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DECOMPOSITION SPACES IN COMBINATORICS

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Abstract. A decomposition space (also called unital 2-Segal space) is a simplicial object satisfying an exactness condition weaker than the Segal condition: just as the Segal condition expresses (up to homotopy) composition, the new condition expresses decomposition. It is a general framework for incidence (co)algebras. In the present contribution, after establishing a formula for the section coefficients, we survey a large supply of examples, emphasising the notion's firm roots in classical combinatorics. The first batch of examples, similar to binomial posets, serves to illustrate two key points: (1) the incidence algebra in question is realised directly from a decomposition space, without a reduction step, and reductions are often given by CULF functors; (2) at the objective level, the convolution algebra is a monoidal structure of species. Specifically, we encounter the usual Cauchy product of species, the shuffle product of $\mathbb{L}$-species, the Dirichlet product of arithmetic species, the Joyal–Street external product of q-species and the Morrison 'Cauchy' product of q-species, and in each case a power series representation results from taking cardinality. The external product of q-species exemplifies the fact that Waldhausen's $S_\bullet$-construction on an abelian category is a decomposition space, yielding Hall algebras. The next class of examples includes Schmitt's chromatic Hopf algebra, the Faà di Bruno bialgebra, the Butcher–Connes–Kreimer Hopf algebra of trees and several variations from operad theory. Similar structures on posets and directed graphs exemplify a general construction of decomposition spaces from directed restriction species. We finish by computing the Möbius function in a few cases, and commenting on certain cancellations that occur in the process of taking cardinality, substantiating that these cancellations are not possible at the objective level.

Contents

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0. Introduction

Decomposition spaces. The notion of decomposition space was introduced by the authors [29, 30, 31] as a general setting for incidence algebras and Möbius inversion, and independently by Dyckerhoff and Kapranov [19], who were motivated by homological algebra, representation theory and geometry. The inherent simplicial nature and the broad scope of applications of the notion prompted a rather abstract categorical and homotopical treatment, with the possible side effect of obscuring its firm roots in combinatorics and its attractive elementary aspects.

The purpose of the present paper is to rectify this possible shortcoming by explaining the combinatorial aspects of the basic theory through many illustrative and natural examples from classical combinatorics. From a theoretical viewpoint, the natural setting for the theory of decomposition spaces is that of simplicial ∞-groupoids, but in fact the notion of decomposition space is interesting even for simplicial sets: there are plenty of natural 'decomposition sets' which are not categories (or posets); some examples can be found in [19]. However, it is our contention that the natural level of generality for decomposition spaces in combinatorics is that of simplicial *groupoids*, simply because many combinatorial objects have symmetries, and these are taken care of elegantly by the groupoid formalism.

From locally finite posets to Möbius categories. To motivate the notion of decomposition space, let us start with incidence coalgebras. Since the work of Joni and Rota [38] we know well that coalgebras in combinatorics arise from the ability to decompose structures into simpler ones. Very often that ability comes from something fancier, namely the ability to actually *compose* structures. A paradigmatic notion of composition is composition of arrows in a category, such as in particular a poset or a monoid. From any locally finite poset, form the free vector space on its intervals, and endow this with a coalgebra structure by defining the comultiplication as

$$\Delta([x,y]) = \sum_{x \leq m \leq y} [x,m] \otimes [m,y].$$

The same construction works for elements in a monoid (with the finite decomposition property [12]). In an appendix to [12], Foata explains how any (reduced) incidence coalgebra of a poset can also be realised as the incidence coalgebra of a monoid, and conversely. However, it seems to be more fruitful to observe as Leroux [51], that both are examples of incidence coalgebras of categories. Recall that a poset can be regarded as a category in which there is at most one arrow between any two given objects. To have an interval $[x, y]$ thus means simply that $x \leq y$, and in categorical terms this means that there is an arrow from x to y. The role of elements in the interval $[x, y]$ is played by the possible two-step factorisations of the arrow $x \to y$. Recall also that a monoid is a category with only one object. The two notions of incidence coalgebras have a common generalisation, namely to locally finite categories, meaning categories in which any given arrow admits only finitely many 2-step factorisations: the incidence coalgebra of such a category is the free vector space on its arrows, with comultiplication given by

$$\Delta(f) = \sum_{ab=f} a \otimes b. \tag{1}$$

The coassociativity is a consequence of the associativity of composition of arrows.

Functoriality. One important point made by Leroux (with Content and Lemay [14]) is that certain functors induce coalgebra homomorphisms. In modern language, these are the *CULF* functors, which stands for *conservative* and *unique lifting of factorisations*. That a functor $F : \mathscr{C} \to \mathscr{D}$ is conservative means that if $F(a)$ is an identity arrow then a was already an identity arrow (see 1.5 below for more precision and discussion). Unique lifting of factorisations means that for an arrow a, there is a one-to-one correspondence between the factorisations of a in $\mathscr{C}$ and the factorisations of $F(a)$ in $\mathscr{D}$.

In the classical theory of posets, often it is not the raw incidence coalgebra that is most interesting, but rather a *reduced* incidence coalgebra, where two intervals are identified if they are equivalent in some specific sense (e.g. isomorphic as abstract posets). As observed in [14], these reductions can quite often be realised by CULF functors. For example, the obvious functor from the poset $(\mathbb{N}, \leq)$ to the monoid $(\mathbb{N}, +)$, sending an 'arrow' $x \leq y$ to the monoid element $y - x$, is CULF and realises a classical reduction: the reduced incidence coalgebra of the poset $(\mathbb{N}, \leq)$ is precisely the raw incidence coalgebra of the monoid $(\mathbb{N}, +)$.

In the general setting of decomposition spaces, virtually *all* reduction procedures become instances of CULF functors, and furthermore, many of them are revealed to be instances of decalage (cf. 1.5.3 below), a general construction in simplicial homotopy theory.

Möbius inversion. Möbius inversion amounts to establishing the convolution invertibility of the zeta function; the inverse is then defined to be the Möbius function [62]. Leroux [51] established a Möbius inversion formula for any *Möbius category*. A category is Möbius when it is locally finite and when for each arrow there are only finitely many ways to write it as a composite of a chain of non-identity arrows. This notion covers both locally finite posets and monoids with the finite-decomposition property. The formula is

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}.$$

Here $\Phi_{\text{even}} = \sum_{k \text{ even}} \Phi_k$, where $\Phi_k(f)$ is the set of decompositions of f into a chain of k composable non-identity arrows. (Similarly for k odd.)

Simplicial viewpoints. The importance of sequences of composable arrows suggests a simplicial viewpoint (see glossary in Appendix B), which is fundamental to the theory of decomposition spaces (and one of the reasons the theory tends to drift into homotopy theory). Recall (see B.1.7) that the nerve of a category $\mathscr{C}$ is the simplicial set

$$N\mathscr{C} : \mathbb{\Delta}^{\text{op}} \to \boldsymbol{Set}$$

whose set of n-simplices is the set of sequences of n composable arrows in $\mathscr{C}$ (allowing identity arrows). The face maps are given by composing arrows (for the inner face maps) and by discarding arrows at the beginning or the end of the sequence (outer face maps). The degeneracy maps are given by inserting an identity map in the sequence.

Leroux's theory can be formulated in terms of simplicial sets, as already exploited by Dür [17], and many of the arguments then rely on certain simple pullback conditions, the first being the Segal condition which characterises categories among simplicial sets (cf. B.2.3). Most importantly in our exploitation of this simplicial viewpoint, the comultiplication (1) can be written in terms of the nerve $N\mathscr{C}$ as a push-pull formula, $\Delta = (d_2, d_0)_! \circ d_1^*$, to be explained below.

Objective method. Möbius inversion is a versatile algebraic counting device. The fact that the formula is always given by an alternating sum illustrates one of the great features of algebra over bijective combinatorics: the existence of additive inverses. On the other hand, it is well appreciated that bijective proofs in general represent deeper insight than purely algebraic proofs.

There is a rather general method for lifting algebraic identities to bijections of sets, which one may try to apply whenever the identity takes place in the vector space spanned by isomorphism classes of objects. This is the so-called objective method, pioneered in this context by Lawvere and Menni [49], working directly with the combinatorial objects rather than their numbers, using linear algebra with coefficients in $\boldsymbol{Set}$ rather than a ring or field.

To illustrate this, observe that a vector in the free vector space on a set S is just a collection of scalars indexed by (a finite subset of) S. The objective counterpart is a family of sets indexed by S, i.e. an object in the slice category $\boldsymbol{Set}_{/S}$. The notion of cardinality has a natural extension to families of finite sets: the cardinality of a family of finite sets indexed by some set B is a B-indexed family of natural numbers, and is in particular an element in the vector space spanned by B. Finiteness issues enter the picture now and should be taken proper care of, see below.

Linear maps at this level are given by spans $S \leftarrow M \to T$, which are, in more abstract terms, the *linear functors*, i.e. functors between slices preserving sums and certain other colimits. Indeed, the pullback formula for composition of spans turns out to correspond precisely to matrix multiplication. Spans have cardinalities, which are linear maps.

The Möbius inversion principle states an equality between certain linear maps (elements in the incidence algebra). At the objective level, such an equality can be expressed as a levelwise bijection of the spans of sets that represents those linear functors. In this way, the algebraic identity is revealed to be the cardinality of a bijection of sets, which carry much more structural information.

Lawvere and Menni [49] established an objective version of the Möbius inversion principle for Möbius categories in the sense of Leroux [51]. A trick is needed to account for the signs: where the algebraic identity states that ζ is convolution invertible with inverse $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$:

$$\zeta * (\Phi_{\text{even}} - \Phi_{\text{odd}}) = \varepsilon,$$

to avoid the minus sign, that term has to be moved to the other side of the equation, and the equivalent statement

$$\zeta * \Phi_{\text{even}} \quad = \quad \varepsilon \quad + \quad \zeta * \Phi_{\text{odd}}$$

can be realised as an explicit bijection of sets [49].

From sets to groupoids. It is useful now to generalise from sets to groupoids, in order to get a better treatment of symmetries. A prominent example illustrating this is the Faà di Bruno coalgebra (treated in detail in 2.4): it ought to be the incidence coalgebra of (a skeleton of) the category of finite sets and surjections but, since finite sets have symmetries, there are too many factorisations, even of identity arrows. This is solved by passing to *fat nerves* (cf. B.2.2). The fat nerve of a category is the simplicial groupoid

$$\mathbf{N}\mathscr{C} : \mathbb{\Delta}^{\text{op}} \to \boldsymbol{Grpd}$$

whose groupoid of n-simplices is the groupoid whose objects are n-sequences of composable arrows, and whose arrows are isomorphisms at each level, as pictured here:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\sim"] & \cdot \arrow[r] \arrow[d, "\sim"] & \cdot \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & \cdot \arrow[d, "\sim"] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot
\end{tikzcd}$$

The slice categories now have to be groupoid slices $\boldsymbol{Grpd}_{/X}$ instead of set slices. Linear algebra works well at this level of generality too (see Appendix A), and there is a notion of homotopy cardinality which is invariant under homotopy equivalence. This approach was initiated by Baez and Dolan [3] and further developed by Baez, Hoffnung and Walker [5]. A cleaner homotopy version of their formalism was introduced in [28], where in particular the notion of homotopy sum is exploited. The upgrade from sets to groupoids is essentially straightforward, as long as the notions involved are taken in a correct homotopy sense, as recalled in Appendix A: bijections of sets are replaced by equivalences of groupoids; the slices playing the role of vector spaces are homotopy slices, the pullbacks and fibres involved in the functors are homotopy pullbacks and homotopy fibres, and the sums are homotopy sums (i.e. colimits indexed by groupoids, just as ordinary sums are colimits indexed by sets).

Decomposition spaces and their incidence (co)algebras. The final abstraction step, which became the starting point for our work [29, 30, 31], and which is where the present paper starts, is to notice that coassociative coalgebras and a Möbius inversion principle can be obtained from simplicial groupoids more general than those satisfying the Segal condition. We call these *decomposition spaces*; Dyckerhoff and Kapranov [19] call them unital 2-Segal spaces. Whereas the Segal condition is the expression of the ability to compose morphisms, the new condition is about the ability to decompose, which of course in general is easier to achieve than composability—indeed every Segal space is a decomposition space (Proposition 1.1.4).

The decomposition space axiom on a simplicial groupoid $X : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$ is expressly the condition needed for a canonical coalgebra structure to be induced on the slice category $\boldsymbol{Grpd}_{/X_1}$. The comultiplication is the linear functor

$$\Delta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1}$$

given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

(with reference to general simplicial notation, reviewed in Appendix B). This can be read as saying that the comultiplication of an edge $f \in X_1$ returns the sum of all pairs of edges (a, b) that are the short edges of a triangle with long edge f. In the case that X is the fat nerve of a category, this is the homotopy sum of all pairs (a, b) of arrows with composite $b \circ a = f$, just as in (1).

Incidence coalgebras, without the need of reduction. It is likely that all incidence (co)algebras can be realised directly (without imposing a reduction) as incidence (co)algebras of decomposition spaces. The decomposition space is found by analysing the reduction step. For example, Dür [17] realises the q-binomial coalgebra as the reduced incidence coalgebra of the category $\boldsymbol{vect}^{\mathrm{inj}}$ of finite dimensional vector spaces over a finite field and linear injections, by imposing the equivalence relation identifying two linear injections if their quotients are isomorphic. Trying to realise the reduced incidence coalgebra directly as a decomposition space immediately leads to Waldhausen's $S_\bullet$-construction, a basic construction in K theory: the q-binomial coalgebra is directly the incidence coalgebra of $S_\bullet(\boldsymbol{vect})$.

Hall algebras. The q-binomial coalgebra fits into a general class of examples: for any abelian category (or even stable ∞-category [29]), the Waldhausen $S_\bullet$-construction is a decomposition space (which is not Segal). Under the appropriate finiteness conditions, the resulting incidence algebras include the Hall algebras, as well as the derived Hall algebras first constructed by Toën [70]. This class of examples plays a key role in the work of Dyckerhoff and Kapranov [18, 19, 20, 21]; we refer to their work for the remarkable richness of the Hall algebra aspects of the theory. See also Bergner et. al [8], Walde [71], and Young [77] for recent contributions in this direction.

Organisation of the paper. In Section 1 we start out with a short, self-contained summary of the basic notions and results of the theory of decomposition spaces, emphasising combinatorial aspects: the definition in Subsection 1.1, their incidence coalgebras in 1.2, and the convolution product in 1.3. In 1.4 we introduce techniques for computing section coefficients, under suitable finiteness conditions, with a closed formula for the case of Segal spaces. In 1.5 we briefly review the notion of CULF functor, relevant because these induce coalgebra homomorphisms. We exploit decalage (a key example of CULF functor) to establish a criterion for local discreteness, essentially the situation in which the section coefficients are integral. We introduce monoidal decomposition spaces as CULF monoidal structures. These induce bialgebras instead of just coalgebras. A running example in this section is Schmitt's Hopf algebra of graphs [65] (called the chromatic Hopf algebra by Aguiar, Bergeron and Sottile [1]), an archetypical example of a coalgebra which cannot be the (raw) incidence coalgebra of a category, but is readily obtained as the incidence coalgebra of a decomposition space. It illustrates well the combinatorial meaning of the decomposition space axiom (Example 1.1.5), the mechanism by which the coalgebra structure arises (1.2.4), and the CULF monoidal structure that makes it a bialgebra (1.5.10).

In Section 2, we first go through some very basic examples, which correspond closely to power series representations of the binomial posets of Doubilet–Rota–Stanley [16], and show how the objective version of these classical incidence algebras amount to monoidal structures on various kinds of species. We emphasise decalage as a general principle behind classical reduction procedures. The case of the Joyal–Street external product of q-species leads to the general treatment of the Waldhausen $S_\bullet$-construction as a decomposition space in 2.3. In 2.4 we revisit the Faà di Bruno bialgebra. Classically it is the reduced incidence bialgebra of the poset of set partitions (reduction modulo type equivalence), but can also be obtained directly from the category of surjections. This suggests that again the reduction step is a decalage, but the relationship turns out to be more subtle: it is a CULF functor but not directly a decalage. In 2.5 we treat examples related to trees and graphs, starting with the Butcher–Connes–Kreimer Hopf algebra of trees [13], another example of an incidence coalgebra which cannot be the (raw) incidence coalgebra of a category. We proceed to treat operadic variations, including incidence coalgebras of general operads, as well as related constructions with directed graphs (cf. Manchon [55] and Manin [56]). We briefly explain how most of the examples treated in this subsection are subsumed in the notion of decomposition spaces from restriction species and directed restriction species, treated in detail elsewhere [32].

In Section 3 we come to Möbius inversion, and need first to recall a few notions from [30]: complete decomposition spaces and nondegeneracy in 3.1, and the notion of locally finite length and the general Möbius inversion formula in 3.2. In 3.3 we compute the Möbius function in a few easy cases, and comment on certain cancellations that occur in the process of taking cardinality, substantiating that these cancellations are not possible at the objective level. This is related to the distinction between bijections and natural bijections.

In Appendix A we provide background on groupoids necessary to understand groupoid slices as the objective analogue of vector spaces, and linear functors and spans as the objective analogue of linear maps. We also explain how to recover the vector space level via taking homotopy cardinality.

In Appendix B we briefly recall the simplicial machinery that is an essential tool in our undertakings, with special emphasis on the relationship with simplicial complexes. In particular we explain the nerve and the fat nerve of a small category, whereby the simplicial setting covers the cases of categories, and in particular posets and monoids.

Note. This work was originally Section 5 of the large single manuscript *Decomposition spaces, incidence algebras and Möbius inversion* [27]. For publication, that manuscript has been split into six papers: [28, 29, 30, 31, 32] and the present paper.

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1. Decomposition spaces and incidence coalgebras

1.1. Segal spaces and decomposition spaces

Segal spaces and decomposition spaces are simplicial groupoids $X : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$ satisfying certain exactness properties. We refer to Appendix B for a glossary on simplicial groupoids.

1.1.1. Segal spaces (Segal groupoids). A simplicial groupoid X is called a *Segal space*, or a *Segal groupoid*, when all squares of the form

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_0"] \arrow[d, "d_{n+1}"'] & X_n \arrow[d, "d_n"] \\ X_n \arrow[r, "d_0"'] & X_{n-1} \end{tikzcd}$$

are (homotopy) pullbacks (see Appendix A.1.5).

The most important such square is

(2)
$$\begin{tikzcd} X_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"'] & X_0 \end{tikzcd}$$

which says that X_2 can be identified with the groupoid $X_1 \times_{X_0} X_1$ of composable pairs of 'arrows'. This is satisfied by the nerve or the fat nerve of a small category.

For a Segal space X, the vector space spanned by $\pi_0 X_1$ has a coalgebra structure analogous to (1).

It turns out [29] that simplicial groupoids other than Segal spaces induce coalgebras. These are the decomposition spaces, which are characterised by a weaker exactness condition than the Segal condition. To give the explicit definitions we need first some simplicial terminology. We refer to Appendix B for notation (which is standard).

1.1.2. Face and degeneracy maps, generic and free maps. The simplex category $\mathbb{\Delta}$ (see Appendix B) has a so-called generic-free factorisation system (a general categorical notion, important in monad theory [74, 75]). An arrow $a : [m] \to [n]$ in Δ is *generic* when it preserves end-points, $a(0) = 0$ and $a(m) = n$; and it is *free* if it is distance preserving, $a(i+1) = a(i)+1$ for $0 \leq i \leq m-1$. The generic maps are generated by the codegeneracy maps $s^i : [n+1] \to [n]$ and by the *inner* coface maps $d^i : [n-1] \to [n]$, $0 < i < n$, while the free maps are generated by the *outer* coface maps $d^\perp := d^0$ and $d^\top := d^n$. Every morphism in Δ factors uniquely as a generic map followed by a free map. Furthermore, it is a basic fact [29] that generic and free maps in Δ admit pushouts along each other, and the resulting maps are again generic and free. For a simplicial groupoid $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$, the images of generic and free maps in Δ are again called generic and free.

1.1.3. Decomposition spaces [29]**.** A simplicial groupoid $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ is called a *decomposition space* when it takes generic-free pushouts to pullbacks.

One can break this down to checking that the following simplicial-identity squares are pullbacks. The diagrams are rendered with the generic maps horizontal and the free maps vertical, and the indices are $n \geq 0$ and $0 \leq k \leq n$:

$$
\begin{tikzcd}
X_{n+1} \arrow[r, "s_{k+1}"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "\lrcorner" near start] & X_{n+2} \arrow[d, "d_\perp"'] & X_{n+3} \arrow[l, "d_{k+2}"'] \arrow[d, "d_\perp"] \arrow[dl, phantom, "\llcorner" near start] \\
X_n \arrow[r, "s_k"'] & X_{n+1} & X_{n+2} \arrow[l, "d_{k+1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{n+1} \arrow[r, "s_k"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "\lrcorner" near start] & X_{n+2} \arrow[d, "d_\top"'] & X_{n+3} \arrow[l, "d_{k+1}"'] \arrow[d, "d_\top"] \arrow[dl, phantom, "\llcorner" near start] \\
X_n \arrow[r, "s_k"'] & X_{n+1} & X_{n+2} \arrow[l, "d_{k+1}"]
\end{tikzcd}
\tag{3}
$$

The most important cases are the four squares that involve $d_1 : X_2 \to X_1$ (corresponding to composition of arrows in a category) and $s_0 : X_0 \to X_1$ (corresponding to the identity arrows in a category):

$$
\begin{tikzcd}
X_2 \arrow[d, "d_\perp"'] & X_3 \arrow[l, "d_2"'] \arrow[d, "d_\perp"] \arrow[dl, phantom, "\llcorner" near start] \\
X_1 & X_2 \arrow[l, "d_1"]
\end{tikzcd}
\quad
\begin{tikzcd}
X_2 \arrow[d, "d_\top"'] & X_3 \arrow[l, "d_1"'] \arrow[d, "d_\top"] \arrow[dl, phantom, "\llcorner" near start] \\
X_1 & X_2 \arrow[l, "d_1"]
\end{tikzcd}
\quad
\begin{tikzcd}
X_1 \arrow[r, "s_1"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "\lrcorner" near start] & X_2 \arrow[d, "d_\perp"] \\
X_0 \arrow[r, "s_0"'] & X_1
\end{tikzcd}
\quad
\begin{tikzcd}
X_1 \arrow[r, "s_0"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "\lrcorner" near start] & X_2 \arrow[d, "d_\top"] \\
X_0 \arrow[r, "s_0"'] & X_1
\end{tikzcd}
\tag{4}
$$

We shall see shortly that the first two pullback squares are essential ingredients in getting coassociativity of the *incidence coalgebra* of X, and the last two pullback squares are essential in getting counitality.

Although the Segal axiom squares are quite different from the decomposition space axioms, it is not difficult to prove the following, which shows that the new setting of decomposition spaces does cover the cases of nerves and fat nerves of categories.

Proposition 1.1.4. *(*[29, Proposition 3.5]*,* [19, Proposition 5.2.6]*) Every Segal space is a decomposition space.*

1.1.5. Example (Schmitt's Hopf algebra of graphs). We give an example of a decomposition space which is not a Segal space, to illustrate the combinatorial meaning of the pullback condition: it is about structures that can be decomposed but not always composed. We shall continue this example in 1.2.4, and see that it corresponds to the Hopf algebra of graphs of Schmitt [65].

We define a simplicial groupoid X by taking X_1 to be the groupoid of graphs (admitting multiple edges and loops), and more generally letting X_k be the groupoid of graphs with an ordered partition of the vertex set into k parts (possibly empty). In particular, X_0 is the contractible groupoid consisting only of the empty graph.

These groupoids form a simplicial object: the outer face maps delete the first or last part of the graph, and the inner face maps join adjacent parts. The degeneracy maps insert an empty part. The simplicial identities are readily checked.

It is clear that X is not a Segal space: for the Segal square (2)

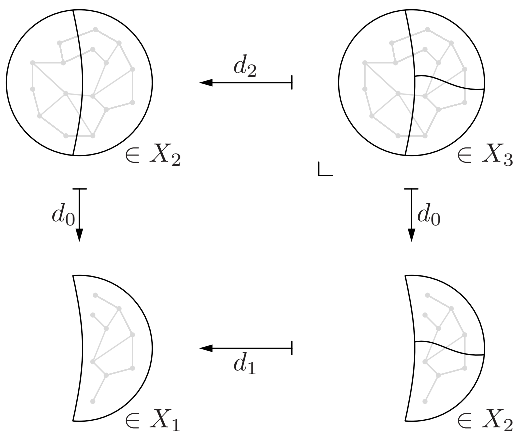
$$\begin{tikzcd} X_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"'] & X_0 \end{tikzcd}$$

to be a pullback would mean that a graph with a two-part partition could be reconstructed uniquely from knowing the two parts individually. But this is not true, because the two parts individually contain no information about the edges going between them.

One can check that it *is* a decomposition space: that the square

$$\begin{tikzcd} X_2 \arrow[d, "d_0"'] & X_3 \arrow[l, "d_2"'] \arrow[d, "d_0"] \arrow[dl, phantom, very near start, "\llcorner"] \\ X_1 & X_2 \arrow[l, "d_1"] \end{tikzcd}$$

is a pullback is to say that a graph with a three-part partition ($\in X_3$) can be reconstructed uniquely from a pair of elements in X_2 with common image in X_1 (under the indicated face maps). The following picture represents elements corresponding to each other in the four groupoids.



The horizontal maps join the last two parts of the partition. The vertical maps forget the first part. Clearly the diagram commutes. To reconstruct the graph with a three-part partition (upper right-hand corner), most of the information is already available in the upper left-hand corner, namely the underlying graph and all the subdivisions except the one between part 2 and part 3. But this information is precisely available in the lower right-hand corner, and their common image in X_1 says precisely how this missing piece of information is to be implanted.

1.2. Incidence coalgebras of decomposition spaces

We now turn to the incidence coalgebra (with groupoid coefficients) associated to any decomposition space, explaining the origin of the decomposition space axioms.

The incidence coalgebra associated to a decomposition space X will be a comonoid object in the symmetric monoidal 2-category $\boldsymbol{LIN}$ (whose objects are groupoid slices and whose morphisms are linear functors—see A.3), and the underlying object is $\boldsymbol{Grpd}_{/X_1}$. Since $\boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1} = \boldsymbol{Grpd}_{/X_1 \times X_1}$, and since linear functors are given by spans, to define a comultiplication functor is to give a span

$$X_1 \leftarrow M \rightarrow X_1 \times X_1.$$

1.2.1. Comultiplication and counit. For X a decomposition space, we can consider the following structure maps on $\boldsymbol{Grpd}_{/X_1}$. The span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1 \tag{5}$$

defines a linear functor, the *comultiplication*

$$\begin{array}{rcl} \Delta : \boldsymbol{Grpd}_{/X_1} & \longrightarrow & \boldsymbol{Grpd}_{/(X_1 \times X_1)} \\ (T \xrightarrow{t} X_1) & \longmapsto & (d_2, d_0)_! \circ {d_1}^*(t). \end{array}$$

Likewise, the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{z} 1 \tag{6}$$

defines a linear functor, the *counit*

$$\begin{array}{rcl} \varepsilon : \boldsymbol{Grpd}_{/X_1} & \longrightarrow & \boldsymbol{Grpd} \\ (T \xrightarrow{t} X_1) & \longmapsto & z_! \circ {s_0}^*(t). \end{array}$$

We proceed to explain that coassociativity follows from the decomposition space axiom. The coalgebra $(\boldsymbol{Grpd}_{/X_1}, \Delta, \varepsilon)$ is called the *incidence coalgebra* of the decomposition space X. (Note that in the classical incidence-algebra literature (e.g. [62], [51]), the counit is denoted δ.)

1.2.2. Coassociativity. The comultiplication and counit maps on $\boldsymbol{Grpd}_{/X_1}$, defined in 1.2.1 for any simplicial groupoid X, become coassociative and counital when the decomposition space axioms hold for X. The desired coassociativity diagram (which should commute up to equivalence)

$$\begin{tikzcd} \boldsymbol{Grpd}_{/X_1} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \boldsymbol{Grpd}_{/X_1 \times X_1} \arrow[d, "\Delta \otimes \mathrm{id}"] \\ \boldsymbol{Grpd}_{/X_1 \times X_1} \arrow[r, "\mathrm{id} \otimes \Delta"'] & \boldsymbol{Grpd}_{/X_1 \times X_1 \times X_1} \end{tikzcd}$$

is induced by the solid spans in the diagram

$$\begin{tikzcd}
X_1 & X_2 \arrow[l, "d_1"'] \arrow[r, "{(d_2,d_0)}"] & X_1 \times X_1 \\
X_2 \arrow[u, "d_1"] \arrow[d, "{(d_2,d_0)}"'] & X_3 \arrow[l, dashed, "d_2"'] \arrow[r, dashed, "{(d_3,d_0d_0)}"'] \arrow[u, dashed, "d_1"] \arrow[d, dashed, "{(d_2d_2,d_0)}"] \arrow[ur, phantom, "\urcorner" very near start] \arrow[dl, phantom, "\llcorner" very near start] & X_2 \times X_1 \arrow[u, "d_1 \times \mathrm{id}"'] \arrow[d, "{(d_2,d_0) \times \mathrm{id}}"] \\
X_1 \times X_1 & X_1 \times X_2 \arrow[l, "\mathrm{id} \times d_1"] \arrow[r, "{\mathrm{id} \times (d_2,d_0)}"'] & X_1 \times X_1 \times X_1 .
\end{tikzcd}$$

Coassociativity will follow from the Beck–Chevalley Lemma A.3.2 if the dashed part of the diagram can be established with pullbacks as indicated. Consider the upper right-hand square: it will be a pullback if and only if its composite with the first projection is a pullback:

$$\begin{tikzcd}
X_2 \arrow[r, "{(d_\top,d_0)}"] & X_1 \times X_1 \arrow[r, "\mathrm{pr}_1"] & X_1 \\
X_3 \arrow[u, "d_1"] \arrow[r, "{(d_\top,d_0d_0)}"'] \arrow[ur, phantom, "\urcorner" very near start] & X_2 \times X_1 \arrow[u, "d_1 \times \mathrm{id}"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur, phantom, "\urcorner" very near start] & X_2 . \arrow[u, "d_1"']
\end{tikzcd}$$

Saying that this composite outer square $d_\top d_1 = d_1 d_\top$ is a pullback is precisely one of the first decomposition space axioms (4).

If one is just interested in coassociativity at the level of π_0, this pullback and its twin, $d_\perp d_2 = d_1 d_\perp$, are all that are needed, as was the case in the work of Toën [70] who dealt with the case where X is the Waldhausen $S_\bullet$ construction of a dg category. On the other hand, it is interesting to analyse when the coassociativity is actually homotopy coherent at the level of groupoid slices. It is proved in [29, Theorem 7.3] that this is true when all the decomposition space axioms hold:

Theorem 1.2.3. *If X is a decomposition space then $\boldsymbol{Grpd}_{/X_1}$ has the structure of strong homotopy comonoid in the symmetric monoidal category $\boldsymbol{LIN}$, with the comultiplication and counit defined by the spans* (5) *and* (6).

1.2.4. Example: Schmitt's Hopf algebra of graphs, continued. The following coalgebra is due to Schmitt [65]. For a graph G with vertex set V (admitting multiple edges and loops), and a subset $U \subset V$, define $G|U$ to be the graph whose vertex set is U, and whose graph structure is induced by restriction (that is, the edges of $G|U$ are those edges of G both of whose incident vertices belong to U). On the vector space spanned by isomorphism classes of graphs, define a comultiplication by the rule

$$\Delta(G) = \sum_{A+B=V} G|A \otimes G|B.$$

This coalgebra is obtained from the decomposition space in Example 1.1.5. Indeed, we have to take X_1 the groupoid of graphs, because the coalgebra is spanned by isomorphism classes of graphs. Since the comultiplication sums over all ways to partition the vertex set into two parts (possibly empty), we must take X_2 to be the groupoid of graphs with a two-part partition of the vertex set. (More generally, X_k

is the groupoid of graphs with an ordered partition of the vertex set into k parts (possibly empty).)

Taking pullback along $d : X_2 \to X_1$ is to consider all possible two-part partitions of a given graph, and taking lowershriek along $(d_2, d_0) : X_2 \to X_1 \times X_1$ is to return the graphs induced by the two parts. In conclusion, this is precisely Schmitt's comultiplication.

1.2.5. Comultiplication of basis elements. We proceed to spell out the effect of the comultiplication on basis elements. The slice $\textbf{\textit{Grpd}}_{/X_1}$ has a canonical basis $\{\ulcorner f \urcorner : 1 \to X_1\}_{f\in\pi_0 X_1}$. Here $\ulcorner f \urcorner : 1 \to X_1$ denotes the map that singles out the element $f \in X_1$, in category theory called the *name* of f. The notion of basis for slices means that every object $T \to X_1$ can be written uniquely as a homotopy sum of names (cf. Lemma A.2.7). Giving $\ulcorner f \urcorner$ as input to the comultiplication, and expanding the result into a homotopy sum of names, we get:

$$\begin{aligned}
\Delta(\ulcorner f \urcorner) &:= \big((X_2)_f \xrightarrow{d_1^*\ulcorner f \urcorner} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1\big) \\
&= \int^{\sigma\in(X_2)_f} \ulcorner d_2\sigma \urcorner \otimes \ulcorner d_0\sigma \urcorner \\
&= \int^{(a,b)\in X_1\times X_1} (X_2)_{f,a,b}\, \ulcorner a \urcorner \otimes \ulcorner b \urcorner \quad \in \ \textbf{\textit{Grpd}}_{/X_1} \otimes \textbf{\textit{Grpd}}_{/X_1}.
\end{aligned} \tag{7}$$

Here $(X_2)_f$ is the fibre of $d_1 : X_2 \to X_1$ over f, and similarly $(X_2)_{f,a,b}$ is the fibre of $(d_1, d_2, d_0) : X_2 \to X_1 \times X_1 \times X_1$ over (f, a, b). Here and throughout, 'fibre' means 'homotopy fibre', cf. A.1.6.

If X is the strict nerve of a category then X_2 is the set of all composable pairs of arrows and $(X_2)_f$ is the subset of those pairs with composite f. In particular, $(X_2)_{f,a,b}$ is then either empty or a singleton, and the comultiplication reduces to the formula (1) from the introduction,

$$\Delta(f) = \sum_{ab=f} a \otimes b.$$

If X is the fat nerve of a category (or more generally X is Segal space, that is, $X_2 \simeq X_1 \times_{X_0} X_1$), then as in the case of the ordinary nerve we see that $X_{f,a,b}$ is empty unless, up to isomorphism, $f = ab$ (a followed by b) and $d_0 a = d_1 b = y$, say.

Recall that $\Omega_b(B)$ denotes the loop groupoid at b in B—actually $\Omega_b(B)$ is just a discrete groupoid, equivalent to the set of elements in the group $\mathrm{Aut}_B(b)$, see A.1.7.

Proposition 1.2.6. *If X is a Segal space, with $f = ab$ as above, then $(X_2)_{f,a,b} = \Omega_y(X_0) \times \Omega_f(X_1)$ and hence*

$$\Delta(\ulcorner f \urcorner) = \int^{(a,b)\in X_1\times X_1} \Omega_y(X_0) \times \Omega_f(X_1)\, \ulcorner a \urcorner \otimes \ulcorner b \urcorner \ \in \ \textbf{\textit{Grpd}}_{/X_1} \otimes \textbf{\textit{Grpd}}_{/X_1}.$$

Proof. Observe that $(X_2)_{a,b}$ can be calculated by the pullback

$$\begin{tikzcd}
(X_2)_{a,b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 = X_1 \times_{X_0} X_1 \arrow[r] \arrow[d, "(d_2,d_0)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d, "\mathrm{diag}"] \\
1 \arrow[r, "\ulcorner a,b \urcorner"'] & X_1 \times X_1 \arrow[r, "d_0 \times d_1"'] & X_0 \times X_0 .
\end{tikzcd} \tag{8}$$

The fibre of the diagonal map is the set of loops (see A.1.7), so we get $(X_2)_{a,b} = \Omega_y(X_0)$ for a and b composable (at y) as we have assumed. Now $(X_2)_{f,a,b}$, the pullback of $\Omega_y(X_0) = (X_2)_{a,b} \to X_1$ along $\ulcorner f \urcorner$, is just $\Omega_y(X_0) \times \Omega_f(X_1)$. □

1.2.7. Local finiteness. As long as we work at the objective level, where all results and proofs are naturally bijective, it is not necessary to impose any finiteness conditions. But in order to be able to take cardinality to recover numerical results (i.e. at the vector-space level), suitable finiteness conditions must be imposed. Intuitively, mimicking the local finiteness for categories, we should require that for each $n \in \mathbb{N}$, the map $X_n \to X_1$ be finite. In the category case, this means that for each arrow $f \in X_1$ and $n \in \mathbb{N}$, there are only finitely many decompositions of f into a sequence of n arrows. Technically, the appropriate definition is the following (from [30]).

A decomposition space $X : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$ is termed *locally finite* if X_1 is locally finite (in the sense of groupoids A.1.4) and both $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps. Then the comultiplication and counit defined above are finite linear functors, and hence (by Proposition A.4.3) descend to slices of finite groupoids

$$\Delta : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}_{/X_1} \otimes \boldsymbol{grpd}_{/X_1} \,, \qquad \varepsilon : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}.$$

We can then take cardinality to obtain comultiplication and counit maps of vector spaces

$$|\Delta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \,, \qquad |\varepsilon| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q} \,.$$

These are coassociative and counital, and

$$\mathcal{I}_X := (\mathbb{Q}_{\pi_0 X_1}, |\Delta| \,, |\varepsilon|)$$

is what we call the *numerical incidence coalgebra* of X.

Remark 1.2.8. If X is the nerve of a poset P, then it is locally finite in the above sense if and only if all intervals $[x, y]$ are finite, which is the usual definition for posets [67]. The points in this interval parametrise precisely the two-stage factorisations of the unique arrow $x \to y$, so this condition amounts to $X_2 \to X_1$ having finite fibre over $x \to y$. (In the poset case, the conditions on X_1 and on $s_0 : X_0 \to X_1$ are automatically satisfied, since everything is discrete.)

Examples of infinite categories which are locally finite are given by free monoids or the free category on a directed graph.

1.3. **Convolution algebras**

1.3.1. Linear dual. If X is a decomposition space, we have seen there is a natural coassociative comultiplication on $\boldsymbol{Grpd}_{/X_1}$, the incidence coalgebra of X, which we see as an 'objectification' of the vector space $\mathbb{Q}_{\pi_0 X_1}$ underlying the classical incidence coalgebra. One may also consider the incidence (or convolution) *algebra*

$\boldsymbol{Grpd}^{X_1}$, which can be obtained from the incidence coalgebra by taking the linear dual (A.3.4). Since $\boldsymbol{Grpd}_{/X_1}$ is the free homotopy-sum completion of X_1 (just as $\mathbb{Q}_{\pi_0 X_1}$ is the 'linear-combination completion' of the set $\pi_0 X_1$), objects in $\boldsymbol{Grpd}^{X_1}$ can be regarded either as presheaves $X_1 \to \boldsymbol{Grpd}$ or as linear functors $\boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$ (see A.3.4). The category $\boldsymbol{Grpd}^{X_1}$ is interpreted as an 'objectification' of the incidence algebra, denoted $\mathcal{I}^X$, which has underlying profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$.

1.3.2. Convolution. The multiplication in the incidence algebra is the convolution product, given as the dual of the comultiplication. Consider two linear functors

$$F, G : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}$$

given by spans $X_1 \leftarrow M \to 1$ and $X_1 \leftarrow N \to 1$. Their tensor product $F \otimes G$ is then given by the span

$$X_1 \times X_1 \leftarrow M \times N \to 1$$

and their *convolution* $F * G$ is the composite of $F \otimes G$ with the comultiplication:

$$F * G : \boldsymbol{Grpd}_{/X_1} \xrightarrow{F \otimes G} \boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1} \xrightarrow{\Delta} \boldsymbol{Grpd}.$$

This is given by the composite span

$$\begin{tikzcd}
X_1 & & \\
X_2 \arrow[u] \arrow[d] & M * N \arrow[l] \arrow[ul] \arrow[d] \arrow[dr] \arrow[dl, phantom, very near start, "\llcorner"] & \\
X_1 \times X_1 & M \times N \arrow[l] \arrow[r] & 1.
\end{tikzcd}$$

The neutral functor for the convolution product is ε.

1.3.3. The zeta functor. The *zeta functor*

$$\zeta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$$

is the linear functor defined by the span

$$X_1 \xleftarrow{=} X_1 \to 1\,.$$

As an element of $\boldsymbol{Grpd}^{X_1}$, this is the terminal presheaf.

Assuming X_1 locally finite then ζ is a finite linear functor and descends to

$$\zeta : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}.$$

Its cardinality $\mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, which can be regarded as an element in the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$, is then the usual zeta function $\pi_0 X_1 \to \mathbb{Q}$ with value 1 on each 1-simplex of X.

1.4. Section coefficients

1.4.1. Section coefficients. If X is a locally finite decomposition space then the homotopy cardinality of the comultiplication at the objective level

$$\begin{array}{rcl} \boldsymbol{grpd}_{/X_1} & \longrightarrow & \boldsymbol{grpd}_{/X_1\times X_1} \\ \ulcorner f \urcorner & \longmapsto & \big((X_2)_f \to X_2 \to X_1 \times X_1\big) \end{array}$$

yields a comultiplication in the category of vector spaces

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \delta_f & \longmapsto & \displaystyle\int^{(a,b)\in X_1\times X_1} |(X_2)_{f,a,b}| \ \delta_a \otimes \delta_b \ = \ \sum_{a,b} c^f_{a,b}\, \delta_a \otimes \delta_b, \end{array}$$

which defines the (numerical) incidence coalgebra $\mathcal{I}_X$. It is just the cardinality of (7), with the *section coefficients*

$$c^f_{a,b} := \frac{|(X_2)_{f,a,b}|}{|\mathrm{Aut}(a)|\,|\mathrm{Aut}(b)|}.$$

Taking cardinality of Proposition 1.2.6 gives the following explicit formula for the section coefficients.

Proposition 1.4.2. *If X is a locally finite Segal space then*

$$c^f_{a,b} = \frac{|\mathrm{Aut}(y)|\,|\mathrm{Aut}(ab)|}{|\mathrm{Aut}(a)|\,|\mathrm{Aut}(b)|}$$

if $d_0 a = d_1 b = y$ say, and $f = ab$; otherwise, $c^f_{a,b} = 0$.

1.4.3. 'Zeroth section coefficients': the counit. Let us also say a word about the zeroth section coefficients, i.e. the computation of the counit. If f is not isomorphic to a degenerate simplex then clearly $|\varepsilon|\,(\delta_f) = 0$. In the case f is degenerate, we just remark on two special cases:

- if X is complete (3.1.1), meaning that s_0 is a monomorphism (A.2.4), then $|\varepsilon|\,(\delta_f) = 1$,
- if $X_0 = *$ then $|\varepsilon|\,(\delta_f) = |\Omega_f(X_1)| = |\mathrm{Aut}(f)|$.

1.4.4. Numerical convolution product. By duality, if X is locally finite, the convolution product descends to the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$ obtained by taking cardinality of $\boldsymbol{grpd}^{X_1}$, defining the (numerical) incidence algebra of X, denoted $\mathcal{I}^X$. It follows from the general theory of homotopy linear algebra (see appendix A.4.5 and [28]) that the cardinality of the convolution product is the linear dual of the cardinality of the comultiplication. Since it is the same span that defines the comultiplication and the convolution product, it is also the exact same matrix that defines the cardinalities of these two maps. It follows that the structure constants for the convolution product (with respect to the pro-basis $\{\delta^f\}$) are the same as the structure constants for the comultiplication (with respect to the basis $\{\delta_f\}$), i.e. the section coefficients.

1.4.5. Example. The strict nerve of a category $\mathscr{C}$ is a decomposition space which is discrete in each degree. The resulting coalgebra at the numerical level (assuming local finiteness) is the coalgebra of Content–Lemay–Leroux [14], and if the category is just a poset, that of Joni and Rota [38].

The objective-level incidence algebra of the strict nerve of $\mathscr{C}$ has the convolution product

$$h^a * h^b = \begin{cases} h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else.} \end{cases} \tag{9}$$

For the fat nerve X of $\mathscr{C}$, we find instead

$$h^a * h^b = \begin{cases} \Omega_y(X_0)\ h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else.} \end{cases} \tag{10}$$

To compute the cardinality of this algebra, note first that the cardinality of the representable h^a is generally different from the canonical basis element δ^a: the formula (25) says

$$|h^a| = |\Omega_a(X_1)|\ \delta^a,$$

leading again to the section coefficients in 1.4.2.

1.4.6. Finite support. The numerical incidence algebra $\mathcal{I}^X$ lives in profinite-dimensional vector spaces, since functions are not required to have finite support—for example, the zeta function does not have finite support for infinite posets or categories. It is also interesting to consider the subalgebra of $\mathcal{I}^X$ consisting of functions with finite support. At the objective level this is the full subcategory $\boldsymbol{grpd}^{X_1}_{\text{fin.sup.}} \subset \boldsymbol{grpd}^{X_1}$, and numerically it is $\mathbb{Q}^{\pi_0 X_1}_{\text{fin.sup.}} \subset \mathbb{Q}^{\pi_0 X_1}$. Of course we have canonical identifications $\boldsymbol{grpd}^{X_1}_{\text{fin.sup.}} \simeq \boldsymbol{grpd}_{/X_1}$, as well as $\mathbb{Q}^{\pi_0 X_1}_{\text{fin.sup.}} \simeq \mathbb{Q}_{\pi_0 X_1}$, but it is important to keep track of which side of duality we are on.

That the decomposition space is locally finite is not the appropriate condition for the convolution and unit to restrict to the functions with finite support. Instead the requirement is that X_1 be locally finite and the maps

$$X_2 \to X_1 \times X_1, \qquad X_0 \to 1$$

be finite. By (8) we know that the former map is finite for any Segal space with X_0 *locally* finite, but for the latter X_0 must actually be finite.

1.4.7. Examples: category algebras. If X is the *strict* nerve of a category $\mathscr{C}$, then the finite-support convolution algebra is precisely the *category algebra* of $\mathscr{C}$. This is an important notion in representation theory (see [73]).

Note that since the strict nerve is a Segal space, the formula for the section coefficients are the same as computed above, giving the familiar formula (9). Similarly the formula for the convolution unit is

$$\varepsilon = \sum_x \delta^{\mathrm{id}_x} = \begin{cases} 1 & \text{for id arrows} \\ 0 & \text{else,} \end{cases}$$

the sum of all indicator functions of identity arrows: for this to be finite we need to require that the category has only finitely many objects.

In the case of the *fat* nerve of a category $\mathscr{C}$, the finiteness condition for having a finite-support convolution is implied by the condition that every object in $\mathscr{C}$ has a finite automorphism group (a condition implied by local finiteness). On the other hand, the convolution unit has finite support precisely when there is only a finite number of isomorphism classes of objects, already a more drastic condition. Compared to the usual category algebra, this 'fat category algebra' has a symmetry factor (cf. (10)):

$$h^a * h^b = \begin{cases} \Omega_y(X_0)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else.} \end{cases}$$

Note that an important source of examples of category algebras are given by the path algebra of a quiver Q (see for example [9]): that is simply the category algebra on the free category on Q. Since there are no automorphisms in a free category, in this case there is no difference between strict and fat nerve.

It should be noted that the finite-support incidence algebras are important also outside the setting of category algebras, namely in the case of the Waldhausen $S_\bullet$-construction (cf. 2.3 below): they are the Hall algebras (see [29]). The finiteness conditions are then homological, namely finite Ext^0 and Ext^1.

1.4.8. Locally discrete decomposition spaces. In the formula in 1.4.2 for the section coefficients there are denominators. In very many examples of importance, however, the section coefficients are actually integral. This happens when the map $d_1 : X_2 \to X_1$ is discrete (and for the zeroth section coefficients one should also require $s_0 : X_0 \to X_1$ to be finite, but this is automatic for complete decomposition spaces 3.1.1, such as (fat) nerves).

We define X to be *locally discrete* when $d_1 : X_2 \to X_1$ and $s_0 : X_0 \to X_1$ are discrete maps.

Remark 1.4.9. In our terminology (1.2.7), 'locally finite' means that $d_1 : X_2 \to X_1$ and $s_0 : X_0 \to X_1$ are finite maps *and* that X_1 is locally finite. To be consistent with this definition, 'locally discrete' should mean $d_1 : X_2 \to X_1$ and $s_0 : X_0 \to X_1$ discrete, *and* X_1 a locally discrete groupoid. If we define a groupoid to be locally discrete if all its hom sets are discrete, then every groupoid is locally discrete, and therefore it is not necessary to mention it in the definition.

1.4.10. Examples. The fat nerve of a category $\mathscr{C}$ is locally discrete if it satisfies anyone of the three conditions

- All the arrows in $\mathscr{C}$ are monos
- All the arrows in $\mathscr{C}$ are epis
- $\mathscr{C}$ is (equivalent to) a category with no isomorphisms other than the identity arrows.

Starting from these three cases, many more examples can be derived in virtue of the following result.

Lemma. 1.4.11. *The following are equivalent for a decomposition space X*

(1) *X is locally discrete.*

(2) $\mathrm{Dec}_\perp(X)$ *is locally discrete.*
(3) $\mathrm{Dec}_\top(X)$ *is locally discrete.*

This result refers to *decalage* (1.5.3), recalled in the next subsection where we also prove the lemma.

As we shall see, examples coming from combinatorics tend to be locally discrete.

1.4.12. A tiny example: the 'hanger category'. The following category is perhaps the smallest example of a category whose fat nerve is *not* locally discrete.

$$\begin{tikzcd}
& \cdot \arrow[loop above, "j"] \arrow[dr, "b"] & \\
\cdot \arrow[ur, "a"] \arrow[rr, "f"'] & & \cdot
\end{tikzcd}$$

in which

$$ab = f \qquad jj = 1 \qquad aj = a \qquad jb = b.$$

It has

$$\Delta(f) = 1 \otimes f + f \otimes 1 + \frac{a \otimes b}{2}$$

since the factorisation ab admits an involution, given by j.

1.5. **CULF functors, coalgebra homomorphisms and bialgebras**

An appropriate notion of morphism between decomposition spaces is that of CULF functors [29], which we briefly recall. Their importance is that they induce coalgebra homomorphisms between the incidence coalgebras. Two main instances of CULF functors are decalage and monoidal structures. As we shall see, decalage accounts for many reduction procedures in classical theory of incidence coalgebras. A CULF monoidal structure on a decomposition space is precisely what makes the incidence coalgebra into a bialgebra.

1.5.1. CULF functors. A simplicial map $F : X \to Y$ is

- *conservative* if it is cartesian with respect to codegeneracy maps (11a).
- *ULF* (for Unique Lifting of Factorisations) if it is cartesian with respect to inner coface maps (11b).
- *CULF* if it is both conservative and ULF, that is, cartesian on all generic maps. We shall use the term *CULF functor* even between simplicial groupoids not assumed to be Segal.

$$\begin{tikzcd}
X_n \arrow[r, "s_i"] \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner\ \text{(a)}"] & X_{n+1} \arrow[d, "F"] \\
Y_n \arrow[r, "s_i"'] & Y_{n+1},
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_{n+1} \arrow[d, "F"'] \arrow[dr, phantom, "\text{(b)}\ \llcorner"] & X_{n+2} \arrow[l, "d_{i+1}"'] \arrow[d, "F"] \\
Y_{n+1} & Y_{n+2} \arrow[l, "d_{i+1}"]
\end{tikzcd}
\qquad (0 \leq i \leq n). \tag{11}$$

If both X and Y are decomposition spaces, then in fact ULF implies CULF [29, Proposition 4.2].

In many examples of decomposition spaces, 1-simplices are thought of as arrows: for simplicial maps between Rezk complete Segal spaces (see B.2.3), conservative means not inverting any arrows, and ULF means inducing a one-to-one correspondence between factorisations of an arrow in X and of its image in Y.

For morphisms of posets, conservative means to preserve $<$, not just $\leq$, while ULF is strictly stronger: it means to induce an isomorphism $[x, x'] \simeq [Fx, Fx']$ on each interval. If the morphism of posets is an inclusion, then ULF is precisely the same as convex (cf. [32]): if two elements belong to the subposet then so do all elements between them. Note that an ULF map of posets does not have to be injective: for example, if X is a discrete poset then *any* map $X \to Y$ is ULF.

Given a simplicial map $F : X \to Y$ between decomposition spaces, the span $X_1 \xleftarrow{=} X_1 \xrightarrow{F_1} Y_1$ defines a linear functor

$$F_{1!} : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/Y_1},$$

which descends to a linear functor $F_{1!} : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}_{/Y_1}$ with cardinality the linear map $\mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}_{\pi_0 Y_1}$ given on the basis by $\delta_f \mapsto \delta_{F_1 f}$.

Lemma. 1.5.2. [29] *If F is CULF, then $F_{1!}$ is a coalgebra homomorphism, meaning that it preserves the comultiplication and counit up to coherent homotopy*

$$(F_{1!} \otimes F_{1!})\, \Delta_X \;\simeq\; \Delta_Y\, F_{1!}, \qquad\qquad \varepsilon_X \;\simeq\; \varepsilon_Y\, F_{1!}.$$

1.5.3. Decalage. An important source of CULF functors is given by decalage. Recall that the *decalage* functor $\mathrm{Dec}_\perp$ on simplicial groupoids forgets the bottom face and degeneracy maps, and shifts the indexing of the groupoids. The unused face map $d_\perp$ provides a natural transformation from the decalage back to the identity functor. We refer to this $d_\perp$ as the *dec map*.

$$\begin{tikzcd}
X & X_0 & X_1 \arrow[l, shift right=3, "d_1"'] \arrow[l, shift left=3, "d_0"] & X_2 \arrow[l, shift right=4, "d_2"'] \arrow[l, "d_1" description] \arrow[l, shift left=4, "d_0"] & X_3 \arrow[l, shift right=6, "d_3" description] \arrow[l, shift right=2, "d_2" description] \arrow[l, shift left=2, "d_1" description] \arrow[l, shift left=6, "d_0" description] & \cdots \\
\mathrm{Dec}_\perp X \arrow[u, "d_\perp"] & X_1 \arrow[u, "d_0"] \arrow[r, "s_1" description] & X_2 \arrow[u, "d_0"] \arrow[l, shift right=3, "d_2"'] \arrow[l, shift left=3, "d_1"] & X_3 \arrow[u, "d_0"] \arrow[l, shift right=4, "d_3"'] \arrow[l, "d_2" description] \arrow[l, shift left=4, "d_1"] & X_4 \arrow[u, "d_0"] \arrow[l, shift right=6, "d_4" description] \arrow[l, shift right=2, "d_3" description] \arrow[l, shift left=2, "d_2" description] \arrow[l, shift left=6, "d_1" description] & \cdots
\end{tikzcd}$$

Similarly, the decalage $\mathrm{Dec}_\top$ forgets the top face and degeneracy maps.

Decalage also plays an important role at the theoretical level, as exemplified by the following result.

Lemma. 1.5.4. *([29]) A simplicial groupoid X is a decomposition space if and only if both $\mathrm{Dec}_\top(X)$ and $\mathrm{Dec}_\perp(X)$ are Segal spaces and the corresponding dec maps $d_\top$ and $d_\perp$ are CULF.*

In particular for any decomposition space X we have a canonical coalgebra homomorphism from the incidence coalgebra of $\mathrm{Dec}_\perp X$ to that of X, and similarly for $\mathrm{Dec}_\top$. This appears in many examples.

Lemma 1.4.11 above refers to decalage, and we owe the proof.

Proof of Lemma 1.4.11. Just note that the dec map is always essentially surjective, since it admits a degeneracy map as a section. Now the result follows from the following lemma. ☐

Lemma. 1.5.5. *A decomposition space X is locally discrete if it admits an essentially surjective CULF functor $Y \to X$ with Y a locally discrete decomposition space.*

Proof. This is a general fact that in a pullback square of groupoids

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, "f"] \\ B' \arrow[r, "e"'] & B \end{tikzcd}$$

where e is essentially surjective, then f is discrete if and only if f' is discrete. ☐

1.5.6. Bialgebras. Recall that a bialgebra is a coalgebra with a compatible algebra structure, meaning that multiplication and unit are coalgebra homomorphisms. More formally it can be characterised as a monoid object in the category of coalgebras. In Lemma 1.5.2 we saw that a sufficient condition for a simplicial map f between decomposition spaces to induce a coalgebra homomorphism on incidence coalgebras is that f be CULF. Accordingly we define a *monoidal decomposition space* [29] to be a decomposition space Z equipped with an associative unital monoid structure given by CULF functors $m : Z \times Z \to Z$ and $e : 1 \to Z$.

Proposition 1.5.7. *If Z is a monoidal decomposition space then $\mathbf{Grpd}_{/Z_1}$ is naturally a bialgebra, termed its* incidence bialgebra*. Monoidal CULF functors induce bialgebra homomorphisms.*

1.5.8. Extensivity. Classically, a category $\mathscr{C}$ with sums is called *extensive* when the natural functor $\mathscr{C}/A \times \mathscr{C}/B \to \mathscr{C}/(A + B)$ is an equivalence. More generally, a monoidal category $(\mathscr{C}, \otimes, I)$ is called *monoidal extensive* when the natural functor $\mathscr{C}/A \times \mathscr{C}/B \to \mathscr{C}/(A \otimes B)$ is an equivalence. The fat nerve of a monoidal extensive category is always a monoidal decomposition space. As an example, the category $\mathscr{F}$ of finite sets and all maps is extensive in the classical sense. The category of finite sets and surjections inherits the monoidal structure $+$ from $\mathscr{F}$, but it is no longer the categorical sum (since there are no sum injections). It is still monoidal extensive. We shall come back to this particular example in Subsection 2.4.

Lemma. 1.5.9. *(*[29, Lemma 9.3]*) The Dec of a monoidal decomposition space has again a natural monoidal structure, and the dec map preserves this structure.*

1.5.10. Example: the Schmitt Hopf algebra of graphs, continued. The decomposition space of Example 1.1.5 (and 1.2.4) has a canonical monoidal structure given by disjoint union. Recall that X_k is the groupoid of graphs equipped with an ordered partition of the vertex set into k parts (possibly empty). The disjoint union of two such structures is given by taking the disjoint union of the underlying graphs, with new partition given by joining the two ith parts, for each $1 \leq i \leq k$. This clearly defines a simplicial map from $X \times X$ to X. To say that it is CULF is to establish that squares like this is a pullback:

$$\begin{tikzcd} X_1 \times X_1 \arrow[d, "+"'] & X_2 \times X_2 \arrow[l, "d_1"'] \arrow[d, "+"] \arrow[dl, phantom, "\lrcorner"] \\ X_1 & X_2 \arrow[l, "d_1"] \end{tikzcd}$$

But this is clear: a pair of graphs with a 2-partition each can be uniquely reconstructed if we know what the two underlying graphs are (an element in $X_1 \times X_1$) and we know how the disjoint union is partitioned (an element in X_2)—provided of course that we can identify the disjoint union of those two underlying graphs with the underlying graph of the disjoint union (which is to say that the data agree down in X_1). It follows that the resulting incidence coalgebra is also a bialgebra. (Furthermore, this bialgebra has a canonical grading, by the number of vertices, and with respect to this grading it is connected, since the only zero-vertex graph is the empty graph. It is well known that connected graded bialgebras are Hopf [23].)

2. Examples

It is characteristic for the classical theory of incidence (co)algebras of posets that most often it is necessary to impose an equivalence relation on the set of intervals in order to arrive at the interesting 'reduced' incidence (co)algebras. This equivalence relation may be simply isomorphism of posets, or equality of length of maximal chains as in binomial posets [16], or it may be more subtle order-compatible relations [17], [65]. Content, Lemay and Leroux [14] remarked that in some important cases the relationship between the original incidence coalgebra and the reduced one amounts to a CULF functor, although they did not make this notion explicit. From our global simplicial viewpoint, we observe that very often these CULF functors arise from decalage, often of a decomposition space which not a poset and sometimes not even a Segal space.

Recall that for X a locally finite decomposition space, we write $\mathcal{I}_X$ for the incidence coalgebra (with underlying vector space $\mathbb{Q}_{\pi_0 X_1}$), and we write $\mathcal{I}^X$ for the incidence algebra (with underlying profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$).

2.0.11. Decomposition spaces for the classical series. Classically important examples of incidence algebras are power series representations. From the perspective of the objective method, these representations appear as cardinalities of various monoidal structures on species, realised as incidence algebras with groupoid coefficients. We list six examples illustrating some of the various kinds of generating functions listed by Stanley [66] (see also Dür [17]).

(1) Ordinary generating functions, the zeta function being $\zeta(z) = \sum_{k\geq 0} z^k$. This comes from ordered sets and ordinal sum, and the incidence algebra is that of ordered species with the ordinary product.
(2) Exponential generating functions, the zeta function being $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{k!}$. Objectively, there are two versions of this: one coming from the standard

Cauchy product of species, and one coming from the shuffle product of $\mathbb{L}$-species (in the sense of [7]).

(3) Ordinary Dirichlet series, the zeta function being $\zeta(z) = \sum_{k>0} k^{-s}$. This comes from ordered sets with the cartesian product.

(4) 'Exponential' Dirichlet series, the zeta function being $\zeta(z) = \sum_{k>0} \frac{k^{-s}}{k!}$. This comes from the Dirichlet product of arithmetic species [4], also called the arithmetic product [54].

(5) q-exponential generating series, with zeta function $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{[k]!}$. This comes from the Waldhausen $S_\bullet$-construction on the category of finite vector spaces. The incidence algebra is that of q-species with a version of the external product of Joyal–Street [41].

(6) A variation with zeta function $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{|\mathrm{Aut}(\mathbb{F}_q^k)|}$, which arises from q-species with the 'Cauchy' product studied by Morrison [59].

Of these examples, only (1) and (3) have trivial section coefficients and come from a Möbius category in the sense of Leroux. We proceed to the details.

2.1. Additive examples

We start with several easy examples that serve to reiterate the importance of having incidence algebras of posets, monoids and monoidal groupoids on the same footing, connected by CULF functors, and in particular by decalage.

2.1.1. Linear orders and the additive monoid. Let $\mathbf{L}$ denote the nerve of the poset $(\mathbb{N}, \leq)$, and let $\mathbf{N}$ be the nerve of the additive monoid $(\mathbb{N}, +)$. Imposing the equivalence relation 'isomorphism of intervals' on the incidence coalgebra of $\mathbf{L}$ gives that of $\mathbf{N}$, and Content–Lemay–Leroux [14] observed that this reduction is induced by a CULF functor $r : \mathbf{L} \to \mathbf{N}$ sending $a \leq b$ to $b - a$. In fact we have:

Lemma. 2.1.2. *There is an isomorphism of simplicial sets*

$$\mathrm{Dec}_\perp(\mathbf{N}) \xrightarrow{\simeq} \mathbf{L}$$

given in degree k by

$$(x_0, \dots, x_k) \longmapsto [x_0 \leq x_0 + x_1 \leq \cdots \leq x_0 + \cdots + x_k],$$

and the CULF functor r is isomorphic to the dec map

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{N}) \to \mathbf{N}, \qquad (x_0, \dots, x_k) \mapsto (x_1, \dots, x_k).$$

The comultiplication on $\boldsymbol{Grpd}_{/\mathbf{N}_1}$ is given by

$$\Delta(\ulcorner n \urcorner) = \sum_{a+b=n} \ulcorner a \urcorner \otimes \ulcorner b \urcorner$$

and, taking cardinality, the incidence coalgebra $\mathcal{I}_\mathbf{N}$ is the vector space $\mathbb{Q}_\mathbb{N}$ spanned by symbols δ_n with comultiplication $\Delta(\delta_n) = \sum\limits_{a+b=n} \delta_a \otimes \delta_b$. The incidence algebra $\mathcal{I}^\mathbf{N}$ is the profinite-dimensional vector space $\mathbb{Q}^\mathbb{N}$ spanned by the symbols δ^n with convolution product $\delta^a * \delta^b = \delta^{a+b}$, and is isomorphic to the ring of power series in one variable,

$$\begin{array}{rcl} \mathcal{I}^{\mathbf{N}} & \xrightarrow{\simeq} & \mathbb{Q}[[z]] \\ \delta^n & \longmapsto & z^n \\ (\mathbb{N} \xrightarrow{f} \mathbb{Q}) & \longmapsto & \sum f(n)\, z^n . \end{array}$$

2.1.3. Upper dec. In the previous example, and in most of the following, it is more convenient to work with lower dec. Let us just point out what happens with upper dec. Let $\mathbf{L}^{\mathrm{op}}$ denote the nerve of the opposite poset of $(\mathbb{N}, \leq)$, that is, $(\mathbb{N}, \geq)$. There is a CULF functor $r' : \mathbf{L}^{\mathrm{op}} \to \mathbf{N}$ sending $a \geq b$ to $a - b$. We have:

Lemma. 2.1.4. *There is an isomorphism of simplicial sets*

$$\mathrm{Dec}_\top(\mathbf{N}) \xrightarrow{\simeq} \mathbf{L}^{\mathrm{op}}$$

given in degree k by

$$(x_0, \ldots, x_k) \longmapsto [x_0 + \cdots + x_k \geq x_1 + \cdots + x_k \geq \cdots \geq x_{k-1} + x_k \geq x_k],$$

and the CULF functor r' is isomorphic to the dec map

$$d_\top : \mathrm{Dec}_\top(\mathbf{N}) \to \mathbf{N}, \qquad (x_0, \ldots, x_k) \mapsto (x_0, \ldots, x_{k-1}).$$

In the following examples, this contravariance comes in for all upper decs. It will not play any role until Example 2.5.1.

2.1.5. Powers. As a variation of the previous example, fix $k \in \mathbb{N}$ and let $\mathbf{L}^k$ denote the (strict) nerve of the poset $(\mathbb{N}^k, \leq)$ and let $\mathbf{N}^k$ denote the strict nerve of the monoid $(\mathbb{N}^k, +)$. Again there is a CULF functor $\mathbf{L}^k \to \mathbf{N}^k$, and the incidence algebra of $\mathbf{N}^k$ is the power series ring in k variables. The functor is defined by coordinatewise difference, and again it is given by decalage, via a natural identification $\mathbf{L}^k \simeq \mathrm{Dec}_\perp(\mathbf{N}^k)$. The functor does *not* divide out by isomorphism of intervals, unless $k = 1$, since isomorphic intervals also arise by permutation of coordinates, treated next.

2.1.6. Symmetric powers. Let M be a monoid. For fixed $k \in \mathbb{N}$, the power M^k is again a monoid, considered as a decomposition space via its strict nerve X. The symmetric group $\mathfrak{S}_k$ acts on $X_1 = M^k$ by permutation of coordinates, and acts on $X_n = X_1^n = (M^k)^n$ diagonally. There is induced a simplicial groupoid $X/\mathfrak{S}_k$ given by homotopy quotient: in degree n it is the action groupoid $\frac{X_1 \times \cdots \times X_1}{\mathfrak{S}_k}$. Since taking homotopy quotient of a group action is a lowershriek operation, it preserves pullbacks, so it follows that this new simplicial groupoid again satisfies the Segal condition. (It is no longer a monoid, though, since in degree zero we have the one-object groupoid $B\mathfrak{S}_k = */\mathfrak{S}_k$, the classifying space of the group $\mathfrak{S}_k$). In general, the quotient map $X \to X/\mathfrak{S}_k$ is a CULF functor which does not arise from decalage.

We now return to the poset $\mathbf{L}^k = (\mathbb{N}^k, \leq)$ from 2.1.5. The reduced incidence algebra, given by identifying isomorphic intervals, coincides with the incidence coalgebra of $\mathbf{N}^k/\mathfrak{S}_k = (\mathbb{N}^k, +)/\mathfrak{S}_k$. The reduction map is the composite CULF functor

$$\mathbf{L}^k \simeq \mathrm{Dec}_\perp(\mathbf{N}^k) \longrightarrow \mathbf{N}^k \longrightarrow \mathbf{N}^k/\mathfrak{S}_k .$$

2.1.7. Injections and the monoidal groupoid of sets under sum. Let $\mathbf{I}$ be the fat nerve of the category of finite sets and injections, and let $\mathbf{B}$ be the monoidal nerve of the monoidal groupoid $(\mathbb{B},+,0)$ of finite sets and bijections (see B.2.4). Dür [17] noted that imposing the equivalence relation 'having isomorphic complements' on the incidence coalgebra of $\mathbf{I}$ gives the binomial coalgebra. Again, we can see this reduction map as induced by a CULF functor from a decalage:

Lemma. 2.1.8. *There is an equivalence of simplicial groupoids*

$$\mathrm{Dec}_\perp(\mathbf{B}) \xrightarrow{\simeq} \mathbf{I}$$

given in degree k by

$$(x_0,\ldots,x_k) \longmapsto [x_0 \subseteq x_0+x_1 \subseteq \cdots \subseteq x_0+\cdots+x_k],$$

and a CULF functor $\mathbf{I}\to\mathbf{B}$ is given by

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{B}) \to \mathbf{B}, \qquad (x_0,\ldots,x_k) \mapsto (x_1,\ldots,x_k).$$

The isomorphism may also be represented diagrammatically using diagrams reminiscent of those in Waldhausen's $S_\bullet$-construction (cf. Subsection 2.3 below). As an example, both groupoids $\mathbf{I}_3$ and $\mathrm{Dec}_\perp(\mathbf{B})_3 = \mathbf{B}_4$ are equivalent to the groupoid of diagrams

$$\begin{tikzcd}
 & & & x_3 \arrow[d] \\
 & & x_2 \arrow[r] \arrow[d] & x_2+x_3 \arrow[d] \\
 & x_1 \arrow[r] \arrow[d] & x_1+x_2 \arrow[r] \arrow[d] & x_1+x_2+x_3 \arrow[d] \\
x_0 \arrow[r] & x_0+x_1 \arrow[r] & x_0+x_1+x_2 \arrow[r] & x_0+x_1+x_2+x_3
\end{tikzcd}$$

The face maps $d_i : \mathbf{I}_3 \to \mathbf{I}_2$ and $d_{i+1} : \mathbf{B}_4 \to \mathbf{B}_3$ both act by deleting the column beginning x_i and the row beginning x_{i+1}. In particular $d_\perp : \mathbf{I}\to\mathbf{B}$ deletes the bottom row, sending a sequence of injections to the sequence of successive complements (x_1,x_2,x_3). We will revisit this theme in the treatment of the Waldhausen $S_\bullet$-construction.

From Subsection 1.5 we have:

Lemma. 2.1.9. *Both $\mathbf{I}$ and $\mathbf{B}$ are monoidal decomposition spaces under disjoint union, and $\mathbf{I}\simeq \mathrm{Dec}_\perp(\mathbf{B})\to\mathbf{B}$ is a monoidal CULF functor inducing a bialgebra homomorphism $\boldsymbol{Grpd}_{/\mathbf{I}_1} \to \boldsymbol{Grpd}_{/\mathbf{B}_1}$.*

Proposition 1.4.2 gives the comultiplication on $\boldsymbol{Grpd}_{/\mathbf{B}_1}$ as

$$\Delta(\ulcorner S \urcorner) \;=\; \sum_{A,B} \frac{\mathrm{Bij}(A+B,S)}{\mathrm{Aut}(A)\times\mathrm{Aut}(B)} \cdot \ulcorner A \urcorner \otimes \ulcorner B \urcorner \;=\; \sum_{\substack{A,B\subset S\\ A\cup B=S,\ A\cap B=\emptyset}} \ulcorner A \urcorner \otimes \ulcorner B \urcorner.$$

It follows that the convolution product on $\boldsymbol{Grpd}^{\mathbb{B}}$ is just the Cauchy product on groupoid-valued species

$$(F * G)[S] = \sum_{A+B=S} F[A] \times G[B].$$

For the representables, the formula says simply $h^A * h^B = h^{A+B}$.

The decomposition space $\mathbf{B}$ is locally finite, and taking cardinality gives the classical binomial coalgebra $\mathcal{I}_{\mathbf{B}} = \mathbb{Q}_{\pi_0 \mathbf{B}_1}$, spanned by symbols δ_n with

$$\Delta(\delta_n) = \sum_{a+b=n} \frac{n!}{a!\,b!}\, \delta_a \otimes \delta_b.$$

As a bialgebra we have $(\delta_1)^n = \delta_n$ and one recovers the comultiplication from $\Delta(\delta_n) = \big(\delta_0 \otimes \delta_1 + \delta_1 \otimes \delta_0\big)^n$.

Dually, the incidence algebra $\mathcal{I}^{\mathbf{B}}$ is the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 \mathbf{B}_1}$ spanned by symbols δ^n with convolution product

$$\delta^a * \delta^b = \frac{(a+b)!}{a!\,b!}\, \delta^{a+b}.$$

This is isomorphic to the algebra $\mathbb{Q}[[z]]$, where δ^n corresponds to $z^n/n!$ and the cardinality of a species F corresponds to its exponential generating series.

2.1.10. Finite ordered sets, and the shuffle product of $\mathbb{L}$-species. Let $\mathbf{OI}$ denote (the fat nerve of) the category of finite ordered sets and monotone injections. The resulting incidence coalgebra can be reduced by identifying two monotone injections if they have isomorphic complements, in analogy with Example 2.1.7, yielding in this case the *shuffle coalgebra*. Again, this reduction is an example of decalage. Consider the decomposition space $\mathbf{Z}$ with $\mathbf{Z}_n = \mathbf{OI}_{/\underline{n}}$, the groupoid of arbitrary maps from a finite ordered set S to $\underline{n}$, or equivalently of n-shuffles of S. This provides a direct construction of the shuffle coalgebra. This example is subsumed in the theory of restriction species, developed in [32]. The section coefficients are the binomial coefficients, but we may now note that on the objective level the convolution algebra is the shuffle product of $\mathbb{L}$-species (cf. [7]).

There is a natural identification

$$\mathbf{OI} \xrightarrow{\simeq} \mathrm{Dec}_\perp \mathbf{Z},$$

which takes a sequence of monotone injections to the list of successive complements. There is also a CULF functor $\mathbf{Z} \to \mathbf{B}$ that takes an n-shuffle to the underlying n-tuple of subsets, and the decalage of this functor is the CULF functor $\mathbf{OI} \to \mathbf{I}$ given by forgetting the order (see [29, Example 4.5]). Combining with Lemma 2.1.9, we get altogether this commutative diagram of monoidal decomposition spaces and monoidal CULF functors,

$$\begin{tikzcd}
\mathbf{OI} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Dec}_\perp(\mathbf{Z}) \arrow[r, "d_\perp"] \arrow[d] & \mathbf{Z} \arrow[d] \\
\mathbf{I} \arrow[r, "\simeq"] & \mathrm{Dec}_\perp(\mathbf{B}) \arrow[r, "d_\perp"'] & \mathbf{B}.
\end{tikzcd}$$

2.1.11. Words. Let A be a fixed set, an alphabet. The comma category $\mathbf{OI}_{/A}$ is the category of finite words in A and subword inclusions, cf. Lothaire [52] (see also Dür [17]). Again it is naturally identified with the decalage of the A-*coloured shuffle decomposition space* $\mathbf{Z}_A$, which in degree k is the groupoid of A-words (of arbitrary length) equipped with a not-necessarily-order-preserving map to $\underline{k}$. Precisely, the objects are spans of sets

$$\underline{k} \leftarrow \underline{n} \rightarrow A.$$

The dec map $\mathbf{OI}_{/A} \simeq \mathrm{Dec}_\perp \mathbf{Z}_A \rightarrow \mathbf{Z}_A$ takes a subword inclusion to its complement word. The incidence algebra $\mathcal{I}^{\mathbf{Z}_A}$ is the Lothaire shuffle algebra of words. Again, it all amounts to observing that A-words admit a forgetful monoidal CULF functor to 1-words, which is just the decomposition space $\mathbf{Z}$ from before, and that this in turn admits a monoidal CULF functor to $\mathbf{B}$.

Note the difference between $\mathbf{Z}_A$ and the free monoid on A: the latter is like allowing only the trivial shuffles, where the subword inclusions are only concatenation inclusions. In terms of the structure maps $\underline{n} \rightarrow \underline{k}$, the free-monoid nerve allows only monotone maps, whereas the shuffle decomposition space allows arbitrary set maps.

2.2. Multiplicative examples

2.2.1. Divisibility poset and multiplicative monoid. In analogy with 2.1.1, let $\mathbf{D}$ denote the (strict) nerve of the divisibility poset $(\mathbb{N}^\times, |)$, and let $\mathbf{M}$ be the strict nerve of the multiplicative monoid $(\mathbb{N}^\times, \cdot)$. Imposing the equivalence relation 'isomorphism of intervals' on the incidence coalgebra of $\mathbf{D}$ gives that of $\mathbf{M}$, and Content–Lemay–Leroux [14] observed that this reduction is induced by the CULF functor $r : \mathbf{D} \rightarrow \mathbf{M}$ sending $d|n$ to n/d. In fact we have:

Lemma. 2.2.2. *There is an isomorphism of simplicial sets*

$$\mathrm{Dec}_\perp(\mathbf{M}) \xrightarrow{\simeq} \mathbf{D}$$

given in degree k by

$$(x_0, x_1, \ldots, x_k) \longmapsto [x_0|x_0x_1|\ldots|x_0x_1\cdots x_k],$$

and the CULF functor r is isomorphic to the dec map

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{M}) \rightarrow \mathbf{M}, \qquad (x_0, \ldots, x_k) \mapsto (x_1, \ldots, x_k).$$

This example can be obtained from Example 2.1.1 directly, since $\mathbf{M} = \prod_p \mathbf{N}$ and $\mathbf{D} = \prod_p \mathbf{L}$, where the (weak) product is over all primes p. Now $\mathrm{Dec}_\perp$ is a right adjoint, so preserves products, and Lemma 2.2.2 follows from Lemma 2.1.1.

We can use the general formula 1.4.2: since there are no nontrivial automorphisms, the convolution product is $\delta^m * \delta^n = \delta^{mn}$, and the incidence algebra is isomorphic to the Dirichlet algebra:

$$\begin{array}{rcl} \mathcal{I}^{\mathbf{D}} & \longrightarrow & \{\sum_{k>0} a_k k^{-s}\} \\ \delta^n & \longmapsto & n^{-s} \\ f & \longmapsto & \sum_{n>0} f(n) n^{-s}. \end{array}$$

2.2.3. Arithmetic species. The Dirichlet coalgebra (2.2.1) also has a fatter version: consider now instead the monoidal groupoid $(\mathbb{B}^\times, \times, 1)$ of non-empty finite sets under the cartesian product, and it monoidal nerve $\mathbf{A}$ with $\mathbf{A}_k := (\mathbb{B}^\times)^k$, as in B.2.4, where this time the inner face maps take the cartesian product of two adjacent factors, and the outer face maps project away an outer factor.

The resulting coalgebra structure is

$$\Delta(S) = \sum_{A\times B \simeq S} A \otimes B.$$

Some care is due to interpret this correctly: the homotopy fibre of $d_1 : \mathbf{A}_2 \to \mathbf{A}_1$ over S is the groupoid whose objects are triples (A, B, ϕ) consisting of sets A and B equipped with a bijection $\phi : A \times B \xrightarrow{\sim} S$, and whose morphisms are pairs of isomorphisms $\alpha : A \xrightarrow{\sim} A'$, $\beta : B \xrightarrow{\sim} B'$ forming a commutative square with ϕ and ϕ'.

The corresponding incidence algebra $\boldsymbol{grpd}^{\mathbb{B}^\times}$ with the convolution product is the algebra of arithmetic species [4] under the Dirichlet product (called the arithmetic product of species by Maia and Méndez [54]).

Clearly we are in the locally finite situation; the section coefficients are given directly by 1.4.2, and we find

$$\delta^m * \delta^n = \frac{(mn)!}{m!n!}\,\delta^{mn}.$$

From this we see that the incidence algebra $\mathcal{I}^{\mathbf{A}}$ is isomorphic to the Dirichlet algebra, namely

$$\begin{array}{rcl} \mathcal{I}^{\mathbf{A}} & \longrightarrow & \{\sum_{k>0} a_k k^{-s}\} \\ \delta^m & \longmapsto & \dfrac{m^{-s}}{m!} \\ f & \mapsto & \sum_{n>0} f(n) \dfrac{k^{-s}}{n!}; \end{array}$$

these are the 'exponential' (or modified) Dirichlet series (cf. Baez–Dolan [4]). So the incidence algebra zeta function in this setting is

$$\zeta = \sum_{k>0} \delta^k \mapsto \sum_{k>0} \frac{k^{-s}}{k!}$$

(which is not the usual Riemann zeta function).

2.3. **Linear examples and the Waldhausen $S_\bullet$-construction**

In this subsection, we are concerned with linear versions of the additive examples: instead of starting with finite sets and injections, we look at vector spaces over a finite field, and their linear injections. This is a richer setting: in particular, there is now an essential difference between quotients and complements, which at the level of decomposition spaces is the difference between the Waldhausen $S_\bullet$-construction and the monoidal nerve of direct sums, as we shall see.

2.3.1. $\mathbb{F}_q$-vector spaces. Let $\mathbb{F}_q$ denote a finite field with q elements. Let $\mathbf{W}$ denote the fat nerve of the category $\boldsymbol{vect}^{\mathrm{inj}}$ of finite-dimensional $\mathbb{F}_q$-vector spaces and $\mathbb{F}_q$-linear injections. From this decomposition space we immediately get a coalgebra, but it is not the most interesting.

2.3.2. Direct sums of $\mathbb{F}_q$-vector spaces and 'Cauchy' product of q-species. A coalgebra which is the q-analogue of $\mathbf{B}$ can be obtained from the monoidal groupoid $(\boldsymbol{vect}^{\mathrm{iso}}, \oplus, 0)$. Denote by $\mathbf{M}$ the monoidal nerve of $(\boldsymbol{vect}^{\mathrm{iso}}, \oplus, 0)$, in the sense of B.2.4. The fibre of $d_1 : \mathbf{M}_2 \to \mathbf{M}_1$ over a vector space V is the groupoid consisting of triples (A, B, ϕ) where ϕ is a linear isomorphism $A \oplus B \xrightarrow{\sim} V$. This groupoid projects to $\boldsymbol{vect}^{\mathrm{iso}} \times \boldsymbol{vect}^{\mathrm{iso}}$: the fibre over (A, B) is discrete, of cardinality $|\mathrm{Aut}(V)|$, giving altogether the section coefficient

$$c^n_{k,n-k} = \frac{\left|\mathrm{Aut}(\mathbb{F}_q^n)\right|}{\left|\mathrm{Aut}(\mathbb{F}_q^k)\right| \left|\mathrm{Aut}(\mathbb{F}_q^{n-k})\right|} = q^{k(n-k)} \binom{n}{k}_q .$$

At the objective level, this convolution product corresponds to the 'Cauchy' product of q-species in the sense of Morrison [59].

If we let δ_n denote the cardinality of the name of an n-dimensional vector space V, the resulting coalgebra $\mathcal{I}_{\mathbf{M}}$ therefore has comultiplication:

$$\Delta(\delta_n) = \sum_{k \leq n} q^{k(n-k)} \binom{n}{k}_q \cdot \delta_k \otimes \delta_{n-k}.$$

In analogy with the discrete case discussed in 2.1.7–2.1.8 there is a canonical simplicial map $\mathrm{Dec}_\perp(\mathbf{M}) \to \mathbf{W}$, given by sending an $(n+1)$-tuple of vector spaces $(V_0, \ldots, V_n)$ to the sequence of inclusions

$$V_0 \hookrightarrow V_0 \oplus V_1 \hookrightarrow \cdots \hookrightarrow V_0 \oplus \cdots \oplus V_n.$$

But in contrast to Lemma 2.1.8, this simplicial map is not an equivalence: the inverse, which in the discrete case was 'taking complements', does not exist in the linear case (or if it is constructed artificially, for example by reference to euclidean structure, it will mess with the isomorphisms). Let us actually compute $\mathrm{Dec}_\perp(\mathbf{M})$.

2.3.3. Complements as retractions. Let $\mathbf{W}^{\mathrm{retr}}$ denote the fat nerve of the category whose objects are finite-dimensional $\mathbb{F}_q$-vector spaces and whose morphisms are retracted injections (linear of course)

$$\begin{tikzcd} V \arrow[r, shift right, tail, "i"'] & V' \arrow[l, shift right, two heads, "r"'] \end{tikzcd}$$

Such retracted injections have canonical complements, namely $\ker(r)$. The following analogue of Lemma 2.1.8 is now straightforward to establish.

Lemma. 2.3.4. *There is a canonical equivalence of simplicial groupoids*

$$\mathrm{Dec}_\perp(\mathbf{M}) \xrightarrow{\simeq} \mathbf{W}^{\mathrm{retr}}$$

given in degree k by

$$(V_0, \ldots, V_k) \longmapsto [V_0 \subseteq V_0 \oplus V_1 \subseteq \cdots \subseteq V_0 \oplus \cdots \oplus V_k]$$

inducing a CULF functor $\mathbf{W}^{\mathrm{retr}} \to \mathbf{M}$.

The discussion shows that altogether $\mathbf{M}$ is not the most interesting viewpoint. We now change perspective from complements to quotients, getting to the more important power series representation with factor $[n]!$ instead of $\left|\mathrm{Aut}(\mathbb{F}_q^n)\right|$, and realise $\mathbf{W}$ as a decalage, in analogy with Lemma 2.1.8.

2.3.5. q-binomials. With reference to the incidence coalgebra of $\mathbf{W}$, impose the equivalence relation identifying two injections if their cokernels are isomorphic. This gives the q-binomial coalgebra (see Dür [17, 1.54]).

The same coalgebra can be obtained without reduction as follows. Put $\mathbf{V}_0 = *$, let $\mathbf{V}_1$ be the maximal groupoid of ***vect***, and let $\mathbf{V}_2$ be the groupoid of short exact sequences. The span

$$\begin{array}{ccccc} \mathbf{V}_1 & \longleftarrow & \mathbf{V}_2 & \longrightarrow & \mathbf{V}_1 \times \mathbf{V}_1 \\ E & \longleftarrow\!\!\shortmid & [E' \to E \to E''] & \longmapsto & (E', E'') \end{array}$$

(together with the span $\mathbf{V}_1 \leftarrow \mathbf{V}_0 \to 1$) defines a coalgebra structure on $\boldsymbol{grpd}_{/\mathbf{V}_1}$ which (after taking cardinality) is the q-binomial coalgebra, without further reduction. The groupoids and maps involved are part of a simplicial groupoid $\mathbf{V} : \mathbb{\Delta}^{\mathrm{op}} \to$ ***Grpd***, namely the Waldhausen $S_\bullet$-construction of ***vect***, studied in more detail below, where we'll see that this is a decomposition space but not a Segal space. The lower dec of $\mathbf{V}$ is naturally equivalent to the fat nerve $\mathbf{W}$ of the category of injections, and the dec map $d_\perp$ is the reduction map of Dür.

We calculate the section coefficients of $\mathbf{V}$. From Proposition 1.4.2 we have the following formula for the section coefficients (which is precisely the standard formula for the *Hall numbers*, as explained further in 2.3.10):

$$c^n_{k,n-k} = \frac{|\mathrm{SES}_{k,n,n-k}|}{\left|\mathrm{Aut}(\mathbb{F}_q^k)\right| \left|\mathrm{Aut}(\mathbb{F}_q^{n-k})\right|}.$$

Here $\mathrm{SES}_{k,n,n-k}$ is the groupoid of short exact sequence with specified vector spaces of dimensions k, n, and $n-k$. This is just a discrete space, and it has $(q-1)^n q^{\binom{k}{2}} q^{\binom{n-k}{2}} [n]!$ elements. Indeed, there are $(q-1)^k q^{\binom{k}{2}} \frac{[n]!}{[n-k]!}$ choices for the injection $\mathbb{F}_q^k \hookrightarrow \mathbb{F}_q^n$, and then $(q-1)^n q^{\binom{n}{2}} [n]!$ choices for identifying the cokernel with $\mathbb{F}_q^{n-k}$. Some q-yoga yields altogether the q-binomials as section coefficients:

$$= \binom{n}{k}_q.$$

This description gives an isomorphism of algebras (cf. Goldman–Rota [34], Dür [17])

$$\begin{array}{ccc} \mathcal{I}^{\mathbf{V}} & \longrightarrow & \mathbb{Q}[[z]] \\ \delta^k & \longmapsto & \dfrac{z^k}{[k]!}. \end{array}$$

Clearly this algebra is commutative. However, an important new aspect is revealed on the objective level: here the convolution product is the external product of q-species of Joyal-Street [41]. They show (working with vector-space valued q-species), that this product has a natural non-trivial braiding (which of course reduces to commutativity upon taking cardinality).

2.3.6. Waldhausen $S_\bullet$-construction of an abelian category. The decomposition space with the short exact sequences leading to the Hall algebra is an example of Waldhausen's $S_\bullet$-construction [72], a centrepiece of modern K theory. We briefly explain this.

The Waldhausen $S_\bullet$-construction of an abelian category $\mathscr{A}$ is a simplicial groupoid $S_\bullet\mathscr{A}$, with the following explicit description. $S_0\mathscr{A}$ is a point, $S_1\mathscr{A}$ is the maximal groupoid in $\mathscr{A}$, and $S_2\mathscr{A}$ is the groupoid of short exact sequences in $\mathscr{A}$. More generally, $S_n\mathscr{A}$ is the groupoid of staircase diagrams like the following (picturing $n = 4$):

$$\begin{tikzcd}
& & & A_{34} \\
& & A_{23} \arrow[r, tail] & A_{24} \arrow[u, two heads] \\
& A_{12} \arrow[r, tail] & A_{13} \arrow[r, tail] \arrow[u, two heads] & A_{14} \arrow[u, two heads] \\
A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[u, two heads] & A_{03} \arrow[r, tail] \arrow[u, two heads] & A_{04} \arrow[u, two heads]
\end{tikzcd}$$

in which each sequence $A_{ij} \to A_{ik} \to A_{jk}$ is exact. The face map d_i deletes all objects containing an i index. The degeneracy map s_i repeats the ith row and the ith column.

A sequence of composable monomorphisms $(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$ determines, up to canonical isomorphism, short exact sequences $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk} = A_{ij}/A_{ik}$ with $A_{0i} = A_i$. Hence the whole diagram can be reconstructed up to isomorphism from the bottom row. (Similarly, since epimorphisms have uniquely determined kernels, the whole diagram can also be reconstructed from the last column.)

We have $s_0(*) = 0$, and

$$d_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (A_2/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1),$$
$$s_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n).$$

The simplicial maps d_i, s_i for $i \geq 1$ are more straightforward: the simplicial set $\mathrm{Dec}_\perp(S_\bullet\mathscr{A})$ is just the fat nerve of $\mathscr{A}^{\mathrm{mono}}$.

Lemma. 2.3.7. *The projections $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathscr{A}^{\mathrm{mono}})$ and $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathscr{A}^{\mathrm{epi}})$ are equivalences of groupoids.*

More precisely (with reference to the fat nerve):

Proposition 2.3.8. *These equivalences assemble into levelwise simplicial equivalences*

$$\mathrm{Dec}_\perp(S_\bullet\mathscr{A}) \simeq \mathbf{N}(\mathscr{A}^{\mathrm{mono}})$$
$$\mathrm{Dec}_\top(S_\bullet\mathscr{A}) \simeq \mathbf{N}(\mathscr{A}^{\mathrm{epi}}).$$

Theorem 2.3.9. [19, Theorem 7.3.3], [29, 10.10] *The Waldhausen $S_\bullet$-construction of an abelian category $\mathscr{A}$ is a decomposition space.*

2.3.10. Hall algebras. The finite-support incidence algebra of a decomposition space X was mentioned in 1.4.6 (see [30, 7.15] for more details). In order for it to admit a cardinality, the required assumption is that X_1 be locally finite, and that $X_2 \to X_1 \times X_1$ be a finite map. In the case of $X = S_\bullet(\mathscr{A})$ for an abelian category $\mathscr{A}$, this translates into the condition that Ext^0 and Ext^1 be finite (which in practice means 'finite dimension over a finite field'). The finite-support incidence algebra in this case is the *Hall algebra* of $\mathscr{A}$ (cf. Ringel [61]; see also [63], although these sources twist the multiplication by the so-called Euler form).

Hall algebras were one of the main motivations for Dyckerhoff and Kapranov [19] to introduce 2-Segal spaces. We refer to their work for development of this important topic, recommending the lecture notes of Dyckerhoff [18] as a starting point.

2.4. **Faà di Bruno bialgebra and variations**

The Faà di Bruno bialgebra, originating with composition of power series, was constructed combinatorially by Doubilet [15] by imposing a *type-equivalence* relation on the incidence coalgebra of the partition poset. Joyal [39] observed that it can also be realised directly from the category of finite sets and surjections, without the need of a reduction step. Both constructions, and in particular the relationship between them, can be cast elegantly in the framework of decomposition spaces, serving to illustrate many of the characteristic aspects of the theory, such as the use of groupoids and the role of decalage.

2.4.1. Faà di Bruno from the partition poset. Fix a finite set of each cardinality, denoted $\underline{0}, \underline{1}, \underline{2}$, etc. Let $\mathcal{P}(\underline{n})$ denote the poset of partitions of the set $\underline{n}$; we write $\rho \leq \pi$ when partition ρ refines partition π. The *partition poset* is by definition the disjoint union of all these, $\mathcal{P} := \sum_{n\in\mathbb{N}} \mathcal{P}(\underline{n})$. The nerve of $\mathcal{P}$ defines a coalgebra (which is furthermore a bialgebra, with multiplication given by disjoint union). More interesting is the reduction of this bialgebra modulo type equivalence. An interval $[\rho, \pi]$ in a poset $\mathcal{P}(\underline{n})$ is said to have *type* $1^{\lambda_1}2^{\lambda_2}\cdots$ if λ_i is the number of blocks of ρ that consist of exactly i blocks of π. Declare two intervals equivalent if they have the same type. The resulting bialgebra is the Faà di Bruno bialgebra.

2.4.2. Partitions as surjections. A partition ρ of $\underline{n}$ can be realised as a surjection $\underline{n} \twoheadrightarrow \underline{k}$, where $\underline{k}$ is the set of blocks. An interval $[\rho, \pi]$, from $\underline{n} \twoheadrightarrow \underline{k}$ to $\underline{n} \twoheadrightarrow \underline{p}$ say, is then realised as a commutative triangle

$$
\begin{tikzcd}[row sep=small]
 & \underline{k} \arrow[dd, "f"] \\
\underline{n} \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & \underline{p}
\end{tikzcd}
\tag{12}
$$

and the type of $[\rho, \pi]$ is $1^{\lambda_1}2^{\lambda_2}\cdots$ if f has λ_i fibres of cardinality i. Identifying type-equivalent intervals amounts to forgetting the ambient set $\underline{n}$, as two intervals have the same type if and only if they are represented by isomorphic comparison surjections:

$$
\begin{tikzcd}
\underline{k} \arrow[r, "\cong"] \arrow[d, two heads, "f"'] & \underline{k}' \arrow[d, two heads, "f'"] \\
\underline{p} \arrow[r, "\cong"] & \underline{p}'
\end{tikzcd}
\tag{13}
$$

Refinement of intervals is precisely factorisation of such comparison surjections, but some care is needed to count these factorisations correctly. Note first that any two isomorphic surjections have the same sets of factorisations. The formula for comultiplication is

$$\Delta(E \twoheadrightarrow B) = \sum_{E \twoheadrightarrow S \twoheadrightarrow B} (E \twoheadrightarrow S) \otimes (S \twoheadrightarrow B).$$

Here the sum is over isomorphism classes of factorisations $E \twoheadrightarrow S \twoheadrightarrow B$. In detail, consider the *factorisation groupoid* $\mathrm{Fact}(E \twoheadrightarrow B)$, whose objects are factorisations of $E \twoheadrightarrow B$ into two surjections $E \twoheadrightarrow S \twoheadrightarrow B$, and whose morphisms are bijections $S \simeq S'$ making the two triangles commute:

$$
\begin{tikzcd}
 & S \arrow[dr, two heads] \arrow[dd, "\simeq"] & \\
E \arrow[ur, two heads] \arrow[dr, two heads] & & B \,. \\
 & S' \arrow[ur, two heads] &
\end{tikzcd}
\tag{14}
$$

Then the above sum is over $\pi_0(\mathrm{Fact}(E \twoheadrightarrow B))$, the set of connected components of the factorisation groupoid. This is Joyal's construction of the Faà di Bruno bialgebra [39] (again the algebra structure is by disjoint union).

This construction fits well into the decomposition space language. The Faà di Bruno bialgebra is simply the incidence bialgebra of the monoidal decomposition space $\mathbf{S}$ given as the fat nerve of the category of finite sets and surjections. Indeed, it has as $\mathbf{S}_0$ the groupoid of finite sets and bijection, as $\mathbf{S}_1$ the groupoids whose objects are surjections and whose morphisms are squares like (13). $\mathbf{S}_2$ is the groupoid whose objects are composable pairs of surjections, and whose morphisms are diagrams

$$
\begin{tikzcd}
\underline{k} \arrow[r, two heads] \arrow[d, "\sim"'] & \underline{p} \arrow[r, two heads] \arrow[d, "\sim"'] & \underline{q} \arrow[d, "\sim"'] \\
\underline{k}' \arrow[r, two heads] & \underline{p}' \arrow[r, two heads] & \underline{q}' .
\end{tikzcd}
$$

The homotopy fibre of $d_1 : \mathbf{S}_2 \to \mathbf{S}_1$ over $f \in \mathbf{S}_1$ is equivalent to the subgroupoid whose objects are composable pairs composing to f and whose morphisms are diagrams (14). It follows readily that the incidence bialgebra is precisely the Faà di Bruno bialgebra à la Joyal.

(Note that in the groupoid setting, there is no need to restrict to a skeleton of the category of finite sets and surjections. We may as well work with the whole groupoid, without making choices. The homotopy equivalences take care of 'dividing out', while keeping the correct automorphism data.)

2.4.3. Relationship via decalage. Having interpreted partitions as surjections, and refinements as factorisations of surjections, one may suspect that the partition poset is the decalage of the surjections nerve. This is almost correct, but not quite: there is a subtle difference related to symmetries (pointed out by Mark Weber), which in turn originates in the fact that the partition poset is based on chosen fixed sets $\underline{n}$. Getting this straight is a nice opportunity to see some CULF functors:

We first analyse the relationship between partitions and surjections. Two surjections $\underline{n} \twoheadrightarrow \underline{p}$ and $\underline{n} \twoheadrightarrow \underline{p}'$ represent the same partition if there is an isomorphism $p \xrightarrow{\sim} p'$ making the triangle commute. Consider the category $\mathscr{C}$ whose objects are surjections between the standard sets $\underline{n}$ and whose morphisms are triangles like (12) (not allowing non-identity isos at the domain $\underline{n}$) . Since there is at most one arrow between any two surjections (because surjections are epimorphisms), this category is equivalent to a poset, and indeed equivalent to the partition poset $\mathcal{P}$. (It does contain non-trivial isomorphisms, but only between distinct surjections.) Since this category $\mathscr{C}$ and the partition poset are equivalent as categories, their fat nerves are (levelwise) equivalent decomposition spaces, and therefore define equivalent coalgebras. (Note that for a strict poset, the fat nerve is the same thing as the strict nerve.)

This category $\mathscr{C}$ sits inside a bigger category $\mathscr{D}$ with the same objects (surjections), but where maps between two surjections are allowed to have a non-identity bijection between the domains, instead of just an identity arrow. The categories $\mathscr{C}$ and $\mathscr{D}$ are not equivalent, and their fat nerves are not equivalent, and their bialgebras are not isomorphic—all because of the different amount of symmetry they sport at the surjection domains. However, it is clear that they have exactly the same type reduction, since the type reduction precisely throws away the surjection domains.

Lemma. 2.4.4. *The inclusion functor $\mathscr{C} \to \mathscr{D}$ is CULF.*

Essentially this is for the same reason as the type reduction argument: CULF-ness is about factorisations of the codomain maps, and this is not affected by what happens at the domain.

Now that in $\mathscr{D}$ we have symmetries built in naturally, there is no reason to restrict to the skeleton anymore. As an equivalent $\mathscr{D}$ we can take the same description but allow the objects to be surjections between arbitrary finite sets, instead of just those chosen sets $\underline{n}$. Note that this bigger category $\mathscr{D}$ has a natural interpretation in terms of partitions: suppose we want a notion of partition, but do not wish to restrict attention to those chosen sets $\underline{n}$. We would then have to say when two partitions are considered the same, and more generally what should be the notion of morphism of partitions: the natural notion is to have a bijection at the domain that preserves block membership, i.e. bijections f such such that if t_1 and t_2 belong to the same block, then also ft_1 and ft_2 belong to the same block. It is clear that this is precisely a refinement. This is a category rather than a poset: it mixes the poset structure with the invertible maps given by renaming of set elements. Note that a partition in $\mathscr{D}$ has more automorphisms than in $\mathscr{C}$: for example a $(2, 2)$-partition has an automorphism group of order 8, namely 4 possibilities to permute within the blocks, and 2 possibilities of interchanging the blocks.

With these extra symmetries we have

Lemma. 2.4.5. *The fat nerve of* $\mathscr{D}$ *is naturally equivalent to* $\operatorname{Dec}_\perp \mathbf{S}$.

Altogether:

Proposition 2.4.6. *Type reduction, and the relationship between the partition poset and the surjections nerve is given by the string of CULF functors*

$$N\mathcal{P} \simeq \mathbf{N}\mathscr{C} \longrightarrow \mathbf{N}\mathscr{D} \simeq \operatorname{Dec}_\perp \mathbf{S} \xrightarrow{d_\perp} \mathbf{S}.$$

The composite sends a partition to its set of blocks, and sends a refinement to the corresponding surjection as in (12).

The verifications are straightforward. It should be noted that these CULF functors are all monoidal, and hence induce bialgebra homomorphisms (by Proposition 1.5.7).

2.4.7. Faà di Bruno section coefficients. We work with the decomposition space $\mathbf{S}$. Since $\mathbf{S}$ is monoidal, to describe its section coefficients, it is enough describe the comultiplication of connected surjections, that is, surjections with codomain 1: a general surjection is (equivalent to) a disjoint union of connected surjections. Our general formula 1.4.2 gives

$$\Delta(\underline{n} \overset{f}{\twoheadrightarrow} \underline{1}) = \sum_{\substack{a:\underline{n}\twoheadrightarrow\underline{k}\\ b:\underline{k}\twoheadrightarrow\underline{1}}} \frac{|\mathrm{Aut}(\underline{k})| \cdot |\mathrm{Aut}(ab)|}{|\mathrm{Aut}(a)| \cdot |\mathrm{Aut}(b)|} \ulcorner a \urcorner \otimes \ulcorner b \urcorner.$$

The order of the automorphism group of $\underline{k}$ and of a surjection $\underline{k} \twoheadrightarrow \underline{1}$ is $k!$, and for a general surjection $a : \underline{n} \twoheadrightarrow \underline{k}$ of type $1^{\lambda_1} 2^{\lambda_2} \cdots$,

$$|\mathrm{Aut}(a)| \;=\; \prod_{j=1}^{\infty} \lambda_j!(j!)^{\lambda_j}$$

and hence

$$\Delta(\underline{n} \overset{f}{\twoheadrightarrow} \underline{1}) = \sum_{\substack{a:\underline{n}\twoheadrightarrow\underline{k}\\ b:\underline{k}\twoheadrightarrow\underline{1}}} \frac{n!}{\prod_{j=1}^{k} \lambda_j!(j!)^{\lambda_j}} \ulcorner a \urcorner \otimes \ulcorner b \urcorner.$$

The section coefficients, called the Faà di Bruno section coefficients, are the coefficients $\binom{n}{\lambda;k}$ of the Bell polynomials, cf. [23, (2.5)].

2.4.8. A decomposition space for the Faà di Bruno *Hopf* algebra. The Faà di Bruno *Hopf* algebra is obtained by further reduction, classically stated as identifying two intervals in the partition poset if they are isomorphic as posets. This is equivalent to forgetting the value of λ_1. There is also a decomposition space that yields this Hopf algebra directly, obtained by quotienting the decomposition space $\mathbf{S}$ by the same equivalence relation. This means identifying two surjections (or sequences of composable surjections) if one is obtained from the other by taking disjoint union with a bijection. One may think of this as 'levelled forests modulo linear trees'. It is straightforward to check that this reduction respects the simplicial identities so as to define a simplicial groupoid, that it is a monoidal decomposition space, and that the quotient map from $\mathbf{S}$ is monoidal and CULF.

2.4.9. Ordered surjections. Let **OS** denote the fat nerve of the category of finite ordered set and monotone surjections. It is a monoidal decomposition space under ordinal sum. Hence to describe the resulting comultiplication, it is enough to say what happens to a connected ordered surjection, say $f : \underline{n} \twoheadrightarrow \underline{1}$, which we denote simply n: since there are no automorphisms around, we find

$$\Delta(n) = \sum_{k=1}^{n} \sum_{a} a \otimes k$$

where the second sum is over the $\binom{n-1}{k-1}$ possible surjections $a : n \twoheadrightarrow k$. The resulting bialgebra is essentially the (dual) Landweber–Novikov bialgebra in algebraic topology [58] (see also [6]), the noncommutative Faà di Bruno bialgebra in combinatorics [10], and the Dynkin–Faà di Bruno bialgebra in numerical analysis [60]; it also comes up in number theory [35]. See [25] and [48] for recent perspectives.

2.5. **Trees and graphs**

Various bialgebras of trees and graphs can be realised as incidence bialgebras of decomposition spaces which are not Segal. This means that one can decompose but not compose, as already exemplified in the running example with graphs 1.1.5. In each case the lack of composability is caused by the decomposition destroying info that would have been needed to define a composition. As we shall see (in 2.5.3), it is sometimes possible to 'remedy' this to get instead a decomposition space which is Segal, at the price of giving up connectedness of the bialgebra. In the examples based on graphs and trees, this involves keeping 'open-ended' edges, and is intimately related to the theory of operads and related structures (2.5.7).

All the examples of decomposition spaces in this subsection are monoidal under disjoint union, and hence the resulting coalgebras are bialgebras.

2.5.1. Butcher–Connes–Kreimer Hopf algebra. A *rooted tree* is a connected and simply connected graph with a specified root vertex; a forest is a disjoint union of rooted trees. The Butcher–Connes–Kreimer Hopf algebra of rooted trees [13] is the free algebra on the set of isomorphism classes of rooted trees, with comultiplication defined by summing over certain admissible cuts c:

$$\Delta(T) = \sum_{c \in \text{adm.cuts}(T)} P_c \otimes R_c \,.$$

An admissible cut c is a splitting of the set of nodes into two subsets, such that the second forms a subtree R_c containing the root node (or is the empty forest); the first subset, the complement 'crown', then forms a subforest P_c, regarded as a monomial of trees. (The order of the two factors is dictated by an operadic viewpoint, where leaves are 'in' and the root is 'out', and is further justified in 2.5.7 below.)

Dür [17] (Ch.IV, §3) gave an incidence-coalgebra construction of the Butcher–Connes–Kreimer coalgebra by starting with the category $\mathscr{C}$ of forests and root-preserving inclusions, generating a coalgebra (in our language the incidence coalgebra of the fat nerve of $\mathscr{C}$), and imposing the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

Note that to be precise, one must use $\mathscr{C}^{\mathrm{op}}$ instead of $\mathscr{C}$:

$$\mathbf{R} := \mathbf{N}(\mathscr{C}^{\mathrm{op}}).$$

From the viewpoint of the incidence coalgebra this 'op' affects the comultiplication only by reversing the order of the tensor factors. We shall see shortly that the 'op' originates in an upper-dec construction (compare 2.1.3).

We can obtain the Butcher–Connes–Kreimer coalgebra directly from a decomposition space: let $\mathbf{H}_1$ denote the groupoid of forests, and let $\mathbf{H}_2$ denote the groupoid of forests with an admissible cut. More generally, $\mathbf{H}_0$ is defined to be a point, and $\mathbf{H}_k$ is the groupoid of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid in which the inner face maps forget a cut, and the outer face maps project away stuff: $d_\perp$ deletes the crown (everything above the top-most cut) and $d_\top$ deletes the bottom layer (the part of the forest below the bottom-most cut). It is readily seen that $\mathbf{H}$ is not a Segal space: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that $\mathbf{H}_2$ is not equivalent to $\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_1$. It is straightforward to check that it *is* a decomposition space, in fact a symmetric monoidal one under disjoint union, and it is also clear from its construction that the resulting bialgebra is the Butcher–Connes–Kreimer Hopf algebra. Note that the decomposition space is graded by the number of nodes (which is precisely the length filtration 3.2.1), and that it is connected since the empty forest is the only forest with zero nodes.

To explain the relationship between the two constructions, note that admissible cuts are essentially the same thing as root-preserving forest inclusions: then the cut is interpreted as the division between the included forest and the forest induced on the nodes in its complement. In this way we see that $\mathbf{H}_k$ is the groupoid of $k-1$ consecutive root-preserving inclusions. Furthermore, there is a natural identification

$$\operatorname{Dec}_\top \mathbf{H} \simeq \mathbf{R} = \mathbf{N}(\mathscr{C}^{\mathrm{op}}),$$

where the 'op' occurs since we are dealing with an upper dec, as in 2.1.3. Under this identification, the dec map $\operatorname{Dec}_\top \mathbf{H} \to \mathbf{H}$, always a (symmetric monoidal) CULF functor, realises precisely Dür's reduction: on $\mathbf{R}_1 \to \mathbf{H}_1$ it sends a root-preserving forest inclusion to its crown, that is, its complement. More generally, on $\mathbf{R}_k \to \mathbf{H}_k$ it sends a sequence of forest inclusions $F_0 \subset F_1 \subset \cdots \subset F_k$ to

$$F_1 \smallsetminus F_0 \subset \cdots \subset F_k \smallsetminus F_0.$$

2.5.2. Restriction species and directed restriction species [32]**.** The Butcher–Connes–Kreimer example shares important characteristics with the graph example of Schmitt, our running example in Section 1 (Examples 1.1.5, 1.2.4, 1.5.10), but where in the graph example there are no constraints on the nature of the cuts, in the tree example, only certain order-respecting cuts are deemed admissible.

Both examples can be subsumed in big families of decomposition spaces, which can be treated uniformly, namely decomposition spaces of restriction species, in the sense of Schmitt [64] (see also [2]), and decomposition spaces of directed restriction species, introduced and studied in [32]. Here we content ourselves with outlining the idea.

A *restriction species* [64] is simply a presheaf of the category $\mathbb{I}$ of finite sets and injections. Compared to a classical species [39], a restriction species is thus functorial not only on bijections but also on injections, meaning that a given structure on a set induces such structure also on any subset.

Given a restriction species $R : \mathbb{I}^{\text{op}} \to \boldsymbol{Set}$, a coalgebra is obtained on the set of isomorphism classes of R-structures with comultiplication

$$\Delta(X) = \sum_{A+B=S} X|A \otimes X|B, \qquad X \in R[S]$$

and counit sending only the empty structures to 1. (This is the construction of Schmitt [64].)

It is preferable to work with groupoid-valued species as in [3], rather than the traditional set-valued species. Given a (groupoid-valued) restriction species $R : \mathbb{I}^{\text{op}} \to \boldsymbol{Grpd}$, we construct a simplicial groupoid $\mathbf{R}$ where $\mathbf{R}_k$ is the groupoid of R-structures with an ordered partition of the underlying set into k parts (possibly empty). Functoriality on generic maps is clear, by joining adjacent parts or inserting an empty part. Functoriality on free maps is about projecting away outer parts, and is possible precisely because R is a restriction species. This simplicial groupoid can be shown to be a decomposition space, and the resulting incidence coalgebra is the Schmitt coalgebra [32]. Furthermore, morphisms of restriction species induce CULF functors and hence coalgebra homomorphisms. A great many species are actually restriction species (such as various classes of graphs, matroids, and posets), providing in this way a large supply of decomposition spaces (which are not Segal spaces).

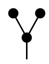
The Butcher–Connes–Kreimer example is subsumed in a large class of examples coming from *directed restriction species*, a notion introduced in [32]. Where ordinary restriction species are presheaves on finite sets and injections, directed restriction species are presheaves on the category of finite posets and convex injections. The definition formalises the idea of considering only decompositions compatible with the poset structure in a certain way, as exemplified clearly by the notion of admissible cut.

2.5.3. Operadic trees and P-trees. There is an important variation on the Butcher–Connes–Kreimer Hopf algebra (but it is only a bialgebra): instead of considering combinatorial trees one considers operadic trees (i.e. trees with open incoming edges, and an open-ended root edge). More generally one can consider P-trees for a finitary polynomial endofunctor P, i.e. trees whose nodes are decorated by the operations of P. For details on this setting, see [42, 43, 44], [26]; it suffices here to note that the notion of P-tree covers many kinds of structured trees, such as planar trees, binary trees, effective trees, linear trees, words, and a large class of inductive data types (W-types).

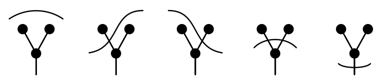
For operadic trees, when copying over the description to get a simplicial groupoid X where X_k is the groupoid of forests with $k-1$ compatible admissible cuts, there are two important differences, both due to the fact that the cuts cannot remove the edges, since this might violate the local structure of the tree, (e.g. being binary)—the cut leaves a trace of the edge on each side of the cut, in the form of an open-ended edge. One difference is that X_0 is not just a point: it is the groupoid of node-less forests. The second difference is that unlike $\mathbf{H}$, the simplicial groupoid X is a Segal space; this follows from the Key Lemma of [26] (see [48] for an abstract viewpoint). The reason is that the 'half-edges' left by the cut constitute enough data to reconstruct a tree with a cut from its bottom tree and crown by grafting. More precisely, the Segal maps $X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ return the layers seen in between the cuts, and they are easily seen to be equivalences: given the layers separately, and a match of their boundaries, one can glue them together to reconstruct the original forest, up to isomorphism. In this sense the operadic-forest decomposition space X is a 'category' with node-less forests as objects, and arbitrary forests as morphisms, a forest being seen as a morphism from its leaves to its roots. In this perspective, the decomposition space $\mathbf{H}$ of combinatorial forests is obtained from X by throwing away the object information, i.e. the data governing the composability constraints. These two differences are crucial in the work on Green functions and Faà di Bruno formulae in [26, 46, 48].

There is a functor from operadic trees or P-trees to combinatorial trees which is *taking core* [44]: it amounts to forgetting the P-decoration and shaving off all open-ended edges. This defines a monoidal CULF functor $X \to \mathbf{H}$ which realises a bialgebra homomorphism from the bialgebra of operadic trees or P-trees to the Butcher–Connes–Kreimer Hopf algebra of combinatorial trees.

2.5.4. Note about symmetries. One *cannot* obtain the same bialgebra of trees (either the combinatorial or the operadic) by taking isomorphism classes in each groupoid X_k: doing so would destroy symmetries that constitute an essential ingredient in the Butcher–Connes–Kreimer bialgebra. Indeed, define a simplicial set Y in which $Y_k = \pi_0(X_k)$, the set of isomorphism classes of forests with k compatible admissible cuts. Consider the tree T



belonging to X_1. The fibre of $d_1 : X_2 \to X_1$ over T is the (discrete) groupoid of all possible cuts in this tree:



The thing to notice here is that while the second and third cuts are isomorphic as abstract cuts, and therefore get identified in $Y_2 = \pi_0(X_2)$, this isomorphism does not fix the underlying tree T. This means that in the formula for comultiplication of T as an element of X_1 both cuts appear, and there is a total of 5 terms, whereas in the formula for comultiplication of T as an element of Y there will be only 4 terms. (Put in another way, the functor $X \to Y$ given by taking components is not CULF.)

It seems that there is no way to circumvent this discrepancy directly at the isomorphism class level: attempts involving ingenious decorations by natural numbers and actions by symmetric groups will almost certainly end up amounting to actually working at the groupoid level, and the conceptual clarity of the groupoid approach seems much preferable.

2.5.5. Non-commutative versions. The Butcher–Connes–Kreimer Hopf algebra of combinatorial trees admits a natural non-commutative version, first studied by Foissy [24]. It is defined in exactly the same way, but with *ordered* forests of *planar* combinatorial trees. In this case, the decomposition space is monoidal but not symmetric monoidal, giving naturally a non-commutative bialgebra.

The same modification can be applied in the operadic case. Planar operadic trees are precisely M-trees for M the free-monoid monad. More generally, to have planar structure on P-trees is to have a cartesian natural transformation $P \Rightarrow M$ (see [33] for details); in this situation there is a non-commutative bialgebra of ordered forests of P-trees.

2.5.6. Free multicategories [33]. Continuing the previous example, for any polynomial endofunctor P cartesian over M, the groupoid of P-trees is (essentially) discrete, which is to say that it is equivalent to the set of isomorphism classes of P-trees (because the planar structure encoded in the cartesian natural transformation to M fixes the automorphisms). This set is the set of operations of the free monad on P [33], [42]. Thinking of P as specifying a signature, we can equivalently think of P-trees as operations for the free (coloured) operad on that signature, or as the multi-arrows of the free multicategory on P regarded as a multigraph. To a multicategory there is associated a monoidal category [37], whose object set is the free monoid on the set of objects (colours). The decomposition space of P-trees is naturally identified with the (fat) nerve of the (monoidal) category associated to the multicategory of P-trees. (The adjective 'fat' is in parenthesis here because it could be omitted: the categories involved here have no invertible arrows (other than the identities), because the multicategory is free.)

2.5.7. Polynomial monads and operads. The decomposition space of P-trees for P a polynomial endofunctor (2.5.3) can be regarded as the decomposition space associated to the free monad on P. In fact the construction works for any (cartesian, discrete-finitary) polynomial monad, not just free ones, as we now proceed to explain. This construction has been generalised and subsumed in a more comprehensive setting of relative two-sided bar constructions in [48]. Presently we outline, in a more heuristic manner, the construction of a monoidal decomposition space from any coloured operad, and from it a commutative bialgebra. For the numerical version, some finiteness conditions must be assumed.

Coloured operads can be encoded as polynomial monads [76]. The combinatorial data of the endofunctor R underlying the monad is a diagram of groupoids

$$I \leftarrow E \rightarrow B \rightarrow I$$

where I is the set (or more generally, groupoid) of colours, B is the groupoid of operations (more precisely the action groupoid of the action of the symmetric groups on the operations), and E is the groupoid of operations with a marked input slot. It follows that $E \rightarrow B$ is a finite map; the fibre over an operation is the set of its input slots. The operad substitution law then amounts to a cartesian monad structure on R, i.e. cartesian natural transformations $R \circ R \Rightarrow R \Leftarrow \mathrm{Id}$ subject to axioms.

Following the graphical interpretation given in [47], one can regard I as the groupoid of decorated unit trees (i.e. trees without nodes), and B as the groupoid of corollas (i.e. trees with exactly one node) decorated with B on the node and I on the edges, compatibly. The arity of a corolla labeled by $b \in B$ is then the cardinality of the fibre E_b.

We can now form a simplicial groupoid X in which X_0 is the groupoid of disjoint unions of decorated unit trees, X_1 is the groupoid of disjoint unions of decorated corollas, and where more generally X_n is the groupoid of R-forests of height n. For example, X_2 is the groupoid of R-forests of height 2, which equivalently can be described as configurations consisting of a disjoint unions of bottom corollas whose leaves are decorated with other corollas, in such a way that the roots of the decorating corollas match the leaves of the bottom corollas. This groupoid can more formally be described as the free symmetric monoidal category on $R(B)$ (the endofunctor R applied to B). Similarly, X_n is the free symmetric monoidal category on $R^{n-1}(B)$. The outer face maps project away the top or bottom layer in a level-n forest. For example $d_0 : X_1 \to X_0$ sends a disjoint union of corollas to the disjoint union of their root edges, while $d_1 : X_1 \to X_0$ sends a disjoint union of corollas to the forest consisting of all their leaves. The generic face maps (i.e. inner face maps) join two adjacent layers by means of the monad multiplication on R. The degeneracy maps insert unary corollas by the unit of the monad. Associativity of the monad law ensures that this simplicial groupoid is actually a category object and a Segal space [48]. The operation of disjoint union makes this a symmetric monoidal decomposition space, and altogether an incidence bialgebra results from the construction.

The example (2.5.3) of P-trees (for P a polynomial endofunctor) and admissible cuts is an example of this construction, namely corresponding to the free monad on P: indeed, the operations of the free monad on P form the groupoid of P-trees, which now plays the role of B. Level-n trees in which each node is decorated by objects in B is the same thing as P-trees equipped with $n-1$ compatible admissible cuts, and grafting of P-trees (as prescribed by the generic face maps in 2.5.3) is precisely the monad multiplication in the free monad on P.

It should be stressed that while the decomposition space of a free operad is always automatically locally finite, the case of a general operad is not automatically so. This condition must be imposed separately if numerical examples are to be extracted.

Another subexample of this is the case where the monad is the terminal reduced monad Comm, which is the free-commutative-semimonoid monad. In this case, the resulting category object in groupoids is equivalent to the fat nerve of the category of surjections (as in 2.4), so the associated bialgebra is the classical Faà di Bruno bialgebra. The main achievement of [48] is to show that the Faà di Bruno formula for the comultiplication in the classical Faà di Bruno bialgebra generalises to incidence bialgebras of arbitrary operads and polynomial monads (the free case having been established previously in [26]).

2.5.8. Progressive graphs and free PROPs. The constructions in 2.5.3 readily generalise from trees to progressive graphs (although the attractive polynomial interpretation does not). By a progressive graph we understand a finite directed graph with a certain number of open input edges, a certain number of open output edges, and prohibited to contain an oriented cycle (see [45] for a categorical formalism). In particular, the set of vertices of a progressive graph has a natural poset structure. The progressive graphs form a groupoid $\mathbf{G}_1$. We allow graphs without vertices, these form a groupoid $\mathbf{G}_0$. Let $\mathbf{G}_2$ denote the groupoid of progressive graphs with an *admissible cut*: by this we mean a partition of the set of vertices into two disjoint parts, a down-set V_1 and an up-set V_2. This partition determines a set of edges, called the *cut*, consisting of the edges that connect a vertex in V_1 with a vertex in V_2, the out-edges of V_0, the in-edges of V_2, and the edges of G that are both in-edges and out-edges. The two vertex sets V_1 and V_2 induce new progressive graphs $G|V_1$ and $G|V_2$, by including all edges incident to the given vertex set, and including in both cases also the whole cut set, which becomes the new set of output edges for $G|V_1$ and the new set of input edges for $G|V_2$. Similarly, let $\mathbf{G}_k$ denote the groupoid of progressive graphs with $k-1$ compatible admissible cuts, just like we did for forests. It is clear that this defines a simplicial groupoid $\mathbf{G}$, easily verified to be a decomposition space and in fact a Segal space. The progressive graphs form the set of operations of the free PROP with one generator in each input/output degree (m, n). Decorating data for progressive graphs are called tensor schemes in [40], and the progressive graphs decorated by a tensor scheme form the set of operations of the free (coloured) PROP on the tensor scheme. The resulting decomposition space is naturally identified with the fat nerve of the underlying (symmetric monoidal) category. In fact, from this viewpoint, the construction works for any PROP, not just free ones, in analogy with the passage from trees and free operads to arbitrary operads (2.5.7). Note that disjoint union (or the monoidal structure underlying any PROP) makes the resulting incidence coalgebras into bialgebras.

Bialgebras of progressive graphs have been studied in the context of Quantum Field Theory by Manchon [55]. Certain decorated progressive graphs, and the resulting bialgebra have been studied by Manin [56], [57] in the theory of computation: his graphs are decorated by operations on partial recursive functions and switches.

3. Möbius inversion

3.1. Completeness, and Möbius inversion at the objective level

We are interested in the invertibility of the zeta functor (see 1.3.3) under the convolution product (see 1.3.2). Unfortunately, at the objective level it can practically *never* be convolution invertible, because the inverse μ should always be given by an alternating sum

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}.$$

We do not have minus sign available, but the sign-free equation

$$\zeta * \Phi_{\text{even}} = \varepsilon + \zeta * \Phi_{\text{odd}}$$

will hold, as we proceed to recall. In the category case (cf. [14, 49]), Φ_{even} is given by the even-length chains of non-identity arrows, that is, by the non-degenerate simplices of even dimension, and similarly for Φ_{odd}. To make sense of this for more general decomposition spaces we need to recall, from [30], the notion of completeness.

A simplex in any simplicial groupoid is degenerate when it is in the image of a degeneracy map. ‘Nondegenerate’ should mean to be in the complement of the image, but this is only well behaved for monomorphisms of groupoids, i.e. maps that are fully faithful as functors, see A.2.4.

3.1.1. Completeness and non-degeneracy. A decomposition space is *complete* if $s_0 : X_0 \to X_1$ is mono [30]. It follows that all other degeneracy maps in X are also mono (see [29]).

For a complete decomposition space X we define $\vec{X}_n \subset X_n$ to be the full subgroupoid of nondegenerate n-simplices, i.e. not in the image of any of the degeneracy maps. More importantly, in a decomposition space one can measure whether a simplex is nondegenerate on its principal edges: it is nondegenerate if and only if all its principal edges are [30, Corollary 2.14]. Hence it really just boils down to defining nondegenerate 1-simplices: define $\vec{X}_1 \subset X_1$ to be the complement of the monomorphism $s_0 : X_0 \to X_1$.

3.1.2. Examples and non-example. Clearly, every discrete decomposition space (such as strict nerves) is complete, since any map between sets which admits a retraction is a monomorphism. Also every Rezk-complete Segal space is complete in the sense of 3.1.1. In particular, fat nerves of categories are complete.

To see an example of a non-complete decomposition space, let G be a nontrivial group, and write BG for the same group considered as a groupoid with one object. Now consider the simplicial groupoid X with $X_n = (BG)^n$. Here $s_0 : 1 \to BG$ is not a monomorphism, as the trivial subgroupoid of BG is not a full subgroupoid.

3.1.3. 'Phi' functors. We define Φ_n to be the linear functor given by the span

$$X_1 \longleftarrow \vec{X}_n \longrightarrow 1.$$

If $n = 0$ then $\vec{X}_0 = X_0$ by convention, and Φ_0 is given by the span

$$X_1 \longleftarrow X_0 \longrightarrow 1.$$

That is, Φ_0 is the linear functor ε. Note that $\Phi_1 = \zeta - \varepsilon$, and is denoted η in the classical literature [14, 62]. The minus sign makes sense here, since X_0 and $\vec{X}_1$, representing ε and Φ_1, define complementary full subgroupoids of X_1, representing ζ.

Computing convolution with the functors Φ_n is really about knowing how the groupoids $\vec{X}_n$ behave under various pullbacks. This is carried out in detail in [30], leading to the following results.

Lemma. 3.1.4. [30, Lemma 3.6] *For a complete decomposition space, we have*

$$\Phi_n \quad = \quad (\Phi_1)^n \quad = \quad (\zeta - \varepsilon)^n,$$

the nth convolution product of Φ_1 *with itself.*

Proposition 3.1.5. *For a complete decomposition space* X*, the square*

$$\begin{tikzcd} \vec{X}_1 + \vec{X}_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \times \vec{X}_1 \arrow[r] & X_1 \times X_1 \end{tikzcd}$$

is a pullback.

These are special cases of [30, Lemma 3.5]. The proposition can be read as saying that if a 2-simplex σ has its second principal edge nondegenerate, then there are two possibilities for the first principal edge: either it is degenerate and the whole simplex σ is determined by the second principal edge (an element of $\vec{X}_1$), or it is nondegenerate and the whole simplex σ is nondegenerate (an element of $\vec{X}_2$).

From this lemma and its higher-dimensional analogues, it is not difficult to prove the following key result.

Proposition 3.1.6. [30, Proposition 3.7] *The linear functors Φ_n satisfy the following explicit equivalences of linear functors*

$$\zeta * \Phi_n \quad = \quad \Phi_n + \Phi_{n+1} \quad = \quad \Phi_n * \zeta.$$

Now let

$$\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n.$$

Theorem 3.1.7. [30, Theorem 3.8] *For a complete decomposition space, the following Möbius inversion principle holds (explicit equivalences of linear functors):*

$$\begin{array}{rcccc} & \zeta * \Phi_{\text{even}} & = & \varepsilon & + & \zeta * \Phi_{\text{odd}}, \\ = & \Phi_{\text{even}} * \zeta & = & \varepsilon & + & \Phi_{\text{odd}} * \zeta. \end{array}$$

Proof. This follows immediately from the proposition: all four linear functors are in fact equivalent to $\sum_{r \geq 0} \Phi_r$. □

For these results there is no need for finiteness conditions: there in no problem in taking infinite sums of groupoids. In the following subsection, however, we must impose finiteness conditions before we can take cardinality and recover Möbius inversion at the level of vector spaces and (co)algebras over $\mathbb{Q}$.

3.2. **Length and Möbius decomposition spaces**

If X is a complete and locally finite decomposition space, then by Proposition A.4.3 the linear functors

$$\Phi_r : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$$

are finite for each $r \geq 0$ and descend to linear functors

$$\Phi_r : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}.$$

This is not enough to guarantee finiteness of the sum of all those Φ_r and hence allow the Möbius inversion formula to descend to the vector-space level. For this we also need to assume that for each $f \in X_1$, there is an upper bound on the dimension of a nondegenerate n-simplex with long edge f. This condition is important in its own right, as it is the condition for the existence of a length filtration 3.2.1, useful in many applications. When X is the nerve of a category, the condition says that for each arrow f, there is an upper bound on the number of non-identity arrows in a sequence of arrows composing to f. We are led to the following definition.

3.2.1. Length. A complete decomposition space X is *of locally finite length* if, for each $a \in X_1$, the fibres $F_a^{(n)}$ of $d_1^{n-1} : \vec{X}_n \to X_1$ over a are empty for n sufficiently large.

The *length* of a is the greatest n for which $F_a^{(n)} \neq \varnothing$; this induces a filtration on the incidence coalgebra. If X is a Segal space, it is the longest factorisation of a into nondegenerate $a_i \in \vec{X}_1$.

3.2.2. Example. The incidence coalgebra of $(\mathbb{N}^2, +)/\mathfrak{S}_2$ (see 2.1.6) is the simplest example we know of in which the length filtration does not agree with the coradical filtration (see Sweedler [69] for this notion). The elements $(1,1)$ and $(2,0) \simeq (0,2)$ are clearly of length 2. On the other hand, the element

$$P := (1,1) - (2,0) - (0,2)$$

is primitive, meaning

$$\Delta(P) = (0,0) \otimes P + P \otimes (0,0)$$

and is therefore of coradical filtration degree 1. (Note that in $(\mathbb{N}^2, +)$ it is not true that P is primitive: it is the symmetrisation that makes the $(0,1)$ terms cancel out in the computation, to make P primitive.)

3.2.3. Möbius condition. A complete decomposition space X is *Möbius* if it is locally finite and of locally finite length, that is, for each a, $F_a^{(n)}$ is finite and eventually empty.

Note that for posets, 'locally finite' already implies 'locally finite length', so the Möbius condition is not needed separately in the poset case. If X is the strict nerve of a category, then it is Möbius in our sense if and only if it is Möbius in the sense of Leroux [51].

Classically, it is known that a Möbius category in the sense of Leroux does not have non-identity invertible arrows [49, Lemma 2.4]. Similarly (cf. [30, Corollary 8.7]), if a Möbius decomposition space X is a Segal space, then it is Rezk complete (meaning that all invertible arrows are degenerate, cf. B.2.3).

Lemma. 3.2.4. *A complete decomposition space X is Möbius if and only if X_1 is locally finite and the restricted composition map*

$$\sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$$

is finite.

Thus, if X is Möbius, the linear functors Φ_{even} and Φ_{odd} also descend to

$$\Phi_{\text{even}}, \Phi_{\text{odd}} : \boldsymbol{grpd}_{/X_1} \to \boldsymbol{grpd}$$

and their cardinalities are elements $|\Phi_{\text{even}}|, |\Phi_{\text{odd}}| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$ of the incidence algebra. We can therefore take the cardinality of the abstract Möbius inversion formula of Theorem 3.1.7:

Theorem 3.2.5. *If X is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{\mathrm{even}}| - |\Phi_{\mathrm{odd}}|$:*

$$|\zeta| * |\mu| = |\varepsilon| = |\mu| * |\zeta| .$$

3.3. Möbius functions and cancellation

We compute the Möbius functions in some of our examples. While the formula $\mu = \Phi_{\mathrm{even}} - \Phi_{\mathrm{odd}}$ seems to be the most general and uniform expression of the Möbius function, it is often not the most economical. At the numerical level, it is typically the case that much more practical expressions for the Möbius functions can be computed with different techniques. The formula $\Phi_{\mathrm{even}} - \Phi_{\mathrm{odd}}$ should not be dismissed on these grounds, though: it must be remembered that it constitutes a natural 'bijective' account, valid at the objective level, in contrast to many of the elegant cancellation-free expressions in the classical theory which are often the result of formal algebraic manipulations, often power-series representations.

Comparison with the economical formulae raises the question whether these too can be realised at the objective level. This can be answered (in a few cases) by exhibiting an explicit cancellation between Φ_{even} and Φ_{odd}, which in turn may or may not be given by a *natural* bijection.

Once a more economical expression has been found for some Möbius decomposition space X, it can be transported back along any CULF functor $f : Y \to X$ to yield also more economical formulae for Y.

3.3.1. Natural numbers. For the decomposition space $\mathbf{N}$ (see 2.1.1), the incidence algebra is $\boldsymbol{grpd}^{\mathbb{N}}$, spanned by the representables h^n, and with convolution product

$$h^a * h^b = h^{a+b}.$$

To compute the Möbius functor, we have

$$\Phi_{\mathrm{even}} = \sum_{r \text{ even}} (\mathbb{N} \smallsetminus \{0\})^r,$$

hence $\Phi_{\mathrm{even}}(\underline{n})$ is the set of ordered compositions of the ordered set $\underline{n}$ into an even number of parts, or equivalently

$$\Phi_{\mathrm{even}}(\underline{n}) = \{\underline{n} \twoheadrightarrow \underline{r} \mid r \text{ even } \},$$

the set of monotone surjections. In conclusion, with an abusive sign notation, the Möbius functor is

$$\mu(\underline{n}) = \sum_{r \geq 0} (-1)^r \{\underline{n} \twoheadrightarrow \underline{r}\}.$$

At the numerical level, this formula simplifies to

$$\mu(n) = \sum_{r \geq 0} (-1)^r \binom{n-1}{r-1} = \begin{cases} 1 & \text{for } n = 0 \\ -1 & \text{for } n = 1 \\ 0 & \text{else,} \end{cases}$$

(remembering that $\binom{-1}{-1} = 1$, and $\binom{k}{-1} = 0$ for $k \geq 0$).

On the other hand, since clearly the incidence algebra is isomorphic to the power series ring under the identification $|h^n| = \delta^n \leftrightarrow z^n \in \mathbb{Q}[[z]]$, and since the zeta function corresponds to the geometric series $\sum_n x^n = \frac{1}{1-x}$, we find that the Möbius function is $1-x$. This corresponds to the functor $\delta^0 - \delta^1$.

At the objective level, there is indeed a cancellation of groupoids taking place. It amounts to an equivalence of the Phi-groupoids restricted to $n \geq 2$:

$$\begin{tikzcd} \Phi_{\text{even}|r\geq 2} \arrow[rr, "\sim"] \arrow[dr] & & \Phi_{\text{odd}|r\geq 2} \arrow[dl] \\ & \mathbb{N}_{\geq 2} & \end{tikzcd}$$

which cancels out most of the stuff, leaving us with the much more economical Möbius function

$$\delta^0 - \delta^1$$

supported on $\mathbb{N}_{\leq 1}$. Since $\mathbb{N}$ is discrete, this equivalence (just a bijection) can be established fibrewise:

For each $n \geq 2$ there is a natural fibrewise bijection

$$\Phi_{\text{even}}(n) \simeq \Phi_{\text{odd}}(n).$$

To see this, encode the elements $(x_1, x_2, \ldots, x_k)$ in $\Phi_{\text{even}}(n)$ (and $\Phi_{\text{odd}}(n)$) as binary strings of length n and starting with 1 as follows: each coordinate x_i is represented as a string of length x_i whose first bit is 1 and whose other bits are 0, and all these strings are concatenated. In other words, thinking of the element $(x_1, x_2, \ldots, x_k)$ as a ordered partition of the ordered set n, in the binary representation the 1-entries mark the beginning of each part. (The binary strings must start with 1 since the first part must begin at the beginning.) For example, with $n = 8$, the element $(3, 2, 1, 1, 1) \in \Phi_{\text{odd}}(8)$, is encoded as the binary string 10010111. Now the bijection between $\Phi_{\text{even}}(n)$ and $\Phi_{\text{odd}}(n)$ can be taken to simply flip the second bit in the binary representation. In the example, 10010111 is sent to 11010111, meaning that $(3, 2, 1, 1, 1) \in \Phi_{\text{odd}}(8)$ is sent to $(1, 2, 2, 1, 1, 1) \in \Phi_{\text{even}}(8)$. Because of this cancellation which occurs for $n \geq 2$ (we need the second bit in order to flip), the difference $\Phi_{\text{even}} - \Phi_{\text{odd}}$ is the same as $\delta_0 - \delta_1$, which is the cancellation-free formula.

The minimal solution $\delta^0 - \delta^1$ can also be checked immediately at the objective level to satisfy the defining equation for the Möbius function:

$$\zeta * \delta^0 = \zeta * \delta^1 + \delta^0$$

This equation says

$$\begin{tikzcd} \mathbb{N} \times \{0\} \arrow[d, "\text{add}"'] & = & (\mathbb{N} \times \{1\}) + \{0\} \arrow[d, "\text{add+incl}"'] \\ \mathbb{N} & & \mathbb{N} \end{tikzcd}$$

In conclusion, the classical formula lifts to the objective level.

3.3.2. Finite sets and bijections. Already for the next example (2.1.7), that of the monoidal groupoid $(\mathbb{B}, +, 0)$, whose incidence algebra is the algebra of species under the Cauchy convolution product (cf. [2]), the situation is more subtle.

Similarly to the previous example, we have $\Phi_r(S) = \mathrm{Surj}(S, \underline{r})$, but this time we are dealing with arbitrary surjections, as S is just an abstract set. Hence the Möbius functor is given by

$$\mu(S) = \sum_{r \geq 0} (-1)^r \, \mathrm{Surj}(S, \underline{r}).$$

Numerically, the incidence algebra is just the power series ring $\mathbb{Q}[[z]]$ (cf. 2.1.7). Since this time the zeta function is the exponential $\exp(z)$, the Möbius function is the series $\exp(-z)$, corresponding to

$$\mu(n) = (-1)^n.$$

The economical Möbius function suggests the existence of the following equivalence at the groupoid level:

$$\mu(S) = \int^r (-1)^r h^r(S) \;\simeq\; \mathbb{B}_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{odd}}(S),$$

where

$$\mathbb{B}_{\mathrm{even}} = \sum_{r \text{ even}} \mathbb{B}_{[r]} \quad \text{and} \quad \mathbb{B}_{\mathrm{odd}} = \sum_{r \text{ odd}} \mathbb{B}_{[r]}$$

are the full subgroupoids of $\mathbb{B}$ consisting of the even and odd sets, respectively. However, it seems that such an equivalence is not possible, at least not over $\mathbb{B}$: while we are able to exhibit a bijective proof, this bijection is *not* natural, and hence does not assemble into a groupoid equivalence.

Proposition 3.3.3. *For a fixed set S, there are monomorphisms $\mathbb{B}_{\mathrm{even}}(S) \hookrightarrow \Phi_{\mathrm{even}}(S)$ and $\mathbb{B}_{\mathrm{odd}}(S) \hookrightarrow \Phi_{\mathrm{odd}}(S)$, and a residual bijection*

$$\Phi_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{even}}(S) = \Phi_{\mathrm{odd}}(S) - \mathbb{B}_{\mathrm{odd}}(S).$$

This is not *natural in S, though, and hence does not constitute an isomorphism of species, only an equipotence of species* [7].

Corollary 3.3.4. *For a fixed S there is a bijection*

$$\mu(S) \simeq \mathbb{B}_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{odd}}(S)$$

but it is not *natural in S.*

Proof of the Proposition. The map $\mathbb{B}_{\mathrm{even}} \to \mathbb{B}$ is a monomorphism of groupoids (A.2.4), so for each set S of even cardinality there is a single element to subtract from $\Phi_{\mathrm{even}}(S)$. The groupoid Φ_{even} has as objects finite sets S equipped with a surjection $S \twoheadrightarrow \underline{k}$ for some even k. If S is itself of even cardinality n, then among such partitions there are $n!$ possible partitions into n parts. If there were given a total order on S, among these $n!$ n-block partitions, there is one for which the order of S agrees with the order of the n parts. We would like to subtract that one and then establish the required bijection. This can be done fibrewise: over a given n-element set S, we can establish the bijection by choosing first a bijection $S \simeq \underline{n} = \{1, 2, \ldots, n\}$, the totally ordered set with n elements.

For each n, there is an explicit bijection

$$\{\textit{surjections } p : \underline{n} \twoheadrightarrow \underline{k} \mid k \textit{ even}, p \textit{ not the identity map}\}$$

$$\leftrightarrow$$

$$\{\textit{surjections } p : \underline{n} \twoheadrightarrow \underline{k} \mid k \textit{ odd}, p \textit{ not the identity map}\}$$

Indeed, define first the bijection on the subsets for which $p^{-1}(1) \neq \{1\}$, i.e. the element 1 is not alone in the first block. In this case the bijection goes as follows. If the element 1 is alone in a block, join this block with the previous block. (There exists a previous block as we have excluded the case where 1 is alone in block 1.) If 1 is not alone in a block, separate out 1 to a block on its own, coming just after the original block. Example

$$(34, 1, 26, 5) \leftrightarrow (134, 26, 5)$$

For the remaining case, where 1 is alone in the first block, we just leave it alone, and treat the remaining elements inductively, considering now the case where the element 2 is not alone in the second block. In the end, the only case not treated is the case where for each j, we have $p^{-1}(j) = \{j\}$, that is, each element is alone in the block with the same number. This is precisely the identity map excluded explicitly in the bijection. (Note that for each n, this case only appears on one of the sides of the bijection, as either n is even or n is odd.) □

In fact, already subtracting the groupoid $\mathbb{B}_{\text{even}}$ from Φ_{even} is not possible naturally. We would have first to find a monomorphism $\mathbb{B}_{\text{even}} \hookrightarrow \Phi_{\text{even}}$ over $\mathbb{B}$. But the automorphism group of an object $\underline{n} \in \mathbb{B}$ is $\mathfrak{S}_n$, whereas the automorphism group of any overlying object in Φ_{even} is a proper subgroup of $\mathfrak{S}_n$. In fact it is the subgroup of those permutations that are compatible with the surjection $\underline{n} \twoheadrightarrow \underline{k}$. So locally the fibration $\Phi_{\text{even}} \to \mathbb{B}$ is a group monomorphism, and hence it cannot have a section. So in conclusion, we cannot even realise $\mathbb{B}_{\text{even}}$ as a full subgroupoid in Φ_{even}, and hence it doesn't make sense to subtract it.

One may note that it is not logically necessary to be able to subtract the redundancies from Φ_{even} and Φ_{odd} in order to find the economical formula. It is enough to establish directly (by a separate proof) that the economical formula holds, by actually convolving it with the zeta functor. At the object level the simplified Möbius function would be the groupoid

$$\mathbb{B}_{\text{even}} - \mathbb{B}_{\text{odd}}.$$

We might try to establish directly that

$$\zeta * \mathbb{B}_{\text{even}} = \zeta * \mathbb{B}_{\text{odd}} + \varepsilon.$$

This should be a groupoid equivalence over $\mathbb{B}$. But again we can only establish this fibrewise. This time, however, rather than exploiting a non-natural total order, we can get away with a non-natural base-point. On the left-hand side, the fibre over an n-element set S, consists of an arbitrary set and an even set whose disjoint union is S. In other words, it suffices to give an even subset of S. Analogously, on the right-hand side, it amounts to giving an odd subset of S—or in the special case of $S = \emptyset$, we also have the possibility of giving that set, thanks to the summand ε. This is possible, non-naturally:

For a fixed nonempty set S, there is an explicit bijection between even subsets of S and odd subsets of S.

Indeed, fix an element $s \in S$. The bijection consists of adding s to the subset U if it does not belong to U, and removing it if it already belongs to U. Clearly this changes the parity of the set.

Again, since the bijection involves the choice of a basepoint, it seems impossible to lift it to a natural bijection.

3.3.5. Finite vector spaces. We calculate the Möbius function in the incidence algebra of the Waldhausen decomposition space of $\mathbb{F}_q$-vector spaces, cf. 2.3.5. In this case, Φ_r is the groupoid of strings of $r-1$ nontrivial injections. The fibre over V is the discrete groupoid of strings of $r-1$ nontrivial injections whose last space is V. This is precisely the set of nontrivial r-flags in V, i.e. flags for which the r consecutive codimensions are nonzero. In conclusion,

$$\mu(V) = \sum_{r=0}^{n} (-1)^r \{ \text{ nontrivial } r\text{-flags in } V\}.$$

(That's in principle a groupoid, but since we have fixed V, it is just a discrete groupoid: a flag inside a fixed vector space has no automorphisms.)

The number of flags with codimension sequence p is the q-multinomial coefficient

$$\binom{n}{p_1, p_2, \dots, p_r}_q.$$

In conclusion, at the numerical level we find

$$\mu(V) = \mu(n) = \sum_{r=0}^{n} (-1)^r \sum_{\substack{p_1+\cdots+p_r=n \\ p_i>0}} \binom{n}{p_1, p_2, \dots, p_r}_q.$$

On the other hand, it is classical that from the power-series representation (2.3.5) one gets the numerical Möbius function

$$\mu(n) = (-1)^n q^{\binom{n}{2}}.$$

While the equality of these two expressions can easily be established at the numerical level (for example via a zeta-polynomial argument, cf. below), we do not know of an objective interpretation of the expression $\mu(n) = (-1)^n q^{\binom{n}{2}}$. Realising the cancellation on the objective level would require first of all to being able to impose extra structure on V in such a way that among all nontrivial r-flags, there would be $q^{\binom{r}{2}}$ special ones!

3.3.6. Faà di Bruno. Recall (from 2.4) that the incidence bialgebra of the fat nerve of the monoidal category of finite sets and surjections is the Faà di Bruno bialgebra. Since clearly ζ and ε are multiplicative, also μ is multiplicative, i.e. determined by its values on the connected surjections. The general formula gives

$$\mu(\underline{n} \twoheadrightarrow \underline{1}) = \sum_{r=0}^{n} (-1)^n \boldsymbol{Tr}(n, r)$$

where $\boldsymbol{Tr}(n,r)$ is the (discrete) groupoid of n-leaf r-level trees with no trivial level (in fact, more precisely, strings of r nontrivial surjections composing to $n \twoheadrightarrow 1$), and where the minus sign is abusive notation for splitting into even and odd.

On the other hand, classical theory (see Doubilet–Rota–Stanley [16]) gives the following 'connected Möbius function':

$$\mu(n) = (-1)^{n-1}(n-1)!.$$

In conjunction, the two expressions yield the following combinatorial identity:

$$(-1)^{n-1}(n-1)! \;=\; \sum_{r=0}^{n} (-1)^r \left|\boldsymbol{Tr}(n,r)\right|.$$

We do not know how to realise the cancellation at the objective level. This would require first developing a bit further the theory of monoidal decomposition spaces and incidence bialgebras, a task we plan to take up in the near future.

3.3.7. Zeta polynomials. For a complete decomposition space X, we can classify the r-simplices according to their degeneracy type, writing

$$X_r = \sum_{k=0}^{r} \binom{r}{k} \vec{X}_k,$$

where the binomial coefficient is an abusive shorthand for that many copies of $\vec{X}_k$, embedded disjointly into X_r by specific degeneracy maps (see [30, 2.6] for details). Now we fibre over a fixed arrow $f \in X_1$, to obtain

$$(X_r)_f = \sum_{k=0}^{\infty} \binom{r}{k} (\vec{X}_k)_f, \tag{15}$$

where we have now allowed ourselves to sum to infinity, but for fixed f of finite length it is still a finite sum.

The '*zeta polynomial*' of a decomposition space X is the function

$$\begin{aligned} \zeta^r(f) : X_1 \times \mathbb{N} &\longrightarrow \boldsymbol{Grpd} \\ (f,r) &\longmapsto (X_r)_f \end{aligned}$$

assigning to each arrow f and $r \in \mathbb{N}$ the ∞-groupoid of r-simplices with long edge f. For fixed $f \in X_1$ of finite length ℓ, this is a polynomial in r, as witnessed by the expression (15). In this case, at the numerical level, we can substitute $r = -1$ into it to find:

$$\zeta^{-1}(f) = \sum_{k=0}^{\infty} (-1)^k \Phi_k(f)$$

Hence $\zeta^{-1}(f) = \mu(f)$, as the notation suggests.

In some cases there is a polynomial formula for $\zeta^r(f)$. For example, in the case $X = (\mathbb{N}, +)$ of 2.1.1 we find $\zeta^r(n) = \binom{n+r-1}{n}$, and therefore $\mu(n) = \binom{n-2}{n}$, in agreement with the other calculations (of this trivial example). In the case $X = (\mathbb{B}, +)$ of 2.1.7, we find $\zeta^r(n) = r^n$, and therefore $\mu(n) = (-1)^n$ again.

Sometimes, even when a formula for $\zeta^r(n)$ cannot readily be found, the (-1)-value can be found by a power-series representation argument. For example in the case of the Waldhausen $S_\bullet$ construction of ***vect*** (2.3.5), we have that $\zeta^r(n)$ is the set of r-flags of $\mathbb{F}_q^n$ (allowing trivial steps). We have

$$\zeta^r(n) = \sum_{\substack{p_1+\cdots+p_r=n \\ p_i \geq 0}} \frac{[n]!}{[p_1]!\cdots[p_r]!},$$

and therefore

$$\sum_{n=0}^{\infty} \zeta^r(n) \frac{z^n}{[n]!} = \left(\sum_{n=0}^{\infty} \frac{z^n}{[n]!} \right)^r,$$

Now $\zeta^{-1}(n)$ can be read off as the nth coefficient in the inverted series $\big(\sum_{n=0}^{\infty} \frac{z^n}{[n]!}\big)^{-1}$. In the case at hand, these coefficients are $(-1)^n q^{\binom{n}{2}}$, as we already saw.

Once a more economical Möbius function has been found for a decomposition space X, it can be exploited to yield more economical formulae for any decomposition space Y with a CULF functor to X. This is the content of the following straightforward lemma:

Lemma. 3.3.8. *Suppose that for the complete decomposition space X we have found a Möbius inversion formula*

$$\zeta * \Psi_0 = \zeta * \Psi_1 + \varepsilon.$$

Then for every decomposition space CULF over X, say $f : Y \to X$, we have the same formula

$$\zeta * f^*\Psi_0 = \zeta * f^*\Psi_1 + \varepsilon$$

for Y.

3.3.9. Length. In most of the examples treated, the length filtration 3.2.1 is actually a grading. Recall from [30, 6.20] that this amounts to having a simplicial map from X to the nerve of $(\mathbb{N}, +)$. In the rather special situation when this is CULF, the economical Möbius function formula

$$\mu = \delta^0 - \delta^1$$

for $(\mathbb{N}, +)$ induces the same formula for the Möbius functor of X. This is of course a very restrictive condition; in fact, for nerves of categories, this happens only for free categories on directed graphs (cf. Street [68]). For such categories, there is for each $n \in \mathbb{N}$ a linear span δ^n consisting of all the arrows of length n. In particular, δ^0 is the span $X_1 \leftarrow X_0 \to 1$ (the inclusion of the vertex set into the set of arrows), and δ^1 is the span $X_1 \leftarrow E \to 1$, the inclusion of the original set of edges into the set of all arrows. The simplest example is the free monoid on a set S, i.e. the monoid of words in the alphabet S. The economical Möbius function is then $\delta^0 - \delta^1$, where $\delta^1 = \sum_{s\in S} \delta^s$. In the power series ring, with a variable z_s for each letter $s \in S$, it is the series $1 - \sum_{s\in S} z_s$.

3.3.10. Decomposition spaces over B (2.1.7). Similarly, if a decomposition space X admits a CULF functor $\ell : X \to \mathbf{B}$ (which may be thought of as a 'length function with symmetries') then at the numerical level and at the objective level, locally for each object $S \in X_1$, we can pull back the economical Möbius 'functor' $\mu(n) = (-1)^n$ from $\mathbf{B}$ to X, yielding the numerical Möbius function on X

$$\mu(f) = (-1)^{\ell(f)}.$$

An example of this is the coalgebra of graphs 1.2.4 of Schmitt [64]: the functor from the decomposition space of graphs to $\mathbf{B}$ which to a graph associates its vertex set is CULF. Hence the Möbius function for this decomposition space is

$$\mu(G) = (-1)^{|V(G)|}.$$

In fact this argument works for any restriction species [32].

We finish with a kind of non-example which raises certain interesting questions.

Example 3.3.11. Consider the strict nerve of the category

$$\begin{tikzcd} x \arrow[loop left, "e"] \arrow[r, bend right, "r"'] & y \arrow[l, bend right, "s"'] \end{tikzcd}$$

in which $r \circ s = \mathrm{id}_y$, $s \circ r = e$ and $e \circ e = e$. This decomposition space X is clearly locally finite, so it defines a vector-space coalgebra, in fact a finite-dimensional one. One can check by linear algebra (see Leinster [50, Ex.6.2]), that this coalgebra has Möbius inversion. On the other hand, X is not of locally finite length, because the identity arrow id_y can be written as an arbitrary long string $\mathrm{id}_y = r \circ s \circ \cdots \circ r \circ s$. In particular X is not a Möbius decomposition space. So we are in the following embarrassing situation: on the objective level, X has Möbius inversion (as it is complete), but the formula does not have a cardinality. At the same time, at the numerical level Möbius inversion exists nevertheless. Since inverses are unique if they exist, it is therefore likely that the infinite Möbius inversion formula of the objective level admits some drastic cancellation at this level, yielding a finite formula, whose cardinality is the numerical formula. Unfortunately, so far we have not been able to pinpoint such a cancellation.

Appendix A. Groupoids

A.1. Homotopy theory of groupoids

We briefly recall the needed basic notions of groupoids and their homotopy cardinalities. The manuscript in preparation [11] will become a suitable reference for this material. For homotopy cardinality we refer to [28] (which however deals with ∞-groupoids instead of groupoids).

A.1.1. Groupoids. A *groupoid* is a small category in which all the arrows are invertible. A map of groupoids is just a functor. Let ***Grpd*** denote the category of groupoids and maps.

Intuitively we consider groupoids as sets with built-in symmetries. While a group models symmetry automorphisms of one object, groupoids model automorphisms and isomorphisms between several objects.

A.1.2. Homotopy equivalences. A homotopy between two maps of groupoids is just a natural transformation of functors. A map of groupoids $f : X \to Y$ is called a *homotopy equivalence* when there exists a pseudo-inverse $g : Y \to X$, meaning that the two composites are homotopic to the identities: $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Just as for categories, homotopy equivalences can also be characterised as functors that are essentially surjective and fully faithful.

Homotopy equivalence is the appropriate notion of sameness for groupoids, and it is important that all the notions involved be invariant under homotopy equivalence.

We adopt the convention that all notions in the paper are the homotopy invariant ones: outside this appendix we will usually say *equivalence, finite, discrete, trivial, cartesian, pullback, fibre, sum, colimit* and *monomorphism* instead of 'homotopy equivalence', 'homotopy finite', 'homotopy discrete', etc, for the notions defined below. It is essential that the word 'homotopy' is understood throughout.

A.1.3. Connectedness, discreteness. A groupoid X is *connected* if $\mathrm{obj}(X)$ is non-empty and the set $\mathrm{Hom}_X(x, y)$ is non-empty for all $x, y \in X$. A maximal connected subgroupoid of X is termed a *component* of X and denoted $[x]$ or $X_{[x]}$, where x is some object in the component. The set of components is denoted $\pi_0(X)$. We denote by $\pi_1(X, x)$ the automorphism group $\mathrm{Aut}_X(x) = \mathrm{Hom}_X(x, x)$. A groupoid X is *homotopy discrete* if $\pi_1(X, x)$ is trivial for all x, and *contractible* if it is homotopy discrete and also connected. This means homotopy equivalent to a point, i.e. the terminal groupoid 1.

A.1.4. Finiteness. A groupoid X is *locally finite* if $\pi_1(X, x)$ is finite for every x, and is *(homotopy) finite* if in addition $\pi_0(X)$ is finite. We denote by ***grpd*** the category of finite groupoids.

A.1.5. Pullbacks. The *homotopy fibre product* of maps $f : G \to B$ and $g : E \to B$ is the groupoid $H = G \times_B E$ whose objects are triples (x, y, ϕ) consisting of $x \in G$, $y \in E$, and $\phi : fx \to gy$ in B, and whose arrows $(x', y', \phi') \to (x, y, \phi)$ are pairs $(\beta, \varepsilon) \in \mathrm{Hom}_G(x', x) \times \mathrm{Hom}_E(y', y)$ such that $\phi \circ f(\beta) = g(\varepsilon) \circ \phi'$. There are canonical projections p, q,

$$
\begin{tikzcd}
H \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "g"] \\
G \arrow[r, "f"'] & B.
\end{tikzcd}
\tag{16}
$$

The diagram does not commute on the nose, but the third components of objects $a = (x, y, \phi)$ provide a natural isomorphism $\{\phi : fp(a) \cong gq(a)\}$. We say a square (16) is *homotopy cartesian* or a *homotopy pullback* if H is homotopy equivalent to the homotopy fibre product $G \times_B E$ given explicitly above. The map p is sometimes termed the *pullback of g along f* and denoted $f^*(g)$.

A.1.6. Fibres. The *homotopy fibre* E_b of a map $p : E \to B$ over an object b of B is the homotopy pullback of p along the map $\ulcorner b \urcorner : 1 \to B$ that picks out the element

b:

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ 1 \arrow[r, "\ulcorner b \urcorner"'] & B \end{tikzcd}$$

A.1.7. Loops. The *loop groupoid* $\Omega_b B$ of a groupoid B at an object b is given by the homotopy pullback $1 \times_B 1$ of the inclusion $\ulcorner b \urcorner : 1 \to B$ along itself. This is discrete: it has $\mathrm{Aut}_B(b)$ as its set of objects, and only the identity isomorphisms.

A.2. **Slices and the fundamental equivalence**

A.2.1. Slices. We shall need homotopy slices, sometimes called weak slices. First recall the usual notion of slice category: If $\mathscr{C}$ is a category, and $I \in \mathscr{C}$, then the usual slice category $\mathscr{C}/I$ is the category whose objects are morphisms $X \to I$ in $\mathscr{C}$ and whose arrows are commutative triangles

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & I. & \end{tikzcd}$$

We are concerned instead with groupoid-enriched categories $\mathscr{C}$, i.e. categories such that the arrows between each pair of objects X, Y define a groupoid $\mathrm{Map}(X, Y)$ instead of just a set, and the composition law is given by groupoid maps instead of just functions. Thus, between two parallel arrows $X \rightrightarrows Y$ there may be invertible 2-cells. The *homotopy slice category* $\mathscr{C}_{/I}$ then has as objects the morphisms $X \to I$; its arrows are triangles with a 2-cell

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \Rightarrow & X' \arrow[dl] \\ & I. & \end{tikzcd} \tag{17}$$

The basic example is $\mathscr{C} = \boldsymbol{Grpd}$ with 2-cells given by homotopies between maps (that is, the natural isomorphisms).

A.2.2. Homotopy sums and Grothendieck construction. For a map $p : E \to B$, each isomorphism $\beta : b' \to b$ in B induces an equivalence of homotopy fibres $\beta_* : E_{b'} \to E_b$, sending an object $(1, e, \phi : pe \cong b')$ to $(1, e, \beta\phi : pe \cong b)$. Thus the homotopy fibres of $p : E \to B$ form a B-indexed family of groupoids, that is, a functor $E_{(-)}$ from B to the category $\boldsymbol{Grpd}$ of groupoids.

The *homotopy sum* of any B-indexed family of groupoids $E : B \to \boldsymbol{Grpd}$ is the groupoid given by the *homotopy colimit* of this functor, which may be defined by the *Grothendieck construction* and denoted $\int^{b \in B} E_b$. Its objects are pairs (b, e) with $b \in B$ and of $e \in E_b$, and isomorphisms $(b', e') \to (b, e)$ are pairs (β, ε) of isomorphisms $\beta : b' \to b$ in B and $\varepsilon : \beta_* e' \to e$ in E_b.

The Grothendieck construction of any family $E : B \to \boldsymbol{Grpd}$ comes equipped with a canonical projection to B whose homotopy fibres give back the original family

E up to homotopy equivalence. Conversely, for any map $E \to B$, the homotopy sum of its homotopy fibres E_b is homotopy equivalent, over B, to E. Thus we have

Theorem A.2.3 (Fundamental Equivalence)**.** *There is a canonical equivalence between the categories of groupoids over a fixed groupoid B and that of groupoid-valued functors from B,*

$$\boldsymbol{Grpd}_{/B} \;\simeq\; \boldsymbol{Grpd}^{B}$$

given by taking homotopy fibres and the Grothendieck construction.

A.2.4. Monomorphisms. A map $E \to B$ is a *homotopy monomorphism* if each homotopy fibre E_b is empty or contractible. Up to homotopy equivalence, such a map is the inclusion of some collection of connected components of B, that is, the Grothendieck construction of an indicator function $B \to \{\varnothing, 1\} \subset \boldsymbol{Grpd}$. Note that in general neither $\ulcorner b \urcorner : 1 \to B$ nor the diagonal map $B \to B \times B$ are homotopy monic.

A.2.5. Finite maps. A map is *homotopy finite* if each homotopy fibre is homotopy finite. A pullback of any homotopy monic or finite map is again homotopy monic or finite.

A.2.6. Families. The homotopy sum of an I-indexed family in $\boldsymbol{Grpd}^B$ is defined as the homotopy sum of the corresponding object of $\boldsymbol{Grpd}^{I\times B}$, composed with the projection,

$$E \longrightarrow I \times B \longrightarrow B.$$

Homotopy sums of I-indexed families in $\boldsymbol{Grpd}_{/B}$ are defined similarly. We regard the maps $\ulcorner b \urcorner : 1 \to B$, for $[b] \in \pi_0 B$ as a *basis* of $\boldsymbol{Grpd}_{/B}$, in analogy with vector spaces. *Scalar multiples* $A\ulcorner b \urcorner$ of basis elements in $\boldsymbol{Grpd}_{/B}$ are given by $A \to 1 \xrightarrow{\ulcorner b \urcorner} B$.

Lemma. A.2.7. *Any $f : E \to B$ in $\boldsymbol{Grpd}_{/B}$ may be expressed as a linear combination of basis elements as follows*

$$f \;\simeq\; \int^{e\in E} \ulcorner f(e) \urcorner \;\simeq\; \int^{b\in B} E_b \ulcorner b \urcorner.$$

A.3. **Linear functors**

A.3.1. Basic slice adjunction. Taking homotopy pullback along a morphism of groupoids $f : B' \to B$ defines a functor between the slice categories

$$f^* : \boldsymbol{Grpd}_{/B} \to \boldsymbol{Grpd}_{/B'}.$$

This has a homotopy left adjoint, given by postcomposition,

$$f_! : \boldsymbol{Grpd}_{/B'} \to \boldsymbol{Grpd}_{/B}.$$

The homotopy adjointness is expressed by natural equivalences of mapping groupoids

$$\mathrm{Map}_{/B}(f_! E', E) \simeq \mathrm{Map}_{/B'}(E', f^* E). \tag{18}$$

Moreover,

Lemma. A.3.2 (Beck–Chevalley)**.** *For any homotopy pullback square* (16)*, the functors*

$$q_! \, p^*, \; g^* f_! : \boldsymbol{Grpd}_{/G} \to \boldsymbol{Grpd}_{/E}$$

are naturally homotopy equivalent.

A.3.3. Spans and linear functors. A pair of groupoid maps $A \xleftarrow{r} G \xrightarrow{f} B$ is termed a *span* between A and B, and induces a functor between the slice categories by pullback and postcomposition

$$f_! \, r^* : \boldsymbol{Grpd}_{/A} \longrightarrow \boldsymbol{Grpd}_{/B}.$$

A functor $\boldsymbol{Grpd}_{/A} \longrightarrow \boldsymbol{Grpd}_{/B}$ is *linear* if it is homotopy equivalent to one arising from a span in this way. By the Beck–Chevalley Lemma A.3.2, composites of linear functors are linear,

$$\begin{tikzcd}
& & H \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\vee" near start] & & \\
& G \arrow[dl, "r"'] \arrow[dr, "f"] & & E \arrow[dl, "g"'] \arrow[dr, "s"] & \\
A & & B & & C
\end{tikzcd}$$

$$(sq)_!(rp)^* : \boldsymbol{Grpd}_{/A} \xrightarrow{f_! r^*} \boldsymbol{Grpd}_{/B} \xrightarrow{s_! g^*} \boldsymbol{Grpd}_{/C} \, .$$

We write $\boldsymbol{LIN}$ for the monoidal 2-category of all slice categories $\boldsymbol{Grpd}_{/B}$ and linear functors between them, with the tensor product induced from the cartesian product

$$\boldsymbol{Grpd}_{/A} \otimes \boldsymbol{Grpd}_{/B} \; := \; \boldsymbol{Grpd}_{/A \times B}.$$

The neutral object is $\boldsymbol{Grpd} \simeq \boldsymbol{Grpd}_{/1}$, playing the role of the ground field.

A.3.4. Duality. The functor category $\boldsymbol{Grpd}^S$ is the *linear dual* of the slice category $\boldsymbol{Grpd}_{/S}$, since there is an equivalence (see [28, §2.11])

$$\boldsymbol{Grpd}^S \;\; \simeq \;\; \boldsymbol{LIN}(\boldsymbol{Grpd}_{/S}, \boldsymbol{Grpd}).$$

A span $A \leftarrow G \to B$ defines both a linear functor $\boldsymbol{Grpd}_{/A} \to \boldsymbol{Grpd}_{/B}$ and the dual linear functor $\boldsymbol{Grpd}^B \to \boldsymbol{Grpd}^A$. In particular the span $1 \leftarrow G \to S$ may be thought of as an element of $\boldsymbol{Grpd}_{/S}$, and its transpose $S \leftarrow G \to 1$ as an element of $\boldsymbol{Grpd}^S$.

There is a canonical pairing

$$\boldsymbol{Grpd}_{/S} \times \boldsymbol{Grpd}^S \to \boldsymbol{Grpd}$$

$$\langle \, \ulcorner t \urcorner \, , \; h^s \, \rangle = \mathrm{Hom}(s,t) = \begin{cases} \Omega_s(S) & (s \cong t) \\ \varnothing & (s \ncong t) \end{cases} \tag{19}$$

The maps $\ulcorner t \urcorner : 1 \to S$ (or the spans $1 \leftarrow 1 \to S$) form the canonical basis of the slice category, and the representable functors $h^s = \mathrm{Hom}(s,-) : S \to \boldsymbol{Grpd}$ (or the spans $S \leftarrow 1 \to 1$) form the canonical basis for the dual.

A.4. **Cardinality**

A.4.1. Cardinality of groupoids. Recall that the cardinality of a finite groupoid G is given by

$$|X| \ := \sum_{[x]\in\pi_0(X)} \frac{1}{|\pi_1(X,x)|} \ \in \ \mathbb{Q}.$$

where the norm signs on the right refer to the order of the group. Homotopy equivalent groupoids have the same cardinality. For any component of a locally finite groupoid B we have

$$\left|B_{[b]}\right| \ = \ |\pi_1(B,b)|^{-1} \ = \ |\Omega_b(B)|^{-1}. \tag{20}$$

For any function $q : \pi_0 B \to \mathbb{Q}$, we use the notation

$$\int^{b\in B} q(b) \ := \sum_{[b]\in\pi_0 B} |B_{[b]}|\, q(x) \ = \sum_{[b]\in\pi_0 B} \frac{q(b)}{|\pi_1(B,b)|}.$$

This is chosen to resemble the Grothendieck construction notation since for any map $E \to B$ from a finite groupoid we have, by [28, Lemma 3.5],

$$|E| = \int^{b\in B} |E_b|.$$

The case of the map $\ulcorner b \urcorner : 1 \to B$ is just equation (20).

A.4.2. Global cardinality. A span $A \xleftarrow{r} G \xrightarrow{f} B$, and the corresponding linear functor $\boldsymbol{Grpd}_{/A} \to \boldsymbol{Grpd}_{/B}$, are termed *finite* if the map r is finite. We have [28, Proposition 4.3],

Proposition A.4.3. *Let A, B, G be locally finite groupoids and $A \leftarrow G \to B$ a finite span. Then the induced finite linear functor $\boldsymbol{Grpd}_{/A} \to \boldsymbol{Grpd}_{/B}$ restricts to*

$$\boldsymbol{grpd}_{/A} \to \boldsymbol{grpd}_{/B}. \tag{21}$$

To a slice category $\boldsymbol{grpd}_{/A}$, with A locally finite, we associate the vector space $\mathbb{Q}_{\pi_0 A}$ with canonical basis $\{\delta_a\}_{[a]\in\pi_0 A}$. To the finite linear functor (21), we associate the linear map

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 A} & \longrightarrow & \mathbb{Q}_{\pi_0 B} \\ \delta_a & \mapsto & \displaystyle\sum_{[b]\in\pi_0 B} \left|B_{[b]}\right| |G_{a,b}|\, \delta_b \ = \ \int^{b\in B} |G_{a,b}|\, \delta_b \end{array} \tag{22}$$

where $G_{a,b}$ are the fibres of the map $G \to A \times B$ defined by the span. This process is functorial [28, Proposition 8.2], and defines what we call *meta* or *global cardinality*

$$\|\ \ \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$$

from the category $\underrightarrow{\boldsymbol{lin}}$ of slice categories $\boldsymbol{grpd}_{/A}$ (A locally finite) and finite linear functors.

A.4.4. Local cardinality. To each object $p : G \to B$ in $\boldsymbol{grpd}_{/B}$ (B locally finite) we can associate a vector $|p : G \to B|$ in $\mathbb{Q}_{\pi_0 B}$, called the *relative* or *local cardinality*,

$$|p| \;:=\; \sum_{[b]\in\pi_0 B} \big|B_{[b]}\big|\, |G_b|\, \delta_b \;=\; \int^{b\in B} |G_b|\, \delta_b$$

Note that p determines a finite linear functor via $1 \leftarrow G \to B$, and the local cardinality $|p|$ is just the image of 1 under the global cardinality $\mathbb{Q} \to \mathbb{Q}_{\pi_0 B}$. It follows that local cardinality respects the action of finite linear functors L,

$$|L(p)| \;=\; \|L\|(|p|).$$

The local cardinality of the basis object $\ulcorner b \urcorner : 1 \to B$ in $\boldsymbol{grpd}_{/B}$ is just the basis vector δ_b in $\mathbb{Q}_{\pi_0 B}$, by (20).

To simplify notation we will write $|L|$ for $\|L\|$ when the meaning is clear from the context, and say just cardinality rather than meta, global, relative or local cardinality.

A.4.5. Cardinality of the dual. Dually we define cardinality of finite-groupoid valued functors (see A.3.4) as a map

$$|\;\;| : \boldsymbol{grpd}^S \to \|\boldsymbol{grpd}^S\| = \mathbb{Q}^{\pi_0 S}$$

where $\mathbb{Q}^{\pi_0 S}$ is the function space, the profinite dimensional vector space spanned by the characteristic functions δ^s.

Finite spans $A \leftarrow G \to B$ define linear maps $\boldsymbol{grpd}^B \to \boldsymbol{grpd}^A$, whose cardinality is defined using the same matrix as in (22) above:

$$\begin{array}{rcl} \mathbb{Q}^{\pi_0 B} & \longrightarrow & \mathbb{Q}^{\pi_0 A} \\ \delta^b & \mapsto & \displaystyle\sum_{[a]\in\pi_0 A} \big|B_{[b]}\big|\, |G_{a,b}|\, \delta^a \end{array} \tag{23}$$

An element $g \in \boldsymbol{grpd}^S$ is represented by a finite span $S \leftarrow G \to 1$ (using the fundamental equivalence) and has cardinality

$$|g| \;=\; \|(S \leftarrow G \to 1)\| \left(\delta^1\right) \;=\; \sum_{[s]\in\pi_0 S} |g(s)|\, \delta^s. \tag{24}$$

The cardinality of the representable functor h^s in $\boldsymbol{grpd}^S$ is thus

$$|h^s| \;=\; \|(S \leftarrow 1 \to 1)\| \left(\delta^1\right) \;=\; |\Omega_s(S)|\, \delta^s \tag{25}$$

and the 'objective pairing' (19) is consistent with the classical pairing

$$\langle |\ulcorner t \urcorner| \,, |h^s| \rangle \;=\; \langle \delta_t, |\Omega_s(S)|\, \delta^s \rangle \;=\; \langle \delta_t, \delta^s \rangle\, |\Omega_s(S)| \;=\; |\langle \ulcorner t \urcorner, h^s \rangle|\,.$$

Appendix B. Simplicial groupoids and fat nerves

In this appendix, we provide some background material on simplicial groupoids and fat nerves. The general notion of simplicial set (originally termed a *complete semi-simplicial complex*) has been widely used in homotopy theory since the work of Eilenberg, Kan and others in the 1950s, owing its utility on one hand to the fact that simplicial sets are a model for topological spaces up to homotopy by way of the singular functor, and on the other hand because it receives a fully faithful functor from the category of small categories, namely the nerve (see B.1.7 below). The theory of ∞-categories, the common generalisation of spaces up to homotopy and categories, exploits the simplicial setting in a crucial way.

Any poset can naturally be regarded as a category, hence we may talk about posets in terms of their nerves. In combinatorics, however, it is common to view posets as simplicial *complexes* instead of simplicial sets, associating to a poset its order complex. The simplicial complexes that arise in this way have a canonical order on each simplex, and due to this they can be regarded as special kinds of simplicial sets, characterised by the property that n-simplices are completely determined by their vertex sets. Although such simplicial sets are of a simple kind, the subcategory they form is not as nice as the category of simplicial sets (which is a presheaf topos). For the purposes of the present undertakings, it is crucial to work with simplicial sets.

In this short appendix we recall the basic definitions, contrasting with simplicial complexes.

B.1. **Simplicial sets and nerves**

B.1.1. The simplex category (the topologist's Delta). Let $\mathbb{\Delta}$ be the *simplex category*, whose objects are the finite nonempty standard ordinals

$$[n] = \{0 < 1 < \cdots < n\},$$

and whose arrows are the order-preserving maps between them. These maps are generated by the injections $\partial^i : [n-1] \to [n]$ that skip the value i, termed *coface* maps, and the surjections $\sigma^i : [n{+}1] \to [n]$ that repeat i, termed *codegeneracy* maps. The obvious relations between these generators are called the *cosimplicial identities* (dual to the simplicial identities below).

B.1.2. Simplicial sets. A *simplicial set* is a functor $X : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Set}$. One writes X_n for the image of $[n]$, and d_i, s_i for the images of ∂^i, σ^i. The elements of X_n are called *n-simplices*.

Explicitly, a simplicial set X is thus a sequence of sets X_n $(n \geq 0)$ together with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracy maps* $s_i : X_n \to X_{n+1}$, $(0 \leq i \leq n)$,

$$\begin{tikzcd}
X_0 & X_1 \arrow[l, shift right=2, "d_1"'] \arrow[l, shift left=2, "d_0"] & X_2 \arrow[l, shift right=4, "d_2"'] \arrow[l, "d_1"] \arrow[l, shift left=4, "d_0"] & X_3 \arrow[l, shift right=6, "d_3"'] \arrow[l, shift right=2, "d_2"] \arrow[l, shift left=2, "d_1"] \arrow[l, shift left=6, "d_0"] & \cdots
\arrow[from=1-1, to=1-2, "s_0"]
\arrow[from=1-2, to=1-3, shift right=2, "s_1"]
\arrow[from=1-2, to=1-3, shift left=2, "s_0"]
\arrow[from=1-3, to=1-4, shift right=4, "s_2"]
\arrow[from=1-3, to=1-4, "s_1"]
\arrow[from=1-3, to=1-4, shift left=4, "s_0"]
\end{tikzcd}$$

subject to the *simplicial identities*: $d_i s_i = d_{i+1} s_i = 1$ and

$$d_i d_j = d_{j-1} d_i, \quad d_{j+1} s_i = s_i d_j, \quad d_i s_j = s_{j-1} d_i, \quad s_j s_i = s_i s_{j-1}, \qquad (i < j).$$

B.1.3. Simplicial maps. A *simplicial map* $F : X \to Y$ between simplicial sets is given by a sequence of maps $(F_n : X_n \to Y_n)_{n \geq 0}$ commuting with face and degeneracy maps, that is, a natural transformation between the functors X and Y. A simplicial map is *cartesian* with respect to an order-preserving map $[m] \to [n]$ in $\mathbb{\Delta}$ if the associated naturality square is a pullback of groupoids (see 1.5.1 for examples).

B.1.4. Simplicial complexes. A *simplicial complex* K consists of a set V of vertices together with a collection $S(K)$ of nonempty subsets of V, termed the *simplices* of K, satisfying

- All the one-element subsets of V are simplices of K.
- Any non-empty subset of a simplex of K is a simplex of K.

A map between simplicial complexes is a function between their vertex sets such that the image of each simplex is a simplex.

B.1.5. Locally ordered simplicial complexes. Certain simplicial complexes can be regarded as simplicial sets, but some ordering is necessary so as to have well-defined face maps. We call a simplicial complex *locally ordered* if there is specified a linear order on each simplex, in such a way that all inclusions preserve these orders. (The terminology *hierarchical simplicial complex* is used by Ehrenborg [22].) A map of locally ordered simplicial complexes is a map of simplicial complexes whose restriction to each simplex is order preserving. This defines a category $\boldsymbol{LOSC}$.

To each locally ordered simplicial complex K there is associated canonically a simplicial set X whose n-simplices are sequences $(v_0, v_1, \ldots, v_n)$ of elements in V, permitting repetitions, which as a set is required to form a simplex $F \in S(K)$, and which as a sequence is required to be non-decreasing for the linear order in the simplex F. This can be described more formally as follows. Each linear order $[n] \in \mathbbe{\Delta}$ can be regarded as a locally ordered simplicial complex, defining in fact a functor $\mathbbe{\Delta} \to \boldsymbol{LOSC}$. Now the simplicial set X assigned to K simply has

$$X_n := \operatorname{Hom}_{\boldsymbol{LOSC}}([n], K).$$

This automatically accounts for the face and degeneracy maps, simply induced by precomposition with the coface maps and codegeneracy maps $[m] \to [n]$ in $\mathbbe{\Delta}$.

This assignment defines a fully faithful functor from locally ordered simplicial complexes to simplicial sets. (Note that allowing repetition in the sequences is necessary for the assignment to be functorial in maps of locally ordered simplicial complexes, because these are allowed to send a simplex to a simplex of lower dimension.)

B.1.6. The order complex and the nerve of a poset. The *order complex* of a poset P is the simplicial complex whose vertices are the elements of P and whose n-simplices are those subsets that form n-chains $v_0 < \cdots < v_n$ in the poset. The order complex is naturally locally ordered since each simplex is a total order, and its associated simplicial set is usually termed the *nerve* of the poset. The definition of the nerve extends to more general categories as follows.

B.1.7. Strict nerve. The *nerve* of a category $\mathscr{C}$ is the simplicial set

$$N\mathscr{C} : \mathbbe{\Delta}^{\mathrm{op}} \to \boldsymbol{Set}$$

whose set of n-simplices is the set of sequences of n composable arrows in $\mathscr{C}$ (allowing identity arrows). The face maps are given by composing arrows (for the inner face maps) and by discarding arrows at the beginning or the end of the sequence (outer face maps). The degeneracy maps are given by inserting an identity map in the sequence. By regarding the total order $[n]$ as a category, we see that a sequence of n

composable arrows in $\mathscr{C}$ is the same thing as a functor $[n] \to \mathscr{C}$, and more formally the n-simplices can be described as

$$(N\mathscr{C})_n = \mathrm{Fun}([n], \mathscr{C}),$$

and in particular we see that the face and degeneracy maps of $N\mathscr{C}$ are given simply by precomposition with the coface and codegeneracy maps in $\mathbb{\Delta}$.

B.2. **Simplicial groupoids, fat nerves, and Segal spaces**

B.2.1. Simplicial groupoids. For any category $\mathscr{E}$, one can talk about simplicial objects $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathscr{E}$. Thus, in the case of the category of groupoids, a *simplicial groupoid* is a sequence of groupoids X_n, $n \geq 0$, and face and degeneracy maps $d_i : X_n \to X_{n-1}$, $s_i : X_n \to X_{n+1}$, $(0 \leq i \leq n)$, subject to the simplicial identities above.

B.2.2. Fat nerve of a small category. Important examples of simplicial groupoids are given by the *fat nerve* of a small category $\mathscr{C}$. Here X_n is the groupoid of all composable sequences $a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} a_n$ of n arrows in $\mathscr{C}$, that is,

$$X_n = \{\text{functors } \alpha : [n] \to \mathscr{C}\}.$$

In the case of the classical *strict nerve* this is just a set, or a discrete groupoid; in the *fat nerve*, X_n includes all natural isomorphisms $\alpha \to \alpha'$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\sim"] & \cdot \arrow[r] \arrow[d, "\sim"] & \cdot \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & \cdot \arrow[d, "\sim"] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot
\end{tikzcd}$$

This can be described succinctly in categorical terms, in terms of the functor category, but allowing only invertible natural transformations:

$$(\mathbf{N}\mathscr{C})_n = \mathrm{Fun}([n], \mathscr{C})^{\mathrm{iso}}.$$

As in the previous cases (B.1.5, B.1.7), this automatically accounts for face and degeneracy maps by precomposition. In particular, $d_0 : X_1 \to X_0$ assigns to an arrow its codomain, and $d_1 : X_1 \to X_0$ assigns to an arrow its domain.

Since X_2 is by definition the groupoid of composable pairs of arrows, we have $X_2 \simeq X_1 \times_{X_0} X_1$. Here the fibre product is

$$\begin{tikzcd}
X_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "d_1"] \\
X_1 \arrow[r, "d_0"'] & X_0
\end{tikzcd} \tag{26}$$

expressing the composability condition: only those pairs of arrows such that the target of the first matches the source of the second.

In particular, $d_1 : X_2 \to X_1$ is the composition map. Also, $d_0 : X_2 \to X_1$ assigns to a composable pair the second arrow, and $d_2 : X_2 \to X_1$ assigns to a composable pair the first arrow. (Here we are referring to the order of composition, as in a-followed-by-b, and not the order used when writing this as $b \circ a$.)

Note that when $\mathscr{C}$ is just a poset, then it has no invertible arrows. Therefore, the notions of strict and fat nerve coincide.

B.2.3. Rezk complete Segal spaces. A simplicial groupoid is a *Segal space* if $X_2 \simeq X_1 \times_{X_0} X_1$, as in (26), and in general the canonical Segal map

$$X_n \longrightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an equivalence for each $n \geq 1$.

Consider the contractible groupoid generated by one isomorphism $0 \xrightarrow{\sim} 1$, and its strict nerve J. A Segal space X is *Rezk complete* if the map $J \to *$ induces an equivalence of groupoids $\mathrm{Map}(*, X) \to \mathrm{Map}(J, X)$, which in terms means that $s_0 : X_0 \to X_1$ is fully faithful and has as its essential image the arrows that admit left and right quasi-inverses. More intuitively, the condition expresses the idea that up to homotopy there are no other weakly invertible arrows than the identities.

The Rezk complete Segal spaces are precisely those simplicial groupoids that are levelwise-equivalent to the fat nerve of a category.

B.2.4. Monoidal groupoids. A monoidal groupoid is a monoidal category $(\mathscr{C}, \otimes, I)$ which happens to be a groupoid. For these, one can define the *monoidal nerve*, which is essentially a simplicial groupoid $X : \mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$. One takes X_0 to be a singleton, takes X_1 to be the groupoid itself, and more generally let X_n be the n-fold cartesian product

$$X_n = \underbrace{\mathscr{C} \times \cdots \times \mathscr{C}}_{n \text{ factors}}.$$

The outer face maps just project away an outer factor. The inner face maps use the monoidal structure $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ on two adjacent factors. The degeneracy maps insert a unit object. All this is completely canonical, given the monoidal structure. The only problem is that the simplicial identities do not hold on the nose, due to the fact that the monoidal structure is not assumed to be strict. The diagram is therefore not literally speaking a simplicial groupoid, but only a pseudo-functor $\mathbb{\Delta}^{\mathrm{op}} \to \boldsymbol{Grpd}$.

While this may be a slight annoyance sometimes, it is not actually important for the purpose of this work: for the sake of defining a homotopy-coherently coassociative coalgebra structure on $\boldsymbol{Grpd}_{/X_1}$, a pseudo-functor is just as good as a strict functor. Another thing is that one can alternatively invoke strictification theorems (see Mac Lane [53, §XI.3, Theorem 1]): any monoidal category is equivalent to a strict monoidal category. The monoidal nerve of the strictification of a monoidal groupoid is then a simplicial groupoid on the nose, equivalent to the original monoidal nerve.

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