_k(\bar{z}_k;\; \vec{z}_{(k+1,n)}) \; , \\ \vec{z}_{(1,k-1)} = (z_1, \dots, z_{k-1}) \; , \quad \vec{z}_{(k+1,n)} = (z_{k+1}, \dots, z_n) \; ,\end{gathered}$$

is defined by means of dressed reflection matrices

$$\begin{gathered}K_k(x; \vec{z}_{(1,k-1)}) = R_{k,k-1}(x, z_{k-1}) \cdots R_{k1}(x, z_1)\; K_k(x)\; R_{k1}(z_1, \bar{x}) \cdots R_{k,k-1}(z_{k-1}, \bar{x}) = \\ = \hat{R}^{-1}_{k-1}(z_{k-1}, x) \cdots \hat{R}^{-1}_2(z_2, x)\, \hat{R}^{-1}_1(z_1, x)\; K_1(x)\; \hat{R}_1(z_1, \bar{x})\, \hat{R}_2(z_2, \bar{x}) \cdots \hat{R}_{k-1}(z_{k-1}, \bar{x}) = \\ = \hat{R}^{-1}_{k-1}(z_{k-1}, x)\, K_{k-1}(x; \vec{z}_{(k-2)})\, \hat{R}_{k-1}(z_{k-1}, \bar{x}) \; ,\end{gathered} \tag{5.11}$$

$$\begin{gathered}\overline{K}_k(\bar{x}; \vec{z}_{(k+1,n)}) = R_{k+1,k}(z_{k+1}, \bar{x}) \cdots R_{nk}(z_n, \bar{x})\; \overline{K}_k(\bar{x})\; R_{kn}(\tilde{\bar{x}}, z_n) \cdots R_{k,k+1}(\tilde{\bar{x}}, z_{k+1}) = \\ \hat{R}_k(z_{k+1}, \bar{x}) \cdots \hat{R}_{n-1}(z_n, \bar{x})\; \overline{K}_n(\bar{x})\; \hat{R}_{n-1}(\tilde{\bar{x}}, z_n) \cdots \hat{R}_k(\tilde{\bar{x}}, z_{k+1}) = \\ = \hat{R}_k(z_{k+1}, \bar{x})\, \overline{K}_{k+1}(\bar{x}; \vec{z}_{(k+2,n)})\, \hat{R}_k(\tilde{\bar{x}}, z_{k+1}) \; .\end{gathered} \tag{5.12}$$

To write expression (5.11) for the matrix $K_k(x;\vec{z}_{(1,k-1)})$ we take into account the unitarity condition for the R-operator $\hat{R}_k(x,z)=\hat{R}_k^{-1}(z,x)$.

For rational and trigonometric R-matrices (5.5) the involutions σ and $\bar{\sigma}$ could be defined as in (5.6)

$$\text{rational case}: \ \ \sigma=\sigma_a \ , \quad \bar{\sigma}=\sigma_{a'} \ \ ; \quad \text{trigonometric case}: \ \ \sigma=\sigma_b^{tri} \ , \quad \bar{\sigma}=\sigma_{b'}^{tri} \ ;$$

and we respectively obtain

$$\tilde{\bar{x}}=\sigma_{a'}(\sigma_a(x))=(a'-a)+x \ , \quad \tilde{\bar{x}}=\sigma_{b'}^{tri}(\sigma_b^{tri}(x))=\frac{b'}{b}\,x \ , \tag{5.13}$$

i.e., for the rational case the spectral parameter $\tilde{\bar{x}}$ is a shift of x by a constant $(a'-a)$, while for the trigonometric case the parameter $\tilde{\bar{x}}$ is a multiplication of x by a constant b'/b. In view of this, for rational and trigonometric cases the operator D_z (4.13), (4.14) can be considered as finite difference derivatives. Note that $\bar{\sigma}\sigma\neq\sigma\bar{\sigma}$.

One can write eqs. (5.10) in the form of quantum Knizhnik-Zamolodchikov equations (see (4.16):

$$\mathsf{A}_k(z_1,\dots,z_k,\dots,z_n) \ \ \Psi(z_1,\dots,z_k,\dots,z_n)=\Psi(z_1,\dots,z_k,\dots,z_n) \ , \tag{5.14}$$

where we interpret Ψ (5.9) as a wave function and introduce connections

$$\begin{gathered}\mathsf{A}_k(z_1,\dots,z_k,\dots,z_n)=\mathcal{A}_k(z_1,\dots,z_k,\dots,z_n)\ D_{z_k}=K_k(z_k;\,\vec{z}_{(1,k-1)})\cdot\overline{K}_k(\bar{z}_k;\,\vec{z}_{(k+1,n)})\ D_{z_k}= \\ =K_k(z_k;\,\vec{z}_{(1,k-1)})\cdot\overline{\mathsf{K}}_k(\bar{z}_k;\,\vec{z}_{(k+1,n)}) \ .\end{gathered} \tag{5.15}$$

In the right hand side of (5.15) we use the dressed reflection matrix (5.11) for $x=z_k$ which can be written in the representations (2.8) and (4.12) as the following

$$\begin{gathered}K_k(z_k;\,\vec{z}_{(1,k-1)})=\hat{R}_{k-1}^{-1}(z_{k-1},z_k)\cdots\hat{R}_1^{-1}(z_1,z_k)\ K_1(z_k)\ \hat{R}_1(z_1,\bar{z}_k)\cdots\hat{R}_{k-1}(z_{k-1},\bar{z}_k)= \\ =\rho(T_{k-1}^{-1}\cdots T_1^{-1}\,T_0\,T_1\cdots T_{k-1})\cdot(s_{k-1}\cdots s_1\,s_0\,s_1\cdots s_{k-1})=\rho(\bar{a}_k)\cdot\mathbf{s}(a_k) \ ,\end{gathered} \tag{5.16}$$

where $\bar{a}_k=T_{k-1}^{-1}\cdots T_1^{-1}T_0T_1\cdots T_{k-1}$ and elements a_k were defined in (2.5). Besides this we also define new dressed reflection matrix

$$\begin{gathered}\overline{\mathsf{K}}_k(\bar{x};\,\vec{z}_{(k+1,n)})=\overline{K}_k(\bar{x};\,\vec{z}_{(k+1,n)})\ D_x=\hat{R}_k(z_{k+1},\bar{x})\cdot\overline{\mathsf{K}}_{k+1}(\bar{x};\,\vec{z}_{(k+2,n)})\cdot\hat{R}_k(x,z_{k+1})= \\ =\hat{R}_k(z_{k+1},\bar{x})\cdots\hat{R}_{n-1}(z_n,\bar{x})\ \overline{K}_n(\bar{x})\ \cdot D_x\ \cdot\hat{R}_{n-1}(x,z_n)\cdots\hat{R}_k(x,z_{k+1}) \ ,\end{gathered} \tag{5.17}$$

which includes the finite difference operator D_x (4.13). In the representations (2.8) and (4.12), for $x=z_k$, the matrix (5.17) can be written as the following

$$\overline{\mathsf{K}}_k(\bar{z}_k;\,\vec{z}_{(k+1,n)})=(s_{k-1}\cdots s_1\,s_0\,s_1\cdots s_{k-1})\cdot\rho(T_k\cdots T_{n-1}\,T_n\,T_{n-1}\cdots T_k)=\mathbf{s}(a_k)\cdot\rho(b_k) \ , \tag{5.18}$$

where a_k and b_k were defined in (2.5). To obtain relations (5.16) and (5.18) we have used formulas (4.14) and

$$\bar{z}_k=(s_{k-1}\cdots s_1\,s_0\,s_1\cdots s_{k-1})\,z_k\,(s_{k-1}\cdots s_1\,s_0\,s_1\cdots s_{k-1}) \ .$$

Finally, using (5.16) and (5.18) one can write connections (5.15) in the form

$$\mathsf{A}_k(z_1,\dots,z_n)=\rho(\bar{a}_k)\cdot\rho(b_k)=\rho(J_k) \ . \tag{5.19}$$

Applying equation (5.14) twice (for two different indices k and r) we deduce the consistency condition

$$[\mathsf{A}_k,\ \mathsf{A}_r]\ \ \Psi(z_1,\dots,z_n)=0 \ ,$$

and our conjecture is that the connections A_k, explicitly given in (5.15) and (5.19), are flat:

$$[\mathsf{A}_k,\ \mathsf{A}_r]=0 \ . \tag{5.20}$$

One can prove this identity directly by using the fact that connections A_k (5.19) are the images of the commuting elements $J_k \in B_n(C^{(1)})$ (see Proposition 2.1). Note that commutativity (5.20) of connections A_k (5.15), where matrix $K_k(z_k;\, \vec{z}_{(1,k-1)})$ is taken in the form (5.16), is valid even for the case when R-matrix is not satisfies unitarity condition. So, we have proved the following statement:

Theorem 5.1. *Connections A_k which were defined in (5.15), (5.16), (5.17) are flat (5.20) for any matrices R, K and $\overline{K}$ satisfying eqs. (4.9),(4.10) and (4.11) and any involutive operations $\sigma, \bar{\sigma}$.*

Remark 1. One can think about boundary operators B^α and $\overline{B}_{\alpha'}$ in (5.4), (5.7) and (5.9) as about boundary states $|B^\alpha\rangle \in \mathcal{H}$ and $\langle\overline{B}_{\alpha'}| \in \mathcal{H}^*$ with the same conditions as in (5.4), (5.7). In this case the operator (5.9) is represented as the matrix element

$$[\Psi^\alpha_{\beta'}]^{i_n \dots i_1}(z_n, \dots, z_2, z_1) = \langle\overline{B}_{\beta'}|\; A^{i_n}(z_n) \cdots A^{i_2}(z_2)\; A^{i_1}(z_1)\; |B^\alpha\rangle \;, \tag{5.21}$$

and the equation (5.14), with the wave function Ψ which is given in (5.21), is nothing but the quantum Knizhnik-Zamolodchikov (q-KZ) equations for the system with nontrivial boundary conditions. One can put $\tilde{V} = \tilde{V}'$, $\beta' = \alpha$ in (5.21) and sum over α. As a result we obtain the following form of the solution of q-KZ equation

$$\Psi^{i_n \dots i_1}(z_n, \dots, z_2, z_1) = \mathrm{Tr}_{\mathcal{H}}\Big(\; A^{i_n}(z_n) \cdots A^{i_2}(z_2)\; A^{i_1}(z_1)\; \rho\;\Big) \;, \tag{5.22}$$

where $\rho = |B^\alpha\rangle\langle\overline{B}_\alpha|$ can be considered as a density matrix.

Remark 2. For systems with periodic boundary conditions one can deduce q-KZ equations by using the same method as was used above for the systems with nontrivial boundary conditions and open boundaries. Consider the function (5.22) with any operator ρ and require that this operator satisfies commutation relations with generators $A^i(x)$:

$$A^i(x)\, \rho = Q^i_j(x)\, \rho\, A^i(\tilde{\tilde{x}}) \;, \quad \tilde{\tilde{x}} = \bar{\sigma}(\sigma(x)) \;. \tag{5.23}$$

Here functions $Q^i_j(x)$ are components of a numerical matrix. Taking into account (5.23) we obtain the following periodicity condition for the wave function (5.22)

$$\begin{gathered} \Psi^{i_n \dots i_1}(z_n, \dots, z_2, z_1) = \\ = \mathrm{Tr}_{\mathcal{H}}\Big(\; A^{i_n}(z_n) \cdots A^{i_3}(z_3)\; A^{i_2}(z_2)\, Q^{i_1}_{j_1}(z_1)\; \rho\; A^{j_1}(\tilde{\tilde{z}}_1)\Big) = Q^{i_1}_{j_1}(z_1)\; \Psi^{j_1 i_n \dots i_2}(\tilde{\tilde{z}}_1, z_n, \dots, z_2) \;. \end{gathered} \tag{5.24}$$

The associativity equation $A^{i_1}(x)(A^{i_2}(y)\;\, \rho) = (A^{i_1}(x)A^{i_2}(y)\;\,)\rho$ requires consistency condition for matrix $Q^i_j(x)$

$$R_{12}(z_1, z_2)Q_1(z_1)Q_2(z_2) = Q_1(z_1)\, Q_2(z_2)\, R_{12}(\tilde{\tilde{z}}_1, \tilde{\tilde{z}}_2) \;. \tag{5.25}$$

We also require the condition

$$R_{12}(\tilde{\tilde{z}}_1, \tilde{\tilde{z}}_2) = R_{12}(z_1, z_2) \quad \Leftrightarrow \quad D_{z_1}\, D_{z_2}\, R_{12}(z_1, z_2) = R_{12}(z_1, z_2)\, D_{z_1}\, D_{z_2} \;,$$

which is obtained automatically for the rational and trigonometric cases, when involutions $\bar{\sigma}$, σ are fixed as in (5.13). In this case equation (5.25) is written as

$$(D_{z_1}\, Q_1(z_1))\, (D_{z_2}\, Q_2(z_2))\, R_{12}(z_1, z_2) = R_{12}(z_1, z_2)\, (Q_1(z_1)\, D_{z_1})\, (Q_2(z_2)\, D_{z_2}) \;.$$

Now we again pick up the generator $A^{i_k}(z_k)$ in the right hand side of (5.22) push this generator to the right with the help of (5.1), then use relation (5.23) and cyclic property of the trace and finally place the operator $A^{i_k}(\tilde{\tilde{z}}_k)$ on its initial k-th position. As a result we obtain equation

$$\Psi(z_n, \dots, z_2, z_1) = \mathsf{A}_k(\vec{z}_{(1,n)})\; \Psi(z_n, \dots, z_2, z_1) \;, \tag{5.26}$$

where $\vec{z}_{(1,n)} = (z_1, \dots, z_n)$ and $\mathsf{A}_k(\vec{z}_{(1,n)})$ is the flat connection for q-KZ equation in the periodic case [11]:

$$\begin{aligned} \mathsf{A}_k(\vec{z}_{(1,n)}) = \quad & R_{k,k-1}(z_k, z_{k-1}) \cdots R_{k,2}(z_k, z_2) R_{k,1}(z_k, z_1)\, Q_k(z_k)\, D_{z_k} \cdot \\ & \cdot R_{kn}(z_k, z_n) R_{k,n-1}(z_k, z_{n-1}) \cdots R_{k,k+1}(z_k, z_{k+1}) \;. \end{aligned} \tag{5.27}$$

Here the finite difference operator D_{z_k} is the same as in (4.13). Using for the periodic braid group elements T_i the same R-matrix representation (4.12) we write connection (5.27) as (cf. (2.12))

$$\begin{gathered}\mathsf{A}_k(\vec{z}_{(1,n)}) = \rho(T_{k-1}\cdots T_1)\cdot X\cdot \rho(T_{n-1}^{-1}\cdots T_k^{-1})\;,\\ X := Q_1(z_1)\,D_{z_1}\,\hat{\mathsf{s}}_1\cdots\hat{\mathsf{s}}_{n-1}\;,\end{gathered} \tag{5.28}$$

where $\hat{\mathsf{s}}_k = P_{k,k+1}\,\mathsf{s}_k$ and we have used unitarity conditions $T_i^2 = 1$. We have (for simplicity we write T_i instead of $\rho(T_i)$)

$$\begin{gathered}T_i\,\hat{\mathsf{s}}_{i+1}\,\hat{\mathsf{s}}_i = \hat{\mathsf{s}}_{i+1}\,\hat{\mathsf{s}}_i\,T_{i+1}\;,\quad T_{i+1}\,\hat{\mathsf{s}}_i\,\hat{\mathsf{s}}_{i+1} = \hat{\mathsf{s}}_i\,\hat{\mathsf{s}}_{i+1}\,T_i\;,\\ X\,T_i = T_{i+1}\,X\;,\quad (i=1,\dots,n-2)\;,\\ T_1\cdot X^2 = T_1 Q_1\,D_{z_1}Q_2\,D_{z_2}(\hat{\mathsf{s}}_1\cdots\hat{\mathsf{s}}_{n-1})^2 = Q_1\,D_{z_1}Q_2\,D_{z_2}\,T_1\,(\hat{\mathsf{s}}_1\cdots\hat{\mathsf{s}}_{n-1})^2 =\\ = Q_1\,D_{z_1}Q_2\,D_{z_2}\,T_1\,(\hat{\mathsf{s}}_2\hat{\mathsf{s}}_1)\cdot(\hat{\mathsf{s}}_3\hat{\mathsf{s}}_2)\cdots(\hat{\mathsf{s}}_{n-1}\hat{\mathsf{s}}_{n-2}) =\\ = Q_1\,D_{z_1}Q_2\,D_{z_2}\,(\hat{\mathsf{s}}_2\hat{\mathsf{s}}_1)\cdots(\hat{\mathsf{s}}_{n-1}\hat{\mathsf{s}}_{n-2})\,T_{n-1} = X^2\,T_{n-1}\;.\end{gathered} \tag{5.29}$$

One can check that the element

$$T_n := X^{-1}\,T_1\cdot X = X\,T_{n-1}\,X^{-1}\;, \tag{5.30}$$

satisfies periodic braid relations

$$T_n\,T_{n-1}\,T_n = T_{n-1}\,T_n\,T_{n-1}\;,\quad T_n\,T_1\,T_n = T_1\,T_n\,T_1\;.$$

Let T_1 be unitary operator $T_1^2 = 1$. In this case the connection (5.28) satisfies the periodicity condition

$$\mathsf{A}_k(\vec{z}_{(1,n)}) = T_{k-1}\cdots T_1\cdot X\cdot T_{n-1}^{-1}\cdots T_k^{-1} = T_{k-1}\cdots T_2\cdot X\cdot T_n^{-1}T_{n-1}^{-1}\cdots T_k^{-1}\;.$$

Proposition 5.2 [11]. *For the periodic chain the connections (5.28)are flat, i.e. satisfy (5.20).*
Proof. The proof is the same as the proof of the Proposition 2.1 in Section **2**. ∎

Remark 3. Consider operator $T_{V\mathcal{V}}(x) \in \mathrm{End}(V\otimes\mathcal{V})$ which satisfies the intertwining relations

$$\mathcal{R}_{\mathcal{V}\mathcal{V}'}(x,y)\;T_{1\mathcal{V}}(x)\;T_{1\mathcal{V}'}(y) = T_{1\mathcal{V}'}(y)\;T_{1\mathcal{V}}(x)\;\mathcal{R}_{\mathcal{V}\mathcal{V}'}(x,y)\quad\in\quad \mathrm{End}(V\otimes\mathcal{V}\otimes\mathcal{V}') \tag{5.31}$$

$$R_{12}^{-1}(x,y)\;T_{1\mathcal{V}}(x)\;T_{2\mathcal{V}}(y) = T_{2\mathcal{V}}(y)\;T_{1\mathcal{V}}(x)\;R_{12}^{-1}(x,y)\quad\in\quad \mathrm{End}(V\otimes V\otimes\mathcal{V})\;, \tag{5.32}$$

where we denote by $\mathcal{V}'$ the second copy of the vector space $\mathcal{V}$, the numbers $1,2$ numerate vector spaces V, and the matrix $R_{12}(x,y)\in\mathrm{End}(V\otimes V)$, as well as the matrix $\mathcal{R}(x,y)\in\mathrm{End}(\mathcal{V}\otimes\mathcal{V}')$, satisfy the Yang-Baxter equation (4.2). Consider the transfer-matrix

$$\tau(z_1,\dots,z_n) = \mathrm{Tr}_{\mathcal{V}}\Big(T_{n\mathcal{V}}(z_n)\cdots T_{2\mathcal{V}}(z_2)T_{1\mathcal{V}}(z_1)\;\rho_{\mathcal{V}}\Big)\;, \tag{5.33}$$

where the operator $\rho_{\mathcal{V}}\in\mathrm{End}(\mathcal{V})$ is such that

$$\mathcal{R}_{\mathcal{V}\mathcal{V}'}(x,y)\rho_{\mathcal{V}}\rho_{\mathcal{V}'} = \rho_{\mathcal{V}}\rho_{\mathcal{V}'}\mathcal{R}_{\mathcal{V}\mathcal{V}'}(x,y)\;. \tag{5.34}$$

Then we have

Proposition 5.3. *Transfer-matrices $\tau(z_1,\dots,z_n)$ and $\tau(z'_1,\dots,z'_n)$, defined in (5.33), are commutative generating functions*

$$[\tau(z_1,\dots,z_n),\;\tau(z'_1,\dots,z'_n)] = 0\;, \tag{5.35}$$

if parameters $(z_1,\dots,z_n)$, $(z'_1,\dots,z'_n)$ and the matrix $\mathcal{R}(x,y)$ are such that

$$\mathcal{R}(z_n,z'_n) = \mathcal{R}(z_k,z'_k)\quad \forall k=1,2,\dots,n-1\;. \tag{5.36}$$

Proof. Let $\mathcal{V}'$ be the second copy of the space $\mathcal{V}$. Then we have

$$\tau(z_1,\ldots,z_n)\,\tau(z'_1,\ldots,z'_n) = \mathrm{Tr}_{\mathcal{V}\mathcal{V}'}\Big(T_{n\mathcal{V}}(z_n)T_{n\mathcal{V}'}(z'_n)\cdots T_{1\mathcal{V}}(z_1)T_{1\mathcal{V}'}(z'_1)\;\rho_{\mathcal{V}}\rho_{\mathcal{V}'}\Big) =$$
$$= \mathrm{Tr}_{\mathcal{V}\mathcal{V}'}\Big(\mathcal{R}^{-1}_{\mathcal{V}\mathcal{V}'}(z_n,z'_n)\cdot T_{n\mathcal{V}'}(z'_n)T_{n\mathcal{V}}(z_n)\cdots T_{1\mathcal{V}'}(z'_1)T_{1\mathcal{V}}(z_1)\cdot \mathcal{R}_{\mathcal{V}\mathcal{V}'}(z_1,z'_1)\;\rho_{\mathcal{V}'}\rho_{\mathcal{V}}\Big) =$$
$$= \mathrm{Tr}_{\mathcal{V}\mathcal{V}'}\Big(T_{n\mathcal{V}'}(z'_n)T_{n\mathcal{V}}(z_n)\cdots T_{1\mathcal{V}'}(z'_1)T_{1\mathcal{V}}(z_1)\;\rho_{\mathcal{V}'}\rho_{\mathcal{V}}\Big) = \tau(z'_1,\ldots,z'_n)\,\tau(z_1,\ldots,z_n)\;,$$

where $\mathrm{Tr}_{\mathcal{V}\mathcal{V}'} = \mathrm{Tr}_{\mathcal{V}}\mathrm{Tr}_{\mathcal{V}'}$ and we have used relations (5.31), (5.34). ■

Note that for the rational (or trigonometric) R-matrices, when we have $R(x,y) = R(x-y)$ (or $R(x,y) = R(x/y)$), relation (5.36) is fulfilled for the choice $z_k - z'_k = x - y$ (or $z_k/z'_k = x/y$) for all k, where x and y are two fixed parameters. For example, in the trigonometric case the commutative transfer-matrix can be taken in the form $\tau(x;z_1,\ldots,z_n) = \tau(x\,z_1,\ldots,x\,z_n)$ and commutativity condition (5.35) is written as

$$[\tau(x;z_1,\ldots,z_n),\;\tau(y;z_1,\ldots,z_n)] = 0\;.$$

Now, in addition to the relation (5.34), we require that the operator $\rho_{\mathcal{V}}$ satisfies (cf. (5.23)):

$$T_{1\mathcal{V}}(x)\;\rho_{\mathcal{V}}\;Q_1 = \rho_{\mathcal{V}}\;Q_1\;T_{1\mathcal{V}}(\tilde{\tilde{x}})\;,\quad Q\in\mathrm{End}(V) \tag{5.37}$$

where for the invertible matrix Q we have (cf. 5.25)

$$R_{12}(x,y)Q_1Q_2 = Q_1Q_2R_{12}(\tilde{\tilde{x}},\tilde{\tilde{y}})\;.$$

Equation (5.37) serves twisted periodic boundary conditions of the type (5.24) for the transfer-matrix (5.33).

At the end of this Section we formulate the following statement.

Proposition 5.4. *Flat connections (5.27) commute with the transfer-matrix (5.33)*

$$\mathsf{A}_k(z_1,\ldots,z_n)\;\tau(z_1,\ldots,z_n) = \tau(z_1,\ldots,z_n)\;\mathsf{A}_k(z_1,\ldots,z_n)\;. \tag{5.38}$$

Proof. Take the operator $T_{k\mathcal{V}}(z_k)$ (in the right hand side of (5.33)) and use the same procedure as in Remark 2. for the cyclic moving of $T_{k\mathcal{V}}(z_k)$. After direct calculations with the use of the relations (5.37), (5.32) and identity

$$\tau(z_1,\ldots,\tilde{\tilde{z}}_k,\ldots,z_n) = D_{z_k}\;\tau(z_1,\ldots,z_k,\ldots,z_n)\;D^{-1}_{z_k}\;,$$

we deduce relation (5.38). ■

Consequence. By using the statement of the Proposition 5.4 we deduce the following result. Let $\Psi(z_n,\ldots,z_1)$ be any solution of the periodic quantum Knizhnik-Zamolodchikov equation (5.26). Then, the vector

$$\Psi'(z_n,\ldots,z_1) = \tau(z_1,\ldots,z_n)\cdot\Psi(z_n,\ldots,z_1)\;,$$

is also a solution of the periodic quantum Knizhnik-Zamolodchikov equation (5.26).

6 Flat connections for Birman–Murakami–Wenzl algebra.

1. Birman–Murakami–Wenzl algebra. Definition and basic relations.

The *Birman–Murakami–Wenzl algebra* $BMW_n(q,\nu)$ was defined in [1], [30] and [31]. It is generated over $\mathbb{C}$ by invertible elements $\mathsf{T}_1,\ldots,\mathsf{T}_{n-1}$ with the following defining relations

$$\mathsf{T}_i\,\mathsf{T}_{i+1}\,\mathsf{T}_i = \mathsf{T}_{i+1}\,\mathsf{T}_i\,\mathsf{T}_{i+1}\;,\quad \mathsf{T}_i\,\mathsf{T}_j = \mathsf{T}_j\,\mathsf{T}_i\quad\text{for}\;\;|i-j|>1\;, \tag{6.1}$$

$$\kappa_i\,\mathsf{T}_i = \mathsf{T}_i\,\kappa_i = \nu\,\kappa_i\;, \tag{6.2}$$

$$\kappa_i\,\mathsf{T}^{\varepsilon}_{i-1}\,\kappa_i = \nu^{-\varepsilon}\,\kappa_i\;\;,\;\;\kappa_i\,\mathsf{T}^{\varepsilon}_{i+1}\,\kappa_i = \nu^{-\varepsilon}\,\kappa_i\;\;\text{with}\;\varepsilon=\pm1\;, \tag{6.3}$$

where

$$\kappa_i := 1 - \frac{\mathsf{T}_i - \mathsf{T}_i^{-1}}{q - q^{-1}} \ . \tag{6.4}$$

Here q and ν are complex parameters of the algebra which we assume generic in the sequel; in particular, the definition (6.4) makes sense, the denominator in the right hand side does not vanish. Note that the algebra $BMW_n(q,\nu)$ with defining relations (6.1)–(6.3) possesses the automorphism $\rho_2(\mathsf{T}_i) = \mathsf{T}_{n-i}$ (cf. (2.4)).

The quotient of the algebra $BMW_n(q,\nu)$ by the ideal generated by the elements $\kappa_1, \dots, \kappa_n$ (in fact, this ideal is generated by any one of these elements, say, κ_1) is isomorphic to the Hecke algebra $H_n(q)$. It is also well-known that the braid group $\mathcal{B}_n$ (of type A) embeds in the BMW_n algebra $\mathcal{B}_n \hookrightarrow BMW_n$. We shall often omit the parameters in the notation for the algebras and write simply BMW_n and H_n.

Let

$$\mu = \frac{q - q^{-1} + \nu^{-1} - \nu}{q - q^{-1}} = \frac{(q^{-1} + \nu)(q - \nu)}{\nu(q - q^{-1})} \ . \tag{6.5}$$

The following relations can be derived from (6.1)–(6.3):

$$\kappa_i^2 = \mu\, \kappa_i \tag{6.6}$$

then, with $\varepsilon = \pm 1$,

$$\begin{aligned}
\kappa_i\, \mathsf{T}_{i+\varepsilon}\, \mathsf{T}_i \ &= \ \mathsf{T}_{i+\varepsilon}\, \mathsf{T}_i\, \kappa_{i+\varepsilon} \ , && (6.7)\\
\kappa_i\, \kappa_{i+\varepsilon}\, \kappa_i \ &= \ \kappa_i \ , && (6.8)\\
\big(\mathsf{T}_i - (q-q^{-1})\big)\kappa_{i+\varepsilon}\big(\mathsf{T}_i - (q-q^{-1})\big) \ &= \ \big(\mathsf{T}_{i+\varepsilon} - (q-q^{-1})\big)\kappa_i\big(\mathsf{T}_{i+\varepsilon} - (q-q^{-1})\big) \ , && (6.9)\\
\mathsf{T}_{i+\varepsilon}\, \kappa_i\, \mathsf{T}_{i+\varepsilon} \ &= \ \mathsf{T}_i^{-1}\, \kappa_{i+\varepsilon}\, \mathsf{T}_i^{-1} \ , && (6.10)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_i\, \mathsf{T}_{i+\varepsilon}\, \mathsf{T}_i \ &= \ \kappa_i\, \kappa_{i+\varepsilon} \ , && (6.11)\\
\kappa_i\, \mathsf{T}_{i+\varepsilon}^{-1}\, \mathsf{T}_i^{-1} \ &= \ \kappa_i\, \kappa_{i+\varepsilon} \ , && (6.12)\\
\kappa_{i+\varepsilon}\, \kappa_i\, \big(\mathsf{T}_{i+\varepsilon} - (q-q^{-1})\big) \ &= \ \kappa_{i+\varepsilon}\, \big(\mathsf{T}_i - (q-q^{-1})\big) \ , && (6.13)
\end{aligned}$$

together with their images under the anti-automorphism ρ_a of the algebra BMW_n defined on the generators by

$$\rho_a(\mathsf{T}_i) = \mathsf{T}_i \ , \quad \rho_a(\mathsf{T}_i\, \mathsf{T}_k) = \mathsf{T}_k\, \mathsf{T}_i \ , \quad \rho_a(\mathsf{T}_i\, \mathsf{T}_j\, \mathsf{T}_k) = \mathsf{T}_k\, \mathsf{T}_j\, \mathsf{T}_i \ , \quad \dots . \tag{6.14}$$

2. Baxterized elements.
The *baxterized elements* $T_i(u,v)$ are defined by

$$T_i(u,v) := \mathsf{T}_i + \frac{q - q^{-1}}{v/u - 1} + \frac{q - q^{-1}}{1 + \nu^{-1} q v/u} \kappa_i \equiv T_i(u/v) \ , \tag{6.15}$$

see [2], [17], [24] and [32]. They are rational functions in complex variables u and v which are called *spectral variables*. The elements $T_i(u,v)$ depend on the ratio of the spectral parameters; for us it is more convenient to have both spectral variables in the notation (6.15) for the baxterized element. However for brevity we shall denote sometimes the baxterized elements by $T_i(u/v) \equiv T_i(u,v)$ (with one argument only).

The baxterized elements satisfy the braid relation of the form

$$T_i(u_2,u_3) T_{i+1}(u_1,u_3) T_i(u_1,u_2) = T_{i+1}(u_1,u_2) T_i(u_1,u_3) T_{i+1}(u_2,u_3) \ . \tag{6.16}$$

The inverses of the baxterized elements are given by

$$T_i(v,u)^{-1} = T_i(u,v)\, f(u,v) \ , \tag{6.17}$$

where

$$f(u,v) = \frac{(u-v)^2}{(u - q^2 v)(u - q^{-2} v)} = f(v,u) \ . \tag{6.18}$$

3. Jucys–Murphy elements.

The Jucys–Murphy elements of the algebra BMW_n are defined by

$$y_1 = 1 \; , \quad y_{k+1} = \mathsf{T}_k \dots \mathsf{T}_2 \, \mathsf{T}_1^2 \, \mathsf{T}_2 \dots \mathsf{T}_k \; , \quad k = 1, \dots, n-1 \; . \tag{6.19}$$

The elements $y_1, \dots, y_n$ pairwise commute and satisfy the identities

$$\kappa_j \, y_{j+1} \, y_j = y_j \, y_{j+1} \, \kappa_j = \nu^2 \, \kappa_j \; . \tag{6.20}$$

The Jucys–Murphy elements were originally used for constructing idempotents for the symmetric groups in [26], [33]. Analogues of the Jucys–Murphy elements can be defined for a number of important algebras related to the symmetric group rings (e.g., the Hecke and Brauer algebras); they turn out to generate maximal commutative subalgebras in these rings (see [18], [20], [36], [37] and references therein). The commutative subalgebra, generated by the Jucys–Murphy elements $y_1, \dots, y_n$, of the generic algebra BMW_n is maximal as well; it follows from the results in [21],[28].

4. Affine BMW algebras of type C (see, e.g., [21] and references therein).

Affine Birman-Murakami-Wenzl algebras $BMW_n(C)$ of type C are extensions of the algebras BMW_n. The algebra $BMW_n(C)$ is generated by the elements $\{\mathsf{T}_1, \dots, \mathsf{T}_{n-1}\}$ with relations (6.1), (6.2), (6.3), (6.4) and the affine element $\mathsf{T}_0 = y_1 \neq 1$ which satisfies

$$\begin{gathered} \mathsf{T}_1 \, \mathsf{T}_0 \, \mathsf{T}_1 \, \mathsf{T}_0 = \mathsf{T}_0 \, \mathsf{T}_1 \, \mathsf{T}_0 \, \mathsf{T}_1 \; , \quad [\mathsf{T}_k, \, \mathsf{T}_0] = 0 \quad \text{for} \;\; k > 1 \; , \\ \kappa_1 \, \mathsf{T}_0 \, \mathsf{T}_1 \, \mathsf{T}_0 \, \mathsf{T}_1 = \hat{z} \, \kappa_1 = \mathsf{T}_1 \, \mathsf{T}_0 \, \mathsf{T}_1 \, \mathsf{T}_0 \, \kappa_1 \; , \end{gathered} \tag{6.21}$$

$$\kappa_1 \, \mathsf{T}_0^k \, \kappa_1 = \hat{z}^{(k)} \kappa_1 \quad , \quad k = 1, 2, 3, \dots \; , \tag{6.22}$$

where $\hat{z}$, $\hat{z}^{(k)}$ are central elements. Initially, the affine version of the Brauer algebras (which are the special limit $q \to 1$ of $BMW_n(C)$), was introduced by M. Nazarov [34]. Note that the central elements $\{\hat{z}^{(k)}\}$ generate an infinite dimensional abelian subalgebra in $BMW_{n+1}(C)$ and we denote this subalgebra as $BMW_0(C)$.

Consider the set of affine elements (cf. with elements a_i in (2.5))

$$y_1 = \mathsf{T}_0 \; , \quad y_{k+1} = \mathsf{T}_k \, y_k \, \mathsf{T}_k \; , \quad k = 1, 2, \dots, n-1 \; .$$

The elements y_k $(k = 1, 2, \dots, n)$ generate a commutative subalgebra Y_n in $BMW_n(C)$.
Proposition 6.1 [15],[19] *For the affine BMW algebra the element*

$$L_j(u) = \frac{u - y_j}{cu \, y_j - 1} \; , \qquad c = -\nu \, q^{-1} \, \hat{z}^{-1} \; , \tag{6.23}$$

is the baxterized solution of the reflection equation

$$T_j(u,v) \, L_j(u) \, T_j(v,\bar{u}) \, L_j(v) = L_j(v) \, T_j(u,\bar{v}) \, L_j(u) \, T_j(u,v) \; , \qquad (j = 1, \dots, n-1) \; , \tag{6.24}$$

where $\bar{u} = 1/(c \, u)$.
Proof. The formula (6.24) is checked by direct calculations. ■

Since we have $T_j\big(u,v\big) = T_j\big(\bar{v},\bar{u}\big)$, the equation (6.24) is a realization of the reflection equation (4.10) if we identify

$$L_j(v) \to K_j(v) \; , \quad T_j\big(u,v\big) \to \hat{R}_j(u,v) \; , \quad \bar{x} = \sigma(x) = \frac{1}{c \, x} \; .$$

5. Affine BMW algebras of type $C^{(1)}$.

The algebra $BMW_n(C^{(1)})$ is generated by the elements $\{T_0, T_1, \dots, T_n\}$ and is associated to the Coxeter graph (2.3) of type $C^{(1)}$. The algebra $BMW_n(C^{(1)})$ is an extension of the affine algebra $BMW_n(C)$ (we add new generator T_n). We require that the algebra $BMW_n(C^{(1)})$ possesses the automorphism ρ_2 which is defined in (2.4). Thus, applying automorphism ρ_2 to the relations (6.21), we obtain relations for the affine element T_n in the form

$$
\begin{gathered}
\mathsf{T}_{n-1}\,\mathsf{T}_n\,\mathsf{T}_{n-1}\,\mathsf{T}_n = \mathsf{T}_n\,\mathsf{T}_{n-1}\,\mathsf{T}_n\,\mathsf{T}_{n-1}\;, \quad [\mathsf{T}_k,\,\mathsf{T}_n] = 0 \quad \text{for } \; k < n-1\;, \\
\kappa_{n-1}\,\mathsf{T}_n\,\mathsf{T}_{n-1}\,\mathsf{T}_n\,\mathsf{T}_{n-1} = \hat{z}'\,\kappa_{n-1} = \mathsf{T}_{n-1}\,\mathsf{T}_n\,\mathsf{T}_{n-1}\,\mathsf{T}_n\,\kappa_{n-1}\;, \\
\kappa_{n-1}\,\mathsf{T}_n^k\,\kappa_{n-1} = \hat{z}'^{(k)}\kappa_{n-1} \quad , \quad k = 1,2,3,\dots\,.
\end{gathered}
\tag{6.25}
$$

where $\hat{z}' = \rho_2(\hat{z})$, $\hat{z}'^{(k)} = \rho_2(\hat{z}^{(k)})$ (as well as $\hat{z}$, $\hat{z}^{(k)}$) are the central elements in the algebra $BMW_n(C^{(1)})$.

Consider the set of affine elements (cf. with elements b_i in (2.5))

$$
\bar{y}_n = \mathsf{T}_n = \rho_2(\mathsf{T}_0)\;, \quad \bar{y}_{k-1} = \mathsf{T}_{k-1}\,\bar{y}_k\,\mathsf{T}_{k-1} = \rho_2(y_{n-k+2})\;, \quad k = 2,\dots,n\;.
$$

The elements $\bar{y}_k$ $(k = 1,\dots,n)$ generate a commutative subalgebra $\bar{Y}_n$ in $BMW_n(C^{(1)})$.

Since the element (6.23) is a solution of the reflection elution (6.24), the element

$$
\bar{L}_j(u) = \frac{u - \bar{y}_j}{c'\,u\,\bar{y}_j - 1} = \rho_2(L_{n-j+1}) \;\in\; BMW_n(C^{(1)})\;, \qquad c' = -\nu\,q^{-1}\,\hat{z}'^{-1} = \rho_2(c)\;,
\tag{6.26}
$$

is the baxterized solution of the dual reflection equation which is obtained as the image of (6.24) under the automorphism ρ_2

$$
T_j(u,v)\,\bar{L}_{j+1}(u)\,T_j(v,\tilde{u})\,\bar{L}_{j+1}(v) = \bar{L}_{j+1}(v)\,T_j(u,\tilde{v})\,\bar{L}_{j+1}(u)\,T_j(u,v)\;, \qquad (j = 1,\dots,n-1)\;,
\tag{6.27}
$$

where $\tilde{u} = 1/(u\,c')$. Taking into account relations (6.17) and identities

$$
T_j(\tilde{v},u) = T_j(\tilde{u},v)\;, \quad \bar{L}_j(\tilde{u}) = \frac{1}{c'}\;\bar{L}_j(u)^{-1}
$$

we write (6.27) in the form

$$
T_j(v,u)\,\bar{L}_{j+1}(\tilde{u})\,T_j(\tilde{v},u)\,\bar{L}_{j+1}(\tilde{v}) = \bar{L}_{j+1}(\tilde{v})\,T_j(\tilde{u},v)\,\bar{L}_{j+1}(\tilde{u})\,T_j(\tilde{u},\tilde{v})\;, \qquad (j = 1,\dots,n-1)\;.
\tag{6.28}
$$

The equation (6.28) can be represented as the reflection equation (4.11) if we identify

$$
\bar{L}_j(\tilde{v}) \to K_j(v)\;, \quad T_j\big(u,v\big) \to \hat{R}_j(u,v)\;, \quad \tilde{x} = \bar{\sigma}(x) = \frac{1}{c'\,x}\;.
$$

6. Embedding of the braid group $B_n(C^{(1)})$ into the algebra $BMW_n(C^{(1)})$.

Let $\{z_1,\dots,z_n\}$ be a set of spectral parameters. Consider the Weyl group generated by the operators s_i (see (2.8)) and the elements

$$
T_i(z_i,z_{i+1})\;, \;\; L_1(z_1)\;, \;\; \bar{L}_n(z_n) \;\;\in\;\; BMW_n(C^{(1)})\;.
$$

Then we have the following statement.

Proposition 6.2 The map ρ_b of the affine braid group $B_n(C^{(1)})$ into $BMW_n(C^{(1)})$ defined as (cf. (4.12))

$$
\rho_b(T_i) \;=\; s_i\,T_i(z_i,z_{i+1}) \quad (i = 1,\dots,n-1)\;, \quad \rho_b(T_0) \;=\; L_1(z_1)\,s_0\;, \quad \rho_b(T_n) \;=\; s_n\,\overline{L}_n(z_n)\;,
\tag{6.29}
$$

is the representation of $B_n(C^{(1)})$.

Proof. One can directly check that $\rho_b(T_i)$ $(i = 0,\dots,n)$ satisfy defining relations in (2.1), (2.2) if $T_k(z_k,z_{k+1})$, $L_1(z_1)$ and $\overline{L}_n(z_n)$ satisfy relations (6.16), (6.24) and (6.27), respectively. ∎

Corollary. The map ρ_c of the affine braid group $B_n(C)$ into $BMW_n(C)$

$$
\rho_c(T_i) \;=\; s_i\,T_i(z_i,z_{i+1}) \quad (i = 1,\dots,n-1)\;, \quad \rho_c(T_0) \;=\; L_1(z_1)\,s_0\;,
\tag{6.30}
$$

is the representation of $B_n(C)$.

7. Flat connections for the algebra $BMW_n(C^{(1)})$.

Flat connections for the algebra $BMW_n(C^{(1)})$ are defined as images $\rho_b(J_i)$ and $\rho_b(\overline{J}_i)$ of the elements J_i and $\overline{J}_i$ (see (2.5)) which form the commuting sets of elements in affine braid group $B_n(C^{(1)})$. The explicit formulas are (cf. (5.15))

$$\mathsf{A}_k(z_1,\dots,z_n) = \rho_b(J_k) = K_k(z_k;z_1,\dots,z_{k-1})\cdot \overline{\mathsf{K}}_k(\bar z_k;z_{k+1},\dots,z_n)\ , \tag{6.31}$$

where (cf. (5.16), (5.17))

$$\begin{aligned} K_k(z_k;\,\vec z_{(1,k-1)}) &= T^{-1}_{k-1}(z_{k-1},z_k)\cdots T_1^{-1}(z_1,z_k)\,L_1(z_k)\,T_1(z_1,\bar z_k)\cdots T_{k-1}(z_{k-1},\bar z_k) = \\ &= \rho_b(T^{-1}_{k-1}\cdots T_1^{-1}\,T_0\,T_1\cdots T_{k-1})\cdot(s_{k-1}\cdots s_1\,s_0\,s_1\cdots s_{k-1}) = \rho_b(\bar a_k)\cdot \mathsf{s}(a_k)\ , \end{aligned} \tag{6.32}$$

$$\begin{gathered} \overline{\mathsf{K}}_k(\bar z_k;\,\vec z_{(k+1,n)}) = \\ = T_k(z_{k+1},\bar z_k)\cdots T_{n-1}(z_n,\bar z_k)\,\overline{L}_n(\bar z_k)\ \cdot D_{z_k}\ \cdot T_{n-1}(z_k,z_n)\cdots T_k(z_k,z_{k+1}) = \mathsf{s}(a_k)\cdot\rho_b(b_k)\ , \end{gathered} \tag{6.33}$$

where the finite difference operator D_{z_k} was defined in (4.13) with

$$\tilde{\bar z} = \bar\sigma(\sigma(z)) = \bar\sigma\Big(\frac{1}{c\,z}\Big) = \frac{c}{c'}\ z\ .$$

We stress that $L_j(u) = D_u$ and $\bar L_{j+1}(u) = D_u^{-1}$ (as well as $L_j(u)=1$ and $\bar L_{j+1}(u)=1$) are solutions of the reflection equations (6.24) and (6.27), respectively. For example, we can substitute solution $\bar L_{j+1}(u)=1$ into (6.33) and reduce the flat connection (6.31) into the form

$$\mathsf{A}_k(z_1,\dots,z_n) = K_k(z_k;z_1,\dots,z_{k-1})\cdot \overline{\mathsf{K}}'_k(\bar z_k;z_{k+1},\dots,z_n)\ , \tag{6.34}$$

where

$$\overline{\mathsf{K}}'_k(\bar z_k;\,\vec z_{(k+1,n)}) = T_k(z_{k+1},\bar z_k)\cdots T_{n-1}(z_n,\bar z_k)\cdot D_{z_k}\ \cdot T_{n-1}(z_k,z_n)\cdots T_k(z_k,z_{k+1})\ ,$$

8. Braid–Hecke algebra $\mathcal{BH}_n(q,\nu)$.

The *Braid–Hecke algebra* $\mathcal{BH}_n(q,\nu)$, as far as we know, was introduced in [6],[7]. It is generated over $\mathbb{C}$ by the invertible *braid type generators*,

$$\mathsf{T}_1,\cdots,\mathsf{T}_{n-1},$$

subject the following defining relations

$$\mathsf{T}_i\,\mathsf{T}_{i+1}\,\mathsf{T}_i = \mathsf{T}_{i+1}\,\mathsf{T}_i\,\mathsf{T}_{i+1}\ ,\quad \mathsf{T}_i\,\mathsf{T}_j = \mathsf{T}_j\,\mathsf{T}_i\quad\text{for}\ \ |i-j|>1\ , \tag{6.35}$$

$$\kappa_i\,\mathsf{T}_i = \mathsf{T}_i\,\kappa_i = \nu\,\kappa_i\ , \tag{6.36}$$

$$\mathsf{T}_{i\pm1}\,\mathsf{T}_i\ \kappa_{i\pm1} - \kappa_i\,\kappa_{i\pm1} = \mathsf{T}_i\,\mathsf{T}_{i\pm1}\,\kappa_i - \kappa_{i\pm1}\,\kappa_i, \tag{6.37}$$

$$\kappa_i\,\kappa_{i\pm1}\,\kappa_i - \kappa_i = \kappa_{i\pm1}\,\kappa_i\,\kappa_{i\pm1} - \kappa_{i\pm1}, \tag{6.38}$$

where

$$\kappa_i := 1 - \frac{\mathsf{T}_i - \mathsf{T}_i^{-1}}{q-q^{-1}}\ . \tag{6.39}$$

Here q and ν are complex parameters of the algebra which we assume generic in the sequel; in particular, the definition (6.39) makes sense, the denominator in the right hand side does not vanish. Note that the algebra $BMW_n(q,\nu)$ is the quotient of the algebra $\mathcal{BH}_n(1,\nu)$ by the two-sided ideal generated by the *tangle relations*

$$\kappa_i\,\kappa_{i\pm1}\,\kappa_i - \kappa_i = 0,\quad \kappa_{i\pm1}\,\kappa_i\,\kappa_{i\pm1} - \kappa_{i\pm1} = 0. \tag{6.40}$$

It is easy to see that

$$(\mathsf{T}_i - q)(\mathsf{T}_i + q^{-1})(\mathsf{T}_i - \nu) = 0,\quad \kappa_i^2 = \frac{(q-\nu)(q^{-1}+\nu)}{\nu(q-q^{-1})}\ \ \kappa_i.$$

It follows from (6.38), that the elements $\{\kappa_1, \ldots, \kappa_{n-1}\}$ generate the Hecke algebra $\mathcal{H}_n(p)$ corresponding to a parameter p such that

$$p + p^{-1} = \frac{(q-\nu)(q^{-1}+\nu)}{\nu(q-q^{-1})}.$$

Note that the algebra $\mathcal{BH}_n(q,\nu)$ with defining relations (6.35)–(6.38) possesses the automorphism $\rho_2(\mathsf{T}_i) = \mathsf{T}_{n-i}$ (cf. (2.4)). It is well-known that $\dim(BMW_n) = (2n-1)!!$. As for the algebra $\mathcal{BH}_n(q,\nu)$, it is known [7] that it has finite dimension, but as far as we know, the exact value of its dimension is still unknown.

- The baxterized elements $\{T_i(u,v),\ i = 1, \ldots, n-1\}$ are defined by (cf (6.16), $(z := u/v)$),

$$(\nu + q\ z)\ T_i(z) = q\ z\ \mathsf{T}_i + \nu\ \mathsf{T}_i^{-1} + (q-q^{-1})\ \frac{z(q+\nu)}{z-1}\ \equiv (\nu + q\ z)\ T_i(u,v).$$

- The Jucys–Murphy elements $\{y_i,\ i = 1, \ldots, n-1\}$ of the algebra $\mathcal{BH}_n(q,\nu)$ are defined by (6.19). The JM elements $y_2, ldots, y_{n-1}$ pairwise commute and satisfy the identities

$$\kappa_j\ y_{j+1}\ y_j = y_j\ y_{j+1}\ \kappa_j, \quad 1 \le j < n-1.$$

- Affine braid-Hecke algebra $\mathcal{BH}_n(C)$ of type C is an extension of the algebra $\mathcal{BH}_n(q,\nu)$ by the the affine element $\mathsf{T}_0 = y_0 \neq 1$, subject to the set of "crossing relations" (6.21) and (6.22). One can check that the set of elements

$$y_1 = \mathsf{T}_0, \quad y_{k+1} := \mathsf{T}_k\ y_k\ \mathsf{T}_k, \quad k = 1, \ldots, n-1$$

generate a commutative subalgebra in $\mathcal{BH}_n(C)$.

- (Markov trace, cf [7]) The family of algebras $\{\mathcal{BH}_n(q,\nu) \hookrightarrow \mathcal{BH}_{n+1}(q,\nu)\}_{n\ge 1}$ can be provided with (unique) set of homomorphisms

$$Tr_{n+1} : \mathcal{BH}_{n+1}(q,\nu) \longrightarrow \mathcal{BH}_n(q,\nu)$$

which satisfy the conditions stated in Proposition 7.2.

Summarizing, the all properties of the algebra $\mathcal{BH}_n(q,\nu)$ stated in the item **8** allow the use of the methods developed in Sections 6 and 7, to construct families of *commutative subalgebras* in the algebra $\mathcal{BH}_n(q,\nu)$, as well as $\mathcal{BH}_n(q,\nu)$-valued flat connections and associated qKZ equations. Details will appear.

7 Sklyanin's transfer-matrices for affine BMW algebra.

In this Section, to simplify formulas we make the redefinition of all spectral parameters $z \to c^{-1/2}z$. In this case the baxterized element (6.15) does not changed (since it depends on the ratio of spectral parameters) and statement of the Proposition **6.1** reads as the following. For the affine BMW algebra $BMW_n(C^{(1)})$ the element

$$L_j(c^{-1/2}u) = \frac{c^{-1/2}u - y_j}{c^{1/2}u\, y_j - 1} \equiv y_j(u)\ , \qquad y_j(u)\cdot y_j(u^{-1}) = c^{-1}\ , \tag{7.1}$$

where $c = -\nu\, q^{-1}\, \hat{z}^{-1}$, is the baxterized solution of the reflection equation

$$T_j(u/v)\, y_j(u)\, T_j(v\,u)\, y_j(v) = y_j(v)\, T_j(v\,u)\, y_j(u)\, T_j(u/v)\ , \qquad (j = 1, \ldots, n-1)\ . \tag{7.2}$$

7.1 Sklyanin's transfer-matrix elements for the algebra $BMW_n(C)$

In this Subsection we generalize to the BMW algebra case results obtained in [16],[14] for the Hecke algebra case.

Definition 7.1 Let $\vec{z}_{(k)} = (z_1, \ldots, z_k)$ be k parameters and $y_1(x) \in BMW_n(C^{(1)})$ is any *local* (i.e., $[y_1(x), T_k] = 0$, $\forall k > 1$) solution of the reflection equation (7.2) with $j = 1$:

$$T_1\,(x/z)\ y_1(x)\, T_1(x\,z)\, y_1(z) = y_1(z)\, T_1(x\,z)\, y_1(x)\, T_1\,(x/z)\ , \tag{7.3}$$

where solution $y_1(z)$ is given in (7.1) for $j=1$. Define the elements (cf. (6.32))

$$\begin{array}{c} y_k(x;\vec{z}_{(k-1)}) = T_{k-1}(\frac{x}{z_{k-1}})\cdots T_2(\frac{x}{z_2})T_1(\frac{x}{z_1})\, y_1(x)\, T_1(xz_1)T_2(xz_2)\cdots T_{k-1}(xz_{k-1}) = \\ = T_{k-1}(\frac{x}{z_{k-1}})y_{k-1}(x;\vec{z}_{(k-2)})T_{k-1}(xz_{k-1})\ , \end{array} \tag{7.4}$$

which we call "baxterized" Jucys–Murphy elements.

Proposition 7.1 *The "baxterized" Jucys–Murphy element (7.4) is a solution of the reflection equation*

$$T_k\,(x/z)\; y_k(x;\vec{z}_{(k-1)})\, T_k(x\,z)\, y_k(z;\vec{z}_{(k-1)}) = y_k(z;\vec{z}_{(k-1)})\, T_k(x\,z)\, y_k(x;\vec{z}_{(k-1)})\, T_k\,(x/z)\ , \tag{7.5}$$

Proof. The case $k=1$ of the equation (7.5) corresponds to our assumption that $y_1(x)$ satisfies the equation (7.3). The general case follows by induction using the definition (7.4) of elements $y_k(x;\vec{z}_{(k-1)})$. ∎

For example, in the case of the affine BMW algebra $BMW_n(C^{(1)})$, one can use the local solution (7.1) for $j=1$ (recall that $y_1 = \mathsf{T}_0$):

$$y_1(x) = \frac{c^{-1/2}x - y_1}{c^{1/2}\,x\,y_1 - 1} = \frac{c^{-1/2}x - \mathsf{T}_0}{c^{1/2}\,x\,\mathsf{T}_0 - 1}\ , \qquad y_1(1) = -c^{-1/2}\ . \tag{7.6}$$

Further we consider only one-boundary affine BMW algebra $BMW_n(C)$ of type C which is obtained as the projection $\mathsf{T}_n = 1$ from the two-boundary affine BMW algebra $BMW_n(C^{(1)})$ (see paragraph 4 in Section **6**).

Consider the following inclusions of the subalgebras $BMW_1(C) \subset BMW_2(C) \subset \cdots \subset BMW_{n+1}(C)$:

$$\{\mathsf{T}_0;\mathsf{T}_1,\dots,\mathsf{T}_{k-1}\} \in BMW_k(C) \subset BMW_{k+1}(C) \ni \{\mathsf{T}_0;\mathsf{T}_1,\dots,\mathsf{T}_{k-1},\mathsf{T}_k\}\ .$$

For the subalgebras $BMW_{k+1}(C)$ we introduce linear mapping (quantum trace)

$$\mathrm{Tr}_{(k+1)}:\ \ BMW_{k+1}(C) \to BMW_k(C)\ , \quad (k=1,2,\dots,n)\ ,$$

which is defined by the formula

$$\kappa_{k+1}\, X_{k+1}\, \kappa_{k+1} = \frac{1}{\nu}\, Tr_{(k+1)}(X_{k+1})\, \kappa_{k+1}\ , \quad \forall X_{k+1} \in BMW_{k+1}(C)\ . \tag{7.7}$$

Proposition 7.2 *For the map $Tr_{(k+1)}$: $BMW_{k+1}(C) \to BMW_k(C)$ we have the following properties* ($\forall X_k, X'_k \in BMW_k(C)\ ,\ \forall Y_{k+1} \in BMW_{k+1}(C)$)

$$\begin{array}{c} Tr_{(k+1)}(\mathsf{T}_k) = 1\ ,\ \ Tr_{(k+1)}(\mathsf{T}_k^{-1}) = \nu^2\ ,\ \ \mathrm{Tr}_{(k+1)}(X_k) = \nu\,\mu\, X_k\ , \\ Tr_{(k+1)}(\kappa_k) = \nu\ ,\ \ \mathrm{Tr}_{(1)}(\mathsf{T}_0^k) = \nu\, \hat{z}^{(k)}\ , \end{array} \tag{7.8}$$

$$Tr_{(k+1)}(\mathsf{T}_k\, X_k\, \mathsf{T}_k^{-1}) = Tr_{(k)}(X_k) = Tr_{(k+1)}(\mathsf{T}_k^{-1}\, X_k\, \mathsf{T}_k)\ , \tag{7.9}$$

$$Tr_{(k+1)}(\mathsf{T}_k\, X_k\, \kappa_k) = Tr_{(k+1)}(\kappa_k\, X_k\, \mathsf{T}_k)\ , \tag{7.10}$$

$$\begin{array}{c} \mathrm{Tr}_{(k+1)}(X_k \cdot Y_{k+1} \cdot X'_k) = X_k \cdot \mathrm{Tr}_{(k+1)}(Y_{k+1}) \cdot X'_k\ \ , \\ \mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(\mathsf{T}_k \cdot Y_{k+1}) = \mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(Y_{k+1} \cdot \mathsf{T}_k)\ . \end{array} \tag{7.11}$$

Proof. Eqs. (7.8) follow from (6.3), (6.6), (6.8) and (6.22). Using (6.11), (6.12) and (6.8) we have

$$\begin{array}{c} \frac{1}{\nu}\, Tr_{(k+1)}(\mathsf{T}_k\, X_k\, \mathsf{T}_k^{-1})\kappa_{k+1} = \kappa_{k+1}\, \mathsf{T}_k\, \mathsf{T}_{k+1}\, X_k\, \mathsf{T}_{k+1}^{-1}\, \mathsf{T}_k^{-1}\, \kappa_{k+1} = \\ = \kappa_{k+1}\, \kappa_k\, X_k\, \kappa_k\, \kappa_{k+1} = \frac{1}{\nu}\, Tr_{(k)}(X_k)\, \kappa_{k+1}\, \kappa_k\, \kappa_{k+1} = \frac{1}{\nu}\, Tr_{(k)}(X_k)\, \kappa_{k+1}\ , \end{array}$$

which is equivalent to the first equality in (7.9) (second equality in (7.9) can be proved analogously). Eq. (7.10) can be proved in the following way

$$\begin{array}{c} \kappa_{k+1}\, \mathsf{T}_k\, X_k\, \kappa_k\, \kappa_{k+1} = \kappa_{k+1}\, \kappa_k\, \mathsf{T}_{k+1}^{-1}\, X_k\, \kappa_k\, \kappa_{k+1} = \kappa_{k+1}\, \kappa_k\, X_k\, \mathsf{T}_{k+1}^{-1}\, \kappa_k\, \kappa_{k+1} = \\ = \kappa_{k+1}\, \kappa_k\, X_k\, \mathsf{T}_k\, \kappa_{k+1}\ . \end{array}$$

The first eq. in (7.11) is evident and the proof of second eq. in (7.11) is the following. First of all for any $Y'_{k+1} \in BMW_{k+1}(C)$ we have

$$\kappa_{k+2}\kappa_k\kappa_{k+1}\, Y'_{k+1}\, \kappa_{k+1}\kappa_k\kappa_{k+2} = \frac{1}{\nu}\kappa_{k+2}\kappa_k\kappa_{k+1}\mathrm{Tr}_{(k+1)}(Y'_{k+1})\,\kappa_k\kappa_{k+2} =$$
$$= \frac{1}{\nu}\kappa_{k+2}\kappa_k\mathrm{Tr}_{(k+1)}(Y'_{k+1})\,\kappa_k = \frac{1}{\nu^2}\kappa_{k+2}\kappa_k\mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(Y'_{k+1})\;.$$

Then, using this equation and relations (6.11), (6.12) we obtain

$$\tfrac{1}{\nu^2}\kappa_{k+2}\kappa_k\mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(Y_{k+1}\,\mathsf{T}_k) = \kappa_{k+2}\kappa_k\kappa_{k+1}\,Y_{k+1}\mathsf{T}_k\,\kappa_{k+1}\kappa_k\kappa_{k+2} = \kappa_{k+2}\kappa_k\kappa_{k+1}\,Y_{k+1}\mathsf{T}_{k+1}^{-1}\,\kappa_{k+2}\kappa_k =$$
$$= \kappa_{k+2}\kappa_k\kappa_{k+1}\,Y_{k+1}\mathsf{T}_{k+2}\,\kappa_{k+1}\,\kappa_{k+2}\kappa_k = \kappa_{k+2}\kappa_k\kappa_{k+1}\,\mathsf{T}_{k+2}Y_{k+1}\,\kappa_{k+1}\,\kappa_{k+2}\kappa_k =$$
$$= \kappa_{k+2}\kappa_k\kappa_{k+1}\,\mathsf{T}_kY_{k+1}\,\kappa_{k+1}\,\kappa_{k+2}\kappa_k = \tfrac{1}{\nu^2}\kappa_{k+2}\kappa_k\mathrm{Tr}_{(k)}\mathrm{Tr}_{(k+1)}(\mathsf{T}_k\,Y_{k+1})\;.$$

■

Below we use the following identities for baxterized elements (6.15):

$$\begin{array}{c} T_n(x)\,T_n(y) = \frac{(q-q^{-1})(1-xy)}{(1-x)(1-y)}T_n(x\,y) + 1 + \frac{(q-q^{-1})\,\nu\,(xy-1)(\nu xy+q^3)}{(\nu y+q)(\nu x+q)(\nu xy+q)}\kappa_n \;\;\Rightarrow \\ T_n(x) = T_n(y) + \frac{(q-q^{-1})(x-y)}{(y-1)(x-1)} + \frac{(q-q^{-1})\nu q(x-y)}{(\nu y+q)(\nu x+q)}\kappa_n\;, \end{array} \tag{7.12}$$

Note that identity (6.17) is a consequence of the first relation in (7.12) if we substitute there $y = x^{-1}$ and take into account $(1-xy)\,T_n(x\,y) \;\stackrel{y\to x^{-1}}{\longrightarrow}\; (q-q^{-1})$.

Using the properties (7.11) of the map $\mathrm{Tr}_{_{(n+1)}}$ and relations (7.12), one can prove the Lemma.

Lemma 7.1 *For all $X_k \in BMW_k(C)$ and all spectral parameters x and z the following identity is true:*

$$\begin{array}{c} \mathrm{Tr}_{_{(k+1)}}\Big(T_k(x)\cdot X_k\cdot T_k(z)\Big) = \frac{(q-1/q)(\nu^2xz-q^2)}{(x\nu+q)(z\nu+q)}\mathrm{Tr}_{_{(k+1)}}\Big(T_k\cdot X_k\cdot\kappa_k\Big) + \\ +\frac{(q^2\nu^2xz-\nu^2xz+xz\nu^2q^{-2}+q\nu(x+z)+q^2)}{(x\nu+q)(z\nu+q)}\,\mathrm{Tr}_{_{(k)}}(X_k) - \frac{(q-1/q)(xz\nu^2-q^2)(xz\nu q^2+xz\nu^2q+(x+z)\nu-xz\nu+q)}{((z\nu+q)(x\nu+q)q(z-1)(x-1))}\,X_k\;, \end{array} \tag{7.13}$$

where $T_k(x)$ and $T_k(z)$ are Baxterized elements (6.15).

Proof. Direct calculations with the help of properties (7.8) – (7.11). ■

From eq. (7.13), for $x\,z = q^2\nu^{-2}$, we obtain the "crossing-unitarity relation"

$$\mathrm{Tr}_{_{(k+1)}}\Big(T_k(x)\cdot X_k\cdot T_k\left(q^2\nu^{-2}/x\right)\Big) = \frac{1}{F(x)}\,\mathrm{Tr}_{_{(k)}}(X_k)\;, \tag{7.14}$$

where $F(x) = \frac{(x\nu+q)^2}{(x\nu+q^3)(x\nu+q^{-1})}$. Note that identity (7.14) was obtained in [15] for slightly different definition of the baxterized elements (6.15).

Proposition 7.3 (see also [15], [16]). *Let $y_k(x) \in BMW_k(C)$ be any solution of the RE (7.5). The operators*

$$\tau_{k-1}(x) = \mathrm{Tr}_{_{(k)}}\,(y_k(x)) \in BMW_{k-1}(C)\;, \tag{7.15}$$

form a commutative family of operators

$$\Big[\tau_{k-1}(x)\,,\;\tau_{k-1}(z)\Big] = 0 \quad (\forall x,z)\;, \tag{7.16}$$

in the subalgebra $BMW_{k-1}(C) \subset BMW_n(C)$ $(k < n)$.

Proof. Using properties (7.9), (7.11) and relations (7.14), (7.5) we find

$$\tau_{k-1}(x)\,\tau_{k-1}(z) = Tr_{_{(k)}}\,(y_k(x)\,\tau_{k-1}(z)) =$$
$$= F(x\,z)\,\mathrm{Tr}_{_{(k)}}\,\Big(y_k(x)\,\mathrm{Tr}_{_{(k+1)}}\Big(T_k(x\,z)\,y_k(z)T_k(q^2(\nu^2xz)^{-1})\Big)\Big) =$$
$$= F(x\,z)\,\mathrm{Tr}_{_{(k)}}\mathrm{Tr}_{_{(k+1)}}\Big(T_k(x/z)\,y_k(x)\,T_k(xz)\,y_k(z)\,T_k^{-1}(x/z)\,T_k(q^2(\nu^2xz)^{-1})\Big) =$$
$$= F(x\,z)\,\mathrm{Tr}_{_{(k)}}\mathrm{Tr}_{_{(k+1)}}\,\big(y_k(z)\,T_k(xz)\,y_k(x)\,T_k(q^2(\nu^2xz)^{-1})\big) =$$
$$= \mathrm{Tr}_{_{(k)}}\,(y_k(z)\,\tau_{k-1}(x)) = \tau_{k-1}(z)\,\tau_{k-1}(x)\;,$$

where $F(x)$ was defined in (7.14).
■

Now we consider the operators (7.15)

$$\tau_n(x;\vec{z}_{(n)}) = \mathrm{Tr}_{(n+1)}\left(y_{n+1}(x;\vec{z}_{(n)})\right) \in BMW_n(C)\ , \tag{7.17}$$

where solution $y_{n+1}(x) \in BMW_{n+1}(C)$ of the reflection equation is taken in the form (7.4). We stress that the elements (7.17) are nothing but the analogs of Sklyanin's transfer-matrices [40] and the coefficients in the expansion of $\tau_n(x;\vec{z}_{(n)})$ over the variable x (for the homogeneous case $z_k = 1$) are the Hamiltonians for the open Birman-Murakami-Wenzel chain models with nontrivial boundary conditions.

For example let us redefine all baxterized elements in (6.15)

$$T_i(x) \ \rightarrow\ \tilde{T}_i(x) = (1-x)\,T_i(x) = (1-x)\Big(\mathsf{T}_i + \frac{(q-q^{-1})x}{x+\nu^{-1}q}\,\kappa_i\Big) + (q-q^{-1})x\ , \tag{7.18}$$

such that the new elements $\tilde{T}_i(x)$ satisfies conditions

$$\tilde{T}_i(x)\Big|_{x=1} = (q-q^{-1})\ \ ,\qquad \partial_x\,\tilde{T}_i(x)\Big|_{x=1} = -\mathsf{T}_i - \frac{(q-q^{-1})}{1+\nu^{-1}q}\,\kappa_i + (q-q^{-1})\ .$$

$$\tilde{T}_i(u/v)\,\tilde{T}_i(v/u) = \frac{(vq^2-u)\,(uq^2-v)}{q^2uv}\ . \tag{7.19}$$

Now we respectively redefine the Sklyanin's transfer-matrix element (7.17) as the following

$$\tilde{\tau}_n(x;\vec{z}_{(n)}) = \prod_{i=1}^{n}\big((1-x/z_i)\,(1-xz_i)\big)\mathrm{Tr}_{(n+1)}\left(y_{n+1}(x;\vec{z}_{(n)})\right) = \mathrm{Tr}_{(n+1)}\left(\tilde{y}_{n+1}(x;\vec{z}_{(n)})\right)\ , \tag{7.20}$$

where $\tilde{y}_{n+1}(x;\vec{z}_{(n)})$ is given by (7.4) with substitution $T_i(x)\to\tilde{T}_i(x)$ and $k\to n+1$. I.e., we have

$$\begin{aligned}\tilde{y}_k(x;\vec{z}_{(k-1)}) &= \tilde{T}_{k-1}(\tfrac{x}{z_{k-1}})\cdots\tilde{T}_2(\tfrac{x}{z_2})\tilde{T}_1(\tfrac{x}{z_1})\,y_1(x)\,\tilde{T}_1(xz_1)\tilde{T}_2(xz_2)\cdots\tilde{T}_{k-1}(xz_{k-1}) = \\ &= \tilde{T}_{k-1}(\tfrac{x}{z_{k-1}})\tilde{y}_{k-1}(x;\vec{z}_{(k-2)})\tilde{T}_{k-1}(xz_{k-1})\ .\end{aligned} \tag{7.21}$$

Using "unitarity condition" (7.19) we represent baxterized Jucys-Murphy elements (7.21) in the form

$$\begin{aligned}\tilde{y}_k(x;\vec{z}_{(k-1)}) &= \Big(\prod_{i=1}^{k-1}\tfrac{(xq^2-z_i)\,(z_iq^2-x)}{q^2xz_i}\Big)\tilde{y}'_k(x;\vec{z}_{(k-1)})\ , \\ \tilde{y}'_k(x;\vec{z}_{(k-1)}) &\equiv \tilde{T}^{-1}_{k-1}(\tfrac{z_{k-1}}{x})\cdots\tilde{T}^{-1}_1(\tfrac{z_1}{x})\,y_1(x)\,\tilde{T}_1(xz_1)\tilde{T}_2(xz_2)\cdots\tilde{T}_{k-1}(xz_{k-1})\ .\end{aligned} \tag{7.22}$$

We will use this form below.

Then, for the homogeneous case $z_i = 1$ $(\forall i)$, we consider the coefficient

$$\frac{c^{1/2}(q-q^{-1})^{1-2n}}{2\nu\mu}\Big(\partial_x\,\tilde{\tau}_n(x;z_i=1)\,|_{x=1}\Big) = \sum_{i=1}^{n-1}\Big(\mathsf{T}_i + \frac{(q-q^{-1})}{1+\nu^{-1}q}\,\kappa_i\Big) + \frac{(q-q^{-1})}{2}\,\frac{c\mathsf{T}_0^2-1}{(c^{1/2}\mathsf{T}_0-1)^2} + \text{constant}\ ,$$

in the expansion of the generating function $\tilde{\tau}_n(x;z_i=1)$ for commutative elements. This coefficient gives (up to an additional constant) the element

$$\mathcal{H} = \frac{(q-q^{-1})}{2}\,\frac{c\mathsf{T}_0^2-1}{(c^{1/2}\mathsf{T}_0-1)^2} + \sum_{i=1}^{n-1}\Big(\mathsf{T}_i + \frac{(q-q^{-1})}{1+\nu^{-1}q}\,\kappa_i\Big)\ \ \in\ \ BMW_n(C)\ ,$$

being the local Hamiltonian for the open BMW chain model with nontrivial boundary condition for the first site of the chain.

Consider the expansion of $\tau_n(x;\vec{z}_{(n)})$ over x for the inhomogeneous case:

$$\tau_n(x;\vec{z}_{(n)}) = \sum_{k=-\infty}^{\infty} \Phi_k(\vec{z}_{(n)})\, x^k \ \in BMW_n(C) \ . \tag{7.23}$$

According to the Proposition 7.3, for fixed parameters $\vec{z}_{(n)} = (z_1,\dots,z_n)$, the elements $\Phi_k(\vec{z}_{(n)})$ generate a commutative subalgebra $\hat{\mathcal{B}}_n(\vec{z}_{(n)}) \subset BMW_n(C)$. These elements are interpreted as Hamiltonians for the inhomogeneous open Hecke chain models. Following [29] we call the subalgebras $\hat{\mathcal{B}}_n(\vec{z}_{(n)})$ as Bethe subalgebras of the affine algebra $BMW_n(C)$.

7.2 Bethe subalgebras for affine BMW algebra and q-KZ connections

In this Section and below we will use the normalized baxterized elements (7.18): $\widetilde{T}_k(x) = (1-x)T_k(x)$. Consider the transfer-matrix operator (7.20) and fix the spectral parameter as $x = z_k$, where $1 \le k \le n$ (analogous results can be obtained if instead we take $x = z_k^{-1}$). In view of relation $T_k(x/z_k)|_{x=z_k} = (q-q^{-1})$ we deduce for the transfer-matrix operator (7.20)

$$\begin{gathered}
B_k(\vec{z}) = \tfrac{1}{(q-q^{-1})}\tau_n(z_k;\vec{z}_{(n)}) = \mathrm{Tr}_{(n+1)}\Big(\underline{\tilde{T}_n(\tfrac{z_k}{z_n})\cdots\tilde{T}_{k+1}(\tfrac{z_k}{z_{k+1}})}\tilde{T}_{k-1}(\tfrac{z_k}{z_{k-1}})\cdots\tilde{T}_1(\tfrac{z_k}{z_1})\,y_1(z_k)\cdot \\
\cdot\tilde{T}_1(z_kz_1)\cdots\tilde{T}_{k-1}(z_kz_{k-1})\tilde{T}_k(z_k^2)\tilde{T}_{k+1}(z_kz_{k+1})\cdots\tilde{T}_n(z_kz_n)\Big) = \\
\mathrm{Tr}_{(n+1)}\Big(\tilde{T}_{k-1}(\tfrac{z_k}{z_{k-1}})\cdots\tilde{T}_1(\tfrac{z_k}{z_1})\,y_1(z_k)\cdot \\
\cdot\tilde{T}_1(z_kz_1)\cdots\tilde{T}_{k-1}(z_kz_{k-1})\cdot\underline{\tilde{T}_n(\tfrac{z_k}{z_n})\cdots\tilde{T}_{k+1}(\tfrac{z_k}{z_{k+1}})}\tilde{T}_k(z_k^2)\tilde{T}_{k+1}(z_kz_{k+1})\cdots\tilde{T}_n(z_kz_n)\Big) = \\
\mathrm{Tr}_{(n+1)}\Big(\tilde{T}_{k-1}(\tfrac{z_k}{z_{k-1}})\cdots\tilde{T}_1(\tfrac{z_k}{z_1})\,y_1(z_k)\cdot\tilde{T}_1(z_kz_1)\cdots\tilde{T}_{k-1}(z_kz_{k-1})\cdot \\
\cdot\tilde{T}_k(z_kz_{k+1})\cdots\tilde{T}_{n-1}(z_kz_n)\tilde{T}_n(z_k^2)\tilde{T}_{n-1}(\tfrac{z_k}{z_n})\cdots\tilde{T}_{k+1}(\tfrac{z_k}{z_{k+2}})\tilde{T}_k(\tfrac{z_k}{z_{k+1}})\Big) \ .
\end{gathered} \tag{7.24}$$

Now we use relations (7.8) to obtain

$$\mathrm{Tr}_{(n+1)}\Big(\tilde{T}_n(z^2)\Big) = \frac{(q^2 - z^2\nu^2)(z^2\nu + q^{-1})}{(z^2\nu + q)} \equiv N(z^2) \ .$$

Then for (7.24) we deduce

$$\begin{gathered}
B_k(\vec{z}) = N(z_k^2)\,\Big(T_{k-1}(\tfrac{z_k}{z_{k-1}})\cdots T_1(\tfrac{z_k}{z_1})\ \cdot y_1(z_k)\,T_1(z_kz_1)\cdots T_{k-1}(z_kz_{k-1})\Big)\ \cdot \\
\cdot\Big(T_k(z_kz_{k+1})\cdots T_{n-1}(z_kz_n)\cdot T_{n-1}(\tfrac{z_k}{z_n})\cdots T_k(\tfrac{z_k}{z_{k+1}})\Big) = \\
= N(z_k^2)\,\tilde{y}_k(z_k,\vec{z}_{(1,k-1)})\cdot\overline{y}_k(z_k,\vec{z}_{(k+1,n)}) = N(z_k^2)\,\Big(\prod_{i=1}^{k-1}\tfrac{(z_kq^2-z_i)\,(z_iq^2-z_k)}{q^2z_kz_i}\Big)\mathsf{A}'_k(\vec{z}) \ ,
\end{gathered} \tag{7.25}$$

where

$$\mathsf{A}'_k(\vec{z}) = \tilde{y}'_k(z_k;\vec{z}_{(k-1)})\cdot\overline{y}_k(z_k,\vec{z}_{(k+1,n)}) \ , \tag{7.26}$$

$$\overline{y}_k(z_k,\vec{z}_{(k+1,n)}) = \tilde{T}_k(z_kz_{k+1})\cdots\tilde{T}_{n-1}(z_kz_n)\cdot\tilde{T}_{n-1}(\tfrac{z_k}{z_n})\cdots\tilde{T}_{k+1}(\tfrac{z_k}{z_{k+2}})\tilde{T}_k(\tfrac{z_k}{z_{k+1}}) \ ,$$

and elements $\tilde{y}_k(x,\vec{z}_{(1,k-1)})$, $\tilde{y}'_k(x,\vec{z}_{(1,k-1)})$ were defined in (7.21), (7.22).

Operators (7.25) are equal to the transfer-matrix operator $\tau_n(x;\vec{z}_{(n)})$ evaluated at the points $x = z_k$. Thus, by definition the operators $\{B_1(\vec{z}),\dots,B_n(\vec{z})\}$ form a commutative set of elements in the algebra $BMW_n(C)$:

$$[B_k(\vec{z})\,,\,B_r(\vec{z})] = 0 \qquad (\forall k,r = 1,\dots,n) \ . \tag{7.27}$$

Thus, operators $\{B_1(\vec{z}),\dots,B_n(\vec{z})\}$ for fixed parameters $\{z_1,\dots,z_n\}$ can be considered as generators of the Bethe subalgebra in $BMW_n(C)$.

The validity of the identities (7.27) can be shown in different way. For this, in view of (7.25), we need to prove the commutativity of the set of elements $\mathsf{A}'_k(\vec{z}) \in BMW_n(C)$ which can be interpreted as analogs of flat connections (6.31) for quantum Knizhnik-Zamolodchikov equations. Taking into account (6.30) we see that the map $\tilde{\rho}_c$: $B_n(C) \rightarrow BMW_n(C)$:

$$\tilde{\rho}_c(T_i) \;=\; s_i\,\tilde{T}_i(z_i, z_{i+1}) \quad (i = 1, \ldots, n-1)\;, \quad \tilde{\rho}_c(T_0) \;=\; y_1(z_1)\, s_0\;, \tag{7.28}$$

where s_0 is defined in (2.8) with $\sigma(z_1) = 1/z_1$, is the representation of $B_n(C)$. Then we have the following statement.

Proposition 7.4 *The flat connections $\mathsf{A}'_i(\vec{z})$ (7.26) are images $\tilde{\rho}_c(\mathsf{J}_i)$ of the pairwise commuting elements*

$$\mathsf{J}_i = (T_{i-1}^{-1} \cdots T_1^{-1} T_0 T_1 \cdots T_{i-1})(T_i \cdots T_{n-1} \cdot T_{n-1} \cdots T_i) \in B_n(C)\;,$$

which are obtained by the projection $T_n \to 1$ from the elements $J_i \in B_n(C^{(1)})$ given in (2.5).
Proof. The formula $\mathsf{A}'_i(\vec{z}) = \tilde{\rho}_c(\mathsf{J}_i)$ can be checked directly with the use of definition (7.26) of $\mathsf{A}'_i(\vec{z})$ and formulas (7.28) for the map $\tilde{\rho}_c$. ■

Remark. Using the special limit in (6.31), one can deduce the BMW analog of the Cherednik's connections

$$A_k(\vec{z}) = T_{k-1}\Big(\frac{z_k}{z_{k-1}}\Big) \cdots T_1\Big(\frac{z_k}{z_1}\Big) \cdot y_1^{-1} \mathsf{T}_1^{-1} \cdots \mathsf{T}_{n-1}^{-1}\, D_{z_k} \cdot T_{n-1}\Big(\frac{z_k}{z_n}\Big) \cdots T_k\Big(\frac{z_k}{z_{k+1}}\Big) \;\in\; BMW_n(C)\;, \tag{7.29}$$

which were presented for the Hecke algebra case in [4] (see there eq. (4.12) in Section 4.2). The finite difference operator D_{z_k} is given in (4.13) with $\tilde{\tilde{z}} = \frac{c}{c'} z$. In (7.29) we have to take into account that Cherednik's affine elements Y_k are related to ours by $Y_k = y_k^{-1}$.

To rewrite our expression (6.31) to the Cherednik's one (7.29) we need to convert the factor

$$L_1(z_k)\, T_1(cz_k z_1) \cdots T_{k-1}(cz_k z_{k-1}) \cdot T_k(cz_k z_{k+1}) \cdots T_{n-1}(cz_k z_n)\, \bar{L}_n(\frac{1}{cz_k})\;, \tag{7.30}$$

entered into the expression (6.31) to the factor $y_1^{-1} \mathsf{T}_1^{-1} \cdots \mathsf{T}_{n-1}^{-1}$. It can be done if we first make in (6.31) the redefinition of all spectral parameters $z_r \to t z_r$ and then consider the limit $t \to \infty$. To do this we note that only the product (7.30) in (6.31) will be dependent on t, where we have to use limits

$$\lim_{t\to\infty} T_r(t^2 cz_k z_r) = \mathsf{T}_r - (q - q^{-1}) + (q - q^{-1})\kappa_r = \mathsf{T}_r^{-1}\;,$$

$$\lim_{t\to\infty} L_1(tz_k) = \frac{1}{c}\, y_1^{-1} = \frac{1}{c}\, \mathsf{T}_0^{-1}\;, \quad \lim_{t\to\infty} \bar{L}_n\Big(\frac{1}{tcz_k}\Big) = \bar{y}_n = \mathsf{T}_n\;.$$

Here we used the expressions for baxterized elements (6.15), (6.23) and (6.26). Thus, the limit of the factor (7.30) is

$$y_1^{-1} \cdot \mathsf{T}_1^{-1} \cdots \mathsf{T}_{n-1}^{-1} \cdot \bar{y}_n = \mathsf{T}_0^{-1} \cdot \mathsf{T}_1^{-1} \cdots \mathsf{T}_{n-1}^{-1} \cdot \mathsf{T}_n \equiv \mathsf{X}\;,$$

and for the limit of the connection (6.31) we obtain expression

$$A'_k(\vec{z}) = T_{k-1}\Big(\frac{z_k}{z_{k-1}}\Big) \cdots T_1\Big(\frac{z_k}{z_1}\Big) \cdot \mathsf{X}\, D_{z_k} \cdot T_{n-1}\Big(\frac{z_k}{z_n}\Big) \cdots T_{k+1}\Big(\frac{z_k}{z_{k+2}}\Big) T_k\Big(\frac{z_k}{z_{k+1}}\Big) \;\in\; BMW_n(C^{(1)})\;,$$

which is a generalization of (7.29). The projection $\mathsf{T}_n \to 1$ for connection $A'_k(\vec{z})$ gives the BMW analog of the Cherednik's connection (7.29).

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