ОБРАБОТКА ИЗОБРАЖЕНИЙ И РАСПОЗНАВАНИЕ ОБРАЗОВ

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ALGEBRA AND GEOMETRY OF MULTICHANNEL IMAGES. PART 1. HYPERCOMPLEX MODELS OF RETINAL IMAGES



Introduction

We develop a conceptual framework and design methodologies for multichannel image processing systems with assessment capability. The term multichannel (i.e., hyperspectral, multicolor) image is used for an image with more than one component. An RGB image is an example of a color image featuring three separate image components: R (red), G (green), and B (blue). We know that primates and animals with different evolutionary histories have multi-channel visual systems of different dimensionality. For example, the human brain uses three channel (RGB) retinal images, reptile and tortoise brains use five channel multicolor images, and shrimps use ten channel multicolor images. The multichannel images are composed of a series of images $f_{\lambda_0}(x,y), f_{\lambda_1}(x,y),..., f_{\lambda_{K-1}}(x,y)$ in different optical bands at wavelengths $\lambda_0,\lambda_1,...,\lambda_{K-1}$, called the spectral channels, where K is the number of different optical channels. A multichannel retinal images can be considered as a n–D K-component (vector–valued) functions (Cronin, Marschal, 1989):

$$\mathbf{f}(\mathbf{x}) = \left(f_0(\mathbf{x}), f_1(\mathbf{x}),..., f_{K-1}(\mathbf{x}) \right) : \mathbf{R}^n \to \mathbf{V}^K \qquad (1)$$

with values into K-D perceptual space $\mathbf{V}^K$ (dichromatic $\mathbf{V}^2$, color $\mathbf{V}^3_{rgb}$, or multichannel $\mathbf{V}^K$), where $\mathbf{x} = (x_1, x_2,..., x_n) \in \mathbf{R}^n$, $n = 2,3,....$ The following cases are very interesting for us:

1) 2–D and 3–D bichromatic images

$$\mathbf{f}(x_1,x_2) = \left(f_0(x_1,x_2), f_1(x_1,x_2) \right) : \mathbf{R}^2 \to \mathbf{V}^2,$$

$$\mathbf{f}(x_1,x_2,x_3) = \left(f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3) \right) : \mathbf{R}^3 \to \mathbf{V}^2.$$

2) 2–D and 3–D trichromatic (color) images

$$\mathbf{f}(x_1,x_2) = \left(f_0(x_1,x_2), f_1(x_1,x_2), f_2(x_1,x_2) \right) : \mathbf{R}^2 \to \mathbf{V}^3_{rgb},$$

$$\mathbf{f}(x_1,x_2,x_3) = \left(f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3), f_1(x_1,x_2,x_3) \right) : \mathbf{R}^3 \to \mathbf{V}^3_{rgb}.$$

3) 2–D and 3–D K-channel images

$$\mathbf{f}(x_1,x_2) = \left(f_0(x_1,x_2), f_1(x_1,x_2),..., f_{K-1}(x_1,x_2) \right) : \mathbf{R}^2 \to \mathbf{V}^K,$$

$$\mathbf{f}(x_1,x_2,x_3) = \left(f_0(x_1,x_2,x_3), f_1(x_1,x_2,x_3),..., f_{K-1}(x_1,x_2,x_3) \right) : \mathbf{R}^3 \to \mathbf{V}^K.$$

4) 2–D and 3–D bichromatic binocular images (Luneburg, 1948, 1950)

$$\vec{\mathbf{f}}(x_1,x_2)=\left(\mathbf{f}_L(x_1,x_2),\mathbf{f}_R(x_1,x_2)\right):\mathbf{R}^2\to\mathbf{V}_L^2\oplus\mathbf{V}_R^2,$$

$$\vec{\mathbf{f}}(x_1,x_2,x_3)=\left(\mathbf{f}_L(x_1,x_2,x_3),\mathbf{f}_R(x_1,x_2,x_3)\right):\mathbf{R}^3\to\mathbf{V}_L^2\oplus\mathbf{V}_R^2.$$

5) 2–D and 3–D trichromatic (color) binocular images:

$$\vec{\mathbf{f}}(x_1,x_2)=\left(\mathbf{f}_L(x_1,x_2),\mathbf{f}_R(x_1,x_2)\right):\mathbf{R}^2\to\mathbf{V}_{rgb,L}^3\oplus\mathbf{V}_{rgb,R}^3,$$

$$\vec{\mathbf{f}}(x_1,x_2,x_3)=\left(\mathbf{f}_L(x_1,x_2,x_3),\mathbf{f}_R(x_1,x_2,x_3)\right):\mathbf{R}^3\to\mathbf{V}_{rgb,L}^3\oplus\mathbf{V}_{rgb,R}^3.$$

6) 2–D and 3–D K-channel images

$$\vec{\mathbf{f}}(x_1,x_2)=\left(\mathbf{f}_L(x_1,x_2),\mathbf{f}_R(x_1,x_2)\right):\mathbf{R}^2\to\mathbf{V}_L^K\oplus\mathbf{V}_R^K,$$

$$\vec{\mathbf{f}}(x_1,x_2,x_3)=\left(\mathbf{f}_L(x_1,x_2,x_3),\mathbf{f}_R(x_1,x_2,x_3)\right):\mathbf{R}^3\to\mathbf{V}_L^K\oplus\mathbf{V}_R^K,$$

where $\mathbf{f}_L,\mathbf{f}_R$ are left and right images, respectively.

For processing and recognition of 2-D, 3-D and n-D images, we turn the perceptual spaces into corresponding hypercomplex algebras (and call them *perceptual algebras*). We give algebraic models for two general levels (retina and Visual Cortex) of visual systems using different hypercomplex and Clifford algebras. In the algebraic-geometrical approach, each multichannel pixel is considered not as a K-D vector, but as a K-D hypercomplex number. We will interpret multichannel retinal images as multiplet-valued signals

$$\mathbf{f}(\mathbf{x})=\left(f_0(\mathbf{x}),f_1(\mathbf{x}),\ldots,f_{K-1}(\mathbf{x})\right)=f_0(\mathbf{x})1+f_1(\mathbf{x})\varepsilon^1+\ldots+f_{K-1}(\mathbf{x})\varepsilon^{K-1} \tag{2}$$

which take values in the so called the retinal multiplet) algebras $Alg_k^{Vis}=Alg_k^{Vis}\left(\mathbf{R}\middle|1,\varepsilon^1,\varepsilon^2,..,\varepsilon^{k-1}\right)$, where $\varepsilon^K=-1,0,+1$ and $1,\varepsilon^1,...,\varepsilon^{K-1}$ are hyperimaginary (multircolor) units with the commutative multiplication rules

$$\varepsilon^s\cdot\varepsilon^r=\varepsilon^r\cdot\varepsilon^s=\begin{cases}\varepsilon^{r\oplus s(\mathrm{mod}\,K)}, & \text{if } \varepsilon^N=+1,\\ \mathrm{Hev}(l-m)\varepsilon^{r\oplus s(\mathrm{mod}\,K)}, & \text{if } \varepsilon^N=0,\\ \mathrm{Sign}(l-m)\varepsilon^{r\oplus s(\mathrm{mod}\,K)}, & \text{if } \varepsilon^N=-1,\end{cases}.$$

where $l\$\ m$ is addition modulo K,

$$\mathrm{Sign}(x)=\begin{cases}+1, & x\geq 0\\ -1, & x<0,\end{cases}\quad \mathrm{Hev}(x)=\begin{cases}+1, & x\geq 0\\ 0, & x<0,\end{cases}$$

are the signum and Heaviside functions.

We interpret a multichannel images in Visual Cortex (VC) as the following hypercomplex–valued signals:

$$\mathbf{f}(\mathbf{x})=\left(f_1(\mathbf{x}),f_2(\mathbf{x}),\ldots,f_K(\mathbf{x})\right)=f_1(\mathbf{x})J_1+f_2(\mathbf{x})J_2+\ldots+f_K(\mathbf{x})J_K, \tag{3}$$

which take values in the Clifford algebra $Alg_{2^K}^{Vis(u,v,w)}\left(\mathbf{R}\middle|1,J_1,J_2,...,J_K\right)=Alg_{2^K}^{Vis(u,v,w)}$, where $J_1,J_2,...,J_K$ are hyperimaginary units with the following non-commutative multiplication rules

$$J_rJ_s=-J_sJ_r,\quad \text{for}\quad s,r=1,2,...,K.$$

$$J_s^2=\begin{cases}-1, & s=1,2,...,u,\\ 0, & s=u+1,u+2,...,u+v,\\ +1, & s=u+v+1,u+v+2,...,u+v+w,\end{cases}$$

where $u+v+w=K$.

In this context, the full machinery of ordinary grey-level signal processing theory can be transposed into multichannel image processing one.

Our hypotheses are (Labunets, 2003):

1. Brain of primates operates with hypercomplex numbers during retinal image processing. In the algebraic approach, each pixel is considered not as a multi–dimensional vector, but as a multi–dimensional (hypercomplex) number. For this reason, we assume that the human retina uses 3-D hypercomplex (triplet) numbers.

2. Brain uses different algebras for Retina and for VC levels. Multichannel images appear on the retina as functions (2) with values in a multiplet K-D algebra (in particular, in K–cycle algebra), where K is the number of spectral channels. But multichannel images in a human VC are functions (3) with values in a K-D Clifford algebra.

3. Visual systems of animals with different evolutionary history use different hypercomplex algebras for color and multicolor image processing.

Algebraic models of perceptual spaces and bichromatic images

2–D bichromatic images

$$\mathbf{f}(x_1,x_2)=\left(f_0(x_1,x_2),\, f_1(x_1,x_2)\right):\mathbf{R}^2\to\mathbf{V}^2$$

have two attributes: spatial and visual 2–D spaces $\mathbf{R}^2$ and $\mathbf{V}^2$, respectively. According to (Doran, 1994; Labunets, 2003), we provide these spaces with the algebraic frame of space $Alg_2^{Sp}\left(\mathbf{R}\,|\,1,I\right)$ and visual $Alg_2^{Vis}\left(\mathbf{R}\,|\,1,J\right)$ algebras of 2–D generalized complex and 2–D bichromatic numbers, respectively, *i.e.*,

$$\begin{aligned}&\mathbf{R}^2\to Alg_2^{Sp}\left(\mathbf{R}\,|\,1,I\right):=\mathbf{R}+\mathbf{R}I=\left\{\mathbf{z}=x_1+Ix_2\,\middle|\,x_1,x_2\in\mathbf{R}\right\},\\&\mathbf{V}^2\to Alg_2^{Vis}\left(\mathbf{R}\,|\,1,J\right):=\mathbf{R}+\mathbf{R}J=\left\{\mathrm{Z}=r+Jg\,\middle|\,r,g\in\mathbf{R}\right\},\end{aligned}\tag{4}$$

where I and J are a spatial and bichromatic visual imaginary units, respectively. These algebras are called the *spatial* and *perceptual bichromatic algebras* of spaces $\mathbf{R}^2$ and $\mathbf{V}^2$, respectively.

There are three spatial algebras

- If $I^2\equiv I_-^2=-1$, then $Alg_2^{Sp}\left(\mathbf{R}\middle|1,I_-\right)=\left\{\mathbf{z}=x+I_-y\middle|x,y\in\mathbf{R};I_-^2=-1\right\}$ is the field of *complex spatial numbers*, where $I_-=i$ is the ordinary imaginary unit.
- If $I^2\equiv I_+^2=+1$, then $Alg_2^{Sp}\left(\mathbf{R}\middle|1,I_+\right)=\left\{\mathbf{z}=x+I_+y\middle|x,y\in\mathbf{R};I_+^2=-1\right\}$ is the ring *of double spatial numbers.* where $I_+=e$ is the ordinary double unit.
- *If* $I^2\equiv I_0^2=0$, *then* $\mathrm{Alg}^{Sp}\left(\mathbf{R}\middle|1,I_0\right)=\left\{\mathbf{z}=x+I_0y\middle|x,y\in\mathbf{R};I_0^2=0\right\}$ *is the ring of dual spatial numbers.* where $I_0\equiv\varepsilon$ is the ordinary dual unit.

There are three perceptual algebras, too:

- If $J^2\equiv J_-^2=-1$, then $Alg_2^{Vis}\left(\mathbf{R}\middle|1,J_-\right)=\left\{\mathrm{Z}=r+J_-g\middle|r,g\in\mathbf{R};J_-^2=-1\right\}$ is the field of *complex bichromatic numbers*, where J_- □ i is similar to the ordinary imaginary unit.
- If $J^2\equiv J_+^2=+1$, then $Alg_2^{Vis}\left(\mathbf{R}\middle|1,J_-\right)=\left\{\mathrm{Z}=r+J_+g\middle|r,g\in\mathbf{R};J_+^2=1\right\}$ *is the ring of double bichromatic numbers*, where J_+ □ e is similar to the ordinary double unit.
- *If* $J^2\equiv J_0^2=0$, *then* $Alg_2^{Vis}\left(\mathbf{R}\middle|1,J_0\right)=\left\{\mathrm{Z}=r+J_0g\middle|r,g\in\mathbf{R};J_0^2=0\right\}$ *is the ring of dual bichromatic numbers*, where J_+ □ e is similar to the ordinary dual unit.

There are nine algebraic models of 2–D bichromatic images $\mathbf{f}(\mathbf{z}):Alg_2^{Sp}\left(\mathbf{R}\,|\,1,I\right)\to Alg_2^{Vis}\left(\mathbf{R}\,|\,1,J\right)$:

${}^{-,-}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_-) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_-)$	${}^{-,0}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_-) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_0)$	${}^{-,+}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_-) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_+)$
${}^{0,-}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_0) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_-)$	${}^{0,0}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_0) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_0)$	${}^{0,+}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_0) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_+)$
${}^{+,-}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_+) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_-)$	${}^{+,0}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_+) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_0)$	${}^{+,+}\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}\,|\,1,I_+) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J_+)$

When one speaks about all six algebras simultaneously, then they are denoted by Alg_2^{Ret} or

$$Alg_2^{\mathrm{Ret}}(\mathbf{R}|1,B) = \begin{cases} Alg_2^{\mathrm{Sp}}(\mathbf{R}|1,B), & B = I, \\ Alg_2^{\mathrm{Vis}}(\mathbf{R}|1,B), & B = J. \end{cases}$$

In Alg_2^{Ret} we introduce conjugation operation, which maps every element $Z = a + Bb$ to the element $\bar{Z} = \overline{a + Bb} = a - Bb$.

Definition 1. *Let* $Z = a + Bb$ *then a quadratic form* $N(Z) := \|Z\| = Z\bar{Z} = a^2 - B^2 b^2$ *is called the pseudonorm of the number* $\mathrm{Z} = a + Bb$*.*

It is easy to check that $N(Z_1 Z_2) = N(Z_1)N(Z_2)$. The arithmetic value of the square root of the norm $|Z| = \sqrt{N(Z)} = \sqrt{Z\bar{Z}}$ is called the *modulus* of the number Z and is considered as a distance to the point Z from the origin.

Now, 2-D algebras $Alg_2^{Sp} \equiv Alg_2^{Sp}(\mathbf{R}\,|\,1,I)$ and $Alg_2^{Vis} \equiv Alg_2^{Vis}(\mathbf{R}\,|\,1,J)$ are easily turned into pseudometric spaces (spatial and perceptual geometries):

$$Alg_2^{Sp}(\mathbf{R}\,|\,1,J) \to \mathbf{Geo}_2^{Sp(s_1,s_2)} = \left\langle \mathbf{R}^2, \rho(\mathbf{z}_1,\mathbf{z}_2) \right\rangle,$$

$$Alg_2^{Vis}(\mathbf{R}\,|\,1,J) \to \mathbf{Geo}_2^{Vis(s_1,s_2)} = \left\langle \mathbf{V}^2, \rho(\mathrm{Z}_1,\mathrm{Z}_2) \right\rangle,$$

if one defines pseudometric

$$\rho(Z_1,Z_2) := \sqrt{(Z_2 - Z_1)\overline{(Z_2 - Z_1)}} = \sqrt{(a_2-a_1)^2 - J^2(b_2-b_1)^2} =$$

$$= \sqrt{s_1(a_2-a_1)^2 + s_2(b_2-b_1)^2} = \begin{cases} \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}, & Z \in Alg_2^{Vis}(\mathbf{R}|B_-), \\ \sqrt{(a_2-a_1)^2 - (b_2-b_1)^2}, & Z \in Alg_2^{Vis}(\mathbf{R}|B_+), \\ |a_2 - a_1|, & Z \in Alg_2^{Vis}(\mathbf{R}|B_0), \end{cases}$$

where $Z_1 = a_1 + Bb_1$, $Z_2 = a_2 + Bb_2$ and two left superscripts (s_1,s_2) in $\mathbf{R}_2^{\mathrm{Ret}(s_1,s_2)}$ denote a signature of the pseudometric ($s_1 = +1$, $s_2 = -1,0,+1$). The algebras $Alg_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B)$ of generalized complex numbers (spatial and bichromatic) are transformed into three 2-D pseudometric spaces $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$ as follows:

- The 2-D Euclidian geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,+)} = \mathbf{R}_2^{\mathrm{Ret}(+,+)} = \left\langle Alg_2^{\mathrm{Ret}}(\mathbf{R}|B_-);\rho \right\rangle$ (spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,+)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,+)}$).
- The 2-D Minkowskian geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,-)} = \mathbf{R}_2^{\mathrm{Ret}(+,-)} = \left\langle Alg_2^{\mathrm{Ret}}(\mathbf{R}|B_-);\rho \right\rangle$ (spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,-)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,-)}$).
- The 2-D Galilean geometry $\mathbf{Geo}_2^{\mathrm{Ret}(+,0)} = \mathbf{R}_2^{\mathrm{Ret}(+,0)} = \left\langle Alg_2^{\mathrm{Ret}}(\mathbf{R}|B_0);\rho \right\rangle$ (spatial $\mathbf{Geo}_2^{\mathrm{Sp}(+,0)}$ and perceptual $\mathbf{Geo}_2^{\mathrm{Vis}(+,0)}$).

Definition 2. *The set of all points in the generalized complex plane* $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$ *satisfying the equation* $|Z|^2 = a^2 - B^2 b^2 = R^2$ *is called the* $\mathbf{Geo}_2^{\mathrm{Ret}(s_1,s_2)}$*-circle of the radius* R *centered at the origin.*

Example 1. Let $Alg_2^{\mathrm{Ret}}(\mathbf{R}\,|\,1,B) \equiv Alg_2^{Sp}(\mathbf{R}\,|\,1,I)$, then there are three types of circles:

- $\mathbf{Geo}_2^{\mathrm{Sp}(+,+)}$ –circle is the classical Euclidean circle (Fig. 1–a),
- $\mathbf{Geo}_2^{\mathrm{Sp}(+,-)}$ –circle is the Minkowskian (hyperbolic) circle (Fig. 1-b) and
- $\mathbf{Geo}_2^{\mathrm{Sp}(+,0)}$ –circle is the Galilean circle (two parallel lines) (Fig. 1-c).

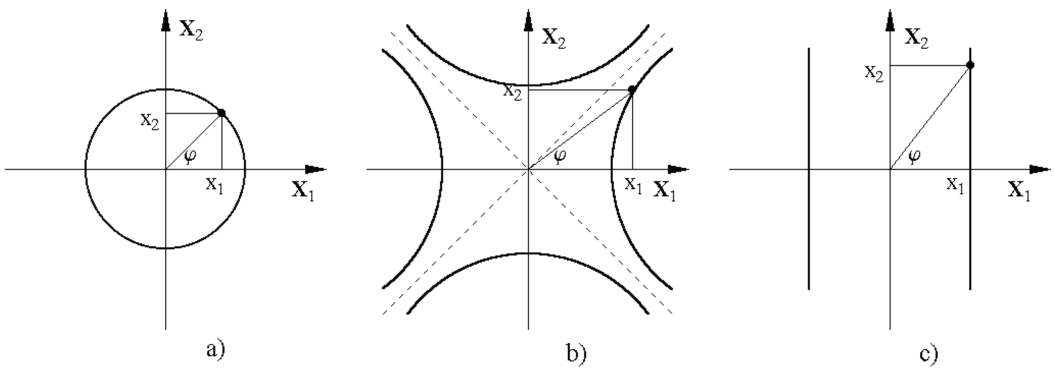


Figure 1: Circles in a) 2–D Euclidean space $\mathbf{Geo}_2^{\mathrm{Sp}(+,+)}$, b) in 2–D Minkowskian space $\mathbf{Geo}_2^{\mathrm{Sp}(+,-)}$ and c) in 2–D Galilean space $\mathbf{Geo}_2^{\mathrm{Sp}(+,0)}$

Let $Z = a + Bb$ be a bichromatic number (spatial or bichromatic), then the number $Z_0 = Z/|Z|$ has the unit modulus if $|Z| = R \neq 0$. It is easily to see, that

$$Z = |Z| \cdot \left(\frac{a}{|Z|} + B\frac{b}{|Z|} \right) = R \cdot (\mathbf{cos}\alpha + B \cdot \mathbf{sin}\alpha) = R \cdot e^{B\theta},$$

where $\mathbf{cos}\alpha$ and $\mathbf{sin}\alpha$ are the Euclidean, Minkowskian (hyperbolic) or Galilean trigonometric functions.

Definition 3. *Bichromatic images* $\mathbf{f}(\mathbf{z}) : Alg_2^{Sp}(\mathbf{R}\,|\,1,I) \to Alg_2^{Vis}(\mathbf{R}\,|\,1,J)$ *are interpreted as* $Alg_2^{Vis}(\mathbf{R}\,|\,J)$*-valued signals of the complex variables* $\mathbf{z} \in Alg_2^{Sp}(\mathbf{R}\,|\,1,I)$:

$$\mathbf{f}(\mathbf{z}) = f_0(x_1 + Ix_2) + J \cdot f_1(x_1 + Ix_2). \qquad (4)$$

Definition 4. *Transformations*

$$\mathbf{z}' = \mathbf{z} + \mathbf{w}, \quad \mathbf{z}' = \lambda\mathbf{z}, \quad \mathbf{z}' = e^{I\varphi_{sp}}\mathbf{z},$$

$$\mathrm{Z}' = \mathrm{Z} + \mathrm{W}, \quad \mathrm{Z}' = \mu\mathrm{Z}, \quad \mathrm{Z}' = e^{J\theta_{ch}}\mathrm{Z},$$

where $\mathbf{z}, \mathbf{z}', \mathbf{w} \in Alg_2^{Sp}(\mathbf{R}\,|\,1,I)$ *and* $\mathrm{Z}, \mathrm{Z}', \mathrm{W} \in Alg_2^{Vis}(\mathbf{R}\,|\,1,J)$ *are called the translations, scalings and rotations of the physical* $\mathbf{Geo}_2^{Sp(s_1,s_2)}$ *and bichromatic* $\mathbf{Geo}_2^{Vis(s_1,s_2)}$ *spaces, respectively.*

They form the following groups:

- *translation spatial* $\mathbf{Tr}\left(\mathbf{Geo}_2^{Sp(s_1,s_2)}\right)$ *and bichromatic* $\mathbf{Tr}\left(\mathbf{Geo}_2^{Vis(s_1,s_2)}\right)$ *groups,*
- *scaling spatial* $\mathbf{Sc}\left(\mathbf{Geo}_2^{Sp(s_1,s_2)}\right)$ *and bichromatic* $\mathbf{Sc}\left(\mathbf{Geo}_2^{Vis(s_1,s_2)}\right)$ *groups,*
- *rotation spatial* $\mathbf{Rot}\left(\mathbf{Geo}_2^{Sp(s_1,s_2)}\right)$ *and bichromatic* $\mathbf{Rot}\left(\mathbf{Geo}_2^{Vis(s_1,s_2)}\right)$ *groups..*

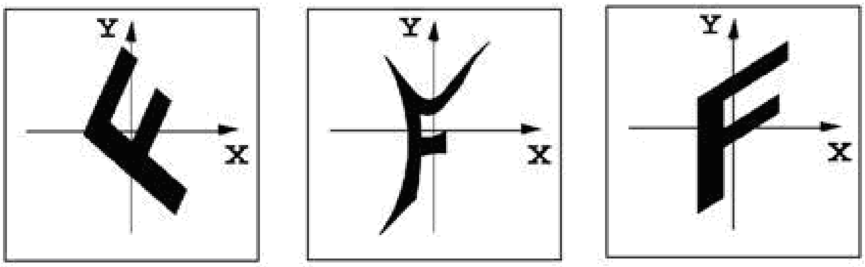


Figure 2: Rotation in a) 2–D Euclidean space $\mathbf{Geo}_2^{Sp(+,+)}$, b) in 2–D Minkowskian space $\mathbf{Geo}_2^{Sp(+,-)}$ and c) in 2–D Galilean space $\mathbf{Geo}_2^{Sp(+,0)}$

Changes in both the physical and perceptual spaces can be treated in the language of spatial and bichromatic algebras as the actions of some space and perceptual transformation groups. These distortions will be caused by

- space transformations (translation $\mathbf{z}'=\mathbf{z}+\mathbf{w}$, rotation $\mathbf{z}'=e^{I\varphi_{sp}}\mathbf{z}$, dilatation $\mathbf{z}'=\lambda\mathbf{z}$) and
- bichromatic transformations (bichromatic translation $\mathbf{f}+\mathrm{W}$, hue transformation $e^{J\theta_{ch}}\mathbf{f}$, and transformation of saturation $\mu\cdot\mathbf{f}$).

If $\mathbf{f}(\mathbf{z})$ are the initial bichromatic 2-D image then

$$^{\mu,\theta_{ch},\mathrm{W}}\mathbf{f}_{\lambda,\varphi_{sp},\mathbf{w}}(\mathbf{z})=\mu e^{J\theta_{ch}}\cdot\mathbf{f}\left(\lambda e^{J\varphi_{sp}}\mathbf{z}+\mathbf{w}\right)+\mathrm{W} \qquad (5)$$

is its spatial and bichromatic distorted version. Spatial distorted by rotation versions of the initial 2–D grey–level image ”F” for different spatial geometries are shown on Fig.2.

Algebraic models of color perceptual spaces and color images

The color retinal images are vector-valued functions $\mathbf{f}(\mathbf{x}):\mathbf{R}^n\to\mathbf{V}_{rgb}^3$, where $\mathbf{V}_{rgb}^3$ is the tri-chromatic (color) RGB-space. We will interpret color images as triplet-valued signals $\mathbf{f}(\mathbf{x})=f_r(\mathbf{x})1+f_g(\mathbf{x})\varepsilon_{col}^1+f_b(\mathbf{x})\varepsilon_{col}^2$ (see Fig. 3) which take values in the *triplet* (in the so called *color*) algebra $Alg_3^{Vis}\left(\mathbf{R}\middle|1,\varepsilon,\varepsilon^2\right):=\mathbf{R}1_{col}+\mathbf{R}\varepsilon_{col}^1+\mathbf{R}\varepsilon_{col}^2$, where $1_{col},\varepsilon_{col}^1,\varepsilon_{col}^2$ are hyperimaginary (color) units, and $\varepsilon_{col}^3=\pm1,0$ [6] We will denote them by $1,\varepsilon^1,\varepsilon^2$. There are three visual (perceptual) algebras.

If $\varepsilon^3=\varepsilon_-^3=-1$, then

$$Alg_3^{Vis}\left(\mathbf{R}\middle|1,\varepsilon_-,\varepsilon_-^2\right):=\mathbf{R}1+\mathbf{R}\varepsilon_-^1+\mathbf{R}\varepsilon_-^2=\left\{\mathrm{C}=r1+g\varepsilon_-^1+b\varepsilon_-^2\middle|r,g,b\in\mathbf{R}\right\} \qquad (6)$$

is the *triplet algebra* of *color acycle numbers*.

If $\varepsilon^3=\varepsilon_+^3=+1$, then

$$Alg_3^{Vis}\left(\mathbf{R}\middle|1,\varepsilon_+,\varepsilon_+^2\right):=\mathbf{R}1+\mathbf{R}\varepsilon_+^1+\mathbf{R}\varepsilon_+^2=\left\{\mathrm{C}=r1+g\varepsilon_+^1+b\varepsilon_+^2\middle|r,g,b\in\mathbf{R}\right\} \qquad (7)$$

is *the triplet algebra* of *color cycle numbers*.

If $\varepsilon^3=\varepsilon_0^3=0$, then

$$Alg_3^{Vis}\left(\mathbf{R}\middle|1,\varepsilon_0,\varepsilon_0^2\right):=\mathbf{R}1+\mathbf{R}\varepsilon_0^1+\mathbf{R}\varepsilon_0^2=\left\{\mathrm{C}=r1+g\varepsilon_0^1+b\varepsilon_0^2\middle|r,g,b\in\mathbf{R}\right\} \qquad (8)$$

is the *triplet nilpotent algebra* of *color nilpotent numbers.*

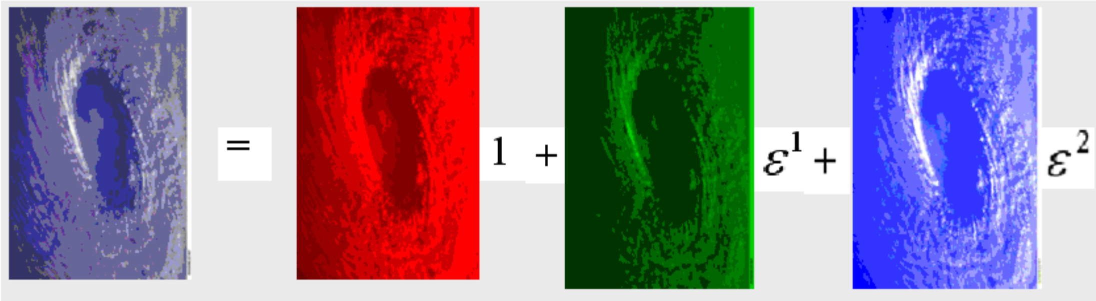


Figure 3: In classical approaches every color pixel is associated to a point of a 3-D color RGB vector space and in the algebraic approach, each pixel is considered as a triplet number. It is called triplet RGB-format

Color cycle numbers of the form $\mathrm{C} = x1 + y\varepsilon + z\varepsilon^2$ $(\varepsilon^3 = 1)$ were considered by Greaves (1847). According to Ch. Greaves, these numbers are called the *triplet numbers*. We shall call them the *color numbers*. The addition and product of two triplet numbers $\mathrm{C}_1 = (r_1 + g_1\varepsilon + b_1\varepsilon^2)$ and $\mathrm{C}_1 = (r_2 + g_2\varepsilon + b_2\varepsilon^2)$ are given by

$$\begin{aligned} \mathrm{C} + \mathrm{C}_2 &= (r_1 + g_1\varepsilon + b_1\varepsilon^2) + (r_2 + g_2\varepsilon + b_2\varepsilon^2) = \\ &= (r_1 + r_2) + (g_1 + g_2)\varepsilon + (b_1 + b_2)\varepsilon^2, \\ \mathrm{C}_1 \cdot \mathrm{C}_2 &= (r_1 + g_1\varepsilon + b_1\varepsilon^2)\cdot(r_2 + g_2\varepsilon + b_2\varepsilon^2) = \\ &= (r_1r_2 + g_1b_2 + b_1g_2) + (r_1g_2 + r_2g_1 + b_1b_2)\varepsilon + (r_1b_2 + g_1g_2 + r_2b_1)\varepsilon^2. \end{aligned} \tag{9}$$

It is easy to see that the triplet product is isomorphic to 3–point cyclic convolution

$$\begin{aligned} \mathrm{C}_1\mathrm{C}_2 &= (r_1 + g_1\varepsilon + b_1\varepsilon^2)\cdot(r_2 + g_2\varepsilon + b_2\varepsilon^2) \equiv \\ &\equiv (r_1, g_1, b_1) * (r_2, g_2, b_2) = \\ &= (r_1r_2 + g_1b_2 + b_1g_2,\ r_1g_2 + r_2g_1 + b_1b_2,\ r_1b_2 + g_1g_2 + r_2b_1). \end{aligned} \tag{10}$$

The *triplet conjugate* of $\mathrm{C} = (r + g\varepsilon + b\varepsilon^2)$ is defined as $\bar{\mathrm{C}} = \overline{r + g\varepsilon + b\varepsilon^2} = r + g\varepsilon^2 + b\varepsilon^1$. The norm $\|\mathrm{C}\|_2$ and modulus $|\mathrm{C}|_2$ are given by

$$\begin{aligned} \|\mathrm{C}\|_2 &= \mathrm{C}\bar{\mathrm{C}} = (r + g\varepsilon + b\varepsilon^2)(r + g\varepsilon^2 + b\varepsilon) = \\ &= (r^2 + g^2 + b^2) - (rg + rb + gb), \\ |\mathrm{C}|_2 &= \sqrt{\|\mathrm{C}\|_2} \equiv \sqrt{\mathrm{C}\bar{\mathrm{C}}} = \sqrt{(r^2 + g^2 + b^2) - (rg + rb + gb)}. \end{aligned} \tag{11}$$

Greaves (1847) showed that each triplet has three norms

$$\begin{aligned} \|\mathrm{C}\|_1 &= |r + g + b|, \\ \|\mathrm{C}\|_2 &= (r^2 + g^2 + b^2) - (rg + rb + gb), \\ \|\mathrm{C}\|_3 &= \|\mathrm{C}\|_1 \|\mathrm{C}\|_2 \equiv r^3 + g^3 + b^3 - 3rgb. \end{aligned} \tag{12}$$

If pseudodistance $\rho(\mathrm{C}, \mathrm{D})$ between two triplet numbers C and D is defined as modulus of their difference $\mathrm{C} - \mathrm{D} = \mathrm{U} = r + g\varepsilon + b\varepsilon^2$:

$$\rho_1(\mathrm{C},\mathrm{D}) = |\mathrm{C}-\mathrm{D}|_1 = |\mathrm{U}|_1 = |r+g+b|,$$
$$\rho_2(\mathrm{C},\mathrm{D}) = |\mathrm{C}-\mathrm{D}|_2 = |\mathrm{U}|_2 = \sqrt[2]{(r^2+g^2+b^2)-(rg+rb+gb)}, \quad (13)$$
$$\rho_3(\mathrm{C},\mathrm{D}) = |\mathrm{C}-\mathrm{D}|_3 = |\mathrm{U}|_3 = \sqrt[3]{r^3+g^3+b^3-3rgb},$$

then the algebra $\mathrm{A}\mathbf{l}\mathrm{g}_3^{Vis}\left(\mathbf{R}|1,\varepsilon,\varepsilon^2\right)$ of triplet numbers is transformed into three 3-D pseudo-metric spaces (color geometries) designed as

$$\mathbf{Geo}_3^{Vis1} = \left\langle\left\langle \mathrm{A}_3\left(\mathbf{R}|1,\varepsilon,\varepsilon^2\right) \mid |r+g+b| \right\rangle\right\rangle,$$
$$\mathbf{Geo}_3^{Vis2} = \left\langle\left\langle \mathrm{A}_3\left(\mathbf{R}|1,\varepsilon,\varepsilon^2\right) \middle| \sqrt[2]{(r^2+g^2+b^2)-(rg+rb+gb)} \right\rangle\right\rangle, \quad (14)$$
$$\mathbf{Geo}_3^{Vis3} = \left\langle\left\langle \mathrm{A}_3\left(\mathbf{R}|1,\varepsilon,\varepsilon^2\right) \middle| \sqrt[3]{r^3+g^3+b^3-3rgb} \right\rangle\right\rangle.$$

Definition 5. *The sets of all points in 3D color geometries* $\mathbf{Geo}_3^{Vis1}, \mathbf{Geo}_3^{Vis2}, \mathbf{Geo}_3^{Vis3}$ *satisfying the equations*

$$\|\mathrm{C}\|_1^1 = r+g+b = R,$$
$$\|\mathrm{C}\|_2^2 = (r^2+g^2+b^2)-(rg+rb+gb) = R, \quad (15)$$
$$\|\mathrm{C}\|_3^3 = \|\mathrm{C}\|_1^1\|\mathrm{C}\|_2^2 = r^3+g^3+b^3-3rgb = R$$

are called $\mathbf{Geo}_3^{Vis1}, \mathbf{Geo}_3^{Vis2}, \mathbf{Geo}_3^{Vis3}$*-spheres of radius* R*, centered at the origin and denoted as* $\mathrm{S}_2^1(R) \in \mathbf{Geo}_3^{Vis1}$, $\mathrm{S}_2^2(R) \in \mathbf{Geo}_3^{Vis2}$, $\mathrm{S}_2^3(R) \in \mathbf{Geo}_3^{Vis3}$ (see, for example, $\mathbf{Geo}_3^{Vis3}$-sphere on Fig.4).

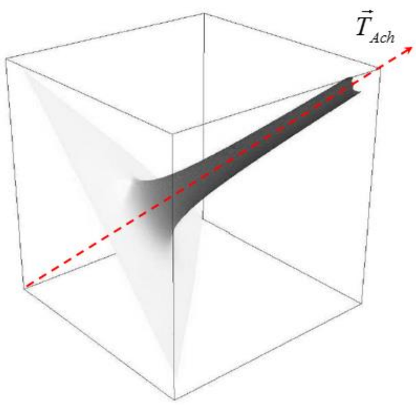


Figure 4: $\mathbf{Geo}_3^{Vis3}$ – sphere $r^3+g^3+b^3-3rgb=1$. (This surface is also called the *Appell sphere* having the achromatic line $\vec{T}_{Ach}$ as axis)

From the above figure one can see, that the line $r=g=b$ is asymptotic axis $\vec{T}_{Ach}$. It is called *achromatic axis*, because it passed through all the achromatic points (*i.e.*, those with $r=g=b$). Obviously, all planes $\pi(a_{lu}):\left\{(r,g,b) \mid \frac{1}{3}r+\frac{1}{3}g+\frac{1}{3}b=a_{lu}\right\}$ are perpendicular to $\vec{T}_{Ach}$, where $a_{lu} \in \mathbf{R}^+$ is a luminance, In the international recommendation for the high definition television standard the following weights for calculating luminance from the red, green and blue components are given: $\pi_{lu}^{TV} : \left\{ (r,g,b) \,|\, 0.2126r + 0.7152g + 0.0722b = a_{lu} \right\}$. This plane is not perpendicular to the achromatic line $\vec{T}_{Ach}$. By this reason the intersection of this plane with the Appell sphere is an ellipse. The regular set of ellipses obtained as intersections the plane $a_{lu} = 0.2126r + 0.7152g + 0.0722b = const$ with regular comb of Appell spheres are MacAdams ellipses (see Fig. 5).

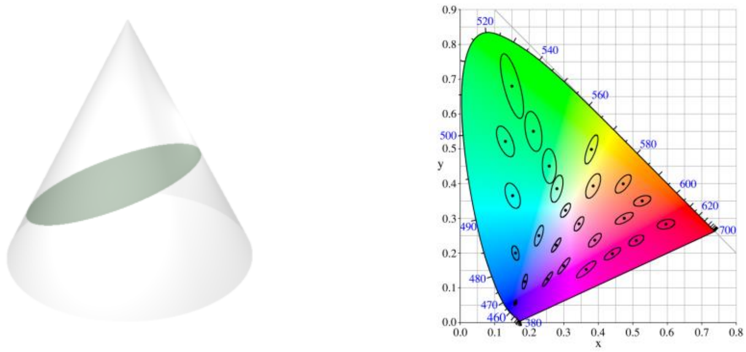


Figure 5: MacAdam ellipses plotted on the CIE 1931 XY-chromaticity diagram.

Greaves gave algebraic and geometrical interpretations of triplet algebra. With geometrical point of view, the color numbers $x + y\varepsilon + z\varepsilon^2$ are points of 3-D color space. With algebraical point of view, the color algebra is the direct sum of the real **R** and complex **C** fields: $Alg_3^{Vis} = \mathbf{R}\cdot \mathrm{e}_{lu} + \mathbf{C}\cdot \mathbf{E}_{ch} = \mathbf{R} \oplus \mathbf{C}$, where $\mathbf{e}_{lu} = (1+\varepsilon+\varepsilon^2)/3$, $\mathbf{E}_{ch} = (1+\omega_3\varepsilon^2+\omega_3^2\varepsilon)/3$ are so called orthogonal «real» and «complex» idempotents ($\mathbf{e}_{lu}^2 = \mathbf{e}_{lu}$, $\mathbf{E}_{ch}^2 = \mathbf{E}_{ch}$, $\mathbf{e}_{lu}\mathbf{E}_{ch} = \mathbf{E}_{ch}\mathbf{e}_{lu} = 0$), respectively, and $\omega_3 := \exp(2\pi/3)$. Therefore, every color number $\mathrm{C} = x + y\varepsilon + z\varepsilon^2$ is a linear combination $\mathrm{C} = a_{lu}\cdot \mathbf{e}_{lu} + Z_{ch}\cdot \mathbf{E}_{ch} = (a_{lu}, Z_{ch})$ of the «real» $a_{lu}\cdot \mathbf{e}_{lu}$ and «complex» parts $Z_{ch}\cdot \mathbf{E}_{ch}$ in the idempotent basis $\{\mathbf{e}_{lu}, \mathbf{E}_{ch}\}$. We will call the real numbers $a_{lu} \in \mathbf{R}$ the luminance (intensity) numbers, and we will call the complex numbers $Z_{ch} \in \mathbf{C}$ the chromaticity numbers. For this reason, we can consider a color image in the two presentations (formats):

$$\mathbf{f}(\mathbf{z}) = f_R(\mathbf{z})1 + f_G(\mathbf{z})\varepsilon + f_B(\mathbf{z})\varepsilon^2,$$
$$\mathbf{f}(\mathbf{z}) = f_{lu}(\mathbf{z})\mathbf{e}_{lu} + \mathbf{f}_{ch}(\mathbf{z})\mathbf{E}_{ch} = \left(f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z}) \right). \quad (16)$$

The first presentation is called the (R,G,B)-format and the second presentation is called the “luminance-chrominance” (LC) format. This format defines every pixel in terms of luminance real-valued (grey-level) part $f_{lu}(\mathbf{z})$ and complex-valued chrominance part $\mathbf{f}_{ch}(\mathbf{z})$, where $|\mathbf{f}_{ch}(\mathbf{z})|$ is saturation and $\mathbf{arg}(\mathbf{f}_{ch}(\mathbf{z}))$ is hue of $\mathbf{f}(\mathbf{z})$. In the second form we have separated the color image into two terms: the *luminance* (intensity) term $f_{lu}(\mathbf{z})$ and the *chromacity term* $\mathbf{f}_{ch}(\mathbf{z})$ (color information), represented on Fig. 6.

Changes in perceptual space of reality such as intensity, color or illumination can be treated in the language of triplet algebra as the action of some transformation groups in the perceptual color space (color algebra) $\mathbf{V}_{RGB}^3 \equiv Alg_3^{Vis}\left(\mathbf{R}\,|\,1,\varepsilon,\varepsilon^2\right)$ (Labunets, 1996; Labunets-Rundblad et al., 2000; Labunets et al., 2002).

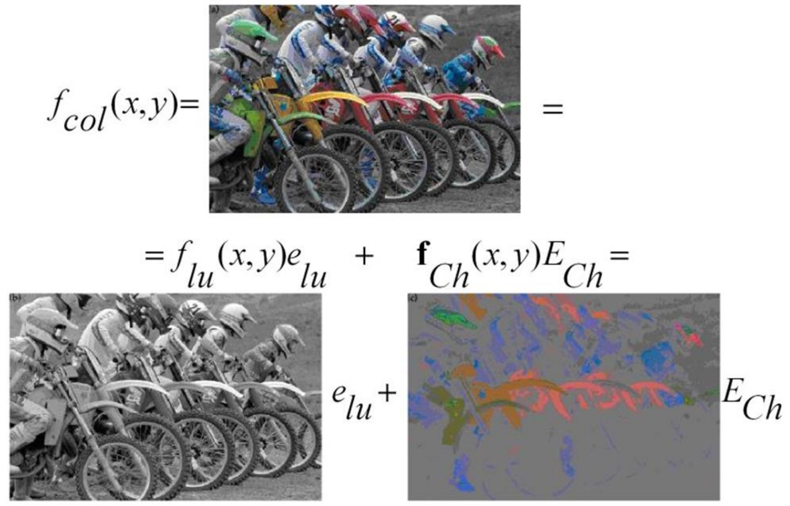


Figure 6: The "luminance-chrominance" (LC)-format

1. For example, if $\mathrm{A}=(a_{lu},Z_{ch})=(a_{lu},|Z_{ch}|e^{i\varphi})$, where $a_{lu}>0$, then the following transformation

$$\mathbf{f}(\mathbf{z})\to\mathrm{A}\cdot\mathbf{f}(\mathbf{z})=(a_{lu},Z_{ch})\cdot(f_{lu}(\mathbf{z}),\mathbf{f}_{ch}(\mathbf{z}))= \\ =(a_{lu},|Z_{ch}|e^{i\varphi})\cdot(f_{lu}(\mathbf{z}),\mathbf{f}_{ch}(\mathbf{z}))=(a_{lu}f_{lu}(\mathbf{z}),|Z_{ch}|e^{i\varphi}\mathbf{f}_{ch}(\mathbf{z})) \quad (17)$$

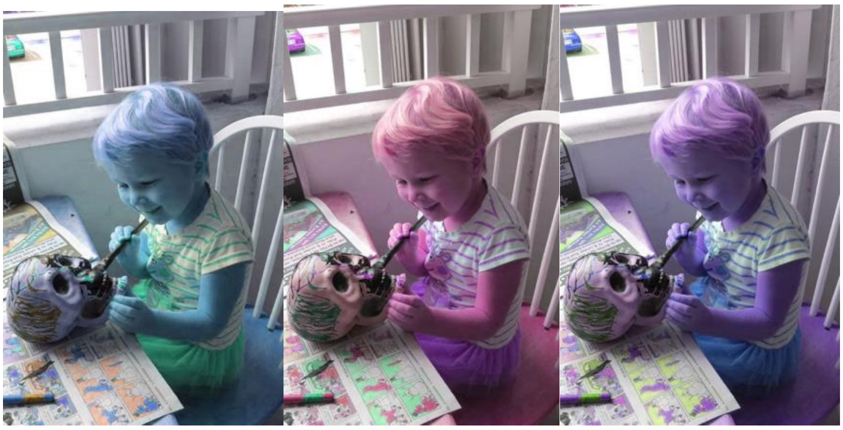
changes luminance, hue and saturation of the initial image. The set of all such transformations forms the *luminance-chromatic group* $\mathbf{LCG}(Alg_3^{Vis}(\mathbf{R}|\varepsilon))=\{(a_{lu},Z_{ch})|(a_{lu}\in\mathrm{R}^{+})\&(Z_{ch}\in\mathbf{C})\}$.

2. Let $\mathrm{A}=(a_{lu},Z_{ch})=(1,e^{i\varphi})$, then the following transformations of color image

$$\mathbf{f}(\mathbf{z})\to\mathrm{A}\cdot\mathbf{f}(\mathbf{z})=(1,e^{i\varphi})\cdot(f_{lu}(\mathbf{z}),\mathbf{f}_{ch}(\mathbf{z}))=(f_{lu}(\mathbf{z}),e^{i\varphi}\mathbf{f}_{ch}(\mathbf{z})) \quad (18)$$

change only hue of the initial image (see Fig. 7). The set of all such transformations forms the *hue orthogonal group* $\mathbf{HOG}(Alg_3^{Vis}(\mathbf{R}|\varepsilon))=\{(1,e^{i\varphi})|e^{i\varphi}\in\mathbf{C}\}$.

a) $\varphi=0$ *b*) $\varphi=\pi/12$ *c*) $\varphi=\pi/6$



$d)\ \varphi = \pi/4$ $e)\ \varphi = -\pi/12$ $f)\ \varphi = -\pi/6$

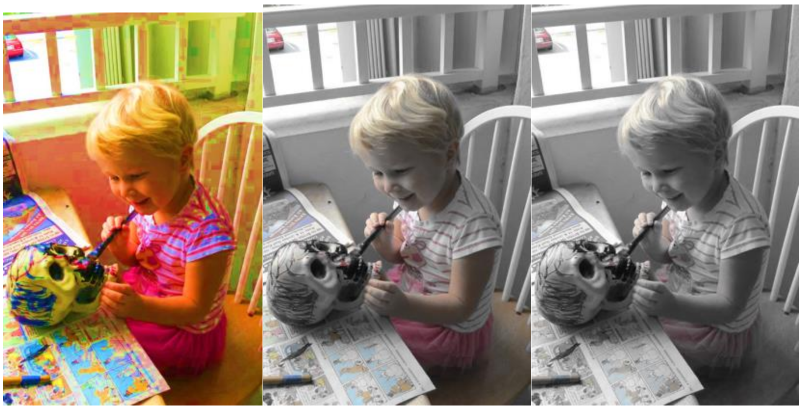
Figure 7: Hue distorted versions of the initial 3–D color image
$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, e^{i\varphi})\cdot(f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z})) = (f_{lu}(\mathbf{z}), e^{i\varphi}\mathbf{f}_{ch}(\mathbf{z}))$ a) initial image "*Yorick*" $(\varphi = 0)$, $b)\ \varphi = \pi/12$, $c)\ \varphi = \pi/6$, $d)\ \varphi = \pi/4$, $e)\ \varphi = -\pi/12$, $f)\ \varphi = -\pi/6$

3. Let now $\mathrm{A} = (1, s)$, $s > 0$, then the following transformations of color image

$$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, s)\cdot(f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z})) = (f_{lu}(\mathbf{z}), s\mathbf{f}_{ch}(\mathbf{z})) \qquad (19)$$

change only saturation of the initial image (see Fig. 8). The set of all such transformations forms the *saturation group* $\mathbf{SaG}\left(Alg_3^{Vis}(\mathbf{R}|\varepsilon_{col})\right) = \left\{(1, s) \,|\, s \in \mathbf{R}^+\right\}$.

$a)\ s = 1$ $b)\ s = 1.3$ $c)\ s = 1.6$



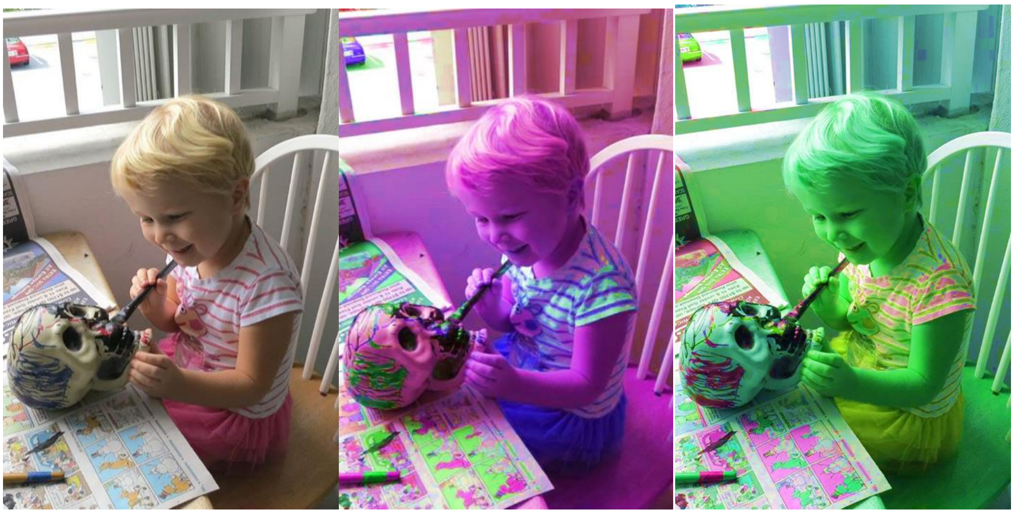
$d)\ s = 2$ $e)\ s = 0.6$ $f)\ s = 0.3$

Figure 8: Saturation distorted versions of the initial 3–D color image
$\mathbf{f}(\mathbf{z}) \to \mathrm{A}\cdot\mathbf{f}(\mathbf{z}) = (1, s)\cdot(f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z})) = (f_{lu}(\mathbf{z}), s\mathbf{f}_{ch}(\mathbf{z}))$, a) initial image "*Yorick*" $(s = 1)$, $b)\ s = 1.3$, $c)\ s = 1.6$ $d)\ s = 2$, $e)\ s = 0.6$, $f)\ s = 0.3$

4. If $A = (1, Z_{ch}) = (1, se^{i\varphi})$, then transformation

$$\mathbf{f}(\mathbf{z}) \to A \cdot \mathbf{f}(\mathbf{z}) = (1, se^{i\varphi}) \cdot (f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z})) = (f_{lu}(\mathbf{z}), se^{i\varphi}\mathbf{f}_{ch}(\mathbf{z})) \quad (20)$$

changes both hue and saturation of the initial image (see Fig. 9). The set of all such transformations forms the *chromatic group* $\mathbf{ChG}\left(Alg_3^{Vis}(\mathbf{R}|\varepsilon_{col})\right) = \left\{(1, se^{i\varphi}) \middle| (e^{i\varphi} \in \mathbf{C}) \& (s \in \mathbf{R}^+)\right\}$.



a) $s = 1,\ \varphi = 0$ *b*) $s = 1.3,\ \varphi = -\pi/12$ *c*) $s = 1.6,\ \varphi = +\pi/12$

Figure 9: Chromatic distorted versions of the initial 3–D color image $\mathbf{f}(\mathbf{z}) \to A \cdot \mathbf{f}(\mathbf{z}) = (1, se^{i\varphi}) \cdot (f_{lu}(\mathbf{z}), \mathbf{f}_{ch}(\mathbf{z})) = (f_{lu}(\mathbf{z}), se^{i\varphi}\mathbf{f}_{ch}(\mathbf{z}))$ a) initial image "*Yorick*" ($s = 1,\ \varphi = 0$), *b*) $s = 1.3,\ \varphi = -\pi/12$, *c*) $s = 1.6,\ \varphi = +\pi/12$

Multiplet algebras for multi-channel image processing

The n-D multichannel images are interpreted as K-D vectors

$$\mathbf{f}(\mathbf{x}) = (f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_{K-1}(\mathbf{x})) : \mathbf{R}^n \to \mathbf{V}^K.$$

We will interpret them as multiplet-valued signals

$$\mathbf{f}(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x})\varepsilon^1 + f_2(\mathbf{x})\varepsilon^2 + .. + f_K(\mathbf{x})\varepsilon^{K-1} \quad (21)$$

which take values in the multiplet algebra $Alg_K^{Vis}(\mathbf{R}|1, \varepsilon, ..., \varepsilon^{K-1}) = \mathbf{R}1 + \mathbf{R}\varepsilon^1 + \ldots + \mathbf{R}\varepsilon^{K-1}$, where $\mathbf{x} \in \mathbf{R}^n$ and $1, \varepsilon^1, \ldots, \varepsilon^{K-1}$ ($\varepsilon^K = +1, 0, -1$) are the *multicolor hyperimaginary units*. Multiplet numbers are represented in its basic form by

$$M = a_0 + a_1\varepsilon^1 + a_2\varepsilon^2 + \ldots + a_{K-1}\varepsilon^{K-1},\ a_i \in \mathbf{R}. \quad (22)$$

with three multiplication rules $\varepsilon^K = +1, 0, -1$ (Labunets et al., 2002; Labunets-Rundblad et al., 2001*a,b*). They form three algebras

$$
\begin{aligned}
Alg_k^{+,Vis}(\mathbf{R}) &\equiv Alg_k^{+,Vis}\left(\mathbf{R}\middle|1,\varepsilon_+^1,\varepsilon_+^2,\ldots,\varepsilon_+^{K-1}\right)=\mathbf{R}+\mathbf{R}\varepsilon_+^1+\mathbf{R}\varepsilon_+^2+\ldots+\mathbf{R}\varepsilon_+^{K-1},\\
Alg_k^{-,Vis}(\mathbf{R}) &\equiv Alg_k^{-,Vis}\left(\mathbf{R}\middle|1,\varepsilon_-^1,\varepsilon_-^2,\ldots,\varepsilon_-^{K-1}\right)=\mathbf{R}+\mathbf{R}\varepsilon_-^1+\mathbf{R}\varepsilon_-^2+\ldots+\mathbf{R}\varepsilon_-^{K-1},\\
Alg_k^{0,Vis}(\mathbf{R}) &\equiv Alg_k^{0,Vis}\left(\mathbf{R}\middle|1,\varepsilon_0^1,\varepsilon_0^2,\ldots,\varepsilon_0^{K-1}\right)=\mathbf{R}+\mathbf{R}\varepsilon_0^1+\mathbf{R}\varepsilon_0^2+\ldots+\mathbf{R}\varepsilon_0^{K-1}.
\end{aligned}
\tag{23}
$$

The addition of muliplet numbers M_1 and M_2 are given by

$$
\begin{aligned}
&\mathrm{M} = \mathrm{M}_1+\mathrm{M}_2 =\\
&=\left(a_0+a_1\varepsilon+a_2\varepsilon^2+\ldots+a_{K-1}\varepsilon^{K-1}\right)+\left(b_0+b_1\varepsilon+b_2\varepsilon^2+\ldots+b_{K-1}\varepsilon^{K-1}\right)=\\
&\left(a_0+b_0\right)+\left(a_1+b_1\right)\varepsilon+\left(a_2+b_2\right)\varepsilon^2+\ldots+\left(a_{K-1}+b_{K-1}\right)\varepsilon^{K-1}.
\end{aligned}
\tag{24}
$$

The product of two muliplet numbers M_1 and M_2 are given by

$$
\begin{aligned}
&\mathrm{M} = \mathrm{M}_1\cdot\mathrm{M}_2 =\\
&=\left(a_0+a_1\varepsilon_-+a_2\varepsilon_-^2+\ldots+a_{K-1}\varepsilon_-^{K-1}\right)\cdot\left(b_0+b_1\varepsilon_-+b_2\varepsilon_-^2+\ldots+b_{K-1}\varepsilon_-^{K-1}\right)=\\
&=\left(\sum_{n=0}^{K-1}a_n\varepsilon_-^n\right)\cdot\left(\sum_{m=0}^{K-1}b_m\varepsilon_-^m\right)=\sum_{m=0}^{K-1}\sum_{n=0}^{K-1}a_nb_m\varepsilon_-^{n+m}=\sum_{l=0}^{K-1}\left(\sum_{m=0}^{K-1}a_{l\$ m}b_m\right)\varepsilon_-^l=\sum_{l=0}^{K-1}c_l\varepsilon_-^l,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\mathrm{M} = \mathrm{M}_1\cdot\mathrm{M}_2 =\\
&=\left(a_0+a_1\varepsilon_++a_2\varepsilon_+^2+\ldots+a_{K-1}\varepsilon_+^{K-1}\right)\cdot\left(b_0+b_1\varepsilon_++b_2\varepsilon_+^2+\ldots+b_{K-1}\varepsilon_+^{K-1}\right)=\\
&=\left(\sum_{n=0}^{K-1}a_n\varepsilon_+^n\right)\cdot\left(\sum_{m=0}^{K-1}b_m\varepsilon_+^m\right)=\sum_{m=0}^{K-1}\sum_{n=0}^{K-1}a_nb_m\varepsilon_+^{n+m}=\sum_{l=0}^{K-1}\left(\sum_{m=0}^{K-1}\mathrm{Sign}(l-m)a_{l\$ m}b_m\right)\varepsilon_+^l=\sum_{l=0}^{K-1}c_l\varepsilon_+^l,\\
&,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\mathrm{M} = \mathrm{M}_1\cdot\mathrm{M}_2 =\\
&=\left(a_0+a_1\varepsilon_0+a_2\varepsilon_0^2+\ldots+a_{K-1}\varepsilon_0^{K-1}\right)\cdot\left(b_0+b_1\varepsilon_0+b_2\varepsilon^2+\ldots+b_{K-1}\varepsilon_0^{K-1}\right)=\\
&=\left(\sum_{n=0}^{K-1}a_n\varepsilon_0^n\right)\cdot\left(\sum_{m=0}^{K-1}b_m\varepsilon_0^m\right)=\sum_{m=0}^{K-1}\sum_{n=0}^{K-1}a_nb_m\varepsilon_0^{n+m}=\sum_{l=0}^{K-1}\left(\sum_{m=0}^{K-1}\mathrm{Hev}(l-m)a_{l\$ m}b_m\right)\varepsilon_0^l=\sum_{l=0}^{K-1}c_l\varepsilon_0^l.
\end{aligned}
\tag{27}
$$

It is easy to see that the multiplet products are isomorphic to the K–point cyclic, acyclic and nilpotent convolutions, respectively,

$$
c_l=\sum_{m=0}^{K-1}a_{l\$ m}b_m,\quad c_l=\sum_{m=0}^{K-1}\mathrm{Sign}(l-m)a_{l\$ m}b_m,\quad c_l=\sum_{m=0}^{K-1}\mathrm{Hev}(l-m)a_{l\$ m}b_{mm}.
\tag{28}
$$

One can show that two algebras $Alg_K^{+,Vis}(\mathbf{R})$ and $Alg_K^{-,Vis}(\mathbf{R})$ are the direct sums of the real and complex fields:

$$
Alg_K^{+,Vis}(\mathbf{R})\equiv\mathbf{R}^{K_{lu}}\oplus\mathbf{C}^{K_{ch}}=
\begin{cases}
\mathbf{R}\cdot\mathbf{e}_{lu}^1+\mathrm{R}\cdot\mathbf{e}_{lu}^2+\sum\limits_{j=1}^{\frac{K}{2}-1}\mathbf{C}\cdot\mathbf{E}_{ch}^j, & \text{if } K \text{ even},\\
\mathbf{R}\cdot\mathbf{e}_{lu}^1+\sum\limits_{j=1}^{\frac{K-1}{2}}\mathbf{C}\cdot\mathbf{E}_{ch}^j, & \text{if } K \text{ odd},
\end{cases}
\tag{29}
$$

$$Alg_K^{-,Vis}(\mathbf{R}) \equiv \mathbf{R}^{K_{lu}} \oplus \mathbf{C}^{K_{ch}} = \begin{cases} \sum_{j=1}^{\frac{K}{2}} \mathbf{C}\cdot\mathbf{E}_{ch}^{j}, & \text{if } K \text{ even}, \\ \mathbf{R}\cdot\mathbf{e}_{lu}^{1} + \sum_{j=1}^{\frac{K-1}{2}} \mathbf{C}\cdot\mathbf{E}_{ch}^{j}, & \text{if } K \text{ odd}, \end{cases} \quad (30)$$

where $\mathbf{e}_{lu}^{i}$ and $\mathbf{E}_{ch}^{j}$ are "real" and "complex" orthogonal idempotents, respectively, such that

$$\left(\mathbf{e}_{lu}^{i}\right)^2 = \mathbf{e}_{lu}^{i}, \quad \left(\mathbf{E}_{ch}^{j}\right)^2 = \mathbf{E}_{ch}^{j}, \quad \mathbf{e}_{lu}^{i}\mathbf{E}_{ch}^{j} = \mathbf{E}_{ch}^{j}\mathbf{e}_{lu}^{i} \quad (31)$$

for all i and j.

Let $K_{lu} = 0,1,2$ and $K_{ch} = \frac{K}{2}, \frac{K}{2}-1, \frac{K-1}{2}$. Every multiplet $\mathrm{M} \in Alg_K^{+,Vis}(\mathbf{R})$ can be represented as a linear combination of K_{lu} «scalar» parts and K_{ch} «complex» parts:

$$\mathrm{M} = \sum_{i=1}^{K_{lu}} \left(a_i \cdot \mathbf{e}_{lu}^{i}\right) + \sum_{j=1}^{K_{ch}} \left(\mathbf{z}_j \cdot \mathbf{E}_{ch}^{j}\right). \quad (32)$$

The real numbers $a_i \in \mathbf{R}$ are called the *multi-intensity ones* and complex numbers $\mathbf{z}_j \in \mathbf{C}$ are called the *multi-chromacity ones*. In such representation two main arithmetic operations have very simple form:

$$\mathrm{M}_1 + \mathrm{M}_2 = \left(\sum_{i=1}^{K_{lu}} a_i \cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} \mathbf{z}_j \cdot \mathbf{E}_{ch}^{j}\right) + \left(\sum_{i=1}^{K_{lu}} b_i \cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} \mathbf{w}_j \cdot \mathbf{E}_{ch}^{j}\right) = \\ = \left(\sum_{i=1}^{K_{lu}} (a_i + b_i)\cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} (\mathbf{z}_j + \mathbf{w}_j)\cdot \mathbf{E}_{ch}^{j}\right), \quad (33)$$

$$\mathrm{M}_1 \cdot \mathrm{M}_2 = \left(\sum_{i=1}^{K_{lu}} a_i \cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} \mathbf{z}_j \cdot \mathbf{E}_{ch}^{j}\right) \cdot \left(\sum_{i=1}^{K_{lu}} b_i \cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} \mathbf{w}_j \cdot \mathbf{E}_{ch}^{j}\right) = \\ = \left(\sum_{i=1}^{K_{lu}} (a_i \cdot b_i)\cdot \mathbf{e}_{lu}^{i} + \sum_{j=1}^{K_{ch}} (\mathbf{z}_j \cdot \mathbf{w}_j)\cdot \mathbf{E}_{ch}^{j}\right), \quad (34)$$

Multiplet algebras possess divisors of zero and form number rings.

Definition 6. *A 2-D multichannel image* $\mathbf{f}(\mathbf{z}): Alg_2^{Sp}(\mathbf{R}) \to Alg_K^{Vis}(\mathbf{R})$ *of the forms*

$$\mathbf{f}(\mathbf{z}) = f_0(\mathbf{z}) + f_1(\mathbf{z})\varepsilon^1 + \ldots + f_{K-1}(\mathbf{z})\varepsilon^{K-1}, \quad (35)$$

and

$$\mathbf{f}(\mathbf{z}) = \sum_{i=1}^{K_{lu}} \left[f_{lu}^{i}(\mathbf{z}) \cdot \mathbf{e}_{lu}^{i} \right] + \sum_{j=1}^{K_{ch}} \left[\mathbf{f}_{ch}^{j}(\mathbf{z}) \cdot \mathbf{E}_{ch}^{j} \right] = \\ = \left(f_{lu}^{1}(\mathbf{z}), f_{lu}^{2}(\mathbf{z}), \ldots, f_{lu}^{K_{lu}}(\mathbf{z});\ \mathbf{f}_{ch}^{1}(\mathbf{z}), \mathbf{f}_{ch}^{2}(\mathbf{z}), \ldots, \mathbf{f}_{ch}^{K_{ch}}(\mathbf{z}) \right) \quad (36)$$

are called a multiplet-valued images in the multiplet and the multiluminance-chrominance (GLC) formats, respectively (Luneburg, 1948, 1950; Labunets, 2003).

The second format defines every pixel in terms of K_{lu} luminance real-valued parts $\left(f_{lu}^{1}(\mathbf{z}), f_{lu}^{2}(\mathbf{z}), \ldots, f_{lu}^{K_{lu}}(\mathbf{z})\right)$ and K_{Ch} chrominance complex-valued parts $\left(\mathbf{f}_{ch}^{1}(\mathbf{z}), \mathbf{f}_{ch}^{2}(\mathbf{z}), \ldots, \mathbf{f}_{ch}^{K_{ch}}(\mathbf{z})\right)$, where $\left|\mathbf{f}_{ch}^{1}(\mathbf{z})\right|,\left|\mathbf{f}_{ch}^{2}(\mathbf{z})\right|,\ldots,\left|\mathbf{f}_{ch}^{K_{ch}}(\mathbf{z})\right|$ are multisaturations and $\arg\left\{\mathbf{f}_{ch}^{1}(\mathbf{z})\right\},\arg\left\{\mathbf{f}_{ch}^{2}(\mathbf{z})\right\},\ldots,\arg\left\{\mathbf{f}_{ch}^{K_{ch}}(\mathbf{x})\right\}$ are multihues of the multichannel image $\mathbf{f}(\mathbf{x})$.

Changes in perceptual space $Alg_{K}^{Vis}(\mathbf{R})$ of multispectral reality such as multi-intensity and multicolor can be treated in the language of multiplet algebra as the action of a multiplet number on an image $\mathbf{f}(\mathbf{z})\rightarrow \mathrm{M}\,:\mathbf{f}(\mathbf{z})$. This transformation change multi-luminancies, multi-hues and multi-saturations of the initial multicolor image. For example, if

$$\mathrm{M}=\left(a_{lu}^{1},a_{lu}^{2},\ldots,a_{lu}^{K_{lu}};\ Z_{ch}^{1},Z_{ch}^{2},..,Z_{ch}^{K_{ch}}\right)= =\left(a_{lu}^{1},a_{lu}^{2},\ldots,a_{lu}^{K_{lu}};\ \left|Z_{ch}^{1}\right|e^{i\varphi_{ch}^{1}},\left|Z_{ch}^{2}\right|e^{i\varphi_{ch}^{2}},..,\left|Z_{ch}^{K_{ch}}\right|e^{i\varphi_{ch}^{K_{ch}}}\right) \quad (37)$$

then the following transformation

$$\mathbf{f}(\mathbf{z})\rightarrow \mathrm{M}\,:\mathbf{f}(\mathbf{z})= =\left(\sum_{i=1}^{K_{lu}}\left[a_{lu}^{i}\cdot\mathbf{e}_{lu}^{i}\right]+\sum_{j=1}^{K_{ch}}\left[\left|Z_{ch}^{j}\right|e^{i\varphi_{ch}^{j}}\cdot\mathbf{E}_{ch}^{j}\right]\right)\cdot\left(\sum_{i=1}^{K_{lu}}\left[f_{lu}^{i}(\mathbf{z})\cdot\mathbf{e}_{lu}^{i}\right]+\sum_{j=1}^{K_{ch}}\left[\mathbf{f}_{ch}^{j}(\mathbf{z})\cdot\mathbf{E}_{ch}^{j}\right]\right)= \quad (38)$$

$$=\left(\sum_{i=1}^{K_{lu}}\left[a_{lu}^{i}f_{lu}^{i}(\mathbf{z})\cdot\mathbf{e}_{lu}^{i}\right]+\sum_{j=1}^{K_{ch}}\left[\left|Z_{ch}^{j}\right|e^{i\varphi_{ch}^{j}}\mathbf{f}_{ch}^{j}(\mathbf{z})\cdot\mathbf{E}_{ch}^{j}\right]\right)$$

changes multiluminancies, multihues and multisaturations of the initial multicomponent image. The set of all such transformations forms the *multiluminance-multichromatic group*

$$\mathbf{MLCG}\left(\mathrm{A}_{k}\left(\mathbf{R}\middle|1,\varepsilon^{1},\varepsilon^{2},\ldots,\varepsilon^{K-1}\right)\right)= =\left\{\left(a_{lu}^{1},a_{lu}^{2},\ldots,a_{lu}^{K_{lu}};\ Z_{ch}^{1},Z_{ch}^{2},..,Z_{ch}^{K_{ch}}\right)\middle|\left(a_{lu}^{1},a_{lu}^{2},\ldots,a_{lu}^{K_{lu}}\in\mathrm{R}^{+}\right)\&\left(Z_{ch}^{1},Z_{ch}^{2},..,Z_{ch}^{K_{ch}}\in\mathbf{C}\right)\right\}. \quad (39)$$

We suppose that the human brain can use hypercomplex algebra for mental changing multiluminancies, multihues and multisaturations of images (for example, in a dream), which are contained in the brain memory on the so-called «screen of mind».

Conclusion

We developed a novel algebraic approach based on hypercomplex algebras to algebraic models of multichannel images. It is our aim to show that the use of hypercomplex algebras fits more naturally to the problems of image processing than the use of color vector spaces does. One can argue that Nature has, through evolution, also learned to utilize properties of hypercomplex numbers. Thus, a brain might have the ability to operate as a hypercomplex algebra computing device.

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