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Layer structure of De Bruijn and Kautz digraphs. An application to deflection routing

J. Fàbrega^{1,2} J. Martí-Farré^{1,3} X. Muñoz^{1,4}

*Departament de Matemàtiques
Universitat Politècnica de Catalunya
Barcelona, Spain*

Abstract

In the main part of this paper we present polynomial expressions for the cardinalities of some sets of interest of the nice distance-layer structure of the well-known De Bruijn and Kautz digraphs. More precisely, given a vertex v , let $S_i^*(v)$ be the set of vertices at distance i from v . We show that $|S_i^*(v)| = d^i - a_{i-1}d^{i-1} - \dots - a_1d - a_0$, where d is the degree of the digraph and the coefficients $a_k \in \{0, 1\}$ are explicitly calculated. Analogously, let w be a vertex adjacent from v such that $S_i^*(v) \cap S_j^*(w) \neq \emptyset$ for some j . We prove that $|S_i^*(v) \cap S_j^*(w)| = d^i - b_{i-1}d^{i-1} - \dots - b_1d - b_0$, where the coefficients $b_t \in \{0, 1\}$ are determined from the coefficients a_k of the polynomial expression of $|S_i^*(v)|$. An application to deflection routing in De Bruijn and Kautz networks serves as motivation for our study. It is worth-mentioning that our analysis can be extended to other families of digraphs on alphabet or to general iterated line digraphs.

Keywords: De Bruijn and Kautz digraphs, General iterated line digraphs, Deflection routing.

1 Motivation

Deflection routing [1] is a routing scheme for bufferless networks based on the idea that if a packet cannot be sent through a certain link due to congestion, it is deflected through any other available link (instead of being buffered in the node queue) and will be rerouted to destination from the node at which the packet arrives. The efficiency of this protocol depends on the decision criteria used to deflect packets when there exists a collision as well as on the network topology. This kind of routing is nowadays interesting in the context of optical networks [7,8] because it is not possible to buffer data without optical to electrical conversion.

In [5] an analytical model for evaluating the performance of deflection routing schemes under different deflection criteria based on Markov chains is proposed. A Markov chain is defined with states $0, 1, \dots, D$, corresponding to the possible distances that a packet may be to its destination, where D stands for the diameter of the network. The transition probabilities in the Markov chain depend on the deflection criteria as well as on the network topology.

If the network topologies under consideration correspond to digraphs on alphabet [4] or, more generally, to some families of iterated line digraphs [6], a careful study of the vertex layer structure can be performed, allowing us to formulate explicit expressions for the deflection probabilities. In this paper we consider the case of De Bruijn and Kautz digraphs $B(d, D)$ and $K(d, D)$ [2] as models for the network topology. In particular, we are interested in the computation of the following two probabilities, which appear in the Markov chain described in [5]:

- *Input probability* $\mathbb{P}_{\text{in}}(i)$: Given a uniform random vertex v , let $\mathbb{P}_{\text{in}}(i)$ be the probability that another uniformly random choosed vertex w , $w \neq v$, is at distance i from v .
- *Deflection probability* $\mathbb{P}_{\text{d}}(i, j)$: If a packet is deflected when it is visiting a vertex at distance i to the destination vertex z , $\mathbb{P}_{\text{d}}(i, j)$ is the probability that the new distance to z after the deflection has occurred is j .

In this paper we present a convenient characterization of the layer structure

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² Email: josep.fabrega@upc.edu

³ Email: jaume.marti@upc.edu

⁴ Email: xavier.munoz@upc.edu

of De Bruijn and Kautz digraphs (Section 3) that is used to compute the given probabilities (Section 4). Because of the lack of space we omit proofs which are technical and tedious as well as algorithmic aspects that will be presented in a full paper.

2 Layer structure of $B(d, D)$ and $K(d, D)$

We will use the well-known sequence representation of the vertices of $B(d, D)$ and $K(d, D)$. Each vertex of $B(d, D)$ corresponds to a sequence $v = v_1v_2 \cdots v_D$, where each element v_k belongs to an alphabet A of d symbols, and vertex v is adjacent to the d vertices $w = v_2 \cdots v_D v_{D+1}$, $v_{D+1} \in A$. Analogously, each vertex of $K(d, D)$ corresponds to a sequence $v = v_1v_2 \cdots v_D$, where now $v_k \neq v_{k+1}$, $1 \leq k < D$, and the base alphabet A has $d + 1$ symbols, $d \geq 2$. Vertex v is adjacent to the d vertices $w = v_2 \cdots v_D v_{D+1}$, $v_{D+1} \in A$, $v_{D+1} \neq v_D$. The digraphs $B(d, D)$ and $K(d, D)$ are d -regular and have diameter D .

In order to describe the layer structure of the vertex set V of these digraphs in a way convenient to compute input and deflection probabilities in the corresponding networks we introduce the following definitions. Given $v \in V$, let $S_i(v)$ be the set of vertices for which there exists a walk from v of length i , $i \geq 0$, and let $S_i^*(v)$ be the set of vertices at distance i from v , $0 \leq i \leq D$. Moreover, for $0 \leq k \leq i$ let $S_{k,i}(v) = S_k(v)$ if $S_k(v) \subset S_i(v)$ and $S_k(v) \not\subset S_j(v) \subset S_i(v)$ for $k < j < i$, and let $S_{k,i}(v) = \emptyset$ otherwise. The next result follows easily.

Proposition 2.1 *Let $v \in V$. Then*

- (i) $|S_i(v)| = d^i$ for $0 \leq i \leq D$.
- (ii) If $G = B(d, D)$ and $i \geq D$, then $S_i(v) = V$.
- (iii) If $G = K(d, D)$ and $i \geq D + 1$, then $S_i(v) = V$.
- (iv) If $k \leq i < D$, then either $S_k(v) \subset S_i(v')$ or $S_k(v) \cap S_i(v') = \emptyset$. Moreover, $S_k(v) \subset S_i(v')$ if and only if $v_{k+1} = v'_{i+1}$, $v_{k+2} = v'_{i+2}$, \dots , $v_{D+k-i} = v'_D$.

The following proposition provides a description of the layers $S_i^*(v)$ and a polynomial expression of its cardinality.

Proposition 2.2 *If $v \in V$, then*

- (i) $S_i^*(v) = S_i(v) \setminus \left(\bigcup_{k=0}^{i-1} S_{k,i}(v) \right)$.
- (ii) $|S_i^*(v)| = d^i - a_{i-1}d^{i-1} - \dots - a_1d - a_0$, where $a_k \in \{0, 1\}$ and $a_k = 1$ if

and only if $S_{k,i}(v) \neq \emptyset$.

In a similar way, we can give a precise polynomial description of $|S_i^*(v) \cap S_j^*(w)|$ when w is a vertex adjacent from v .

Proposition 2.3 *Let w be a vertex adjacent from v such that $S_i^*(v) \cap S_j^*(w) \neq \emptyset$ for some j , $i \leq j < D$. Then $|S_i^*(v) \cap S_j^*(w)| = d^i - b_{i-1}d^{i-1} - \dots - b_1d - b_0$, where the coefficients $b_t \in \{0, 1\}$ are determined from the coefficients a_k of the polynomial expression of $|S_i^*(v)|$. More precisely,*

- (i) *If $a_{i-1} = 1$, then $b_k = a_k$ for all k .*
- (ii) *If $a_{i-1} = 0$ and either $S_{i-1,j}(w) = \emptyset$ or $\emptyset \neq S_{i-1,j}(w) \not\subset S_i(v)$, then $b_k = a_k$ for all k .*
- (iii) *If $a_{i-1} = 0$ and $\emptyset \neq S_{i-1,j}(w) \subset S_i(v)$, then $b_{i-1} = 1$, and for $k < i - 1$ we have: $b_k = 0$ if $a_k = 0$, $b_k = 0$ if $a_k = 1$ and $S_k(v) \subset S_{i-1,j}(w)$, or $b_k = 1$ if $a_k = 1$ and $S_k(v) \not\subset S_{i-1,j}(w)$.*

3 Input and deflection probabilities

In this section we use Propositions 2.2 and 2.3 to compute the input and deflection probabilities in $B(d, D)$ and $K(d, D)$ networks.

Let $V = \mathcal{V}_1 \cup \dots \cup \mathcal{V}_l$ be the partition induced by the equivalence relation defined by $v = v_1v_2\dots v_D \sim v' = v'_1v'_2\dots v'_D$ if and only if there exists a permutation σ of the symbol alphabet A such that $\sigma(v_k) = v'_k$, $1 \leq k \leq D$. The classes \mathcal{V}_r correspond to the different sequence structures of the vertices.

Proposition 3.1

- (i) $|\mathcal{V}_r|$ is easily computed from d and the number s of distinct symbols in the sequence representation of $v \in \mathcal{V}_r$.
- (ii) If n_s is the number of vertex classes \mathcal{V}_r such that $|\mathcal{V}_r| = m_s$, then n_s does not depend on d and it can be computed recursively using the equality $\sum_s n_s m_s = n$.
- (iii) If $v, v' \in \mathcal{V}_r$ then $|S_i^*(v)| = |S_i^*(v')|$.

Given a vertex v selected at random (uniformly) from V , let $\mathbb{P}_{\text{in}}(i)$ be the (input) probability that a uniformly random selected vertex from $V \setminus \{v\}$ is at distance i from v . We have $\mathbb{P}_{\text{in}}(i) = \sum_r \mathbb{P}_{\text{in}}(i \mid v \in \mathcal{V}_r) \mathbb{P}(v \in \mathcal{V}_r)$, where $\mathbb{P}_{\text{in}}(i \mid v \in \mathcal{V}_r) = |S_i^*(v)|/n - 1$ and $\mathbb{P}(v \in \mathcal{V}_r) = |\mathcal{V}_r|/n$. In this way we obtain the following result.

Theorem 3.2 *The input probability $\mathbb{P}_{\text{in}}(i)$ can be expressed as*

$$\mathbb{P}_{\text{in}}(i) = \frac{1}{n(n-1)} \sum_{r=1}^l |\mathcal{V}_r| \left(d^i - a_{i-1}^{(r,i)} d^{i-1} - \dots - a_1^{(r,i)} d - a_0^{(r,i)} \right)$$

where $a_k^{(r,i)} \in \{0, 1\}$. If $v \in \mathcal{V}_r$, then $a_k^{(r,i)} = 1$ if and only if $S_{k,i}(v) \neq \emptyset$.

Notice that $\mathbb{P}_{\text{in}}(i | v \in \mathcal{V}_r) = \Theta(1/d^{D-i})$ independently of the vertex class \mathcal{V}_r , and, hence, $\mathbb{P}_{\text{in}}(i) = \Theta(1/d^{D-i})$.

If a packet is deflected when it is visiting a vertex at distance i to the destination vertex z , the probability $\mathbb{P}_{\text{d}}(i, j)$ that the new distance to z after a deflection has occurred is j can also be calculated as $\mathbb{P}_{\text{d}}(i, j) = \sum_r \mathbb{P}_{\text{d}}(i, j | v \in \mathcal{V}_r) \mathbb{P}(v \in \mathcal{V}_r)$, where $\mathbb{P}_{\text{d}}(i, j | v \in \mathcal{V}_r) = (1/(d-1)) |S_i^*(v) \cap S_j^*(w)| / |S_i^*(v)|$. Finally we have:

Theorem 3.3 *The deflection probabilities $\mathbb{P}_{\text{d}}(i, j)$, $1 \leq i \leq j < D$, are given by*

$$\mathbb{P}_{\text{d}}(i, j) = \frac{1}{n(d-1)} \sum_{r=1}^l c^{(r,i,j)} |\mathcal{V}_r| p^{(r,i,j)}$$

where

$$p^{(r,i,j)} = \frac{d^i - b_{i-1}^{(r,i,j)} d^{i-1} - \dots - b_1^{(r,i,j)} d - b_0^{(r,i,j)}}{d^i - a_{i-1}^{(r,i)} d^{i-1} - \dots - a_1^{(r,i)} d - a_0^{(r,i)}}$$

and $a_k^{(r,i)}, b_k^{(r,i,j)}, c^{(r,i,j)} \in \{0, 1\}$. Moreover, let $v \in \mathcal{V}_r$ and let w be the vertex adjacent from v given by $w = v_2 \dots v_D v_{i+(D-j)}$. Then

- (i) $c^{(r,i,j)} = 0$ if and only if $S_i(v) \subset S_{t,j}(w)$ for some t , $i \leq t < j$.
- (ii) $a_k^{(r,i)} = 1$ if and only if $S_{k,i}(v) \neq \emptyset$ and $b_k^{(r,i,j)}$ is determined from $a_k^{(r,i)}$.

4 Final remarks

The probabilities given in Theorems 3.2 and 3.3 can be used to calculate the efficiency of deflection routing in De Bruijn and Kautz networks by means of the Markov model mentioned in Section 2.1, which can be found in [5].

We emphasize that our analysis of the layer structure of the digraph and the efficiency of deflection routing in the corresponding network topology can be extended to other families of digraphs on alphabet or to general iterated line digraphs such that, for instance, generalized De Bruijn cycles [3].

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