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Bayesian estimation of the Rayleigh distribution under different loss function

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In a Rayleigh distribution, We are interested in the estimation of the parameter and some reliability characteristics, as the reliability and the failure rate functions. We used the Bayesian approach under different loss function (squared loss and Linex loss) with a type II censored data. The prior law of the parameter is non-informative prior then a natural conjugated prior. The estimators of σ, $S(t)$ and $h(t)$ are obtained with the exact analytic expression, the posterior risks are calculated in each case. A simulation study was carried out as well as real data analysis. A comparison between the different estimators from there posterior risks leads us to conclude that the best estimator is obtained under the Linex loss function.

keywords: Rayleigh distribution, Bayesian estimation, Posterior risk, Linex loss function.

1 Introduction

Lord Rayleigh (1880) introduced the Rayleigh distribution in connection with a problem in the field of acoustics. Since then, extensive work has taken place related to this distribution in different areas of science and technology. Its has some nice relations with some of well known distribution like Weibull, Chi-square or extreme distributions. The origin and other aspects of this distribution can be found in Siddiqui.
The Rayleigh distribution is a special case of the two parameter Weibull distribution and a suitable model for life testing studies. Polovko, (1968), Dyer and Whisenand,(1973),

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demonstrated the importance of this distribution in electro vacuum devices and communication engineering.
The probability density function (pdf), the reliability function and the failure rate function, respectively are given by:

$$f(x,\sigma) = \frac{x}{\sigma^2} exp(-\frac{x^2}{2\sigma^2}), \;\; x > 0, \;\; \sigma > 0 \tag{1}$$

$$S(x,\sigma) = exp(-\frac{x^2}{2\sigma^2}) \tag{2}$$

and

$$h(x,\sigma) = \frac{x}{\sigma^2} \tag{3}$$

Where $\sigma > 0$ is the parameter. An important characteristics of the Rayleigh distribution is that its failure rate is an increasing linear function of time. This means that when the failure times are distributed according to the Rayleigh law, an intense aging of the equipment takes place. Then as time increases the reliability function decreases at a much higher rate than in the case of exponential distribution.
Several authors have studied the Rayleigh distribution, Howlader and Hossain (1995) studied the problem of the estimation of the parameter and the reliability function with censored data and squared loss function. Dyer and Whisenand, (1973), provided the best linear unbiased estimator of σ based on complete sample, censored sample and selected order statistics. Bayesian estimation and prediction problems for the Rayleigh distribution based on doubly censored sample have been considered by Balakrishnan, (1989), Fernandez, (2000), Raqab and Madi, (2011). Bayesian estimation problems for the Rayleigh distribution based on progressively censored sample have been considered by Kim and Han, (2009), Raqab and Madi,(2011), and Dey and Dey, (2014).
In this paper, we study the Bayesian estimator of the parameter, reliability function and the failure rate function under squared loss function and asymmetric loss function in presence of censored type II sample with non-informative prior density and conjugate prior.

Notation

$\widehat{\sigma}_{ML}$: Maximum likelihood estimator of the parameter σ.

$\widehat{\sigma}_{SV}$: Bayesian estimator of the parameter σ under the squared loss function with a vague prior.

$R(\widehat{\sigma}_{SV})$: Posterior risk of $\widehat{\sigma}_{SV}$.

$\widehat{\sigma}_{LV}$: Bayesian estimator of the parameter σ under the Linex loss function with a vague prior.

$R(\widehat{\sigma}_{LV})$: Posterior risk of $\widehat{\sigma}_{LV}$.

$S_{ML}(t)$: Maximum likelihood estimator of the reliability function.

$S_{SV}(t)$: Bayesian estimator of the reliability function under the squared loss function with a vague prior.

$R(S_{ML}(t))$: Posterior risk of $S_{ML}(t)$.

$S_{LV}(t)$: Bayesian estimator of the reliability function under the Linex loss function with a vague prior.

$R(S_{LV}(t))$: Posterior risk of $S_{LV}(t)$.

$h_{ML}(t)$: Maximum likelihood estimator of the failure rate function.

$h_{SV}(t)$: Bayesian estimator of the failure rate function under the squared loss function with a vague prior.

$R(h_{SV}(t))$: Posterior risk of $h_{SV}(t)$.

$h_{LV}(t)$: The Bayesian estimator of the failure rate function under the Linex loss function with a vague prior.

$R(h_{LV}(t))$: Posterior risk of $h_{LV}(t)$.

$\widehat{\sigma}_{CS}$: Bayesian estimator of the parameter σ under the squared loss function with a natural conjugated prior.

$R(\widehat{\sigma}_{CS})$: Posterior risk of $\widehat{\sigma}_{CS}$.

$\widehat{\sigma}_{CL}$: Bayesian estimator of the parameter σ under the Linex loss function with natural conjugated prior.

$R(\widehat{\sigma}_{CL})$: Posterior risk of $\widehat{\sigma}_{CL}$.

$S_{CS}(t)$: Bayesian estimator of the reliability function under squared loss function with natural conjugated prior.

$R(S_{CS}(t))$: Posterior risk of $S_{CS}(t)$.

$S_{CL}(t)$: Bayesian estimator of the reliability function under the Linex loss function with natural conjugated prior.

$R(S_{CL}(t))$: Posterior risk of $S_{CL}(t)$.

$h_{CS}(t)$: Bayesian estimator of the failure rate function under Linex loss function with natural conjugated prior.

$R(h_{CS}(t))$: Posterior risk of $h_{CS}(t)$.

$h_{CL}(t)$: Bayesian estimator of the failure rate function under the Linex loss function with natural conjugated prior.

$R(h_{CL}(t))$: Posterior risk of $h_{CL}(t)$.

2 Bayesian estimation with non-informative prior

Let $(X_{(1)}, X_{(2)}, ..., X_{(r)}, ..., X_{(n)})$ a sample of size n censored in $X_{(r)}$ The likelihood writes:

$$L(x \mid \sigma) \propto \frac{1}{\sigma^{2r}} exp(-\frac{T_r}{2\sigma^2})$$

or

$$T_r = \sum_{i=1}^{r} x_i^2 + (n-r)x_r^2$$

In the Bayesian context, when we have few or no information of the parameter, we use vague priors. The most popular is due to Jeffreys et al. defined as follows:

$$\pi_1(\sigma) = |I_1(\sigma)|^{\frac{1}{2}} = |-E\frac{\partial^2 lnf}{\partial\sigma^2}| \propto \frac{1}{\sigma}$$

The posterior density is then:

$$\pi_1(\sigma|x) = \frac{L(x|\sigma)\pi_1(\sigma)}{\int_0^\infty L(x|\sigma)\pi_1(\sigma)d\sigma} = \frac{(T_r)^r}{\Gamma(r)} \frac{1}{2^{r-1}} \sigma^{-2r-1} exp(-\frac{T_r}{2\sigma^2})$$

Or x is the vector of observations.

2.1 Loss functions

2.1.1 Squared loss function

Let $\widehat{\theta}$ the estimator of θ, the squared loss function defined by $L_1(\theta, \widehat{\theta}) = (\theta - \widehat{\theta})^2$ is proposed by Legendre (1805) and Gauss (1810), it is widely used in literature. The Bayesien estimator of θ is then the posterior mean, let $\widehat{\theta}_B = E(\theta|x)$.

2.1.2 Linex loss function

The Linex (Linear Exponential) loss function is dominantly and widely used because it is a natural extension of squared loss function. It was originally introduced by Varian Varian, (1975), and got a lot of popularity due to Zellner Zellner, (1986).

The mathematical form of Linex loss function may simply be expressed by:

$$L(\Delta) \propto e^{a\Delta} - a\Delta - 1 \ \ , a \neq 0.$$

Where $\Delta = (\widehat{\theta} - \theta)$, $\widehat{\theta}$ is an estimate of θ.

Consider the following convex loss function:

$$L(\Delta) \propto e^{a\Delta} - a\Delta - 1, a \neq 0 \tag{4}$$

The signe of a and its absolute magnitude represent the direction and the degree of asymmetry. For $a \to 0$, we find the squared loss function. Varian (1975) considered the loss function in (4) for $\Delta_1 = \widehat{\theta} - \theta$; the function $L(\Delta_1)$ is called the Linex loss function. Let $E_p(L(\Delta_1))$ the posterior expectation of $L(\Delta_1)$, the Bayesian estimator of θ under this loss function is denoted $\widehat{\theta}_{LB}$ it corresponds to the value of $\widehat{\theta}$ that minimizes $E_p(L(\Delta_1))$. (E_p is the mean with respect to the posterior density).

$$E_p(L(\Delta_1)) = expa(\widehat{\theta})E_p(exp(-a\theta)) + aE_p(\theta) - a\widehat{\theta} - 1 \tag{5}$$

We derive the above expression with respect to $\widehat{\theta}$ and we equal to zero

$$\frac{\partial E_p(L(\Delta_1))}{\partial \widehat{\theta}} = a(exp(a\widehat{\theta}))E_p(exp(-a\theta)) - a = 0$$

The solution of this equation is:

$$\widehat{\theta}_{LB} = -\frac{1}{a} ln(E_p(e^{-a\theta}))$$

Consider the loss function $L(\Delta_2)$; or $\Delta_2 = (\frac{\widehat{\theta}}{\theta})^2 - 1$; this loss function has been used by several authors whose Zellner (2006) and (2009). We minimize the posterior expectation $E_p(L(\Delta_2))$:

$$\begin{aligned} E_p(L(\Delta_2)) &= E_p[exp((\frac{\widehat{\theta}}{\theta})^2 - 1) - a((\frac{\widehat{\theta}}{\theta})^2 - 1) - 1] \\ &= exp(-a)E_p(exp(\frac{\widehat{\theta}}{\theta})^2) - a(\frac{\widehat{\theta}}{\theta})^2 - 1 \end{aligned}$$

We derive $E_p(L(\Delta_2))$ with respect to $\widehat{\theta}$ and we equal to zero, we obtain the Bayesian estimator $\widehat{\theta}_{LB}$ of θ under the loss function $L(\Delta_2)$

$$\frac{\partial E_p(L(\Delta_2))}{\partial \widehat{\theta}} = 2a(exp(-a))E_p(\frac{\widehat{\theta}}{\theta^2} exp(a(\frac{\widehat{\theta}}{\theta})^2)) - 2aE_p(\frac{\widehat{\theta}}{\theta^2}) = 0$$

$\widehat{\theta}_{LB}$ is then the solution of the equation

$$E_p[\frac{1}{\theta^2} exp(a(\frac{\widehat{\theta}_{LB}}{\theta^2}))] = exp(a)E_p(\frac{1}{\theta^2}) \tag{6}$$

2.2 Estimation of the parameter σ

The maximum likelihood estimator denoted $\widehat{\sigma}_{ML}$ is obtained by solving the equation $\frac{\partial lnL(x,\sigma)}{\partial \sigma} = 0$; then:

$$\widehat{\sigma}_{ML} = \sqrt{\frac{T_r}{2r}} \tag{7}$$

With respect to the squared loss function, and with a vague prior on σ; the estimator of σ denoted $\widehat{\sigma}_{SW}$ is obtained with calculate its expectation with respect to the posterior density:

$$\widehat{\sigma}_{SV} = \int_0^\infty \sigma\pi(\sigma|x)d\sigma = \frac{(T_r)^r}{\Gamma(r)} \frac{1}{2^{r-1}} \int_0^\infty \sigma^{-2r} exp(-\frac{T_r}{2\sigma^2})d\sigma = \frac{\Gamma(r-\frac{1}{2})}{\Gamma(r)} (\frac{T_r}{2})^{\frac{1}{2}} \tag{8}$$

The posterior risk of the parameter σ is given by

$$R(\widehat{\sigma}_{VQ}) = (\frac{T_r}{2})(\frac{\Gamma(r-1)}{\Gamma(r)} - \frac{\Gamma^2(r-\frac{1}{2})}{\Gamma^2(r)}). \quad (9)$$

With respect to squared loss function $L(\Delta_2)$ and the vague prior, the estimator of σ denoted $\widehat{\sigma}_{LV}$ is the solution of equation given in (5) or:

$$E_p[\frac{\widehat{\sigma}_{LV}}{\sigma^2} exp(a(\frac{\widehat{\sigma}^2_{LV}}{\sigma^2}))] = \frac{(T_r)^r}{\Gamma(r)} \frac{\widehat{\sigma}_{LV}}{2^{r-1}} \int_0^\infty \frac{1}{\sigma^{2r+3}} exp(-\frac{1}{2\sigma^2}(T_r - 2a\widehat{\sigma}^2_{LV}))d\sigma$$

and

$$exp(a)E_p(\frac{\widehat{\sigma}_{LV}}{\sigma^2}) = \frac{(T_r)^r}{\Gamma(r)} \frac{\widehat{\sigma}_{LV}}{2^{r-1}} \int_0^\infty \frac{1}{\sigma^{2r+3}} exp(-\frac{1}{2\sigma^2(T_r)})d\sigma$$

After some algebraic manipulations, we obtain:

$$\widehat{\sigma}_{LV} = [\frac{T_r}{2a}(1 - exp(-\frac{a}{r+1}))]^{\frac{1}{2}} \quad (10)$$

the posterior risk of the parameter σ under the Linex loss function is given by:

$$R(\widehat{\sigma}_{LV}) = a(\widehat{\sigma}_{SV} - \widehat{\sigma}_{LV}). \quad (11)$$

2.3 Estimation of the reliability function

for obtain the estimator $S_{ML}(t)$ of the reliability $S(t)$, we replace σ with $\widehat{\sigma}_{ML}$ in the expression of $S(t)$, then

$$S_{ML}(t) = exp(-\frac{rt^2}{T_r}) \quad (12)$$

The Bayesien estimator of $S(t)$ with respect to the squared loss function and the vague prior is $S_{SV}(t)$

$$S_{SV}(t) = \int_0^\infty S(t)\pi_1(\sigma|x)d\sigma = (\frac{T_r}{T_r + t^2})^r \quad (13)$$

The posterior risk of the reliability function under the squared loss function is given by:

$$R(S_{SV}(t)) = (\frac{T_r}{T_r + 2t^2})^r - (\frac{T_r}{T_r + t^2})^{2r} \quad (14)$$

The Bayesian estimator of $S(t)$ with respect to the Linex loss function is denoted $S_{LV}(t)$, for calculate, we make the following change of variable: $S(t) = exp(-\frac{t^2}{2\sigma^2}) = \gamma \Rightarrow \sigma = (-\frac{t^2}{2ln\gamma})^{\frac{1}{2}}$; we write the posterior density according to γ

$$\pi_1(\gamma|x) = \frac{(T_r)^r}{\Gamma(r)} \frac{1}{2^{r-1}} \frac{1}{2}(-\frac{t^2}{2ln\gamma})^{\frac{1}{2}}(\frac{t^2}{2\gamma(ln\gamma)^2})(-\frac{t^2}{2ln\gamma})^{-r-\frac{1}{2}}(\gamma)^{\frac{T_r}{t^2}}$$

$$= (\frac{T_r}{t^2})^r \frac{1}{\Gamma(r)}(\gamma)^{\frac{T_r}{t^2}-1}(-ln\gamma)^{r-1};\ 0 \le \gamma \le 1$$

We used the loss function $L(\Delta_1)$, The Bayesian estimator γ_{LV} of γ, is

$$\gamma_{LV} = -\frac{1}{a}lnE_p(exp(-a\gamma)) = -\frac{1}{a}ln[(\frac{T_r}{t^2})^r \frac{1}{\Gamma(r)} \int_0^\infty exp(-a\gamma)(\gamma)^{\frac{T_r}{t^2}-1}(-ln\gamma)^{r-1}d\gamma]$$

$$= -\frac{1}{a}ln[\sum_{j=0}^{k} \frac{(-a)^j}{j!}(1+j\frac{t^2}{T_r})^{-r}] \quad (15)$$

The last result is obtained by using a development of $(exp(-a\gamma))$ to order k in a neighborhood of zero and making the change variable $u = (-ln\gamma)$ to the calculation of this integral.
The posterior risk of the reliability function under the Linex loss function is given by

$$R(S_{LV}(t)) = a(S_{SV}(t) - S_{LV}(t)) \quad (16)$$

2.4 Estimation of the failure rate function

The maximum likelihood estimator of $h(t)$ denoted $h_{ML}(t)$ is obtained when we replace σ with $\widehat{\sigma}_{ML}$ in the expression of $h(t)$, then:

$$h_{ML}(t) = 2r\frac{t}{T_r} \quad (17)$$

With respect to the squared loss function and a vague prior of σ, the Bayesian estimator of $h(t)$ denoted $h_{SV}(t)$ is:

$$h_{SV}(t) = \int_0^\infty h(t)\pi_1(\sigma|x)d\sigma = \frac{(T_r)^r}{\Gamma(r)}\frac{t}{2^{r-1}}\int_0^\infty \sigma^{-2r-3}exp(-\frac{T_r}{2\sigma^2})d\sigma$$

$$h_{SV}(t) = 2r\frac{t}{T_r} \quad (18)$$

The posterior risk of the failure rate function under the squared loss function is:

$$R(h_{SV}(t)) = \frac{4t^2\Gamma(r+2)}{\Gamma(r)}(T_r)^{-2} - (2r)^2\frac{t^2}{(T_r)^2} \quad (19)$$

Remark : The estimator of $h(t)$ obtained by maximum likelihood estimation and with the Bayesian approach with the non-informative prior and squared loss function are identical.
The asymmetric loss functions $L(\Delta_1)$ and $L(\Delta_2)$ are not appropriate for a simple analytic form of a Bayesian estimator $h(t)$; that is why, we define $\Delta = (\frac{\theta}{\widehat{\theta}} - 1)$ and we replace in the loss function given by (4), then we take the posterior expectation, we drift and we equal to zero to find the value of θ denoted $\widehat{\theta}_{LV}$ which minimizes

$$E_p(L(\Delta)) = E_p(exp(a(\frac{\theta}{\widehat{\theta}} - 1))) - aE_p(\frac{\theta}{\widehat{\theta}} - 1) - 1$$

$$\frac{\partial E_p(L(\Delta))}{\partial\widehat{\theta}} = exp(-a)E_p(-a\frac{\theta}{\widehat{\theta}^2}exp(\frac{\theta}{\widehat{\theta}})) + aE_p(\frac{\theta}{\widehat{\theta}}^2) = 0$$

The Bayesian estimator $\widehat{\theta}_{LV}$ with respect to the loss function $L(\Delta)$ is then the solution of the equation:

$$exp(-a)E_p(\theta exp(a(\frac{\theta}{\widehat{\theta}_{VL}}))) = E_p(\theta) \tag{20}$$

We place $h(t) = \theta$, let $\frac{t}{\sigma^2} = \theta \Rightarrow \sigma = (\frac{t}{\theta})^{\frac{1}{2}}$, we write the posterior density according to θ:

$$\pi_1(\theta|x) = \frac{(T_r)^r}{\Gamma(r)}\frac{1}{2^{r-1}}(\frac{t}{\theta})^{-r-\frac{1}{2}}exp(-\frac{T_r}{2t}\theta)\frac{1}{2}(\frac{t}{\theta})^{-\frac{1}{2}}(\frac{t}{\theta^2})$$

$$= (\frac{T_r}{2t})^r\frac{1}{\Gamma(r)}\theta^{r-1}exp(-\frac{T_r}{2t}\theta);\;\; \theta \geq 0.$$

We remark that the posterior law of θ is a gamma $G(r, \frac{T_r}{2t})$.
We solve the equation (20), or $\widehat{\theta}_{LV}$ is the estimator of the failure rate $h(t)$ and we denoted $h_{LV}(t)$

$$exp(-a)(\frac{T_r}{2t})^r\frac{1}{[\frac{T_r}{2t} - \frac{a}{h_{VL}(t)}]^{r+1}} = \frac{2t}{T_r}$$

$$h_{LV}(t) = a\frac{2t}{T_r}[1 - exp(-\frac{a}{r+1})]^{-1} \tag{21}$$

The posterior risk of the failure rate function under the Linex loss function is given by:

$$R(h_{LV}(t)) = a(h_{LV}(t) - h_{SV}(t)) \tag{22}$$

3 Bayesian estimation with a natural conjugated prior

The naturel conjugated prior is defined as:

$$\pi_2(\sigma) \propto \frac{1}{\sigma^{\alpha+1}}exp(-\frac{\beta}{2\sigma^2});\;\; \alpha, \beta > 0 \tag{23}$$

The posterior law is then:

$$\pi_2(\sigma|x) = \frac{(T_r + \beta)^{r+\frac{\alpha}{2}}}{2^{r+\frac{\alpha}{2}-1}\Gamma(r + \frac{\alpha}{2})}\frac{1}{\sigma^{2r+\alpha+1}}exp(-\frac{1}{2\sigma^2}(T_r + \beta))$$

Remark

For $\alpha = 0$ and $\beta = 0$, we obtain the case of non-informative prior.

3.1 Estimation of the parameter σ

Always with respect to the squared loss function but with a natural conjugated prior of σ; the estimator of σ denoted $\widehat{\sigma}_{CS}$ is obtained when we calculate its expectation with respect to the posterior density:

$$\widehat{\sigma}_{CS} = \int_0^{\infty} \sigma\pi_2(\sigma|x)dx = \sqrt{\frac{T_r+\beta}{2}}\frac{\Gamma(r+\frac{\alpha}{2}-\frac{1}{2})}{\Gamma(r+\frac{\alpha}{2})} \quad (24)$$

The posterior risk of the parameter σ is given by the following formula:

$$R(\widehat{\sigma}_{CS}) = \frac{T_r+\beta}{2}(\frac{\Gamma(r+\frac{\alpha}{2}-1)}{\Gamma(r+\frac{\alpha}{2})} - \frac{\Gamma^2(r-\frac{\alpha}{2}-\frac{1}{2})}{\Gamma^2(r-\frac{\alpha}{2})}) \quad (25)$$

The Bayesian estimator of σ with respect to $L(\Delta_2)$ and the natural conjugated prior of σ, σ denoted $\widehat{\sigma}_{CL}$ is the solution of the equation:

$$E_p[\frac{\widehat{\sigma}_{CL}}{\sigma^2}exp(a(\frac{\widehat{\sigma}^2_{CL}}{\sigma^2}))] = 2^{r+\frac{\alpha}{2}}\widehat{\sigma}_{CL}[\frac{\Gamma(r+\frac{\alpha}{2}+1)}{(T_r+\beta-2a\widehat{\sigma}^2_{CL})^{r+\frac{\alpha}{2}+1}}]$$

$$exp(a)E_p(\frac{\widehat{\sigma}_{CL}}{\sigma^2}) = \widehat{\sigma}_{CL}e^a\frac{\Gamma(r+\frac{\alpha}{2}+1)}{(T_r+\beta)^{r+\frac{\alpha}{2}+1}}$$

$$\widehat{\sigma}_{CL} = [\frac{T_r+\beta}{2a}(1-exp(-\frac{a}{r+\frac{\alpha}{2}+1}))]^{\frac{1}{2}} \quad (26)$$

The posterior risk of the parameter σ under the Linex loss function is given by:

$$R(\widehat{\sigma}_{CL}) = a(\widehat{\sigma}_{CS} - \widehat{\sigma}_{CL}) \quad (27)$$

3.2 Estimation of the reliability function

The Bayesian estimator of $S(t)$ with respect to the squared loss function and the natural conjugated prior is $S_{CS}(t)$

$$S_{CS}(t) = \int_0^{\infty} S(t)\pi_2(\sigma|x)d\sigma = (\frac{T_r+\beta}{T_r+\beta+t^2})^{r+\frac{\alpha}{2}}$$

$$S_{CS}(t) = (\frac{T_r+\beta}{T_r+\beta+t^2})^{r+\frac{\alpha}{2}} \quad (28)$$

The posterior risk of the reliability function under the squared loss function is given by:

$$R(S_{CS}(t)) = (\frac{T_r+\beta}{T_r+\beta+2t^2})^{r+\frac{\alpha}{2}} - (\frac{T_r+\beta}{T_r+\beta+t^2})^{2r+\alpha} \quad (29)$$

with a natural conjugated prior, the Bayesian estimator with respect to the Linex loss function is denoted $S_{CL}(t)$, for calculate, we use the following variable change: $S(t) = exp(-\frac{t^2}{2\sigma^2}) = \gamma \Rightarrow \sigma = (-\frac{t^2}{2ln\gamma})^{\frac{1}{2}}$; we write the posterior density according to γ.

$$\gamma_{CQ} = \frac{-1}{a}lnE_p(exp(-a\gamma))$$

$$= \frac{-1}{a} ln[\frac{(T_r+\beta)^{r+\frac{\alpha}{2}}}{\Gamma(r+\frac{\alpha}{2})} \frac{1}{(t^2)^{r+\frac{\alpha}{2}}} \int_0^{\infty} exp(-a\gamma)(\gamma)^{\frac{T_r+\beta}{t^2}-1}(-ln\gamma)^{r+\frac{\alpha}{2}-1} d\gamma]$$

$$= -\frac{1}{a} ln[\sum_0^k \frac{(-a)^j}{j!}(1+\frac{jt^2}{T_r+\beta})^{(-r-\frac{\alpha}{2})}] \tag{30}$$

The posterior risk of the reliability function under the Linex loss function is given by

$$R(S_{CL}(t)) = a(S_{CS}(t) - S_{CL}(t)) \tag{31}$$

3.3 Estimation of the failure rate function

with respect to the squared loss function, the Bayesian estimator is given by:

$$h_{CS}(t) = \int h(t,\sigma)\pi_2(\sigma|t)dt = \frac{2(r+\frac{\alpha}{2})}{T_r+\beta} t \tag{32}$$

The posterior risk of the failure rate function is given by:

$$R(h_{CS}(t)) = (\frac{t}{2})^2 \frac{\Gamma(r+\frac{\alpha}{2}+2)}{\Gamma(r+\frac{\alpha}{2})}(T_r+\beta)^{-2} - h_{CS}(t)^2 \tag{33}$$

Under the Linex loss function, the Bayesian estimator of the failure rate function is given by the following formula:

$$h_{CL}(t) = a\frac{2t}{T_r+\beta}[1-exp(-\frac{a}{r+\frac{\alpha}{2}+1})]^{-1} \tag{34}$$

The posterior risk of the failure rate function under the Linex loss function is:

$$R(h_{CL}(t)) = a(h_{CS}(t) - h_{CL}(t)) \tag{35}$$

4 Simulations

In this section, we propose to study the performance of the Bayesian estimators of the reliability function and the parameter under some various loss functions with respect to the MLE. An exhaustive Monte Carlo comparative study is performed using the loss functions given in the previous sections.

Firstly, we take $(\alpha, \beta) = (1, 2)$ and we generate the natural conjugated prior of σ, given by expression (23) and we deduce the values of σ (we obtain that $\sigma = \Gamma(\frac{1}{2}) = 1.7724$ the initial value of σ)

We generate $N = 10000$ sample size of n of the rayleigh distribution of the parameter σ (we used the Inversion method) witch have the cumulative distribution function (CDF) given by: $F(x,\sigma) = 1-S(x,\sigma) = 1-exp(-\frac{x^2}{2\sigma^2})$, we take tree values of n ($n = 20, 30, 50$) and two values of r ($r = 10, 15$) for get increasing censured rate.

We calculate the maximum likelihood estimators of σ, $S(t)$ and $h(t)$ (denoted ML), then, we calculate the Bayesian estimators under the squared loss function (denoted S) and under the Linex loss function for three values of a ($LI1$ ($a = -0.5$), $LI2$ ($a = -1$), $LI3$ ($a = 1$)).

For each estimator, we calculate mean squared error (MSE) given by the expression below, ϕ considered σ, $S(t)$, and $h(t)$, $\widehat{\phi}$ his estimator respectively

$$MSE(\phi) = \frac{1}{N}\sum_{i=1}^{N}(\phi - \widehat{\phi})^2.$$

The values of MSE are given between brackets in the first cologne of the table 1.

Tn the case of Bayesian estimators, we calculate the posterior risk (denoted PR) (the expression analytic of PR are given by (9), (11), (14), (16), (19), (22), (25), (27), (29), (31), (33), (35)).

In the following tables, we present the results of the Monte-Carlo study, in the first table the prior law is a vague prior, but in the second table we consider the natural conjugated prior.

Table 1: **Bayesian estimators of the Rayleigh distribution with a vague prior**

(n,r)	ϕ	ML(MSE)	S(PR)	LI1(PR)	LI2(PR)	LI3(PR)
(20,10)	σ	1.7494(0.0005)	1.8187(0.0021)	1.6872(0.0072)	1.7084(0.0041)	1.6272(0.0210)
	$S(t)$	0.9058(0.0048)	0.9063(0.0049)	0.8841(0.0023)	0.8402(0.0017)	0.6825(0.0235)
	$h(t)$	0.2651(0.0006)	0.2651(0.0006)	0.2850(0.0021)	0.2791(0.0016)	0.3067(0.0046)
(30,10)	σ	1.7513(0.0044)	1.8106(0.0023)	1.6990(0.0069)	1.7081(0.0041)	1.6310(0.0199)
	$S(t)$	0.9059(0.0048)	0.9063(0.0049)	0.8841(0.0023)	0.8403(0.0002)	0.6827(0.0253)
	$h(t)$	0.2652(0.0007)	0.2652(0.0007)	0.2859(0.0021)	0.2788(0.0016)	0.3048(0.0043)
(50,10)	σ	1.7478(0.0006)	1.8170(0.0019)	1.6856(0.0075)	1.7090(0.0040)	1.6332(0.0193)
	$S(t)$	0.9059(0.0006)	0.9065(0.0049)	0.8845(0.0023)	0.8398(1.46e-05)	0.6824(0.0235)
	$h(t)$	0.2656(0.0007)	0.2656(0.0007)	0.2856(0.0021)	0.2780(0.0015)	0.3048(0.0043)
(20,15)	σ	1.7557(0.0002)	1.8012(0.0008)	1.7157(0.0032)	1.7285(0.0019)	1.6760(0.0092)
	$S(t)$	0.9086(0.0052)	0.9089(0.0053)	0.8867(0.0025)	0.8105(10-05)	0.6838(0.0231)
	$h(t)$	0.2565(0.0003)	0.2565(0.0003)	0.2684(0.0008)	0.2645(0.0006)	0.2812(0.0018)
(30,15)	σ	1.7625(9e-05)	1.8082(0.0012)	1.7200(0.0027)	1.7238(0.0018)	1.6710(0.0102)
	$S(t)$	0.9033(0.0053)	0.9096(0.0054)	0.8871(0.0026)	0.8426(4e-05)	0.6837(0.0231)
	$h(t)$	0.2543(0.0002)	0.2543(0.0002)	0.2671(0.0008)	0.2642(0.0006)	0.2830(0.0019)
(50,15)	σ	1.7551(0.0002)	1.8006(0.0007)	1.7119(0.0036)	1.7299(0.0018)	1.67421(0.0096)
	$S(t)$	0.9085(0.0003)	0.9088(0.0052)	0.8864(0.0025)	0.8427(4e-05)	0.6838(0.0237)
	$h(t)$	0.2566(0.0003)	0.2566(0.0003)	0.2693(0.0009)	0.2451(0.0006)	0.2818(0.0018)

Table 2: **Bayesian estimators Rayleigh with a natural conjugated prior**

(n,r)	ϕ	S(PR)	LI1(PR)	LI2(PR)	LI3(PR)
(20,10)	σ	1.7846(0.0788)	1.6615(0.0805)	1.6831(0.0767)	1.6123(0.0891)
	S(t)	0.9033(0.0054)	0.9011(0.0051)	0.8903(0.0037)	0.9640(0.0180)
	h(t)	0.2740(0.0102)	0.2936(0.0133)	0.2858(0.0118)	0.3119(0.0172)
(30,15)	σ	1.7867(0.0773)	1.6634(0.0787)	1.6824(0.0761)	1.6103(0.0900)
	S(t)	0.9037(0.0055)	0.9014(0.0051)	0.8903(0.0037)	0.9636(0.0179)
	h(t)	0.2729(0.0098)	0.2924(0.0128)	0.2859(0.0119)	0.3127(0.0174)
(50,10)	σ	1.7879(0.0781)	1.6646(0.0791)	1.6823(0.0793)	1.6120(0.0889)
	S(t)	0.9037(0.0055)	0.9015(0.0052)	0.8900(0.0037)	0.9640(0.0179)
	h(t)	0.2729(0.01000)	0.2924(0.0131)	0.2868(0.0124)	0.3117(0.0168)
(20,15)	σ	1.7856(0.0516)	1.7012(0.0518)	1.7074(0.0524)	1.6655(0.0576)
	S(t)	0.9077(0.0056)	0.9053(0.0053)	0.8932(0.0037)	0.9691(0.0186)
	h(t)	0.2599(0.0055)	0.2725(0.0067)	0.2707(0.0064)	0.2846(0.0028)
(30,15)	σ	1.7856(0.0516)	1.7012(0.0518)	1.7074(0.0524)	1.6655(0.0576)
	S(t)	0.9070(0.0055)	0.9046(0.0052)	0.8932(0.0037)	0.9689(0.0186)
	h(t)	0.2620(0.0055)	0.2747(0.0068)	0.2706(0.0065)	0.2852(0.0081)
(50,15)	σ	1.7782(0.0528)	1.6942(0.0540)	1.7113(0.0517)	1.6603(0.0574)
	S(t)	0.9069(0.0056)	0.9045(0.0052)	0.8936(0.0038)	0.9686(0.0185)
	h(t)	0.2423(0.0057)	0.2750(0.0069)	0.2695(0.0064)	0.2860(0.0081)

After this simulation study, we conclude that the estimator of the parameter has a minimum risk when we used the squared loss function and the best estimator of the reliability function is obtained with Linex loss function ($a = -1$).

5 Data Analysis

We apply the proposed methods to areal data set presented in Lawless. The data arose in test on the endurance of deep-groove bearings and are originally discussed by Lieblein and Zelen. They are the number of revolutions to failure for each of $n = 23$ bearings in the life test. Raqab and Madi indicated that a one parameter Rayleigh distribution acceptable for these data. Here we consider $n = 23$ deep-groove ball bearing failure times. The 23 failure times are:

0.1788, 0.2892, 0.3300, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.7304.

The maximum likelihood estimator of the parameter σ is equal to $\widehat{\sigma}_{ML} = 0.9175$
For different values of $t = 0.25, 0.5, 0.75, 1$, we obtain the maximum likelihood estimators of the reliability function that are $S_{ML}(t) = 0.9635$, 0.8620, 0.7160, 0.5521

The maximum likelihood estimator of the failure rate function for different values of t are: $\widehat{h}_{ML}(t) = 0.2969,\ 0.5939,\ 0.8908,\ 1.1878$
The Bayesian estimators of the parameters σ, the reliability function and the failure rate function with a wave prior, then with a natural conjugated prior are given in the following tables:

Table 3: **Bayesian estimators of the Rayleigh distribution with a vague prior and real data**

t	ϕ	S(PR)	LI1(PR)	LI2(PR)	LI3(PR)
0.25	σ	0.9451(0.0187)	0.9001(0.0449)	0.8921(0.0264)	0.8686(0.0765)
	h(t)	0.5939(0.0271)	0.6170(0.0231)	0.6282(0.0171)	0.6170(0.0231)
0.75	S(t)	0.7190(0.0042)	0.7144(0.0045)	0.7189($2.14*10^{-5}$)	0.7378(0.0188)
	h(t)	0.8908(0.0610)	0.9255(0.0346)	0.9423(0.0257)	0.9940(0.1031)
1	S(t)	0.5594(0.0078)	0.5603(0.0008)	0.5609(0.0007)	0.5628(0.0033)
	h(t)	1.1878(0.1085)	1.2340(0.0462)	1.2564(0.0343)	1.3275(0.1375)

Table 4: **Bayesian estimators of Rayleigh distribution with a natural conjugated prior and real data**

t	ϕ	S(PR)	LI1(PR)	LI2(PR)	LI3(PR)
0.25	σ	0.9473(0.0181)	0.6075(0.3397)	0.6051(0.1710)	0.5980(0.3493)
	S(t)	0.9638(0.0654)	0.9065(0.0581)	0.9176(0.0231)	0.9852(0.0214)
	h(t)	0.2949(0.0064)	0.3059(0.0110)	0.3113(0.0082)	0.3277(0.0328)
0.5	S(t)	0.8635(0.1843)	0.7124(0.1511)	0.7173(0.0731)	0.7124(0.1511)
	h(t)	0.5898(0.0257)	0.6117(0.0220)	0.6226(0.0164)	0.6118(0.0220)
0.75	S(t)	0.7205(0.2371)	0.4772(0.2432)	0.4776(0.1214)	0.4783(0.2421)
	h(t)	0.8847(0.0579)	0.9178(0.0331)	0.9339(0.0246)	0.9833(0.0986)
1	S(t)	0.5614(0.2015)	0.2731(0.2882)	0.2723(0.1445)	0.2697(0.2916)
	h(t)	1.1796(0.1030)	1.2237(0.0441)	1.2452(0.0328)	1.3111(0.1315)

6 Conclusion

In this paper, we studied the problem of Bayesian parameter estimation, reliability function and failure rate function in Rayleigh model with typeII censored data. The interest of this work is that the analytical expression of these different estimators and their posteriori risks could be explicitly given under each loss function (Linex and quadratic).

Data analysis and simulation lead to the conclusion that the estimators $h(t)$ and $R(t)$ are better under a Linex loss function for t relatively large.

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