

Roth's solvability criteria for the matrix equations $AX - \widehat{X}B = C$ and $X - A\widehat{X}B = C$ over the skew field of quaternions with an involutive automorphism $q \mapsto \hat{q}$ *

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Abstract

The matrix equation $AX - XB = C$ has a solution if and only if the matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar. This criterion was proved over a field by W.E. Roth (1952) and over the skew field of quaternions by Huang Liping (1996). H.K. Wimmer (1988) proved that the matrix equation $X - AXB = C$ over a field has a solution if and only if the matrices $\begin{bmatrix} A & C \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$ are simultaneously equivalent to $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$. We extend these criteria to the matrix equations $AX - \widehat{X}B = C$ and $X - A\widehat{X}B = C$ over the skew field of quaternions with a fixed involutive automorphism $q \mapsto \hat{q}$.

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1 Introduction

Let \mathbb{H} be the skew field of quaternions with a fixed involutive automorphism $h \mapsto \hat{h}$; that is, a bijection $\mathbb{H} \rightarrow \mathbb{H}$ (possibly, the identity) such that

$$\widehat{\widehat{h + k}} = \widehat{\hat{h} + \hat{k}}, \quad \widehat{\widehat{hk}} = \widehat{\hat{h}\hat{k}}, \quad \widehat{\widehat{\hat{h}}} = h$$

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for all $h, k \in \mathbb{H}$. If $H = [h_{ij}]$ is a quaternion matrix, then we write $\widehat{H} := [\widehat{h}_{ij}]$. We prove two criteria of solvability of quaternion matrix equations (see Theorems 1 and 2):

- $AX - \widehat{X}B = C$ has a solution if and only if $\widehat{S}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for some nonsingular S ;
- $X - A\widehat{X}B = C$ has a solution if and only if $\begin{bmatrix} A & C \\ 0 & I \end{bmatrix} R = \widehat{S} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} R = S \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$ for some nonsingular S and R .

In order to prove them, we represent these quaternion matrix equations by complex matrix equations using the injective homomorphism

$$a + bi + cj + dk \mapsto \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix} \quad (1)$$

of \mathbb{H} to the matrix ring $\mathbb{C}^{2 \times 2}$, and then we use known criteria of solvability of complex matrix equations.

In Sections 1.1 and 1.2, we give a brief exposition of some results on Roth theorems and their generalizations.

1.1 Roth's theorems over a field

Roth [21] proved two criteria of solvability of matrix equations over a field:

$$AX - YB = C \text{ has a solution if and only if } \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are equivalent (i.e., have equal rank),} \quad (2)$$

and

$$AX - XB = C \text{ has a solution if and only if } \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are similar.} \quad (3)$$

The criterion (3) is known as *Roth's removal rule*.

Wimmer [25] (see also Yusun [28]) proved that

$$X - AXB = C \text{ over a field has a solution if and only if } \begin{bmatrix} A & C \\ 0 & I \end{bmatrix} R = S \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} R = S \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \text{ for some nonsingular } S \text{ and } R. \quad (4)$$

The necessities in (2)–(4) are clear; for example, if X is a solution of $AX - XB = C$, then

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX - XB \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix};$$

see also (20). The sufficiencies in (2)–(4) are surprising: if $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent or similar, then the transforming matrices can be taken to be upper triangular.

W.E. Roth proved (2) and (3) using canonical forms. Flanders and Wimmer [5] gave invariant proofs, which are presented in the books [18, Theorem 44.3] and [15, Section 12.5]. Other proofs of Roth’s theorems were given by R. Feinberg (1975), J.K. Baksalary and R. Kala (1979), R. Hartwig (1983), Jiong Sheng Li (1984) A.J.B. Ward (1993, 1999), F. Gerrish and A.J.B Ward (1998, 2000), Yu.A Al’pin and S.N. Il’in (2006), and M. Lin and H.K. Harald (2011). Guralnick [7, 8] and Gustafson [9] extended Roth’s theorems to matrices and sets of matrices over commutative rings. Rosenblum [20] showed that Roth’s theorem is not in general valid for bounded operators on a Hilbert space, but it is valid for selfadjoint operators. Fuhrmann and Helmke [6] pointed out that Roth’s theorem (3) is also about existence of complementary subspaces.

Statements (2)–(4) are the most elegant criteria for existence of solutions of $AX - YB = C$, $AX - XB = C$, and $X - AXB = C$, though one can write each of these matrix equations as a system of linear equations $Mx = c$ and formulate criteria for existence and uniqueness of solutions via M and c ; see [15, Section 12.3] and [14]. However, the obtained conditions are not convenient since the system of linear equations $Mx = c$ is large and can be ill-conditioned. Note that the complex matrix equation $AX - XB = C$ has a unique solution if and only if A and B have no common eigenvalues and the complex matrix equation $X - AXB = C$ has a unique solution if and only if $\lambda\mu \neq 1$ for all eigenvalues λ of A and μ of B ; see [15, Section 12.3].

Dmytryshyn and Kågström [4, Theorem 6.1] extended Roth’s criterions to the systems

$$A_i X_{i'} M_i - N_i X_{i''}^{\varepsilon_i} B_i = C_i, \quad i = 1, \dots, s$$

of matrix equations with unknown matrices X_1, \dots, X_t over a field \mathbb{F} of characteristic not 2, in which all $i', i'' \in \{1, \dots, t\}$ and each $X_{i''}^{\varepsilon_i}$ is either $X_{i''}$, or $X_{i''}^T$, or $X_{i''}^*$ (if $\mathbb{F} = \mathbb{C}$).

Bevis, Hall, and Hartwig [1, 2] proved that

$$AX - \bar{X}B = C \text{ over } \mathbb{C} \text{ (in which } \bar{X} \text{ is the complex conjugate of } X \text{) has a solution if and only if } \bar{S}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ for some nonsingular } S. \quad (5)$$

The theory of equations of the form $AXM - N\bar{X}B = C$ is summarized in Wu and Zhang’s book [26].

1.2 Quaternion matrix equations

The quaternion matrix equations $AX - XB = C$ and $X + AXB = C$ are studied in L. Rodman's book [19, Section 5.11]. Solutions of $AX - XB = C$ are analyzed by Bolotnikov [3]. Liping [16]¹ studies the quaternion matrix equation $AXB + CXD = E$.

Liping [16, Corollary 3] proved that

$$AX - XB = C \text{ over } \mathbb{H} \text{ has a solution if and only if } \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are similar.} \quad (6)$$

Our proofs of Theorems 1 and 2 are close to his proof; we extend (3)–(6) to the matrix equations $AX - \tilde{X}B = C$ and $X - A\tilde{X}B = C$ over \mathbb{H} with a fixed involutive automorphism $q \mapsto \hat{q}$, which can be the identity.

Most authors study matrix equations over \mathbb{H} with the identity automorphism or the involutive automorphism

$$h = a + bi + cj + dk \quad \mapsto \quad \tilde{h} := j^{-1}hj = a - bi + cj - dk. \quad (7)$$

Jiang and Ling [10, Theorem 3.2] proved that $AX - \tilde{X}B = C$ over \mathbb{H} with the automorphism (7) has a solution if and only if $\begin{bmatrix} A_\sigma & C_\sigma \\ 0 & B_\sigma \end{bmatrix}$ and $\begin{bmatrix} A_\sigma & 0 \\ 0 & B_\sigma \end{bmatrix}$ are similar over \mathbb{R} , where

$$A_\sigma := \begin{bmatrix} A_1 & A_2 & -A_3 & A_4 \\ A_2 & -A_1 & -A_4 & -A_3 \\ A_3 & -A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & -A_1 \end{bmatrix} \in \mathbb{R}^{4n \times 4n}$$

is the real representation of a quaternion matrix $A = A_1 + A_2i + A_3j + A_4k \in \mathbb{H}^{n \times n}$. Jiang and Wei [11, 12] obtained expressions for exact solutions of $X - AXB = C$ and $X - A\tilde{X}B = C$ in terms of the coefficients of characteristic polynomials; explicit solutions of these equations were also obtained by Song, Chen, and Liu [22, 23]. Yuan and Liao [27] studied $X - A\tilde{X}B = C$ using the complex representation of quaternion matrices.

1.3 Involutive automorphisms of \mathbb{H}

Klimchuk and Sergeichuk [13, Lemma 1] proved the following lemma.

Lemma 1. *Each nonidentical involutive automorphism of \mathbb{H} has the form*

$$h = a + bi + cj + dk \quad \mapsto \quad i^{-1}hi = a + bi - cj - dk \quad (8)$$

in a suitable set of orthogonal imaginary units $i, j, k \in \mathbb{H}$.

¹Huang Liping also publishes as Liping Huang, Li-Ping Huang, and Li Ping Huang.

Two advantages of the automorphism (8) over (7), which will be used in the next sections, are indicated in [13]:

- If $h \mapsto \hat{h}$ is (8) and $h \in \mathbb{H}$ is represented in the form $h = u + vj$ with $u, v \in \mathbb{C}$, then $\hat{h} = u - vj$; compare with $\tilde{h} = \bar{u} + \bar{v}j$. By Lemma 1, each involutive automorphism has the form

$$a + bi + cj + dk \quad \mapsto \quad a + bi + \varepsilon(cj + dk) \quad (9)$$

for some $\varepsilon \in \{1, -1\}$, up to reselection of the orthogonal imaginary units i, j, k . This admits to study equations over \mathbb{H} with the identity automorphism and with (8) simultaneously; see [13, Section 3] and the proofs of Theorems 1 and 2.

- If $h \mapsto \hat{h}$ is (9), then

each square quaternion matrix is \wedge -similar to a complex matrix (two quaternion matrices A and B are said to be \wedge -similar (10) if $\widehat{S}^{-1}AS = B$ for some nonsingular quaternion matrix S).

A canonical form of a quaternion matrix

- under similarity was given by Wiegmann [24] (see also [29] and [19, Theorem 5.5.3]),
- under \wedge -similarity with $h \mapsto \hat{h}$ defined in (8) was given in [13, Theorem 3],
- under \wedge -similarity with $h \mapsto \hat{h}$ defined in (7) was given by Liping [17, Theorem 3].

The canonical forms (a) and (b) (as distinct from (c)) are complex matrices, which ensures (10).

2 Roth's theorem for the quaternion matrix equation $AX - \widehat{X}B = C$

The following theorem generalizes (3) and (6).

Theorem 1. *Let $h \mapsto \hat{h}$ be an involutive automorphism of the skew field of quaternions. The quaternion matrix equation $AX - \widehat{X}B = C$ has a solution if and only if*

$$\widehat{S}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (11)$$

for some nonsingular S .

Proof. \implies . If X is a solution of $AX - \widehat{X}B = C$, then (11) holds with $S = \begin{bmatrix} I & -\widehat{X} \\ 0 & I \end{bmatrix}$ since

$$\begin{bmatrix} I & -\widehat{X} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX - \widehat{X}B \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

\impliedby . Due to Lemma 1, we suppose that the automorphism $h \mapsto \widehat{h}$ is of the form (9). Let (11) hold.

Case 1: A and B are complex matrices. Write

$$C = C_1 + C_2j, \quad S = S_1 + S_2j,$$

in which C_1, C_2, S_1, S_2 are complex matrices. Then

$$M_1 := \begin{bmatrix} A & C_1 \\ 0 & B \end{bmatrix}, \quad M_2 := \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix}, \quad N := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (12)$$

are complex matrices too, $M := \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = M_1 + M_2j$, and $\widehat{S} = S_1 + \varepsilon S_2j$.

By (9) and (11),

$$(M_1 + M_2j)(S_1 + S_2j) = (S_1 + \varepsilon S_2j)N. \quad (13)$$

Applying to (13) the injective homomorphism (1), we get

$$\begin{bmatrix} M_1 & M_2 \\ -\bar{M}_2 & \bar{M}_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} = \begin{bmatrix} S_1 & \varepsilon S_2 \\ -\varepsilon \bar{S}_2 & \bar{S}_1 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \bar{N} \end{bmatrix}.$$

Then

$$J \begin{bmatrix} M_1 & M_2 \\ -\bar{M}_2 & \bar{M}_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} = J \begin{bmatrix} S_1 & \varepsilon S_2 \\ -\varepsilon \bar{S}_2 & \bar{S}_1 \end{bmatrix} J J \begin{bmatrix} N & 0 \\ 0 & \bar{N} \end{bmatrix}$$

with

$$J := \begin{bmatrix} I & 0 \\ 0 & \varepsilon I \end{bmatrix} \quad (14)$$

gives

$$\begin{bmatrix} M_1 & M_2 \\ -\varepsilon \bar{M}_2 & \varepsilon \bar{M}_1 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \varepsilon \bar{N} \end{bmatrix}.$$

Therefore, the matrices

$$\begin{bmatrix} M_1 & M_2 \\ -\varepsilon \bar{M}_2 & \varepsilon \bar{M}_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} N & 0 \\ 0 & \varepsilon \bar{N} \end{bmatrix}$$

are similar. We substitute (12) obtaining

$$\left[\begin{array}{cc|cc} A & C_1 & 0 & C_2 \\ 0 & B & 0 & 0 \\ \hline 0 & -\varepsilon\bar{C}_2 & \varepsilon\bar{A} & \varepsilon\bar{C}_1 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{cc|cc} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ \hline 0 & 0 & \varepsilon\bar{A} & 0 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right],$$

then apply the similarity transformation given by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad (15)$$

and get the complex matrices

$$\left[\begin{array}{cc|cc} A & 0 & C_1 & C_2 \\ 0 & \varepsilon\bar{A} & -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \\ \hline 0 & 0 & B & 0 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{cc|cc} A & 0 & 0 & 0 \\ 0 & \varepsilon\bar{A} & 0 & 0 \\ \hline 0 & 0 & B & 0 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right],$$

which are similar. By Roth's theorem (3), the complex matrix equation

$$\begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} - \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}$$

has a solution. Equating the (1,1) and (1,2) entries on both the sides, we find

$$AZ_1 - Z_1B = C_1, \quad AZ_2 - \varepsilon Z_2\bar{B} = C_2. \quad (16)$$

Interchanging these equations, taking their complex conjugates, and multiplying them by $\pm\varepsilon$, we obtain

$$-\varepsilon\bar{A}\bar{Z}_2 + \bar{Z}_2B = -\varepsilon\bar{C}_2, \quad \varepsilon\bar{A}\bar{Z}_1 - \varepsilon\bar{Z}_1\bar{B} = \varepsilon\bar{C}_1. \quad (17)$$

Write (16) and (17) in matrix form:

$$\begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} - \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}.$$

Then

$$J \begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} - J \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} J J \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = J \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}$$

with J defined in (14) gives

$$\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} - \begin{bmatrix} Z_1 & \varepsilon Z_2 \\ -\varepsilon \bar{Z}_2 & \bar{Z}_1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\bar{C}_2 & \bar{C}_1 \end{bmatrix}.$$

Due to the homomorphism (1), the quaternion matrix $Z_1 + Z_2j$ is a solution of $AX - \widehat{X}B = C$.

Case 2: A and B are quaternion matrices. Let $X = PYQ$, where P and Q are some nonsingular quaternion matrices and Y is a new unknown matrix. Substituting $X = PYQ$ into $AX - \widehat{X}B = C$, we get

$$APYQ - \widehat{P}\widehat{Y}\widehat{Q}B = C.$$

Multiply it by \widehat{P}^{-1} on the left and by Q^{-1} on the right:

$$\widehat{P}^{-1}AP \cdot Y - \widehat{Y} \cdot \widehat{Q}BQ^{-1} = \widehat{P}^{-1}CQ^{-1}. \quad (18)$$

Choose P and Q such that $\widehat{P}^{-1}AP$ and $\widehat{Q}BQ^{-1}$ are real matrices, which is possible due to (10).

By Case 1, (18) has a solution if

$$\begin{bmatrix} \widehat{P}^{-1}AP & \widehat{P}^{-1}CQ^{-1} \\ 0 & \widehat{Q}BQ^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widehat{P}^{-1}AP & 0 \\ 0 & \widehat{Q}BQ^{-1} \end{bmatrix}$$

are \wedge -similar. These matrices are \wedge -similar since they are equal to

$$\begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & \widehat{Q} \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & \widehat{Q} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix},$$

which are \wedge -similar to $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, which are \wedge -similar by (11). Thus, (18) has a solution, and so $AX - \widehat{X}B = C$ has a solution too. \square

3 Roth's theorem for the quaternion matrix equation $X - A\widehat{X}B = C$

The following theorem is the quaternion version of Wimmer's theorem (4).

Theorem 2. *Let $h \mapsto \widehat{h}$ be an involutive automorphism of the skew field of quaternions. The quaternion matrix equation $X - A\widehat{X}B = C$ has a solution if and only if there exist nonsingular quaternion matrices S and R such that*

$$\begin{bmatrix} A & C \\ 0 & I \end{bmatrix} R = \widehat{S} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} R = S \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}. \quad (19)$$

Proof. \implies . If X is a solution of $X - A\widehat{X}B = C$, then (19) holds with $R = \begin{bmatrix} I & \widehat{X}B \\ 0 & I \end{bmatrix}$ and $S = \begin{bmatrix} I & \widehat{X} \\ 0 & I \end{bmatrix}$ since

$$\begin{aligned} \begin{bmatrix} A & X - A\widehat{X}B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & \widehat{X}B \\ 0 & I \end{bmatrix} &= \begin{bmatrix} A & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \\ \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & \widehat{X}B \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & \widehat{X}B \\ 0 & B \end{bmatrix} = \begin{bmatrix} I & \widehat{X} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}. \end{aligned} \quad (20)$$

\impliedby . Due to Lemma 1, we suppose that the automorphism $h \mapsto \widehat{h}$ is of the form (9). Let (19) hold.

Case 1: A and B are complex matrices. Write

$$C = C_1 + C_2j, \quad S = S_1 + S_2j, \quad R = R_1 + R_2j,$$

where $C_1, C_2, S_1, S_2, R_1, R_2$ are complex matrices. Then

$$M_1 := \begin{bmatrix} A & C_1 \\ 0 & I \end{bmatrix}, \quad M_2 := \begin{bmatrix} 0 & C_2 \\ 0 & 0 \end{bmatrix}, \quad N := \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad L := \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \quad (21)$$

are complex matrices too, $M := \begin{bmatrix} A & C \\ 0 & I \end{bmatrix} = M_1 + M_2j$, and $\widehat{S} = S_1 + \varepsilon S_2j$. By (19), $MR = \widehat{S}N$ and $LR = SL$. Applying to them the injective homomorphism (1), we get

$$\begin{aligned} \begin{bmatrix} M_1 & M_2 \\ -\bar{M}_2 & \bar{M}_1 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ -\bar{R}_2 & \bar{R}_1 \end{bmatrix} &= \begin{bmatrix} S_1 & \varepsilon S_2 \\ -\varepsilon \bar{S}_2 & \bar{S}_1 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \bar{N} \end{bmatrix}, \\ \begin{bmatrix} L & 0 \\ 0 & \bar{L} \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ -\bar{R}_2 & \bar{R}_1 \end{bmatrix} &= \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & \bar{L} \end{bmatrix}. \end{aligned} \quad (22)$$

By the first equation,

$$J \begin{bmatrix} M_1 & M_2 \\ -\bar{M}_2 & \bar{M}_1 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ -\bar{R}_2 & \bar{R}_1 \end{bmatrix} = J \begin{bmatrix} S_1 & \varepsilon S_2 \\ -\varepsilon \bar{S}_2 & \bar{S}_1 \end{bmatrix} J J \begin{bmatrix} N & 0 \\ 0 & \bar{N} \end{bmatrix}$$

with J defined in (14), which gives

$$\begin{bmatrix} M_1 & M_2 \\ -\varepsilon \bar{M}_2 & \varepsilon \bar{M}_1 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ -\bar{R}_2 & \bar{R}_1 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ -\bar{S}_2 & \bar{S}_1 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & \varepsilon \bar{N} \end{bmatrix}.$$

This equation and (22) ensure that the matrices in the pairs

$$\left(\begin{bmatrix} M_1 & M_2 \\ -\varepsilon \bar{M}_2 & \varepsilon \bar{M}_1 \end{bmatrix}, \begin{bmatrix} L & 0 \\ 0 & \bar{L} \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} N & 0 \\ 0 & \varepsilon \bar{N} \end{bmatrix}, \begin{bmatrix} L & 0 \\ 0 & \bar{L} \end{bmatrix} \right)$$

are simultaneously equivalent. Substituting (21), we get

$$\left(\left[\begin{array}{cc|cc} A & C_1 & 0 & C_2 \\ 0 & I & 0 & 0 \\ \hline 0 & -\varepsilon\bar{C}_2 & \varepsilon\bar{A} & \varepsilon\bar{C}_1 \\ 0 & 0 & 0 & \varepsilon I \end{array} \right], \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & \bar{B} \end{array} \right] \right)$$

and

$$\left(\left[\begin{array}{cc|cc} A & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & \varepsilon\bar{A} & 0 \\ 0 & 0 & 0 & \varepsilon I \end{array} \right], \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & \bar{B} \end{array} \right] \right).$$

Multiplying them by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$$

on the left and by (15) on the right, we obtain the pairs

$$\left(\left[\begin{array}{cc|cc} A & 0 & C_1 & C_2 \\ 0 & \varepsilon\bar{A} & -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array} \right], \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & B & 0 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right] \right)$$

and

$$\left(\left[\begin{array}{cc|cc} A & 0 & 0 & 0 \\ 0 & \varepsilon\bar{A} & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array} \right], \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & B & 0 \\ 0 & 0 & 0 & \varepsilon\bar{B} \end{array} \right] \right),$$

whose matrices are simultaneously equivalent. By Wimmer's criterion (4), the complex matrix equation

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}$$

has a solution. Equating the (1,1) and (1,2) entries on both the sides, we find

$$Z_1 - AZ_1B = C_1, \quad Z_2 - \varepsilon AZ_2\bar{B} = C_2. \quad (23)$$

Interchanging these equations, taking their complex conjugates, and multiplying them by $\pm\varepsilon$, we obtain

$$-\varepsilon\bar{Z}_2 + \bar{A}\bar{Z}_2B = -\varepsilon\bar{C}_2, \quad \varepsilon\bar{Z}_1 - \varepsilon\bar{A}\bar{Z}_1\bar{B} = \varepsilon\bar{C}_1. \quad (24)$$

Write (23) and (24) in matrix form:

$$\begin{bmatrix} Z_1 & Z_2 \\ -\varepsilon\bar{Z}_2 & \varepsilon\bar{Z}_1 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ -\varepsilon\bar{Z}_2 & \varepsilon\bar{Z}_1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}.$$

Then

$$J \begin{bmatrix} Z_1 & Z_2 \\ -\varepsilon\bar{Z}_2 & \varepsilon\bar{Z}_1 \end{bmatrix} - J \begin{bmatrix} A & 0 \\ 0 & \varepsilon\bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ -\varepsilon\bar{Z}_2 & \varepsilon\bar{Z}_1 \end{bmatrix} J J \begin{bmatrix} B & 0 \\ 0 & \varepsilon\bar{B} \end{bmatrix} = J \begin{bmatrix} C_1 & C_2 \\ -\varepsilon\bar{C}_2 & \varepsilon\bar{C}_1 \end{bmatrix}$$

with J defined in (14) gives

$$\begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & Z_1 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} Z_1 & \varepsilon Z_2 \\ -\varepsilon\bar{Z}_2 & \bar{Z}_1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \bar{B} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ -\bar{C}_2 & \bar{C}_1 \end{bmatrix}.$$

Due to the homomorphism (1), the quaternion matrix $Z_1 + Z_2j$ is a solution of $X - A\bar{X}B = C$.

Case 2: A and B are quaternion matrices. Let $X = PYQ$, where P and Q are some nonsingular quaternion matrices and Y is a new unknown matrix. Substituting $X = PYQ$ into $X - A\bar{X}B = C$, we get

$$PYQ - A\widehat{P}\widehat{Y}\widehat{Q}B = C.$$

Multiply it by P^{-1} on the left and by Q^{-1} on the right:

$$Y - P^{-1}A\widehat{P} \cdot \widehat{Y} \cdot \widehat{Q}BQ^{-1} = P^{-1}CQ^{-1}. \quad (25)$$

Choose P and Q such that $P^{-1}A\widehat{P}$ and $\widehat{Q}BQ^{-1}$ are complex matrices, which is possible due to (10).

By Case 1, (25) has a solution if

$$\begin{aligned} \begin{bmatrix} P^{-1}A\widehat{P} & P^{-1}CQ^{-1} \\ 0 & I \end{bmatrix} R' &= \widehat{S}' \begin{bmatrix} P^{-1}A\widehat{P} & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & \widehat{Q}BQ^{-1} \end{bmatrix} R' &= S' \begin{bmatrix} I & 0 \\ 0 & \widehat{Q}BQ^{-1} \end{bmatrix} \end{aligned} \quad (26)$$

for some nonsingular quaternion matrices R' and S' . Write these equations in the form:

$$\begin{aligned} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & C \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{P} & 0 \\ 0 & Q^{-1} \end{bmatrix} R' &= \widehat{S}' \begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{P} & 0 \\ 0 & Q^{-1} \end{bmatrix}, \\ \begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & \widehat{Q} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \widehat{P} & 0 \\ 0 & Q^{-1} \end{bmatrix} R' &= S' \begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & \widehat{Q} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \widehat{P} & 0 \\ 0 & Q^{-1} \end{bmatrix}. \end{aligned}$$

Setting

$$R := \begin{bmatrix} \widehat{P} & 0 \\ 0 & Q^{-1} \end{bmatrix} R' \begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & Q \end{bmatrix}, \quad S := \begin{bmatrix} \widehat{P} & 0 \\ 0 & \widehat{Q}^{-1} \end{bmatrix} S' \begin{bmatrix} \widehat{P}^{-1} & 0 \\ 0 & \widehat{Q} \end{bmatrix}, \quad (27)$$

we get (19).

Thus, if the condition (19) holds for some matrices R and S , then we can define R' and S' from (27) and obtain the equalities (26). They ensure the solvability of (25), and so the solvability of $X - A\widehat{X}B = C$. \square

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