SULLIVAN MINIMAL MODELS OF OPERAD ALGEBRAS

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ABSTRACT. We prove the existence of Sullivan minimal models of operad algebras, for a quite wide family of operads in the category of complexes of vector spaces over a field of characteristic zero. Our construction is an adaptation of Sullivan's original step by step construction to the setting of operad algebras. The family of operads that we consider includes all operads concentrated in degree 0 as well as their minimal models. In particular, this gives Sullivan minimal models for algebras over Com, Ass and Lie, as well as over their minimal models Com_{∞} , Ass_{∞} and Lie_{∞} . Other interesting operads, such as the operad Ger encoding Ger encoding Ger algebras, also fit in our study.

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1. Introduction

The classical construction of Sullivan minimal models of commutative differential graded algebras over a field \mathbf{k} of characteristic zero, is done step by step by a process of "attaching cells", called KS-extensions (from Koszul-Sullivan) or $Hirsch\ extensions$. The data of these KS-extensions is encoded in a graded vector space together with a linear differential, whereas the multiplication of the algebra comes for free, thanks to the notion of free algebra. With this in mind, it is natural to ask whether the cell attachment construction can be extrapolated to a more general context. An obvious candidate is the category of P-algebras, where P is an operad in the category of complexes of \mathbf{k} -vector spaces, where both complexes and free P-algebras are available.

While P-algebras can behave very badly, in the sense that operations with negative degrees can undo the work of previous steps in a cell attachment procedure, many interesting operads given in nature (i.e. geometry, topology and physics) behave badly, but in a somewhat tame way that we precise here:

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Let P be an operad in cochain complexes of **k**-vector spaces. We will always assume that P is connected $(P(1) = \mathbf{k})$ and that it is either reduced (P(0) = 0) or unitary $(P(0) = \mathbf{k})$. Let $r \ge 0$ be an integer. We say that P is r-tame if for all $n \ge 2$, we have that

$$P(n)^q = 0 \text{ for all } q \le (1-n)(1+r) .$$

Note that r-tame implies (r+1)-tame. Examples of 0-tame operads are: the three graces $\mathcal{A}ss$, $\mathcal{C}om$ and $\mathcal{L}ie$, every operad concentrated in degree 0 and the operads $\mathcal{A}ss_{\infty}$, $\mathcal{C}om_{\infty}$ and $\mathcal{L}ie_{\infty}$. More generally, minimal models of reduced r-tame operads are r-tame. An example of 1-tame operad is $\mathcal{G}er$, the one encoding Gerstenhaber algebras.

In the category of P-algebras, there is a notion of free P-algebra generated by a graded \mathbf{k} -vector space. From this notion, we define KS-extensions of free P-algebras analogously to the rational homotopy setting of Com-algebras. We say that a P-algebra \mathcal{M} is a Sullivan minimal P-algebra if it is the colimit of a sequence of KS-extensions starting from P(0), ordered by non-decreasing positive degrees. A Sullivan minimal model of a P-algebra \mathcal{M} is a Sullivan minimal P-algebra \mathcal{M} , together with a morphism $f: \mathcal{M} \longrightarrow A$ of P-algebras whose underlying map of cochain complexes induces an isomorphism in cohomology; i.e., a quasi-isomorphism of P-algebras. P

As in the rational homotopy setting, we require cohomological connectedness for our algebras. A P-algebra A is called 0-connected if $H^i(A) = 0$ for all i < 0 and $H^0(A) \cong P(0)$. Let $r \ge 0$. Then A is called r-connected if, in addition, $H^1(A) = \cdots = H^r(A) = 0$. We prove:

Theorem 4.6. Let $r \geq 0$. Let P be an r-tame operad. Then every r-connected P-algebra A has a Sullivan minimal model $f: \mathcal{M} \longrightarrow A$ with $\mathcal{M}^0 = P(0)$ and $\mathcal{M}^i = 0$ for all i < r with $i \neq 0$. Furthermore, if A is (r+1)-connected and $H^*(A)$ is of finite type, then \mathcal{M} is of finite type.

Note that in the particular case $P = \mathcal{C}om$ we recover Sullivan's theorem of minimal models for commutative differential graded algebras over \mathbf{k} . We also obtain Sullivan minimal models for 0-connected P-algebras, when P is one of the operads $\mathcal{A}ss$, $\mathcal{L}ie$, $\mathcal{C}om_{\infty}$, $\mathcal{A}ss_{\infty}$ or $\mathcal{L}ie_{\infty}$ among others. Furthermore, the above result gives Sullivan minimal models for 1-connected $\mathcal{G}er$ -algebras. All these minimal models are unique:

Theorem 5.3. Let P be an r-tame operad and let A be an r-connected P-algebra. Let $f: \mathcal{M} \longrightarrow A$ and $f': \mathcal{M}' \longrightarrow A$ be two Sullivan minimal models of A. Then there is an isomorphism $g: \mathcal{M} \longrightarrow \mathcal{M}'$, unique up to homotopy, such that $fg \simeq f'$.

Remarks 1.1. A few remarks are in order:

- (1) Relation with existing Sullivan minimal models. Sullivan's classical construction of minimal models for commutative differential graded algebras has been adapted to several other algebraic settings. Examples are Quillen's models of differential graded Lie algebras [Qui69], the models for chain differential graded (Lie) algebras of Baues-Lemaire [BL77] and Neisendorfer [Nei78], the theory of Leibnitz algebras of [Liv98b] and, more closely related to our approach, the theory of minimal models of chain P-algebras, where P is a Koszul operad concentrated in degree 0, developed by Livernet in her PhD Thesis [Liv98a]. As we show in Section 7, our results are equally valid for cochain and chain algebras, after minor modifications are taken into account. In particular, our work generalizes all of the above mentioned studies.
- (2) Koszul duality theory. For Koszul quadratic P-algebras, there is a theory of quasi-free resolutions which give minimal models in some situations (see [LV12], [Mil11], [Mil12]). While there is a certain overlap of algebras for which both Koszul duality theory and our Sullivan algorithm for algebras over tame operads apply, let us mention some notable differences. First, to know whether

¹Warning: here and elsewhere along the paper, f and all algebra morphisms are morphisms in the strict sense, not ∞ -morphisms!

an operad is Koszul or not, can prove to be very difficult (see [MSS02], Remark 3.98). The theory developed in this paper doesn't require operads to be Koszul, not even quadratic. In particular, there is no restriction on the height of the relations among their generators. In contrast, we do impose some restrictions on the arity-degree range of the elements of the operad. This condition is straightforward to verify. Second, while Koszul duality theory applies to quadratic algebras satisfying certain conditions, our algorithm applies to all sufficiently connected P-algebras, once the operad P is proven to be tame. Furthermore, we produce minimal models for both unitary, $P(0) = \mathbf{k}$, and non-unitary, P(0) = 0 algebras, while Koszul duality theory applies only to the latter case. Lastly, let us mention that in Koszul duality theory, minimal models are constructed via the cobar resolution of the associated coalgebra, while, in this paper, we give "step by step" minimal models, following Sullivan's classical approach. This may be useful, for instance, to compute partial minimal models up to a certain degree and extract homotopical information.

- (3) Kadeishvili's models. There are many results in the literature about "minimal models" for operad algebras in the ∞ -sense. Prominently, Kadeishvili [Kad80] defined minimal models of A_{∞} -algebras as A_{∞} -algebras with trivial differential. Similarly, there is the Homotopy Transfer Theorem for P_{∞} -algebras (see [LV12]) and the theory of minimal models for operad algebras developed in [CL10]. As it is well-known, minimal models à la Kadeishvili do not correspond to minimal models à la Sullivan, the main differences being that for the first ones, morphisms are ∞ -morphisms and minimality is a vanishing condition on the differential, while for the later, morphisms are strict and minimality involves freeness and a certain behavior of the (not-necessarily trivial) differential. However, a characterizing property is shared by the two approaches: every quasi-isomorphism between minimal algebras is an isomorphism.
- (4) Minimal models of operads. Every reduced operad P in the category of complexes of **k**-vector spaces such that $H(P)(1) = \mathbf{k}$ has a minimal model (defined as a free operad whose differential is decomposable). Here we study minimal models of the algebras, and not of the operads themselves. However, there is a relation between the two problems that we address at the end of this paper. The idea, is that one can consider the category of algebras above all operads as a fibered category. We show that minimal objects in this category are given by those objects that are both minimal on the fiber and the base. This provides a global invariance of our minimal models.

We explain the contents of the paper. In Section 2 we collect well-known results on operads and operad algebras. In Section 3 we review the basic homotopy theory of operad algebras. In Section 4 we introduce r-tame operads and prove the existence of minimal models for algebras over operads. We also show that the minimal model of every r-tame operad is r-tame, and give some examples. Section 5 deals with the uniqueness of our minimal models. In Section 6 we study the fibred category of algebras over all operads and give global minimal models in this case. Lastly, in Section 7 explain the case of chain operad algebras (with homological degree).

2. Preliminaries

In this first section, se recall some main constructions for operads and operad algebras in the category of cochain complexes of vector spaces over a field of characteristic 0 and fix notation. For preliminaries on operads, we refer to [MSS02], [LV12], [Fre09] and [KM95]. We refer to [GM13], [FHT01] and the original paper of Sullivan [Sul77] for a review of rational homotopy theory.

Throughout this paper, we will let \mathbf{k} denote a field of characteristic 0.

Operads in cochain complexes. We will consider unital symmetric operads in the category of unbounded cochain complexes of vector spaces over **k**. Denote by **Op** the category of such operads.

Given an operad P in \mathbf{Op} we will denote by

$$\gamma_{l;m_1,\ldots,m_l}^P: P(l)\otimes P(m_1)\otimes\cdots\otimes P(m_l)\longrightarrow P(m),$$

with $m = m_1 + \cdots + m_l$ its structure morphisms and by $\eta : \mathbf{k} \longrightarrow P(1)$ its unit. These morphisms satisfy equivariance, associativity, and unit axioms (see [MSS02], Definition I.4).

The initial object in **Op** is the *identity operad* I given by $I(1) = \mathbf{k}$ concentrated in degree 0 and I(m) = 0 for all $m \neq 1$.

An operad P is called *unitary* if $P(0) = \mathbf{k}$ is concentrated in degree 0. It is called *reduced* if P(0) = 0. We will say that P is *connected* if $P(1) = \mathbf{k}$ is concentrated in degree 0. In this paper we will always consider connected operads that are either unitary or reduced.

2.1. **Operad algebras.** Let $P \in \mathbf{Op}$ be an operad. Denote by \mathbf{Alg}_P the category of P-algebras. For a P-algebra A, we will denote by $\theta_A(l): P(l) \otimes_{\Sigma_l} A^{\otimes l} \longrightarrow A$ its structure morphisms. These are subject to natural associativity and unit constraints (see [KM95]).

Note that I-algebras are just cochain complexes of vector spaces.

Since every P-algebra has an underlying cochain complex, we have a notion of quasi-isomorphism in \mathbf{Alg}_P given by those morphisms of P-algebras whose underlying morphism of cochain complexes induces an isomorphism in cohomology. Denote by \mathcal{W} the class of quasi-isomorphisms of \mathbf{Alg}_P .

We next recall some constructions in the category of P-algebras that will be used in the sequel.

- 2.2. Functorial properties. (c.f. [LV12], 5.2.14). Every morphism of operads $F: P \to Q$ induces a reciprocal image or restriction of scalars functor $F^*: \mathbf{Alg}_Q \longrightarrow \mathbf{Alg}_P$ defined on objects $B \in \mathbf{Alg}_Q$ by the compositions $\theta_{F^*B}(l) = \theta_B(l) \circ (F(l) \otimes \mathrm{id}_B^{\otimes l}) : P(l) \otimes_{\Sigma_l} B^{\otimes l} \longrightarrow B$. This functor, which is analogous to the restriction functor for modules, admits a left adjoint $F_!: \mathbf{Alg}_P \longrightarrow \mathbf{Alg}_Q$, called direct image or extension of scalars functor.
- 2.3. **Tensor product.** Let $P,Q \in \mathbf{Op}$ be two operads. Their pointwise tensor product is the operad $P \otimes Q$ whose arity l is the cochain complex $P(l) \otimes Q(l)$. Given a P-algebra A and a Q-algebra B, their tensor product as cochain complexes $A \otimes B$ has a natural structure of $(P \otimes Q)$ -algebra. The operad Com being the unit of our tensor product of operads, one has $P \otimes Com = P$ and hence the tensor product $A \otimes K$ of any P-algebra A with a Com-algebra K is always a P-algebra. This gives a bifunctor $\mathbf{Alg}_P \times \mathbf{Alg}_{Com} \longrightarrow \mathbf{Alg}_P$ defined on objects by $(A,K) \mapsto A \otimes K$.
- 2.4. Free algebras. (c.f. [LV12], Section 5.2.5) Let $P \in \mathbf{Op}$ be an operad and let V be a graded vector space. The free P-algebra generated by V is the P-algebra

$$P\langle V\rangle = \bigoplus_{m\geq 0} \left(P(m) \otimes_{\Sigma_m} V^{\otimes m}\right)$$

with the structure maps $\theta(m): P(m) \otimes_{\Sigma_m} P\langle V \rangle^{\otimes m} \longrightarrow P\langle V \rangle$ given by the composition of the shuffle isomorphism followed by the structure morphisms γ of P:

$$P(l) \otimes \left(P(m_1) \otimes_{\Sigma_{m_1}} V^{\otimes m_1}\right) \otimes \cdots \otimes \left(P(m_l) \otimes_{\Sigma_{m_l}} V^{\otimes m_l}\right)$$

$$\cong \bigvee_{Sh} Sh$$

$$P(l) \otimes \left(P(m_1) \otimes \cdots \otimes P(m_l)\right) \otimes_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_l}} V^{\otimes (m_1 + \cdots + m_l)}$$

$$\bigvee_{\gamma_{l; m_1, \cdots, m_l \otimes 1}} V^{\otimes (m_1 + \cdots + m_l)}$$

$$P(m_1 + \cdots + m_l) \otimes V^{\otimes (m_1 + \cdots + m_l)}$$

By the universal property of the free P-algebra ([LV12]), for any linear map $f: V \longrightarrow A$ of degree 0, there exists a unique morphism of P-algebras $P(V) \longrightarrow A$ that restricted to V agrees with f.

- 2.5. Cone of a morphism. Given a morphism $f: A \to B$ of P-algebras, we denote by C(f) the cone of f. This is the cochain complex given by $C(f)^n = A^{n+1} \oplus B^n$ with differential d(a,b) = (-da, -fa + db). The morphism f is a quasi-isomorphism of P-algebras if and only if $H^*(C(f)) = 0$.
- 2.6. **Basic homotopy theory of operads.** The following results are well-known ([Mar96], [MSS02], cf. [GNPR05]), so here we just recall the results in the way we will use them.

Definition 2.1. Let n > 1 be an integer. Let $P \in \mathbf{Op}$ be free as graded operad, $P = \Gamma(V)$. An arity n principal extension of P is the free graded operad

$$P \sqcup_d \Gamma(V') := \Gamma(V \oplus V')$$
,

where V' is a Σ_n -module of zero differential and $d:V'\to ZP(n)$ a cochain map of Σ_n -modules. The differential on $P\sqcup_d\Gamma(V')$ is built upon the differential of P, d and the Leibnitz rule.

Lemma 2.2. Let $P \sqcup_d \Gamma(V')$ be a principal extension of a free operad $P = \Gamma(V)$, and let $F: P \to Q$ be a morphism of operads. A morphism $F': P \sqcup_d \Gamma(V') \longrightarrow Q$ extending F is uniquely determined by a morphism of Σ_n -modules $\Phi: V' \to Q(n)$ satisfying $d\Phi = Fd$.

Lemma 2.3. Let $I: P \longrightarrow P \sqcup_d \Gamma(V)$ be an arity n principal-extension and

a solid commutative diagram of operad morphisms, where W is a surjective quasi-isomorphism. Then, there is an operad morphism G' making both triangles commute.

Definition 2.4. A Sullivan operad is the colimit of a sequence of principal extensions of arities ≥ 2 , starting from 0.

Remark 2.5. The terminology "Sullivan operad" is not standard. The choice of this name will be made obvious in Section 6 when we study minimal models of *P*-algebras over variable operads.

Proposition 2.6. Let R be a Sullivan operad. For every solid diagram of operads

$$C \xrightarrow{F} Q$$

in which W is a surjective quasi-isomorphism, there exists G making the diagram commute.

Corollary 2.7. The homotopy relation between morphisms of operads is an equivalence relation for those morphisms whose source is a Sullivan operad.

Denote by [P,Q] the set of homotopy classes morphisms of operads $F:P\to Q$.

Corollary 2.8. Let R be a Sullivan operad. Any quasi-isomorphism $W: P \longrightarrow Q$ of operads induces a bijection $W_*: [R, P] \longrightarrow [R, Q]$.

3. Basic homotopy theory of operad algebras

Throughout this section we let $P \in \mathbf{Op}$ be a fixed operad in the category of cochain complexes of vector spaces over \mathbf{k} . We first review KS-extensions of P-algebras and prove that they satisfy the lifting property with respect to surjective quasi-isomorphisms. Then, we give some main properties of homotopies between morphisms of P-algebras.

Remark 3.1. In general, in order to define extensions one would require the not easy notion of *tensor product* of *P*-algebras (see [Hin01], [SU04], [MS06], [Lod11], for instance). Fortunately, in our case it suffices to consider tensor products of *free* (non-differential) algebras.

Definition 3.2. Let n > 0 be an integer. Let $A \in \mathbf{Alg}_P$ be free as graded algebra $A = P\langle V \rangle$. A degree n KS-extension of A is the free graded P-algebra

$$A \sqcup_d P\langle V' \rangle := P\langle V \oplus V' \rangle$$
,

where V' is a graded vector space of homogeneous degree n and $d: V' \to Z^{n+1}(A)$ a **k**-linear map. The differential on $A \sqcup_d P\langle V' \rangle$ is built upon the differentials of A, P, d and the Leibnitz rule.

Lemma 3.3. $A \sqcup_d P\langle V' \rangle$ is a P-algebra.

Proof. It suffices to see that the differential on $A \sqcup_d P\langle V' \rangle$ is compatible with the structure maps $\theta(l)$ defined via the Shuffle isomorphism and the structure morphisms of P. We prove it on a component

$$\theta := (\gamma \otimes 1) \circ Sh : P(l) \otimes \left(P(m_1) \otimes_{\Sigma_{m_1}} V^{\otimes m_1} \right) \otimes \cdots \otimes \left(P(m_l) \otimes_{\Sigma_{m_l}} V^{\otimes m_l} \right) \longrightarrow P(V).$$

Note first, that we may write $Sh = (Sh_1 \otimes Sh_2)$, where Sh_1 acts on elements of P and Sh_2 acts on elements of V. Note as well that $Sh \circ d = d \circ Sh$. We get:

$$d \circ \theta = d \circ (\gamma(Sh_1) \otimes Sh_2) = d\gamma(Sh_1) \otimes Sh_2 \oplus (-1)^x \gamma(Sh_1) \otimes d(Sh_2),$$

where x is a sign function given by the Shuffle. Likewise:

$$\theta \circ d = (\gamma \otimes 1) \circ Sh \circ d = \gamma \circ d(Sh_1 \otimes Sh_2) = (\gamma \otimes 1)(dSh_1 \otimes Sh_2 \oplus (-1)^x Sh_1 \otimes dSh_2) = d\theta.$$

We have the following universal property for KS-extensions:

Lemma 3.4. Let $A \sqcup_d P\langle V' \rangle$ be a KS-extension of a free P-algebra $A = P\langle V \rangle$, and let $f: A \to B$ be a morphism of P-algebras. A morphism $f': A \sqcup_d P\langle V' \rangle \longrightarrow B$ extending f is uniquely determined by a linear map $\varphi: V' \to B$ of degree 0 satisfying $d\varphi = fd$.

Proof. By the universal property of free algebras we get $f': P\langle V \oplus V' \rangle \longrightarrow B$. To prove that it is compatible with the differentials of $A \sqcup_d P\langle V' \rangle$ and B, it suffices to check this on the restriction to V. We have $f' \circ d|_V = f \circ d = d \circ \varphi = d \circ f'|_V$.

KS-extensions satisfy the lifting lemma with respect to surjective quasi-isomorphisms:

Lemma 3.5. Let $i: A \longrightarrow A \sqcup_d P\langle V \rangle$ be a KS-extension of degree n and

$$A \xrightarrow{f} B$$

$$\downarrow \downarrow g' \nearrow \downarrow \downarrow w$$

$$A \sqcup_d P\langle V \rangle \xrightarrow{g} C$$

a solid commutative diagram of P-algebra morphisms, where w is a surjective quasi-isomorphism. Then, there is a P-algebra morphism g' making both triangles commute.

Proof. Consider the solid diagram of **k**-vector spaces

$$Z^{n}(C(1_{B}))$$

$$\downarrow^{1 \oplus w}$$

$$V \xrightarrow{\lambda} Z^{n}(C(w))$$

where $\lambda = (f \circ d, g_{|V})$. Since w is a surjective quasi-isomorphism, this is well defined and $1 \oplus w$ is surjective. Therefore there exists a dotted arrow $\mu = (\alpha, \beta)$ making the diagram commute. It is straightforward to see that the image of the linear map $(d_V, \beta) : V \longrightarrow A^{n+1} \oplus B^n$ is included in $Z^p(C(\phi))$. According to the universal property of KS-extensions of Lemma 3.4, we may obtain g' as the morphism induced by $g|_A$ together with $\beta : V \longrightarrow B^{n+1}$.

Definition 3.6. A Sullivan P-algebra is the colimit of a sequence of KS-extensions of non-negative degrees, starting from P(0).

Proposition 3.7. Let C be a Sullivan P-algebra. For every solid diagram of P-algebras

$$C \xrightarrow{g} A$$

$$\downarrow w$$

$$C \xrightarrow{f} B$$

in which w is a surjective quasi-isomorphism, there exists g making the diagram commute.

Proof. Assume that $C' \longrightarrow C = C' \sqcup_d P\langle V \rangle$ is a KS-extension of degree n, and that we have constructed $g': C' \to A$ such that wg' = f', where f' denotes the restriction of f to f'. The existence of g extending g' now follows from Lemma 3.5.

The following are standard consequences of Proposition 3.7. The proofs are straightforward adaptations of the analogous results in the setting of Com-algebras (see Section 11.3 of [GM13], see also Section 2.3 of [Cir15] for proofs in the abstract setting of categories with a functorial path).

Denote by $\mathbf{k}[t, dt]$ the Com-algebra with a generator t in degree zero, a generator dt in degree one, and d(t) = dt. We have the unit ι and evaluations δ^0 and δ^1 at t = 0 and t = 1 respectively, which are morphisms of Com-algebras satisfying $\delta^0 \circ \iota = \delta^1 \circ \iota = 1$.

Definition 3.8. A functorial path in the category of P-algebras is defined as the functor

$$-[t,dt]:\mathbf{Alg}_P\longrightarrow\mathbf{Alg}_P$$

given on objects by $A[t, dt] = A \otimes \mathbf{k}[t, dt]$ and on morphisms by $f[t, dt] = f \otimes \mathbf{k}[t, dt]$, together with the natural transformations

$$A \xrightarrow{\iota_A} A[t, dt] \xrightarrow{\delta_A^0} A \; ; \; \delta_A^k \circ \iota_A = 1_A$$

given by $\delta_A^k = 1_A \otimes \delta^k : A[t, dt] \to A \otimes \mathbf{k} = A$ and $\iota_A = 1_A \otimes \iota : A = A \otimes \mathbf{k} \longrightarrow A[t, dt]$.

Note that the map ι_A is a quasi-isomorphism of P-algebras while the maps δ_A^0 and δ_A^1 are surjective quasi-isomorphisms of P-algebras.

The functorial path gives a natural notion of homotopy between morphisms of P-algebras:

Definition 3.9. Let $f, g: A \longrightarrow B$ be two morphisms of P-algebras. An homotopy from f to g is given by a morphism of P-algebras $h: A \to B[t, dt]$ such that $\delta_B^0 \circ h = f$ and $\delta_B^1 \circ h = g$. We use the notation $h: f \simeq g$.

The homotopy relation defined by a functorial path is reflexive and compatible with the composition (see for example [KP97, Lemma I.2.3]. Furthermore, the symmetry of Com-algebras $\mathbf{k}[t, dt] \longrightarrow \mathbf{k}[t, dt]$ given by $t \mapsto 1 - t$ makes the homotopy relation into a symmetric relation. However, the homotopy relation is not transitive in general. As in the rational homotopy setting of Com-algebras, we have:

Corollary 3.10. The homotopy relation between morphisms of P-algebras is an equivalence relation for those morphisms whose source is a Sullivan P-algebra.

Denote by [A, B] the set of homotopy classes of morphisms of P-algebras $f: A \to B$.

Corollary 3.11. Let C be a Sullivan P-algebra. Any quasi-isomorphism $w: A \longrightarrow B$ of P-algebras induces a bijection $w_*: [C, A] \longrightarrow [C, B]$.

4. Sullivan minimal models

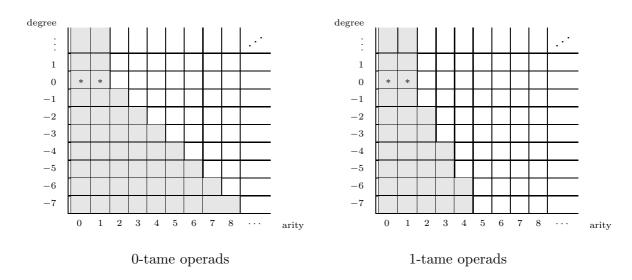
In this section, we prove the existence of Sullivan minimal models of P-algebras, for a quite wide family of operads in the category of cochain complexes of k-vector spaces.

We first introduce the notion of r-tame operad. For this class of operads, r-connected P-algebras will have Sullivan minimal models.

Definition 4.1. Let $r \geq 0$ be an integer. An operad $P \in \mathbf{Op}$ is called *r-tame* if for all $n \geq 2$,

$$P(n)^q = 0 \text{ for all } q \le (1-n)(1+r) .$$

Note that r-tame implies (r+1)-tame for all $r \geq 0$. Below we represent the condition for being a r-tame operad, for r=0 and r=1. Elements of r-tame operads are allowed to be non-zero in the arity-degree range determined by the blank squares below, except for the identity id $\in P(1) = \mathbf{k}$, and $P(0) \in \{0, \mathbf{k}\}$ which are denoted by * and live in arity-degree (1, 0) and (0, 0) respectively.



Definition 4.2. Let $r \ge 0$ be an integer. A Sullivan r-minimal P-algebra is the colimit of a sequence of KS-extensions starting from P(0), ordered by non-decreasing degrees bigger than r:

$$P(0) \longrightarrow \mathcal{M}[1] = P\langle V[1] \rangle \longrightarrow \mathcal{M}[2] = \mathcal{M}[1] \sqcup_d P\langle V[2] \rangle \longrightarrow \cdots$$

with $r < \deg(V[n]) \le \deg(V[n+1])$ for all $n \ge 1$. A Sullivan r-minimal model for a P-algebra A is a Sullivan r-minimal P-algebra \mathcal{M} together with a quasi-isomorphism $f : \mathcal{M} \longrightarrow A$.

As in the rational homotopy setting, to prove the existence of Sullivan minimal models we will restrict to the case when our P-algebras are cohomologically connected (which we will call connected for short from now on).

Definition 4.3. A P-algebra A is called 0-connected if $H^i(A) = 0$ for all i < 0 and $H^0(A) \cong P(0)$. Let $r \geq 0$. Then A is called r-connected if, in addition, $H^1(A) = \cdots = H^r(A) = 0$.

For the construction of Sullivan minimal models we will use the following two lemmas. The first of these lemmas ensures that free P-algebras generated by positively-graded vector spaces, are positivelygraded when P is tame.

Lemma 4.4. Let $r \geq 0$ be an integer. Let $V = \bigoplus_{i > r} V^i$ be a graded vector space with degrees > r. If P is r-tame then $P\langle V \rangle$ is non-negatively graded with $P\langle V \rangle^0 = \mathbf{k}$ and $P\langle V \rangle^k = 0$ for all $0 < k \le r$. In particular, P(V) is r-connected.

Proof. Let $k \in \mathbb{Z}$. The degree k-part of P(V) may be written as

$$P\langle V \rangle^k \cong P(0)^k \oplus \left(\sum_{i>r} P(1)^{k-i} \otimes V^i \right) \oplus \left(\sum_{\substack{n \geq 2, \\ i_1, \dots, i_n > r}} P(n)^{q_n} \otimes V^{i_1} \otimes \dots \otimes V^{i_n} \right) ,$$

where $q_n = k - i_1 - \dots - i_n \le k - n(1+r)$. Since $P(0)^k = 0$ for all $k \ne 0$ and $P(1)^{k-i} = 0$ for all $k \neq i$, it suffices to see that for all $n \geq 2$ and all k < r we have $P(n)^{q_n} = 0$. Since P is r-tame, it suffices to prove that $q_n \leq q_n^* := (1-n)(1+r)$. Let $n \geq 2$ be fixed and assume that k < r. Then

$$q_n = k - i_1 - \dots - i_n \le k - n(1+r) \le (r-1) - n(1+r) = q_n^* - 2 < q_n^*.$$

The second lemma studies the shows the good behavior r-tame operads with KS-extensions and is inspired in Lemma 10.4 of [GM13].

Lemma 4.5. Let $V = \bigoplus_{r < i \le p} V^i$ be a graded vector space with $0 \le r < i \le p$. Let V' be a graded vector space of homogeneous degree p and let P be an r-tame operad. Then:

- (1) $P\langle V \oplus V' \rangle^k = P\langle V \rangle^k$ for all k < p and $P\langle V \oplus V' \rangle^p = P\langle V \rangle^p \oplus V'$. (2) If $V^{r+1} = 0$ then $P\langle V \oplus V' \rangle^{p+1} = P\langle V \rangle^{p+1}$.

Proof. For all $k \in \mathbb{Z}$ we may write

$$\frac{P\langle V \oplus V' \rangle^k}{P\langle V \rangle^k} \cong \left(\sum_{n \geq 1} P(n)^{q_n} \otimes V'^{\otimes n} \right) \oplus \left(\sum_{\substack{n \geq 2, 1 \leq j \leq n-1 \\ r < i_1 \leq \dots \leq i_j \leq p}} P(n)^{q'_n} \otimes V^{i_1} \otimes \dots \otimes V^{i_j} \otimes V'^{\otimes (n-j)} \right),$$

where $q_n = k - pn$ and $q'_n = k - i_1 - \dots - i_j - p(n - j)$. We first show that for $n \ge 2$ and $k \le p$, we have $P(n)^{q_n} = P(n)^{q'_n} = 0$. Since P is r-tame, it suffices to see that both q_n and q'_n are smaller or equal than $q_n^* := (1 - n)(r + 1)$. Since p < r, we have

$$q_n = k - pn \le p(1-n) \le (1-n)(1+r) = q_n^*$$
.

Note that q'_n attains its maximum when k = p, j = n - 1 and $i_1 = \cdots = i_j = r + 1$. Then

$$q'_n \le p + (1-n)(1+r) - p = q_n^*$$
.

This proves that for $k \leq p$ we have

$$\frac{P\langle V \oplus V' \rangle^k}{P\langle V \rangle^k} \cong P(1)^{k-p} \otimes V'.$$

Now (1) follows from the fact that $P(1)^{k-p} = 0$ for all $k \neq p$ and $P(1)^0 = \mathbf{k}$.

Assume that $V^{r+1}=0$. Then in the above formula for $P(V\oplus V')^{p+1}/P(V)^{p+1}$ we have

$$q_n = p + 1 - pn = p(1 - n) + 1 \le (1 - n)(1 + r) + 1 = q_n^* + 1 < q_n^*.$$

Note that now q'_n attains its maximum when j = n - 1 and $i_1 = \cdots = i_j = r + 2$. Then

$$q'_n = p + 1 - i_1 - \dots - i_j - p(n - j) \le p + 1 + (r + 2)(1 - n) - p = q_n^* + (2 - n) \le q_n^*$$

Therefore all the contributions vanish and (2) is satisfied.

Theorem 4.6. Let $r \geq 0$. Let P be an r-tame operad. Then every r-connected P-algebra A has a Sullivan r-minimal model $f: \mathcal{M} \longrightarrow A$ with $\mathcal{M}^0 = P(0)$ and $\mathcal{M}^i = 0$ for all i < r with $i \neq 0$. Furthermore, if A is (r+1)-connected and $H^*(A)$ is of finite type, then \mathcal{M} is of finite type.

Proof. We follow the steps of the classical proof of existence of Sullivan minimal models for *Com*algebras. We will construct, inductively over the degree $n \geq 0$, a sequence of free *P*-algebras $\mathcal{M}[n]$ together with morphism of *P*-algebras $f_n : \mathcal{M}[n] \longrightarrow A$ satisfying the following conditions:

- (a_n) The *P*-algebra $\mathcal{M}[n]$ is a composition of KS-extensions of degree n of $\mathcal{M}[n-1]$ and the morphism f_n extends f_{n-1} .
- (b_n) The map $H^i f_n$ is an isomorphism for all $i \leq n$ and a monomorphism for i = n + 1.

Then the morphism $f: \bigcup_n f_n: \bigcup_n \mathcal{M}[n] \longrightarrow A$ will be a minimal model for A. Indeed, condition (a_n) implies that \mathcal{M} is minimal and that $\mathcal{M}^n = \mathcal{M}[k]^n$ for all $k \geq n$. From (b_{n+1}) it follows that $H^n(Cf) = H^n(Cf_{n+1}) = 0$. Therefore f is a quasi-isomorphism.

Let $\mathcal{M}[0] = \cdots = \mathcal{M}[r] = H^0(A)$ and define $f_r : \mathcal{M}[r] \longrightarrow A$ by taking a section of the projection $Z^0(A) \twoheadrightarrow H^0(A)$. Then $H^i(f_r)$ is an isomorphism for $i \leq r$ and a monomorphism for i = r + 1. Therefore conditions (a_r) and (b_r) are trivially satisfied.

Assume that, for all i < n we have a morphism of P-algebras $f_i : \mathcal{M}[i] \longrightarrow A$ satisfying (a_i) and (b_i) . Condition (b_n) is equivalent to the vanishing of $H^i(Cf_{n-1})$ for all i < n. Let $V[n,0] := H^n(Cf_n)$ and consider it as a graded vector space of homogeneous degree n. Take a section of the projection $Z^n(Cf_n) \twoheadrightarrow V[n,0]$ to obtain a linear differential $d:V[n,0] \to Z^{n+1}\mathcal{M}[n-1]$ and a linear map $\varphi:V \longrightarrow A^n$ such that $d\varphi = f_{n-1}d$. We then let

$$\mathcal{M}[n,0] := \mathcal{M}[n-1] \sqcup_d P\langle V[n,0] \rangle$$

and denote by $f_{n,0}: \mathcal{M}[n,0] \longrightarrow A$ the extension of f_{n-1} by φ .

By Lemma 4.4, $\mathcal{M}[n,0]$ is an r-connected P-algebra. Furthermore, by (1) of Lemma 4.5 we have that $\mathcal{M}[n,0]^k = \mathcal{M}[n-1]^k$ for all k < n and $\mathcal{M}[n,0]^n = \mathcal{M}[n-1]^n \oplus V[n,0]$. In particular, we have $H^i f_{n,0} = H^i f_{n-1}$ for all i < n. Hence by induction hypothesis, $H^i f_{n,0}$ is an isomorphism for all i < n. We next prove that $H^n f_{n,0}$ is an isomorphism. Denote by $j_0 : \mathcal{M}[n-1] \longrightarrow \mathcal{M}[n,0]$ the inclusion. Then we have $H^i(Cj_0) = 0$ for all i < n and $H^n(Cj_0) = V[n,0]$: this follows, as in Lemma 10.4 of [GM13]), using Lemma 4.5. The morphism of cones (id, f_{n-1}) : $C(j_0) \longrightarrow C(f_{n-1})$ induces

a morphism of long exact sequences in cohomology. By the five lemma, it follows that $H^n f_{n,0}$ is an isomorphism.

In degree n+1 we obtain an inclusion $\operatorname{Ker}(H^{n+1}f_{n,0}) \subset \operatorname{Ker}(H^{n+1}j_0)$. To make $H^{n+1}f_{n,0}$ into a monomorphism, we let

$$V[n,1] = \text{Ker}(H^{n+1}f_{n,0}) = H^n(Cf_{n,0})$$

and consider it as a vector space of homogeneous degree n. We repeat the above process by letting

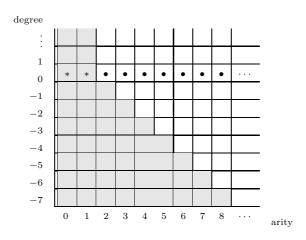
$$\mathcal{M}[n,i] = \mathcal{M}[n,i-1] \sqcup_d P\langle V[n,i] \rangle$$

until Ker $(H^{n+1}f_{n,i}) = 0$. If this never happens, we let $\mathcal{M}[n] := \bigcup_i \mathcal{M}[n,i]$ and define $f_n : \mathcal{M}[n] \longrightarrow A$ by $f|_{\mathcal{M}[n,i]} = f_{n,i}$. Since Ker $(H^{n+1}f_{n,i}) \subset \text{Ker } (H^{n+1}j_i)$, where $j_i : \mathcal{M}[n,i-1] \longrightarrow \mathcal{M}[n,i]$ denotes the inclusion, it follows that Ker $(H^{n+1}f_n) = 0$. Since $H^i f_{n,i}$ is an isomorphism for each $i \leq n$, it follows that $H^n f_n$ is an isomorphism and (b_n) is satisfied. This ends the inductive step.

If A is (r+1)-connected, then by Lemma 4.5 we have that $\mathcal{M}[n,0]^{n+1} = \mathcal{M}[n-1]^{n+1}$. This implies that $\operatorname{Ker}(H^{n+1}f_{n,0}) = 0$ and hence $\mathcal{M}[n] = \mathcal{M}[n,0] = \mathcal{M}[n-1] \sqcup_d P\langle V[n,0] \rangle$. If $H^*(A)$ has finite type, then V[n,0] is finite dimensional and $\mathcal{M}[n]$ has finite type.

Let us review a few examples where Theorem 4.6 applies.

The operads Ass, Com and Lie encoding differential graded associative, commutative and Lie algebras respectively are generated by operations in arity-degree (2,0). Therefore they are concentrated in degree 0. We have:

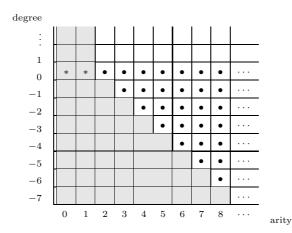


The operads Ass, Com and Lie are 0-tame

The above operads have minimal models, encoding the infinity-versions of their algebras. These are depicted in the following table.

Corollary 4.7. Let P be one of the operads Ass, Com, Lie, Com_{∞} , Ass_{∞} or Lie_{∞} . Then every 0-connected P-algebra has a Sullivan minimal model. Also, every 1-connected P-algebra with finite type cohomology has a Sullivan minimal model of finite type.

More generally, every reduced operad P such that $H(P)(1) = \mathbf{k}$ has a minimal model (see Theorem 3.125 in [MSS02]). Since r-tame operads satisfy $P(1) = \mathbf{k}$, we may consider minimal models of reduced r-tame operads. We have:



The operads Ass_{∞} , Com_{∞} and Lie_{∞} are 0-tame

Proposition 4.8. Let $P \in \mathbf{Op}$ be a reduced r-tame operad. Then its minimal model is r-tame.

Proof. We review the steps in the construction of minimal models of reduced operads with $H(P)(1) = \mathbf{k}$, of Theorem 3.125 in [MSS02]. The algorithm runs by induction over the arity. It starts by letting M_2 freely generated by elements of P(2). In the next step, M_3 is defined by adding to M_2 , elements of $P(3)^q$ in arity-degree (3, q), together with elements of $M_2(3)^q$ in arity-degree (3, q - 1). For our purposes, it is not necessary to know neither which elements we are adding nor what are their differentials. We only need to keep track of their possible arities and degrees. Assume inductively that for all i < n we have M_i satisfying:

- (1_i) For all $k \ge 2$, $M_i(k)^q = 0$ for all $q \le (1+r)(1-k)$.
- (2_i) $M_i(i+1)^q = 0$ for q = (1+r)(-i) + 1.

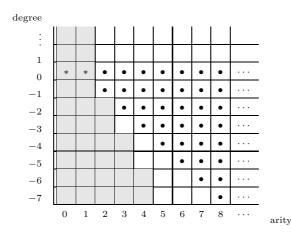
In particular, elements of M_n are allowed to have arity-degree (k, > (r+1)(1-k)) with $2 \le k \ne n$ and (n, > (r+1)(1-n)+1). The algorithm tells us that M_n is generated by elements of M_{n-1} , together with elements of $P(n)^q$ in arity-degree (n, q) and elements of $M_{n-1}(n)^q$ in arity-degree (n, q-1). The only new arity-degree that we consider in M_n is (n, (r+1)(1-n)+1). Therefore to prove (1_n) , it suffices to check that if (m, q) is such that q > (r+1)(1-m), then (m', q') := a(m, q) + b(n, 2-n) is an admissible arity-degree of M_n , in the sense that q' > (r+1)(1-m') for all a, b non-negative integers with a+b>0. We have:

$$q' = aq + b(r+1)(1-n) + b > a(r+1)(1-m) + b(r+1)(1-n) = (r+1)(a+b-am-bn) \ge (r+1)(1-m')$$
. This proves (1_n) . We now prove (2_n) . Note that since $P(1) = \mathbf{k}$, the only new arity-degrees with arity equal to $n+1$ are of the type $(n+1,q) = (1,0) + (n,(r+1)(1-n)+1)$. But then $q > (1+r)(-n)+1$ and hence (2_n) is satisfied. The minimal model of P is the limit of all M_n . Since (1_n) is satisfied for all n , this proves that M is r -tame.

Corollary 4.9. Let $P \in \mathbf{Op}$ be a reduced r-tame operad and let $P_{\infty} \longrightarrow P$ be a minimal model of P. Then every r-connected P_{∞} -algebra has a Sullivan minimal model. Also, every (r+1)-connected P_{∞} -algebra with finite type cohomology has a Sullivan minimal model of finite type.

An example of 1-tame operad is given by the operad encoding $Gerstenhaber \ algebras$: these are graded-commutative algebras with a Lie bracket of degree -1 satisfying the Poisson identity. The ordinary multiplication has arity-degree (2,0), while the Lie bracket has arity-degree (2,-1). We have:

Corollary 4.10. Every 1-connected Ger-algebra (resp. Ger_{∞} -algebra) has a Sullivan minimal model and every 2-connected Ger-algebra (resp. Ger_{∞} -algebra) with finite type cohomology has a Sullivan minimal model of finite type.



The Gerstenhaber operad Ger is 1-tame

5. Uniqueness of the minimal model

In this section we prove the uniqueness of Sullivan minimal models. The proof is parallel to that in the setting of Com-algebras. As in the previous section, the key ingredient is Lemma 4.5.

Lemma 5.1. Let P be an r-tame operad and let $f: A \longrightarrow \mathcal{M}$ be a morphism of r-connected P-algebras, with \mathcal{M} a Sullivan r-minimal P-algebra. Then there exists a morphism of P-algebras $g: \mathcal{M} \longrightarrow A$ such that $fg = \mathrm{id}_{\mathcal{M}}$.

Proof. We rewrite the proof of Gómez-Tato (see Lemma 4.4 of [GT93]) for *Com*-algebras in the *P*-algebra setting (see also Theorem 14.11 of [FHT01] and [Roi94b], [Roi93] for a categorical version).

By definition, we may write $\mathcal{M} = \mathcal{M}' \sqcup_d P\langle V \rangle$ where \mathcal{M} is a free P-algebra generated by a graded vector space V' of degrees $r < i \le p$ and V is a graded vector space of homogeneous degree p, with p > 0. Assume inductively that we have a morphism of P-algebras $g' : \mathcal{M}' \longrightarrow A$ such that $fg' = 1_{\mathcal{M}'}$. Then g' is injective. The morphism f induces a morphism of cochain complexes (not of P-algebras!)

$$\overline{f}: A/q'(\mathcal{M}') \longrightarrow \mathcal{M}/\mathcal{M}'$$

which is a quasi-isomorphism. By Lemma 4.5 we have that $(\mathcal{M}/\mathcal{M}')^{p-1} = 0$ and that $(\mathcal{M}/\mathcal{M}')^p = V$. This gives a surjection at the level of cocycles

$$\pi: Z^p(A/g'(\mathcal{M}')) \twoheadrightarrow Z^p(\mathcal{M}/\mathcal{M}') = V.$$

We obtain a linear map $\varphi: V \longrightarrow A^p$ such that $f\varphi = 1_V$ to a morphism $g: \mathcal{M} \longrightarrow A$ by taking sections of the projections $A \longrightarrow A/g'(\mathcal{M}')$ and π and considering the composition

$$V = (\mathcal{M}/\mathcal{M}')^p \longrightarrow Z^p(A/g'(\mathcal{M}') \hookrightarrow (A/g'(\mathcal{M}'))^p \longrightarrow A^p.$$

For a proof of this last fact taking elements and checking that everything works fine see the proof of Theorem 14.11 in [FHT01]. By Lemma 3.4, the map φ extends f' to a morphism $f: \mathcal{M} \longrightarrow A$.

As a classical consequence of Lemma 5.1 we have:

Lemma 5.2. Let P be an r-tame operad and let $f: \mathcal{M} \longrightarrow \mathcal{M}'$ be a quasi-isomorphism of Sullivan r-minimal P-algebras. Then f is an isomorphism.

Proof. By Lemma 5.1 we have a morphism $g: \mathcal{M}' \longrightarrow \mathcal{M}$ such that $fg = \mathrm{id}_{\mathcal{M}'}$. By the two out of three property, g is also a quasi-isomorphism. Again, by Lemma 5.1 we have a morphism $g': \mathcal{M} \longrightarrow \mathcal{M}$ such that $gg' = \mathrm{id}_{\mathcal{M}}$. Therefore g is both injective and surjective and hence an isomorphism and we have $f = g^{-1}$.

Finally, we get the main theorem of this section.

Theorem 5.3. Let P be an r-tame operad and let A be an r-connected P-algebra. Let $f: \mathcal{M} \longrightarrow A$ and $f': \mathcal{M}' \longrightarrow A$ be two Sullivan r-minimal models of A. Then there is an isomorphism $g: \mathcal{M} \longrightarrow \mathcal{M}'$, unique up to homotopy, such that $fg \simeq f'$.

Proof. By Corollary 3.11 we obtain g, uniquely defined up to homotopy, such that $fg \simeq f'$. By Lemma 5.2, g is an isomorphism.

Corollary 5.4. Let P be an r-tame operad. The category \mathbf{Alg}_P^r of r-connected P-algebras is a Sullivan category in the sense of [GNPR10]. In particular, the inclusion induces an equivalence of categories

$$\mathbf{SMin}_P^r/\simeq \stackrel{\sim}{\longrightarrow} \mathbf{Alg}_P^r[\mathcal{W}^{-1}]$$
.

Here \mathbf{SMin}_P^r/\simeq stands for the category of r-connected Sullivan minimal P-algebras, whose morphisms are homotopy classes of morphisms of P-algebras, and $\mathbf{Alg}_P^r[\mathcal{W}^{-1}]$ is the localized category of r-connected P-algebras with respect to the class of quasi-isomorphisms.

Proof. For every P-algebra A, the choice of a minimal model \mathcal{M} gives a well-defined functor $A \mapsto \mathcal{M}$ between the homotopy categories, which is the quasi-inverse of the functor induced by the inclusion. \square

6. Algebras over variable operads

Let P be an operad and A a P-algebra. Given two minimal models $F: P_{\infty} \to P$ and $F': P'_{\infty} \to P$, we may consider the reciprocal images $F^*(A)$ and $F'^*(A)$ of A in the categories of P_{∞} -algebras and P'_{∞} -algebras respectively. In this section, we compare the minimal models of these reciprocal images. This problem is better understood in the fibered category of algebras over all operads, which we next introduce.

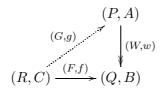
Definition 6.1. Denote by **Alg** the category whose objects are pairs (P,A) with $P \in \mathbf{Op}$ and $A \in \mathbf{Alg}_P$ and whose morphisms $(F,f):(P,A) \longrightarrow (Q,B)$ are given by a morphism $F:P \longrightarrow Q$ of operads, together with a morphism $f:A \longrightarrow F^*(B)$ of P-algebras. The composition of morphisms $(F,f):(P,A) \longrightarrow (Q,B)$ and $(G,g):(Q,B) \longrightarrow (R,C)$ is defined by $(G,g) \circ (F,f):=(G \circ F,F^*(g) \circ f)$. Objects in **Alg** will be called algebras (over variable operads).

Following the main theorem of [Roi94a] and taking into account the remarks of [Sta12], one can produce a Quillen model category structure on \mathbf{Alg} , from the ones on \mathbf{Op} and \mathbf{Alg}_P (see [BM03], [Hin97]). However, since here we are only interested on minimal models, we don't need the whole power of a Quillen model structure. As we have seen, in order to talk about and prove existence of minimal models it suffices to consider weak equivalences (quasi-isomorphisms). If, on top, we want to study uniqueness, we also need a notion of homotopy.

Definition 6.2. A morphism $(F, f): (P, A) \longrightarrow (Q, B)$ in **Alg** is said to be a *quasi-isomorphism* if $F: P \longrightarrow Q$ is a quasi-isomorphism of operads and $f: A \longrightarrow F^*(B)$ is a quasi-isomorphism of P-algebras.

Definition 6.3. We will say that a pair $(R, C) \in \mathbf{Alg}$ is a Sullivan algebra if R is a Sullivan operad and C is a Sullivan R-algebra.

Proposition 6.4. Let (R,C) be a Sullivan algebra. Then for every solid diagram in Alg



where (W, w) is a surjective quasi-isomorphism, there exists (G, g) making the diagram commute.

Proof. Since R is a Sullivan operad, by Lemma 2.3 there exists a morphism $G: R \longrightarrow P$ such that $W \circ G = F$. Consider the solid diagram of P-algebras

$$G^{*}(A)$$

$$g \xrightarrow{f} \downarrow^{G^{*}(w)}$$

$$C \xrightarrow{f} F^{*}(B)$$

Note that since $G^*W^* = F^*$, this is well-defined. Since C is a Sullivan R-algebra and $G^*(w)$ is a quasi-isomorphism, there is a morphism g making the diagram commute.

Definition 6.5. A functorial path in the category **Alg** is defined as the functor

$$-[t, dt] : \mathbf{Alg} \longrightarrow \mathbf{Alg}$$

given on objects by (P, A)[t, dt] = (P[t, dt], A[t, dt]) and on morphisms by (F, f)[t, dt] = (F[t, dt], f[t, dt]), together with the natural transformations

$$(P,A) \xrightarrow{(I,\iota)} (P[t,dt], A[t,dt]) \xrightarrow[(\Delta^0,\delta^0)]{(\Delta^1,\delta^1)} (P,A) \quad ; \quad (\Delta^k,\delta^k) \circ (I,\iota) = \mathrm{id}_{(P,A)} \ .$$

Note that if $F: P \longrightarrow Q$ is a morphism of operads then $F[t, dt]^* = F^*[t, dt]$.

The path gives a natural notion of homotopy between morphisms in **Alg**. As in Section 3, the following are classical consequences of Proposition 6.4. Again, we refer to [Cir15] for proofs in the abstract setting of categories with a functorial path.

Corollary 6.6. The homotopy relation between morphisms in Alg is an equivalence relation for those morphisms whose source is a Sullivan algebra.

Denote by [(P,A),(Q,B)] the set of homotopy classes of morphisms of algebras $(F,f):(P,A)\to (Q,B)$.

Corollary 6.7. Let (R,C) be a Sullivan algebra. Any quasi-isomorphism $(W,w):(P,A)\longrightarrow (Q,B)$ induces a bijection $(W,w)_*:[(R,C),(P,A)]\longrightarrow [(R,C),(Q,B)]$.

We now study the existence and uniqueness of minimal models in **Alg**. For the rest of this section we let $r \ge 0$ be an integer.

Definition 6.8. We will say that $(P_{\infty}, \mathcal{M}) \in \mathbf{Alg}$ is a Sullivan r-minimal algebra if P_{∞} is a minimal operad which is r-tame and A is a Sullivan r-minimal P_{∞} -algebra.

Theorem 6.9. Let P be a reduced r-tame operad and let A be an r-connected P-algebra. Then (P, A) has a Sullivan r-minimal model.

Proof. By Theorem 3.125 of [MSS02], every reduced operad $P \in \mathbf{Op}$ with $H(P)(1) = \mathbf{k}$ has a minimal model $F: P_{\infty} \longrightarrow P$. Since $P(1) = \mathbf{k}$ this hypothesis is clearly satisfied. Furthermore, P_{∞} is r-tame by Proposition 4.8. Since $F^*(A)$ is an r-connected P_{∞} -algebra, by Theorem 4.6 there is a Sullivan minimal P_{∞} -algebra \mathcal{M} together with a quasi-isomorphism $f: \mathcal{M} \longrightarrow F^*(A)$. The morphism $(F, f): (P_{\infty}, \mathcal{M}) \longrightarrow (P, A)$ is a Sullivan r-minimal model of (P, A).

Lemma 6.10. Let $(F, f) : (P_{\infty}, \mathcal{M}) \longrightarrow (P'_{\infty}, \mathcal{M}')$ be a quasi-isomorphism of Sullivan r-minimal algebras. Then (F, f) is an isomorphism.

Proof. Since $F: P_{\infty} \to P'_{\infty}$ is a quasi-isomorphism of minimal operads, it is an isomorphism (see Theorem 3.119. of [MSS02]). Therefore F^* preserves Sullivan minimal algebras and hence $f: \mathcal{M} \to F^*\mathcal{M}'$ is also an isomorphism.

Remark 6.11. Note that Corollary 6.7 together with Lemma 6.10 make Sullivan minimal algebras in **Alg**, *minimal* in an abstract categorical sense (c.f. [Roi94b], [Roi93], [GNPR10]).

Theorem 6.12. Let A be an r-connected P-algebra. Let

$$(F,f):(P_{\infty},\mathcal{M})\longrightarrow (P,A) \ and \ (F',f'):(P'_{\infty},\mathcal{M}')\longrightarrow (P,A)$$

be two Sullivan r-minimal models of (P, A). Then there is an isomorphism

$$(G,g):(P_{\infty},\mathcal{M})\longrightarrow (P'_{\infty},\mathcal{M}'),$$

unique up to homotopy, such that $(F, f) \circ (G, g) \simeq (F', f')$.

Proof. By Corollary 6.7 we obtain (G, g), uniquely defined up to homotopy, such that $(F, f) \circ (G, g) \simeq (F', f')$. By Lemma 5.2, (G, g) is an isomorphism.

Denote by \mathbf{Alg}^r the category whose objects are pairs (P, A) where P is a reduced r-tame operad and A is an r-connected P-algebra.

Corollary 6.13. The category \mathbf{Alg}^r is a Sullivan category in the sense of [GNPR10]. In particular, the inclusion induces an equivalence of categories

$$\mathbf{SMin}^r/\simeq \stackrel{\sim}{\longrightarrow} \mathbf{Alg}^r[\mathcal{W}^{-1}]$$
.

Here \mathbf{SMin}^r/\simeq stands for the category of Sullivan r-minimal algebras, whose morphisms are homotopy classes of morphisms in \mathbf{Alg} , and $\mathbf{Alg}^r[\mathcal{W}^{-1}]$ is the localized category of \mathbf{Alg}^r with respect to the class of quasi-isomorphisms.

7. Chain operad algebras

In this section, we verify that our results are also valid for chain operad algebras, i.e., algebras over operads in the category of chain complexes of **k**-vector spaces (with homological grading).

Note that the proofs of Sections 3, 5 and 6 don't depend on any specific behavior of the degree of differentials. In particular, all statements and proofs admit automatic translations to the chain setting just by replacing the word cochain by the word chain everywhere in the text, together with the following minor changes:

- (1) In the definition of a KS-extension (see Definition 3.2) of a free P-algebra A by a graded vector space V' of degree n, the linear map from v' to A goes to $d: V \longrightarrow Z_{n-1}(A)$ (instead of Z^{n+1}).
- (2) The cone of a morphism $f: A \longrightarrow B$ is in the chain setting given by $C(f)_n = A_{n-1} \oplus B_n$ with d(a,b) = (-da,db-f(a)).
- (3) In the definition of the algebra $\mathbf{k}[t, dt]$, dt has degree -1.

We next revise the construction of Sullivan minimal models of Section 4. Let us remark that in the chain setting, we keep the same definition of r-tame operad is in Definition 4.1. We also keep the same definition of Sullivan minimal P-algebra as a colimit of a sequence of KS-extensions ordered by non-decreasing degrees. Note that the key Lemmas 4.4 and 4.5 are still valid in the chain setting, since neither the statements nor the proofs involve any differentials.

Theorem 7.1. Let $r \geq 0$. Let P be an r-tame operad in chain complexes (with homological degree). Then every r-connected P-algebra A has a Sullivan r-minimal model $f: \mathcal{M} \longrightarrow A$ with $\mathcal{M}_0 = P(0)$ and $\mathcal{M}_i = 0$ for all i < r with $i \neq 0$. Furthermore, if A is (r+1)-connected and $H_*(A)$ is of finite type, then \mathcal{M} is of finite type.

Proof. The proof is analogous to that of Theorem 4.6 with minor changes, as done by Neisendorfer in [Nei78] in the case of chain Lie algebras. Let $\mathcal{M}[0] = H_0(A)$ and define $f_0 : \mathcal{M}[0] \longrightarrow A$ by taking a section of the projection $Z_0(A) \twoheadrightarrow H_0(A)$. Then $H_i f_0$ is trivially an isomorphism for i < 0 and an epimorphism for i = 0.

Assuming we have constructed f_{n-1} : $\mathcal{M}[n-1] \longrightarrow A$ with $\mathcal{M}[n-1]$ a Sullivan minimal P-algebra generated in degrees < n and f_{n-1} a morphism such that $H_i f_{n-1}$ is an isomorphism for i < n-1 and an epimorphism for i = n-1, we build $\mathcal{M}[n]$ in two steps:

- (1_n) The map $f'_n: \mathcal{M}[n]' \longrightarrow A$ is obtained from $f_{n-1}: \mathcal{M}[n-1] \longrightarrow A$ after killing the kernel of f_{n-1} in degree n. This is done by successively attaching KS-extensions of degree n-1 (in the (r+1)-connected case, only one KS-extension is needed).
- (2_n) The map $f:\mathcal{M}[n] \longrightarrow A$ is obtained from $f'_n:\mathcal{M}[n]' \longrightarrow A$ after killing the cokernel of f'_n in dimension n+1. This is done by a attaching KS-extension of degree n+1 with trivial differential.

Now, the resulting Sullivan P-algebra $\mathcal{M} = \bigcup_n \mathcal{M}[n]$ is not minimal, since KS-extensions are not ordered by degree. We next show that steps (2_n) and (1_{n+1}) can actually be permuted. Consider the sequence

$$\cdots \longrightarrow \mathcal{M}[n]' \longrightarrow \mathcal{M}[n] = \mathcal{M}[n]' \sqcup_0 P\langle U_{n+1}\rangle \longrightarrow \mathcal{M}[n+1]' = \mathcal{M}[n] \sqcup_d P\langle V_n\rangle \longrightarrow \cdots$$

Since the differential on U_{n+1} is trivial, it suffices to show that

$$d: V_n \longrightarrow Z_{n-1}(\mathcal{M}[n-1] \setminus P\langle U_{n+1} \rangle)$$
.

This is a direct consequence of the fact that $\mathcal{M}[n+1]_{n-1} = \mathcal{M}[n]'_{n-1}$, by Lemma 4.5.

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