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Extreme Weights in Steinhaus Triangles

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Abstract

Let $\{0 = w_0 < w_1 < w_2 < \ldots < w_m\}$ be the set of weights of binary Steinhaus triangles of size n, and let W_i be the set of sequences in $\mathbb{F}_2^n$ that generate triangles of weight w_i. In this paper we obtain the values of w_i and the corresponding sets W_i for $i \in \{2, 3, m\}$, and we conjecture the answer for $i = m - 1$.

Keywords: Steinhaus triangles, boolean Pascal triangles, balanced Steinhaus triangles, weights of triangles.

1 Introduction

Let $\mathbb{F}_2$ be the field of order 2 and $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n$, that we will write $\mathbf{x} = x_0x_1 \ldots x_n$. The *derivative* of $\mathbf{x}$ is the sequence $\partial\mathbf{x} = (x_0 + x_1, x_1 + x_2, \ldots, x_{n-2} + x_{n-1})$. If $\mathbf{y} \in \mathbb{F}_2^{n-1}$, the sequence $\mathbf{x} \in \mathbb{F}_2^n$ such that $\partial\mathbf{x} = \mathbf{y}$ is called the *primitive* of $\mathbf{y}$.

We define $\partial^0\mathbf{x} = \mathbf{x}$, $\partial^1\mathbf{x} = \partial\mathbf{x}$ and $\partial^i\mathbf{x} = \partial\partial^{i-1}\mathbf{x}$, for $2 \leq i \leq n - 1$. The *Steinhaus triangle* of the sequence $\mathbf{x}$ is the sequence $T(\mathbf{x})$ formed by $\mathbf{x}$

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and its derivatives: $T(\mathbf{x}) = (\mathbf{x}, \partial\mathbf{x}, \ldots, \partial^{n-1}\mathbf{x})$. For $i \in \{0, \ldots, n-1\}$, the component $\partial^i\mathbf{x}$ of $T(\mathbf{x})$ is the i-th *row* of the triangle. In Figure 1 we can see a graphical example of $T(0001001)$.

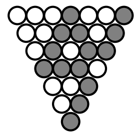

Fig. 1. Steinhaus triangle $T(\mathbf{x})$ of the sequence $\mathbf{x} = 0001001$. The grey and white circles represent ones and zeros, respectively

In 1958, H. Steinhaus [8] asked for which sequences $\mathbf{x} \in \mathbb{F}_2^n$ the triangle $T(\mathbf{x})$ is balanced, that is, $T(\mathbf{x})$ has as many zeroes as ones. He observed that no sequence of length $n \equiv 1, 2 \pmod 4$ produces a balanced triangle, so the problem was to decide if they exist for lengths $n \equiv 0, 3 \pmod 4$. H. Harborth [6] answered the question in the affirmative by constructing examples of such sequences. S. Eliahou et al. studied binary sequences generating balanced triangles with some additional condition: sequences of length n, all of whose initial segments of length $n - 4t$ for $0 \leq t \leq n/4$ generate balanced triangles [3], symmetric and anti-symmetric sequences [4], and sequences with zero sum [5]. F.M. Malyshev and E.V. Kutyreva [7] estimated the number of Steinhaus triangles (which they call Boolean Pascal triangles) of sufficiently large size n containing a given number $\omega \leq kn$ $(k > 0)$ of ones. Steinhaus triangles appear also in the context of cellular automata, and there A. Barbé [1] has studied some properties related to symmetries. In [2] we characterized Steinhaus triangles with rotational and dihedral symmetry.

The *weight* of a sequence $\mathbf{x} = x_0 \ldots x_{n-1}$ is the number $|\mathbf{x}|$ of ones the sequence contains. The *weight* $|T(\mathbf{x})|$ of the triangle $T(\mathbf{x})$ is $|\mathbf{x}| + |\partial\mathbf{x}| + \ldots + |\partial^{n-1}\mathbf{x}|$, the sum of the weights of its rows.

The set $S(n)$ of Steinhaus triangles of size n is an $\mathbb{F}_2$-vector space and the mapping $T\colon \mathbb{F}_2^n \to S(n)$ defined by $\mathbf{x} \mapsto T(\mathbf{x})$ is an isomorphism. Then, the vector space $S(n)$ can be seen as a linear code of length $n(n+1)/2$ and dimension n. In general, it is difficult to find the weight distribution of the words of a linear code; in particular, this seems to be the case for the code $S(n)$. Here, we focus in the smallest and largest values of the weight distribution of $S(n)$. To be precise, let $0 = w_0 < w_1 < w_2 < \ldots < w_{m-1} < w_m$ be all the weights of the triangles of $S(n)$. For $i \in \{0, \ldots, m\}$, an *i-sequence* is a sequence $\mathbf{x}$ such that $|T(\mathbf{x})| = w_i$. We denote by W_i the set of i-sequences. Obviously, it exists only one triangle with weight 0, generated by the sequence

Fig. 2. The sequences $r(\mathbf{x})$, $\ell(\mathbf{x})$ and $i(\mathbf{x})$ for $\mathbf{x} = 0001001$.

$00\ldots0$. H. Harborth [6] observed that $w_1 = n$ and $w_m = \lceil n(n+1)/3 \rceil$, and gave sequences reaching these values. The other weights are not know. Our goal is to determine w_2, w_3 and w_{m-1}, and the corresponding sets W_2, W_3 and W_{m-1}. Moreover, we determine exactly the set W_m. In this moment we have only a conjecture about de values of w_{m-1} and the elements in W_{m-1} that we present at the end.

If we apply to the graphical representation of a Steinhaus triangle a rotation of 120 degrees, of 240 degrees or a symmetry respect to the height of the inferior vertex, we obtain a new triangles with the same weight. We denote by r, l and i these three movements, and they generate the dihedral group D_6 that acts on $\mathbb{F}_2^n$ as follows (see Figure 2): given $\mathbf{x}$, $r(\mathbf{x})$ is the right-side sequence of the Steinhaus triangle $T(\mathbf{x})$, $l(\mathbf{x})$ is the left-side sequence of $T(\mathbf{x})$, and $i(\mathbf{x})$ is obtained reading $\mathbf{x}$ from right to left.

In relation of the notation here, we shall use a dot to represent concatenation of two sequences, thus, $101 \cdot 01 = 10101$. The expression $\overline{x_1 x_2 \ldots x_p}$ stands for the infinite sequence obtained by repeating $x_1 x_2 \ldots x_p$, and $\overline{x_1 x_2 \ldots x_p}[n]$ is the sequence formed by the first n entries of $\overline{x_1 x_2 \ldots x_p}$.

In this paper we determine the values of w_i and sets W_i, for $i = 1, 2, m$, and we conjecture the case $i = m-1$. We state the results only for sufficiently large n. For the smaller values of n we have also the results but we omit them because of length restrictions. Most proofs are by induction, following different restrictions and tricks in each case.

2 2-sequences

To find the 2-sequences we considerer the primitives of the sequences of weight w_1 and we define

$$\mathbf{b}_1 = \overline{10}[n],\ \mathbf{b}_2 = 01 \cdot \overline{0}[n-2],\ \mathbf{b}_3 = \overline{0}[n-2] \cdot 11,$$
$$\mathbf{b}_4 = \overline{01}[n],\ \mathbf{b}_5 = 11 \cdot \overline{0}[n-2],\ \mathbf{b}_6 = \overline{0}[n-2] \cdot 10.$$

As we can see in Figure 3, there are one or two equivalence classes with respect to the action of D_6 on $\mathbb{F}_2^n$, depending on the parity of n.

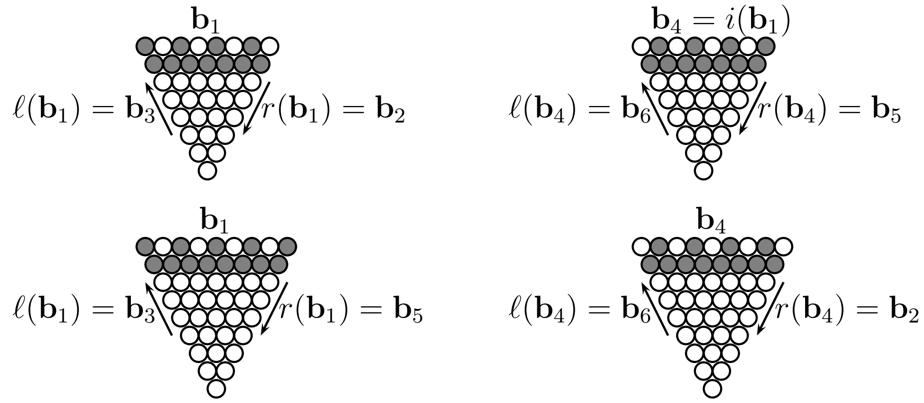

Fig. 3. The sequences $\mathbf{b}_i$ for $n = 8$ in the top (case n even), and $n = 9$ in the bottom (case n odd).

Theorem 2.1 *Let $n \geq 8$. Then $w_2 = \lfloor 3n/2 \rfloor - 1$ and the set W_2 is the following.*

(i) *If n is even, $W_2 = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6\}$.*

(ii) *If n is odd, $W_2 = \{\mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_6\}$.*

3 3-sequences

From the results of the previous section we can solve easily the problem of determining w_3 and W_3 when n is odd.

Theorem 3.1 *If $n \geq 7$ is odd, then $w_3 = \lfloor 3n/2 \rfloor$ and $W_3 = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$.*

From now on in this section we study the case when n is even. As in the other section, we consider the primitives of the sequences of weight w_2 in defining

$$\mathbf{c}_1 = \overline{0011}[n],\ \mathbf{c}_2 = 101 \cdot \overline{0}[n-3],\ \mathbf{c}_3 = \overline{0}[n-3] \cdot 100,$$
$$\mathbf{c}_4 = \overline{1100}[n],\ \mathbf{c}_5 = 001 \cdot \overline{0}[n-3],\ \mathbf{c}_6 = \overline{0}[n-3] \cdot 101.$$

Theorem 3.2 *Let $n \geq 10$ be an even integer.*

(i) *If $n \equiv 0 \pmod 4$, then $w_3 = 2n - 3$ and $W_3 = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6\}$.*

(ii) *If $n \equiv 2 \pmod 4$, then $w_3 = 2n - 4$ and $W_3 = \{\mathbf{c}_1, \mathbf{c}_3, \mathbf{c}_5\}$.*

In order to prove the last theorem we need to calculate the value or some bound of the weight of the triangles generated by the canonical basis of $\mathbb{F}_2^n$.

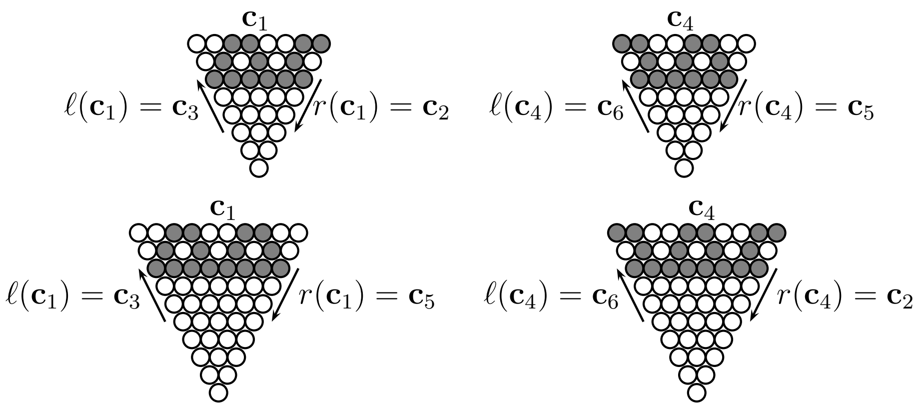

Fig. 4. The sequences $\mathbf{c}_i$ for $n = 8 \equiv 0 \pmod 4$ and $n = 10 \equiv 2 \pmod 4$.

Consider the vectors of the canonical basis of $\mathbb{F}_2^n$: $\mathbf{e}_0 = 1 \cdot \overline{0}[n-1]$; $\mathbf{e}_k = \overline{0}[k] \cdot 1 \cdot \overline{0}[n-k-1]$, $1 \le k \le (n-1)/2$.

Proposition 3.3 *Let n be the length of the considered sequences.*

(i) $|T(\mathbf{e}_0)| = n$ *and* $|T(\mathbf{e}_1)| = \lfloor (3n-2)/2 \rfloor$.

(ii) *If* $n \equiv 2 \pmod 4$, *then* $|T(\mathbf{e}_2)| = 2n-4$; *otherwise the weight is* $2n-3$.

(iii) *If* $n \equiv 3 \pmod 4$, *then* $|T(\mathbf{e}_3)| = 9(n-3)/4$; *otherwise the weight is* $\lfloor (9n-20)/4 \rfloor$.

(iv) *If* $n \ge 9$ *and* $4 \le k \le (n-1)/2$, *then* $|T(\mathbf{e}_k^{(n)})| \ge 2n-3$.

4 m-sequences

For $n \ge 2$, define

$$\mathbf{z}_1 = \overline{110}[n], \quad \mathbf{z}_2 = \overline{011}[n], \quad \mathbf{z}_3 = \overline{101}[n].$$

Theorem 4.1 *Let $n \ge 2$. The maximum weight of the Steinhaus triangles is $w_m = \lceil n(n+1)/3 \rceil$. Moreover,*

(i) *if* $n \equiv 0, 2 \pmod 3$, *then* $W_m = \{\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}\}$;

(ii) *if* $n \equiv 1 \pmod 3$, *then* $W_m = \{\mathbf{z_1}, \mathbf{z_3}\}$.

5 $(m-1)$-sequences

Theorem 5.1 *Let $n \ge 5$ and $n \equiv 1 \pmod 3$. Then $w_{m-1} = w_m - 1$ and $W_{m-1} = \{\mathbf{z}_2\}$.*

In this moment we have not a proof for the cases $n \equiv 0, 2 \pmod 3$. We have some evidences about the result and we thing we are very close to prove it.

Conjecture 5.2 *Let* $n \geq 11$. *Then* $w_{m-1} = w_m - \lfloor n/3 \rfloor$.

(i) *If* $n \equiv 2 \pmod 3$ *then* W_{m-1} *is the equivalence class of* $\overline{100}[n]$ *by the action of the dihedral group* D_6.

(ii) *If* $n \equiv 0 \pmod 3$, *then* W_{m-1} *is union of the equivalence class of* $\overline{100}[n]$ *and* $\overline{010}[n]$ *by the action of the dihedral group* D_6.

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