Error Bound of the Multilevel Adaptive Cross Approximation (MLACA)

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***Abstract*—An error bound of the multilevel adaptive cross approximation (MLACA), which is a multilevel version of the adaptive cross approximation-singular value decomposition (ACA-SVD), is rigorously derived. For compressing an off-diagonal submatrix of the method of moments (MoM) impedance matrix with a binary tree, the *L*-level MLACA includes *L*+1 steps, and each step includes 2^L ACA-SVD decompositions. If the relative Frobenius norm error of the ACA-SVD used in the MLACA is smaller than ε, the rigorous proof in this communication shows that the relative Frobenius norm error of the *L*-level MLACA is smaller than $(1+\varepsilon)^{L+1}-1$. In practical applications, the error bound of the MLACA can be approximated as $\varepsilon(L+1)$, because ε is always $\ll 1$. The error upper bound can be used to control the accuracy of the MLACA. To ensure an error of the *L*-level MLACA smaller than ε for different *L*, the ACA-SVD threshold can be set to $(1+\varepsilon)^{\frac{1}{L+1}}-1$, which approximately equals $\varepsilon/(L+1)$ for practical applications.**

***Index Terms*—error bound, method of moments (MoM), low-rank decomposition, multilevel adaptive cross approximation (MLACA).**

I. Introduction

Adaptive cross approximation (ACA)-based algorithms [1]-[9] become more and more popular in solving the method of moments (MoM) [10] impedance matrix equations obtained from discretizing integral equations due to their efficient, adaptive and kernel-independent properties. These algorithms can adaptively and efficiently compress the rank-deficient impedance submatrix related to two well-separated blocks into a product of matrices requiring much less memory. As a result, the computational time and storage cost of the MoM can be significantly decreased.

The conventional ACA algorithm [1], [2] can reduce the computational time and memory requirement of the MoM to $O(N^{4/3}\log N)$ [2] for analyzing moderate electrical size targets with iterative solvers, where N is the number of unknowns. However, for very large targets, the complexities of the conventional ACA become as high as $O(N^3)$ and $O(N^2)$ [2] for CPU time and storage, respectively. The adaptive cross approximation-singular value decomposition (ACA-SVD) [3], [4] can optimize the storage of the ACA by transforming the ACA decomposition into a truncated SVD. Nevertheless, the ACA-SVD does not improve the asymptotic complexity of the conventional ACA. To mitigate this problem, the multilevel adaptive cross approximation (MLACA) [5] has been proposed, which is a multilevel version of the ACA-SVD with the aid of the butterfly algorithm [11]. As a result, it can achieve $O(N^2)$ computational complexity and $O(N\log^2 N)$ storage complexity for very large targets [5].

For compressing an off-diagonal submatrix of the MoM impedance matrix, in the single-level ACA-SVD algorithm, the ACA-SVD decomposition [3], [4] of the impedance submatrix is directly performed. The decomposition error can be easily estimated, as the threshold of the used ACA-SVD. Different from the single-level ACA-SVD algorithm, the *L*-level MLACA [5] algorithm includes *L*+1 steps, and each step involves 2^L ACA-SVD decompositions if a binary tree is used to partition the target under analysis. Since there are $2^L(L+1)$ ACA-SVD decompositions in the *L*-level MLACA, a question is raised here: how large does the error of the MLACA become? In [5], the authors intuitively reason that the maximum error of the *L*-level MLACA in the worst case does not exceed $\varepsilon(L+2)$, where ε is the error of the ACA-SVD used in the MLACA. However, a rigorous derivation of this conclusion is not provided in [5]. In this communication, an upper bound of the relative Frobenius norm [12] error of the MLACA is rigorously derived. The proof shows that the relative Frobenius norm error of the *L*-level MLACA is smaller than $(1+\varepsilon)^{L+1}-1$ when the relative Frobenius norm error of the ACA-SVD is smaller than ε. The error bound of the MLACA can be approximated as $\varepsilon(L+1)$, because ε is always $\ll 1$ in practical applications. Consequently the threshold of the ACA-SVD can be set to $(1+\varepsilon)^{\frac{1}{L+1}}-1 \approx \varepsilon/(L+1)$ to ensure an error of the *L*-level MLACA below ε for various *L*.

II. Formulations

First, the MLACA algorithm is briefly illustrated in Section II-A. Then, in Section II-B, two inequalities and one equality are proved, which are used in the derivation of the error bound of the MLACA. Finally, the bound of the relative Frobenius norm error of the MLACA is derived in detail in Section II-C.

A. MLACA

To apply the MLACA algorithm [5] on the MoM impedance matrix, the geometry of the target under analysis needs to be split hierarchically into blocks at different levels. As in [5], the binary tree is employed here. The impedance submatrices associated with well-separated blocks at the peer level can be efficiently compressed by the MLACA.

We assume that the target is divided into l' levels by the binary tree, where level l' is the finest level, and consider the MoM impedance submatrix $\mathbf{B}^{(0)}$ representing the interactions of the basis functions in two well-separated blocks at level l. The *L*-level MLACA can be used to compress $\mathbf{B}^{(0)}$. It contains

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L+1 steps, where $L = l' - l$. At the pth step ($0 \le p \le L$), the rows of $\mathbf{B}^{(p)}$ are first joined according to the partitioning at Level $l' - p$, and then the columns of $\mathbf{B}^{(p)}$ are split according to the partitioning at Level $l + p$. This operation identifies a total of 2^L new submatrices $\hat{\mathbf{B}}_i^{(p)}$ ($i = 1, 2, 3, \cdots, 2^L$) which are compressed into $\mathbf{A}_i^{(p+1)}\mathbf{B}_i^{(p+1)}$ by the ACA-SVD [3], [4], [9], as shown in Fig. 1. Then, $\mathbf{B}^{(p)}$ can be approximated with

$$\mathbf{B}^{(p)} \approx \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}, \qquad (1)$$

where $\mathbf{A}^{(p+1)}$ is composed of the left singular vectors of the $\hat{\mathbf{B}}_i^{(p)}$ and $\mathbf{B}^{(p+1)}$ includes the products of the singular values and the right singular vectors of the $\hat{\mathbf{B}}_i^{(p)}$. When all L+1 steps of the MLACA are completed, $\mathbf{B}^{(0)}$ is compressed into a product of L+2 sparse matrices as follows

$$\mathbf{B}^{(0)} \approx \mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{A}^{(3)}\cdots\mathbf{A}^{(L+1)}\mathbf{B}^{(L+1)} = \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}, \qquad (2)$$

where $\mathbf{A}^{(p)}$ for $1 \le p \le (L+1)$ and $\mathbf{B}^{(L+1)}$ have 2^L submatrices. All the submatrices of $\mathbf{A}^{(p)}$ are orthogonal matrices.

Fig. 1 shows the pictorial representation of the compression process of $\mathbf{B}^{(0)}$ by the three-level MLACA. Finally, $\mathbf{B}^{(0)}$ is decomposed into the product of $\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{A}^{(3)}\mathbf{A}^{(4)}\mathbf{B}^{(4)}$ as shown in Fig. 2.

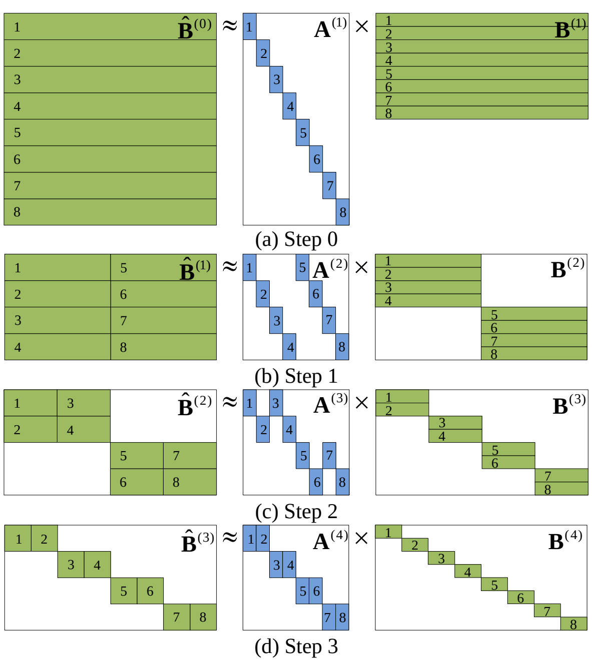

Fig.1. The pictorial representation of the three-level MLACA. (a) Step 0; (b) Step 1; (c) Step 2; (d) Step 3.

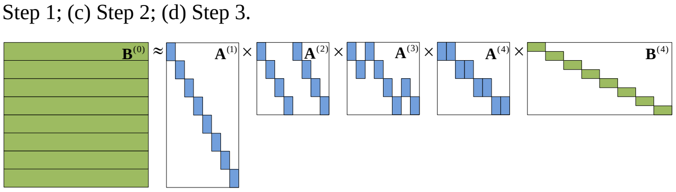

Fig. 2. $\mathbf{B}^{(0)}$ is compressed into the product of $\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{A}^{(3)}\mathbf{A}^{(4)}\mathbf{B}^{(4)}$ by the three-level MLACA.

Assuming that the relative Frobenius norm error of the ACA-SVD used in the MLACA is not larger than ε, the purpose of this communication is to give an upper bound of the relative error (3) of the MLACA for compressing $\mathbf{B}^{(0)}$,

$$\frac{\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F}, \qquad (3)$$

where $\|\bullet\|_F$ stands for the Frobenius norm [12]. The Frobenius norm of $\mathbf{D} \in \mathbb{C}^{m\times n}$ is defined as

$$\|\mathbf{D}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|d_{ij}|^2}, \qquad (4)$$

where d_{ij} is the element in the ith row and jth column of $\mathbf{D}$.

B. Lemmas

In this subsection we prove three lemmas that we need for the MLACA error upper bound derived in the next subsection.

Lemma 1. $\left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F \le \varepsilon\left\|\mathbf{B}^{(p)}\right\|_F$ *for* $0 \le p \le L$.

Proof. At the pth step of the MLACA [5] for $0 \le p \le L$, the ith submatrix of $\hat{\mathbf{B}}^{(p)}$, $\hat{\mathbf{B}}_i^{(p)}$, for $i = 1, 2, 3, \cdots, 2^L$ is compressed by the truncated ACA-SVD [3], [4], [9] into

$$\hat{\mathbf{B}}_i^{(p)} \approx \mathbf{U}_i^{(p)}\mathbf{S}_i^{(p)}\mathbf{V}_i^{(p)}, \qquad (5)$$

where $\mathbf{U}_i^{(p)}$ is the truncated left singular matrix of $\hat{\mathbf{B}}_i^{(p)}$, $\mathbf{V}_i^{(p)}$ is the truncated right singular matrix of $\hat{\mathbf{B}}_i^{(p)}$, and $\mathbf{S}_i^{(p)}$ is a diagonal matrix and contains the retained singular values of $\hat{\mathbf{B}}_i^{(p)}$. $\mathbf{U}_i^{(p)}$ and $\mathbf{V}_i^{(p)}$ are orthogonal matrices.

We recall that we assume the relative Frobenius norm error of (5) to be smaller than ε, thus

$$\frac{\left\|\hat{\mathbf{B}}_i^{(p)} - \mathbf{U}_i^{(p)}\mathbf{S}_i^{(p)}\mathbf{V}_i^{(p)}\right\|_F}{\left\|\hat{\mathbf{B}}_i^{(p)}\right\|_F} \le \varepsilon. \qquad (6)$$

Then, we have

$$\left\|\hat{\mathbf{B}}_i^{(p)} - \mathbf{A}_i^{(p+1)}\mathbf{B}_i^{(p+1)}\right\|_F \le \varepsilon\left\|\hat{\mathbf{B}}_i^{(p)}\right\|_F, \qquad (7)$$

where $\mathbf{A}_i^{(p+1)}$ and $\mathbf{B}_i^{(p+1)}$ denote the ith submatrix of $\mathbf{A}^{(p+1)}$ and $\mathbf{B}^{(p+1)}$, respectively.

$$\mathbf{A}_i^{(p+1)} = \mathbf{U}_i^{(p)}, \qquad (8)$$

$$\mathbf{B}_i^{(p+1)} = \mathbf{S}_i^{(p)}\mathbf{V}_i^{(p)}. \qquad (9)$$

According to the definition of the Frobenius norm [12], the square of the Frobenius norm of a matrix equals the sum of the squares of the Frobenius norm of its submatrices. Thus,

$$\left\|\hat{\mathbf{B}}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F^2 = \sum_{i=1}^{2^L}\left\|\hat{\mathbf{B}}_i^{(p)} - \mathbf{A}_i^{(p+1)}\mathbf{B}_i^{(p+1)}\right\|_F^2. \qquad (10)$$

Substituting (7) into (10) yields

$$\left\|\hat{\mathbf{B}}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F^2 \le \varepsilon^2\sum_{i=1}^{2^L}\left\|\hat{\mathbf{B}}_i^{(p)}\right\|_F^2 = \varepsilon^2\left\|\hat{\mathbf{B}}^{(p)}\right\|_F^2. \qquad (11)$$

Taking the square root of both sides of (11), we find

$$\left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F \le \varepsilon\left\|\mathbf{B}^{(p)}\right\|_F, \qquad (12)$$

where it has been used that $\hat{\mathbf{B}}^{(p)} = \mathbf{B}^{(p)}$.

Lemma 1 states that if the relative Frobenius norm error of the truncated ACA-SVD is smaller than ε, the relative Frobenius norm error of the pth step in the MLACA is also bounded by ε.

Lemma 2. $\left\|\mathbf{A}^{(p)}\mathbf{B}^{(p)}\right\|_F = \left\|\mathbf{B}^{(p)}\right\|_F$ *for* $1 \le p \le (L+1)$.

Proof. The square of $\left\|\mathbf{A}^{(p)}\mathbf{B}^{(p)}\right\|_F$ can be rewritten as

$$\left\|\mathbf{A}^{(p)}\mathbf{B}^{(p)}\right\|_F^2 = \sum_{i=1}^{2^L}\left\|\mathbf{A}_i^{(p)}\mathbf{B}_i^{(p)}\right\|_F^2 . \tag{13}$$

Because $\mathbf{A}_i^{(p)}$ is orthogonal for $i = 1,2,3,\cdots,2^L$, due to the orthogonal invariance of the Frobenius norm [12], we have

$$\left\|\mathbf{A}_i^{(p)}\mathbf{B}_i^{(p)}\right\|_F = \left\|\mathbf{B}_i^{(p)}\right\|_F . \tag{14}$$

Substituting (14) into (13), we obtain

$$\left\|\mathbf{A}^{(p)}\mathbf{B}^{(p)}\right\|_F^2 = \sum_{i=1}^{2^L}\left\|\mathbf{B}_i^{(p)}\right\|_F^2 = \left\|\mathbf{B}^{(p)}\right\|_F^2 . \tag{15}$$

Taking the square root of both sides of (14), we have

$$\left\|\mathbf{A}^{(p)}\mathbf{B}^{(p)}\right\|_F = \left\|\mathbf{B}^{(p)}\right\|_F . \tag{16}$$

Lemma 2 asserts that $\mathbf{A}^{(p)}$ does not change the Frobenius norm of $\mathbf{B}^{(p)}$ when multiplying $\mathbf{B}^{(p)}$ by $\mathbf{A}^{(p)}$, even though $\mathbf{A}^{(p)}$ is not an orthogonal matrix in general.

Furthermore, lemma 2 implies:

Corollary 1.

$$\left\|\mathbf{A}^{(p)}\mathbf{M}\right\|_F = \left\|\mathbf{M}\right\|_F , \tag{17}$$

if the matrix $\mathbf{M}$ has the same sparsity pattern as $\mathbf{B}^{(p)}$.

Lemma 3. $\left\|\mathbf{B}^{(p+1)}\right\|_F \Big/ \left\|\mathbf{B}^{(p)}\right\|_F \le 1+\varepsilon$ *for* $0 \le p \le L$.

Proof. $\left\|\mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F$ can be rewritten as

$$\left\|\mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F = \left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)} - \mathbf{B}^{(p)}\right\|_F . \tag{18}$$

Due to the triangle inequality of a matrix norm [12], we have

$$\left\|\mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F \le \left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F + \left\|\mathbf{B}^{(p)}\right\|_F . \tag{19}$$

With Lemma 1 and 2, this leads to

$$\left\|\mathbf{B}^{(p+1)}\right\|_F \le \varepsilon\left\|\mathbf{B}^{(p)}\right\|_F + \left\|\mathbf{B}^{(p)}\right\|_F . \tag{20}$$

Or equivalently,

$$\frac{\left\|\mathbf{B}^{(p+1)}\right\|_F}{\left\|\mathbf{B}^{(p)}\right\|_F} \le 1+\varepsilon . \tag{21}$$

C. Error Bound of MLACA

In this subsection, the Frobenius norm error bound of the MLACA is derived in detail. First, the numerator of (3) is considered, which can be rewritten as

$$\begin{aligned}
&\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F \\
&= \left\|\mathbf{B}^{(0)} + \sum_{p=1}^{L}\left(\left(\prod_{q=1}^{p}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p)} - \left(\prod_{q=1}^{p}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p)}\right)\right. \\
&\quad \left. - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F \\
&= \left\|\sum_{p=0}^{L}\left(\prod_{q=1}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F
\end{aligned} . \tag{22}$$

Due to the triangle inequality of a matrix norm [12], we have

$$\begin{aligned}
&\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F \\
&\le \sum_{p=0}^{L}\left\|\left(\prod_{q=1}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F
\end{aligned} . \tag{23}$$

From Fig. 2, it can be clearly seen that the sparsity pattern of $\mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}$ is equal to that of $\mathbf{B}^{(p)}$. It is because they are essentially the same matrices. The only difference is that the blocks of $\mathbf{B}^{(p)}$ are vertically split and then compressed by the ACA-SVD to obtain $\mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}$. Furthermore, from Fig. 2, we also can conclude that the sparsity pattern of $\mathbf{A}^{(p+1)}\mathbf{M}$ is equal to that of $\mathbf{B}^{(p)}$, if the matrix $\mathbf{M}$ has the same sparsity pattern as $\mathbf{B}^{(p+1)}$. By recursively applying this conclusion, we can find that both $\left(\prod_{q=n+1}^{p}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p)}$ and $\left(\prod_{q=n+1}^{p+1}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p+1)}$ have the same sparsity pattern as $\mathbf{B}^{(n)}$ for $1 \le n \le p$. Thus,

$$\begin{aligned}
&\left(\prod_{q=n+1}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right) \\
&= \left(\prod_{q=n+1}^{p}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p)} - \left(\prod_{q=n+1}^{p+1}\mathbf{A}^{(q)}\right)\mathbf{B}^{(p+1)}
\end{aligned} \tag{24}$$

also has the same sparsity pattern as $\mathbf{B}^{(n)}$. Due to corollary 1, we find

$$\begin{aligned}
&\left\|\left(\prod_{q=1}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F \\
&= \left\|\left(\prod_{q=2}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F \\
&= \left\|\left(\prod_{q=3}^{p}\mathbf{A}^{(q)}\right)\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F \\
&\cdots \\
&= \left\|\mathbf{A}^{(p)}\left(\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right)\right\|_F \\
&= \left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F
\end{aligned} . \tag{25}$$

Substituting (25) into (23) and using Lemma 1, we obtain

$$\begin{aligned}
\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F &\le \sum_{p=0}^{L}\left\|\mathbf{B}^{(p)} - \mathbf{A}^{(p+1)}\mathbf{B}^{(p+1)}\right\|_F \\
&\le \varepsilon\sum_{p=0}^{L}\left\|\mathbf{B}^{(p)}\right\|_F
\end{aligned} . \tag{26}$$

Then, substituting (26) into (3), the relative Frobenius norm error of the MLACA satisfies

$$\frac{\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1}\mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F} \le \varepsilon\sum_{p=0}^{L}\frac{\left\|\mathbf{B}^{(p)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F} . \tag{27}$$

If $p = 0$, $\left\|\mathbf{B}^{(p)}\right\|_F \Big/ \left\|\mathbf{B}^{(0)}\right\|_F = 1$. If $1 \le p \le L$, $\left\|\mathbf{B}^{(p)}\right\|_F \Big/ \left\|\mathbf{B}^{(0)}\right\|_F$ can be rewritten as

$$\frac{\left\|\mathbf{B}^{(p)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F} = \frac{\left\|\mathbf{B}^{(1)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F}\frac{\left\|\mathbf{B}^{(2)}\right\|_F}{\left\|\mathbf{B}^{(1)}\right\|_F}\cdots\frac{\left\|\mathbf{B}^{(p)}\right\|_F}{\left\|\mathbf{B}^{(p-1)}\right\|_F} = \prod_{q=0}^{p-1}\frac{\left\|\mathbf{B}^{(q+1)}\right\|_F}{\left\|\mathbf{B}^{(q)}\right\|_F} . \tag{28}$$

According to Lemma 3, this yields

$$\frac{\left\|\mathbf{B}^{(p)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F} \le \left(1+\varepsilon\right)^p \tag{29}$$

for $0 \le p \le L$.

Finally, substituting (29) into (27), we obtain

$$\begin{aligned}\frac{\left\|\mathbf{B}^{(0)} - \left(\prod_{p=1}^{L+1} \mathbf{A}^{(p)}\right)\mathbf{B}^{(L+1)}\right\|_F}{\left\|\mathbf{B}^{(0)}\right\|_F} &\le \varepsilon \sum_{p=0}^{L} \left(1+\varepsilon\right)^p . \\ &= \left(1+\varepsilon\right)^{L+1} - 1\end{aligned} \tag{30}$$

Eq. (30) states that for the *L*-level MLACA, which includes $L+1$ steps and compresses $\mathbf{B}^{(0)}$ into a product of $L+2$ sparse matrices, the relative Frobenius norm error is not larger than $(1+\varepsilon)^{L+1}-1$, where ε is the relative Frobenius norm error of the ACA-SVD used in the MLACA. In consequence, if we set the threshold of the ACA-SVD to $(1+\varepsilon)^{\frac{1}{L+1}}-1$, it is guaranteed that the error of the *L*-level MLACA is smaller than ε for any *L*. In practical applications, ε is always $\ll 1$. Thus, we can approximate $(1+\varepsilon)^{L+1}-1$ with $\varepsilon(L+1)$ as the error bound of the MLACA.

III. Conclusion

In this communication, an error upper bound of the MLACA for compressing a MoM impedance submatrix associated with two well-separated blocks has been rigorously derived. If the relative Frobenius norm error of the ACA-SVD used in the MLACA is not larger than ε, the proof shows that the relative Frobenius norm error of the *L*-level MLACA does not exceed $(1+\varepsilon)^{L+1}-1$. The error bound can be approximated as $\varepsilon(L+1)$, because $\varepsilon \ll 1$ for practical applications. Thanks to this error upper bound, the error of the MLACA can be easily controlled.

References

[1] M. Bebendorf, “Approximation of bound element matrices,” *Numer. Math.*, vol. 86, no. 4, pp. 565-589, 2000.

[2] K. Zhao, M. N. Vouvakis, and J.-F. Lee, “The adaptive cross approximation algorithm for accelerated method of moments computations of EMC,” *IEEE Trans. Electromagn. Compat.*, vol. 47, no. 4, pp. 763-773, 2005.

[3] M. Bebendorf and S. Kunis, “Recompression techniques for adaptive cross approximation,” *J. Integ. Equat. Appl.* , vol. 21, no. 3, pp. 331-357, 2009.

[4] A. Heldring, J. M. Rius, J. M. Tamayo, J. Parrón, and E. Ubeda, “Multiscale compressed block decomposition for fast direct solution of method of moments linear system,” *IEEE Trans. Antennas Propag.*, vol. 59, no. 2, pp. 526-536, Feb. 2011.

[5] J. M. Tamayo, A. Heldring, and J. M. Rius, “Multilevel adaptive cross approximation (MLACA),” *IEEE Trans. Antennas Propag.*, vol. 59, no. 12, pp. 4600-4608, Dec. 2011.

[6] A. Heldring, J. M. Tamayo, C. Simon, E. Ubeda, and J. M. Rius, “Sparsified adaptive cross approximation algorithm for accelerated method of moments computations,” *IEEE Trans. Antennas Propag.*, vol. 61, no. 1, pp. 240-246, Jan. 2013.

[7] X. Chen, C. Gu, Z. Niu, and Z. Li, “Fast adaptive cross-sampling scheme for the sparsified adaptive cross approximation,” *IEEE Antennas Wirel. Propag. Lett.*, vol. 13, pp. 1061-1064, 2014.

[8] X. Chen, C. Gu, Z. Li, and Z. Niu, “Sparsified Multilevel Adaptive Cross Approximation,” in *2014 IEEE Asia-Pacific Conference on Antennas and Propagation*, Jul. 2014, pp. 971-973.

[9] X. Chen, C. Gu, J. Ding, Z. Li, and Z. Niu, “Multilevel fast adaptive cross-approximation algorithm with characteristic basis functions,” *IEEE Trans. Antennas Propag.*, vol. 63, no. 9, pp. 3994-4002, Sep. 2015.

[10] W. C. Gibson, *The Method of Moments in Electromagnetics*. Boca Raton, FL: CRC Press, 2007.

[11] E. Michielssen and A. Boag, “A multilevel matrix decomposition algorithm for analyzing scattering from large structures,” *IEEE Trans. Antennas Propag.*, vol. 44, no. 8, pp. 1086-1093, Aug. 1996.

[12] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.