Spherical quadrilaterals with three non-integer angles

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Abstract

A spherical quadrilateral is a bordered surface homeomorphic to a closed disk, with four distinguished boundary points called corners, equipped with a Riemannian metric of constant curvature 1, except at the corners, and such that the boundary arcs between the corners are geodesic. We discuss the problem of classification of these quadrilaterals and perform the classification up to isometry in the case that one corner of a quadrilateral is integer (i.e., its angle is a multiple of π) while the angles at its other three corners are not multiples of π. The problem is equivalent to classification of Heun's equations with real parameters and unitary monodromy, with the trivial monodromy at one of its four singular point.

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1 Introduction

A *circular polygon* is a simply connected surface whose boundary consists of arcs of circles.

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More precisely, let $\mathbf{S}$ be the Riemann sphere. A circle C is the set of fixed points of an anti-conformal involution of $\mathbf{S}$. This definition does not depend on the metric but only on the conformal structure of $\mathbf{S}$. Let $\mathbf{D}$ be the closed unit disk, and $A = (a_0, a_1 \ldots, a_{n-1})$ a sequence of boundary points enumerated according to the positive orientation of $\partial\mathbf{D}$. Let $f : \mathbf{D} \to \mathbf{S}$ be a continuous function analytic in the open disk U, which is a local homeomorphism on $\mathbf{D} \setminus A$, and such that each *side* $(a_j, a_{j+1}) \subset \partial\mathbf{D}$ is mapped by f into some circle C_j; the circles are not necessarily distinct. The triple $(\mathbf{D}, A, f)$ is called a circular polygon, f is called the *developing map*, and the points of A are called *corners*.

Two polygons $(\mathbf{D}, A, f)$ and $(\mathbf{D}, A', g)$ are *equivalent* if there exist conformal maps $\phi : \mathbf{S} \to \mathbf{S}$ and $\psi : \mathbf{D} \to \mathbf{D}$, $\psi(a_j) = a'_j$, $0 \leq j \leq n-1$, such that $\phi \circ f = g \circ \psi$. Note that equivalence preserves the linear order of the corners, not only their cyclic order.

At each corner a_j the (interior) angle $\alpha_j \geq 0$ is defined. We measure angles in multiples of π, so angle α_j contains $\pi\alpha_j$ radians.

If $\mathbf{S}$ is equipped with the standard spherical metric, and the circles C_j are geodesic (great circles), the polygon is called a *spherical polygon*.

Two spherical polygons are considered equal (isometric) if they are equivalent, and in addition, ϕ is an isometry of the sphere. Notice that isometry of polygons with vertices $(a_0, \ldots, a_{n-1})$ and $(a'_0, \ldots, a'_{n-1})$, according to our definition must send a_j to a'_j.

One can give an alternative definition of a spherical polygon: it is a surface homeomorphic to the closed disk, equipped with a Riemannian metric of constant curvature $+1$ except at the corners $a_0, a_1, \ldots, a_{n-1}$ on the boundary, and such that the sides (a_j, a_{j+1}) are geodesic, and the metric has conical singularities at the corners. Two spherical polygons in the sense of this definition are considered equal if there is an orientation-preserving isometry respecting the labels of the corners.

The equivalence of the two definitions follows from the consideration of the developing map: if z_0 is an interior point of the spherical polygon in the sense of the second definition, there is a neighborhood of this point and an isometry f from this neighborhood to the sphere with the standard spherical metric. Analytic continuation of f gives the developing map.

Notice that great circles C_j corresponding to a spherical polygon satisfy the following

Condition C. *Every two circles either coincide or intersect at two points* *transversally.*

This paper is a part of the series whose goal is classification of spherical quadrilaterals up to isometry. A complete classification of spherical 2-gons is contained in [11], and of spherical triangles in [2, 5], see also [8]. Polygons with two non-integer angles were studied in [3, 4]. In this paper we study spherical polygons, especially quadrilaterals, with three non-integer angles. The general context and history of the question are described in [4]. We also mention Schilling's thesis [10] where circular quadrilaterals with one angle 2π were studied in great detail.

It will be convenient to use the upper half-plane $\overline{\mathbf{H}}$ instead of the unit disk $\mathbf{D}$ for the parametrization of a polygon. In that case, developing map f of a circular n-gon satisfies a Schwarz differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right) = R, \tag{1.1}$$

where

$$R(z) = \sum_{j=0}^{n-1} \frac{1-\alpha_j^2}{2(z-a_j)^2} + \frac{c_j}{z-a_j}, \tag{1.2}$$

a_j and c_j are real, $\alpha_j \geq 0$, and c_j satisfy three linear relations

$$\sum_{j=0}^{n-1} c_j = 0, \quad \sum_{j=0}^{n-1} a_j c_j = -\sum_{j=0}^{n-1} \frac{1-\alpha_j^2}{2}, \quad \sum_{j=0}^{n-1} a_j^2 c_j = -\sum_{j=0}^{n-1} a_j(1-\alpha_j^2).$$

These three conditions ensure that the point at ∞ is not singular. Equations (1.1) and (1.2) give a parametrization of all circular polygons with corners a_j and angles α_j. When a_j and α_j are fixed, this set depends on $n-3$ real parameters, which are called *accessory parameters*. The condition that the polygon is equivalent to a spherical polygon translates into the condition that the monodromy of the equation (1.1) is conjugate to a subgroup of the projective unitary group $PSU(2)$ (we say that the monodromy is *unitarizable*); this condition imposes $n-3$ equations on the accessory parameters.

Thus our problem of classifying spherical polygons is equivalent to the study of real solutions of these $n-3$ equations. We mainly restrict ourselves to the case $n=4$ and three non-integer angles, but in the next section we prove a statement which holds for every n.

The position of four corners of a quadrilateral up to a conformal automorphism of the disk depends on one parameter which is called the *modulus*. One accessory parameter must be determined from the condition that the monodromy is unitarizable. So in general we can expect finitely many quadrilaterals with prescribed angles and prescribed modulus. Our two main problems are, first, to describe the necessary conditions on the angles for a quadrilateral with the given angles to exist, for some value of the modulus; and second, when those conditions are satisfied, to find upper and lower bounds for the number of quadrilaterals with the fixed angles and modulus.

The plan of the paper is the following.

In section 2 we discuss conditions that the angles of a spherical quadrilateral with three non-integer angles must satisfy. We also write explicitly the equation on the accessory parameter for this case. This equation is algebraic and we determine its degree. This degree gives an upper estimate for the number of quadrilaterals with prescribed modulus and angles.

In the following sections we classify these spherical quadrilaterals up to isometry, and obtain a lower estimate for the number of quadrilaterals with prescribed modulus and angles.

2 Conditions on the angles and equation for the accessory parameter

1. Consider a finite collection of circles on the sphere $\mathbf{S}$ satisfying Condition C, such that their union is connected and there are no triple intersections.

These circles define a cell decomposition of the sphere whose 0-, 1- and 2-cells are called *vertices*, *edges* and *faces*.

To each pair (F, v) where F is a face and $v \in \partial F$ is a vertex on the boundary of F an angle $\alpha \in (0, 1)$ is assigned. It is the interior angle of the circular polygon F at v, measured in multiples of π.

Let us assign an arbitrary orientation to each circle. Then to each pair (F, v) we assign a sign $+$ or $-$ by the following rule: if the orientations of the two boundary edges of F adjacent at v are consistent we assign $+$, otherwise we assign $-$.

Let $f : \mathbf{D} \to \mathbf{S}$ be the developing map of a circular polygon with corners a_j whose sides are mapped to the circles of our collection.

The f-preimage of the cell decomposition of the sphere is called a *net*; it is a cell decomposition of the closed disk. The net is a combinatorial object: two nets are equivalent if there is a homeomorphism of $\mathbf{D}$ which sends one to another and respects the labels of the corners.

At every corner v of the polygon we have an angle, which is the sum of the angles assigned to $(f(F), f(v))$ over all faces F adjacent to v. This angle is an integer when the two sides of the polygon adjacent to v are mapped to the same circle, and non-integer otherwise.

The signs assigned to faces $f(F)$ for F adjacent to v make an alternating sequence. For a corner with non-integer angle the length of this sequence is odd, and we assign to v the sign $s(v) = +$ if the sequence begins and ends with $+$, otherwise $s(v) = -$.

With this setting we have

Proposition 2.1 *The number N of signs $s(v) = -$ assigned to non-integer corners is congruent modulo 2 to*

$$\sigma_0 := \sum_{j:\alpha_j \notin \mathbf{Z}} [\alpha_j] + \sum_{j:\alpha_j \in \mathbf{Z}} (\alpha_j - 1). \tag{2.1}$$

Proof. Consider a corner v with integer angle α. Orientation of the sides changes at v if and only if $\alpha - 1$ is odd. So the number of changes of orientation at the corners with integer angles is congruent modulo 2 to

$$\sum_{j:\alpha_j \in \mathbf{Z}} (\alpha_j - 1).$$

Now consider a corner v with non-integer angle α. It is easy to see that the sides of the polygon change orientation at v if $[\alpha]$ is even and $s(v) = -$, or if $[\alpha]$ is odd and $s(v) = +$. In the other two cases the orientation of the sides does not change at v.

We conclude that the number

$$\sum_{j:\alpha_j \in \mathbf{Z}} (\alpha_j - 1) + \sum_{j:\alpha_j \notin \mathbf{Z}} [\alpha_j] + N$$

is congruent modulo 2 to the number of changes of orientation as we trace the boundary of the net, but this last number is evidently even. Thus the number of signs $s(v) = -$ assigned to the corners is congruent to σ_0.

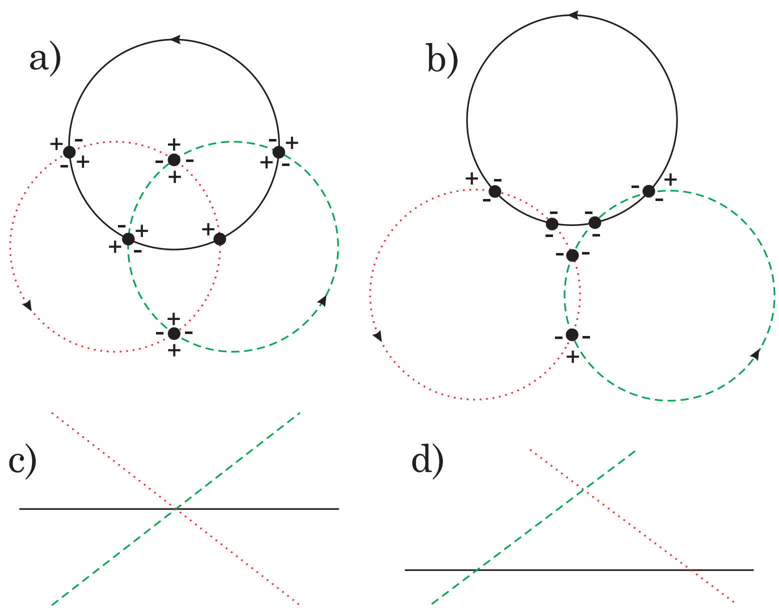

Figure 1: a) Generic unitarizable case, b) and d) Non-unitarizable cases, c) Exceptional case.

2. Consider three circles on the sphere such that every two of them intersect, and no two are tangent to each other. There are four possible topologically distinct configurations shown in Fig. 1.

Indeed, if there is a triple intersection, we can send this point to ∞ and obtain a configuration of three lines. There are two possible configurations of three lines, shown in Fig. 1c and Fig. 1d.

If there is no triple intersection, one of the circles C_j either separates the two intersection points of the other two circles or does not separate them. If it does separate, then every circle separates the intersection points of the other two circles. So we have two possibilities shown in Fig. 1a and Fig. 1b.

It is easy to see that the triple of circles is equivalent to a geodesic triple exactly in the cases a) and c). The case c) will be called *exceptional*.

For a circular polygon whose sides are mapped by the developing map into a generic configuration of three circles (that is the circles satisfy Condition C and are not exceptional) we give a criterion in terms of the angles for the case a).

Proposition 2.2 *Let Q be a circular polygon with non-integer angles θ, θ',*

θ'' and the rest of the angles integers. Suppose that the images of its sides under the developing map satisfy Condition C. Then Q is equivalent to a spherical polygon if and only if it is either exceptional, or

$$\Sigma := \cos^2 \pi\theta + \cos^2 \pi\theta' + \cos^2 \pi\theta'' + 2(-1)^\sigma \cos \pi\theta \cos \pi\theta' \cos \pi\theta'' < 1, \quad (2.2)$$

where

$$\sigma = \sum_{j:\alpha_j \in \mathbf{Z}} (\alpha_j - 1).$$

Proof. Suppose that there are no triple intersections. Then Proposition 2.1 is applicable. Let us choose the orientation of the circles as shown in Fig. 1. This means that we choose a triangular face in the cell decomposition, orient its boundary counterclockwise, and this gives the orientation of all three circles. Then we put signs according to the rule described above. The original triangle gets signs $+, +, +$ on its corners. Notice that each pair of circles intersects twice, and at each intersection we have four angles, two of them with $+$ and two with $-$. All angles formed by two circles which are marked $+$ are equal, and all angles which are marked with $-$ are equal. The sum of an angle marked with $+$ and an angle marked $-$ in the configuration of two circles equals 1.

Consider the functions

$$\phi^\pm(\beta, \beta', \beta'') = \cos^2 \pi\beta + \cos^2 \pi\beta' + \cos^2 \pi\beta'' \pm 2 \cos \pi\beta \cos \pi\beta' \cos \pi\beta''.$$

Notice that for any integers m, n, k we have

$$\phi^\pm(\beta, \beta', \beta') = \phi^\pm(\beta + m, \beta' + n, \beta'' + k) \quad \text{if} \quad m + n + k \quad \text{is even,}$$

and

$$\phi^\pm(\beta, \beta', \beta'') = \phi^\mp(\beta + m, \beta' + n, \beta'' + k) \quad \text{if} \quad m + n + k \quad \text{is odd.}$$

Moreover,

$$\phi^\pm(\beta, \beta', \beta'') = \phi^\mp([\beta] + 1 - \{\beta\}, \beta', \beta''),$$

and similarly for β', β''.

For the angles β, β', β'' of a triangular face in a) we have $\phi^+(\beta, \beta', \beta'') < 1$, and for a triangular face in b) we have $\phi^+(\beta, \beta', \beta'') > 1$. The angles of these triangles are in $(0, 1)$. Each angle $\theta, \theta', \theta''$ corresponds to a vertex of the cell decomposition of the sphere where a pair of circles intersect. These three pairs are all different. So, to obtain $\theta, \theta', \theta''$ from β, β', β'', we have to add the integer parts $[\theta]$, $[\theta'], [\theta'']$ to β, β', β'', and replace some angles by complementary angles. Notice that for corners marked with $-$ the fractional part equals to an angle marked $-$ on the cell decomposition of the sphere.

Thus passing from β, β', β'' to $\theta, \theta', \theta''$ will require

$$[\theta] + [\theta'] + [\theta''] + N$$

sign changes in $\phi^{\pm}$, where N is the number of corners marked with $-$. By Proposition 2.1, N has the same parity as σ_0. This proves Proposition 2.2.

3. The Schwarz equation (1.1), (1.2) can be reduced to a linear differential equation with regular singularities. In the case of quadrilaterals to which we restrict from now on, this linear differential equation is the Heun equation (2.3) below. Assuming that the quadrilateral is parametrized by the upper half-plane, we can choose the corners at the points 0, 1, a and ∞. (The cyclic order of the corners depends on the position of a on the real line, so the notation α_0, α_1 differs in this section from the rest of the paper.) Then $f = y_1/y_0$, the ratio of two linearly independent solutions of the equation

$$z(z-1)(z-a)\left\{y'' + \left(\frac{1-\alpha_0}{z} + \frac{1-\alpha_1}{z-1} + \frac{1-\alpha_a}{z-a}\right)y'\right\} + \alpha'\alpha'' zy = \lambda y, \quad (2.3)$$

where λ is the accessory parameter,

$$\alpha' - \alpha'' = \alpha_\infty, \quad (2.4)$$

and

$$\alpha_0 + \alpha_1 + \alpha_a + \alpha' + \alpha'' = 2. \quad (2.5)$$

We assume that α_0 is an integer, while $\alpha_1, \alpha_a, \alpha_\infty$ are not integers.

The exponents at 0 are 0 and $\alpha_0 > 0$. There is always one holomorphic solution at 0 which can be normalized by $y_1(z) = z^{\alpha_0} + \ldots$. The second linearly independent solution may in general contain logarithms. Condition C in the introduction is equivalent to the absence of logarithms, and it must be satisfied if the monodromy is unitarizable. To write this condition explicitly, we substitute a power series $y_0(z) = 1 + c_1 z + c_2 z^2 + \ldots$ to (2.3) and try to determine the coefficients. We obtain the recurrence relation

$$a(1-\alpha_0)c_1 - \lambda = 0,$$

$$r_n c_{n+1} - (q_n + \lambda) c_n + p_n c_{n-1} = 0, \quad n \geq 1, \tag{2.6}$$

where

$$r_n = (n+1)(n+1-\alpha_0)a$$
$$q_n = n\left((n-\alpha_0)(1+a) + a(1-\alpha_1) + 1 - \alpha_a\right),$$
$$p_n = (n-1+\alpha')(n-1+\alpha'').$$

As α_0 is an integer, we have $r_{\alpha_0-1} = 0$, so c_{α_0} can be only determined when

$$G_{\alpha_0}(\lambda) := -(q_{\alpha_0-1} + \lambda)c_{\alpha_0-1} + p_{\alpha_0-1}c_{\alpha_0-2} = 0.$$

It is easy to see that G_{α_0} is a polynomial of degree α_0. Thus the equation

$$G_{\alpha_0}(\lambda) = 0 \tag{2.7}$$

gives a necessary and sufficient condition for the absence of logarithms in the solutions of the Heun equation (2.3). Now we have

Theorem 2.3 *Suppose that $\alpha_0 \geq 2$ is a positive integer while the other three angles are not integers, and $a \in \mathbf{R}$.*

Then for the projective monodromy of (2.3) to be unitarizable, it is necessary and sufficient that (2.7) holds and either we have (2.2), or $\Sigma = 1$ in (2.2) and we have the exceptional case (as in Fig. 1c). For every given angles, there are only finitely many values of parameter a corresponding to the exceptional case.

Proof. Suppose that the polygon defined by (2.3) is not exceptional. Necessity of conditions (2.7) and (2.2) has already been established. For sufficiency, notice that all parameters in (2.3) are real, so solutions with real initial conditions at some non-singular point of the real line will map a real neighborhood of this point into the real line. Thus any solution of the Schwarz equation will map this neighborhood into a circle. Condition (2.7) implies that two adjacent sides cannot be tangent unless they coincide. As we assume that the configuration is not exceptional, Proposition 2.2 implies that the configuration of the circles to which the sides belong is equivalent to a configuration of great circles. Therefore the monodromy group, which is generated by products of even numbers of reflections in these circles, is conjugate to a subgroup of $PSU(2)$.

Let us prove now that the number of equivalence classes of exceptional quadrilaterals with prescribed angles is finite. Partitions of the sphere corresponding to such quadrilaterals consist of 3 intersecting lines and, possibly, one vertex on one of these lines. These configurations with prescribed angles between the lines are conformally equivalent. Let us show that there are finitely many possible nets. The area of a spherical polygon is a function of its interior angles, so when the angles are fixed, the area is fixed as well. This gives an upper bound for the number of faces of the net, so we have finitely many nets. This completes the proof.

Spherical polygons correspond to *real* solutions of equation (2.7). Thus for given angles and fixed modulus a, there exist at most α_0 spherical quadrilaterals with these angles and modulus. In the rest of the paper we will prove a lower estimate. This will be done by obtaining a combinatorial classification of all possible quadrilaterals with given angles.

We also mention here an algebraic argument which gives a lower estimate. Three-term recurrence (2.6) shows that (2.7) is the characteristic equation of a finite Jacobi matrix

$$J = \begin{pmatrix} 0 & r_0 & & & & & \\ p_1 & -q_1 & r_1 & & & & \\ & p_2 & -q_2 & r_2 & & & \\ & & & & \dots & & \\ & & & & p_{n-1} & -q_{n-1} & r_{n-1} \\ & & & & & p_n & -q_n \end{pmatrix}. \tag{2.8}$$

For the theory of such matrices we refer to [6].

If $r_j p_{j+1} > 0$, $0 \leq j \leq n-1$, then all eigenvalues are real and simple. If $p_j \neq 0$ for $1 \leq j \leq n$, then we have

$$J^T R = RJ, \tag{2.9}$$

where $R = \text{diag}\,(s_0, \dots, s_n)$, $s_0 = 1$ and

$$s_j = s_{j-1} \frac{r_{j-1}}{p_j}, \quad 1 \leq j \leq n.$$

Suppose that the sequence s_j, $0 \leq j \leq n$ has P positive and N negative terms. Then J has at most $\min\{P, N\}$ pairs of complex conjugate non-real eigenvalues. The reason is that equation (2.9) says that J is Hermitian with respect to the quadratic form R of signature (P, N), and the estimate of the number of non-real eigenvalues follows from Pontrjagin's theorem [9].

This gives an algorithm, which for any given angles and given a provides a lower estimate of the number of spherical quadrilaterals with these parameters. This lower estimate is not always exact. Another lower estimate which will be proved in Section 7 is conjectured to be exact. For example, the choice $\alpha_0 = 4, \alpha_1 = \alpha_\infty = 1/2, \alpha_a = 3/2$ and $a < 0$ gives by this algorithm that equation (2.7) has at most two non-real solutions, while the result we obtain in Section 7 implies that all solutions are real in this case.

We will need the following observation. If one sets $a = 0$ in the recurrence relations (2.6), we obtain $r_j = 0$, $0 \leq j \leq n - 1$, so the Jacobi matric J is lower triangular. The eigenvalues in this case are $-q_j = j(\alpha_0 + \alpha_a) - j - 1)$, $0 \leq j \leq n = \alpha_0 - 1$. They are all real and distinct, because α_a is not integer. It follows that for a sufficiently small, all solutions of (2.7) are real. We state this as

Proposition 2.4 *When the modulus of a quadrilateral is sufficiently large or small, that is a in (2.3) is sufficiently close to* 0 *or* 1*, all solutions of (2.7) are real, and there exist exactly α_0 quadrilaterals with given angles and given modulus.*

3 Introduction to nets

A spherical n-gon Q (or a *spherical polygon* when n is not specified) has its corners labeled $a_0, \ldots, a_{n-1}$ in the counterclockwise order on the boundary of Q. This defines linear order on the corners of Q. When $n = 2$, 3 and 4, we call Q a spherical *digon*, *triangle* and *quadrilateral*, respectively. The angles at the corners of Q are measured in multiples of π. A corner is integer if it has an integer, i.e., multiple of π, angle. For $n = 1$, there is a unique 1-gon with the angle $\alpha_0 = 1$ at its single corner. For convenience, we often drop "spherical" and refer simply to n-gons, polygons, etc.

Let Q be a spherical polygon and $f : Q \to \mathbf{S}$ its developing map. The images of the sides (a_j, a_{j+1}) of Q are contained in geodesics (great circles) on $\mathbf{S}$. These circles define a *partition* $\mathcal{P}$ of $\mathbf{S}$ into vertices (intersection points of the circles) edges (arcs of the circles between the vertices) and faces (components of the complement to the circles). Two sides of Q meeting at its corner are mapped by f into the same circle if and only if the corner is integer.

The *order* of a corner is the integer part of its angle. A *removable* corner is an integer corner of order 1. A polygon Q with a removable corner is isometric to a polygon with a smaller number of corners.

A polygon with all integer corners is called *rational*. (The developing map extends to a rational function in this case, which explains the name). All sides of a rational polygon map to the same circle.

Definition 3.1 Preimage of $\mathcal{P}$ defines a cell decomposition $\mathcal{Q}$ of Q, called the *net* of Q. The corners of Q are vertices of $\mathcal{Q}$. In addition, $\mathcal{Q}$ may have *side vertices* and *interior vertices*. If the circles of $\mathcal{P}$ are in general position, interior vertices have degree 4, and side vertices have degree 3. Each face F of $\mathcal{Q}$ maps one-to-one onto a face of $\mathcal{P}$. An edge e of $\mathcal{Q}$ maps either onto an edge of $\mathcal{P}$ or onto a part of an edge of $\mathcal{P}$. The latter possibility may happen when e has an end at an integer corner of Q. The adjacency relations of the cells of $\mathcal{Q}$ are compatible with the adjacency relations of their images in $\mathbf{S}$. The net $\mathcal{Q}$ is completely defined by its 1-skeleton, a connected planar graph. When it does not lead to confusion, we use the same notation $\mathcal{Q}$ for that graph.

If C is a circle of $\mathcal{P}$, then the intersection $\mathcal{Q}_C$ of $\mathcal{Q}$ with the preimage of C is called the *C-net* of Q. Note that the intersection points of $\mathcal{Q}_C$ with preimages of other circles of $\mathcal{P}$ are vertices of $\mathcal{Q}_C$. A *C-arc* of Q (or simply an arc when C is not specified) is a non-trivial path γ in the 1-skeleton of $\mathcal{Q}_C$ that may have a corner of Q only as its endpoint. If γ is a subset of a side of Q then it is a *boundary arc*. Otherwise, it is an *interior arc*. The *order* of an arc is the number of edges of $\mathcal{Q}$ in it. An arc is *maximal* if it is not contained in a larger arc. Each side L of $\mathcal{Q}$ is a maximal boundary arc. The *order* of L is, accordingly, the number of edges of $\mathcal{Q}$ in L.

Definition 3.2 We say that Q is *reducible* if it contains a proper sub-polygon Q' such that the corners of Q' are some (possibly, all) of the corners of Q. Otherwise, Q is *irreducible*. The net of a reducible polygon Q contains an interior arc with the ends at two distinct corners of Q. We say that Q is *primitive* if it is irreducible and its net does not contain an interior arc that is a loop.

Definition 3.3 Two irreducible polygons Q and Q' are *combinatorially equivalent* if there is a homeomorphism $Q \to Q'$ preserving the labels of their corners and mapping the corners of Q to the corners of Q', and the net $\mathcal{Q}$ of Q to the net $\mathcal{Q}'$ of Q'.

Two rational polygons Q and Q' with all sides mapped to the same circle C of $\mathcal{P}$ are combinatorially equivalent if there is a homeomorphism $Q \to Q'$ preserving the labels of their corners and mapping the net $\mathcal{Q}_C$ of Q to the net $\mathcal{Q}'_C$ of Q'.

If Q and Q' are reducible and represented as the union of two polygons Q_0 and Q_1 (resp., Q'_0 and Q'_1) glued together along their common side, then Q and Q' are combinatorially equivalent when there is a homeomorphism $Q \to Q'$ inducing combinatorial equivalence between Q_0 and Q'_0, and between Q_1 and Q'_1.

Thus an equivalence class of nets is a combinatorial object. It is completely determined by the labeling of the corners and the adjacency relations. We'll call such an equivalence class "a net" when this would not lead to confusion.

Conversely, given labeling of the corners and a partition $\mathcal{Q}$ of a disk with the adjacency relations compatible with the adjacency relations of $\mathcal{P}$, a spherical polygon with the net $\mathcal{Q}$ can be constructed by gluing together the cells of $\mathcal{P}$ according to the adjacency relations of $\mathcal{Q}$.

In what follows we classify all equivalence classes of nets in the case when $n = 4$ (that is, when Q is a quadrilateral) and $\mathcal{P}$ is defined by three transversal great circles (that is, Q has one integer corner and three non-integer corners). In this case, the boundary of each 2-cell of the net $\mathcal{Q}$ of Q consists of three segments mapped to the arcs of distinct circles, with the vertices at the common endpoints of each two segments and, possibly, at the integer corner of Q.

By the Uniformization Theorem, each spherical quadrilateral is conformally equivalent to a closed disk with four marked points on the boundary. Thus conformal class of a quadrilateral Q depends on one parameter, the *modulus* of Q. In section 7 below we will study whether for given permitted angles of a quadrilateral an arbitrary modulus can be achieved. This will be done by the method of continuity, and for this we'll need some facts about deformation of spherical quadrilaterals.

4 Classification of primitive quadrilaterals

Let us consider a partition $\mathcal{P}$ of the Riemann sphere $\mathbf{S}$ by three transversal great circles defined by the images of the sides of a quadrilateral Q with one

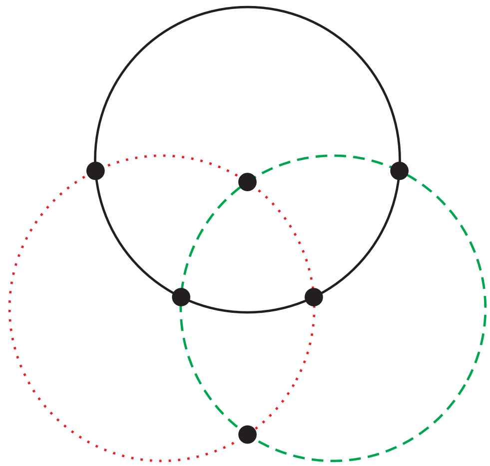

Figure 2: Partition $\mathcal{P}$ of the Riemann sphere by three circles.

integer corner (see Fig. 2).

We assume that with the corner a_0 of Q selected so that the integer corner of Q is labeled a_3. Then the sides L_3 and L_4 of Q are mapped to the same circle C of $\mathcal{P}$ (solid line in Fig. 2). Let C' and C'' be the circles of $\mathcal{P}$ to which the sides L_1 and L_2 of Q are mapped (dotted and dashed lines in Fig. 2).

Lemma 4.1 *The integer corner a_3 of an irreducible quadrilateral Q does not map to a vertex of $\mathcal{P}$.*

Proof. Suppose that a_3 maps to the intersection of C and C' (the case of the intersection of C and C'' follows by symmetry). Consider the union G of all faces of $\mathcal{Q}$ adjacent to a_3. Each of these faces is a triangle with two sides adjacent to a_3 mapped to C and C', and a side opposite a_3 mapped to C''. Thus G is a triangle with an integer corner at a_3. Its base γ is an arc of $\mathcal{Q}$ mapped to C''. Note that, since Q is irreducible, all vertices of $\mathcal{Q}$ on γ are interior vertices of $\mathcal{Q}$, except the two endpoints of γ which are on the boundary of Q. It follows that, for each edge e of γ, there is a face F_e of $\mathcal{Q}$ adjacent to e and not adjacent to a_3. The union of all faces F_e is a triangle G' with the base γ and a vertex p common for all F_e. Since p is connected to a_3 by an interior arc, it cannot be a corner of Q. This implies that the angle α_3 of Q at a_3 cannot be greater than 1, since p cannot have degree greater than 4. If $\alpha_3 = 1$ then p cannot be a boundary vertex of $\mathcal{Q}$, since then the whole boundary of Q would map to C which would contradict the assumption that

Q has three non-integer corners. Also, p cannot be an interior vertex of $\mathcal{Q}$, since then both ends of γ would have degree at least 4, thus they should be both corners of Q, contradicting the assumption that Q is irreducible.

Lemma 4.2 *The net $\mathcal{Q}$ of a primitive quadrilateral Q does not have interior vertices mapped to $C' \cap C''$.*

Proof. Let q be an interior vertex of $\mathcal{Q}$ mapped to $C' \cap C''$. Then there are four faces of $\mathcal{Q}$ adjacent to q. The boundary Γ of the union of those faces is a circle mapped to C. Since Q cannot be bounded by Γ, there is an interior edge of $\mathcal{Q}$ in Γ. Let γ be a maximal interior arc in Γ. Since Q is primitive, γ has two distinct ends, which must be corners of Q (the corner a_1 maps to $C' \cap C''$, and the corner a_3 cannot map to the intersection of two circles of $\mathcal{P}$ by Lemma 4.1). This contradicts the assumption that Q is primitive.

Lemma 4.3 *The net $\mathcal{Q}$ of a primitive quadrilateral Q does not have interior arcs of order greater than 2. Any maximal interior arc of $\mathcal{Q}$ of order 2 which is mapped to C has one of its ends at the corner a_3 of Q. Any maximal interior arc of $\mathcal{Q}$ of order 1 which is mapped to C has one of its ends at a corner of Q.*

Proof. We show first that a maximal interior arc $\gamma \subset \mathcal{Q}_C$ containing an interior vertex q of $\mathcal{Q}$ which is mapped to C' has one end at the corner a_3 of Q. If both edges of γ adjacent to q would have other ends not at a_3 then, using the same argument as in the proof of Lemma 4.2, one can show that the boundary Γ of the union of the four faces of $\mathcal{Q}$ adjacent to q either would be a loop or would contain an interior arc connecting two corners of Q, in contradiction with the assumption that Q is primitive. Thus q is connected to a_3 by an edge of γ. This implies that γ has order 2, otherwise it would contain more than one interior vertex of $\mathcal{Q}$, and only one such vertex can be connected to a_3.

Now we consider a maximal interior arc γ of order 1 (i.e., a single edge of $\mathcal{Q}$) mapped to C. If none of its ends would be at a corner of Q then each face F of $\mathcal{Q}$ adjacent to γ would have its sides other than γ on the sides L_1 and L_2 of Q. Thus F must have the corner a_1 as its vertex, but only one of the two faces of $\mathcal{Q}$ adjacent to γ may be also adjacent to a_1.

Lemma 4.4 *At least two non-integer corners of a primitive quadrilateral Q have order zero.*

Proof. It is enough to show that if the corner a_0 of Q has positive order then its corners a_1 and a_2 have order zero. If a_0 has positive order then there is a maximal interior arc γ of $\mathcal{Q}_C$ and a maximal interior arc γ' of $\mathcal{Q}_{C'}$, each of them having one end at a_0, such that $\mathcal{Q}$ has a face F adjacent to L_1 and γ, and a face F' adjacent to γ and γ'. Due to Lemma 4.2, γ' has order 1, and the other end q' of γ' is on L_2. Thus γ also have order 1, and its other end q is on L_2. This implies that a_1 has order zero, as it is a vertex of F. This also implies that a_2 has order zero. Otherwise, the same arguments will show that a maximal interior arc with one end at a_2 must have the other end on L_1, which is impossible.

Theorem 4.5 *Any primitive quadrilateral with one integer corner is equivalent to one of the following:*
$X_{\mu\nu}$ *with* $\mu \geq 0,\ \nu \geq 0$, *the angles* $[\alpha_0] = [\alpha_1] = [\alpha_2] = 0,\ \alpha_3 = \mu + \nu + 1$, μ *(resp.,* ν*) interior arcs of the net having one end at* a_3 *and another end on* L_1 *(resp.,* L_2*) (Fig. 3a and Fig. 3b);*
$\bar{X}_{\mu\nu}$ *with* $\mu + \nu \geq 2$ *even, the angles* $[\alpha_0] = [\alpha_2] = [\alpha_3] = 0,\ [\alpha_1] = (\mu + \nu)/2$, μ *(resp.,* ν*) interior arcs of the net having one end at* a_1 *and another end on* L_4 *(resp.,* L_3*) (Fig. 3c);*
$R_{\mu\nu}$ *with* $\mu > 0,\ \nu \geq 0$, *the angles* $[\alpha_0] = \mu,\ \alpha_3 = \nu + 1,\ [\alpha_1] = [\alpha_2] = 0$, 2μ *(resp.,* ν*) interior arcs of the net having one end at* a_0 *(resp.,* a_3*) and another end on* L_2 *(Fig. 4a and Fig. 4b);*
$\bar{R}_{\mu\nu}$ *with* $\mu \geq 0,\ \nu > 0$, *the angles* $\alpha_3 = \mu + 1,\ [\alpha_2] = \nu,\ [\alpha_0] = [\alpha_1] = 0$, μ *(resp.,* 2ν*) interior arcs of the net having one end at* a_3 *(resp.,* a_2*) and another end on* L_1 *(Fig. 4c and Fig. 4d);*
$U_{\mu\nu}$ *with* $\mu > 0,\ \nu > 0$, *the angles* $\alpha_3 = \mu + 1,\ [\alpha_1] = \nu,\ [\alpha_0] = [\alpha_2] = 0$, μ *(resp.,* 2ν*) interior arcs of the net having one end at* a_3 *(resp.,* a_1*) and another end on* L_1 *(resp.,* L_3*) (Fig. 5a and Fig. 5b);*
$\bar{U}_{\mu\nu}$ *with* $\mu > 0,\ \nu > 0$, *the angles* $[\alpha_1] = \mu,\ \alpha_3 = \nu + 1,\ [\alpha_0] = [\alpha_2] = 0$, 2μ *(resp.,* ν*) interior arcs of the net having one end at* a_3 *(resp.,* a_1*) and another end on* L_4 *(resp.,* L_2*) (Fig. 5c and Fig.5d);*
$V_{\mu\nu}$ *with* $\mu > 0,\ \nu > 0$, *the angles* $\alpha_3 = \mu + 1,\ [\alpha_1] = \nu,\ [\alpha_0] = [\alpha_2] = 0$, μ *(resp.,* $2\nu - 1$*) interior arcs of the net having one end at* a_3 *(resp.,* a_1*) and another end on* L_1 *(resp.,* L_3*), and one interior arc having one end at* a_1 *and another end on* L_1 *(Fig. 6a);*

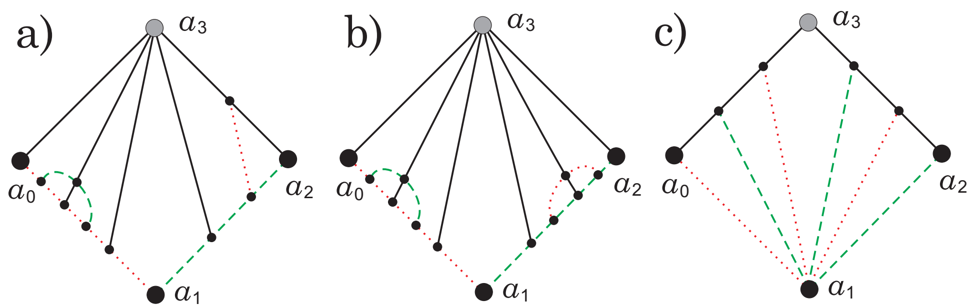

Figure 3: Primitive quadrilaterals of types X and $\bar{X}$.

$\bar{V}_{\mu\nu}$ *with* $\mu > 0,\ \nu > 0$, *the angles* $[\alpha_1] = \mu,\ \alpha_3 = \nu + 1,\ [\alpha_0] = [\alpha_2] = 0$, ν *(resp.,* $2\mu - 1$*) interior arcs of the net having one end at* a_3 *(resp.,* a_1*) and another end on* L_2 *(resp.,* L_4*), and one interior arc having one end at* a_1 *and another end on* L_2 *(Fig. 6b);*
$\bar{Z}_{\mu\nu}$ *with* $\mu > 0,\ \nu > 0$, *the angles* $\alpha_3 = \mu + \nu + 1,\ [\alpha_0] = [\alpha_2] = 0.\ [\alpha_1] = 1$, μ *(resp.,* ν*) interior arcs of the net having one end at* a_3 *and another end on* L_1 *(resp.,* L_2*), one interior arc having one end at* a_1 *and another end on* L_1*, and one interior arc having one end at* a_1 *and another end on* L_2 *(Fig. 6c).*

Remark 4.6 *The nets of primitive quadrilaterals in Theorem 4.5 may contain interior arcs with both ends on the sides of the quadrilateral (see Figs. 3-6). Location of these arcs is uniquely determined by the arcs of the net having one end at a corner of the quadrilateral.*

Proof. If all non-integer corners have order 0 then any maximal interior arc of $\mathcal{Q}$ mapped to C has one end at a_3. Let μ (resp., ν) be the number of such arcs having the other end on L_1 (resp, L_2). Then Q is equivalent to $X_{\mu\nu}$.

If the corners a_1 and a_2 of Q have order 0 but the order of a_0 is greater than 0, then any maximal interior arc of $\mathcal{Q}$ mapped to C has one end either at a_0 or at a_3. Any such arc having one end at a_0 must have order 1, and its other end must be on L_2. Also, any interior arc of $\mathcal{Q}$ mapped to C' with one end at a_0 must have order 1 and its other end on L_2, since it is the side of a face of $\mathcal{Q}$ adjacent to an interior arc mapped to C with one end at a_0. Since the order of a_0 is greater than 0, any maximal interior arc of $\mathcal{Q}$ mapped to C with one end at a_3 must have its other end on L_2. The interior arcs of $\mathcal{Q}$ (if any) having both ends on the sides of Q but not at its corners are completely determined by the arcs having one end at a corner of Q. If ν is

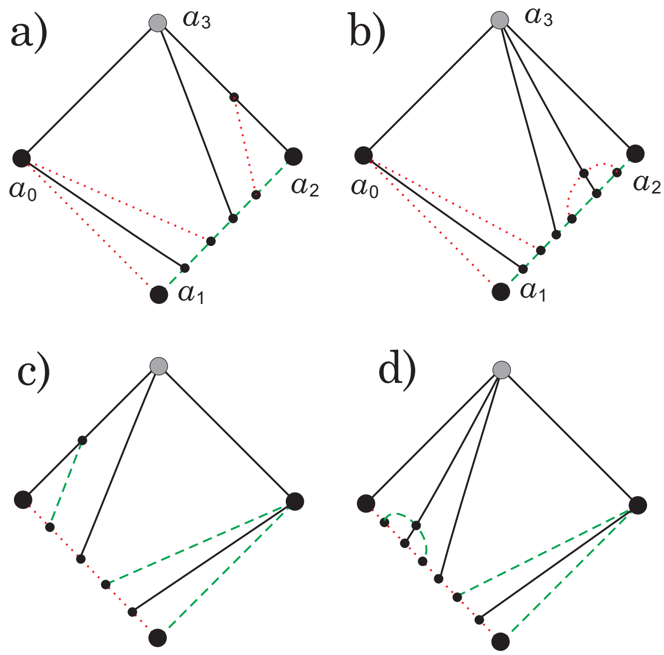

Figure 4: Primitive quadrilaterals of types R and $\bar{R}$.

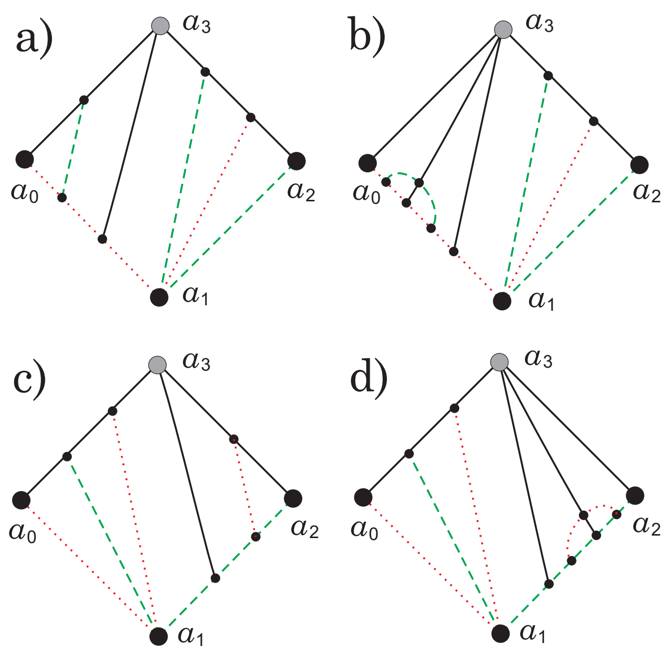

Figure 5: Primitive quadrilaterals of types U and $\bar{U}$.

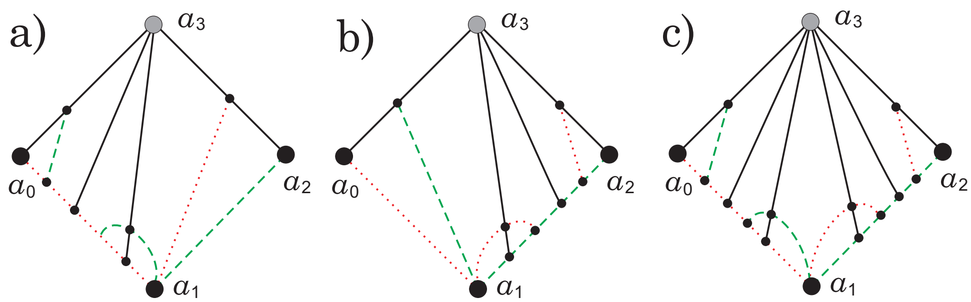

Figure 6: Primitive quadrilaterals of types V, $\bar{V}$ and Z.

the number of maximal interior arcs of $\mathcal{Q}$ with one end at a_3, and $\mu > 0$ is the number of interior arcs mapped to C having one end at a_0 (which is equal to the number of interior arcs mapped to C' having one end at a_0) then Q is equivalent to $R_{\mu\nu}$. Similarly, if a_0 and a_1 have order 0 but the order of a_2 is greater than 0 then Q is equivalent to $\bar{R}_{\mu\nu}$ for some $\mu \geq 0$ and $\nu > 0$.

If the corners a_0 and a_2 and a_3 of Q have order 0 but the order of a_1 is greater than 0, then any maximal interior arc of Q having one end at a_1 has order 1 and its other end is either on L_3 or L_4. The number of such arcs is even. There are no other arcs in $\mathcal{Q}$. If μ (resp., ν) is the number of interior arcs of $\mathcal{Q}$ having one end at a_1 and another end on L_4 (resp., L_3) then Q is equivalent to $\bar{X}_{\mu\nu}$.

Consider now the case when a_0 and a_2 have order 0 but the orders of both a_1 and a_3 are greater than 0. Suppose first that all maximal interior arcs of $\mathcal{Q}$ mapped to C have one end at a_3 and the other end on L_1. Let $\mu > 0$ be the number of these arcs. Let $\nu > 0$ be the number of maximal interior arcs of $\mathcal{Q}$ having one end at a_1 mapped to C'. Then there are exactly ν maximal interior arcs with one end at a_1 mapped to C''. There are two possibilities. If all these arcs have the other end on L_3 then Q is equivalent to $U_{\mu\nu}$. Alternatively, one of these arcs mapped to C'' may have the other end either on L_1 or (when $\mu = 1$) on L_4. Then Q is equivalent to $V_{\mu\nu}$. The interior arcs of $\mathcal{Q}$ (if any) having both ends on the sides of Q but not at its corners are completely determined by the arcs having one end at a corner of Q. Similarly, if all maximal interior arcs of $\mathcal{Q}$ mapped to C have one end at a_3 and another end on L_2 then Q is equivalent to either $\bar{U}_{\mu\nu}$ or $\bar{V}_{\mu\nu}$ for some $\mu > 0$ and $\nu > 0$.

Finally, suppose that $\mu > 0$ (resp., $\nu > 0$) of maximal interior arcs mapped to C have one end at a_3 and the other end on L_1 (resp., L_2). Then there are exactly two interior arcs having one end at a_1, one of them, mapped to C', has the other end either on L_2 or (if $\nu = 1$) on L_3, and the other arc, mapped to C'', has the other end either on L_1 or (if $\mu = 1$) on L_4. In that case Q is equivalent to $Z_{\mu\nu}$.

Remark 4.7 *Sometimes it will be convenient to use notation $\bar{X}_{0,0}$, $R_{0,0}$, $\bar{R}_{0,0}$, $U_{0,0}$, $\bar{U}_{0,0}$ for the quadrilaterals equivalent to $X_{0,0}$. We may also use notation $U_{\mu\nu}$ and $\bar{U}_{\mu\nu}$ with either $\mu = 0$ or $\nu = 0$. Then $U_{\mu,0}$ is equivalent to $X_{\mu,0}$, and $U_{0,\nu}$ is equivalent to $\bar{X}_{0,2\nu}$ for $\nu > 0$. Similarly, $\bar{U}_{0,\nu}$ is equivalent to $X_{0,\nu}$, and $\bar{U}_{\mu,0}$ is equivalent to $\bar{X}_{2\mu,0}$ for $\mu > 0$. Also, $V_{0,\nu}$ is equivalent to $\bar{X}_{1,2\nu-1}$, $\bar{V}_{\mu,0}$ is equivalent to $\bar{X}_{2\mu-1,1}$.*

5 Classification of irreducible quadrilaterals

Lemma 5.1 *Let γ be a loop of the net $\mathcal{Q}$ of an irreducible quadrilateral Q with three non-integer corners. Then γ contains either the integer corner a_3 or the non-integer corner a_1 of Q.*

Proof. A loop γ bounds a disk $D \subset Q$ (shaded area of Fig. 7) consisting of 4 faces of $\mathcal{Q}$ with a common vertex p. If γ would not contain any corner then the 4 faces of $\mathcal{Q}$ adjacent to γ outside D would also have a common vertex, thus Q would be a sphere, a contradiction. Thus γ contains a corner of Q. If γ contains a_0 then it maps either to C or to C'. We assume that γ maps to C, the other case being treated similarly. Then D contains part of a maximal arc γ' of $\mathcal{Q}$ with one end at a_1 that maps to C', and part of a maximal arc γ'' that maps to C'' (see Fig. 7).

We want to show that γ' is a loop. Since γ' contains a_0, and Q is irreducible, γ' cannot contain any other corner of Q. Let $q \neq a_0$ be the intersection point of γ' and γ (see Fig. 7). Then the union of faces F and F' of $\mathcal{Q}$ adjacent to the segment $[a_0, q]$ of γ outside D have a segment of γ' outside D connecting q with a_1 as part of its boundary.

Next, we want to show that γ'' is a loop. Let p' be the intersection point of γ'' with γ inside the disk bounded by γ', and let $q' \neq p'$ and $q'' \neq p$ be the intersection points of γ'' with γ and γ', respectively (see Fig. 7). Then the face F'' of $\mathcal{Q}$ adjacent to the segments $[q, q']$ of γ and $[q, q'']$ of γ' has a segment $[q', q'']$ of γ'' in its boundary. As $[q', q'']$ is a segment of the boundary of a single face F'' mapped to C'', it cannot contain a corner of Q. Thus it is a single edge of $\mathcal{Q}$. The boundary of F'' cannot contain a corner of Q, since

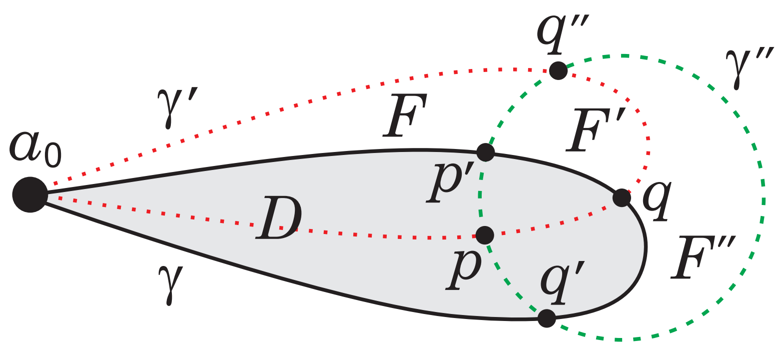

Figure 7: Illustration of the proof of Lemma 5.1.

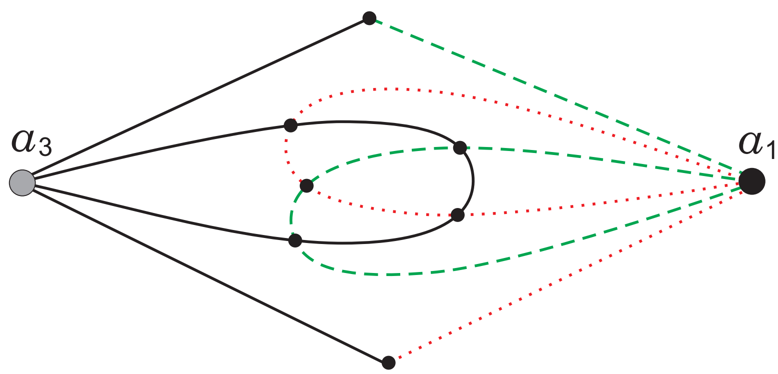

Figure 8: Pseudo-diagonal connecting corners a_1 and a_3 of Q.

all vertices of $\mathcal{Q}$ on this boundary belong either to γ or to γ', and those two loops already contain the corner a_0. Thus the loop γ'' contains no corners, a contradiction.

It follows that γ cannot contain a_0. A similar argument shows that γ cannot contain a_2. Thus γ contains either a_1 or a_3.

Proposition 5.2 *Every loop of the net $\mathcal{Q}$ of an irreducible quadrilateral Q with three non-integer corners is part of a pseudo-diagonal (Fig. 8) connecting its integer corner a_3 with its non-integer corner a_1.*

Proof. The same arguments as in the proof of Lemma 5.1 show that a loop containing a_1 should be part of a pair (γ', γ'') where γ' is a loop mapping to C' and γ'' is a loop mapping to C'', both loops containing a_1. In addition, there should be a loop γ mapped to C intersecting both loops γ' and γ''.

The loop γ must contain a corner of Q, and the integer corner a_3 is the only possibility (any other corner is rejected by the same argument as in the proof of Lemma 5.1).

Conversely, if γ is a loop of $\mathcal{Q}$ mapped to C and containing a_3 then it bounds a disk $D \subset Q$ so that the arcs intersecting inside D are parts of two loops, γ' mapped to C' and γ'' mapped to C'', both containing a_1. This can be shown by the arguments similar to those in the proof of Lemma 5.1.

Corollary 5.3 *Each irreducible spherical quadrilateral Q with three non-integer corners is either primitive or can be obtained from a primitive polygon $\tilde{Q}$ of any type except $R_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ with a face F of its net containing the corners a_1 and a_3 of $\tilde{Q}$ are on its boundary, by adding any number of pseudo-diagonals connecting a_1 with a_3 inside F.*

We use notation $U^\kappa_{\mu\nu}$, $\bar{U}^\kappa_{\mu\nu}$, $X^\kappa_{\mu\nu}$, $\bar{X}^\kappa_{\mu\nu}$, $V^\kappa_{\mu\nu}$, $\bar{V}^\kappa_{\mu\nu}$, $Z^\kappa_{\mu\nu}$ for an irreducible quadrilateral obtained from one of the primitive quadrilaterals in Theorem 4.5 by adding κ pseudo-diagonals.

6 Classification of reducible quadrilaterals

To classify reducible quadrilaterals, we need digons D_n, and primitive triangles T_n, E_n, M_n and $\bar{M}_n$ (see Fig. 9). We also need a triangle ∇ with all its angles equal 1 and all its sides mapped to the circle C corresponding to the sides L_3 and L_4 of Q. Geometrically, ∇ is just a disk bounded by C with three distinct points on C marked as its corners.

A digon D_n is a union of n disks, each bounded by the same circle of $\mathcal{P}$, glued together over the segments of their boundaries, each of them having its ends at the same two points of the circle, so that their union is homeomorphic to a disk. The angles at the two corners of D_n are equal n. The two boundary arcs of D_n are called its sides. We use the same notation D_n for a digon having its sides on any of the three circles of $\mathcal{P}$, and for any possible sizes of its two sides. We use notation D_0 for an empty digon.

A primitive triangle T_n (see Fig. 9a) has an integer corner with the angle $n \geq 1$ mapped to an intersection point of the two circles C' and C'' of $\mathcal{P}$ corresponding to the sides L_1 and L_2 of Q. The side of T_n opposite its integer corner maps to the circle C of $\mathcal{P}$ corresponding to the sides L_3 and L_4 of Q.

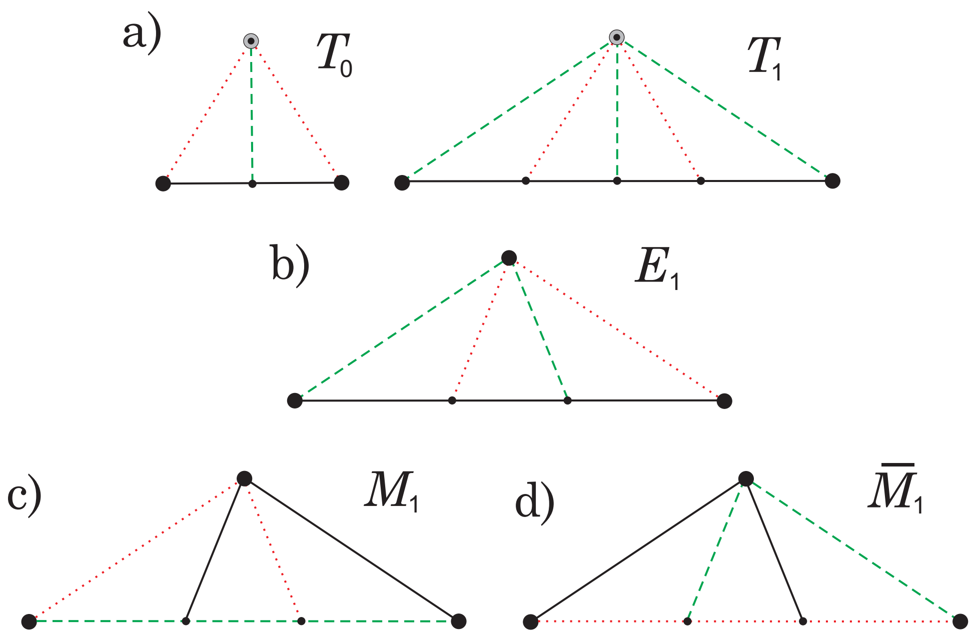

Figure 9: Primitive triangles that appear in reducible quadrilaterals.

We use the same notation T_n for a triangle with two sides mapped either to C' or to C''.

A primitive triangle E_n (see Fig. 9b) has a non-integer corner with the integer part of the angle equal n mapped to an intersection point of the two circles C' and C'' of $\mathcal{P}$ corresponding to the sides L_1 and L_2 of Q. The side of E_n opposite its integer corner maps to the circle C of $\mathcal{P}$ corresponding to the sides L_3 and L_4 of Q. The cyclic order of the sides of E_n is consistent with the cyclic order of the sides of Q.

A primitive triangle M_n (resp., $\bar{M}_n$, see Fig. 9c and Fig. 9d) has a non-integer corner with the integer part of the angle equal $n \geq 1$ mapped to an intersection point of the circle C' of $\mathcal{P}$ corresponding to the side L_1 of Q (resp., the circle C'' corresponding to the side L_2) with the circle C corresponding to the sides L_3 and L_4. The cyclic order of the sides of M_n and $\bar{M}_n$ is consistent with the cyclic order of the sides of Q.

If there is no arc of the net $\mathcal{Q}$ of a reducible quadrilateral Q connecting its opposite corners then Q contains an irreducible quadrilateral $\tilde{Q}$, with digons D_i, D_k, D_l, D_m attached to its four sides (see Fig. 10). A non-empty disk may be attached only to a "short" side of $\tilde{Q}$ (that is, to a side shorter than

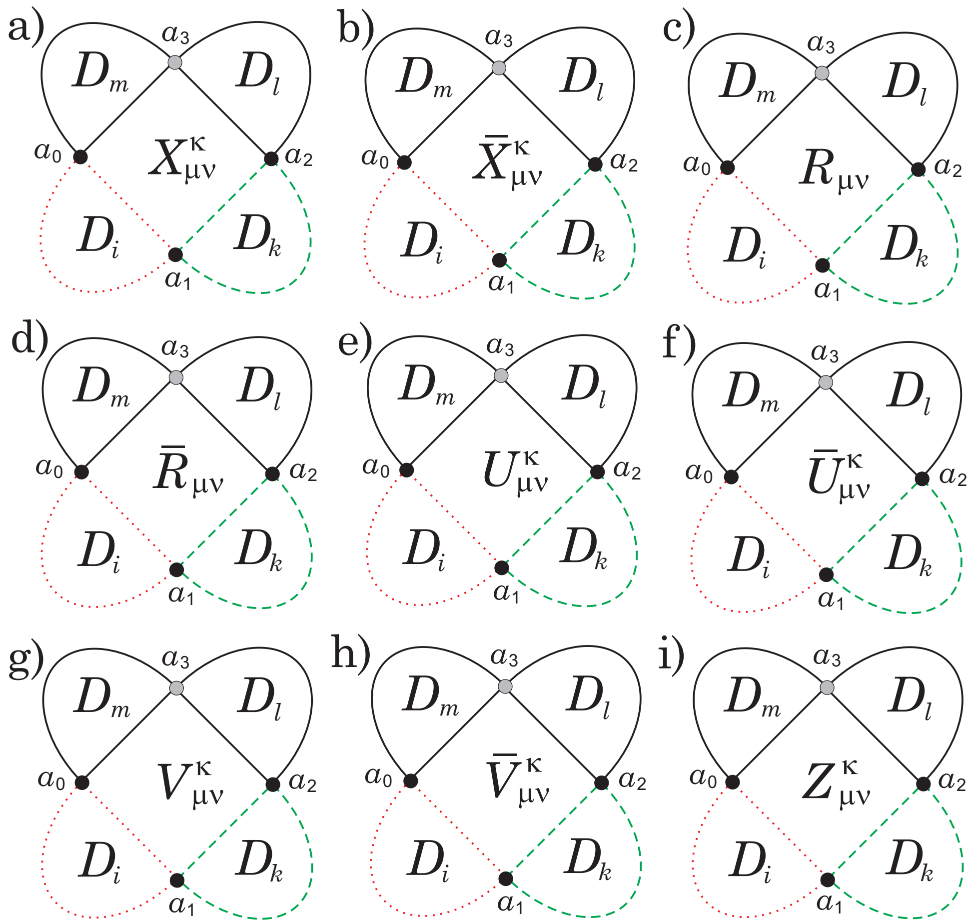

Figure 10: Reducible quadrilaterals with a generic integer corner.

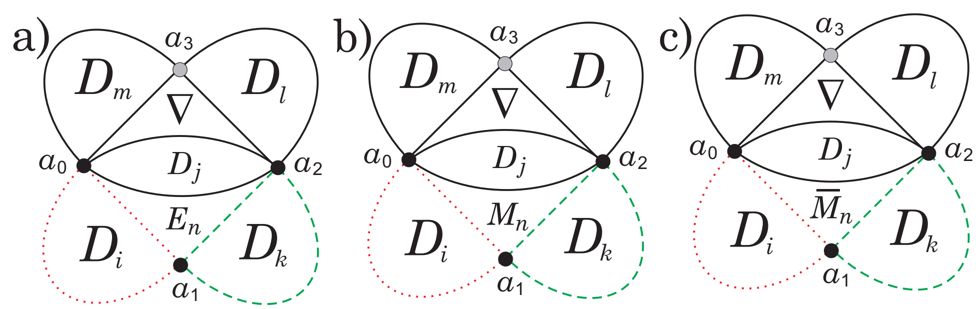

Figure 11: Reducible quadrilaterals with a generic integer corner (cont).

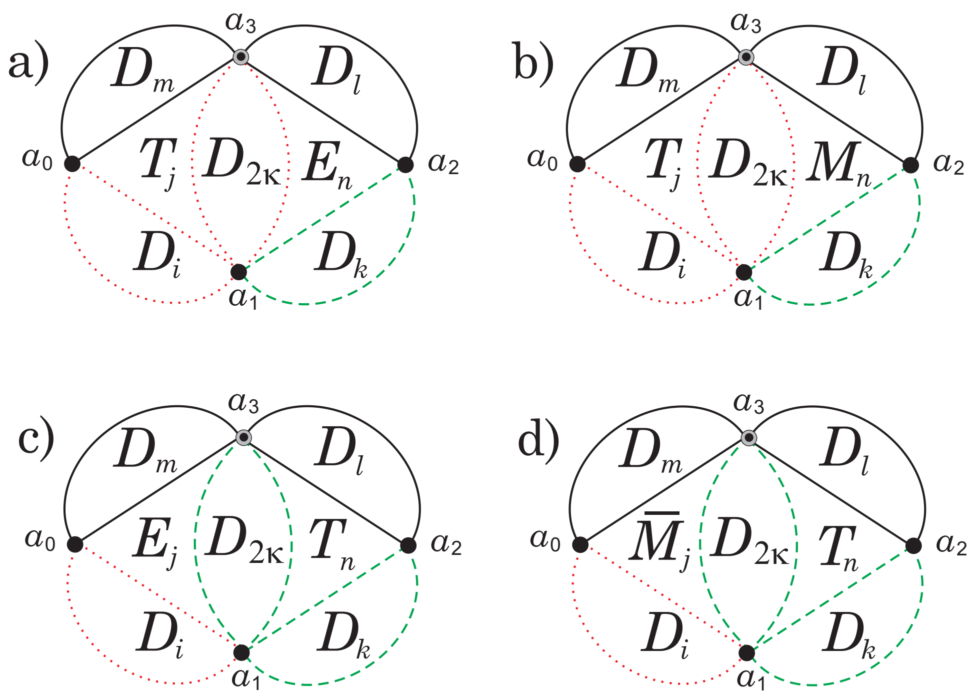

Figure 12: Reducible quadrilaterals with a non-generic integer corner.

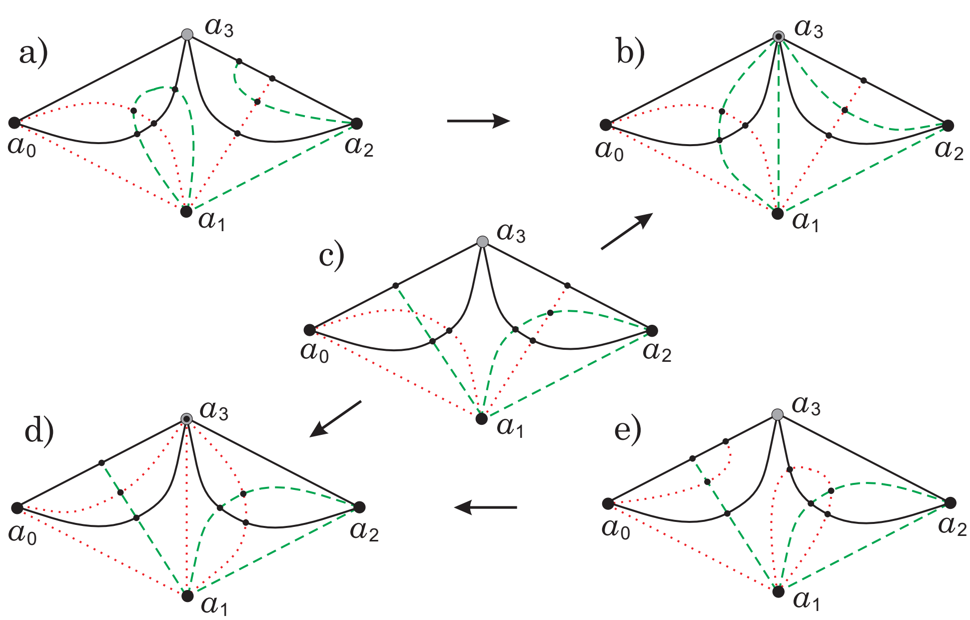

Figure 13: An ab-chain of quadrilaterals of length 2.

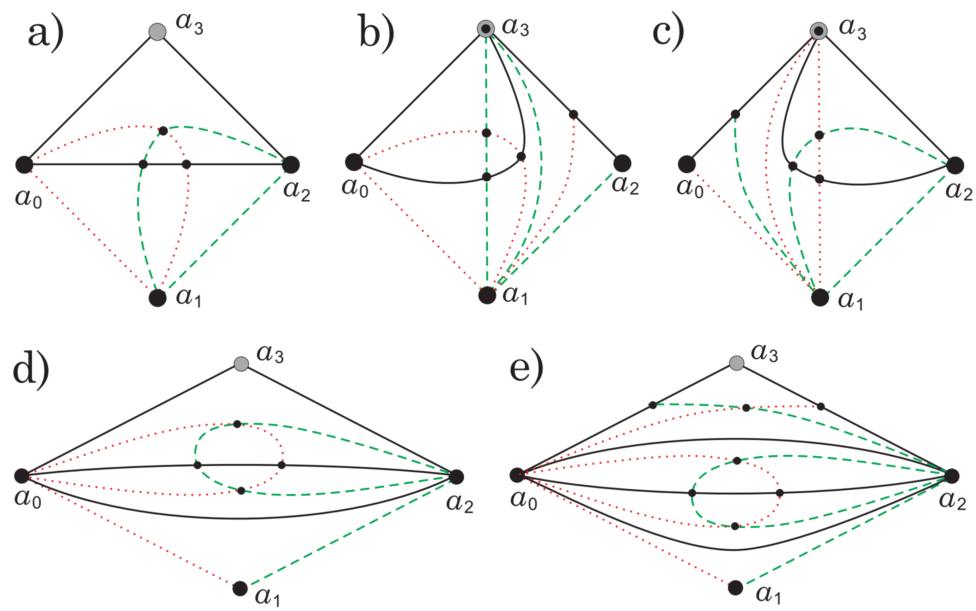

Figure 14: Examples of reducible quadrilaterals.

the full circle):
If $\tilde{Q} = X^{\kappa}_{\mu\nu}$ then $i > 0$ only if $\mu \leq 1$ and $k > 0$ only if $\nu \leq 1$.
If $\tilde{Q} = \bar{X}^{\kappa}_{\mu\nu}$ then $m > 0$ only if $\mu \leq 3$ and $l > 0$ only if $\nu \leq 3$.
If $\tilde{Q} = R_{\mu\nu}$ then $k > 0$ only if $\mu = 1$ and $\nu = 0$.
If $\tilde{Q} = \bar{R}_{\mu\nu}$ then $i > 0$ only if $\mu = 0$ and $\nu = 1$.
If $\tilde{Q} = U^{\kappa}_{\mu\nu}$ then $i > 0$ only if $\mu = 1$ and $l > 0$ only if $\nu = 1$.
If $\tilde{Q} = \bar{U}^{\kappa}_{\mu\nu}$ then $m > 0$ only if $\mu = 1$ and $k > 0$ only if $\nu = 1$.
If $\tilde{Q} = V^{\kappa}_{\mu\nu}$ then $i > 0$ only if $\mu = 1$ and $l > 0$ only if $\nu \leq 2$.
If $\tilde{Q} = \bar{V}^{\kappa}_{\mu\nu}$ then $m > 0$ only if $\mu \leq 2$ and $k > 0$ only if $\nu = 1$.
If $\tilde{Q} = Z^{\kappa}_{\mu\nu}$ then $i > 0$ only if $\mu = 1$ and $k > 0$ only if $\nu = 1$.

If the corners a_0 and a_2 of Q are connected by an arc of $\mathcal{Q}$ then Q is a union of a triangle ∇ and one of the triangles E_n (for $0 \leq 1 \leq 1$), M_n (for $n \geq 1$), or $\bar{M}_n$ (for $n \geq 1$) with a digon D_j inserted between them and digons D_i, D_k, D_l, D_m attached to their other sides (see Fig. 11). Examples of such quadrilaterals are given in Fig. 14a,d,e.

Note that the union of ∇ and E_1 (see Fig. 14a) contains also arcs of $\mathcal{Q}$ connecting a_1 with a_0 and with a_2, and is equivalent to one of the quadrilaterals either in Fig. 10c,d or in Fig. 11b,c.

The integer corner a_3 of Q is "generic" – it may map to a point on C that either belongs or does not belong to another circle of $\mathcal{P}$. A non-empty disk may be attached only to a "short" side of a triangle M_n or $\bar{M}_n$:
If $n > 1$ then $k = 0$ in Fig. 11b.
If $n > 1$ then $i = 0$ in Fig. 11c.

If the corners a_1 and a_3 of Q are connected by an arc of $\mathcal{Q}$ then Q is a union of two primitive triangles (see Fig. 12) with a digon $D_{2\kappa}$ inserted between them and and digons D_i, D_k, D_l, D_m attached to their other sides.

We assume $n \geq 1$ in Fig. 12b, and $j \geq 1$ in Fig. 12d. The integer corner a_3 of such a quadrilateral Q is "non-generic" – it must map to an intersection of C with another circle of $\mathcal{P}$. A non-empty disk may be attached only to a "short" side of any triangle:
In Fig. 12a, $m = 0$ if $j > 1$, and $l = 0$ if $n > 1$.
In Fig. 12b, $m = 0$ if $j > 1$, and $k = 0$ if $n > 1$.
In Fig. 12c, $m = 0$ if $j > 1$, and $l = 0$ if $n > 1$.
In Fig. 12d, $i = 0$ if $j > 1$, and $l = 0$ if $n > 1$.

Note that a digon with the odd angle cannot be inserted between the two triangles in Fig. 12 since both sides of the digon should be arcs of $\mathcal{Q}$ of length 1. One could consider also the union of triangles M_1 and T_n, (or T_j and $\bar{M}_1$) with a digon $D_{2\kappa+1}$ inserted between them (see Fig. 14b and Fig. 14c), but the resulting quadrilateral would be equivalent to one of the quadrilaterals in Fig. 12 with a digon $D_{2\kappa}$ inserted between two triangles.

Theorem 6.1 *For any non-negative integers A_0, A_1, A_2 and a positive integer A_3 there exists a spherical quadrilateral with $[a_0] = A_0$, $[a_1] = A_1$, $[a_2] = A_2$, and $a_3 = A_3$.*

Proof. Consider first the case $A_3 = 1$. If $A_1 \geq A_0 + A_2$ then a quadrilateral $\bar{X}^0_{\mu\nu}$ with $\mu+\nu = 2(A_1-A_0-A_2)$ and digons D_i and D_k attached to its sides L_1 and L_2 (see Fig. 10b) where $i = A_0$ and $k = A_2$ has the required integer parts of its angles. If $A_0+A_1+A_2$ is even and (A_0-1, A_1, A_2-1) satisfy the triangle inequality then the union of ∇ and E_0 with a digon D_j inserted between them and digons D_i and D_k attached to its sides L_1 and L_2 (see Fig. 12a) where $j = (A_0 + A_2 - A_1)/2 - 1$, $i = (A_0 + A_1 - A_2)/2$, $k = (A_1 + A_2 - A_0)/2$ has the required integer parts of its angles. If $A_0 + A_1 + A_2$ is odd and $(A_0 - 1, A_1 - 1, A_2 - 1)$ satisfy the triangle inequality then the union of ∇ and E_1 with a digon D_j inserted between them and digons D_i and D_k attached to its sides L_1 and L_2 (see Fig. 12a) where $j = (A_0+A_2-A_1-1)/2$,

$i=(A_0+A_1-A_2-1)/2$, $k=(A_1+A_2-A_0-1)/2$ has the required integer parts of its angles. If either $A_0 \geq A_1+A_2+1$ or $A_2 \geq A_0+A_1+1$ then the union of ∇ and either M_n or $\bar{M}_n$ with D_j inserted between them and either D_i attached to L_1 or D_k attached L_2 (see Fig. 12a and Fig. 12b) has the required integer parts of its angles.

Let now $A_3 \geq 2$. If $A_1+A_3 \leq A_0+A_2$ then, attaching digons D_l and D_m to the sides L_3 and L_4 of the union of ∇ and either E_n or M_n or $\bar{M}_n$, for some l and m such that $l+m=A_3-1$, we can reduce the problem to the case when $A_3=1$ and $A_1<A_0+A_2$.

Suppose now that $A_1+A_3=A_0+A_2+1$. Attaching digons D_i, D_k, D_l, D_m to the sides of the quadrilateral $X_{0,0}$ (see Fig. 10a) we can get a quadrilateral with the required integer parts of the angles. Moreover, reducing (resp., increasing) by 1 the indices i and l, and increasing (resp., reducing) by 1 the indices k and m, we obtain a quadrilateral with the same angles. Thus we may always assume that either $\min(i,l)=0$ or $\min(k,m)=0$. If, for example, $i=0$ or $k=0$, we may replace $X_{0,0}$ by either $X_{\mu,0}$ or $X_{0,\nu}$ and increase A_3. Similarly, if $m=0$ or $l=0$, we may replace $X_{0,0}$ by either $\bar{X}_{\mu,0}$ or $\bar{X}_{0,\nu}$ and increase A_1. It is easy to check that this way we can obtain all remaining values of the integer parts of the angles.

Remark 6.2 *The primitive quadrilateral $\tilde{Q}$ contained in a reducible quadrilateral Q is not unique. The quadrilateral Q shown in Fig. 13c can be represented as one of the following:*
X^0_{11} *with $i=k=1$ and $l=m=0$ (see Fig. 10a);*
$\bar{X}^0_{22}$ *with $i=k=0$ and $l=m=1$ (see Fig. 10b);*
U^0_{11} *with $i=l=1$ and $k=m=0$ (see Fig. 10e);*
$\bar{U}^0_{11}$ *with $i=k=0$ and $l=m=1$ (see Fig. 10f).*
The quadrilateral Q in Fig. 14a is the union of a triangle ∇ and a triangle E_1. The same quadrilateral can be represented as a quadrilateral R_{10} with $i=l=m=0$ and $k=1$, or as a quadrilateral $\bar{R}_{01}$ with $i=1$ and $k=l=m=0$ (see Fig. 10c,d).

Lemma 6.3 *The following reducible quadrilaterals are equivalent:*
1) $X^\kappa_{\mu,1}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10a, $k>0$, and $U^\kappa_{\mu,1}$ with digons D_i, D_{k-1}, D_{l+1}, D_m attached as in Fig. 10e.
2) $X^\kappa_{1,\nu}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10a, $i>0$, and $\bar{U}^\kappa_{1,\nu}$ with digons D_{i-1}, D_k, D_l, D_{m+1} attached as in Fig. 10f.
3) $\bar{X}^\kappa_{2\mu,2}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10b, $l>0$, and $\bar{U}^\kappa_{\mu,1}$

with digons D_i, D_{k+1}, D_{l-1}, D_m attached as in Fig. 10f.
4) $\bar{X}^\kappa_{2,2\nu}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10b, $m > 0$, and $U^\kappa_{1,\nu}$ with digons D_{i+1}, D_k, D_l, D_{m-1} attached as in Fig. 10e.
5) $\bar{X}^\kappa_{2\mu-1,3}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10b, $l > 0$, and $\bar{V}^\kappa_{\mu,1}$ with digons D_i, D_{k+1}, D_{l-1}, D_m attached as in Fig. 10h.
6) $\bar{X}^\kappa_{3,2\nu-1}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10b, $m > 0$, and $V^\kappa_{1,\nu}$ with digons D_{i+1}, D_k, D_l, D_{m-1} attached as in Fig. 10g.
7) $Z^\kappa_{\mu,1}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10i, $k > 0$, and $V^\kappa_{\mu,1}$ with digons D_i, D_{k+1}, D_{l-1}, D_m attached as in Fig. 10g.
8) $Z^\kappa_{1,\nu}$ with digons D_i, D_k, D_l, D_m attached as in Fig. 10i, $i > 0$, and $\bar{V}^\kappa_{1,\nu}$ with digons D_{i-1}, D_k, D_l, D_{m+1} attached as in Fig. 10h.
9) The union of ∇ and E_1 with a digon D_j, $j > 0$, inserted between them and digons D_i, D_k, D_l, D_m attached as in Fig. 11a, and either the union of ∇ and M_1 with a digon D_{j-1} inserted between them and digons D_i, D_{k+1}, D_l, D_m attached as in Fig. 11b, or the union of ∇ and $\bar{M}_1$ with a digon D_{j-1} inserted between them and digons D_{i+1}, D_k, D_l, D_m attached as in Fig. 11c.
10) The union of ∇ and E_1 digons D_i, D_k, D_l, D_m attached as in Fig. 11a, $j = 0$, and either $R_{1,0}$ with digons D_i, D_{k+1}, D_l, D_m attached as in Fig. 10c, or $\bar{R}_{0,1}$ with digons D_{i+1}, D_k, D_l, D_m attached as in Fig. 10d.

This can be shown by comparing the nets of the quadrilaterals in each of the ten cases.

7 Chains of quadrilaterals

We use the results on deformation of quadrilaterals from [4].

Let Γ be a family of curves in some region $D \subset \mathbf{C}$. Let $\lambda \geq 0$ be a measurable function in D. We define the λ-length of a curve γ by

$$\ell_\lambda(\gamma) = \int_\gamma \lambda(z)|dz|,$$

if the integral exists, and $\ell_\lambda(\gamma) = +\infty$ otherwise. Then we set

$$L_\lambda(\Gamma) = \inf_{\gamma\in\Gamma} \ell_\lambda(\gamma),$$

and

$$A_\lambda(D) = \int_D \lambda^2(z)dm,$$

where dm is the Euclidean area element. Then the extremal length of Γ is defined as

$$L(\Gamma) = \inf_{\lambda} \frac{L_{\lambda}^{2}(\Gamma)}{A_{\lambda}(D)}.$$

The extremal length is a conformal invariant. Extremal distance between two closed sets is defined as the extremal length of the family of all curves in D that connect these two sets. For a flat rectangle with vertices a_0, a_1, a_2, a_3 conformally equivalent to the quadrilateral Q, the extremal distance between its sides $[a_1, a_2]$ and $[a_3, a_0]$ is equal to $|[a_0, a_1]|/|[a_1, a_2]|$ [1].

In addition to the extremal distance, we consider the ordinary intrinsic distances between the pairs of opposite sides. They are defined as the infima of spherical lengths of curves contained in our quadrilateral and connecting the two sides of a pair.

Lemma 7.1 *([4, Lemma 15.1] Consider a sequence of spherical quadrilaterals whose developing maps f are at most p-valent with a fixed integer p. If the intrinsic distance between the sides $[a_1, a_2]$ and $[a_3, a_0]$ is bounded from below, while the intrinsic distance between the sides $[a_0, a_1]$ and $[a_2, a_3]$ tends to zero, then the extremal distance between the sides $[a_1, a_2]$ and $[a_3, a_0]$ tends to $+\infty$.*

Let Q be a quadrilateral with non-integer corners a_0, a_1 and a_2, and with integer corner a_3 which is mapped to a point $X \in C$ that is not a vertex of $\mathcal{P}$. Let $\mathcal{Q}$ be the net of Q. When X approaches a vertex X_0 of $\mathcal{P}$, and the combinatorial class of Q is fixed, we have the following possibilities.

(a) Q degenerates so that the intrinsic distance between its opposite sides L_1 and L_3 tends to zero, while the intrinsic distance between its sides L_2 and L_4 does not tend to zero. This happens when $\mathcal{Q}$ has an arc of order 1 connecting a_3 with a point on L_1 mapped to X_0, but does not have an arc of order 1 connecting a_3 to a point on L_2 mapped to X_0.

(b) Q degenerates so that the intrinsic distance between it opposite sides L_2 and L_4 tends to zero, while the intrinsic distance between its sides L_1 and L_3 does not tend to zero. This happens when $\mathcal{Q}$ has an arc of order 1 connecting a_3 with a point on L_2 mapped to X_0, but does not have an arc of order 1 connecting a_3 to a point on L_1 mapped to X_0.

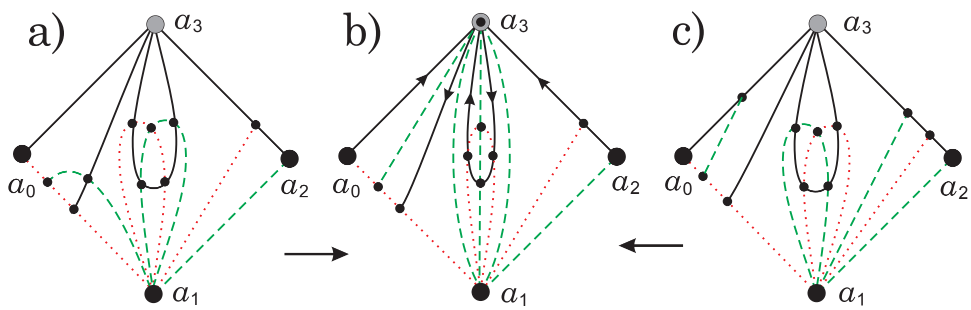

Figure 15: An aa-chain of quadrilaterals of length 1.

(c) Q does not degenerate, but converges to a quadrilateral Q' with the corner a_3 mapped to the vertex X_0 of $\mathcal{P}$. This happens when $\mathcal{Q}$ does not have an arc of order 1 connecting a_3 with a point on one of the sides L_1 and L_2 mapped to X_0.

In the case (c), the quadrilateral Q' must be one of the quadrilaterals shown either in Fig. 11 or in Fig. 12, since a quadrilateral Q containing an irreducible quadrilateral $\tilde{Q}$ cannot have its integer corner a_3 mapped to a vertex of $\mathcal{P}$. If Q is one of the quadrilaterals in Fig. 11 then Q' is equivalent to Q. In this case, the combinatorial type of Q does not change when X passes trough X_0. If Q' is one of the quadrilaterals in Fig. 12 then it is not equivalent to Q, and we say that Q and Q' are *adjacent*. In this case, Q' has two adjacent quadrilaterals with different combinatorial types.

Example 7.2 *The quadrilateral Q in Fig. 15a is $V^1_{1,1}$. As X (the point on C where a_3 maps) approaches $X_0 \in C \cap C''$, the quadrilateral Q converges to the quadrilateral Q' in Fig. 15b which is the union of triangles $\bar{M}_1$ and T_1 with a digon D_2 inserted between them (see Fig. 12d). After X passes X_0, the quadrilateral becomes $U^1_{1,1}$.*

Lemma 7.3 *The following quadrilaterals are adjacent to the quadrilaterals in Fig. 12:*
a) Let Q' be the union of triangles T_j and E_n with a digon $D_{2\kappa}$ inserted between them and digons D_i, D_k, D_l, D_m attached as shown in Fig. 12a. Then Q' is adjacent to the quadrilaterals $\bar{X}^\kappa_{2j,2n}$ and $\bar{X}^\kappa_{2j-1,2n+1}$ with digons D_i, D_k, D_l, D_m attached as shown in Fig. 10b.
b) Let Q' be the union of triangles T_j and M_n with a digon $D_{2\kappa}$ inserted between them and digons D_i, D_k, D_l, D_m attached as shown in Fig. 12b.

Then Q' is adjacent to the quadrilaterals $\bar{U}^\kappa_{jn}$ and $\bar{V}^\kappa_{jn}$ with digons D_i, D_k, D_l, D_m attached as shown in Fig. 10f,h.
c) Let Q' be the union of triangles E_j and T_n with a digon $D_{2\kappa}$ inserted between them and digons D_i, D_k, D_l, D_m attached as shown in Fig. 12c. Then Q' is adjacent to the quadrilaterals $\bar{X}^\kappa_{2j+1,2n-1}$ and $\bar{X}^\kappa_{2j,2n}$ with digons D_i, D_k, D_l, D_m attached as shown in Fig. 10b.
d) Let Q' be the union of triangles $\bar{M}_j$ and T_n with a digon $D_{2\kappa}$ inserted between them and digons D_i, D_k, D_l, D_m attached as shown in Fig. 12d. Then Q' is adjacent to the quadrilaterals U^κ_{jn} and V^κ_{jn} with digons D_i, D_k, D_l, D_m attached as shown in Fig. 10e,g.

This can be shown by comparing the chains of the quadrilaterals in each of the cases a-d.

Definition 7.4 For $k > 0$, a sequence $Q_0, Q'_1, Q_1, \dots, Q'_k, Q_k$ of quadrilaterals with distinct combinatorial types, where any two consecutive quadrilaterals are adjacent, and each of the terminal quadrilaterals Q_0 and Q_k has only one adjacent quadrilateral, is called a *chain* of the *length k*. A quadrilateral Q_0 having no adjacent quadrilaterals is called a chain of the length 0.

If both cases (a) and (b) are possible for degeneration of Q_0 and Q_k then the chain is called an *ab-chain*. If only the case (a) is possible, the chain is an *aa-chain*. If only the case (b) is possible, the chain is a *bb-chain*.

Example 7.5 A chain of quadrilaterals of length 3 is shown in Fig. 16. The quadrilateral Q_0 in Fig. 16a can be represented either as V^1_{11} with a digon D_1 attached to its side L_1 or as $\bar{X}^1_{3,1}$ with a digon D_1 attached to its side L_4. The quadrilateral Q_1 in Fig. 16c can be represented either as U^1_{11} with a digon D_1 attached to its side L_1 or as $\bar{X}^1_{2,2}$ with a digon D_1 attached to its side L_4. The quadrilateral Q_2 in Fig. 16e is $\bar{X}_{13}$ with a digon D_1 attached to its side L_4. The quadrilateral Q_3 in Fig. 16g is $\bar{X}_{04}$ with a digon D_1 attached to its side L_4. The quadrilateral Q'_1 in Fig. 16b is a union of triangles $\bar{M}_1$ and T_0 with a digon D_2 inserted between them and a digon D_1 attached to its side L_1. It can be also represented as a union of triangles E_1 and T_0 with a digon D_2 inserted between them and a digon D_1 attached to its side L_4. The quadrilateral Q'_2 in Fig. 16d is a union of triangles $\bar{T}_0$ and E_1 with a digon D_2 inserted between them and a digon D_1 attached to its side L_4. The quadrilateral Q'_3 in Fig. 16f is a union of triangles E_0 and T_1 with a digon D_2 inserted between them and a digon D_1 attached to its side L_4.

If the point X to which the corner a_3 of Q_0 maps approaches $C \cap C'$, the intrinsic distance between the sides L_1 and L_3 (but not of L_2 and L_4) tends to zero (case a). If X approaches $C \cap C''$, the quadrilateral Q_0 converges to Q_1' (case c). If the point X to which the corner a_3 of Q_1 maps approaches $C \cap C'$ (resp., $C \cap C''$), the quadrilateral Q_1 converges to Q_2' (resp., Q_1'). If the point X to which the corner a_3 of Q_3 maps approaches $C \cap C'$ (resp., $C \cap C''$), the quadrilateral Q_2 converges to Q_2' (resp., Q_3'). If the point X to which the corner a_3 of Q_3 maps approaches $C \cap C''$, the quadrilateral Q_3 converges to Q_3'. If X approaches $C \cap C'$, the intrinsic distance between the sides L_1 and L_3 (but not of L_2 and L_4) tends to zero (case a). Thus the chain in Fig. 16 is an aa-chain.

Lemma 7.6 *A chain of quadrilaterals with one integer angle is an aa-chain (resp., a bb-chain) if and only if it contains a quadrilateral $U^\kappa_{\mu\nu}$ (resp., $\bar{U}^\kappa_{\mu\nu}$) with digons D_i, D_k, D_l, D_m attached as in Fig. 10e,f, with $\mu \geq 1$, $\nu \geq 1$, and $\min(i,l) = 0$ (resp., $\min(k,m) = 0$). Each aa-chain (resp., bb-chain) contains only one quadrilateral satisfying these conditions.*

Proof. Due to reflection symmetry, it is enough to consider only aa-chains containing a quadrilateral Q with the net $U^\kappa_{\mu\nu}$ and digons D_i, D_k, D_l, D_m attached, as in Fig. 10e, satisfying conditions of Lemma 7.6.

If $i = 0$ then the net $\mathcal{Q}$ of $Q = Q_1$ contains an arc of length 1 connecting its corner a_3 with a point on its side L_1. On the other hand, Q is adjacent to a quadrilateral Q_1' which is the union of triangles $\bar{M}_\mu$ and T_ν with a digon $D_{2\kappa}$ inserted between them and digons D_k, D_l, D_m attached (since $i = 0$, digon D_i is empty) as shown in Fig. 12d (see Lemma 7.3). A quadrilateral Q_0 with the net $V^\kappa_{\mu\nu}$ and digons D_k, D_l, D_m attached, as in Fig. 10g is adjacent to Q_1', and its net contains an arc of length 1 connecting its corner a_3 with a point on its side L_1. Thus the chain of length 1 consisting of Q_0, Q_1' and $Q_1 = Q$ is an aa-chain. An example of such a chain is shown in Fig. 15. Note that if $l > 0$ then $\nu = 1$, and the quadrilateral Q is equivalent to a quadrilateral with the net $X^\kappa_{\mu,1}$ and digons D_{k+1}, D_{l-1}, D_m attached to its sides L_2, L_3, L_4.

Suppose now that $i > 0$ but $l = 0$. This is only possible if $\mu = 1$. An example of such a net is shown in Fig. 16c, and it is shown in Example 7.5 that it belongs to an aa-chain. The same arguments as in Example 7.5 show that any quadrilateral with the net $U^\kappa_{1,\nu}$ and digons D_i, D_k, D_m attached

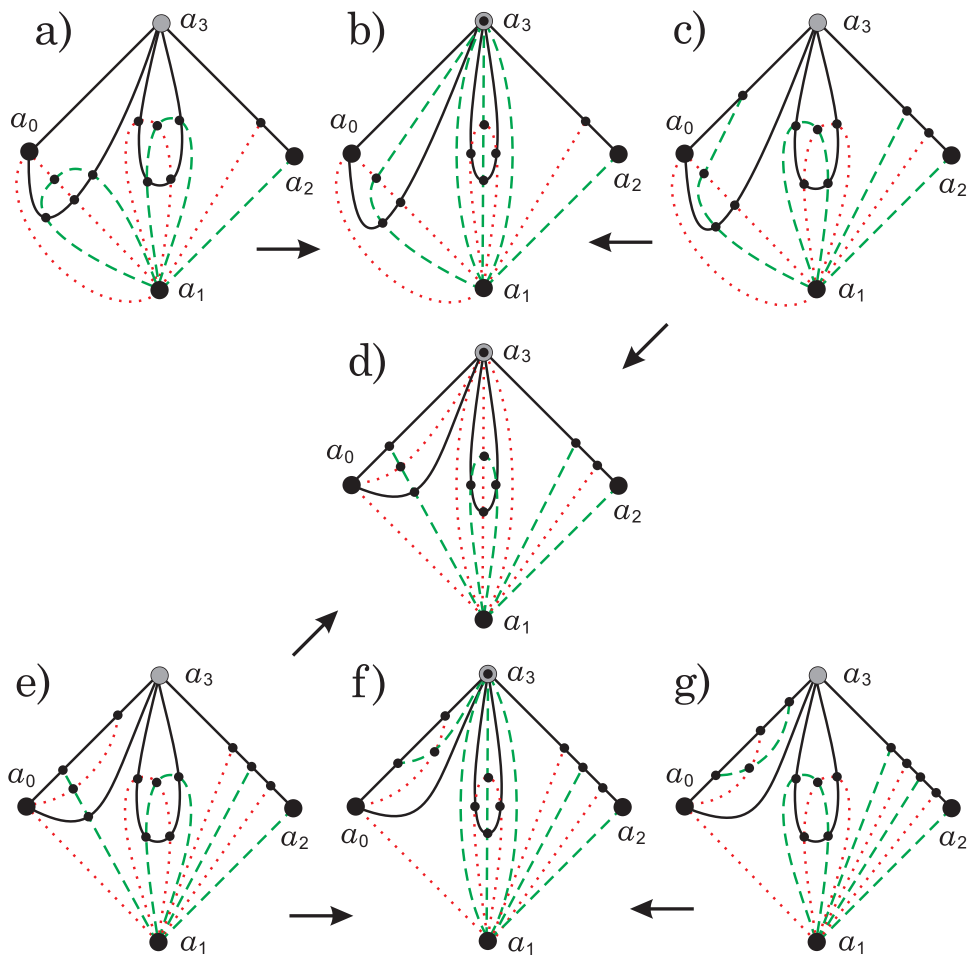

Figure 16: An aa-chain of quadrilaterals of length 3.

to its sides L_1, L_2, L_4, where $i > 0$, which is equivalent to a quadrilateral with the net $\bar{X}^{\kappa}_{2,2\nu}$ and digons D_{i-1}, D_k, D_{m+1} attached to its sides L_1, L_2, L_4, belongs to an aa-chain of length 1, containing also a quadrilateral with the net $V^{\kappa}_{1,\nu}$ with the same digons as those attached to the sides of $U^{\kappa}_{1,\nu}$, and quadrilaterals $\bar{X}^{\kappa}_{1,2\nu+1}$ $\bar{X}^{\kappa}_{0,2\nu+2}$ with the same digons as those attached to the sides of $\bar{X}^{\kappa}_{2,2\nu}$.

It remains to show that any chain that either does not contain a quadrilateral $U^{\kappa}_{\mu\nu}$ with digons D_i, D_k, D_l, D_m attached to its sides, or contains such a quadrilateral with $i > 0$ and $l > 0$, is not an aa-chain. The corresponding statement for $\bar{U}^{\kappa}_{\mu\nu}$ would follow by reflection symmetry.

We start with the chains containing $U^{\kappa}_{\mu\nu}$. Conditions $i > 0$ and $l > 0$ are satisfied only when $\mu = \nu = 1$. Such a quadrilateral (for $\kappa = 0$, $i = l = 1$ and $k = m = 0$) is shown in Fig. 13c. It is an easy exercise to show that it belongs to an ab-chain of length 2 containing also $V^{0}_{1,1}$ with two digons D_1 attached to L_1 and L_3 (equivalent to $\bar{X}^{0}_{3,1}$ with two digons D_1 attached to L_3 and L_4, see Fig. 13a) and $\bar{V}^{0}_{1,1}$ with two digons D_1 attached to L_2 and L_4 (equivalent to $\bar{X}^{0}_{1,3}$ with two digons D_1 attached to L_3 and L_4, see Fig. 13e).

The same arguments show that a quadrilateral $Q = Q_1$ with the net $U^{\kappa}_{1,1}$ and digons D_i, D_k, D_l, D_m attached to its sides (equivalent to a quadrilateral $X^{\kappa}_{2,2}$ with digons D_i, D_{k+1}, D_{l-1}, D_m attached, to a quadrilateral $\bar{X}^{\kappa}_{2,2}$ with digons D_{i-1}, D_k, D_l, D_{m+1} attached, and to a quadrilateral $\bar{U}^{\kappa}_{1,1}$ with digons D_{i-1}, D_{k+1}, D_{l-1}, D_{m+1} attached) belongs to an ab-chain of length 2 containing also a quadrilateral Q_0 with the net $V^{\kappa}_{1,1}$ and the same digons as those attached to the sides of $U^{\kappa}_{1,1}$ (equivalent to $\bar{X}^{\kappa}_{3,1}$ with the same digons as those attached to the sides of $\bar{X}^{\kappa}_{2,2}$), and a quadrilateral Q_2 with the net $\bar{V}^{\kappa}_{1,1}$ and the same digons as those attached to the sides of $\bar{U}^{\kappa}_{1,1}$ (equivalent to $\bar{X}^{\kappa}_{1,3}$ with the same digons as those attached to the sides of $\bar{X}^{\kappa}_{2,2}$).

By reflection symmetry, all chains containing $\bar{U}^{\kappa}_{\mu\nu}$ are either bb-chains or ab-chains. One can easily check that any chain containing a net $V^{\kappa}_{\mu\nu}$ (resp., $\bar{V}^{\kappa}_{\mu\nu}$) with digons D_i, D_k, D_l, D_m attached to its sides as in Fig. 10g,h contains also a net $U^{\kappa}_{\mu\nu}$ (resp., $\bar{U}^{\kappa}_{\mu\nu}$) with the same digons attached to its sides.

Now we are going to show that all chains that do not contain either $U^{\kappa}_{\mu\nu}$ or $\bar{U}^{\kappa}_{\mu\nu}$ are ab-chains.

Any net $X^{\kappa}_{\mu\nu}$ with digons D_l and D_m attached to its sides L_3 and L_4 has arcs of length 1 connecting a_3 with points on both L_1 and L_2, thus it is an ab-chain of length 0. If, in addition, a non-empty digon D_k is attached to its side L_2 then $\nu \le 1$. When $\mu > 0$, this net is equivalent to $U^{\kappa}_{\mu,1}$ with digons D_{k-1}, D_{l+1}, D_m attached. When $\mu = 0$, it is equivalent to $\bar{X}^{\kappa}_{0,2}$ with digons

D_{k-1}, D_{l+1}, D_m attached, and belongs to an ab-chain of length 2 containing also the nets $\bar{X}^\kappa_{1,1}$ and $\bar{X}^\kappa_{2,0}$ with the same digons as those attached to the sides of $\bar{X}^\kappa_{0,2}$. Similarly, the net $X^\kappa_{0,1}$ with digons D_i, D_k, D_l, D_m attached, $k > 0$, is equivalent to $\bar{X}^\kappa_{0,2}$ with digons D_i, D_{k-1}, D_{l+1}, D_m attached, and belongs to an ab-chain of length 2 containing also the nets $\bar{X}^\kappa_{1,1}$ and $\bar{X}^\kappa_{2,0}$ with the same digons as those attached to the sides of $\bar{X}^\kappa_{0,2}$.

All nets $\bar{X}^\kappa_{\mu\nu}$ with digons D_i, D_k, D_l, D_m attached to their sides as in Fig. 10b have been already considered, except those with $l = m = 0$. For given κ, $\mu+\nu$, i and k, such nets belong to a single ab-chain of length $\mu+\nu$.

Each net $R_{\mu,\nu}$ (resp., $\bar{R}_{\mu,\nu}$) with digons D_i, D_k, D_l, D_m attached to its sides as in Fig. 10c,d has arcs of length 1 connecting a_3 with points on both L_1 and L_2, thus it is an ab-chain of length 0. The same is true for each net $Z^\kappa_{\mu\nu}$ with digons D_i, D_k, D_l, D_m attached to its sides as in Fig. 10i.

Each net in Fig. 11 is an ab-chain of length 0. Although its net may not have arcs of length 1 connecting a_3 with points on both L_1 and L_2, the net does not change combinatorial equivalence class when the image X of its corner a_3 passes through an intersection point of C with either C' or C'', unless the arc of ∇ connecting a_3 with either a_0 or a_1 contracts to a point.

This completes the proof of Lemma 7.6.

Lemma 7.7 *For the given angles $\alpha_0,\dots,\alpha_3$ at the corners $a_0,\dots,a_3$ of a quadrilateral Q with one integer corner a_3, the number of nets $U^\kappa_{\mu\nu}$ with digons D_i, D_k, D_l, D_m attached to its sides as in Fig. 10e, so that* $\min(i,l) = 0$, *equals*

$$\left[\min\left(\frac{\alpha_3}{2}, \frac{1+[\alpha_1]}{2}, \frac{\delta}{2}\right)\right] \tag{7.1}$$

where $[\alpha_0] = i+m$, $[\alpha_1] = i+k+2\kappa+\nu$, $[\alpha_2] = k+l$, $\alpha_3 = l+m+2\kappa+\mu+1$ are integer parts of the angles of Q, and $\delta = \max(0, 1+[\alpha_1]+\alpha_3-[\alpha_0]-[\alpha_2])/2$.

Proof. One can establish one-to-one correspondence between the nets in Lemma 7.7 and the corresponding nets in Theorem 17.2 of [4] for a quadrilateral with two opposite integer corners, except the angle at the corner a_2 of the quadrilateral in Theorem 17.2 of [4] would be $1+[\alpha_1]$.

Theorem 7.8 *For given angles $\alpha_0,\dots,\alpha_3$, where α_3 is integer and $\alpha_0,\dots,\alpha_2$ are not integer, and for a fixed modulus, there are at least*

$$a_3 - 2\left[\min\left(\frac{\alpha_3}{2}, \frac{1+[\alpha_1]}{2}, \frac{\delta}{2}\right)\right] \tag{7.2}$$

quadrilaterals, where $\delta = \max(0, 1 + [\alpha_1] + \alpha_3 - [\alpha_0] - [\alpha_2])/2$.

Proof. It follows from Proposition 2.4 that the total number of chains of quadrilaterals with the angles as in Theorem 7.8 is a_3. It follows from Lemmas 7.6 and 7.7 that the number of aa-chains (and the number of bb-chains which is equal to the number of aa-chains by reflection symmetry) is given by (7.1). Thus the number of ab-chains is given by (7.2). For each ab-chain there is a quadrilateral with such chain with any modulus, and these quadrilaterals are distinct for distinct chains.

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