ROTATION SETS AND ALMOST PERIODIC SEQUENCES

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Abstract. We study the rotational behaviour on minimal sets of torus homeomorphisms and show that the associated rotation sets can be any type of line segments as well as non-convex and even plane-separating continua. This shows that restrictions which hold for rotation set on the whole torus are not valid on minimal sets.

The proof uses a construction of rotational horseshoes by Kwapisz to transfer the problem to a symbolic level, where the desired rotational behaviour is implemented by means of suitable irregular Toeplitz sequences.

1. Introduction.

Given a torus homeomorphisms $f : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity, a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ and any set $M \subseteq \mathbb{T}^2$, the *rotation set of F on M* is defined as

$$(1.1)\qquad \rho_M(F) \;=\; \left\{\rho \in \mathbb{R}^2 \;\middle|\; \exists n_i \nearrow \infty,\ z_i \in \pi^{-1}(M) : \lim_{i\to\infty} \left(F^{n_i}(z_i) - z_i\right)/n \;=\; \rho\right\} ,$$

where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denotes the canonical projection. In case $M = \mathbb{T}^2$, the set $\rho(F) = \rho_{\mathbb{T}^2}(F)$ is simply called the *rotation set of F*. It takes a central place in the classification of torus homeomorphisms, since a wealth of dynamical information can be obtained from the shape of $\rho(F)$ (see, for example, [1]–[6] and references therein). A crucial fact in this context is that $\rho(F)$ is always compact and convex [7]. Concerning the rotational behaviour on minimal subsets, it is known that if $\rho(F)$ has non-empty interior, then for every vector $\rho \in \text{int}(\rho(F))$ there exists a minimal set $M_\rho \subseteq \mathbb{T}^2$ with $\rho_{M_\rho}(F) = \{\rho\}$ [8]. Further, if M is minimal, then $\rho_M(F)$ is always compact and connected [9], and examples in [9] show that it can be a line segment of the form $\{0\} \times [a, b]$ with $a < b$.

The aim of this note is to explore more complex rotational behaviour on minimal sets. The bottomline is that apparently no restrictions exist for the associated rotation sets, besides compactness and connectedness. We demonstrate this by means of three types of examples, which are actually all realised by the same torus homeomorphism. Denote by $\text{Homeo}_0(\mathbb{T}^2)$ the set of homeomorphisms of $\mathbb{T}^2$ homotopic to the identity.

Theorem 1.1. *There exists $f \in \text{Homeo}_0(\mathbb{T}^2)$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

(a) *for an open set $V \subseteq \mathbb{R}^2$ and all $v \in V$ there is a minimal set M_v such that $\rho_{M_v}(F)$ is a line segment of positive length contained in $v + \mathbb{R}v^\perp$;*
(b) *for some minimal set M, the associated rotation set $\rho_M(F)$ is plane-separating;*
(c) *for some minimal set M', the associated rotation set $\rho_{M'}(F)$ has non-empty interior.*

The proof of Theorem 1.1 can roughly be outlined as follows. The homeomorphism f is chosen such that it has a *rotational horseshoe* with three symbols and the topology depicted in Figure 1.1. This construction essentially goes back to [10], where it is implemented in much greater generality to show that every rational polygon can occur as the rotation set of a torus homeomorphism. For our purposes, the important fact is that in this situation we obtain an invariant set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\pi(D))$, where $D \subseteq \mathbb{R}^2$ is a topological disk that projects injectively to $\mathbb{T}^2$, and a symbolic coding $h : \Lambda \to \{0, 1, 2\}^{\mathbb{Z}}$ such that $h \circ f = \sigma \circ h$. Moreover, given $z \in \Lambda$, the entry $h(z)_0$ determines whether a lift $\hat{z} \in D$ of $z \in \pi(D)$ remains

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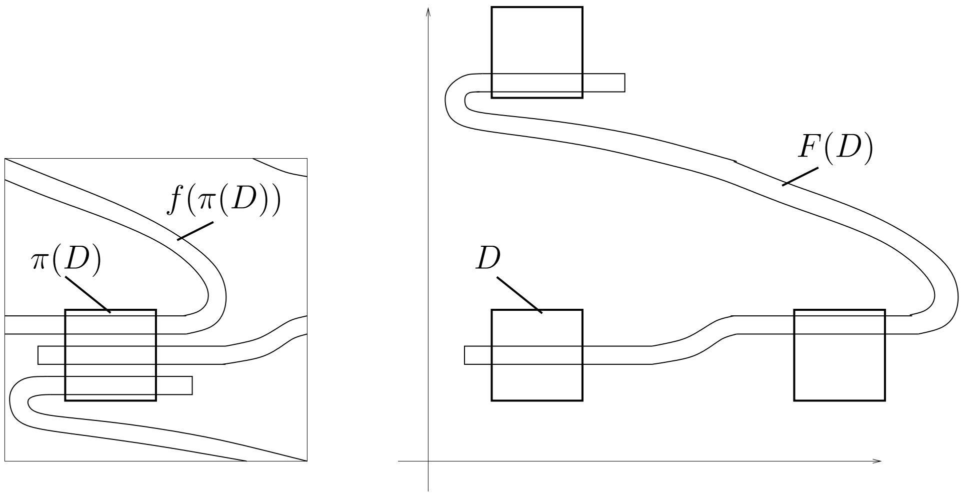

FIGURE 1.1. Geometry of a rotational horseshoe: The horseshoe is located in the topological disk $\pi(D) \subseteq \mathbb{T}^2$ shown on the left. The Markov partition is given by the preimages of the connected components of $\pi(D) \cap f(\pi(D))$. The situation for the lift is depicted on the right, the displacement vectors are $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$.

in D, moves to $D+(1,0)$ or to $D+(0,1)$. Consequently, if we let $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$, then the displacement vector $F^n(z) - z$ differs from the vector $\sum_{i=0}^{n-1} v_{h(z)_i}$ only by an error term that is bounded uniformly in $n \in \mathbb{N}$ and $z \in \Lambda$. Asymptotically, this means that rotation vectors and sets are completely determined by the coding, and the rotational behaviour on minimal sets can be studied on a purely symbolic level. The crucial issue on the technical side then is to construct suitable almost periodic sequences that produce the desired rotation sets. To that end, we work within the class of irregular Toeplitz sequences, which have been used previously to produce a number of interesting examples in topological and symbolic dynamics [11, 12, 13, 14]. In certain aspects, our construction is reminiscent of these more classical ones.

It is well-known that a dynamical situation like the one in Figure 1.1 is stable under perturbations. Hence, our construction immediately yields an open set of torus homeomorphisms that satisfy the assertions of Theorem 1.1. Moreover, it is known that the existence of rotational horseshoes is $\mathcal{C}^0$-generic within an open and dense subset of $\text{Homeo}_0(\mathbb{T}^2)$, see [9]. In order to give a precise statement in our context, we denote by $\mathcal{F}$ the set of those $f \in \text{Homeo}_0(\mathbb{T}^2)$ whose rotation sets have non-empty interior. Then $\mathcal{F}$ is open in the $\mathcal{C}^0$-topology [8], and we have

Theorem 1.2. *The set of $f \in \text{Homeo}_0(\mathbb{T}^2)$ which satisfy the assertions of Theorem 1.1 form an open and dense subset of $\mathcal{F}$.*

In fact, we believe that this set is equal to $\mathcal{F}$ and, that arbitrary continua in the interior of $\rho(F)$ can be realised. This leads to the following

Conjecture 1.3. *Given $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $\text{int}(\rho(F)) \neq \emptyset$ and any continuum $C \subseteq \text{int}(\rho(F))$, there exists a minimal set M_C such that $\rho_{M_C}(F) = C$.*

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2. Rotational horseshoes and the symbolic computation of rotation sets

Rotational horseshoes. We say that $R \subset \mathbb{T}^2$ is a *(topological) rectangle* if it is homeomorphic to the unit square $[0,1]^2$. Given an invariant set $C \subset \mathbb{T}^2$ of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, we say that a family of pairwise disjoint rectangles $\mathcal{R} = \{R_0, \ldots, R_N\}$ is a *partition* of C if $C \subset \bigcup_{i=0}^N R_i$. In this case, we let $\Sigma := \{0, \ldots, N\}^{\mathbb{Z}}$ and denote by $\mathcal{S}$ the set of those sequences $\omega \in \Sigma$ for which there exists $x \in C$ with $f^i(x) \in R_{\omega(i)}$ for all $i \in \mathbb{Z}$. Then $\mathcal{S}$ is compact and invariant under the shift σ on $\mathcal{S}$. If it happens, as in the classical horseshoe construction, that for every sequence $\xi \in \mathcal{S}$ there is a unique $x \in C$ with $f^i(x) \in R_{\omega(i)}$ for all $i \in \mathbb{Z}$, then the map $h_{\mathcal{R}} : \mathcal{S} \to C$ taking ξ to the corresponding x is a conjugacy from $\sigma_{|\mathcal{S}}$ to $f_{|C}$. This happens to be the case for any zero-dimensional hyperbolic set C in $\mathbb{T}^2$ with local product structure. In fact, in this situation the partition $\mathcal{R}$ can be chosen such that it is a *Markov partition*, that is, $\mathcal{S}$ is a subshift of finite type (see [9, 15]). If $\mathcal{S} = \Sigma$, we call C a horseshoe and say it is *rotational* if in addition the following two properties are satisfied: (i) there exists a bounded topological disk $D \subseteq \mathbb{R}^2$ such that $\pi(D) \subseteq \mathbb{T}^2$ is a topological disk and $\bigcup_{i=0}^N R_i \subseteq \pi(D)$; (ii) for each $i = 0, \ldots, N$ there exists a unique vector v_i such that if $z \in D \cap \pi^{-1}(C)$ and $\pi(z) \in R_i$, then $F(z) \in D + v_i$. In other words, in a rotational horseshoe the symbolic coding determines to which copy of D a point is mapped by F. As mentioned before, this allows to compute rotation sets and rotation vectors on a purely symbolic level.

More precisely, given a finite word $w = w_1 \ldots w_m$, let $|w| = m$ be the length of w and $\psi(w) = \sum_{j=1}^m v_{w_j}$. Further, for a closed and σ-invariant set $\mathcal{M} \subset \mathcal{S}$ we define

$$\rho_{\mathcal{M}} = \left\{ \lim_{n\to\infty} \frac{\psi(w^{(n)})}{|w^{(n)}|} \;\middle|\; w^{(n)} \text{ is a subword of some } \omega^{(n)} \in \mathcal{M} \text{ and } |w^{(n)}| \geq n \right\} . \tag{2.1}$$

The following lemma provides the crucial estimate that allows to translate these symbolic to dynamical rotation sets. Given $\omega \in \Sigma$, we let $\omega_{[1,n]} = \omega(1)\omega(2)\ldots\omega(n)$.

Lemma 2.1 ([9], Proposition 2.1)**.** *There exists $r > 0$ so that for any $z \in C$ we have $F^{n+1}(z) - z \in B_r(\psi(h_{\mathcal{R}}^{-1}(z)_{[1,n]}))$.*

As a direct consequence, we obtain

Corollary 2.2. $\rho_{\mathcal{M}} = \rho_{h_{\mathcal{R}}(\mathcal{M})}(F)$.

A sequence $\omega \in \Sigma$ is *almost periodic* if any finite subword occurs infinitely often and the time between two occurrences is uniformly bounded. It is well-known that ω is almost periodic if and only if $\overline{\mathcal{O}_\sigma(\omega)}$ is minimal. Moreover, in this case $\overline{\mathcal{O}_\sigma(\omega)}$ coincides with the set of those sequences $\xi \in \Sigma$ which have exactly the same subwords as ω [16]. Together with Corollary 2.2, this yields the following statement.

Proposition 2.3. *Given an almost periodic sequence $\omega \in \Sigma$, the set $\mathcal{M} = \overline{\mathcal{O}_f(h(\omega))}$ is minimal with respect to f and we have*

$$\rho_{\mathcal{M}}(F) = \left\{ \lim_{n\to\infty} \frac{\psi(w^{(n)}|}{|w^{(n)}|} \;\middle|\; w^{(n)} \textit{ is a subword of } \omega, |w^{(n)}| \geq n \right\} . \tag{2.2}$$

For constructing suitable almost periodic sequences, it is convenient to work only in the one-sided shift space. Due to the following folklore lemma, this is sufficient.

Lemma 2.4. *Suppose $\omega^+ \in \Sigma^+$ is almost periodic and ω is any sequence in Σ whose right side coincides with ω^+. Then $\overline{\mathcal{O}_\sigma(\omega)}$ is minimal and coincides with the set of sequences that have exactly the same finite subwords as ω^+.*

A particular case of almost periodic sequences are Toeplitz sequences. A sequence $\omega^+ \in \Sigma^+$ ($\xi \in \Sigma$) is called a *Toeplitz sequence* if for every $j \in \mathbb{N}$ ($j \in \mathbb{Z}$) there exists $p \in \mathbb{N}$ so that $\omega^+_{j+np} = \omega^+_j$ for all $n \in \mathbb{N}$ ($\omega_{j+np} = \omega_j$ for all $n \in \mathbb{Z}$). In other words, every entry of a Toeplitz sequence occurs periodically. However, since the periods depend on the position, the sequence itself need not be periodic. In fact, aperiodicity is often included in the definition, and we will follow this convention here.

3. Realisation of rotation sets by Toeplitz sequences

3.1. **Preliminary notions.** We fix $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ such that f has a rotational horseshoe C with three symbols and displacement vectors $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$, as in Figure 1.1. Thus, there exists a bounded topological disk $D \subseteq \mathbb{R}^2$ and a partition $\mathcal{R} = \{R_0, R_1, R_2\}$ of C with $\bigcup_{i=0}^2 R_i \subseteq \pi(D)$ such that $F(\pi^{-1}(R_i) \cap D) \subseteq D + v_i$. As before, we denote by $h_{\mathcal{R}}$ the conjugacy between the shift σ on $\Sigma := \{0,1,2\}^{\mathbb{Z}}$ and $f_{|C}$. As we will see in Section 4, the family of such maps is open and dense in the set $\mathcal{F} \subseteq \mathrm{Homeo}_0(\mathbb{T}^2)$ of torus homeomorphisms with non-empty interior rotation sets. According to Corollary 2.2 and Proposition 2.3, our aim is to construct almost periodic sequences whose associated rotation sets are line segments of positive length, separate the plane or have non-empty interior. To that end, we first introduce a general block structure which produces Toeplitz sequences through an inductive construction.

A general block structure. Suppose $(b_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are sequences of positive integers, with d_{n+1} a multiple of d_n for all $n \in \mathbb{N}$. Let $a_1 \in \mathbb{N}$. Slighly abusing notiation, we denote by $[k,l]$ the interval of all integers i with $k \le i \le l$, similarly for open and half-open intervals. Then we recursively define

- $a_{n+1} = (b_n d_n + 1)a_n$
- $\mathcal{A}_n = [1, a_n] + d_n a_n \mathbb{N}$
- $\mathcal{B}_n = \bigcup_{i=1}^n \mathcal{A}_n$
- $\mathcal{C}_n = \mathcal{B}_n \setminus \mathcal{B}_{n-1}$

We call the maximal intervals in $\mathcal{A}_n$ *blocks of level* n. If such a block is not equal to the first block $[1, a_n]$, we call it a *repeated block*. The following facts are easy to check.

(F1) Given $k < k'$, any block of level k' starts and ends with a block of level k.
(F2) If two blocks of levels k and k' are disjoint and $k \le k'$, then the interval between the blocks has length $\ge (d_k - 1)a_k$.
(F3) The asymptotic density of $\mathcal{B}_n$ is at most $\delta_n = \sum_{j=1}^n \frac{1}{d_j}$.
(F4) If J is an interval of integers whose length is a multiple of $a_n d_n$, then $\frac{1}{|J|}|J \cap \mathcal{B}_n| \le \delta_n$. Consequently, given $M \in \mathbb{N}$ and any interval J' of length $\ge a_n d_n / M$ we have $\frac{1}{|J'|}|J' \cap \mathcal{B}_n| \le M\delta_n$. Here $|J|$ denotes the cardinality of a set $J \subseteq \mathbb{N}$.
(F5) If a sequence $\omega = (a_i)_{i\in\mathbb{N}}$ is chosen so that for all $n \in \mathbb{N}$, $j \in [1, a_n]$ and $k \in \mathbb{N}$ it satisfies $a_{j+ka_nd_n} = a_j$, then ω is Toeplitz.

We let $\delta_\infty = \lim_n \delta_n = \sup_n \delta_n$.

3.2. **Line segments.** We first need to specify the open set $V \subseteq \mathbb{R}^2$ in Theorem 1.1. In principle, we could take the whole interior of the simplex Δ spanned by the vectors v_0, v_1 and v_2 defined above. However, for the sake of convenience we let $\alpha = \alpha(v) := \left\langle v_1, \frac{v}{\|v\|} \right\rangle$, $\beta = \beta(v) := \left\langle v_2, \frac{v}{\|v\|} \right\rangle$ and define V as the subset of vectors in Δ for which $\|v\| \le \min\{\alpha, \beta\}$, which will simplify our construction below to some extent.

Given $\omega^+ \in \Sigma^+$, we denote by $\mathcal{M}(\omega^+) = \Omega(\omega)$ the omega-limit set of a sequence $\omega \in \Sigma$ whose right side coincides with ω^+. According to Proposition 2.3, $\mathcal{M}(\omega)$ is a minimal set, and the subwords of sequences in $\mathcal{M}(\omega)$ are exactly the subwords of ω^+. Given $v \in V$, our aim is now to construct a one-sided sequence $\omega_v = (\omega_v(j))_{j\in\mathbb{N}}$ such that $\rho_{\mathcal{M}(\omega_v)}$ is a line segment of positive length contained in $v + \mathbb{R}v^\perp$. To that end, we use the above general block structure with the following specifications. We let $b_n = 1$ and $d_n = 2^{n+t}$ for some integer t such that $\delta_\infty \le \frac{\|v\|}{10\max\{\alpha,\beta\}}$. We start the construction with an integer $a_1 \ge 2M/\|v\| + 1$.

where $M = \|v\| + \max\{\alpha, \beta\}$. Further, we let $D(l,j) = \left\langle \sum_{i=l}^{j} v_{\omega_v(i)}, \frac{v}{\|v\|} \right\rangle - (j-l+1)\|v\|$. The sequence ω_v will be constructed by induction on the sets $\mathcal{C}_n$. To that end, we first define ω_v on $[0, a_n] \cap \mathcal{C}_n$ and then extend it to the whole of $\mathcal{C}_n$ by $a_n d_n$-periodic repetition. On $[0, a_n]$, we choose the entries $\omega_v(j)$ by induction on j according to the following rules.

(I) If n is odd, we let $\iota = 1$, if n is even we let $\iota = 2$.

(II) If neither of $j, j+1, \ldots, j+K$ intersects a block of level $< n$, then we choose $\omega_v(j) \in \{0, \iota\}$ such that $D(1,n)$ is contained in the interval $[0, M]$. If this is true for both possible choices 0 and ι, we let $\omega_v(j) = 0$.

(III) If $B = [m+1, m+a_k] \subseteq [0, a_n]$ is a block of level $k < n$ which is not contained in a larger block of level $< n$, then we choose $\omega_v(m-K+1), \ldots, \omega_v(m)$ such that $D(1,j) \in [-M, M]$ for all $j = m-K+1, \ldots, m$ and $D(1,m) \in [-D(1,a_k), M - D(1,a_k)]$. In order to make this choice unique, we require in addition that $D(1,j)$ always takes the smallest value which is possible under these conditions. This means we put 0 whenever possible, and ι only when necessary.

In order to see that these rules are consistent, note that if $\omega_v(j) = 0$, then $D(1,j) = D(1,j-1) - \|v\|$, if $\omega_v(j) = 1$ then $D(1,j) = D(1,j-1) + \alpha$ and if $\omega_v(j) = 2$ then $D(1,j) = D(1,j-1) + \beta$. In each step, we therefore have the choice to either increase or decrease the value of $D(1,j)$. Thus, if $D(1,j-1) \in [0, M]$, then due to the choice of M we can always choose $\omega_v(j)$ in such a way that $D(1,j) \in [0, M]$ as well. Since rule (III) ensures that $D(1,j-1) \in [0, M]$ whenever $j-1$ is the end of a block of level $< n$, it is possible to follow rule (II) whenever it applies. If $j = m-K+1$, where $m+1$ is the starting point of a block of level $< n$ and $D(1,j-1) \in [0, M]$, then choosing $\omega_v(i) = 0$ for all $i = j, \ldots, m$ would yield $D(m) \leq M - K\|v\| \leq -M$. Thus, by replacing some of the zeros with ι's, it is also possible to meet the requirements of rule (III). Note here that due to the choice of $K = a_1 - 1$ and the spacing of the blocks, the integers $j, \ldots, m$ are not contained in any repeated block of level $< n$. Altogether, this implies that the above algorithm yields a well-defined sequence ω_v. Furthermore, by construction we obtain that $|D(1,j)| \in [0, M]$ whenever j is not contained in a repeated block.

In order to ensure that $\rho_{\overline{\mathcal{O}_\sigma(\omega_v)}} \subseteq v + \mathbb{R}v^\perp$, we need to show that

$$\lim_{n \to \infty} \frac{1}{n} \max\left\{|D(i,j)| \mid |j-i| \leq n\right\} = 0 . \tag{3.1}$$

Since the a_n grow super-exponentially, this will be a direct implication of the following.

Proposition 3.1. *If $0 < j - i \leq a_n$, then $|D(i,j)| \leq 2nM + 1$.*

For the proof, we need to introduce some further notation. We say that $j \in \mathbb{N}$ has *depth d*, and write $\mathrm{depth}(j) = d$, if d is the maximal integer such that $j \in B_d$ and $B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_d$ is a nested sequence of blocks with $\min B_i < \min B_{i+1}$ and $\max B_i > \max B_{i+1}$ for all $i = 1, \ldots, d-1$. Note that the nested sequence could be given by only one block $B_1 = [1, a_n]$, but it always exists since every integer is contained in some initial block. For the same reason, B_1 will always be an initial block and B_2 is the largest repeated block that contains j. Note also that the level of the blocks is decreasing, and if $j \in [1, a_n]$ then $\mathrm{depth}(j) \leq n$. Moreover, if n is the smallest integer such that $j \in [1, a_n]$, then B_1 is equal to $[1, a_n]$.

Lemma 3.2. *We have $|D(1,j)| \leq M\,\mathrm{depth}(j)$ for all $j \in \mathbb{N}$. In particular $|D(1,j)| \leq Mn$ for all $j \in [1, a_n]$.*

Proof. We prove the lemma for all $j \in [1, a_n]$ by induction on n. The statement holds for $j \in [1, a_1]$, since on this interval we apply rule II to all j and consequently $D(1,j) \in [0, M]$. Assume that the estimate holds for all $j \in [1, a_n]$ and let $j' \in [1, a_{n+1}]$. If $\mathrm{depth}(j') = 1$, the statement holds by construction. Note here that if $\omega_v(j)$ is chosen according to rule II, then $|D(1,j)| \in [0, M]$, whereas if we apply rule III then $|D(1,j)| \in [-M, M]$.

Now, assume that *$depth(j') = d$* and the block $B_2 = [m+1, m+a_k] \subsetneq [1, a_{n+1}]$ is of level $k \leq n$. Then m is not contained in any block and thus $|D(1,m)| \leq M$ again by construction.

Further, we have $depth(j'-m) \leq \text{depth}(j')-1$, and consequently

$$|D(m+1,j')| \;=\; |D(1,j'-m)| \;\leq\; M(\text{depth}(j')-1)\ .$$

Note that here $\omega_v(m+1),\ldots,\omega_v(m+a_k)=\omega_v(1),\ldots,\omega_v(a_k)$. Together, we obtain

$$|D(1,j')| \;\leq\; |D(1,m)|+|D(m+1,j')| \;\leq\; C\,\text{depth}(j')$$

as required. □

Proof of Proposition 3.1. Fix $k>0$. We proceed again by induction on n. Assume that the statement holds for all $i,j\in[1,a_l]$, $l\leq n$, and suppose that $i,j\in[1,a_{n+1}]$ with $|D(i,j)|>2Mk+1$. We have to show that $j-i>a_k$.

If i,j are both contained in a repeated block $B=[m+1,m+a_p]$ of level $p\leq n$, then $|D(i,j)|=|D(i-m,j-m)|$ and the induction statement applies. Thus, we may assume that this does not happen. Due to Lemma 3.7 we have

$$|D(i,j)| \;\leq\; |D(1,j)|+|D(1,i-1)| \;\leq\; M(depth(j)+depth(i-1))\ ,$$

so that either j or $i-1$ has depth bigger than k. We distinguish three cases.

First, if both have depth bigger or equal to k, then as they cannot both be contained in a single repeated block, they have to be contained in disjoints blocks of level bigger than or equal to k. However, as two such blocks are at least $(d_k-1)a_k$ apart, the statement follows.

Secondly, assume that $d=depth(i-1)>k$ and $depth(j)\leq k$. Let $B_1\supsetneq\cdots\supsetneq B_d$ be a nested sequence of blocks as in the definition of $depth(i-1)$, with $B_1=[1,a_{n+1}]$ and $B_2=[m+1,m+a_p]$. Since $depth(i-1)>k$, we have $p\geq k$. In the case $i-1\notin[m+a_p-a_k+1,m+a_p]$, we have $j-i>a_k$ as required. Otherwise, we have that

$$\begin{aligned}|D(i,m+a_p)| \;&=\; |D(i-m-a_p+a_k,a_k)|\\ &=\; |D(1,a_k)|+|D(1,i-m-a_p+a_k-1)| \;\leq\; Mk+1\ ,\end{aligned}$$

using $D(1,a_k)\in[0,M]$ and Lemma 3.7. Consequently, we obtain

$$\begin{aligned}|D(i,j)| \;&\leq\; |D(i,m+a_s)+D(m+a_s+1,j)|\\ &=\; |D(i,m+a_s)-D(1,m+a_s)+D(1,j)|\\ &\leq\; Mk+1+M\,\text{depth}(j) \;\leq\; 2Mk+1\ ,\end{aligned}$$

contradicting our assumption. Finally, the case $depth(i-1)\leq k$ and $\text{depth}(j)>k$ can be treated in an analogous way. □

As mentioned above, Proposition 3.1 implies that $\rho_{\mathcal{M}(\omega_v)}\subseteq v+\mathbb{R}v^\perp$, and if we let $M_v=h_{\mathcal{R}}^{-1}(\mathcal{M}(\omega_v))$, then according to Corollary 2.2 the same will be true for the rotation set $\rho_{M_v}(F)$. It remains to show that $\rho_{\mathcal{M}(\omega_v)}$ is a segment of positive length. To that end, we note that for for all $n\in\mathbb{N}$ and $j\in[1,a_n]\cap C_n$ we have $\omega_v(j)\in\{0,\iota\}$, where $\iota=1$ if n is odd and $\iota=2$ if n is even. In the first case, (F4) implies that the fraction of 2's in the interval $[1,a_n]$ is bounded by $\delta=\frac{\|v\|}{10\max\{\alpha,\beta\}}$. At the same time, the requirement that $D(1,a_n)\in[0,M]$ implies that a proportion of $\|v\|/\max\{\alpha,\beta\}$ of symbols in $[1,a_n]$ must be non-zero. This yields that the frequency of 1's in $[1,a_n]$ is greater than 9δ. For even n, we obtain exactly the opposite estimates for the frequencies of 1's and 2's. In the limit $n\to\infty$, this yields the existence of two distinct vectors in $\rho_{\mathcal{M}(\omega_v)}$. This completes the proof of Theorem 1.1(a).

3.3. **Plane separating continua.** For the construction we make again use of the general block structure presented above, this time with the following specifications.

(i) We choose $(d_n)_{n\in\mathbb{N}}$ so that all d_n are even and $\delta_\infty\leq\frac{1}{32}$.

(ii) We choose integers $K\geq 17$ and $L\geq 64$ and let $a_1=(3L+4)K$ and $b_n=(3L+4)K$ for all $n\in\mathbb{N}$. The sequence $(a_n)_{n\in\mathbb{N}}$ is then defined inductively by $a_{n+1}=(b_nd_n+1)a_n$, according to the general block structure introduced above.

(iii) Note that due to the choice of b_n we have $a_{n+1}\geq 8a_nb_n$ for all $n\in\mathbb{N}$, which implies in particular that $\sum_{j=1}^{n-1}a_jd_j\leq a_nd_n/2$.

Then we construct $\omega = (\omega_n)_{n\in\mathbb{N}}$ inductively on the sets $\mathcal{A}_n$ as follows. Suppose ω_j is defined for all $j \in [1, a_n]$, and hence for all $j \in \mathcal{A}_n$ (recall $\omega_{j+ka_nd_n} = \omega_j$ for all $j \in [1, a_n]$ and $k \in \mathbb{N}$). We extend the definition to $[1, a_{n+1}]$, and thus to $\mathcal{A}_{n+1}$, as follows. Let

$$\begin{aligned} p_n &= (L+1)Ka_nd_n - a_nd_n + 1 + \sum_{j=1}^{n} a_jd_j/2 \ , \\ q_n &= \ (L+1)Ka_nd_n + a_nd_n - \sum_{j=1}^{n} a_jd_j/2 - 1 \ . \end{aligned} \tag{3.2}$$

Then divide $[1, a_{n+1}]$ into the following seven intervals (see Figure 3.1).

$I_1^0 = [1, (LKd_n + 1)a_n]$,
$I_1^1 = [(LKd_n + 1)a_n + 1, p_n - 1]$,
$I_1^2 = [p_n, q_n]$,
$I_2^1 = [q_n + 1, (L+2)Kd_na_n]$,
$I_2^0 = [(L+2)Kd_na_n + 1, ((2L+2)Kd_n + 1)a_n]$,
$I_2^2 = [((2L+2)Kd_n + 1)a_n + 1, (2L+4)Kd_na_n]$,
$I_3^0 = [(2L+4)Kd_na_n + 1, a_{n+1}]$.

Due to the choice of p_n and q_n, the following properties are easy to verify.

(PQ1) The intervals I_1^0, I_2^0 and I_3^0 all have the same length $(LKd_n + 1)a_n$ and start and end with a block of level n (and thus with blocks of all levels $k \le n$).

(PQ2) The length of I_1^1 and I_2^1 is between $(K-1)a_nd_n$ and Ka_nd_n. *(Note here that due to (iii) we have $\sum_{j=1}^{n-1} a_jd_j/2 \le a_nd_n/4$.)*

(PQ3) The length of I_1^2 is between $a_nd_n/2$ and a_nd_n.

(PQ4) The interval I_1^2 is concentric around a block B_n of level n.

(PQ5) The distance of p_n and q_n to any block B_k of level $k \le n$ is at least $a_kd_k/4$. *(In order to check this for p_n, note that for each $k \le n$ a block of level k starts at $(L+1)Ka_nd_n - a_nd_n + 1 + \sum_{j=k+1}^{n} a_jd_j/2$ and $\sum_{j=1}^{k} a_jd_j/2 \le 3a_kd_k/4$. A similar comment applies to q_n.)*

Let $I^0 = I_1^0 \cup I_2^0 \cup I_3^0$, $I^1 = I_1^1 \cup I_2^1$, $I^2 = I_1^2 \cup I_2^2$ and $I^* = I_1^1 \cup I_1^2 \cup I_2^1$.

(PQ6) The intervals I^* and I_2^2 both have length $(2Kd_n - 1)a_n$.

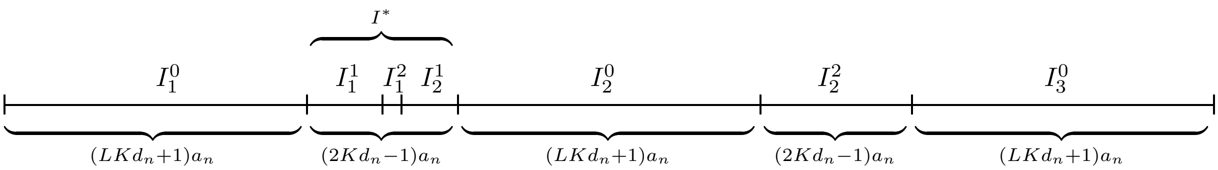

FIGURE 3.1. The configuration of intervals.

We define

$$\omega_j = \begin{cases} 0 & \text{if } j \in I_0 \setminus \mathcal{B}_n \\ 1 & \text{if } j \in I_1 \setminus \mathcal{B}_n \\ 2 & \text{if } j \in I_2 \setminus \mathcal{B}_n \end{cases} \tag{3.3}$$

for all $j \in [1, a_n] \setminus \mathcal{B}_n$ and $\omega_{j+ka_nd_n} = \omega_j$ for all $j \in [1, a_n]$ and $k \in \mathbb{N}$. By induction on $n \in \mathbb{N}$ this yields a sequence $\omega = (\omega_j)_{j\in\mathbb{N}}$, which follows our general block structure introduced above and is, in particular, Toeplitz.

Recall that $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$ are the integer vectors associated to the partition and for every interval $J \subseteq \mathbb{N}$ we write $\rho(J) = \frac{\psi(J)}{|J|}$, where $\psi(J) = \sum_{j\in J} v_{\omega_j}$.

Lemma 3.3. *$\rho([1, a_{n+1}]) \in B_{\frac{1}{8}}(v_0)$ for all $n \in \mathbb{N}_0$.*

Proof. With the above notions, we have that $\{j \in [1, a_{n+1}] : \omega_j \neq 0\} \subseteq ([1, a_{n+1}] \cap \mathcal{B}_n) \cup I^1 \cup I^2$. By (F4), we have $|[1, a_{n+1}] \cap \mathcal{B}_n| \leq \delta_\infty a_{n+1} \leq \frac{1}{16} a_{n+1}$. Further, we have that $|I^1 \cup I^2| \leq \frac{4}{3l} a_{n+1} \leq \frac{1}{16} a_{n+1}$. The statement follows. □

Lemma 3.4. *Suppose $J = [1, m]$ or $J = [m, a_{n+1}]$ for some $n \in \mathbb{N}_0$ and $m \in (1, a_{n+1})$. Then $\rho(J) \in B_{\frac{1}{8}}(v_0)$.*

Proof. We proceed by induction on n. Assume that the statement holds for $n \in \mathbb{N}_0$. Let $J = [1, m] \subset [1, a_{n+1}]$ (the proof for $J = [m, a_{n+1}]$ is similar). We distinguish several cases.

Case 1: $J \subset I_1^0$. Let

$$J \cap \mathcal{B}_n = B_1 \cup \cdots \cup B_j \cup B',$$

where $B' = B_{j+1} \cap J$ and B_i, $i = 1, \ldots, j+1$, are full blocks of level $\leq n$. By the previous lemma $\rho(B_i) \in B_{\frac{1}{8}}(v_0)$ for $i = 1, \ldots, j$. Due to the induction hypothesis $\rho(B') \in B_{\frac{1}{8}}(v_0)$. Hence $\rho(J \cap \mathcal{B}_n) \in B_{\frac{1}{8}}(v_0)$. Since by construction $\rho(J \setminus \mathcal{B}_n) = v_0$, we have $\rho(J) \in B_{\frac{1}{8}}(v_0)$.

Case 2: $J \subseteq I_1^0 \cup I^*$ but $J \not\subseteq I_1^0$. By (F4) we have that $|I_1^0 \cap \mathcal{B}_n| \leq \frac{1}{16} |I_1^0|$, and hence $\rho(I_1^0) \in B_{\frac{1}{16}}(v_0)$. Further $|J \cap I^*| \leq |I^*| \leq \frac{2}{l} |I_1^0| < \frac{1}{16} |I_1^0|$. Thus $\rho(J) \in B_{\frac{1}{8}}(v_0)$.

The remaining cases $J \subseteq I_1^0 \cup I^* \cup I_2^0$, $J \subseteq I_1^0 \cup I^* \cup I_2^0 \cup I_2^2$ and $J \subseteq [1, a_{n+1}]$ can be treated by similar arguments. □

Lemma 3.5. *Suppose that $J \subseteq I_j^i$ is an interval of length $\geq a_n d_n / 2$, where $i = 0, 1, 2$ and $j = 1, 2$ or $i = 0$ and $j = 3$. Then $\rho(J) \in B_{\frac{1}{16}}(v_i)$.*

Proof. By (F4), we obtain that $|J \cap \mathcal{B}_n| / |J| \leq \frac{1}{16}$. Since all free positions in J are filled by i's in the $(n+1)$st step of the construction, this implies the statement. □

Corollary 3.6. *For $i = 0, 1, 2, j = 1, 2$ or $i = 0, j = 3$ we have $\rho(I_j^i) \in B_{\frac{1}{16}}(v_i)$.*

Lemma 3.7. *Let $J \subseteq I \subseteq \mathbb{N}$ be intervals and assume that J has one endpoint in common with I.*

(a) If $I = I_j^0$, $j = 1, 2, 3$, then $\rho(J) \in B_{\frac{1}{8}}(v_0)$.
(b) If $I = I_j^1$, $j = 1, 2$, then $\rho(J) \in B_{\frac{1}{8}}(v_1)$.
(c) If $I = I_j^2$, $j = 1, 2$, then $\rho(J) \in B_{\frac{1}{8}}(v_2)$.
(d) If $I = I^*$, then $\rho(J) \in B_{\frac{1}{8}}(v_1)$.

Proof. *(a)* If $J \subseteq I_1^0$ starts with 1 or $J \subseteq I_3^0$ ends with a_{n+1}, then the statement is contained in Lemma 3.4. However, by construction the configurations of symbols in the intervals I_1^0, I_2^0 and I_3^0 are identical. In order to see this, note that since these have the same length and all start and end with a block of level n by (PQ1), the configuration of the blocks is identical, and all positions not contained in previous blocks are filled by 0's. Hence, by symmetry the statements for I_1^0 and I_3^0 extend to the other intervals.

(b) and (c) The proofs of all cases of (b) and (c) are similar. Hence, we consider only one of them. The crucial observation is the fact that all endpoints of these intervals have distance $\geq a_k d_k / 4$ to any block of level k. For the points p_n and q_n, this is true by construction, see (PQ5). For endpoints of I_1^1, I_2^1 and I_2^2 this holds since the adjacent intervals I_j^0, $j = 1, 2, 3$, start and end with blocks of level n. Hence, the nearest block of any level $k \leq n$ in one of the considered intervals can appear at distance $(d_k - 1) a_k$ to the boundary points.

Assume that $I = I_1^2$ and $J = [p_n, k]$. If J does not intersect any blocks of level $k \leq n$, then $\rho(J) = v_2$. Otherwise, let m be the largest integer such that $J \cap B \neq \emptyset$ for some m-block B. Then by (PQ5) the length of J is at least $a_m d_m / 4$, and due to (F4) we obtain that $|J \cap \mathcal{B}_n| / |J| \leq 4\delta_\infty \leq \frac{1}{8}$, which implies $\rho(J) \in B_{\frac{1}{8}}(v_2)$. As mentioned, the other cases are analogous.

(d) If J is either contained in I_1^1 or I_2^1, then the statement is contained in (b). Otherwise, it follows from the fact that $\rho(I_j^1) \in B_{\frac{1}{16}}(v_1)$ by Corollary 3.6 and $|I_1^2|/|I_j^1| \leq \frac{1}{K-1} \leq \frac{1}{16}$.

□

Proposition 3.8. *Let* $T = \{\lambda v_i + (1-\lambda)v_j \,:\, i,j \in \{0,1,2\}, \lambda \in [0,1]\}$ *and* $S = B_{\frac{1}{8}}(T)$. *Then* $\rho(J) \in S$ *for every* $J = [a,b] \subset \mathbb{N}$.

Proof. Let n be the smallest integer such that J is contained in a block of level $n+1$. We prove the statement by induction on n and may thus assume that J is not entirely contained in any block of level $k \leq n$. Moreover, since the structure inside all blocks of level $n+1$ is the same, we may assume without loss of generality that $J \subseteq [1, a_{n+1}]$. We distinguish several cases.

Case 1. Suppose that J intersects both I^* and I_2^2. In this case J contains I_2^0, and by Corollary 3.6 we have $\rho(I_2^0) \in B_{\frac{1}{16}}(v_0)$. Moreover, $|I^* \cup I_2^2|/|I_2^0| \leq \frac{4}{3l} \leq 1/16$. If J also intersects the intervals I_1^0 and I_3^0, say $J' = J \cap I_1^0$ and $J'' = J \cap I_3^0$, then $\rho(J'), \rho(J'') \in B_{1/8}(v_0)$ by Lemma 3.7(a). Putting everything together, we obtain $\rho(J) \subseteq B_{1/8}(v_0)$.

Case 2. Suppose that J intersects exactly two of the five intervals $I_1^0, I^*, I_2^0, I_2^2, I_3^0$. Since all the subcases are similar, we only treat one and assume J intersects I_1^0 and I^*. Let $J' = J \cap I_1^0$ and $J'' = J \cap I^*$. Then $\rho(J') \in B_{\frac{1}{8}}(v_0)$ by 3.7(a), whereas $\rho(J'') \in B_{\frac{1}{8}}(v_1)$ by Lemma 3.7(d). Consequently, $\rho(J)$ is a convex combination of a vector in $B_{\frac{1}{8}}(v_0)$ and a vector in $B_{\frac{1}{8}}(v_1)$, and therefore belongs to S.

Case 3. Suppose $J \subseteq I^*$ intersects at least two of the intervals I_1^1, I_1^2 and I_2^1. Let $J = J' \cup J'' \cup J'''$, where $J' = J \cap I_1^1$, $J'' = J \cap I_1^2$ and $J''' = J \cap I_2^1$. Then $\rho(J'), \rho(J'') \in B_{\frac{1}{8}}(v_1)$ by Lemma 3.7(b), and $\rho(J'') \in B_{\frac{1}{8}}(v_2)$ by Lemma 3.7(c). Hence, we obtain again that $\rho(J) \in S$.

Case 4. Finally, suppose that J is contained in just one of the seven intervals of the decomposition of $[1, a_{n+1}]$, say $J \subseteq I_j^i$. Then

$$J \cap \mathcal{B}_n \;=\; B' \cup B_1 \cup \ldots \cup B_m \cup B'' \ ,$$

where $B' = J \cap B_0$, $B'' = J \cap B_{m+1}$ and the B_l with $l = 0, \ldots, m+1$ are those maximal blocks contained in $\mathcal{B}_n$ which intersect J, ordered in an increasing way. Since J is not entirely contained in one block, we can use Lemmas 3.3 and 3.4 to see that $\rho(B'), \rho(B_1), \ldots, \rho(B_m), \rho(B'') \in B_{\frac{1}{8}}(v_0)$, and hence $\rho(J \cap \mathcal{B}) \in B_{\frac{1}{8}}(v_0)$. At the same time we have $\rho(J \setminus \mathcal{B}) = v_i$, such that again $\rho(J)$ is contained in S. □

Proposition 3.9. $\rho_{\mathrm{Cl}(\mathcal{O}(\omega_{sp},\sigma))}$ *separates the plane.*

Proof. We have that

$$\rho_{\mathrm{Cl}(\mathcal{O}(\omega_{\mathrm{sp}},\sigma))} = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} k}$$

where $k = \{\rho(J) \mid J \subseteq \mathbb{N}$ is a finite interval with $|J| = n\}$.

Given $0 \leq i < j \leq 2$ we let $S_{ij} = \{\lambda v_i + (1-\lambda)v_j \,:\, \lambda \in [0,1]\}$. For $n \in \mathbb{N}$, let $J_1 = I_1^2 = [p_n, q_n]$ and choose an interval $J_2 \subseteq I_2^2$ which has the same length as J_1 and is concentric around a block of level n. Since this also holds for J_1, we have that the configuration of blocks inside both intervals is the same. Since free positions in both intervals are both filled by 2's, we have that $\rho(J_1) = \rho(J_2) \in B_{\frac{1}{8}}(v_2)$.

Let $M_n \in \mathbb{N}$ be such that $J_2 = J_1 + M_n = \{j + M_n \mid j \in J_1\}$ and let $\rho_i^n = \rho(J_1 + i)$ for $i = 0, \ldots, M_n$. Then all the ρ_i^n belong to S, and the distance between ρ_i^n and ρ_{i+1}^n is at most $2/|J_1|$. Moreover, as i increases from 0 to M_n, the interval $J_1 + i$ will leave I_1^2 in order to enter I_2^1, move on to I_2^0 and eventually enter I_2^2 until it stops at $J_1 + M_n = J_2$.

According to Lemmas 3.5 and 3.7, the corresponding vectors ρ_i^n always remain in S. Further, they start in $B_{\frac{1}{8}}(v_2)$, then move to $B_{\frac{1}{8}}(v_1)$ while remaining in S_{12}, then move to $B_{\frac{1}{8}}(v_0)$ while remaining in S_{01} and finally return to $B_{\frac{1}{8}}(v_2)$ while remaining in S_{02}. Note here that $|J_1| \geq a_n d_n/2$, such that Lemma 3.5 applies whenever $J_1 + i$ is entirely contained in one of the intervals of the decomposition, and otherwise we can always combine two of the statements of Lemma 3.7

Since $|J_1| \nearrow \infty$ as $n \to \infty$, it follows easily from these facts that the upper Hausdorff limit of the sequence of finite sets $\{\rho_0^n, \ldots, \rho_{M_n}^n\} \subseteq k$ contains a continuum $\mathcal{C}$ that separates the two connected components of the complement of S. Since $\mathcal{C} \subseteq \rho_{\mathrm{Cl}(\mathcal{O}(\omega_{\mathrm{sp}},\sigma))}$, this completes the proof. □

This shows Theorem 1.1(b).

3.4. **Non-empty interior.** It remains to construct $\omega' \in \Sigma^+$ such that $\rho_{\mathcal{M}(\omega')}$ has non-empty interior. It turns out that in comparison with the previous cases this is quite easy. We use the same block construction as before, with $b_n = 1$ for all $n \in \mathbb{N}$ and d_n chosen such that $\delta = \delta_\infty < 1/10$. Let $\Delta_\delta = \{sv_1 + tv_2 \mid s, t > \delta,\ s + t < 1 - \delta\}$ and choose a sequence of vectors $\rho_n \in \Delta_\delta$ such that the coordinates of ρ_n are integer multiples of $1/a_n$ and $\{\rho_n \mid n \in \mathbb{N}\}$ is dense in Δ_δ. Then we simply define ω' inductively on $[1, a_n]$ in such a way that $\frac{1}{a_n} \sum_{i=1}^{a_n} v_{\omega'(i)} = \rho_n$ for all $n \in \mathbb{N}$. This is possible, since in each stage of the construction we have $|[1, a_n] \setminus \mathcal{B}_{n-1}| = a_n - |[1, a_n] \cap \mathcal{B}_n| \geq (1 - \delta_n)a_n$. We thus obtain that $\Delta_\delta \subseteq \bigcap_{n\in\mathbb{N}} \overline{\{\rho_k \mid k \geq n\}} \subseteq \rho_{\mathcal{M}(\omega')}$.

This proves Theorem 1.1(c) and thus completes the proof of Theorem 1.1.

4. Abundance.

Finally, in this section we want to prove that the phenomena given by Theorem 1.1 are abundant, in the sense that they occur for an open set in $\mathrm{Homeo}_0(\mathbb{T}^2)$. Recall that we denote $\mathcal{F}$ the set of those $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ having non-empty interior rotation set. The result we want to prove is the following.

Theorem 4.1. *The family $\mathcal{G}$ in Theorem 1.1 contains an open and dense set of $\mathcal{F}$.*

This statement essentially follows from series of results on Axiom A diffeomorphisms which is already collected in [9]. We mainly follow that paper and keep the exposition brief. Recall that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is an axiom A diffeomorphism if the non-wandering set is hyperbolic and contains a dense set of periodic points. We call by $\mathcal{F}_0 \subset \mathcal{F}$ the set of those axiom A maps having zero-dimensional (totally disconnected) non-wandering set. The elements of $\mathcal{F}_0$ are called fitted Axiom A.

Theorem 4.2 ([17])**.** *The set $\mathcal{F}_0$ is dense in $\mathcal{F}$.*

Theorem 4.3 ([18])**.** *For any $f \in \mathcal{F}_0$ there is a C^0-neighborhood $\mathcal{U}(f)$ of f so that for all $g \in \mathcal{U}(f)$ there exists a semiconjugacy h between g and f, that is, a continuous onto map h so that $h \circ g = f \circ h$. Moreover, the semiconjugacy can be chosen in the homotopy class of the identity.*

The last theorem implies in particular that given $g \in \mathcal{U}(f)$ as above, we have $\rho_C(G) = \rho_{h(C)}(F)$ for any closed invariant set C of g. Thus if we prove that $\mathcal{F}_0 \subset \mathcal{G}$, we automatically have that

$$\bigcup_{f\in\mathcal{A}_0} \mathcal{U}(f) \subset \mathcal{G}$$

by means of the last theorem, which proves Theorem 1.2. Thus, the remainder of this section is devoted to showing that $\mathcal{F}_0 \subset \mathcal{G}$.

Recall that a *basic piece* $\Lambda \subset \mathbb{T}^2$ of a diffeomorphism f is a hyperbolic transitive set which is locally maximal. Given a set $X \subset \mathbb{R}^2$ we denote its convex hull by $\mathrm{conv}(X)$. The proof of the following statement can be found in [9, Corollary 5.2].

Theorem 4.4. *Every $f \in \mathcal{F}_0$ has a basic piece Λ so that $\mathrm{Conv}(\rho_\Lambda(F))$ has non-empty interior.*

We denote the basic set given in the last theorem by Λ_{rot}. The following results is a mixture of [9, Lemma 5.2] and the proof of [9, Theorem 5.2]. There is only one new consideration which is not done in [9], which is the following.

In [9, Lemma 5.2] the assumption is that we have a basic piece whose rotation set is not a single point, and the conclusion is the existence of a heteroclinic relation for its lift between a fixed point and an integer translation of it. In our situation given by Theorem 4.4, we have a basic piece whose rotation set has at least two non-colinear vectors, and the conclusion we need is the existence of a fixed point of the lift that has heteroclinic relations with two non-colinear integer translations of itself. However, the proof for this case is completely analogous to that of [9, Lemma 5.2]. Then, applying the same arguments as in the proof of [9, Theorem 5.2], one easily obtains the following result.

Theorem 4.5. *Suppose $f \in \mathcal{F}_0$ and let Λ_{rot} be the basic piece given by Theorem 4.4. Then there exists a positive integer n and an invariant set $\Lambda \subset \Lambda_{rot}$ which admits a Markov partition $\mathcal{R} = \{R_0, R_1, R_2\}$, so that $\bigcup_{i=0}^{2} R_i$ is contained in a topological disk D, $f_{|\Lambda}$ is conjugated via $h_{\mathcal{R}}$ to the full shift on $\{0,1,2\}^{\mathbb{Z}}$, and if we consider lifts $\tilde{R}_0, \tilde{R}_1, \tilde{R}_2$ of R_0, R_1, R_2 we have:*

- $F^n(\tilde{R}_0) \cap \tilde{R}_0 \neq \emptyset$,
- $F^n(\tilde{R}_1) \cap \tilde{R}_1 + v \neq \emptyset$,
- $F^n(\tilde{R}_2) \cap \tilde{R}_2 + w \neq \emptyset$,

where $v, w \in \mathbb{Z}^2 \setminus \{0\}$ are non-colinear.

This implies that any element f in $\mathcal{F}_0$ has a power which already has very similar properties to the ones we used in the constructions in the previous sections. In fact, by consindering an iterate f^n of f and performing a linear change of coordinates on $\mathbb{T}^2$, we may assume that $v = v_1$ and $w = v_2$ (see [10] for details). Therefore f^n has minimal sets with the rotation sets described in Theorem 1.1. However, since $\rho_M(F^n) = n\rho_M(F)$, the same applies to f itself. This completes the proof of Theorem 1.2.

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