

# THREE HOPF ALGEBRAS AND THEIR COMMON SIMPLICIAL AND CATEGORICAL BACKGROUND

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ABSTRACT. We consider three *a priori* totally different setups for Hopf algebras from number theory, mathematical physics and algebraic topology. These are the Hopf algebras of Goncharov for multiple zeta values, that of Connes–Kreimer for renormalization, and a Hopf algebra constructed by Baues to study double loop spaces. We show that these examples can be successively unified by considering simplicial objects, cooperads with multiplication and Feynman categories at the ultimate level. These considerations open the door to new constructions and reinterpretation of known constructions in a large common framework.

## INTRODUCTION

Hopf algebras have long been known to be a highly effective tool in classifying and methodologically understanding complicated structures. In this vein, we start by recalling three Hopf algebra constructions, two of which are rather famous and lie at the center of their respective fields. These are Goncharov’s Hopf algebra of multiple zeta values [Gon05] whose variants lie at the heart of the recent work [Bro15a], for example, and the ubiquitous Connes–Kreimer Hopf algebra of rooted forests [CK98]. The third Hopf algebra predates them but is not as prevalent: it is a Hopf algebra discovered and exploited by Baues [Bau81] to model double loop spaces. We will trace the existence of the first and third of these algebras back to a fact known to experts<sup>1</sup>, namely that simplices form an operad. It is via this simplicial bridge that we can push the understanding of the Hopf algebra of Goncharov to a deeper level and relate it to Baues’ construction which comes from an *a priori* totally different setup. Here, we prove a general theorem, that any simplicial object gives rise to bialgebra.

The Hopf algebra of Connes and Kreimer fits into this picture through a map given by contracting all the internal edges of the trees. This map also furnishes an example *par excellence* of the complications that arise in this story. A simpler example is given by restricting to the sub-Hopf algebra of three-regular trees. In this case the contraction map exhibits the corresponding Hopf algebra as a pull-back of a simplicial object. This relationship is implicit in [Gon05] and is now put into a more general framework.

We show that the essential ingredient to obtain a Hopf structure in all three examples is our notion of cooperad with multiplication. For the experts, we wish to point out that due to different gradings (in the operad degree) this is neither what is known as a Hopf operad nor its dual. We prove a general theorem that states that a cooperad with multiplication

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<sup>1</sup>As one expert put it: “Yes this is well-known, but not to many people”.

always yields a bialgebra. In the general setting these bialgebras are neither unital nor counital. While there is no problem adjoining a unit, the counit is a subtle issue in general and we discuss the conditions for their existence in detail. In the special cases at hand, they do exist however. This is due to the fact that they are free constructions of a cooperad with multiplication from a cooperad with a cooperadic unit. Examples of the latter are furnished for instance by the dual of (partial) unital operads. An upshot of the more general case is that there is a natural ‘depth’ filtration, and we prove that there is always a surjection from a free construction to the associated graded. In particular we prove the following structural theorem, if the bialgebra has a left coalgebra counit, then it is a deformation of its associated graded and moreover this associated graded is a quotient of the free construction of its first graded piece.

Another nice generalization comes about by noticing that just as there are operads and pseudo-operads, there are cooperads and pseudo-cooperads. We show that these dual structures lead to bialgebras and a version of infinitesimal bialgebras. The operations corresponding to the dual of the partial compositions of pseudo-operads are then dual to the infinitesimal action of Brown. In other words they give the Lie-coalgebra structure dual to the pre-Lie structure.

Moving from the constructed bialgebras to Hopf algebras is possible under the extra condition of almost connectedness. If the cooperad satisfies this condition, which technically encompasses the existence of a split bialgebraic counit, then there is a natural quotient of the bialgebra which is connected and hence Hopf. Indeed in the three examples, this quotient is taken, by prescribing values to degenerate expressions.

A further level of complexity is reflected in the fact that there are several variations of the construction of the Connes–Kreimer Hopf algebra based for example on planar labelled trees, labelled trees, unlabelled trees and trees whose external legs have been “amputated” —a term common in physics and the subject. We show, in general, these correspond to non-Sigma cooperads, coinvariants of symmetric cooperads and certain colimits, which are possible in semisimplicial cooperads.

An additional degree of understanding is provided by the insight that the underlying cooperads for the Hopf algebra of Goncharov and Baues are given by a cosimplicial structure. This also allows us to understand the origin of the shuffle product and other relations commonly imposed in theory of multiple zeta values and motives from this angle. For the shuffle product, in the end it is as Broadhurst remarked, the product comes from the fact that we want to multiply the integrals, which are the amplitudes of connected components of disconnected graphs. In simplicial terms this translates to the compatibility of different naturally occurring free monoid constructions, in the form of the Alexander–Whitney map and a multiplication base on the relative cup product. There are more surprising direct correspondences between the extra relations, like the contractibility of a 2-skeleton used by Baues and a relation on multiple zeta values essential for the motivic coaction.

These digressions into mathematical physics bring us to the ultimate level of abstraction and source of Hopf algebras of this type: the Feynman categories of [KW13]. We show that under reasonable assumptions a Feynman category gives rise to a Hopf algebra formed by the free Abelian group of its morphisms. Here the coproduct, motivated by a discussion

with D. Kreimer, is deconcatenation. With hindsight, this type of coproduct goes back at least as far as [JR79] or [Ler75], who considered a deconcatenation coproduct from a combinatorial point of view. Feynman categories are monoidal, and this monoidal structure yields a product. Although it is not true in general for any monoidal category that the multiplication and comultiplication are compatible and form a bialgebra, it is for Feynman categories, and hence also for their opposites. This also gives a new understanding for the axioms of a Feynman category. The case relevant for cooperads with multiplication is the Feynman category of finite sets and surjections and its enrichments by operads. The constructions of the bialgebra then correspond to the pointed free case considered above if the cooperad is the dual of an operad. Invoking opposite categories, one can treat cooperads directly. For this one notices that the opposite Feynman category, that for coalgebras, can be enriched by cooperads. It is here that we can also say that the two constructions of Baues and Goncharov are related by Joyal duality to the operad of surjections.

The construction is more general in the sense that there are other Feynman categories. One of the most interesting examples going deeper into mathematical physics is the Feynman category whose morphisms are graphs. This allows us to obtain the graph Hopf algebras of Connes and Kreimer. Going further, there are also the Hopf algebras corresponding to cyclic operads, modular operad, and new examples based on 1-PI graphs and motivic graphs, which yield the new Hopf algebras of Brown [Bro15a]. Here several general constructions on Feynman categories, such as enrichment, decoration, universal operations, and free construction come into play and give interrelations between the examples.

There are quotients that are obtained by “dividing out isomorphisms”, which amounts to dividing out by certain coideals. This again allows us to distinguish the levels between planar, symmetric, labelled and unlabelled versions. To actually get the Hopf algebras, rather than just bialgebras, one again has to take quotients and require certain connectedness assumptions. Here the conditions become very transparent. Namely, the unit, hidden in the three examples by normalizations, will be given by the unit endomorphism of the monoidal unit  $\mathbb{1}$  of the Feynman category, viz.  $id_{\mathbb{1}}$ . Isomorphisms keep the coalgebra from being conilpotent. Even if there are no isomorphisms, still all identities are group-like and hence the coalgebra is not connected. This explains the necessity of taking quotients of the bialgebra to obtain a Hopf algebra. We give the technical details of the two quotients, first removing isomorphisms and then identifying all identity maps.

The paper is organized as follows. We begin by recalling the three Hopf algebras and their variations in §1. We give all the necessary details and add a discussion after each example indicating its position within the whole theory. In §2, we give the main definition of a cooperad with multiplication and the constructions of bialgebras and Hopf algebras. To be self-contained, we write out the relevant definitions at work in the background at each step. This paragraph also contains a discussion of the filtered and graded cases. This setup is strictly more general than the three examples, which are all of a free type that we define. Given that the origin of the cooperad structure for Goncharov’s and Baues’ Hopf algebras is simplicial, we develop the general theory for the simplicial setting in §3. It is §4 that contains the generalization to Feynman categories. Here we realize the examples in the more general setting and give several pertinent constructions. Having the whole

theory at hand, we give a detailed discussion in §5. To be self-contained the paper also has three appendices. One on graphs, one on coalgebras and Hopf algebras and one on Joyal duality. The latter is of independent interest, since this duality explains the ubiquitous occurrence of two types of formulas, those with repetition and those without repetition, in the contexts of number theory, mathematical physics and algebraic topology. This also explains the two graphical versions used in this type of calculations, polygons vs. trees, which are now just Joyal duals of each other.

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1. PREFACE: THREE HOPF ALGEBRAS

In this section, we will review the construction of the main Hopf algebras which we wish to put under one roof and generalize. After each example we will give a discussion paying special attention to their unique features.

**1.1. Multiple zeta values.** We briefly recall the setup of Goncharov's Hopf algebra of multiple zeta values. Given  $r$  natural numbers  $n_1, \dots, n_{r-1} \geq 1$  and  $n_r \geq 2$ , one considers the real numbers

$$\zeta(n_1, \dots, n_r) := \sum_{1 \leq k_1 \leq \dots \leq k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \tag{1.1}$$

The value  $\zeta(2) = \pi^2/6$ , for example, was calculated by Euler.

Kontsevich remarked that there is an integral representation for these, given as follows. If  $\omega_0 := \frac{dz}{z}$  and  $\omega_1 := \frac{dz}{1-z}$  then

$$\zeta(n_1, \dots, n_r) = \int_0^1 \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_1-1} \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_2-1} \dots \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_r-1} \tag{1.2}$$

Here the integral is an iterated integral and the result is a real number. The *weight* of (1.2) is  $N = \sum_1^r n_i$  and its *depth* is  $r$ .

**Example 1.1.** As was already known by Leibniz,

$$\zeta(2) = \int_0^1 \omega_1 \omega_0 = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$$

One of the main interests is the independence over  $\mathbb{Q}$  of these numbers: some relations between the values come directly from their representation as iterated integrals, see e.g. [Bro12b] for a nice summary. As we will show in Chapter 3 many of these formulas can be understood from the fact that simplices form an operad and hence simplicial objects form a cooperad.

**1.1.1. Formal symbols.** Following Goncharov, one turns the iterated integrals into formal symbols  $\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)$  where the  $a_i \in \{0, 1\}$ . That is, if  $w$  is an arbitrary word in  $\{0, 1\}$  then  $\hat{I}(0; w; 1)$  represents the iterated integral from 0 to 1 over the product of forms according to  $w$ , so that

$$\hat{I}(0; 1, \underbrace{0, \dots, 0}_{n_1-1}, 1, \underbrace{0, \dots, 0}_{n_2-1}, \dots, 1, \underbrace{0 \dots 0}_{n_r-1}; 1)$$

is the formal counterpart of (1.2). The weight is now the length of the word and the depth is the number of 1s. Note that the integrals (1.2) converge only for  $n_r \geq 2$ , but may be extended to arbitrary words using a regularization described e.g. in [Bro12b, Lemma 2.2].

1.1.2. **Goncharov's first Hopf algebra.** Taking a more abstract viewpoint, let  $\mathcal{H}_G$  be the polynomial algebra on the formal symbols  $\hat{I}(a; w; b)$  for elements  $a, b$  and any nonempty word  $w$  in the set  $\{0, 1\}$ , and let

$$\hat{I}(a; \emptyset; b) = \hat{I}(a; b) = 1 \quad (1.3)$$

On  $\mathcal{H}_G$  define a comultiplication  $\Delta$  whose value on a polynomial generator is

$$\begin{aligned} \Delta(\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)) = \\ \sum_{\substack{k \geq 0 \\ 0=i_0 < i_1 < \dots < i_k = n}} \hat{I}(a_{i_0}; a_{i_1}, \dots, a_{i_k}) \otimes \hat{I}(a_{i_0}; a_{i_0+1}, \dots, a_{i_1}) \hat{I}(a_{i_1}; a_{i_1+1}, \dots, a_{i_2}) \cdots \hat{I}(a_{i_{k-1}}; a_{i_{k-1}+1}, \dots, a_{i_k}) \end{aligned} \quad (1.4)$$

**Theorem 1.2.** [Gon05] *If we assign  $\hat{I}(a_0; a_1, \dots, a_m; a_{m+1})$  degree  $m$  then  $\mathcal{H}_G$  with the coproduct (1.4) (and the unique antipode) is a connected graded Hopf algebra.*

**Remark 1.3.** The fact that it is unital and connected follows from (1.3).

**Remark 1.4.** The letters  $\{0, 1\}$  are actually only pertinent insofar as to get multiple zeta values at the end; the algebraic constructions work with any finite set of letters  $S$ . For instance, if  $S$  are complex numbers, one obtains polylogarithms.

1.1.3. **Goncharov's second Hopf algebra and the version of Brown.** There are several other conditions one can impose, which are natural from the point of view of iterated integrals or multiple zeta values, by taking quotients. They are

(1) The shuffle formula

$$\hat{I}(a; a_1, \dots, a_m; b) \hat{I}(a; a_{m+1}, \dots, a_{m+n}; b) = \sum_{\sigma \in \text{III}_{m,n}} \hat{I}(a; a_{\sigma(1)}, \dots, a_{\sigma(m+n)}; b) \quad (1.5)$$

where  $\text{III}_{m,n}$  is the group of  $m, n$  shuffles.

(2) The path composition formula.

$$\forall x \in \{0, 1\} : \hat{I}(a_0; a_1, \dots, a_m; a_{m+1}) = \sum_{k=1}^m \hat{I}(a_0; a_1, \dots, a_k; x) \hat{I}(x; a_{k+1}, \dots, a_m; a_{m+1}) \quad (1.6)$$

(3) The triviality of loops

$$\hat{I}(a; w; a) = 0 \quad (1.7)$$

(4) The inversion formula

$$\hat{I}(a_0; a_1, \dots, a_n; a_{n+1}) = (-1)^n \hat{I}(a_{n+1}, a_n, \dots, a_1; a_0) \quad (1.8)$$

(5) The exchange formula

$$\hat{I}(a_0; a_1, \dots, a_n; a_{n+1}) = \hat{I}(1 - a_{n+1}; 1 - a_n, \dots, 1 - a_1; 1 - a_0) \quad (1.9)$$

here the map  $a_i \mapsto 1 - a_i$  interchanges 0 and 1.

(6) 2–skeleton equation

$$\hat{I}(a_0; a_1; a_2) = 0 \tag{1.10}$$

**Definition 1.5.**  $\tilde{\mathcal{H}}_G$  be the quotient of  $\mathcal{H}_G$  with respect to the following homogeneous relations (1),(2),(3) and (4), let  $\mathcal{H}_B$  be the quotient of  $\mathcal{H}_G$  with respect to (1), (3), (4) and let  $\tilde{\mathcal{H}}_B$  be the quotient by (1),(2),(4),(5) and (6).

Again one can generalize to a finite set  $S$ .

**Theorem 1.6.** [Gon05, Bro12a, Bro12b]  $\Delta$  and the grading descend to  $\tilde{\mathcal{H}}_G$  and using the unique antipode is a graded connected Hopf algebra. Furthermore (1), (2), (3) imply (4).  $\mathcal{H}_B$  and  $\tilde{\mathcal{H}}_B$  are graded connected Hopf algebras as well.

1.1.4. **Discussion.** In the theory of MZVs it is essential that there are two parts to the story. The first is the motivic level. This is represented by the Hopf algebras and comodules over them. The second are the actual real numbers that are obtained through the iterated integrals. The theory is then an interplay between these two worlds, where one tries to get as much information as possible from the motivic level. This also explains the appearance of the different Hopf algebras since the evaluation in terms of iterated integrals factors through these quotients. In our setting, we will be able to explain many of the conditions naturally. The first condition (1.3) turns a naturally occurring non-connected bialgebra into a connected bialgebra and hence a Hopf algebra. The existence of the bialgebra itself follows from a more general construction stemming from cooperad structure with multiplication. One example of this is given by simplicial objects and the particular coproduct (1.4) is of this simplicial type. This way, we obtain the generalization of  $\mathcal{H}_G$ . Condition (1.3) is understood in the simplicial setup in Chapter 3 as the contraction of a 1-skeleton of a simplicial object. The relation (2) is actually related to a second algebra structure, the so-called path algebra structure [Gon05], which we will discuss in the future. The relation (3) is a normalization, which is natural from iterated integrals. The condition (1) is natural within the simplicial setup, coming from the Eilenberg–Zilber and Alexander–Whitney maps and interplay between two naturally occurring monoids. That is we obtain a generalization of  $\mathcal{H}_B$  used in the work of Brown [Bro15b, Bro12a].

The Hopf algebra  $\mathcal{H}_B$  is used in [Bro12b]. The relation (5), in the simplicial case, can be understood in terms of orientations. Finally, the equation (6) corresponds to contracting the 2-skeleton of a simplicial object. It is intriguing that on one hand (6) is essential for the coaction [Bro16] while is essential in a totally different context to get a model for chains on a double loop space [Bau98], see below.

Moreover, in his proofs, Brown essentially uses operators  $D_r$  which we show to be equal to the dual of the  $\circ_i$  map used in the definition of a pseudo-cooperad, see §2.9.1. There is a particular normalization issue with respect to  $\zeta(2)$  which is handled in [Bro15b] by regarding the Hopf comodule  $\mathcal{H}_B \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta^m(2))$  of  $\mathcal{H}_B$ . The quotient by the second factor then yields the Hopf algebra above, in which the element representing  $\zeta(2)$  vanishes.

## 1.2. Connes–Kreimer.



**1.2.1. Rooted forests without tails.** We will consider graphs to be given by vertices, flags or half-edges and their incidence conditions; see Appendix A for details. There are two ways to treat graphs: either with or without tails, that is, free half-edges. In this section, we will recapitulate the original construction of Connes and Kreimer and hence use graphs without tails.

A tree is a contractible graph and a forest is a graph all of whose components are trees. A rooted tree is a tree with a marked vertex. A rooted forest is a forest with one root per tree. A rooted subtree of a rooted tree is a subtree which shares the same root. A rooted subforest of a rooted tree is clearly either a rooted subtree or the empty forest.

**1.2.2. Connes–Kreimer’s Hopf algebra of rooted forests.** We now fix that we are talking about isomorphism classes of trees. In particular, the trees in a forest will have no particular order. Let  $\mathcal{H}_{CK}$  be the free commutative algebra, that is, the polynomial algebra, on rooted trees, over a fixed ground field  $k$ . A forest is thus a monomial in trees and the empty forest is the unit  $1_k$  in  $k$ . We denote the commutative multiplication by juxtaposition and the algebra is graded by the number of vertices.

Given a rooted subtree  $\tau_0$  of a rooted tree  $\tau$ , we define  $\tau \setminus \tau_0$  to be the forest obtained by deleting all of the vertices of  $\tau_0$  and all of the edges incident to vertices of  $\tau_0$  from  $\tau$ : it is a rooted forest given by a collection of trees whose root is declared to be the unique vertex that has an edge in  $\tau$  connecting it to  $\tau_0$ . One also says that  $\tau \setminus \tau_0$  is given by an admissible cut [CK98].

Define the coproduct on rooted trees as:

$$\Delta(\tau) := \tau \otimes 1_k + 1_k \otimes \tau + \sum_{\substack{\tau_0 \text{ rooted subtree of } \tau \\ \tau_0 \neq \tau}} \tau_0 \otimes \tau \setminus \tau_0 \quad (1.11)$$

and extend it multiplicatively to forests,  $\Delta(\tau_1 \tau_2) = \tau_1^{(1)} \tau_2^{(1)} \otimes \tau_1^{(2)} \tau_2^{(2)}$  in Sweedler notation. One may include the primitive terms in the sum by considering also  $\tau_0 = \tau$  and  $\tau_0 = \emptyset$  (the empty rooted subforest of  $\tau$ ), respectively.

**Theorem 1.7.** [CK98] *The comultiplication above together with the grading define a structure of connected graded Hopf algebra.*

Note that, since the Hopf algebra is graded and connected, the antipode is unique.

**1.2.3. Other variants.** There is a planar variant, using planar planted trees. Another variant which is important for us is the one using trees with tails. This is discussed in §2 and §5 and Appendix A. There is also a variant where one uses leaf labelled trees. For this it is easier not to pass to isomorphism classes of trees and just keep the names of all the half edges during the cutting.

Finally there are algebras based on graphs rather than trees, which are possibly super-graded commutative by the number of edges. In this generality, we will need Feynman categories to explain the naturality of the constructions. Different variants of interest to physics and number theory are discussed in §5.



**1.2.4. Discussion.** This Hopf algebra, although similar, is more complicated than the example of Goncharov. This is basically due to three features which we would like to discuss. First, we are dealing with isomorphism classes, secondly, in the original version, there are no tails and lastly there is a sub-Hopf algebra of linear trees. Indeed the most natural bialgebra that will occur will be on planar forests with tails. To make this bialgebra into a connected Hopf algebra, one again has to take a quotient analogous to the normalization (1.3), implemented by the identification of the forests with no vertices (just tails) with the unit in  $k$ . To obtain the commutative, unlabelled case, one has to pass to coinvariants. Finally, if one wants to get rid of tails, one has to be able to ‘amputate’ them. This is an extra structure, which in the case of labelled trees is simply given by forgetting a tail together with its label. Taking a second colimit with respect to this forgetting construction yields the original Hopf algebra of Connes and Kreimer. The final complication is given by the Hopf subalgebra of forests of linear, i.e. trees with only binary vertices. This Hopf subalgebra is again graded and connected. In the more general setting, the connectedness will be an extra check that has to be performed. It is related to the fact that for an operad  $\mathcal{O}$ ,  $\mathcal{O}(1)$  is an algebra and dually for a cooperad  $\check{\mathcal{O}}$ ,  $\check{\mathcal{O}}(1)$  is a coalgebra, as we will explain. If  $\mathcal{O}$  or  $\check{\mathcal{O}}$  is not reduced (i.e. one dimensional generated by a unit, if we are over  $k$ ), then this extra complication may arise and in general leads to an extra connectedness condition.

**1.3. Baues.** The basic starting point for Baues [Bau81] is a simplicial set  $X$ , from which one passes to the chain complex  $C_*(X)$ . It is well known that  $C_*(X)$  is a coalgebra under the diagonal approximation chain map  $\Delta : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$ , and to this coalgebra one can apply the cobar construction:  $\Omega C_*(X)$  is the free algebra on  $\Sigma^{-1}C_*(X)$ , with a natural differential which is immaterial to the discussion at this moment.

The theorem by Adams and Eilenberg–Moore is that if  $\Omega X$  is connected then  $\Omega C_*(X)$  is a model for chains on the based loop space  $\Omega X$  of  $X$ . This raises the question of iterating the construction, but, unlike  $\Omega X$ , which can be looped again,  $\Omega C_*(X)$  is now an algebra and thus does not have an obvious cobar construction. To remedy this situation Baues introduced the following comultiplication map:

$$\Delta(x) = \sum_{\substack{k \geq 0 \\ 0=i_0 < i_1 < \dots < i_k=n}} x_{(i_0, i_1, \dots, i_k)} \otimes x_{(i_0, i_0+1, \dots, i_1)} x_{(i_1, i_1+1, \dots, i_2)} \cdots x_{(i_{k-1}, i_{k-1}+1, \dots, i_k)},$$

where  $x \in X_n$  is an  $(n-1)$ -dimensional generator of  $\Omega C_*(X)$ , and  $x_{(\alpha)}$  denotes its image under the simplicial operator specified by a monotonic sequence  $\alpha$ .

**Theorem 1.8.** [Bau81] *If  $X$  has a reduced one skeleton  $|X|^1 = *$ , then the comultiplication, together with the free multiplication and the given grading, make  $\Omega C_*(X)$  into a Hopf algebra. Furthermore if  $\Omega \Omega |X|$  is connected, i.e.  $|X|$  has trivial 2-skeleton, then  $\Omega \Omega C_*(X)$  is a chain model for  $\Omega \Omega |X|$ .*

**1.3.1. Discussion.** Historically, this is actually the first of the type of Hopf algebra we are considering. With hindsight, this is in a sense the graded and noncommutative version of Goncharov and gives the Hopf algebra of Goncharov a simplicial backdrop. There are several features, which we will point out. In our approach, the existence of the diagonal

(co-product), written by hand in [Bau81], is derived from the fact that simplices form an operad. This can then be transferred to a cooperad structure on any simplicial set. Adding in the multiplication as a free product (as is done in the cobar construction), we obtain a bialgebra with our methods. The structure can actually be pushed back into the simplicial setting, rather than just living on the chains, which then explains the appearance of the shuffle products.

To obtain a Hopf algebra, we again need to identify 1 with the generators of the one skeleton. This quotient passes through the contraction of the one skeleton, where one now only has one generator. This is equivalent to the normalization (1.3). We speculate that the choice of the *chemin droit* of Deligne can be seen as a remnant of this in further analysis. We expect that this gives an interpretation of (1.9). The condition (1.8) can be viewed as an orientation condition, which suggests to work with dihedral instead of non-Sigma operads, see e.g. [KL13]. Again this will be left for the future.

Lastly, the condition (1.10) corresponds to the triviality of the 2-skeleton needed by Baues for the application to double loop spaces. At the moment, this is just an observation, but we are sure this bears deeper meaning.

## 2. HOPF ALGEBRAS FROM COOPERADS WITH MULTIPLICATION

In this section, we give a general construction, which encompasses all the examples discussed in §1. We start by collecting together the results needed about operads, which we will later dualize to cooperads, as these are the main actors. There are many sources for further information about operads. A standard reference is [MSS02] and [Kau04] contains the essentials with figures for the relevant examples.

The construction is more general than we would need for the examples, which all correspond to a free non-connected construction on the dual of an operad, where the free construction furnishes the compatible multiplication. As such they carry additional structure, such as a double grading. These gradings reduce to filtrations in the general case. Another complication is the existence of units and counits. We can prove a structure theorem saying that if the units and counits exist, then we are dealing with a deformation of a quotient of the free connected construction on a cooperad.

### 2.1. Recollections on operads.

**2.1.1. Non- $\Sigma$  pseudo-operads.** Loosely an operad is a collection of “somethings” with  $n$  inputs and one output, like functions of several variables. And just like for functions there are permutations of variables and substitution operations. To make things concrete: consider the category  $g\mathcal{A}b$  of graded Abelian groups with the tensor product  $\otimes_{\mathbb{Z}}$ . This is a symmetric monoidal category, if one adds the so-called associativity constraints  $(G \otimes H) \otimes K \rightarrow G \otimes (H \otimes K) : (g \otimes h) \otimes k \mapsto g \otimes (h \otimes k)$  and the commutativity  $g \otimes h \mapsto (-1)^{|g||h|} h \otimes g$ , where  $|g|$  is the degree of  $g$ . A *non- $\Sigma$  pseudo-operad* in this category is given by a collection  $\mathcal{O}(n)$  of Abelian groups, together with structure maps

$$\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(k + m - 1) \text{ for } 1 \leq i \leq k \quad (2.1)$$

which are associative in the appropriate sense,

$$(- \circ_i -) \circ_j - = \begin{cases} - \circ_i (- \circ_{j-i+1} -) & \text{if } i \leq j < m + i \\ ((- \circ_j -) \circ_{i+n-1} -)\pi & \text{if } 1 \leq j < i. \end{cases}$$

Here  $\pi = (23) : \mathcal{O}(k) \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(k) \otimes \mathcal{O}(n) \otimes \mathcal{O}(m)$ .

We call  $\mathcal{O}$  connected if  $\mathcal{O}(1)$  is  $\mathbb{Z}$  or in general the unit of the monoidal category.

**2.1.2. Pseudo-operads.** If we add the condition that each  $\mathcal{O}(n)$  has an action of the symmetric group  $\mathbb{S}_n$  and that the  $\circ_i$  are equivariant with respect to the symmetric group actions in the appropriate sense, we arrive at the definition of an operad.

**Example 2.1.** As previously mentioned, the most instructive example is that of multivariate functions, given by the collection  $\{End(X)(n) = Hom(X^{\otimes n}, X)\}$ . The  $\circ_i$  act as substitutions, that is,  $f_1 \circ_i f_2$  substitutes the function  $f_2$  into the  $i$ th variable of  $f_1$ . The symmetric group action permutes the variables. The equivariance then states that it does not matter if one permutes first and then substitutes or the other way around, provided that one uses the correct permutation. If one takes  $X$  to be a set or a compact Hausdorff space  $\otimes$  stands for the Cartesian product. If  $X$  is a vector space over  $k$ , then  $\otimes$  is the tensor product over  $k$  and the functions are multilinear. The most commonly known examples are  $X = \mathbf{R}$  considered as a topological space and  $X = V$  a vector space.

**Remark 2.2.** The only thing we needed in the definitions is that the underlying category is symmetric monoidal, in particular there is a monoidal, aka. tensor, product. We obviously need monoidality to write down the structure morphisms. In the axioms, we need to consider the switching and re-bracketing of factors, i.e. the symmetric monoidal structure. The other categories we will consider are  $\mathcal{Set}$  with  $\sqcup$ ,  $\mathcal{Vect}_k$  with  $\otimes_k$ . If one works with Feynman categories, one does not need the symmetric monoidal structure in the non-symmetric case. The associativity is then associativity of morphisms.

**2.1.3. The three main examples.** Here we give the main examples which underlie the three Hopf algebras above. Notice that not all of them directly live in  $\mathcal{Ab}$  or  $\mathcal{Vect}_k$ , but for instance live in  $\mathcal{Set}$ . There are then free functors, which allow one to carry these over to  $\mathcal{Ab}$  or  $\mathcal{Vect}_k$  as needed.

**Example 2.3.** The operad of leaf-labelled rooted trees. We consider the set of rooted trees with  $n$ -labelled leaves, which means that the leaves are labelled by  $\{1, \dots, n\}$ . Given a  $n$ -labelled tree  $\tau$  and an  $m$ -labelled tree  $\tau'$ , we define an  $(m + n - 1)$ -labelled tree  $\tau \circ_i \tau'$  by grafting the root of  $\tau'$  onto the  $i$ th leaf of  $\tau$  to form a new edge. The root of the tree is taken to be the root of  $\tau$  and the labelling first enumerates the first  $i - 1$  leaves of  $\tau$ , then the leaves of  $\tau'$  and finally the remaining leaves of  $\tau$ .

The action of  $\mathbb{S}_n$  is given by permuting the names of the labels.

There are several interesting suboperads, such as that of trees whose vertices all have valence  $k$ . Especially interesting are the cases  $k = 2$  and  $3$ : the linear trees the 3-regular trees. Also of interest are the trees whose vertices have valence at least  $k$ , especially  $k \geq 3$ .

**Example 2.4.** The non- $\Sigma$  operad of leaf-labelled planar planted trees. A planar planted tree is a rooted tree with a linear order at each vertex, (the root flag being the first). This structure gives a linear order to all the leaves, and thus we do not have to give them extra labels for the gluing: there is an unambiguous  $i$ -th leaf for each planar planted tree with  $\geq i$  leaves, and  $\tau \circ_i \tau'$  is the tree obtained by grafting the root flag of  $\tau'$  onto that  $i$ -th leaf.

The suboperads above given by restricting the valency exist as well.

**Example 2.5.** The operad of surjections, also known as planar labelled corollas. Consider  $n$ -labelled planar corollas, that is, rooted trees with one vertex. For an  $n$ -labelled corolla  $\tau$  and an  $m$ -labelled corolla  $\tau'$  define  $\tau \circ_i \tau'$  to be the  $(n + m - 1)$ -labelled planar corolla with the same relabelling scheme as above.

Alternatively we can think of such a corolla as the unique map of ordered sets from the set  $\underline{n} = \{1, \dots, n\}$ , with the order given by the planar structure, to the one element set  $\underline{1} = \{1\}$ . The composition of the maps is now just given by using the composition of the orders according to the labelling scheme above. That is splicing in the orders.

The  $\mathbb{S}_n$  action permutes the labels and acts effectively on the possible orders. There is the non- $\Sigma$  version, in which case we are dealing with unlabelled planar corollas. This is then the non- $\Sigma$  operad of order preserving surjections of the sets  $\underline{n}$  with the natural order.

**Example 2.6.** Simplices form a non- $\Sigma$  operad (see also Proposition 3.3 for another dual operad structure). We consider  $[n]$  to be the category with  $n + 1$  objects  $\{0, \dots, n\}$  and morphisms generated by the chain  $0 \rightarrow 1 \rightarrow \dots \rightarrow n$ . The  $i$ -th composition of  $[m]$  and  $[n]$  is given by the following functor  $\circ_i : [m] \sqcup [n] \rightarrow [m + n - 1]$ . On objects of  $[m] : \circ_i(l) = l$  for  $l < i$  and  $\circ_i(l) = l + n - 1$  for  $l \geq i$ . On objects of  $[n] : \circ_i(l) = i - 1 + l$ . Finally on morphisms: the morphism  $l - 1 \rightarrow l$  of  $[m]$  is sent to the morphism  $l - 1 \rightarrow l$  of  $[m + n - 1]$  for all  $l < i$ , the morphism  $i - 1 \rightarrow i$  of  $[m]$  is sent to the composition of  $i - 1 \rightarrow i \dots \rightarrow i + n - 1$  in  $[m + n - 1]$ , the morphism  $l - 1 \rightarrow l$  of  $[m]$  to  $l + n - 1 \rightarrow l + n$  of  $[m + n - 1]$  for  $l > i$  and finally sends the morphism  $k \rightarrow k + 1$  of  $[n]$  to  $k + i \rightarrow k + 1 + i$ .

In words, one splices the chain  $[n]$  into  $[m]$  by replacing the  $i$ -th link, see Figure 1. This is of course intimately related to the previous discussion of order preserving surjections. In fact the two are related by Joyal duality as we will explain in §3 and Appendix C.

**2.1.4. The  $\circ$ -product aka. pre-Lie structure.** One important structure going back to Gerstenhaber [Ger64] is the following bilinear map:

$$a \circ b := \sum_{i=1}^n a \circ_i b \text{ if } a \text{ has operad degree } n \quad (2.2)$$

This product is neither commutative nor associative but preLie, which means that it satisfies the equation  $(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$ .

An important consequence is that  $[a, b] = a \circ b - b \circ a$  is a Lie bracket.

**Remark 2.7.** One often shifts degrees as in the cobar construction, such that  $\mathcal{O}(n)$  obtains degree  $n - 1$  and the operation obtains degree 1, see [KWZ12] for a full discussion

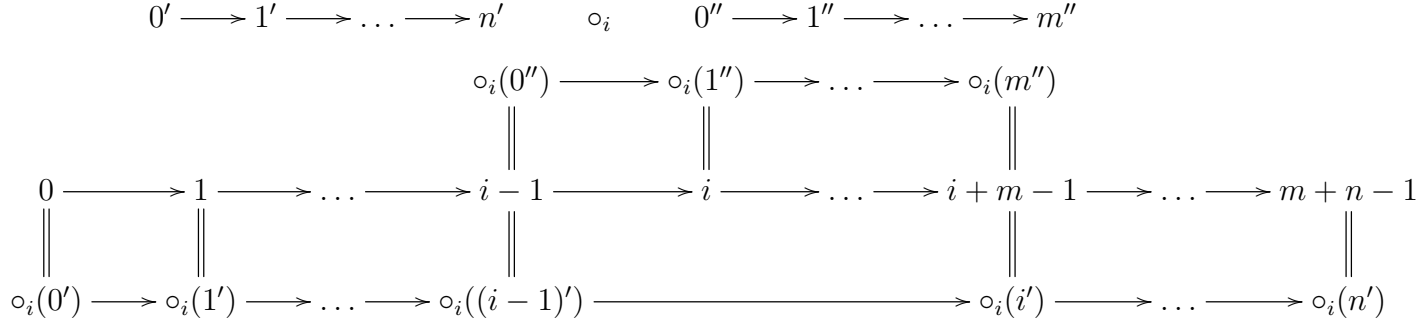


FIGURE 1. Splicing together of simplices. Primes and double primes are mnemonics only

$$a \circ b := \sum_i^n (-1)^{(i-1)(n-1)} a \circ_i b \text{ if } a \text{ has operad degree } n \quad (2.3)$$

The algebra is graded pre-Lie [Ger64] and the commutator is odd Lie.

2.1.5. **(Non- $\Sigma$ ) Operads:**  $\gamma$ . Another almost equivalent way to encode the above data is as follows. A non- $\Sigma$  operad is a collection  $\mathcal{O}(n)$  together with structure maps

$$\gamma = \gamma_{n_1, \dots, n_k} : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_k) \rightarrow \mathcal{O}\left(\sum_{i=1}^k n_i\right) \quad (2.4)$$

Such that map  $\gamma$  is associative in the sense that if  $(n_1, \dots, n_k)$  is a partition of  $n$ , and  $(n_1^i, \dots, n_{l_i}^i)$  are partitions of the  $n_i$ ,  $l = \sum_{i=1}^k l_i$  then

$$\gamma_{n_1, \dots, n_k} \circ id \otimes \gamma_{n_1^1, \dots, n_{l_1}^1} \otimes \gamma_{n_1^2, \dots, n_{l_2}^2} \otimes \dots \otimes \gamma_{n_1^k, \dots, n_{l_k}^k} = \gamma_{n_1^1, \dots, n_{l_1}^1, n_1^2, \dots, n_{l_2}^2, \dots, n_1^k, \dots, n_{l_k}^k} \circ \gamma_{l_1, \dots, l_k} \otimes id^{\otimes l} \circ \pi \quad (2.5)$$

as maps  $\mathcal{O}(k) \otimes \bigotimes_{i=1}^k (\mathcal{O}(l_i) \otimes \bigotimes_{j=1}^{l_i} \mathcal{O}(n_i^j)) \rightarrow \mathcal{O}(n)$ , where  $\pi$  permutes the factors of the  $\mathcal{O}(l_i)$  to the right of  $\mathcal{O}(k)$ . Notice that we chose to index the operad maps, since this will make the operations easier to dualize. The source and target of the map are then determined by the length  $k$  of the index, the indices  $n_i$  and their sum.

For an operad one adds the data of an  $\mathbb{S}_n$  action on each  $\mathcal{O}(n)$  and demands that the map  $\gamma$  is equivariant, again in the appropriate sense, see Example 2.1 or [MSS02, Kau04].

2.1.6. **Morphisms.** Morphisms of (pseudo)-operads  $\mathcal{O}$  and  $\mathcal{P}$  are given by a family of morphisms  $f_n : \mathcal{O}(n) \rightarrow \mathcal{P}(n)$  that commute with the structure maps. E.g.  $f_n(a) \circ_i^{\mathcal{P}} f_m(b) = f_{n+m-1}(a \circ_i b)$ . If there is are symmetric group actions, then the maps  $f_n$  should be  $\mathbb{S}_n$  equivariant.

**Example 2.8.** If we consider the operad of rooted leaf labelled trees  $\mathcal{O}$  there is a natural map to the operad of corollas  $\mathcal{P}$  given by  $\tau \mapsto \tau/E(\tau)$ , where  $\tau/E(\tau)$  is the corolla that

results from contracting all edges of  $\tau$ . This works in the planar and non planar version as well as in the pseudo-operad setting, the operad setting and the symmetric setting. This map contracts all linear trees and identifies them with the unit corolla. Furthermore, it restricts to operad maps for the suboperads of  $k$ -regular or at least  $k$ -valent trees.

An example of interest considered in [Gon05] is the map restricted to planar planted 3-regular tress (sometimes called binary). The kernel of this map is the operadic ideal generated by the associativity equations which says that the two possible planar planted binary trees with three tails are equivalent.

**2.1.7. Units.** The two notions of pseudo-operads and operads become equivalent if one adds a unit.

**Definition 2.9.** A *unit* for a pseudo-operad is an element  $u \in \mathcal{O}(1)$  such that  $u \circ_1 b = b$  and  $b \circ_i u = b$  for all  $m$ , for all  $1 \leq i \leq m$  and all  $b \in \mathcal{O}(m)$ .

A unit for an operad is an element  $u \in \mathcal{O}(1)$  such that

$$\gamma(u, a) = a \text{ and } \gamma(a; u, \dots, u) = a \quad (2.6)$$

There is an equivalence of categories between unital pseudo-operads and unital operads. It is given by the following formulas;

$$a \circ_i b = \gamma(a; u, \dots, u, b, u, \dots, u) \text{ } b \text{ in the } i\text{-th place} \quad (2.7)$$

and vice-versa:

$$\gamma(a; b_1, \dots, b_k) = (\dots ((a \circ_k b_k) \circ_{k-1} b_{k-1}) \dots) \circ_1 b_1 \quad (2.8)$$

Morphisms for (pseudo)-operads with units should preserve the unit.

**Remark 2.10.** The component  $\mathcal{O}(1)$  always forms an algebra via  $\gamma : \mathcal{O}(1) \otimes \mathcal{O}(1) \rightarrow \mathcal{O}(1)$ . If there is an operadic unit, then this algebra is unital.

## 2.2. The first act: from non- $\Sigma$ cooperads to bialgebras.

**2.2.1. Non- $\Sigma$  cooperads  $\check{\gamma}$ .** Dualizing the equation for  $\gamma$ , we obtain the notion of a cooperad. That is, there are structure maps for all  $m, k$  and partitions  $(n_1, \dots, n_k)$  of  $m$ ,

$$\check{\gamma}_{n_1, \dots, n_k} : \check{\mathcal{O}}(m) \rightarrow \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k) \quad (2.9)$$

which satisfy the dual relations. That is,

$$\begin{aligned} id \otimes \check{\gamma}_{n_1^1, \dots, n_{i_1}^1} \otimes \check{\gamma}_{n_1^2, \dots, n_{i_2}^2} \otimes \dots \otimes \check{\gamma}_{n_1^k, \dots, n_{i_k}^k} \circ \check{\gamma}_{n_1, \dots, n_k} = \\ \pi \circ \check{\gamma}_{l_1, \dots, l_k} \otimes id^{\otimes l} \circ \check{\gamma}_{n_1^1, \dots, n_{i_1}^1, n_1^2, \dots, n_{i_2}^2, \dots, n_1^k, \dots, n_{i_k}^k} \end{aligned} \quad (2.10)$$

as maps  $\check{\mathcal{O}}(n) \rightarrow \check{\mathcal{O}}(k) \otimes \bigotimes_{i=1}^k (\check{\mathcal{O}}(l_i) \otimes \bigotimes_{j=1}^{l_i} \check{\mathcal{O}}(n_i^j))$ , for any  $k$ -partition  $(n_1, \dots, n_k)$  of  $n$  and  $l_i$ -partitions  $(n_1^i, \dots, n_{l_i}^i)$  of  $n_i$ . Either side of the relation determines these partitions and hence determines the other side. Here  $l = \sum l_i$  and  $\pi$  is the permutation permuting the factors  $\check{\mathcal{O}}(l_i)$  to the left of the factors  $\check{\mathcal{O}}(n_i^j)$ .

**2.2.2. Morphisms.** Morphisms of cooperads  $\check{\mathcal{O}}$  and  $\check{\mathcal{P}}$  are given by a family of morphisms  $f_n : \check{\mathcal{O}}(n) \rightarrow \check{\mathcal{P}}(n)$  that commute with the structure maps.  $\check{\gamma}_{n_1, \dots, n_k}^{\check{\mathcal{P}}} \circ f_n = f_k \otimes f_{n_1} \otimes \cdots \otimes f_{n+k} \circ \check{\gamma}_{n_1, \dots, n_k}^{\check{\mathcal{O}}}$ .

**Remark 2.11.** If the monoidal category in which the cooperad lives is cocomplete and colimits commute with taking tensors, then we can define

$$\check{\gamma} : \check{\mathcal{O}}(m) \rightarrow \lim_k \lim_{(n_1, \dots, n_k) : \sum_{i=1}^k n_i = m} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k). \quad (2.11)$$

**Definition 2.12.** A *non- $\Sigma$  cooperad with multiplication*  $\mu$  is a non- $\Sigma$  cooperad  $(\check{\mathcal{O}}, \check{\gamma})$  together with a family of maps,  $n, m \geq 0$ ,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \rightarrow \check{\mathcal{O}}(n+m),$$

which satisfy the following compatibility equations:

- (1) For any  $n, n' \geq 1$  and partitions  $m_1 + \cdots + m_k = n$  and  $m'_1 + \cdots + m'_{k'} = n'$ , write  $\check{\gamma}$  and  $\check{\gamma}'$  for  $\check{\gamma}_{m_1, \dots, m_k}$  and  $\check{\gamma}_{m'_1, \dots, m'_{k'}}$  respectively, and write  $\check{\gamma}''$  for  $\check{\gamma}_{m_1, \dots, m_k, m'_1, \dots, m'_{k'}}$ , the following diagram commutes

$$\begin{array}{ccc} \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') & \xrightarrow{\pi(\check{\gamma} \otimes \check{\gamma}')} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^k \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \\ \downarrow \mu_{n,n'} & & \downarrow \mu_{k,k'} \otimes \text{id} \\ \check{\mathcal{O}}(n+n') & \xrightarrow{\check{\gamma}''} & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^k \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \end{array} \quad (2.12)$$

Here  $\pi$  is the isomorphism which permutes the  $k+k'+2$  tensor factors according to the  $(k+1)$ -cycle  $(23 \dots k+2)$ .

- (2) If  $m''_1 + \cdots + m''_{k''} = n+n'$  is a partition of  $n+n'$  which does not arise as the concatenation of a partition of  $n$  and a partition of  $n'$  (that is, there is no  $k$  such that  $m''_1 + \cdots + m''_k = n$  and  $m''_{k+1} + \cdots + m''_{k''} = n'$ ) then the composite

$$\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \xrightarrow{\mu_{n,n'}} \check{\mathcal{O}}(n+n') \xrightarrow{\check{\gamma}_{m''_1, \dots, m''_{k''}}} \check{\mathcal{O}}(k'') \otimes \bigotimes_{r''=1}^{k''} \check{\mathcal{O}}(m''_{r''})$$

is zero.

Under the completeness assumption, the  $\mu_{n,m}$  assemble into a map  $\mu$  satisfying the compatibility relation

$$\check{\gamma}(\mu(a \otimes b)) = \mu(\pi(\check{\gamma}(a) \otimes \check{\gamma}(b))) \quad (2.13)$$

where  $\pi$  is the permutation that permutes the first factor of  $\check{\gamma}(b)$  next to the first factor of  $\check{\gamma}(a)$ .

A *morphism of cooperads with multiplication*  $f : \check{\mathcal{O}} \rightarrow \check{\mathcal{P}}$  is a morphism of cooperads which commutes with the multiplication,  $f_{m+n} \mu_{n,m} = \mu_{n,m}(f_n \otimes f_m)$ .



**Assumption 2.13.** In order to simplify the situation, we will make the following assumptions. There is no  $\check{\mathcal{O}}(0)$ . This means that there are only finitely many maps and the limits reduce to finite limits.

In order to write down the multiplication and the comultiplication, we will need to take coproducts over all  $\mathcal{O}(n)$  and then identify them with products. Since the main applications of the Hopf algebras in lie in  $k - Vect$ , we will thus assume:

**Assumption 2.14.** We will further assume that we are in Abelian monoidal categories whose biproduct distributes over tensors. and use  $\oplus$  for the biproduct.

Main examples will be  $Set$ , and the Abelian monoidal categories of (graded) vector spaces  $k - Vect$ , differential graded vector spaces  $dg - Vect$ ,  $Ab$  Abelian groups and  $gAb$  graded Abelian groups.

**Theorem 2.15.** *Let  $\check{\mathcal{O}}$  be a cooperad with compatible multiplication  $\mu$  in a coalgebraic symmetric monoidal category with unit  $\mathbb{1}$ . Then*

$$\mathcal{B} := \bigoplus_n \check{\mathcal{O}}(n)$$

is a (non-unital, non-counital) bialgebra, with multiplication  $\mu$ , and comultiplication  $\Delta$  given by  $(\text{id} \otimes \mu)\check{\gamma}$ :

$$\begin{array}{ccc} \check{\mathcal{O}}(n) & \xrightarrow{\check{\gamma}} & \bigoplus_{\substack{k \geq 1, \\ n = m_1 + \dots + m_k}} \left( \check{\mathcal{O}}(k) \otimes \bigotimes_{r=1}^k \check{\mathcal{O}}(m_r) \right) \\ & \searrow^{(\text{id} \otimes \mu)\check{\gamma}} & \downarrow \text{id} \otimes \mu \\ & & \bigoplus_{k \geq 1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n). \end{array} \quad (2.14)$$

*Morphisms of cooperads with comultiplication induce homomorphisms of bialgebras.*

*Proof.* The multiplication  $\mu$  is associative by definition. The compatibility of  $\mu$  with  $\check{\gamma}$ , together with the associativity of  $\mu$ , shows that  $\mu$  is a morphism of coalgebras,  $\Delta\mu = (\mu \otimes \mu)\pi(\Delta \otimes \Delta)$ :

$$\begin{array}{ccccc} \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') & \xrightarrow{\pi \check{\gamma}^{\otimes 2}} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^k \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) & \xrightarrow{\text{id} \otimes \mu \otimes \mu} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \\ & \downarrow \mu_{n,n'} & & & \downarrow \mu_{k,k'} \otimes \mu_{n,n'} \\ & & & & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \\ & & \text{compatibility} & & \downarrow \mu_{k,k'} \otimes \mu_{n,n'} \\ & & & & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \\ & & & & \downarrow \mu_{k,k'} \otimes \mu_{n,n'} \\ \check{\mathcal{O}}(n+n') & \xrightarrow{\check{\gamma}} & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^k \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) & \xrightarrow{\text{id} \otimes \mu} & \check{\mathcal{O}}(k+k') \otimes \check{\mathcal{O}}(n+n'). \end{array}$$

For the coassociativity, we notice that  $\Delta$  just like  $\check{\gamma}$  can be written in components  $\Delta = \sum_n \Delta_n = \sum_n \sum_k \Delta_{k,n}$  with  $\Delta_{k,n} : \check{\mathcal{O}}(n) \rightarrow \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n)$  and these can be decomposed further as  $\Delta_{k,n} = \sum_{(n_1, \dots, n_k) : \sum n_i = n} \Delta_{n_1, \dots, n_k}$  with  $\Delta_{n_1, \dots, n_k} = (id \otimes \mu^{\otimes k-1}) \circ \check{\gamma}_{n_1, \dots, n_k}$ .

One now has to prove that  $(id \otimes \Delta_{l,n}) \Delta_{k,n} = (\Delta_{k,l} \otimes id) \Delta_{l,n} : \check{\mathcal{O}}(n) \rightarrow \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(l) \otimes \check{\mathcal{O}}(n)$ , which can be done term by term using (2.10) and (2.12).

Explicitly fix a  $k$ -partition  $n_1, \dots, n_k$  of  $n$  an  $l$  partition  $(m_1, \dots, m_l)$  of  $n$ . by compatibility the left hand side vanishes unless  $(m_1, \dots, m_l)$  naturally decomposes into the list  $(n_1^1, \dots, n_{l_1}^1, n_1^2, \dots, n_{l_2}^2, \dots, n_1^k, \dots, n_{l_k}^k)$  where  $n_j^i$  is a partition of  $n_i$ . This yields the  $k$  partition  $(l_1, \dots, l_k)$  of  $l$ . Starting on the rhs that is with  $(m_1, \dots, m_l)$  and  $(l_1, \dots, l_k)$ , we decompose the list  $(m_1, \dots, m_l)$  as above, which determines the  $n_i = \sum_j n_j^i$ . The proof is then:

$$\begin{aligned} (id \otimes \Delta_{m_1, \dots, m_l}) \Delta_{n_1, \dots, n_k} &= (id \otimes id \otimes \mu^{l-1}) (id \otimes [\check{\gamma}_{m_1, \dots, m_l} \circ \mu^{k-1}]) \circ \check{\gamma}_{n_1, \dots, n_k} \\ &= (id \otimes id \otimes \mu^{l-1}) (id \otimes \mu^{k-1} \otimes id^{\otimes l}) \circ \pi \circ (id \otimes \check{\gamma}_{n_1^1, \dots, n_{l_1}^1} \otimes \check{\gamma}_{n_1^2, \dots, n_{l_2}^2} \otimes \dots \otimes \check{\gamma}_{n_1^k, \dots, n_{l_k}^k}) \circ \check{\gamma}_{n_1, \dots, n_k} \\ &= (id \otimes \mu^{k-1} \otimes id) (id \otimes id^{\otimes k} \otimes \mu^{l-1}) (\check{\gamma}_{l_1, \dots, l_k} \otimes id^{\otimes l}) \check{\gamma}_{n_1^1, \dots, n_{l_1}^1, n_1^2, \dots, n_{l_2}^2, \dots, n_1^k, \dots, n_{l_k}^k} \\ &= [(id \otimes \mu^{k-1}) \check{\gamma}_{l_1, \dots, l_k} \otimes id] (id \otimes \mu^{l-1}) \check{\gamma}_{m_1, \dots, m_l} = (\Delta_{l_1, \dots, l_k} \otimes id) \Delta_{m_1, \dots, m_l} \quad (2.15) \end{aligned}$$

where  $\pi$  is the permutation that shuffles all the right factors next to each other as before.  $\square$

**2.2.3. Examples from a free construction.** In this section, we show that there are lots of examples of the structure above. We show that for any cooperad, there exists a non-connected version, which is a cooperad with multiplication and hence furnishes a bialgebra as above. For finiteness, we assume that there is no cooperadic degree 0 part, as above.

Cooperads themselves can be obtained by dualizing operads. Namely, starting with a non- $\Sigma$  operad  $\mathcal{O}$  and let  $\check{\mathcal{O}}$  be its linear dual, that is assuming the existence of inner homs, set  $\check{\mathcal{O}}(n) = (\mathcal{O}(n))^\vee = Hom(\mathcal{O}(n), \mathbb{1})$ . In particular, we can use the examples from 2.1.3. In order to transport *Set* cooperads with multiplication to Abelian categories, we can take the free construction, dual to the forgetful functor [Kel82]. Similarly, we can induce cooperads in different categories, by extending coefficients, say from  $\mathbb{Z}$  to  $\mathbb{Q}$ , and other free constructions.

**Construction 2.16.** Let  $\check{\mathcal{O}}$  be a non- $\Sigma$  cooperad. Consider

$$\check{\mathcal{O}}^{nc}(n) := \bigoplus_{(n_1, \dots, n_k) : \sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k) \quad (2.16)$$

and define  $\mu$  to be the concatenation of tensors:  $\mu(a, b) = a \otimes b$ . This means that  $\mathcal{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$  is the tensor algebra on  $\check{\mathcal{O}} := \bigoplus_n \check{\mathcal{O}}(n)$ . The collection  $\check{\mathcal{O}}^{nc}(n)$  is a non- $\Sigma$  cooperad, by using (2.12) to extend  $\check{\gamma}$  form  $\check{\mathcal{O}}$  to its free tensor algebra  $\mathcal{B}$ .

$$\mathcal{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n) = \bigoplus_{n, k} \bigoplus_{(n_1, \dots, n_k) : \sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k) \quad (2.17)$$

Since  $\check{\mathcal{O}}^{nc}$  as a cooperad with multiplication satisfies the conditions of Theorem 2.15, we obtain:

**Proposition 2.17.**  $\mathcal{B}$ , as defined in (2.17), with tensor multiplication and the associated  $\Delta$  is a (non-unital, non-counital) bialgebra and this association is functorial.

*Proof.* It is clear from the construction that  $\check{\mathcal{O}}^{nc}$  is a cooperad with multiplication. It is also straightforward that any map  $\check{\mathcal{O}} \rightarrow \check{\mathcal{P}}$  of cooperads induces a map  $\check{\mathcal{O}}^{nc} \rightarrow \check{\mathcal{P}}^{nc}$  of cooperads with multiplication and hence bialgebras.  $\square$

**Remark 2.18.**

- (1) This type of non-connected version of (co)–operads is one of the variations for non-connected operads studied in detail in [KWZ12].
- (2) This type of example is also the type of example that comes from the enriched Feynman categories  $\mathfrak{F}_{\mathcal{O}}$ , see [KW13] and §4.
- (3) This example has the several extra properties not present in the general situation. There is an induced double grading by length of the tensor word and cooperadic degree. In general, as we show below, there will just be a depth filtration replacing the tensor length. Furthermore the bialgebra is generated by  $\check{\mathcal{O}}$  as an algebra, that is words of length one. Some of these additional properties will be reappear as necessary conditions to construct units, counits and an antipode on a suitable quotient.

**Example 2.19.** Our main examples of operads of §2.1.3 all define bialgebras by first taking their duals and then performing the free construction. Notice, they are all unital pseudo–operads and hence equivalently are unital operads.

Taking the duals, we view each tree as the characteristic function of itself,  $\tau \leftrightarrow \delta_\tau$  where  $\delta_\tau(\tau) = 1$  and  $\delta_\tau(\tau') = 0$  for all  $\tau' \neq \tau$ . Taking the tensor algebra corresponds to regarding ordered forests.

The operad maps between them induce maps of bialgebras going in the other direction, since we are taking duals. Thus we obtain a morphisms from the bialgebra of forests of corollas to the bialgebra of forests of (binary) trees.

**2.2.4. External graded version.** One obtains an external graded version of the above if one uses the tensor algebra on the suspension. This is analogous to the use of signs in the pre–Lie structure [KWZ12]. An internal grading is already built in.

**2.2.5. Cobar version.** Given an operad, another way to obtain a cooperad is by considering the operadic bar transform. One can then plug this cooperad into the non–connected construction. This is much bigger than just doing the tensor algebra on the dual, see §4.

### 2.3. Intermezzo: A natural filtration and the associated graded.

**Definition 2.20.** We define the *decreasing depth filtration* on a cooperad  $\check{\mathcal{O}}$  as follows:  $a \in F^{\geq p}$  if  $\check{\gamma}(a) \in \bigoplus_{k \geq p} \bigoplus_{(n_1, \dots, n_k): \sum_i n_i = m} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k)$ . So  $\mathcal{B} = F^{\geq 1} \supset F^{\geq 2} \supset \dots$  and  $\bigcap_p F^{\geq p} = 0$ , since we assumed that there is no  $\check{\mathcal{O}}(0)$ .

We define the depth of an element  $a$  to be the maximal  $p$  such that  $a \in F^{\geq p}$ .

This filtration induces a depth filtration  $F^{\geq p}T\mathcal{B}$  on the tensor algebra  $T\mathcal{B}$  by giving  $F^{\geq p_1} \otimes \cdots \otimes F^{\geq p_k}$  depth  $p_1 + \cdots + p_k$ . Note that any element in  $T^p\mathcal{B}$  will have depth at least  $p$ .

**Proposition 2.21.** *The following statements hold for a cooperad with multiplication with empty  $\check{O}(0)$ :*

- a) *The algebra structure is filtered:  $F^{\geq p} \cdot F^{\geq q} \subset F^{\geq p+q}$ .*
- b) *The cooperad structure satisfies  $\check{\gamma}(F^{\geq p}) \subset F^{\geq p} \otimes T^{\geq p}\mathcal{B}$  where  $T^{\geq p}\mathcal{B} = \bigoplus_{i=p}^{\infty} (\mathcal{B})^{\otimes i} \subset F^{\geq p}T\mathcal{B}$  and more precisely  $\check{\gamma}_{n_1, \dots, n_k} : \check{O}(n) \cap F^{\geq p} \rightarrow [\check{O}(k) \otimes \check{O}(n_1) \otimes \cdots \otimes \check{O}(n_k)] \cap F^{\geq p} \otimes F^{\geq k}T\mathcal{B}$ .*
- c) *The coalgebra structure satisfies:  $\Delta(F^{\geq p}) \subset F^{\geq p} \otimes F^{\geq p}$  and more precisely  $\Delta_k(F^{\geq p}) \subset F^{\geq p} \otimes F^{\geq k}$ .*
- d)  *$\check{O}(n) \cap F^{\geq n+1} = \emptyset$ .*

*Proof.* The first statement follows from the compatibility (2.12). The second statement follows from the Lemma 2.22 below. The more precise statement on the right part of the filtration stems from the fact that  $T^k\mathcal{B} \subset F^{\geq k}T\mathcal{B}$ . The third statement then follows from a) and b), since there are at least  $p$  factors on the right before applying the multiplication and the filtration starts at 1. This shows that the right factor is in  $F^p$ . Finally, for  $\check{O}(n)$  the greatest depth that can be achieved happens when all the  $n_i = 1 : i = 1, \dots, k$  and since they sum up to  $n$  this is precisely at  $k = n$ .  $\square$

**Lemma 2.22.** *If  $a^p \in \mathcal{B}$  of depth  $p$  let  $\check{\gamma}_{n_1, \dots, n_k}(a^p) = \sum a_{(p_0)}^{(0)} \otimes a_{(p_1)}^{(1)} \otimes \cdots \otimes a_{(p_k)}^{(k)}$ , where we used Sweedler notation for both the cooperad structure and the depth. Then the terms of lowest depth will satisfy  $p_0 = \sum_{i=1}^k p_i \geq p$ .*

*Proof.* To show the equation, we use coassociativity of the cooperad structure. If we apply  $\text{id} \otimes \check{\gamma}^{\otimes k}$  we get least  $1 + k + \sum_{i=1}^k p_i$  tensor factors from the lowest depth term, since we assumed that  $\check{O}(0)$  is empty. On the other hand applying  $\check{\gamma} \otimes \text{id}^{\otimes k}$  to the terms of lowest depth, we obtain elements with at least  $1 + p_0 + k$  tensor factors. Since elements of higher depth due to equation (2.10) produce more tensor factors these numbers have to agree. Since all the  $p_i \geq 1$  their sum is  $\geq p$ .  $\square$

**2.3.1. The associated graded bialgebra.** We now consider the associated objects  $Gr^p := F^{\geq p}/F^{\geq p+1}$  and denote the image of  $\check{O}(n) \cap F^p$  in  $Gr^p$  by  $\check{O}(n, p)$ . An element of depth  $p$  will have non-trivial image in  $Gr^p$  under this map. We denote the image of an element  $a^p$  of depth  $p$  under this map by  $[a^p]$  and call it the principal part.

We set  $Gr = \bigoplus Gr^p$ , by part d) of 2.21:  $Gr = \bigoplus_p \bigoplus_{n=1}^p \check{O}(n, p)$  and define a grading by giving the component  $\check{O}(n, p)$  the total degree  $n - p$ .

**Corollary 2.23.** *By the Proposition 2.21 above we obtain maps*

- $\mu : Gr^p \otimes Gr^q \rightarrow Gr^{p+q}$  by taking the quotient by  $F^{\geq p+1} \otimes F^{q+1}$  on the left and  $F^{\geq p+q+1}$  on the right

- $\tilde{\gamma}^{p,k} : Gr^p \rightarrow Gr^p \otimes (Gr^1)^{\otimes k}$  by taking the quotient by  $F^{\geq p+1}$  on the left and  $F^{\geq k+1}T\mathcal{B} \cap T^{\otimes k}\mathcal{B}$  on the right. In particular  $\tilde{\gamma}(Gr^1) \subset Gr^1 \otimes TGr^1$
- $\Delta^{p,k} : Gr^p \rightarrow Gr^p \otimes Gr^k$  by taking the quotient by  $F^{\geq p+1}$  on the left and  $F^{\geq k+1}$  on the right.
- $\Delta^p : Gr^p \rightarrow Gr^p \otimes Gr$  via  $\Delta^p k = \sum_k \Delta^{p,k}$
- $\Delta : Gr \rightarrow Gr \otimes Gr$  via  $\Delta = \sum_p \Delta^p$

**Proposition 2.24.** *Gr inherits the structure of a non-unital, non-counital graded bialgebra. Each  $Gr^p$  is a non-counital comodule over  $Gr$ , and  $Gr^1$  is a cooperad.*

*Proof.* Most claims are straightforward from the definitions in the corollary. For the grading we notice the multiplication preserves grading:  $\check{O}(n, p) \otimes \check{O}(m, q) \rightarrow \check{O}(n+m, p+q)$ . For the comultiplication we have that  $\Delta_k(\check{O}(n, p)) \subset \check{O}(k, p) \otimes \check{O}(n-k, p)$ . The degree on the left is  $n-p$  and on the right is  $k-p+n-k = n-p$  and hence the comultiplication also preserves degree.  $\square$

**Example 2.25.** For the free construction  $\check{O}^{nc}$  of §2.2.3 we obtain

$$\mathcal{F}^{\geq p} = \bigoplus_{k \geq p} \bigoplus_{(n_1, \dots, n_k)} \check{O}(n_1) \dots \check{O}(n_k) \quad (2.18)$$

$$Gr^p = \bigoplus_{(n_1, \dots, n_k)} \check{O}(n_1) \dots \check{O}(n_k) \quad (2.19)$$

$$\check{O}^{nc}(n, k) = \bigoplus_{(n_1, \dots, n_k) : \sum_i n_i = n} \check{O}(n_1) \dots \check{O}(n_k) \quad (2.20)$$

This means that the depth of an element of  $\mathcal{B}$  given by an elementary tensor is its length. The associated graded is isomorphic to the  $\mathcal{B}$  which has a double grading by depth and operadic degree. Furthermore  $Gr^1 = \check{O}$  and  $\mathcal{B} = (Gr^1)^{nc} = \check{O}^{nc}$ .

**Corollary 2.26.** *Since  $Gr^1$  is a cooperad  $(Gr^1)^{nc}$  yields a cooperad with multiplication. Multiplication gives a morphism  $(Gr^1)^{nc} \rightarrow Gr$  of cooperads with multiplication preserving the filtrations and hence gives a morphism of bialgebras.*

*Proof.* Indeed the multiplication map gives such a map of algebras, since  $Gr^{nc}$ . The compatibility map (2.13) ensures that this is also a map of cooperads with multiplication. The compatibility with the filtration is clear.  $\square$

**2.4. The second act: Unital and counital bialgebra structure.** The general construction gives a multiplication and a comultiplication which are compatible. What is missing for a bialgebra are the unit and counit. In this section, we will show that there is no problem in adding a unit and that the existence of a bialgebraic counit in the free case  $\check{O}^{nc}$  is equivalent to the existence of a cooperadic counit for  $\check{O}$ . For the general case, the existence of a right cooperadic counit is a necessary condition and such a cooperadic counit determines a bialgebra counit uniquely if it exists. But, the unique candidate does

not automatically work. We give several conditions that are necessary for this, treating the cases of left and right counits separately with care.

The existence of a right bialgebra counit, is equivalent to the cooperad having a right counit, which extends to a multiplicative family.

Having a left coalgebra counit for  $\mathcal{B}$  fixes the structure of the associated graded as a quotient of the free construction on  $Gr^1$  via the map of Corollary 2.26 and  $\mathcal{B}$  is a deformation of this quotient, see Theorem 2.38.

In the free case, we can show that having a bialgebra unit for  $\mathcal{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$  is equivalent to having a cooperadic counit for the underlying  $\check{\mathcal{O}}$ .

**2.4.1. Unit.** As the multiplication preserves operad degree and there is no element of operad degree 0,  $(\mathcal{B}, \mu)$  cannot have a unit. We formally adjoin a unit 1 to  $\mathcal{B}$ :  $\mathcal{B}' = \mathbb{1} \oplus \mathcal{B}$ , we let  $\eta$  be the inclusion of  $\mathbb{1}$  and  $pr$  the projection to  $\mathcal{B}$ . We extend  $\mu$  in the obvious way, and set  $\Delta(1) = 1 \otimes 1$ . This makes  $\mathcal{B}'$  into a unital bialgebra. For the fastidious reader  $1 = id_{\mathbb{1}} \in Hom(\mathbb{1}, \mathbb{1})$ .

**Remark 2.27.** In the free construction, we think of  $\mathbb{1} \subset \mathcal{B}'$  as the tensors of length 0 and in the Feynman category interpretation indeed  $1 = id_{\mathbb{1}}$  where  $\mathbb{1}$  is the empty word. The unit is thus not really in  $\check{\mathcal{O}}(0)$ , but in an additional space.

**2.4.2. Counit and multiplicativity.** We will denote putative counits on  $\mathcal{B}$  by  $\epsilon_{tot} : \mathcal{B} \rightarrow \mathbb{1}$  and decompose  $\epsilon_{tot} = \sum_{k \geq 1} \epsilon_k$  according to the direct sum decomposition on  $\mathcal{B}$ :  $\epsilon_k : \check{\mathcal{O}}(k) \rightarrow \mathbb{1}$  extended to zero on all other components. We will also use the truncated sum  $\epsilon_{\geq p} = \sum_{k \geq p} \epsilon_k$  which is 0 on all  $\check{\mathcal{O}}(k)$  for  $k < p$ .

**Remark 2.28.** There is a 1–1 correspondence between (left/right) counits on  $\mathcal{B}$  and on  $\mathcal{B}'$ . This is given by adding  $\epsilon_0$  on the identity component via the definition  $\epsilon_0 \circ \eta = id$  and vice-versa truncating the extended sum  $\epsilon_{tot} = \sum_{k \geq 0} \epsilon_k$  at  $k = 1$ .

A family of morphisms  $\epsilon_k : \check{\mathcal{O}}(k) \rightarrow \mathbb{1}$  is called multiplicative if  $\kappa \circ (\epsilon_k \otimes \epsilon_l) = \epsilon_{k+l} \circ \mu$ , where  $\kappa : \mathbb{1} \otimes \mathbb{1} \rightarrow \mathbb{1}$  is the unit constraint —e.g. multiplication in the ground field in  $k - Vect$ — which we will omit from now on.

**Lemma 2.29.** *If  $\epsilon_{tot}$  is a counit (left or right) then the  $\epsilon_k$  are a multiplicative family. More generally  $\epsilon_{n_1} \otimes \cdots \otimes \epsilon_{n_k} = \epsilon_{\sum n_i} \circ \mu^{k-1}$  and in particular  $\epsilon_1^{\otimes k} = \epsilon_k \otimes \mu^{k-1}$ . If  $\epsilon_k$  is a any multiplicative family and  $\eta_1$  is a section of  $\epsilon_1$  then  $\mu^{k-1} \circ \eta_1^{\otimes k}$  is a section of  $\epsilon_k$ .*

*Furthermore  $\epsilon_{tot}$  descends to the associated graded.*

*Proof.* The first statement is equivalent to  $\epsilon$  being an algebra morphism. The other equations follow readily. Now  $\epsilon_p(F^{\geq p+1}) = 0$ , since  $\check{\mathcal{O}}(p, p+1) = 0$  and hence each  $\epsilon_p$  descends to  $Gr^p$ . The sum  $\epsilon_{tot}$  then descends as the sum of the  $\epsilon_p$  with each  $\epsilon_p$  defined on the summand  $Gr^p$ .  $\square$

**2.4.3. Recollection on cooperadic counits.** A morphism  $\epsilon : \mathcal{B} \rightarrow \mathbb{1}$  with support in  $\check{\mathcal{O}}(1)$  is a left and right cooperadic counit if it satisfies<sup>2</sup>:

$$\sum_k \epsilon \otimes id^{\otimes k} \circ \check{\gamma} = id \quad (2.21)$$

$$\sum_k id \otimes \epsilon^{\otimes k} \circ \check{\gamma} = id \quad (2.22)$$

**Remark 2.30.** The notion of cooperadic counits is the dual to a unit  $u \in \mathcal{O}(1)$ , thought of as a map of  $u : \mathbb{1} \rightarrow \mathcal{O}(1)$ , where  $\mathbb{1}$  is  $\mathbb{Z}$  for Abelian groups or in general the unit object, e.g.  $k$  for  $\mathcal{Vect}_k$ . Its dual is then a morphism  $\check{u} := \check{\mathcal{O}}(1) \rightarrow \mathbb{1}$ . We will use  $\epsilon : \mathcal{B} \rightarrow \mathbb{1}$  for its extension by 0 on all  $\check{\mathcal{O}}(n) : n \neq 1$ .  $\epsilon$  is a left/right cooperadic counit if it satisfies the diagrams dual to the equations (2.6), that is the equations (2.21) and (2.22).

**Remark 2.31.** Note, if there is only one tensor factor on the right, then the left factor has to be  $\check{\mathcal{O}}(1)$  by definition. If  $\epsilon$  would have support outside  $\check{\mathcal{O}}(1)$ , the  $\check{\gamma}$  would have to vanish on the right side for all elements having that left hand side, which is rather non-generic. This is why we assume  $\epsilon$  vanishes outside  $\check{\mathcal{O}}(1)$ .

#### 2.4.4. Right counits.

**Lemma 2.32.** *If  $\mathcal{B}$  has a right bialgebra counit  $\epsilon_{tot}$ , then  $\epsilon_1$  is a right cooperadic counit. If there are elements of depth greater than one, there can be no left cooperadic counit.*

*Proof.* For the first statement, we verify (2.22) using Lemma 2.29:

$$\sum_k (id \otimes \epsilon_1^{\otimes k}) \circ \check{\gamma} = \sum_k (id \otimes \epsilon_k) \circ \mu^{k-1} \circ \check{\gamma} = (id \otimes \epsilon_{tot}) \circ \Delta = id \quad (2.23)$$

The second statement just says that using  $\epsilon$  on the left, we would need only one tensor factor on the right after applying  $\check{\gamma}$  in order to get an identity, but if  $a \in F^{\geq p}$  then there are at least  $p$  tensor factors, so there can be no left counit on  $F^{\geq 2}$ .  $\square$

A necessary condition for the existence of a right counit for  $\mathcal{B}$  is hence

**Proposition 2.33.**  *$\epsilon_{tot}$  is a right bialgebraic counit if and only if  $\epsilon_1$  is a right cooperadic counit which extends to a multiplicative family  $\epsilon_k$ .*

*Proof.* This follows by reading equation (2.23) right to left.  $\square$

#### 2.4.5. Left counits.

**Proposition 2.34.** *If  $\mathcal{B}$  as a coalgebra has a left counit  $\epsilon_{tot}$ , then  $F^{\geq p} = (F^{\geq 1})^{\geq p}$ , where the latter denotes the sum of the  $k$ -th powers of  $F^{\geq 1}$  with  $k \geq p$ . Moreover, the morphism of cooperads with multiplication and of bialgebras  $(Gr^1)^{nc} \rightarrow Gr$  given by Corollary 2.26 is surjective.*

<sup>2</sup>Here and in the following, we suppress the unit constraints in the monoidal category and tacitly identify  $V \otimes \mathbb{1} \simeq V \simeq \mathbb{1} \otimes V$ .



*Proof.* The inclusion  $F^{\geq p} \supset (F^{\geq 1})^{\geq p}$  is in Proposition 2.21. For the reverse inclusion, let  $a \in F^{\geq p}$ , then after applying  $(\epsilon_{tot} \otimes id) \circ \Delta$  we are left with a sum of products of at least  $p$  factors and hence the reverse inclusion follows.

In the same way, we see that  $Gr^p = (Gr^1)^p$  and that the map in question is surjective.  $\square$

We recall from [Ger64] that a filtered algebra/ring  $(\mathcal{B}, F^{\geq p})$  is predevelopable if there exists for each  $p$  an additive mapping  $q_p : Gr^p \rightarrow F^{\geq p}$  which is a section of  $p_p : F^{\geq p} \rightarrow Gr^p = F^{\geq p}/F^{\geq p+1}$  i.e.  $p_p \circ q_p(a) = a$  for all  $a \in Gr^p$ . It is developable if also  $\bigcap_p F^{\geq p} = 0$  and the ring is complete in the topology induced by the filtration. In our case, due to the assumption there is no  $\check{O}(0)$ , the first condition is true and also since we only took finite sums, the algebra is complete.

**Proposition 2.35.** *If  $\mathcal{B}$  has a left coalgebra counit then  $P_p = (\epsilon_{\geq p} \otimes id) \circ \Delta$  is a projector to  $F^{\geq p}$ . Hence the short exact sequence  $0 \rightarrow F^{\geq p+1} \rightarrow F^{\geq p} \rightarrow Gr^p \rightarrow 0$  splits and  $\mathcal{B}$  is predevelopable.*

*Proof.* If  $\epsilon_{tot}$  is a left coalgebra counit then using multi-Sweedler notation for  $a \in \check{O}(n)$  :  $a = (\epsilon_{tot} \otimes id) \circ \Delta(a) = \sum_k \epsilon_k(a_k^{(0)}) \otimes a_{n_1}^{(1)} \cdots a_{n_k}^{(1)} =: \sum_k a_k$  with  $a_k$  a product of  $k$  factors and hence in  $F^{\geq k}$ . Since  $\epsilon_{\geq p} = 0$  on  $\check{O}(k) : k < p$ , we see that  $P_p(a) = \sum_{k \geq p}^n a_k$  and hence the image of  $P_p$  lies in  $F^{\geq p}$ . If on the other hand  $a \in F^{\geq p}$  then  $a = \sum_k a_k = \sum_{k \geq p} a_k = P_p(a)$ , since all lower terms do not exist as the summation for  $\Delta$  stands at  $p$ .  $\square$

Note that  $T_i(a) = [P_{i-1} \cdots P_1(a)]$  gives the development of  $a$  in  $Gr$  in the notation of [Ger64].

**Corollary 2.36.** *If  $\epsilon_{tot}$  is a left bialgebra unit, then for  $a \in \check{O}(n) \cap F^{\geq p}$  there is a decomposition  $a = \sum_{k \geq p}^n a_k$  with each  $a_k \in F^{\geq k}$  and (after possibly collecting terms) this gives the development of  $a$ .  $\square$*

**Corollary 2.37.** *If  $\epsilon_{tot}$  is a left coalgebra counit for  $\mathcal{B}$ , then  $\epsilon_p$  descends to a well defined map  $Gr^p \rightarrow \mathbb{1}$ . and on  $Gr^p : (\epsilon_p \otimes id) \circ \Delta_p = id$ . Thus  $\epsilon_{tot}$  understood as acting on  $Gr^p$  with  $\epsilon_p$  is a left counit for  $Gr$ . Furthermore  $(\epsilon_k \otimes id) \circ \Delta|_{Gr^p} = \delta_{k,p} id$ .*

*Proof.* First  $\epsilon_p(F^{\geq p+1}) = 0$ , since  $\check{O}(p, p+1) = 0$ . The statements then follows from the development.  $\square$

It is known [Ger64] that if  $\mathcal{B}$  is developable then  $Gr$  is a deformation of  $\mathcal{B}$ . Coupled with the results above one has:

**Theorem 2.38.** *if  $\mathcal{B}$  has a left coalgebra counit, then  $\mathcal{B}$  is a deformation  $Gr$ , which is a quotient of the free construction on  $Gr^1$ .  $\square$*

**2.5. Units and counits for the free case  $\check{\mathcal{O}}^{nc}$ .** In this section, we let  $\check{\mathcal{O}}$  be a cooperad and consider  $\check{\mathcal{O}}^{nc}(n) = \bigoplus_k \bigoplus_{(n_1, \dots, n_k): \sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)$  and its bialgebra  $\mathcal{B} = \bigoplus \check{\mathcal{O}}^{nc}(n)$ .

**Proposition 2.39.** *For the bialgebra  $\mathcal{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$  to have a bialgebraic counit it is sufficient and necessary that  $\check{\mathcal{O}}$  has a cooperadic counit.*

*Proof.* We already know that a right cooperadic counit for  $\check{\mathcal{O}}^{nc}$  is necessary. This yields a right operadic counit for  $\check{\mathcal{O}}$  by restriction to  $Gr^1 = \check{\mathcal{O}}$ . Then for  $a \in \check{\mathcal{O}} = Gr^1$   $a = \epsilon_1 \otimes id \circ \Delta(a) = \sum_k \epsilon_1 \otimes id^{\otimes k} \circ \check{\gamma}$ , since all terms with  $k \neq 1$  vanish and for the term with  $k = 1$   $\Delta = \check{\gamma}$ . Thus  $\epsilon_1$  is also a left cooperadic unit for  $\check{\mathcal{O}}$  (note not for  $\check{\mathcal{O}}^{nc}$ ).

Now assume that  $\epsilon_1$  is a cooperadic counit for  $\check{\mathcal{O}}$ . It follows that  $\epsilon_1$  is a right cooperadic unit for  $\check{\mathcal{O}}^{nc}$  by compatibility. Now since  $\mu = \otimes$ : the extension  $\epsilon_k = \epsilon_1^{\otimes k}$  is multiplicative and hence a right bialgebra counit. It remains to check whether it is bialgebraic. For this it remains to check that it is a left coalgebraic unit. The multiplicativity is clear, so, we only need to check on  $Gr^1$ , that is  $a \in \check{\mathcal{O}}^{nc}(n, 1) = \check{\mathcal{O}}(n)$ . Here we get the equation that  $\epsilon_1$  is a left cooperadic unit for  $\check{\mathcal{O}}$ . □

**Corollary 2.40.** *If  $\mathcal{O}$  has an operadic unit, then  $\check{\mathcal{O}}$  has a cooperadic counit and hence  $\mathcal{B}' = \mathbb{1} \oplus \bigoplus \check{\mathcal{O}}^{nc}(n)$  is a unital, counital bialgebra.*

This encompasses all the examples of §2.1.3.

**2.5.1. Counits summary.** If  $\mathcal{B}$  comes from  $\check{\mathcal{O}}^{nc}$  then having a bialgebra unit  $\epsilon_{tot}$  is equivalent to  $\epsilon_1$  being a cooperad unit on  $\check{\mathcal{O}}$ .

In general, for  $\mathcal{B}$  to have a bialgebra counit, it is necessary, that

- (1)  $\epsilon_1$  is a right cooperadic counit.
- (2)  $F^{\geq p} = (F^{\geq 1})^{\geq p}$ .
- (3)  $P_k = (\epsilon_{\geq k} \otimes id) \circ \Delta$  are projectors onto  $F^{\geq k}$ .
- (4)  $\mathcal{B}$  is developable and a deformation of the associated graded  $Gr$

On the associated graded  $Gr$ . If  $\epsilon_{tot}$  is a putative bialgebra counit

- (1)  $\epsilon_p$  is uniquely determined from  $\epsilon_1$ .
- (2) Lifted to  $(Gr^1)^{nc}$ ,  $\epsilon_1$  is a cooperadic unit, which ensures that the lift of  $\epsilon_{tot}$  is a bialgebra unit.
- (3) For  $\epsilon_{tot}$  to descend to  $Gr$ , it needs to vanish on the kernel of the by (2) surjective map  $\mu^{\otimes p-1} : (Gr^1)^{\otimes p} \rightarrow Gr^p$ .

The first statement holds by Proposition 2.34 and Corollary 2.37 which says that  $Gr^p = (Gr^1)^p$  and hence (2.29) determines  $\epsilon_p$ . Since counits are multiplicative, they lift onto via 2.39.

**Definition 2.41.** In general, we say that a cooperadic right counit  $\epsilon_1$  is *bialgebraic*, if it extends to a bialgebraic counit  $\epsilon_{tot}$  for  $\mathcal{B}$ . If such an  $\epsilon_{tot}$  exists, we will call  $\check{\mathcal{O}}$  bialgebraic.

**2.6. 2nd Intermezzo: sections, pointed and almost connected.** We would like to produce Hopf algebras, by showing that appropriate bialgebras are connected. For this one actually needs distinguished elements, which will be called  $|$  or sections, see Appendix B. Even if these exist, the bialgebra is usually not connected, since the powers  $|^p$  keep it from being so. However, taking a quotient remedies the situation up to a possible problem in the coalgebra  $\check{\mathcal{O}}(1)$ . We now set the stage and do the construction in the next section.

We will also give further necessary conditions for the existence of bialgebraic counits in the pointed case.

**Definition 2.42.** A cooperad  $\check{\mathcal{O}}$  with a right operadic counit  $\epsilon_1$  is called *pointed* if the  $\epsilon_1$  is split, i.e. there is a section  $\eta_1 : \mathbb{1} \rightarrow \check{\mathcal{O}}(1)$  of  $\epsilon_1$ .

We call  $\check{\mathcal{O}}$  *reduced* if it is pointed and  $\eta_1$  is an isomorphism  $\mathbb{1} \simeq \check{\mathcal{O}}(1)$ ; it is then automatically pointed. A bialgebra unit will be called pointed if the associated right cooperadic unit  $\epsilon_1$  is pointed.

We will denote  $| := \eta_1(1)$ . Here strictly speaking  $1 = id_{\mathbb{1}}$ . For pointed cooperads Lemma 2.29 applies and we split each  $\check{\mathcal{O}}(n) = \mathbb{1} \oplus \check{\check{\mathcal{O}}}(n)$  where  $\check{\check{\mathcal{O}}}(n) = \ker(\epsilon_n) = \ker(\epsilon_{tot}|_{\check{\mathcal{O}}(n)})$  and  $\mathbb{1}$  is the component of  $|^n$ . We set  $\bar{B} = \bigoplus \check{\check{\mathcal{O}}}(n)$ .

Notice that this is smaller than the augmentation ideal  $\mathcal{B}^{red} = \ker(\epsilon_{tot})$ . For a pointed cooperad we let  $\check{\mathcal{O}}^{red}(n) = \check{\check{\mathcal{O}}}(n)$  for  $n > 1$  and  $\check{\mathcal{O}}^{red}(1)$  is given by the splitting  $\check{\mathcal{O}}(1) \simeq \mathbb{1} \oplus \check{\mathcal{O}}^{red}(1)$  defined by  $\epsilon_1$  and  $\eta_1$ . We also let  $\mathcal{B}^{red} = \bigoplus \check{\mathcal{O}}^{red}(k)$ .

**Example 2.43.** Any cooperad with multiplication  $\check{\mathcal{O}}^{nc}$  that is the free construction of dual  $\check{\mathcal{O}}$  of a unital operad  $\mathcal{O}$  is pointed if the unit morphism  $u : \mathbb{1} \rightarrow \mathcal{O}(1)$  split via a morphism  $c$ . We call such an operad *split unital*. In the notation above  $\check{u} = \epsilon_1$  and  $\check{c} = \eta_1$ . The element  $|$  is then the dual element to the unit  $u(1) \in \mathcal{O}(1)$ . Here  $| = \check{c}(1) = \eta_1(1)$  and being the dual element means that  $\check{u}(|) = \epsilon_1 \circ \eta_1(1) = (c \circ u)^\vee(1) = 1$ .

Again all of the examples of §2.1.3 have this property.

**Lemma 2.44.** *If  $\mathcal{B}$  has a split bialgebraic counit, then have  $\Delta(|) = | \otimes | + \bar{\Delta}(|)$  with  $\bar{\Delta}(|) \in \check{\mathcal{O}}(1) \otimes \check{\mathcal{O}}(1)$  and hence  $\Delta(|^p) = |^p \otimes |^p +$  terms of lower order in  $|$ . Thus the image of  $|^p$  is not 0 in  $Gr^p$  and we can split  $Gr^p = \mathbb{1} \oplus \overline{Gr^p}$  where  $\mathbb{1}$  is the component if the image of  $|^p$ .*

*Proof.* The first statement follows since  $\epsilon_{tot}$  is a bialgebraic unit. The second statement follows, from the bialgebra compatibility condition.  $\square$

More generally,

**Proposition 2.45.** *Let  $\check{\mathcal{O}}$  be a cooperad with a pointed bialgebraic counit on  $\mathcal{B}$ , then*

$$\begin{aligned} \Delta(|) &= | \otimes | + \bar{\Delta}(|) \text{ with} \\ \bar{\Delta}(|) &\in \check{\mathcal{O}}(1) \otimes \check{\mathcal{O}}(1) \end{aligned} \tag{2.24}$$

$$\begin{aligned} \Delta(|^p) &= |^p \otimes |^p + \bar{\Delta}(|^p) \text{ with} \\ \bar{\Delta}(|^p) &\in \check{\mathcal{O}}(p) \otimes \check{\mathcal{O}}(p) \end{aligned} \tag{2.25}$$

And for  $a \in \check{\mathcal{O}}(n) \cap F^p$

$$\begin{aligned} \Delta(a) &= \sum_{k \geq p}^n |^k \otimes a_k + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ a_k &\in \check{\mathcal{O}}(n), \bar{\Delta}(a) \in \check{\mathcal{B}} \otimes \check{\mathcal{O}}(n) \end{aligned} \quad (2.26)$$

with  $a = \sum_{k \geq p}^n a_k$  and the  $a_k$  are as in Corollary 2.36.

Likewise, in the associated graded case, for  $a \in \check{\mathcal{O}}(n, p)$

$$\begin{aligned} \Delta(a) &= |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ \bar{\Delta}(a) &\in \check{G}r \otimes \check{G}r \end{aligned} \quad (2.27)$$

Again, if these equations hold having a bialgebraic counit  $\epsilon_{tot}$  is equivalent to  $\epsilon_1$  being a right cooperadic counit.

*Proof.* Using Corollary 2.36 and applying  $\epsilon_{tot}$  on the left, we obtain the first term and applying  $\epsilon_{tot}$  on the right, the second term. These are different if  $a \neq |^k$  for some  $k$ . In the case  $a = |^k$  the equation follows from the Lemma above. In general, the remaining terms lie in the reduced space. Replacing  $\mathcal{B}$  with  $Gr$  proves the rest.  $\square$

We also get a practical criterion for a bialgebra counit.

**Corollary 2.46.** *Assume the equations in Propositions 2.45 hold, then having a bialgebraic counit  $\epsilon_{tot}$  is equivalent to  $\epsilon_1$  being a right cooperadic counit.*

*Proof.* By Lemma 2.29, we see that  $\epsilon_k$  is the projection to the factor  $|^k$  of  $\check{\mathcal{O}}(k) = \mathbb{1} \oplus \check{\mathcal{O}}(k)$  and on that factor it is  $\epsilon_1^k \circ \mu^{k-1}$  and hence determined by  $\epsilon_1$ . Now the second term of (2.26) is equivalent to  $\epsilon_{tot}$  being a right bialgebra counit. Furthermore, since this is the term relevant for the right cooperad counit, we obtain the equivalence for the right bialgebra counit. Similarly, applying the given  $\epsilon_{tot}$  as a potential left bialgebra counit, we see that having a left bialgebra counit is equivalent to  $a = \sum_k a_k$ , i.e. the first term in (2.26).  $\square$

**2.7. The final act: Hopf Structure.** In this section, unless otherwise stated, we will assume that  $\check{\mathcal{O}}$  is a multiplicative cooperad with bialgebraic counit.

**Assumption 2.47.** We also assume that tensor and kernels commute. Under this assumption the notions of conilpotent and connected are equivalent.

For example this is the case if we are working in  $k$ -Vect.

**Definition 2.48.** We call a pointed multiplicative cooperad  $\check{\mathcal{O}}$  with bialgebraic counit  $\epsilon_{tot}$  *almost connected* if

- (1) The element  $|$  is group-like:  $\Delta(|) = | \otimes |$
- (2)  $(\check{\mathcal{O}}(1), \eta_1, \epsilon_1)$  is connected as a coalgebra in the sense of Quillen [Qui67] (see Appendix B).

Notice that a reduced  $\check{\mathcal{O}}$  is automatically almost connected, but this is not a necessary condition.

**Lemma 2.49.** *Let  $\check{\mathcal{O}}^{nc}$  be the free construction on the dual  $\check{\mathcal{O}}$  of a split unital operad  $\mathcal{O}$ . Then  $\check{\mathcal{O}}$  is almost reduced if any element  $a \in \mathcal{O}(1)$  is only represented by finite reduced words, that is any decomposition  $a = \prod_{i \in I} a_i$  with all  $\eta_1(a_i) = 0$ ,  $I$  is finite (or empty).*

*Proof.* Recall that the coproduct is dual to multiplication in the monoid, that is, it is decomposition. Being conilpotent then is just equivalent to the given finiteness condition.  $\square$

**Example 2.50.** If the unital operad  $\mathcal{O}$  is reduced, that is  $\mathcal{O}(1) \simeq \mathbb{1}$  it is split, the so is its dual pointed and the free construction on it is almost reduced. This is the case for the surjection and the simplex operads.

But moreover, if a split unital  $\mathcal{O}$  is such that  $\mathcal{O}(1)$  is free of finite rank as a unital monoid, then  $\check{\mathcal{O}}^{nc}$  is almost reduced. This is the case for the operad of trees. It is free of rank 1 with the generator being the rooted corolla with one tail. This linked to the considerations of [Moe01] and those of higher rank to [vdLM06a].

If  $\check{\mathcal{O}}(1)$  contains group-like elements except for the unit, it is not almost reduced. If  $\mathcal{O}(1)$  contains any isomorphisms except for the unit, then  $\check{\mathcal{O}}^{nc}$  is not almost reduced. More precisely, if  $\mathcal{O}(1)$  splits as  $\mathbb{1} \oplus \check{\mathcal{O}}(1)$ , then  $\check{\mathcal{O}}(1)$  may not contain any invertible elements. Indeed, if  $a$  is such an isomorphism it has representatives of infinite length.

**Remark 2.51.** Notice that for an almost connected  $\check{\mathcal{O}}$  the bialgebra  $\mathcal{B}'$  is not connected, since all powers  $|^k$  are group like:  $\Delta(|^k) = |^k \otimes |^k$ ,  $\epsilon_{tot}(|^k) = 1$ .

For a pointed  $\check{\mathcal{O}}$ , let  $\mathcal{I}$  be the two-sided ideal spanned by  $1 - |$ . Set

$$\mathcal{H} := \mathcal{B}' / \mathcal{I} \tag{2.28}$$

Notice that in  $\mathcal{H}$  we have that  $|^k \equiv 1 \pmod{\mathcal{I}}$  for all  $k$ .

**Proposition 2.52.** *If  $\check{\mathcal{O}}$  is connected, then  $\mathcal{I}$  is a coideal and hence  $\mathcal{H}$  is a coalgebra. The unit  $\eta$  descends to a unit  $\bar{\eta} : \mathbb{1} \rightarrow \mathcal{H}$  and the counit  $\epsilon_{tot}$  factors as  $\bar{\epsilon}$  to make  $\mathcal{H}$  into a bialgebra.*

*Proof.*  $\Delta(1 - |) = 1 \otimes 1 - | \otimes | = (1 - |) \otimes (1 + |) \in \mathcal{I} \otimes \mathcal{B}$  and  $\epsilon_{tot}(1 - |) = 1 - 1 = 0$ .  $\square$

**Theorem 2.53.** *If  $\check{\mathcal{O}}$  is almost reduced then  $\mathcal{H}$  is conilpotent and hence admits a unique structure of Hopf algebra.*

*Proof.* Let  $\pi = id - \bar{\epsilon} \circ \bar{\eta}$  be the projection  $\mathcal{H} = \mathbb{1} \oplus \bar{\mathcal{H}} \rightarrow \bar{\mathcal{H}}$  to the augmentation ideal. We have to show that each element lies in the kernel of some  $\pi^{\otimes m} \circ \Delta^m$ . For  $\mathbb{1}$  this is clear, for the image of  $\check{\mathcal{O}}(1)$  this follows from the assumptions, from the Lemma above and the identification of  $1$  and  $|$  in the quotient. Now we proceed by induction on  $n$ , namely, for  $a \in \check{\mathcal{O}}(n)$ , we have that  $\Delta(a) \in \bigoplus_{k,n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n)$ . Since the coproduct is coassociative, we see that all summands with  $k < n$  are taken care of by the induction assumption. This leaves the summands with  $k = n$ . Then the right hand side of the tensor product is the product of elements which are all in  $\check{\mathcal{O}}(1)$ . Since  $\Delta$  is compatible with the multiplication, we are done by the assumption on  $\check{\mathcal{O}}(1)$  and coassociativity.  $\square$

**2.8. The Hopf algebra as a deformation.** Rather than taking the approach above, we can produce the Hopf algebra in two different steps. Without adding a unit, we will first mod out by the two-sided ideal  $\mathcal{C}$  generated by  $|a - a|$ . This forces  $|$  to lie in the centre. We denote the result by  $\mathcal{H}_q := \mathcal{B}/\mathcal{C}$ . Here the image of  $|$  under this quotient is denoted by  $q$ . This allows us to view  $q$  as a deformation parameter and view the quotient as the classical limit  $q \rightarrow 1$ . In this section we assume that  $\check{\mathcal{O}}$  is pointed.

**Proposition 2.54.**  $\mathcal{C}$  is a coideal and hence  $\mathcal{H}_q$  with the induced unit and counit is a bialgebra.

*Proof.*

$\Delta(|a - a|) = |a^{(1)} \otimes |a^{(2)} - a^{(1)}| \otimes a^{(2)}| = |a^{(1)} - a^{(1)}| \otimes |a^{(2)} - a^{(1)}| \otimes |a^{(2)} - a^{(2)}| \subset \mathcal{C} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{C}$  using Sweedler notation. Furthermore  $\epsilon(|a - a|) = \epsilon(a) - \epsilon(a) = 0$ .  $\square$

**Remark 2.55.** If  $\rho_|$  and  $\lambda_|$  are right and left multiplication by  $|$ , then  $\mathcal{C}$  is also a coequalizer in the sequence

$$\mathcal{B} \begin{array}{c} \xrightarrow{\rho_|} \\ \xrightarrow{\lambda_|} \end{array} \mathcal{B} \longrightarrow \mathcal{H} \quad (2.29)$$

Notice that the image of  $|^n$  is  $q^n$  and if we give  $q$  the degree 1, then the grading by operadic degree is preserved as well as the depth filtration and all other filtrations and gradings mentioned above.

By moving all the  $q$ 's to the left the elements in  $\mathcal{H}_q$  can be thought of as polynomials in  $q$  whose coefficients lie in  $\bar{\mathcal{B}}$ , i.e.  $\mathcal{H}_q \subset \bar{\mathcal{B}}[q]$ . The degree of a polynomial is the operadic degree plus the degree of  $q$ .

**Proposition 2.56.**  $\mathcal{H}_q$  is a deformation of  $\mathcal{H}$  given by  $q \rightarrow 1$ .  $\square$

**2.8.1. The  $|$  filtration on  $\mathcal{B}$ .** Let  $\mathcal{J}$  be the two-sided ideal of  $\mathcal{B}$  spanned by  $|$ . Then there is an exhaustive filtration of  $\mathcal{B}$  by the powers of  $\mathcal{J}$ . This filtration survives the quotient by  $\mathcal{C}$  and gives a filtration in powers of  $q$ . Here we can then also view the filtration as a deformation over a formal disc, with the central fiber being the associated graded.

**Example 2.57.** For the free construction  $\check{\mathcal{O}}^{nc}$  we have that as an algebra:

$$\mathcal{H}_q = \bigoplus_d \bigoplus_{n \leq d} q^{n-d} \check{\mathcal{O}}^{nc, red}(n) \simeq T\check{\mathcal{O}}^{red}[q] \quad (2.30)$$

where  $\check{\mathcal{O}}^{nc, red}(n) = \bigoplus_k \bigoplus_{(n_1, \dots, n_k): \sum n_i = n} \check{\mathcal{O}}^{red}(n_1) \otimes \dots \otimes \check{\mathcal{O}}^{red}(n_k)$  and  $\check{\mathcal{O}}^{red} = \ker(\epsilon_1)$ . This is so, since the terms with  $|$  only arise from products with  $\mathbb{1} \subset \check{\mathcal{O}}(1)$ .

The associated graded with respect to  $\mathcal{J}$  is isomorphic to  $\mathcal{H}_q$ .

**2.9. Infinitesimal version.** The filtration above also allows us to obtain the infinitesimal version of the Hopf algebra. This involves the use of pseudo-cooperads that we briefly review.

2.9.1. **Pseudo-cooperads  $\check{\circ}_i$ .** A right cooperadic counit allows one to write the dual operations to the  $\circ_i$ :

$$\check{\circ}_i(a) = (id \otimes \epsilon \otimes \cdots \otimes \epsilon \otimes id \otimes \epsilon \otimes \cdots \otimes \epsilon) \circ (\check{\gamma}(a)) \text{ with } id \text{ in the } i+1\text{-st place.} \quad (2.31)$$

here we again implicitly use the structural isomorphism for the unit.

Dualizing the picture above, a pseudo-cooperad is a collection  $\check{\mathcal{O}}(n)$  of Abelian groups (or objects of a symmetric monoidal category in general) each with an  $\mathbb{S}_n$  action together with structure maps

$$\check{\circ}_i : \check{\mathcal{O}}(k) \rightarrow \bigoplus_{n=i}^k \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(k-n+1) \text{ for } 1 \leq i \leq k \quad (2.32)$$

2.9.2. **Copre-Lie  $\check{\circ}$ .** Summing over all the  $\check{\circ}_i$  we get a map

$$\check{\circ} : \check{\mathcal{O}}(k) \rightarrow \bigoplus_{n=1}^k \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(k-n+1) \quad (2.33)$$

**Lemma 2.58.** *Let  $\check{\mathcal{O}}$  be an cooperad with multiplication and bialgebraic multiplicative right counit, then in  $\mathcal{B}$ :  $\Delta(|^n) = |^n \otimes |^n$  and if  $a \in \check{\mathcal{O}}(n) \cap F^p$  and  $\epsilon_{tot}(a) = 0$ , then*

$$\begin{aligned} \Delta(a) &= |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ \bar{\Delta}(a) &= \sum_{i=1}^n \sum_{k=p}^n a_k^{(i,1)} \otimes |^i a_{n-k+1}^{(i,2)} |^{k-i-1} + \sum_{k=p}^n \mathcal{B}^{red} \otimes \mathcal{J}^{<k-1} \text{ and setting} \\ \check{\circ}_i(a) &= \sum_{k=p}^n a_k^{(i,1)} \otimes a_{n-k+1}^{(i,2)} \end{aligned} \quad (2.34)$$

defines a pseudo-cooperad structure.

*Proof.* The first statement follows from the bialgebra structure. The second statement follows from the fact that  $\epsilon$  is a left and right counit. In general one can count the factors of  $|$  that may occur in  $\Delta_k$ . Applying formula (2.31) then gives the last statement.  $\square$

### 2.9.3. A type of bialgebra from cooperads with multiplication.

**Definition 2.59.** A pseudo-cooperad with multiplication  $\mu$  is a pseudo-cooperad  $\check{\mathcal{O}}$  with a family of maps,  $n, m \geq 0$ ,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \rightarrow \check{\mathcal{O}}(n+m)$$

which together with the comultiplication  $\delta := \check{\circ}$  satisfies the equation

$$\delta \circ \mu = (\text{id} \otimes \mu)(\delta \otimes \text{id}) + (\mu \otimes \text{id})(\text{id} \otimes \delta) \quad (2.35)$$

**Remark 2.60.** Although equation (2.35) is the same equation as that for an infinitesimal bialgebra our  $\delta$  is not coassociative; just like  $\circ$  is not associative, but only pre-Lie. What we do have is what one could call a co-pre-Lie bialgebra.



**Proposition 2.61.** *If  $\check{\mathcal{O}}$  is a non- $\Sigma$  cooperad with multiplication and multiplicative right co-operadic counit. Then the multiplication is also compatible with the non- $\Sigma$  pseudo-cooperad structure.*

*Proof.* Straightforward.  $\square$

**Remark 2.62.** In the example of Connes and Kreimer, this corresponds to making a single cut. In simplicial terms, the dual defines the  $\cup_1$  product. See also §5.2.2.

**2.9.4. Infinitesimal part as  $q^k$  part.** We have seen above that we can get the infinitesimal version. We show that we can recover this it through the following construction giving credence to the name infinitesimal.

From equation 2.34 we see that the degree  $q^{k-1}$  part of  $\Delta_k$  gives the degree  $k$  part of the cooperad structure.

**2.9.5. Infinitesimal version in the Hopf setting.** After passing to the Hopf quotient, the factors of  $|$  are identified with 1 and hence

**Proposition 2.63.** *Let  $\check{\mathcal{O}}$  be a counital non- $\Sigma$  cooperad with multiplication and bialgebraic multiplicative counit. Then the copre-Lie structure*

In the case of Brown [Bro12a] this give the operators  $D_r$  determining the coaction. In general, it is easy to see that

**Proposition 2.64.** *If  $\mathcal{B}$  is almost connected, then in the Hopf quotient, the co-preLie structure induces a co-Lie algebra structure on the indecomposables  $\mathcal{H}_>/\mathcal{H}_>\mathcal{H}_>$ , where  $\mathcal{H}_>$  is reduced version of  $\mathcal{H}$ .*  $\square$

**Example 2.65.** In the free case  $\check{\mathcal{O}}^{nc}$ , the indecomposables are precisely given by  $\check{\mathcal{O}}$  and the co-pre-Lie structure is  $\check{\circ}$ . Moreover if  $\check{\mathcal{O}}$  is the dual of  $\mathcal{O}$  then furthermore the co-Lie structure corresponds to the usual Lie structure of Gerstenhaber.

**2.10. Coinvariants: commutative version.** We now assume that the cooperad is symmetric. To pass to invariants, it will be convenient to pass first to operads in arbitrary sets, see e.g. [MSS02]. This means that for any finite set  $S$  we have an  $\mathcal{O}(S)$  and any isomorphism  $\sigma : S \rightarrow S'$  an isomorphism  $\mathcal{O}(S) \rightarrow \mathcal{O}(S')$ . The composition is then defined for any map  $f : S \rightarrow T$  as a morphism  $\mathcal{O}(S) \otimes \bigotimes_T \mathcal{O}(f^{-1}(t)) \rightarrow \mathcal{O}(S)$  which is equivariant for any diagram of the form

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \sigma' \downarrow & & \downarrow \sigma \\ S' & \xrightarrow{f'} & T' \end{array} \quad (2.36)$$

Notice that if we are only given the  $\check{\mathcal{O}}(n)$  then the extension to finite sets is given by  $\check{\mathcal{O}}(S) := \text{colim}_{f:S \rightarrow \bar{n}} \check{\mathcal{O}}(n)$ . This actually yields an equivalence of categories.

With  $|S| = |S'| = k$ ,  $n_t = |f^{-1}(t)| = |f'^{-1}(\sigma(t))|$ ,  $\sigma'_t : f^{-1}(t) \rightarrow (f')^{-1}(\sigma(t))$  the restriction: the outer square of the diagram below commutes.

$$\begin{array}{ccc}
\mathcal{O}(T) \otimes \bigotimes_T \mathcal{O}(f^{-1}(t)) & \xrightarrow{\gamma_f} & \mathcal{O}(S) \\
\downarrow \sigma \otimes \bigotimes_T \sigma'_t & \swarrow & \downarrow \sigma' \\
\mathcal{O}(k)^{\mathbb{S}_k} \otimes \text{Symm}(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)}) & \xrightarrow{\tilde{\gamma}} & \mathcal{O}(n)^{\mathbb{S}_n} \\
\downarrow & \swarrow & \downarrow \\
\mathcal{O}(T') \otimes \bigotimes_{T'} \mathcal{O}(f^{-1}(t')) & \xrightarrow{\gamma_{f'}} & \mathcal{O}(S')
\end{array}$$

for the other morphisms, let  $\text{Iso}(n, k)$  be the category with objects the surjections  $S \rightarrow T$  with  $|S| = n$  and  $|T| = k$  and morphisms the commutative diagrams of the type (2.36) with  $\sigma, \sigma'$  bijections and  $f, f'$  surjections, and  $\text{Iso}(n)$  the category with objects  $S$ , with  $|S| = n$  and bijections. Then

- (1)  $\text{Symm}$  is the subspace of symmetric tensors,
- (2)  $\lim_{\text{Iso}(n, k)} \mathcal{O} = \mathcal{O}(k)^{\mathbb{S}_k} \otimes \text{Symm}(\bigotimes_{i=1}^k \mathcal{O}(n_i)^{\mathbb{S}(n_i)})$ ,
- (3)  $\lim_{\text{Iso}(n)} \mathcal{O} = \mathcal{O}(n)^{\mathbb{S}_n}$ ,
- (4) and the middle map exists by the universal property of limits.

**Remark 2.66.** These are exactly universal operations in the sense of [KW13]. In order to establish this, we recall that any operad under the equivalence established in [KW13][Example 4.12] can be thought of either an enrichment of the Feynman category of sets and surjections or as a functor from the Feynman category for operads to a target category. As the latter, we obtain universal operations through colimits, see paragraph §?? of [KW13]. On the other hand via the construction in paragraph §4 below.

Dualizing these diagrams we obtain the diagrams

$$\begin{array}{ccc}
\check{\mathcal{O}}(S) & \xrightarrow{\tilde{\gamma}_f} & \check{\mathcal{O}}(T) \otimes \bigotimes_T \check{\mathcal{O}}(f^{-1}(t)) \\
\downarrow \sigma' & \swarrow & \downarrow \sigma \otimes \bigotimes_T \sigma'_t \\
\check{\mathcal{O}}(n)_{\mathbb{S}_n} & \xrightarrow{\tilde{\gamma}} & \check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \check{\mathcal{O}}(n_k)_{\mathbb{S}(k)} \\
\downarrow & \swarrow & \downarrow \\
\check{\mathcal{O}}(S') & \xrightarrow{\gamma} & \check{\mathcal{O}}(S') \otimes \bigotimes_{T'} \check{\mathcal{O}}(f')^{-1}(t')
\end{array}$$

- (1)  $\bigodot$  denotes the symmetric tensor product
- (2)  $\text{colim}_{\text{Iso}(n, k)} \mathcal{O} = \mathcal{O}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \mathcal{O}(n_k)_{\mathbb{S}(k)}$
- (3)  $\text{colim}_{\text{Iso}(n)} \mathcal{O} = \mathcal{O}(n)_{\mathbb{S}_n}$
- (4) and the middle map exists by the universal property of colimits.

**Definition 2.67.** A cooperad with multiplication in finite sets, is a cooperad in finite sets with multiplications  $\mu_{S, T} : \check{\mathcal{O}}(S) \otimes \check{\mathcal{O}}(T) \rightarrow \check{\mathcal{O}}(S \sqcup T)$ , such that the following diagram

commutes.

$$\begin{array}{ccc} \check{O}(S) \otimes \check{O}(T) & \xrightarrow{\mu_{S,T}} & \check{O}(S \sqcup T) \\ \sigma \otimes \sigma' \downarrow & & \downarrow \sigma \sqcup \sigma' \\ \check{O}(S') \otimes \check{O}(T') & \xrightarrow{\mu_{S',T'}} & \check{O}(S' \sqcup T') \end{array} \quad (2.37)$$

and the analogue of (2.13) holds equivariantly.

**Lemma 2.68.** *For a cooperad with multiplication in finite sets the cooperad structure and the multiplication descend to the coinvariants.*  $\square$

Set  $\mathcal{B}_{\mathbb{S}} = \bigoplus_n \check{O}(n)_{\mathbb{S}_n}$ . A bialgebraic counit is called invariant if for all  $a_S \in \check{O}(S)$  and any isomorphism  $\sigma : S \rightarrow S'$ ,  $\epsilon \circ \sigma = \epsilon$ .

**Proposition 2.69.** *With the assumption above,  $\mathcal{B}_{\mathbb{S}}$  is a non-unital, non-counital, bialgebra. If we furthermore assume that an invariant bialgebraic counit for  $\mathcal{B}$  exists then  $\mathcal{B}'_{\mathbb{S}} = k \oplus \mathcal{B}_{\mathbb{S}}$  is a unital and counital bialgebra and  $\mathcal{H} := \mathcal{B}/\bar{\mathcal{I}}$ , where  $\bar{\mathcal{I}}$  is the image of  $\mathcal{I}$  in  $\mathcal{B}'_{\mathbb{S}}$  is a connected commutative Hopf algebra*

2.10.1. **The free example.** In the free example, starting with a symmetric operad, we do not only have to take the sum, but also induce the representation to  $\mathbb{S}_n$  in order to obtain a symmetric cooperad with multiplication. Let

$$\check{O}^{symnc}(n) = \bigoplus_{n,k} \text{Ind}_{(\mathbb{S}(n_1) \times \dots \times \mathbb{S}(n_k)) \wr \mathbb{S}(k)}^{\mathbb{S}_n} \bigoplus_{(n_1, \dots, n_k) : \sum_i n_i = n} \check{O}(n_1) \otimes \dots \otimes \check{O}(n_k) \quad (2.38)$$

**Remark 2.70.** When taking coinvariants, this induction step is cancelled and we only have to take coinvariants with respect to  $\mathbb{S}(n_1) \times \dots \times \mathbb{S}(n_k) \times \mathbb{S}(k)$ . That is  $\mathcal{B} = \bigoplus \check{O}^{symnc}(n)_{\mathbb{S}_n}$

**Proposition 2.71.** *The  $\check{O}^{symnc}(n)$  form a symmetric cooperad with multiplication and  $\mathcal{B} = \bigoplus \check{O}^{symnc}(n)_{\mathbb{S}_n}$  forms a bialgebra, and if  $\check{O}$  has an operadic counit, then  $\mathcal{B}'$  is a unital an non-unital bialgebra. Furthermore if  $\check{O}(1)$  is equivariantly almost connected, then the quotient  $\mathcal{B}'/I$  is a Hopf algebra.*

*Proof.* It is clear that the free multiplication then also satisfies (2.37) and the equivariant version (2.13) holds. A counit for a symmetric cooperad is by definition a morphism  $\check{O}(\{s\}) \rightarrow k$  that is invariant under isomorphism, hence so is its extension. The rest of the statements are proved analogously to the non-symmetric case.  $\square$

**Remark 2.72.** Since now  $\mathcal{H}$  is commutative its dual  $\mathcal{H}^* = U(\text{Prim}(\mathcal{H}^*))$  by the Cartier–Milnor–Moore theorem. We leave this analysis for further study.

**Example 2.73.** We now look at the examples we have treated before

- (1) For the ordered surjections, we get all the surjections, since the permutation action induces any order. These are pictorially represented by forests of nonplanar corollas. Taking coinvariants makes these forrests unlabelled.

- (2) For the leaf labelled trees, the planar trees become non-planar. Taking coinvariants kills the labelling of the leaves. The coproduct is then given by cutting edges with admissible cuts as described originally by Connes and Kreimer.
- (3) This carries on to the graded case like in Baues.

2.11. **Connes–Kreimer quotient.** To obtain the Hopf algebra of Connes and Kreimer on the nose, we have to take one more quotient and make one more assumption.

**Definition 2.74.** A (non- $\Sigma$ ) cooperad has a clipping or amputation structure, if it has a cosemisimplicial structure compatible with the operad structure. That is

- (1) there are maps  $\sigma_i : \check{\mathcal{O}}(n) \rightarrow \check{\mathcal{O}}(n-1)$ , and for  $i \leq j : \sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$
- (2) For all  $n$ , and partition  $(n_1, \dots, n_k)$  of  $n$  and each  $i \in n$ ,  $\check{\gamma}_{n_1, \dots, n_j}(a) = id \otimes id \otimes \dots \otimes \sigma_{i_j} \otimes id \otimes \dots \otimes id$ , where the factor is in the  $j$ -th position. Here  $j$  is the index of the block  $n_j$  of the partition in which  $i$  lives and  $i_j$  is the position within this block.

It follows that  $(id \otimes \sigma_i) \circ \Delta = \Delta \circ \sigma_i$ . For a symmetric cooperad, we also demand compatibility with the permutation group actions.

In this case, we can take the colimit over the directed system given by the  $\sigma_i$ . Let  $\mathcal{B}^{amp} = colim_{\sigma} \mathcal{O}(n)$  and  $\mathcal{H}^{amp}$  resp.  $\bar{\mathcal{H}}^{amp}$  be the respective quotients by the image of  $\mathcal{I}$ . The following is then straightforward.

**Proposition 2.75.**  $\mathcal{H}^{amp}$  is a Hopf algebra and  $\bar{\mathcal{H}}^{amp}$  is a commutative Hopf algebra.

**Example 2.76.** We start with the operad of leaf-labelled trees and consider its dual. The operation  $\sigma_i$  is to forget the  $i$ -th flag. It is clear that the cooperad structure and the coproduct commutes with this operation. Indeed the coproduct and cooperad structure only cut internal edges. When considering the colimit, the representatives are trees without flags, sometimes called amputated trees [Kre99, BBM13]. The coproduct is exactly that of Connes and Kreimer, both in the commutative and noncommutative case. For the latter one considers planar trees. These need not have labels on the flags, since they come in a fixed order.

2.11.1. **Grading.** If we consider the colimit, then the usual grading will not prevail. Instead we can use the grading by the associated coradical degree. We recall that the coradical filtration  $\mathbb{R}$  is compatible with multiplication and comultiplication.

**Definition 2.77.** We call the coradical filtration of  $\mathcal{H}$  well behaved, if  $\Delta_k(\mathbb{R}^i) \subset \bigoplus \bigoplus_{p+q=i-k+1} \mathbb{R}^p \otimes \mathbb{R}^q$ . This is for  $\mathcal{H}, \bar{\mathcal{H}}, \mathcal{H}^{amp}$  and  $\bar{\mathcal{H}}^{amp}$ .

If the coradical filtration is well behaved, then we define the total degree for  $\mathcal{H}^{amp}$  and  $\bar{\mathcal{H}}^{amp}$  to be the sum of the of the coradical degree and the depth minus one. We then have:

**Proposition 2.78.**  $Gr_F Gr_{\mathbb{R}} \mathcal{H}^{amp}$  and  $Gr_F Gr_{\mathbb{R}} \bar{\mathcal{H}}^{amp}$  are graded Hopf algebras.

### 3. COOPERADS FROM SIMPLICIAL OBJECTS

In this section we present an important (but accessible) construction of some cooperads with multiplication. This construction is best expressed in the language of simplicial objects, and so we will first recall some of the basic notions. Some of the examples, however, can be understood with no simplicial background. For an arbitrary set  $S$ , we will see that the set  $X$  of all sequences or words in  $S$  has the structure of a cooperad, and Goncharov's Hopf algebra may be obtained from the case  $S = \{0, 1\}$ . Elements of  $X$  are of course strings of edges in the complete graph (with vertex loops)  $K_S$ , and further geometric intuition can be obtained by considering also strings of triangles or general  $n$ -simplices. In fact our construction defines a cooperad with multiplication, and hence a bialgebra (or Hopf algebra) for  $X$  any (reduced) simplicial set, see Proposition 3.8. In this guise we also recover the Hopf algebra of Baues.

**3.1. Recollections: the simplicial category and simplicial objects.** Let  $\Delta$  be the small category whose objects are the finite non-empty ordinals  $[n] = \{0 < 1 < \dots < n\}$  and whose morphisms are the order-preserving functions between them. Of course, each  $[n]$  can itself be regarded as a small category, with objects  $0, 1, \dots, n$  and a (unique) arrow  $i \rightarrow j$  iff  $i \leq j$ , and order preserving functions are just functors. Thus  $\Delta$  is a full subcategory of the category of small categories.

Among the order-preserving functions  $[m] \rightarrow [n]$  one considers the following generators: the injections  $\partial^i : [n-1] \rightarrow [n]$  which omit the value  $i$ , termed coface maps, and the surjections  $\sigma^i : [n+1] \rightarrow [n]$  which repeat the value  $i$ , termed codegeneracy maps. These maps satisfy certain obvious cosimplicial relations.

For  $D$  a small category, and  $\mathcal{C}$  any category, we can consider the contravariant functors or the covariant functors  $X$  from  $D$  to  $\mathcal{C}$ . For  $D = \Delta$  these are termed the simplicial and the cosimplicial objects in  $\mathcal{C}$ . A functor  $D^{op} \rightarrow \mathcal{C}$  is *representable* if it is  $\text{hom}_D(-, d)$  for some object  $d$ . The Yoneda Lemma gives a bijection between the set of natural transformations  $\text{hom}_D(-, d) \rightarrow X$  and the set  $X(d)$ , and in particular  $d \mapsto \text{hom}_D(-, d)$  defines a full embedding of  $D$  into the functor category  $\mathcal{C}^{D^{op}}$ .

Now a simplicial object is determined by the sequence of objects  $X_n$ , and the face and degeneracy maps  $d_i : X_n \rightarrow X_{n-1}$  and  $s_i : X_n \rightarrow X_{n+1}$ , given by the images of  $[n]$ , and  $\partial^i$  and  $\sigma^i$ , and dually for cosimplicial objects. Maps  $X \rightarrow Y$  of (co)simplicial objects, that is, natural transformations, are just families of maps  $X_n \rightarrow Y_n$  that commute with the (co)face and (co)degeneracy maps.

We write  $\Delta[n]$  for the representable simplicial set  $\text{hom}_\Delta(-, [n])$  so, by Yoneda, simplicial maps  $\Delta[n] \rightarrow X$  are just elements of  $X_n$  and maps  $\Delta[m] \rightarrow \Delta[n]$  are just order preserving maps  $[m] \rightarrow [n]$ . For such a map  $\alpha$  we use the notation  $\alpha^* = X(\alpha) : X_n \rightarrow X_m$  and

$$x_{(\alpha_0, \dots, \alpha_m)} \in X_m$$

to denote the image under  $\alpha^*$  of an  $n$ -simplex  $x$  in a simplicial set  $X$ .

The following result is central to the classical theory of simplicial sets.

**Lemma 3.1.** *Let  $D$  be a small category and  $\mathcal{C}$  a cocomplete category. Any functor  $r : D \rightarrow \mathcal{C}$  has a unique extension along the Yoneda embedding to a functor  $R : \mathbf{Set}^{D^{op}} \rightarrow \mathcal{C}$  with a right adjoint  $N$ ,*

$$\begin{array}{ccc}
 D & \xrightarrow{y} & \mathbf{Set}^{D^{op}} \\
 & \searrow r & \uparrow R \dashv N \\
 & & \mathcal{C}
 \end{array}$$

*If  $r : D \rightarrow \mathcal{C}$  is a monoidal functor between monoidal categories, then  $R$  sends monoidal functors  $D^{op} \rightarrow \mathbf{Set}$  to monoid in  $\mathcal{C}$ .*

The functor  $R$  is sometimes denoted  $(-) \otimes_D r$ , where the tensor over  $D$  is thought of as giving an object of  $D$  for every pair of  $D^{op}$ - and  $D$ -objects in  $\mathcal{C}$ , analogously to the language of tensoring left and right modules or algebras over a ring. The right adjoint  $N$  is termed the *nerve*, and is given on objects by

$$N(C) = \text{hom}_{\mathcal{C}}(r(-), C).$$

If  $D = \Delta$  and  $X$  is a simplicial set then  $R(X)$  is usually called the *realization* of a simplicial set with respect to the models  $r$ . Considering for example the embedding  $r : \Delta \rightarrow \mathbf{Cat}$  we obtain the notion of the simplicial nerve of a category: for  $C$  a small category, there is a natural bijection between the functors from  $[n]$  to  $C$  and the  $n$ -simplices of the nerve  $NC$ ,

$$N(C)_n = \text{hom}_{\mathbf{Cat}}([n], C).$$

**Example 3.2.** Let  $S$  be a set, and let  $X(S)$  be the simplicial set given by the nerve of the contractible  $G(S)$  with object set  $S$ ,

$$X(S) = NG(S).$$

If  $S = [n]$ , for example, we may identify  $G(S)$  with the fundamental groupoid of  $\Delta[n]$ , and

$$X([n]) \cong N\pi_1\Delta[n].$$

Giving a functor from  $[n]$  to the contractible groupoid  $G(S)$  is the same as giving the function on the objects, so an  $n$ -simplex of  $X(S)$  is just a sequence of  $n + 1$  elements of  $S$ ,

$$X(S)_n = S^{n+1} = \{(a_0; a_1, a_2, \dots, a_{n-1}; a_n) : a_i \in S\}.$$

In the case  $S = \{0, 1\}$ , the groupoid  $G(S)$  is



and the  $n$ -simplices of  $X$  are words of length  $n + 1$  in the alphabet  $\{0, 1\}$ .

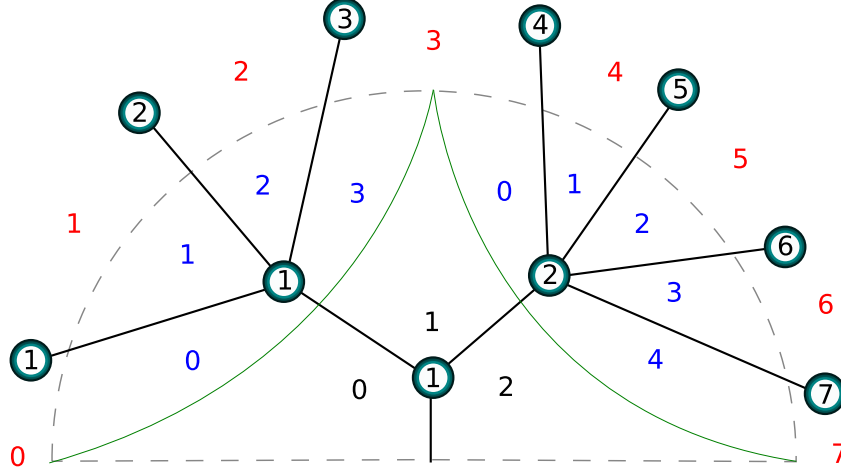


FIGURE 2. An example of a factorization  $\underline{7} \rightarrow \underline{2} \rightarrow \underline{1}$  of order preserving surjections and, reading outwards from the root to the leaves, the corresponding operad structure map  $\gamma_{3,4} : [2] \cup [3] \cup [4] \rightarrow [7]$ .

**3.2. The operad of little ordinals.** The category of small categories, and the category of simplicial sets, can be regarded as monoidal categories with the disjoint union playing the role of the tensor product, and the initial object  $\emptyset$  the neutral object. In this context, we have the following result, compare for example [DK12, Example 3.6.4].

**Proposition 3.3.** *The sequence of finite nonempty ordinals  $([n])_{n \geq 0}$  forms an operad in the category of small categories. For any partition  $n = m_1 + m_2 + \cdots + m_k$ , consider the subset  $\{0 = n_0 < n_1 < n_2 < \cdots < n_k = n\}$  of  $[n]$  given by  $n_r = m_1 + \cdots + m_r$ . Then the structure map*

$$\gamma_{m_1, \dots, m_k} = (\gamma^0, \gamma^1, \dots, \gamma^k) : [k] \cup [m_1] \cup \cdots \cup [m_k] \rightarrow [n]$$

is defined by

$$\gamma^0(i) = n_i \quad (0 \leq i \leq k) \quad \text{and} \quad \gamma^r(j) = n_r + j \quad (0 \leq j \leq m_r, 1 \leq r \leq k).$$

This operad clearly has a unit  $u : \emptyset \rightarrow [1]$ .

This construction is related, via Joyal duality (see Appendix C), to the factorisations of maps  $\underline{n} \rightarrow \underline{1}$  into order preserving surjections  $\underline{n} \rightarrow \underline{k} \rightarrow \underline{1}$ , where  $\underline{n} = \{1, \dots, n\}$ . Under the Joyal duality between end-point preserving ordered maps —see Appendix C—  $[k] \rightarrow [n]$  and ordered maps  $\underline{n} \rightarrow \underline{k}$ , the injection  $\gamma^0 : [k] \rightarrow [n]$  defined in the Proposition corresponds to the order preserving surjection  $\underline{n} \rightarrow \underline{k}$  whose fibres over each  $i$  have cardinality  $m_i$  (see Figure 2).

The image of the operad structure in Proposition 3.3 under the Yoneda embedding gives:

**Corollary 3.4.** *The collection of representable simplicial sets  $(\Delta[n])_{n \geq 0}$  forms a unital operad in the category of simplicial sets.*



If  $X$  is a simplicial set, then the unital operad structure on the sequence  $\Delta[n]$ ,  $n \geq 0$ , induces a counital cooperad structure on the sequence  $X_n = \text{hom}(\Delta[n], X)$ . That is, the sequence  $(X_n)_{n \geq 0}$  forms a counital cooperad with

$$\begin{aligned} X_n &\xrightarrow{\check{\gamma}_{m_1, \dots, m_k}} X_k \times X_{m_1} \times \dots \times X_{m_k} \\ x &\longmapsto (x_{(n_0, n_1, \dots, n_k)}, x_{(n_0, n_0+1, \dots, n_1)}, \dots, x_{(n_{k-1}, n_{k-1}+1, \dots, n_k)}) \end{aligned} \quad (3.1)$$

where  $0 = n_0 < n_1 < n_2 < \dots < n_k = n$  are given by  $n_r = m_1 + \dots + m_r$  as usual. The counit is given by the unique map

$$X_1 \rightarrow \{*\}.$$

More generally:

**Corollary 3.5.** *Let  $X$  be a simplicial object in a category  $\mathcal{C}$  with finite products. Then the sequence  $(X_n)_{n \geq 0}$  forms a counital cooperad in  $\mathcal{C}$ .*

**Example 3.6.** The set of all words in a given alphabet  $S$  is naturally a simplicial set (see Example 3.2 above) and so by Corollary 3.5 it forms a counital cooperad  $X$  in the category of sets. The elements of arity  $n$  in this cooperad are the words of length  $n + 1$  in  $S$ ,

$$X_n = S^{n+1} = \{(a_0; a_1, a_2, \dots, a_{n-1}; a_n) : a_i \in S\}$$

and the operation  $\check{\gamma}_{m_1, \dots, m_k}$  sends such an element  $(a_0; a_1, a_2, \dots, a_{n-1}; a_n)$  to

$$((a_{n_0}; a_{n_1}, \dots; a_{n_k}), (a_{n_0}; a_{n_0+1}, \dots; a_{n_1}), \dots, (a_{n_{k-1}}; a_{n_{k-1}+1}, \dots; a_{n_k}))$$

where  $n_0 = 0$ ,  $n_k = n$  and  $n_r - n_{r-1} = m_r$ .

This construction can also be carried out in an algebraic setting.

**Proposition 3.7.** *Let  $X$  be a simplicial set, and let  $\check{\mathcal{O}}(n)$  be the free abelian group on the set  $X_n$ , for each  $n \geq 0$ . Then  $\check{\mathcal{O}}$  forms a counital cooperad in the category of abelian groups, with the cooperadic structure given by*

$$\begin{aligned} \check{\mathcal{O}}(n) &\xrightarrow{\check{\gamma}} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(m_1) \otimes \dots \otimes \check{\mathcal{O}}(m_k) \\ x &\longmapsto x_{(n_0, n_1, \dots, n_k)} \otimes x_{(n_0, n_0+1, \dots, n_1)} \otimes \dots \otimes x_{(n_{k-1}, n_{k-1}+1, \dots, n_k)} \end{aligned}$$

and the counit given by the augmentation

$$\check{\mathcal{O}}(1) \longrightarrow \mathbb{Z}.$$

*Proof.* This follows by applying free abelian group functor (which carries finite cartesian products of sets to tensor products) to the cooperad structure considered in (3.1).  $\square$

From section 2.2.3 we therefore have

**Proposition 3.8.** *Let  $X$  be a simplicial set. The cooperad structure  $\check{\mathcal{O}}$  on  $(\mathbb{Z}X_n)_{n \geq 1}$  of the previous proposition extends to a structure of a cooperad with (free) multiplication, and hence to a graded bialgebra structure, on the free tensor algebra*

$$\mathcal{B}(X) = \bigoplus_n \check{\mathcal{O}}^{nc}(X)(n) = \bigoplus_{n_1, n_2, \dots \geq 1} \bigotimes_i \mathbb{Z}X_{n_i}$$

generated by  $X$ , where elements of  $\mathbb{Z}X_n$  have degree  $n - 1$ .

**3.2.1. Goncharov's first Hopf algebra.** Let  $S$  be the set  $\{0, 1\}$ . We considered in Example 3.2 the contractible groupoid  $G(S)$  with object set  $S$ , and the simplicial set  $X = X(S)$  given by its simplicial nerve. If we denote the simplices of  $X_n$  by tuples  $(a_0; a_1, \dots, a_{n-1}; a_n)$  as in Example 3.6 and apply Proposition 3.8 we obtain a graded bialgebra

$$\mathcal{B}(X) = \mathbb{Z}[(a_0; a_1, \dots, a_{n-1}; a_n); a_i \in \{0, 1\}]$$

with the coproduct that sends a polynomial generator  $(a_0; a_1, \dots, a_{n-1}; a_n)$  in degree  $n - 1$  to

$$\sum_{0=n_0 < n_1 < \dots < n_k = n} (a_{n_0}; a_{n_1}, \dots; a_{n_k}) \otimes \prod_{i=0}^{k-1} (a_{n_i}; a_{n_{i+1}}, \dots; a_{n_{i+1}})$$

When we identify all generators in degree 0 we obtain Goncharov's connected graded Hopf algebra  $\mathcal{H}_G$ , as in Theorem 1.2.

For any simplicial set  $X$ , let  $C_n(X)$  be the free abelian group on the  $n$ -simplices  $X_n$ . This defines a chain complex  $(C(X), d_X)$  where

$$d_X(x) = \sum_{i=0}^n (-1)^i d_i x.$$

Diagonal approximation makes  $CX$  a differential graded coalgebra,

$$C(X) \longrightarrow C(X \times X) \longrightarrow CX \otimes CX$$

whose classical cobar construction is the tensor algebra on the desuspension of the reduced coalgebra

$$\Omega CX = (T\Sigma^{-1}\overline{CX}, d_\Omega)$$

where the differential  $d_\Omega$  is formed from  $d_X$  and the coproduct. For the moment, however, we merely observe that if one takes the symmetric rather than the tensor algebra then the underlying graded abelian group is isomorphic to Goncharov's  $\mathcal{H}_G$ .

**3.3. Simplicial strings.** For  $(D, \otimes)$  a strict monoidal category, consider  $(\Omega'D, \boxtimes)$  the strict monoidal category generated by  $D$  together with morphisms  $a \boxtimes b \rightarrow a \otimes b$  for objects  $a, b$  of  $D$ , subject to the obvious naturality and associativity relations. In this way a strict monoidal functor on  $\Omega'D$  is exactly a (strictly unital) *lax monoidal* functor on  $D$ : a functor  $F$  on  $D$  together with maps  $Fa \otimes Fb \rightarrow F(a \otimes b)$  satisfying appropriate naturality and associativity conditions.

**Definition 3.9.** Let  $\Delta_{*,*}$  be the strict monoidal category given as the subcategory of  $\Delta$  containing just the generic (that is, end-point preserving) maps  $[m] \rightarrow [n]$ , with the monoidal structure  $[p] \otimes [q] = [p + q]$  given by identifying  $p \in [p]$  and  $0 \in [q]$ .

We define the category of *simplicial strings*  $\Omega\Delta$  to be the strict monoidal category  $\Omega'\Delta_{*,*}$ .

This agrees with Baues' construction in [Bau80, Definition 2.7]. Now a contravariant monoidal functor on the category of simplicial strings is just an oplax monoidal functor on  $\Delta_{*,*}^{op}$ . Explicitly, if  $\mathcal{C}$  is a category with the cartesian monoidal structure, then to

give a monoidal functor  $(\Omega\Delta)^{op} \rightarrow \mathcal{C}$  is to give a functor  $X : \Delta_{*,*}^{op} \rightarrow \mathcal{C}$  together with associative natural transformations  $\mu_{p,q} = (\lambda_{p,q}, \rho_{p,q}) : X_{p+q} \rightarrow X_p \times X_q$ . Note that  $X$  becomes a simplicial object, if we define outer face maps  $X_n \rightarrow X_{n-1}$  by  $d_0 = \rho_{1,n-1}$  and  $d_n = \lambda_{n-1,1}$ . Moreover these determine all maps  $\rho_{p,q}$  and  $\lambda_{p,q}$  via the naturality conditions  $(d_1^{p-1} \times \text{id})\mu_{p,q} = \mu_{1,q}d_1^{p-1}$  and  $(\text{id} \times d_1^{q-1})\mu_{p,q} = \mu_{p,1}d_1^{q-1}$ . Thus we have:

**Proposition 3.10.** *Let  $\mathcal{C}$  be a cartesian monoidal category. Then the following categories are equivalent:*

- *The category of simplicial objects in  $\mathcal{C}$ ,*
- *The category of oplax monoidal functors  $\Delta_{*,*}^{op} \rightarrow \mathcal{C}$ ,*
- *The category of monoidal functors  $(\Omega\Delta)^{op} \rightarrow \mathcal{C}$ .*

*Given a simplicial object  $X$ , the corresponding oplax monoidal functor is given by the restriction of  $X$  to the endpoint preserving maps, with the structure map*

$$(d_{p+1}^q, d_0^p) : X_{p+q} \rightarrow X_p \times X_q.$$

**Definition 3.11.** *An interval object [BT97] (or a segment [BM06]) in a monoidal category  $(\mathcal{D}, \otimes, I)$  is an augmented monoid  $(L, L^{\otimes 2} \xrightarrow{\mu} L, I \xrightarrow{\eta} L, L \xrightarrow{\varepsilon} I)$  together with an *absorbing object*, that is,  $\bar{\eta} : I \rightarrow L$  satisfying  $\mu(\text{id}_L \otimes \bar{\eta}) = \bar{\eta}\varepsilon = \mu(\bar{\eta} \otimes \text{id}_L)$ ,  $\varepsilon\bar{\eta} = \text{id}_I$ .*

To any augmented monoid  $L$  one associates a simplicial object or, under Joyal duality, a covariant functor  $L^\bullet$  on  $\Delta_{*,*}$  with  $L^0 = L^1 = I$ ,  $L^n = L^{\otimes(n-1)}$ ,

$$\begin{aligned} s^0 &= \varepsilon \otimes \text{id}, \quad s^n = \text{id} \otimes \varepsilon, \quad s^i = \text{id} \otimes \mu \otimes \text{id} : L^{\otimes n} \rightarrow L^{\otimes(n-1)}, \\ d^i &= \text{id} \otimes \eta \otimes \text{id} : L^{\otimes(n-2)} \rightarrow L^{\otimes(n-1)}, \end{aligned}$$

If in addition  $L$  has an absorbing object then  $L^\bullet$  has a lax monoidal structure

$$\text{id} \otimes \bar{\eta} \otimes \text{id} : L^{\otimes(p-1)} \otimes L^{\otimes(q-1)} \rightarrow L^{\otimes(p+q-1)}$$

so we obtain a monoidal functor  $L^\bullet : \Omega\Delta \rightarrow \mathcal{D}$ .

**Definition 3.12.** Let  $X$  be a simplicial set, or the corresponding contravariant monoidal functor on the category of simplicial strings (Proposition 3.10). Baues' *geometric cobar construction*  $\Omega_L X$  with respect to an interval object  $L$  in a cocomplete monoidal category  $\mathcal{D}$  is defined as the monoid object in  $\mathcal{D}$  given by the realisation functor (see Lemma 3.1),

$$\Omega_L(X) = X \otimes_{\Omega\Delta} L^\bullet$$

We have four fundamental examples:

- (1) Let  $L = [0, 1]$  be the unit interval in the category of CW complexes, with unit and absorbing objects  $0, 1 : \{*\} \rightarrow [0, 1]$ , and multiplication given by  $\max : [0, 1]^2 \rightarrow [0, 1]$ . Then the geometric cobar construction on a 1-reduced simplicial set is homotopy equivalent to the loop space of the realisation of  $X$ .
- (2) Taking the cellular chains on the previous interval object we gives an interval object  $L$  in the category of chain complexes. In this case  $\Omega_L(X)$  coincides with Adams' cobar construction, which has the same homology as the loop space on  $X$ , if  $X$  is 1-reduced.

- (3) If we forget the boundary maps in example (2) we obtain an interval object  $L$  in the category of graded abelian groups, and  $\Omega_L(X)$  coincides as an algebra with the object  $\mathcal{B}(X)$  of Proposition 3.8: it is just the free tensor algebra whose generators in dimension  $n$  are the  $n + 1$ -simplices of  $X$ .
- (4) Let  $L = \Delta[1]$  in the category of simplicial sets, with unit and absorbing object  $d^1$  and  $d^0 : \Delta[0] \rightarrow \Delta[1]$ , and multiplication  $\mu : \Delta[1]^2 \rightarrow \Delta[1]$  defined by

$$\mu_n([n] \xrightarrow{x} [1], [n] \xrightarrow{x'} [1]) = (i \mapsto \max(x_i, x'_i)).$$

Berger has observed that, up to group completion,  $\Omega_L X$  has the same homotopy type as the simplicial loop group  $GX$  of Kan.

Note that the CW complex given by the simplicial realisation of  $\Delta[1]^2$  does not have the same cellular structure as  $[0, 1]^2$ : to relate examples (1–3) with (4) requires appropriate diagonal approximation and shuffle maps.

In example (3) the multiplication is free, and we have seen that the cooperad structure  $\check{\gamma}$  on the simplicial set  $X$  gives a comultiplication and hence a bialgebra structure on  $\Omega_L(X) = \mathcal{B}(X)$ . Baues showed that essentially the same coproduct gives a differential graded bialgebra structure on  $\Omega_L(X)$  in example (2), and used this to iterate the classical cobar construction to obtain an algebraic model of the double loop space. In example (4) we remain in the category of simplicial sets, and we have the following result:

**Proposition 3.13.** *Let  $X$  be a simplicial set, and  $\Omega_L(X)$  the simplicial monoid given by the geometric cobar construction on  $X$  with respect to the interval object  $L = \Delta[1]$ . Then the cooperad structure  $\check{\gamma}$  on  $X$  induces a map*

$$\Omega_L(X)_n \longrightarrow \prod_{m_1 + \dots + m_k = n} \Omega_L(X)_{k-1} \times \Omega_L(X)_{n-k}$$

for each  $n, k \geq 1$

*Proof.* Let  $Y = \Omega_L(X)$ . For each partition  $m_1 + \dots + m_k = n$  the cooperad structure map  $\check{\gamma}_{m_1, \dots, m_k}$  of (3.1) induces a map  $Y_{n-1} \rightarrow Y_{k-1} \times Y_{n-k}$  as follows. The map  $\gamma_{m_1, \dots, m_k}$  of Proposition 3.3 restricts to give a bijection  $\underline{k-1} \cup \underline{m_1-1} \cup \dots \cup \underline{m_k-1} \rightarrow \underline{n-1}$  and hence an isomorphism

$$\Delta[1]^{n-1} \longrightarrow \Delta[1]^{k-1} \times \Delta[1]^{m_1-1} \times \dots \times \Delta[1]^{m_k-1}.$$

Together with the map  $\check{\gamma}_{m_1, \dots, m_k}$  of (3.1) this defines a map

$$X_n \times \Delta[1]^{n-1} \longrightarrow X_k \times \Delta[1]^{k-1} \times (X_{m_1} \times \Delta[1]^{m_1-1} \times \dots \times X_{m_k} \times \Delta[1]^{m_k-1})$$

which induces the map on  $Y$  as required.  $\square$

**3.4. Comparison with Goncharov's second Hopf algebra.** We have seen above that Goncharov's first Hopf algebra  $\mathcal{H}_G$  and Baues Hopf algebra  $\Omega_L(X)$  are closely related. The differences between Baues' and Goncharov's algebras are as follows

- Baues' Hopf algebra has a differential, and the underlying graded abelian group  $\mathcal{B}(X)$  is the free tensor algebra, that is, a free associative algebra. No differential is given on Goncharov's algebra, which is a free polynomial algebra, that is, a free commutative and associative algebra.
- To obtain a model for the double loop space Baues requires  $X$  to have trivial 2-skeleton (only one vertex, one degenerate edge, and one degenerate 2-simplex), but to construct Goncharov's bialgebra we take  $X$  to be 0-coskeletal (a unique  $n$ -simplex for any  $(n + 1)$ -tuple of vertices). In the latter construction, however, one may still impose the relations  $x \sim 1$  and  $x \sim 0$  for 1- and 2-simplices  $x$  *after* taking the polynomial algebra (compare (1.3) and (1.10) respectively).

For Goncharov's second Hopf algebra  $\tilde{\mathcal{H}}_G$ , and the variants due to Brown, one imposes extra relations such as the shuffle formula (1.5). This has the following natural expression in the language of the cobar construction. Let  $X = X(S)$ , the 0-coskeletal simplicial set with vertex set  $X_0 = S$ . The cobar construction  $\Omega_L X$  is a colimit of copies of  $C(x_{n+1}) = L^{\otimes n}$  for each  $(n + 1)$ -simplex  $x_{n+1} = (s; w_n; s')$ , where  $w_n$  is a word of length  $n$  in the alphabet  $S$ . In a symmetric monoidal category each  $(p, n - p)$ -shuffle corresponds to a natural isomorphism  $L^{\otimes p} \otimes L^{\otimes(n-p)} \rightarrow L^{\otimes n}$  and the content of the shuffle relation is that this isomorphism is also obtained from the shuffle of the letters of a word  $w_p$  with a word  $w_{n-p}$  to obtain a word  $w_n$ .

#### 4. FEYNMAN CATEGORIES

4.1. **Definition of a Feynman category.** Consider the following data:

- (1)  $\mathcal{V}$  a groupoid.
- (2)  $\mathcal{F}$  a symmetric monoidal category, where the monoidal structure is denoted by  $\otimes$ .
- (3)  $\iota : \mathcal{V} \rightarrow \mathcal{F}$  a functor.

Let  $\mathcal{V}^{\otimes}$  be the free symmetric category on  $\mathcal{V}$  (words in  $\mathcal{V}$ ), then  $\iota$  factors:

$$\begin{array}{ccc}
 \mathcal{V} & \xrightarrow{\iota} & \mathcal{F} \\
 & \searrow j & \nearrow \iota^{\otimes} \\
 & \mathcal{V}^{\otimes} & 
 \end{array}$$

For any category  $\mathcal{C}$ , we will denote by  $Iso(\mathcal{C})$  the groupoid underlying  $\mathcal{C}$  that is all the objects and all the isomorphisms of  $\mathcal{C}$ . We will denote the comma category  $(id_{\mathcal{F}}, id_{\mathcal{F}})$  by  $(\mathcal{F} \downarrow \mathcal{F})$  and  $(\mathcal{F} \downarrow \mathcal{V})$  will denote the comma category  $(id_{\mathcal{F}}, \iota)$ .

**Definition 4.1.** A triple  $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$  as above is called a *Feynman category* if

- (i)  $\iota^{\otimes}$  induces an equivalence of symmetric monoidal categories between  $\mathcal{V}^{\otimes}$  and  $Iso(\mathcal{F})$ .
- (ii)  $\iota$  and  $\iota^{\otimes}$  induce an equivalence of symmetric monoidal categories  $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$  and  $Iso(\mathcal{F} \downarrow \mathcal{F})$ .
- (iii) For any  $* \in \mathcal{V}$ ,  $(\mathcal{F} \downarrow *)$  is essentially small.

The first condition says that  $\mathcal{V}$  knows all about the isomorphisms. The third condition is technical to guarantee that certain colimits exist. The second condition, also called the *hereditary condition*, is the key condition. It can be understood as follows:

- (1) In particular, fix  $\phi : X \rightarrow X'$  and fix  $X' \simeq \bigotimes_{v \in I} \iota(*_v)$ : there are  $X_v \in \mathcal{F}$ , and  $\phi_v \in \text{Hom}(X_v, *_v)$  s.t. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \simeq \downarrow & & \downarrow \simeq \\ \bigotimes_{v \in I} X_v & \xrightarrow{\bigotimes_{v \in I} \phi_v} & \bigotimes_{v \in I} \iota(*_v) \end{array} \quad (4.1)$$

- (2) For any two such decompositions  $\bigotimes_{v \in I} \phi_v$  and  $\bigotimes_{v' \in I'} \phi'_{v'}$  there is a bijection  $\psi : I \rightarrow I'$  and isomorphisms  $\sigma_v : X_v \rightarrow X'_{\psi(v)}$  s.t.  $P_\psi^{-1} \circ \bigotimes_v \sigma_v \circ \phi_v = \bigotimes \phi'_{v'}$  where  $P_\psi$  is the permutation corresponding to  $\psi$ .
- (3) These are the only isomorphisms between morphisms.

We call a Feynman category strict, if  $\text{Iso}(\mathcal{F}) = \mathcal{V}^\otimes$  and the monoidal structure is strict. Up to equivalence this can always be achieved.

**4.1.1. Non-symmetric version.** Now let  $(\mathcal{V}, \mathcal{F}, \iota)$  be as above with the exception that  $\mathcal{F}$  is only a monoidal category and  $\mathcal{V}^\otimes$  the free monoidal category and  $\iota^\otimes$  likewise is the corresponding morphism of monoidal categories.

**Definition 4.2.** A non-symmetric triple  $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \iota)$  as above is called a *non- $\Sigma$  Feynman category* if

- (i)  $\iota^\otimes$  induces an equivalence of monoidal categories between  $\mathcal{V}^\otimes$  and  $\text{Iso}(\mathcal{F})$ .
- (ii)  $\iota$  and  $\iota^\otimes$  induce an equivalence of monoidal categories  $\text{Iso}(\mathcal{F} \downarrow \mathcal{V})^\otimes$  and  $\text{Iso}(\mathcal{F} \downarrow \mathcal{F})$ .
- (iii) For any  $* \in \mathcal{V}$ ,  $(\mathcal{F} \downarrow *)$  is essentially small.

**4.1.2. Length and isomorphisms.** Notice that due to (i) every object  $X$  has a unique length, which is the length of a word in  $\mathcal{V}^\otimes$  representing  $X$ . We denote the length of  $X$  by  $|X|$  and define the length of a morphism  $\phi : X \rightarrow Y$  as  $|\phi| = |X| - |Y|$ . Notice that length is additive under composition. In order to be invertible a morphism  $\phi$  has to have length 0.

If the Feynman category is also strict, then by conditions (i), (ii) and strictness  $\phi = \bigotimes \phi_v$  with  $\phi_v \in \text{Mor}(\mathcal{V})$  whose length is 0 and this decomposition is unique once the order of the tensor factors is fixed and is unique in the non-symmetric version. Thus the morphisms of length 0 are precisely tensor products of morphisms in  $\mathcal{F}$  from objects of  $\mathcal{V}$  to objects of  $\mathcal{V}$ , i.e.  $\text{Hom}_{\mathcal{F}}(\mathcal{V}, \mathcal{V})^\otimes$ .

**4.1.3. Examples from graphs.** In order to give examples from graphs, which go beyond the cooperadic structure like the Feynman category of graphs of Connes and Kreimer, we use the language of [KW13, BM08]. Here an essential idea is that although there is a general category of graphs, [BM08], the graphs one considers in the applications and Feynman categories actually are part of morphisms and not the objects. More precisely, they

appear by indexing morphisms from corollas to corollas. The composition of morphisms in this situation then corresponds to inserting graphs into vertices for the indexing graphs, not gluing at leaves. See the appendix and [KW13] for details. Different types of Feynman categories are then given by restricting graphs or decorating graphs in a manner stable with respect to composition. This is detailed in the appendix and the examples in the next section §5.

There are several new examples, that have not yet appeared and are motivated by questions from physics and number theory. The first is that of collections 1–PI graphs, which we call the Broadhurst–Connes–Kreimer Feynman category. Indeed, blowing up a vertex of a 1–PI graphs into a 1–PI graph leaves the defining property (namely that the graph is still connected) invariant. Another way to define 1–PI for non–connected graphs is that all edge cuts decrease the first Betti number or loop number by one. Another new example is Brown’s Hopf algebra of motivic graphs.

**4.1.4. Enriched versions.** There are also enriched versions for Feynman categories. Their definition is a bit more involved and we refer to [KW13] for details. In principle, one needs to replace the comma categories indexed limits throughout and if  $\mathcal{C}$  is not Cartesian monoidal, fix the definition of a groupoid. This is categorically a bit tedious, but in the case of  $k$  vector spaces, this means that the decomposition is unique up to factors, which yield the same tensor product and that all the isomorphisms are indexed by an underlying groupoid. The relevant examples here are actually of a particular type given *a priori* as an enrichment of an underlying Feynman category, see §4.7.1 for the relevant facts.

To reproduce the case of cooperads corresponding to the free construction coming from operads in a monoidal category  $\mathcal{C}$ , see 2.2.3, we will need enrichment over  $\mathcal{C}$ . The following paragraph illustrates this nicely.

**Example 4.3** (Examples from operads). This is the construction relevant for the previous parts of the paper. As proven in [KW13] there is a one-to-one correspondence between equivalence classes of (non- $\Sigma$ ) Feynman categories with trivial  $\mathcal{V}$  and (non- $\Sigma$ ) operads, whose  $\mathcal{O}(1) \simeq \mathbb{1}$ . Here trivial  $\mathcal{V}$  means one object  $\underline{1}$  and its identity. In both the symmetric and non-symmetric case  $\mathcal{V}^{\otimes}$  has objects  $\underline{n} = \underline{1}^{\otimes n}, n \in \mathbb{N}_0$  with  $\underline{0} = \underline{1}^{\otimes 0} = \mathbb{1}$  the unit object. In the symmetric case the morphisms are  $Hom(\underline{n}, \underline{n}) = \mathbb{S}_n$  the symmetric group. In the non- $\Sigma$  case there are only identity morphisms. Thus, up to equivalence  $Iso(\mathcal{F})$  is fixed. By condition (ii) up to equivalence, all morphisms are fixed by the morphisms  $Hom(\underline{n}, \underline{1}) =: \mathcal{O}(n)$ . Given an operad  $\mathcal{O}$  with  $\mathcal{O}(1) \simeq \mathbb{1}$  we denote the corresponding Feynman category by  $\mathfrak{F}_{\mathcal{O}}$ . These are exactly the enrichments of  $\mathfrak{F}_{Surj, <}$  in the non  $\Sigma$  case respectively  $\mathfrak{F}_{Surj}$  in the symmetric case.

Under this correspondence the  $\mathbb{S}_n$  action comes from precomposing and the composition of morphisms corresponds to  $\gamma$ . If the elements of  $\mathcal{O}(n)$  are rooted leaf labelled graphs, the morphisms of  $\mathcal{F}$  are given by disjoint union of graphs (a.k.a. forests) and the composition is given by gluing leaves to roots. The isomorphism condition then reads that  $\mathcal{O}(1)$  only has *id* as an invertible element.

If  $\mathcal{O}(1)$  has isomorphisms these just have to be included into  $\mathcal{V}$ , which is then not trivial, but has one object which has isomorphisms corresponding to the isomorphisms in  $\mathcal{O}(1)$ .



For the moment, we will stick to Feynman categories as defined above and return to enrichment later.

## 4.2. The three main examples.

**4.2.1. The operad of leaf-labelled rooted trees.** Let  $\mathcal{F}$  whose objects are  $\mathbb{N}_0$  have morphisms given by rooted forests. In particular  $Hom(\underline{n}, \underline{m})$  are given by  $n$ -labelled rooted forests with  $n$  roots. The operad is then the one of leaf-labelled rooted trees. In the non- $\Sigma$  version, one uses planar forests/trees and omits labels.

**4.2.2. The operad of surjections.** In the non- $\Sigma$  version  $\mathcal{F} = Surj_{<}$ , which is the wide subcategory of order preserving surjections inside the augmented simplicial category  $\Delta_+$ . For the symmetric version,  $\mathcal{F} = Surj$ , is the skeleton of the category of finite sets and surjections. This which is the wide subcategory of surjections inside the augmented crossed simplicial group  $\Sigma\Delta_+$ .

In both cases  $\mathcal{V}$  is trivial and we will simply write  $\mathfrak{F} = Surj_{<}$  or  $\mathfrak{F} = Surj$ .

**4.2.3. The Feynman category of simplices as the Joyal dual of  $Surj_{<}$ .** There is a very interesting and useful contravariant duality [Joy97] of subcategories of  $\Delta_+$  between  $\Delta$  and the category of intervals, which are the endpoint preserving morphisms in  $\Delta$ . It maps surjections in  $\Delta$  to double base point preserving injections  $inj_{*,*}$ , see Appendix C.1. Thus the category  $inj_{*,*}^{op}$  is again a Feynman category with trivial  $\mathcal{V}$ . But, surprisingly, it is also a Feynman category itself in a suitable interpretation. This is given as the subcategory of the Feynman category FI of [KW13][2.9.3].

**4.3. Bialgebras from Feynman categories.** Given a Feynman category consider the free Abelian group on the set of all morphisms of  $\mathcal{F} : \mathcal{B} = \mathbb{Z}[Mor(\mathcal{F})]$ .

**Assumption 4.4.** Since in the following we will be interested in fixing  $\mathcal{V}$  and fixing  $\mathcal{F}$  only up to equivalence, we will assume first of all that the Feynman category is strict and after using MacLane's coherence theorem [ML98] that  $\mathcal{F}$  is a strict monoidal category, that is, that the associativity and unit constraints are all identities.

Without these assumptions, the structures will all be weak, i.e. unital and associative up to given isomorphisms.

**4.3.1. The product.** With the assumption above,  $\mathcal{B}$  has a unital associative product given by  $\otimes$  with the unit  $id_{\mathbb{1}}$ , i.e. the identity morphisms of the monoidal unit  $\mathbb{1}$  of  $\mathcal{F}$ . This is the free algebra on the unital monoid  $(Mor(\mathcal{F}), \otimes)$ .

**4.3.2. The coproduct.** Suppose that  $\mathcal{F}$  is decomposition finite. This means that for each morphism  $\phi$  of  $\mathcal{F}$  the set  $\{(\phi_0, \phi_1) : \phi = \phi_0 \circ \phi_1\}$  is finite. Then  $\mathcal{B}$  carries a coassociative coproduct given by the dual of the composition. On generators it is given by:

$$\Delta(\phi) = \sum_{\{(\phi_0, \phi_1) : \phi = \phi_0 \circ \phi_1\}} \phi_0 \otimes \phi_1 \quad (4.2)$$

where we have abused notation to denote by  $\phi$  the morphism  $\delta_\phi(\psi)$  that evaluates to 1 on  $\phi$  and zero on all other generators.

A counit on generators is by:

$$\epsilon(\phi) = \begin{cases} 1 & \text{if for some object } X : \phi = id_X \\ 0 & \text{else} \end{cases} \quad (4.3)$$

This coproduct for any Abelian finite decomposition categories appeared in [JR79] and goes back to [Ler75] where it was considered for Moebius categories.

**4.3.3. Graded version.** One can enlarge the setting to the situation, in which the sets of morphisms are graded and composition preserves the grading. In this case, one only need degreewise composition finite. This will be the case for any graded Feynman category [KW13].

**Lemma 4.5.** *In a decomposition finite category for any object  $X$  the sets  $Aut(X)$ , the set of all automorphisms of  $X$ , and  $Iso(X)$ , the set of all objects isomorphic to  $X$  are finite sets.*

*Proof.* We show this by decomposing  $id_X$ .

$$\Delta(id_X) = \sum_{(\phi_L, \phi_R) : \phi_L \circ \phi_R = id_X} \phi_R \otimes \phi_L$$

This includes the term  $\sum_{Y \in Iso(X)} \sum_{\phi \in Iso(X, Y)} \phi \otimes \phi^{-1}$  and furthermore in this subsum the terms coming from  $Aut(X)$  are  $\sum_{\sigma \in Aut(X)} \sigma \otimes \sigma^{-1}$  and hence both sets are finite.  $\square$

**4.3.4. Bialgebra structure.** The product and coproduct above would actually work in any strict monoidal category with finite decomposition. However the compatibility axiom of a bialgebra does not hold in general. One needs to check that  $\Delta \circ \mu = (\mu \otimes \mu) \circ \tau_{2,3} \circ (\Delta \otimes \Delta)$ .

For  $\Delta \circ \mu$  the sum is over diagrams of the type

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{\Phi = \phi \otimes \psi} & Z \otimes Z' \\ & \searrow \Phi_0 \quad \nearrow \Phi_1 & \\ & Y & \end{array} \quad (4.4)$$

where  $\Phi = \Phi_1 \circ \Phi_0$ .

When considering  $(\mu \otimes \mu) \circ \pi_{23} \circ (\Delta \otimes \Delta)$  the diagrams are of the type

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{\phi \otimes \psi} & Z \otimes Z' \\ & \searrow \phi_0 \otimes \psi_0 \quad \nearrow \phi_1 \otimes \psi_1 & \\ & Y \otimes Y' & \end{array} \quad (4.5)$$

where  $\phi = \phi_1 \circ \phi_0$  and  $\psi = \psi_1 \circ \psi_0$ . And there is no reason for there to be a bijection of such diagrams.

The compatibility does hold when dealing with strict Feynman categories due to the hereditary condition. Let us consider the diagrams appearing in the compatibility equation of a bialgebra, see the Theorem below.

**Theorem 4.6.** *For any strict Feynman category which has finite decomposition and is strictly monoidal, the tuple  $(\mathcal{B}, \otimes, \Delta, \epsilon, \eta)$  defines a bialgebra over  $\mathbb{Z}$ .*

*Proof.* We check the compatibility axioms. First,  $\Delta(id_{\mathbb{1}}) = id_{\mathbb{1}} \otimes id_{\mathbb{1}}$ , since there is only one empty morphism  $id_{\emptyset} \in Hom_{\mathcal{V}^{\otimes}}(\emptyset, \emptyset)$  which in  $\mathcal{F}$  is exactly  $id_{\mathbb{1}}$ . This is the only invertible element in  $Hom_{\mathcal{F}}(\mathbb{1}, \mathbb{1})$  due to condition (i) of a Feynman category. Furthermore, there are no maps  $Hom_{\mathcal{V}^{\otimes}}(X, \emptyset)$  for any word of length greater than 0 by condition (ii). Hence  $\Delta(id_{\mathbb{1}})$  only has one summand.

Next,  $\epsilon(\phi \otimes \psi) = \epsilon(\phi)\epsilon(\psi)$ , since  $id_X \otimes id_Y = id_{X \otimes Y}$ , since  $\mathcal{F}$  is strict monoidal. Because of axiom (i) this is then the unique decomposition of  $id_{X \otimes Y}$ . We also have that  $X \otimes \mathbb{1}_{\mathcal{F}} = X$  and hence  $\phi \otimes id_{\mathbb{1}} = \phi$  and the compatibility, since  $m id_{\mathbb{1}} \otimes_{\mathbb{Z}} n id_{\mathbb{1}} = mn id_{\mathbb{1}}$ .

In order to prove that  $\Delta$  is an algebra morphism, we consider the two sums over the diagrams (4.4) and (4.5) above and show that they coincide. First, it is clear that all diagrams of the second type appear in the first sum. Vice versa, given a diagram of the first type, we know that  $Y = Y \otimes Y'$ , since  $\Phi_1$  has to factor by axiom (ii) and the Feynman category is strict. Then again by axiom (ii)  $\Phi_0$  must factor. We see that we obtain a diagram:

$$\begin{array}{ccc}
 \hat{X} \otimes \hat{X}' & \xrightarrow{\simeq\sigma} & X \otimes X' & \xrightarrow{\Phi = \phi \otimes \psi} & Z \otimes Z' & \\
 & & \searrow \Phi_0 & & \nearrow \Phi_1 & \\
 & & & Y = Y \otimes Y' & & \\
 & \searrow \hat{\phi}_0 \otimes \hat{\psi}_0 & & \downarrow \simeq\sigma' & & \nearrow \hat{\phi}_1 \otimes \hat{\psi}_1 \\
 & & & \hat{Y} \otimes \hat{Y}' & & 
 \end{array} \tag{4.6}$$

Now since the Feynman category is strict and non-symmetric, the two isomorphisms also decompose as  $\sigma = \sigma_1 \otimes \sigma_2$ ,  $\sigma' = \sigma'_1 \otimes \sigma'_2$ , so that  $\Phi_0 = \sigma_1'^{-1} \circ \hat{\phi}_0 \circ \sigma_1^{-1} \otimes \sigma_1'^{-1} \circ \hat{\psi}_0 \circ \sigma_1^{-1}$  and  $\Phi_1 = \sigma_2'^{-1} \circ \hat{\phi}_0 \circ \sigma_2^{-1} \otimes \sigma_2'^{-1} \circ \hat{\psi}_1 \circ \sigma_2^{-1}$  and both diagrams sums agree.  $\square$

**Remark 4.7.** We could also already start with a Feynman category augmented over a tensor category  $\mathcal{E}$  where  $\mathcal{E}$  has a faithful functor to  $\mathcal{A}b$ , e.g.  $k\text{-Vect}$ . In this case one should work over the ring  $K = Hom_{\mathcal{F}}(\mathbb{1}, \mathbb{1})$ , see [KW13] for details.

**4.4. Symmetric version.** In the symmetric version there are two relevant constructions. Both involve quotienting by certain isomorphisms.

We let  $\mathcal{B}^{sk}(\mathcal{F}) = \mathbb{Z}[Mor(i^{\otimes} \downarrow i)^{\otimes}]$  that is the free Abelian group on these morphisms, which by assumption carries the structure of a free symmetric monoid. The basic morphisms are by morphisms from words in  $\mathcal{V}^{\otimes}$  to  $\mathcal{V}$ . If  $\mathcal{V}$  is skeletal, then so is  $(i^{\otimes}, i)^{\otimes}$  up to the the  $\mathbb{S}_k$  action permuting the letters of the word. In particular if  $X = \bigotimes_{i=1}^k *_i$  then  $Aut(X)$  in  $\mathcal{B}^{sk}$  is given by  $Aut(*_1) \times \cdots \times Aut(*_k) \wr \mathbb{S}_k$ . Recall that skeletal means that there is only one object per isomorphism class. Up to equivalence of categories, one can

always assume that this is the case. If  $\mathcal{V}$  is skeletal then the objects of the actual skeleton of  $(i^\otimes, i)^\otimes$  would be the symmetric words.

By axiom (ii)  $\mathcal{B}^{sk}(\mathcal{F})$  is Morita equivalent to the monoid on all morphisms. Effectively this means that we are considering tensor products of morphisms and isomorphisms together with free permutations of factors for example an isomorphism  $\hat{X} \otimes \hat{X}' \rightarrow X \otimes X'$  has to be of the form  $\pi \circ \sigma_1 \otimes \sigma_2$  or  $\pi \circ \sigma_1 \otimes \sigma_2 \circ \pi$  where  $\pi$  interchanges two factors and  $\sigma_1, \sigma_2$  are isomorphisms. We also set  $\mathcal{B}_{iso} = \mathcal{B} / \sim$  where  $\sim$  is the equivalence relation on morphisms given by isomorphisms in  $(\mathcal{F} \downarrow \mathcal{F})$ . The equivalence relation  $\sim$ , which exists on any category, means that for given  $f$  and  $g: f \sim g$  if there is a commutative diagram with isomorphisms as vertical morphisms.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \simeq \sigma & & \downarrow \simeq \sigma' \\ X' & \xrightarrow{g} & Y' \end{array}$$

i.e.:  $f = \sigma'^{-1} \circ g \circ \sigma$ . Plugging in  $id_X$  we obtain:

**Lemma 4.8.**  *$id_X \sim f$  if and only if  $f : Y \rightarrow Y'$  is an isomorphism and  $Y \simeq X \simeq Y'$ .*

□

**Remark 4.9.** Notice that this equivalence is stronger than the equivalence studied in [JR79] for the standard reduced incidence category.

We have maps  $\mathcal{B}^{sk} \hookrightarrow \mathcal{B} \rightarrow \mathcal{B}_{iso}$ . Now picking an inverse for the equivalence of  $\mathcal{B}^{sk}$  and  $\mathcal{B}$  the quotient map to  $\mathcal{B}_{iso}$  factors.

**Example 4.10.** It is instructive to study two examples.

The first is a Feynman category  $\mathfrak{F}_{\mathcal{O}}$ .  $\mathcal{B}^{sk}$  in this case is given by the formula for  $\check{\mathcal{O}}^{nc}$  (2.17), while the full  $\mathcal{B}$  is provided by formula (2.38). Finally  $\mathcal{B}_{iso} = \bigoplus_n \bigoplus_{n_1 \leq \dots \leq n_k, \sum n_i = n} \check{\mathcal{O}}(n_i)_{\mathbb{S}_i}$ .

The second is the case of connected or 1-PI graphs. Here  $\mathcal{B}_{iso}$  is given by the free monoid of the isomorphisms classes of connected or 1-PI ghost graphs, i.e. graphs with unlabelled vertices and flags.  $\mathcal{B}$  is given by the morphisms, that is, ghost graphs with all the additional data, and  $\mathcal{B}^{sk}$  is given by the ghost graphs, where now the source morphism is to be picked up to an induction from the wreath product to the full symmetric group.

**Proposition 4.11.** *Consider the deconcatenation coproduct on the free groups  $\mathcal{B}$  generated by the morphisms of a decomposition finite category. Let  $\mathcal{C}$  be the set generated by the relation  $\sim$ , i.e. the ideal generated by all elements of the form  $f - g$  with  $f \sim g$ . Then*

$$\Delta(\mathcal{C}) \subset \mathcal{B} \otimes \mathcal{C} + \mathcal{C} \otimes \mathcal{B} \tag{4.7}$$

and thus descends to  $\mathcal{B}/\mathcal{C}$ . Furthermore extending scalars, so that for all  $x$  the products  $|Iso(X)||Aut(X)|$  are invertible (e.g. in  $\mathbb{Q}$  or a field  $k$  of characteristic 0) defining  $\bar{\epsilon}$  on

generators via

$$\bar{\epsilon}([f]) := \begin{cases} \frac{1}{|Iso(X)||Aut(X)|} & \text{if } [f] = [id_X] \\ 0 & \text{else} \end{cases} \quad (4.8)$$

defines a counit on  $\mathcal{B}/\mathcal{C}$ .

**Remark 4.12.** Note that  $\mathcal{C}$  is not a coideal in general, since for any automorphism  $\sigma_X \in Aut(X) : [\sigma_X] = [id_X]$  and hence  $\epsilon(\mathcal{C}) \not\subseteq ker(\epsilon)$ . Likewise if  $X \simeq Y \xrightarrow{\phi} Y'$  then  $[id_X] = [\phi]$  from Lemma 4.8. If there are no automorphisms and the underlying category is skeletal, then  $\epsilon$  descends as claimed in [JR79].

*Proof.* To compute the coproduct, we break up the sum over the factorizations of  $f$  and  $g$  with  $f \sim g$  into the pieces that correspond to a factorization through a fixed space  $Z$ .

$$\begin{array}{ccc} & & Z \\ & \nearrow^{f_1} & \\ X & \xrightarrow{f} & Y \\ & \searrow_{f_2} & \\ & & Z \\ \downarrow \simeq \sigma' & & \downarrow \simeq \sigma \\ X' & \xrightarrow{g} & Y' \\ & \searrow_{g_1} & \\ & & Z \\ & \nearrow_{g_2} & \end{array} \quad (4.9)$$

Now the term in  $\Delta_{f-g}$  corresponding to  $Z$  is  $\sum_i f_1^i \otimes f_2^i - \sum_j g_1^j \otimes g_2^j$ . Resumming using the identification  $g_1^i = f_1^i \circ \sigma'$  and  $g_2^i = \sigma' \circ f_2^i$  this equals to

$\sum_i (f_1^i \otimes f_2^i - g_1^i \otimes g_2^i) = \sum_i (f_1^i - g_1^i) \otimes g_2^i + \sum_i f_1^i \otimes (f_2^i - g_2^i)$ . This proves the first part.

For the second part notice that  $\Delta([f]) = [\Delta(f)]$  is a sum of terms factoring through an intermediate space  $Z$ . If  $Z \not\simeq X, Y$  then these terms are killed by  $\bar{\epsilon}$  on either side, since there will be no isomorphism in the decomposition. If  $Z \simeq X$ , then any factorization  $\sigma \otimes f \circ \sigma^{-1}$  with  $\sigma \in Iso(X, Z)$  descends to  $[\sigma] \otimes [f \circ \sigma^{-1}] = [id_X] \otimes [f]$ . Since  $Iso(X, Z)$  is a left  $Aut(X)$  torsor, there are exactly  $|Aut(X)||Iso(X)|$  of these terms and  $\bar{\epsilon} \otimes id$  evaluates to  $1 \otimes [f]$  on their sum. By Lemma 4.8 all other decompositions will evaluate to 0 and we obtain that  $\epsilon$  is a left counit. Likewise  $\epsilon$  is a right counit by considering the terms which factor through  $Y' \in Iso(Y)$ .

□

**4.4.1. Free actions.** Suppose now that  $\mathcal{F}$  is skeletal and that the action of  $Aut(Z)$  on  $Hom(X, Z) \times Hom(Z, Y)$  given by  $\rho(\sigma)(f_0, f_1) = (\sigma \circ f_0, f_1 \circ \sigma^{-1})$  is free. Then we can define a reduced coalgebra structure on  $\mathcal{B}/\mathcal{C}$ .

Notice that

$$\begin{aligned}
\Delta([f]) = [\Delta(f)] &= \sum_Z \sum_i [f_0^i \otimes f_1^i] \\
&= \sum_Z \sum_{i_r} \sum_{\sigma \in \text{Aut}(Z)} [\sigma \circ f_0^{i_r}] \otimes [f_1^{i_r} \circ \sigma^{-1}] \\
&= \sum_Z \sum_{i_r} |\text{Aut}(Z)| [f_0^{i_r}] \otimes [f_1^{i_r}]
\end{aligned}$$

where  $f_0^{i_r}$  and  $f_1^{i_r}$  are representatives.

Given the conditions above, we define the reduced coproduct and counit on  $\mathcal{B}/\mathcal{C}$  as  $\Delta^{\text{red}}(f) = \sum_Z \sum_{i_r} [f_0^{i_r}] \otimes [f_1^{i_r}]$  and

$$\epsilon^{\text{red}}([f]) = \begin{cases} 1 & \text{if } [f] = [id_X] \\ 0 & \text{else} \end{cases} \quad (4.10)$$

Collecting the results so far:

**Theorem 4.13.** *Fix a composition finite Feynman category, let  $\mathcal{B}$  and  $\mathcal{B}^{\text{sk}}$  as given above considered as algebras with  $\otimes$  as product and  $id_{\mathbb{1}}$  as the unit. Let  $\mathcal{C}$  be the ideal generated by  $\sim$  in  $\mathcal{B}$  and  $\mathcal{C}^{\text{sk}}$  the respective ideal in  $\mathcal{B}^{\text{sk}}$ . Set  $\mathcal{B}_{\text{iso}} = \mathcal{B}/\mathcal{C}$  and set  $\mathcal{B}_{\text{iso}}^{\text{sk}} = \mathcal{B}^{\text{sk}}/\mathcal{C}^{\text{sk}}$ . Then*

- (1)  $\mathcal{B}$  and  $\mathcal{B}^{\text{sk}}$  are Morita equivalent as algebras and  $\mathcal{B}_{\text{iso}} \simeq \mathcal{B}_{\text{iso}}^{\text{sk}}$ .
- (2) Both  $\mathcal{B}$  and  $\mathcal{B}^{\text{sk}}$  are coalgebras with respect to the deconcatenation coproduct with counit  $\epsilon$ . Furthermore  $\mathcal{B}$  and  $\mathcal{B}^{\text{sk}}$  are unital, counital bialgebras.
- (3) After extending scalars, so that all  $|\text{Aut}(X)|$  are invertible,  $\mathcal{B}_{\text{iso}}^{\text{sk}}$  is a unital counital bialgebra with counit  $\bar{\epsilon}$  and if the  $|\text{Iso}(X)|$  are also invertible  $\mathcal{B}_{\text{iso}}$  is unital counital bialgebra.
- (4) If the action of  $\text{Aut}(Z)$  on  $\text{Hom}(X, Z) \times \text{Hom}(Z, Y)$  is free for all  $X, Y, Z$ , then  $\mathcal{B}_{\text{iso}}$  is a bialgebra with respect to  $(\otimes, \Delta^{\text{red}}, \eta, \epsilon^{\text{red}})$ .  $\square$

**4.4.2. Morphisms of length 0.** Since the length of morphisms is additive under compositions, the morphisms of length 0, i.e.  $\text{Hom}_{\mathcal{F}}(\mathcal{V}, \mathcal{V})^{\otimes}$  form a subbialgebra  $\mathcal{B}_0$ , in all constructions.

Moreover, the morphisms  $\text{Hom}_{\mathcal{F}}(\mathcal{V}, \mathcal{V})$  together with the counit  $\epsilon$  and the unit  $\eta$  form a pointed coalgebra  $\mathcal{B}_{\mathcal{V}}$ , which generates  $\mathcal{B}_0$  as an algebra. The elements of  $\mathcal{B}_{\mathcal{V}}$  split according to whether they are isomorphisms or not. That is, whether or not they lie in  $\text{Mor}(\mathcal{V})$ .

**4.5. Hopf algebras from Feynman categories.** The above bialgebras are usually not connected. There are two obstructions. Each isomorphism class of an object  $X$  gives a unit and, unless  $\mathcal{V}$  is discrete, there are isomorphisms which prevent the identities of the different  $X$  from being group-like elements. Furthermore,  $\mathcal{B}_0$  (or better  $\mathcal{B}_{\mathcal{V}}$ ) could keep things from being connected. This is analogous to the situation for cooperads with multiplication, where, however,  $\mathcal{V}$  is trivial.

If  $\mathcal{V}$  is discrete, quotienting by the ideal  $\mathcal{J} = \langle id_X - id_Y \rangle$  already makes the  $\mathcal{B}/\mathcal{J}$  into a connected bialgebra if  $\mathcal{B}_{\mathcal{V}}$  is connected. In fact  $\mathcal{J}$  is the ideal of group like elements.

Otherwise, we define the ideal  $\bar{\mathcal{J}} = \langle |Aut(X)||Iso(X)|id_X - |Aut(Y)||Iso(Y)|id_Y \rangle$  of  $\mathcal{B}_{iso}$ , and then  $\mathcal{B}_{iso}/\bar{\mathcal{J}}$  is connected if  $\mathcal{B}_{\mathcal{V}}/\bar{\mathcal{J}}$  is connected. In the skeletal case the factors  $|Iso(X)| = 1$  and the appropriate ideal is generated by  $|Aut(X)|id_X - |Aut(Y)|id_Y$ .

**Definition 4.14.**  $\mathfrak{F}$  is called almost connected if  $\mathcal{B}_{\mathcal{V}}/\bar{\mathcal{J}}$  is connected.

**Theorem 4.15.** *Assume that  $\mathfrak{F}$  is composition finite and almost connected. If  $\mathcal{V}$  is discrete, then  $\mathcal{J}$  is a coideal in  $\mathcal{B}$  and  $\mathcal{H} = \mathcal{B}/\mathcal{J}$  with counit induced by  $\epsilon$  and unit  $\eta(1) = [id_{\mathbb{1}_{\mathcal{F}}}]$  is a connected bialgebra, and hence a Hopf algebra. In general,  $\bar{\mathcal{J}}$  is a coideal in  $\mathcal{B}_{iso}$  and  $\mathcal{H} = \mathcal{B}_{iso}/\bar{\mathcal{J}}$ , with counit induced by  $\bar{\epsilon}$  and unit  $\eta(1) = [id_{\mathbb{1}_{\mathcal{F}}}]$ , is a connected bialgebra, and hence a Hopf algebra. The same holds true for the skeletal version. Furthermore in the free case, the analogous statement holds for the reduced coproduct and counit.*

*Proof.* From the proof of Proposition 4.11

$$\Delta(id_X) = \sum_{X', \sigma \in Iso(\mathcal{F})(X, X')} \sigma \otimes \sigma^{-1} = \sum_{X', \sigma \in Hom_{\mathcal{V}}(X, X')} \sigma \otimes \sigma^{-1} \quad (4.11)$$

If  $\mathcal{V}$  is discrete, then the  $id_X$  are group like and  $\epsilon(id_X - id_Y) = 0$ , so that  $\mathcal{J}$  is a coideal. Fixing  $\eta(1) = [id_{\mathbb{1}_{\mathcal{F}}}]$  gives a unit and defines a split counit for the coalgebra structure. It is easy to see that then  $\mathcal{H}$  is conilpotent and hence connected. The reason is that any decomposition which has a morphism on the left or the right that is not of length 0 has a shorter length. The terms with length 0 factors are taken care of by the almost connectedness assumption.

In the case of non-discrete  $\mathcal{V}$  in  $\mathcal{B}_{iso}$ , (4.11) reads  $\Delta([id_X]) = |Aut(X)||Iso(X)|[id_X] \otimes [id_X]$ . The ideal  $\bar{\mathcal{J}}$  is generated by elements  $|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]$ . These also satisfy  $\bar{\epsilon}([id_X] - [id_Y]) = 1 - 1 = 0$ , so that this ideal is again a coideal. Again  $\bar{\eta}$  yields a split counit and the coalgebra part is nilpotent and connected. The skeletal and the reduced case are analogous.  $\square$

**Remark 4.16.** At this point, it is interesting to remark that any morphism  $\phi : X \rightarrow Y : \Delta(\phi) = id_X \otimes \phi + \phi \otimes id_Y + \dots$  and in  $\mathcal{B}_{iso}$  the primitive elements are the  $id_X$ . Hence it is interesting to study the co-radical filtration and the  $([id_X], [id_Y])$ -primitive elements. They correspond to the generators for morphisms in Feynman categories [KW13]. In the main examples they are all tensors of elements of length 1.

## 4.6. Functoriality and opposite category.

**4.6.1. Functoriality.** Let  $F : \mathfrak{F} \rightarrow \mathfrak{F}'$  be a morphism of Feynman categories. In the strict case, this is basically a strict monoidal functor from  $F : \mathcal{F} \rightarrow \mathcal{F}'$  compatible with all the structures, see [KW13][Chapter 1.5]. For a morphism  $\phi \in Mor(\mathcal{F}')$  thought of as a characteristic function  $\phi(\psi) = \delta_{\phi, \psi}$ . We see that

$$F^*(\phi) := \phi \circ F = \sum_{\hat{\phi} \in Mor(\mathcal{F}) : F(\hat{\phi}) = \phi} \hat{\phi}$$



**Proposition 4.17.**  *$F^*$  induces a morphism of algebras. If  $F$  is injective on objects, then  $F^*$  induces a morphism of unital bialgebras. If  $F^*$  is bijective on objects, it induces a morphism of unital and counital bialgebras  $\mathcal{B}_{\mathfrak{F}'} \rightarrow \mathcal{B}_{\mathfrak{F}}$ .*

*Proof.* We have to check the multiplication, but since  $\mathcal{F}$  is strictly monoidal, we get  $F^* \otimes F^*(\phi \otimes \psi) = \phi \circ F \otimes \psi \circ F = (\phi \otimes \psi) \circ F$ . For the coproduct, we get

$$\Delta(F^*\phi) = \sum_{\hat{\phi} \in \text{Mor}(\mathcal{F}): F(\hat{\phi}) = \phi} \sum_{(\hat{\phi}_0, \hat{\phi}_1): \hat{\phi}_1 \circ \hat{\phi}_0 = \hat{\phi}} \hat{\phi}_0 \otimes \hat{\phi}_1 \quad (4.12)$$

$$(F^* \otimes F^*)\Delta(\phi) = \sum_{(\phi_0, \phi_1): \phi_1 \circ \phi_0 = \phi} \sum_{\hat{\phi}_0, \hat{\phi}_1 \in \text{Mor}(\mathcal{F}): F(\hat{\phi}_0) = \phi_0, F(\hat{\phi}_1) = \phi_1} \hat{\phi}_0 \otimes \hat{\phi}_1 \quad (4.13)$$

We now check that the sums coincide. Certainly for any term in the first sum corresponding to decomposition  $\hat{\phi} = \hat{\phi}_1 \circ \hat{\phi}_0$  appears in the second sum, since  $F$  is a functor:  $\phi_1 \circ \phi_0 = F(\hat{\phi}_1) \circ F(\hat{\phi}_0) = F(\hat{\phi}_1 \circ \hat{\phi}_0) = F(\hat{\phi}) = \phi$ . The second sum might be larger, since the lifts need not be composable. If, however,  $F$  is injective on objects, then all lifts of a composition are composable and the two sums agree. The unit agrees, because of the injectivity and uniqueness of the unit object and the triviality of  $\text{Hom}(\mathbb{1}, \mathbb{1})$ . For the co-unit, we need bijectivity. In this case  $\hat{id}_X = id_{\hat{X}} + T$ , with  $\epsilon(T) = 0$ , so that  $\epsilon(F^*\phi) = \epsilon\phi$ , since  $F(id_{\hat{X}}) = id_X$  and hence if  $\phi \neq id_X$ , then there is no  $id_{\hat{X}}$  in the fiber. If the functor is not injective, we might have more objects in the fiber and if it is not surjective  $F^*(id_X)$  can be 0.  $\square$

Recall that in order to get a Hopf algebra, we needed to quotient by the ideal  $\bar{\mathcal{J}}$  defined in §4.5

**Definition 4.18.** We call a functor  $F$  as above Hopf compatible if it is bijective on objects and  $F^*(\bar{\mathcal{J}}_{\mathfrak{F}'}) \subset \bar{\mathcal{J}}_{\mathfrak{F}}$ .

The following is straightforward.

**Proposition 4.19.** *If  $\mathfrak{F}$  and  $\mathfrak{F}'$  are almost connected, a Hopf compatible functor induces a morphism of Hopf algebras  $\mathcal{H}_{\mathfrak{F}'} \rightarrow \mathcal{H}_{\mathfrak{F}}$ .*  $\square$

The following is a useful criterion:

**Proposition 4.20.** *If  $F|_{\mathcal{V}}$  is bijective on objects, surjective on morphisms and  $F$  does not send any non-invertible elements of  $\text{Mor}(\mathcal{F})$  to invertible elements in  $\text{Mor}(\mathcal{F}')$ , then  $F$  is Hopf compatible.*

*Proof.* Suppose the conditions are true and let  $* \in \mathcal{V}'$  with  $F^{-1}(*) = \hat{*}$ . Set  $G = \text{Aut}_{\mathcal{V}'}(*) = \text{Aut}_{\mathcal{F}'}(*)$ ,  $H = \text{Aut}_{\mathcal{V}}(\hat{*}) = \text{Aut}_{\mathcal{F}}(\hat{*})$  and  $K = \ker(F|_G)$ . then  $F^*([id_*]) = \sum_{\phi_k \in K} [\phi_k] = |K|[id_{\hat{*}}]$  and by the condition (i) of Feynman categories the same holds for  $*$  replaced by  $X$  and  $\hat{*}$  replaced by  $\hat{X} = F^{-1}(X)$ . Due to the bijectiveness and third condition  $|Iso(X)| = |Iso(\hat{X})|$  and the statement follows from the orbit formula  $|G| = |K||H|$ .  $\square$

These criteria reflect that the Hopf algebras are very sensitive to invertible elements. It says that that we can identify isomorphisms and are allowed to identify morphisms, but only in each class separately.

**Example 4.21.** An example is provided by the map of operads: rooted 3-regular forrests  $\rightarrow$  rooted corollas. This give a functor of Feynman categories enriching  $\mathfrak{F}_{Surj}$  or in the planar version of  $\mathfrak{F}_{Surj,<}$ . This functor is Hopf compatible thus induces a map of Hopf algebras which is the morphism considered by Goncharov in [Gon05].

**Example 4.22.** Another example is given by the map of rooted forrests with no binary vertices  $\rightarrow$  corollas. The corresponding morphisms of Feynman categories is again Hopf compatible.

However, if we consider the functor of Feynman categories induced by rooted trees  $\rightarrow$  rooted corollas is not Hopf compatible. It sends all morphisms corresponding to binary trees to the identity morphism of the corolla with one input. Thus is maps non-invertible elements to invertible elements. The presence of these extra morphisms in  $\mathcal{H}_{CK}$  is what makes it especially interesting. They also correspond to a universal property [Moe01] and Example 2.50.

**4.6.2. Opposite Feynman category yields the co-opposite bialgebra.** Notice that usually the opposite category of a Feynman category is not a Feynman category, but it still defines a bialgebra. Namely, the constructions above just yield the co-opposite bialgebra structure  $\mathcal{B}^{co-op}$  and Hopf algebra structure  $\mathcal{H}^{co-op}$  if the extra conditions are met.

This means, the multiplication is unchanged but the comultiplication is switched. That is  $\Delta(\phi^{op}) = \sum_{\phi_1 \circ \phi^0 = \phi} \phi_1^{op} \otimes \phi_0^{op}$ .

**4.7. Constructions on Feynman categories.** There are three constructions on Feynman categories that are relevant to these examples.

**4.7.1. Enrichments, plus construction and hyper category  $\mathfrak{F}^{hyp}$ .** The first construction is the plus construction  $\mathfrak{F}^+$  and its quotient  $\mathfrak{F}^{hyp}$  and its equivalent reduced version  $\mathfrak{F}^{hyp,rd}$ , see [KW13]. The main result [KW13][Lemma 4.5] says that for any Feynman category  $\mathfrak{F}$  there exists a Feynman category  $\mathfrak{F}^{hyp}$  and the set of monoidal functors  $\mathcal{O} : \mathcal{F}^{hyp} \rightarrow \mathcal{E}$  is in 1-1 correspondence with enrichments  $\mathcal{F}_{\mathcal{O}}$  of  $\mathcal{F}$  over  $\mathcal{E}$ .

For such an enrichment, one has

$$Hom_{\mathcal{F}_{\mathcal{O}}}(X, Y) = \coprod_{\phi \in Hom_{\mathcal{F}}(X, Y)} \mathcal{O}(\phi) \quad (4.14)$$

And that if  $\phi$  is an isomorphism, then  $\mathcal{O}(\phi) \simeq \mathbb{1}_{\mathcal{E}}$  This generalizes the notion of hyperoperads of [GK98], whence the superscript *hyp*.

There is an equivalent, but slightly smaller category  $\mathfrak{F}^{hyp,rd}$ , we can alternatively use. The relevant example is that  $\mathfrak{F}_{Surj}^{hyp,rd} = \mathfrak{F}_{operads,0}$  that is, operads whose  $\mathcal{O}(1)$  contains only 1 as an invertible element, we will call these operads almost pointed. Thus any such operad  $\mathcal{O} : \mathcal{F}_{operads,0} \rightarrow \mathcal{E}$  gives rise to a Feynman category  $\mathfrak{F}_{surj,\mathcal{O}}$  whose morphisms are determined by

$$Hom_{\mathfrak{F}_{surj,\mathcal{O}}}(n, 1) = \mathcal{O}(n) \quad (4.15)$$

If  $\mathcal{O}(1)$  has more invertible elements, one has to enlarge  $\mathfrak{F}_{surj}$  by choosing the appropriate  $\mathcal{V}$ . In the case of Cartesian  $\mathcal{E}$  this is  $\mathcal{V} = \mathcal{O}(1)^\times$  and in the  $k$ -linear case this is  $\mathcal{V} = \mathcal{O}(1)^\times/k^*$ .

This gives rise to extra isomorphisms and a  $K$ -collection, see [KW13][2.6.4]. This means in particular that any operad gives rise to morphisms of a Feynman category. The dual of the morphisms are then cooperads and the cooperadic and Feynman categorical construction coincide.

The non- $\Sigma$  case is similar. For this one uses  $\mathfrak{F}_{surj,<}$  and then obtains enrichments by non- $\Sigma$  operads. Thus again the cooperadic methods apply and yield the same results as the Feynman category constructions.

**Proposition 4.23.** *In both the symmetric and non-symmetric case, if  $\mathfrak{F}$  is any  $\mathfrak{F}^{hyp}\text{-Op}$   $\mathcal{O}$  gives rise to a unital, counital bialgebra by regarding the morphisms of  $\mathfrak{F}_{\mathcal{O}}$ . If its quotient by the ideal generated by the  $\mathcal{O}(id_X) \simeq \mathbb{1}$  corresponding to  $\mathcal{J}$  is connected, in which case we call  $\mathfrak{F}_{\mathcal{O}}$  almost connected, we obtain a Hopf algebra.  $\square$*

**Remark 4.24.** Applying the constructions of this chapter to  $\mathfrak{F}_{surj,<,\mathcal{O}}$  equivalent to the construction of Chapter 2 in the free case, see 2.2.3. The symmetric case is then equivalent to considering  $\mathfrak{F}_{surj,\mathcal{O}}$ .

The condition of being almost connected then coincides with the definition of almost connected Definition 2.48.

**Remark 4.25.** The construction of identifying all invertible elements in  $\check{\mathcal{O}}(1)$  is exactly the passage from  $\mathfrak{F}^+\text{-Ops}$  to  $\mathfrak{F}^{hyp}\text{-Ops}$ .

**4.7.2. Decoration  $\mathfrak{F}_{dec\mathcal{O}}$ .** This type of modification is discussed in [KL13]. It gives a new Feynman category  $\mathfrak{F}_{dec\mathcal{O}}$  from a pair  $(\mathfrak{F}, \mathcal{O})$  of a Feynman category  $\mathfrak{F}$  and a strong monoidal functor  $\mathcal{O} : \mathcal{F} \rightarrow \mathcal{C}$ . The objects of  $\mathfrak{F}_{dec\mathcal{O}}$  are pairs  $(X, a_X)$ ,  $a_X \in \mathcal{O}(X)$  ( $a_x \in Hom_{\mathcal{E}}(\mathbb{1}, \mathcal{O}(X))$  for the fastidious reader). The morphisms from  $(X, a_X)$  to  $(Y, a_Y)$  are those  $\phi \in Hom_{\mathcal{F}}(X, Y)$  for which  $\mathcal{O}(\phi)(a_X) = a_Y$ .

Looking at the free Abelian group generated by the morphisms turns the operad into a Feynman category and one can apply the results of this chapter.

This construction explains the constructions of chapter 3 as discussed below.

**4.7.3. Universal operations.** It is shown that  $\mathfrak{F}_{\mathcal{V}}$ , which is given by  $\mathcal{F}_{\mathcal{V}} = \text{colim}_{\mathcal{V}\iota}$ , yields a Feynman category with trivial  $\mathcal{V}_{\mathcal{V}}$ . This generalizes the Meta-Operad structure of [Kau07]. The result is again a Feynman category whose morphisms define an operad and hence the free Abelian group yields a cooperad.

Moreover in many situations, see [KW13] the morphisms of the category are weakly generated by a simple Feynman category obtained by “forgetting tails”. The action is then via a foliation operator as introduced in [Kau07]. In fact there is a poly-simplicial structure hidden here, as can be inferred from [BB09].

**4.7.4. Enrichment over  $\mathcal{C}^{op}$  and opposite Feynman category.** Notice that we can regard functors  $\mathfrak{F} \rightarrow \mathcal{C}^{op}$  as co-versions of operads, etc.. In particular if we have a functor

$\mathfrak{F}^{hyp} \rightarrow \mathcal{C}^{op}$ , we get a Feynman category  $\mathfrak{F}_{\mathcal{O}}$  enriched over  $\mathcal{C}^{op}$ . This means that  $\mathfrak{F}_{\mathcal{O}}^{op}$  is enriched over  $\mathcal{C}$ .

**Example 4.26.** In particular, if  $\mathcal{O} : \mathfrak{F}_{Surj}^{hyp} = \mathfrak{F}_{operads,0} \rightarrow \mathcal{C}^{op}$  that is  $\mathcal{O}$  is an almost pointed operad which is split unital in  $\mathcal{C}^{op}$  or an almost pointed cooperad in  $\mathcal{C}$ . Then decorating with  $\mathcal{O}$  gives us  $\mathfrak{F}_{Surj<, \mathcal{O}}$  which is enriched in  $\mathcal{C}^{op}$ . Taking the opposite we get  $\mathcal{F}_{Surj<, \mathcal{O}}^{op}$ . The underlying category is  $Inj_{*,*}$  enriched by  $\check{\mathcal{O}}$ , where  $\check{\mathcal{O}}$  is the cooperad in  $\mathcal{C}$  corresponding to the operad in  $\mathcal{C}^{op}$ . This means that the objects are the natural numbers  $n$  and the morphisms are  $Hom(1, n) = \check{\mathcal{O}}(n)$ . This is the enrichment in which the unique map in  $Hom_{Inj_{*,*}}([1], [n])$  is assigned  $\check{\mathcal{O}}(n)$  in the overlying enriched category.

Putting all the pieces together then yields the following:

**Theorem 4.27.** *Given a cooperad  $\check{\mathcal{O}}$  that is given by a functor  $\mathcal{O} : \mathfrak{F}_{operads,0} \rightarrow \mathcal{C}^{op}$ . Let  $\mathcal{B}_{\check{\mathcal{O}}nc}$  be the bialgebra of Example 2.2.3. And let  $\mathcal{B}_{\mathfrak{F}_{Surj<, \mathcal{O}}^{op}}$  be the bialgebra of the Feynman category discussed above then these two bialgebra coincide.*

*Moreover if  $\mathfrak{F}_{Surj<, \mathcal{O}}$  is almost reduced, the so is  $\check{\mathcal{O}}$  and the corresponding Hopf algebras coincide.  $\square$*

## 5. DISCUSSION OF THE THREE CASES AND MORE EXAMPLES

We will now illustrate the different concepts and constructions by considering the three main cases as well as a few more instructive examples.

### 5.1. Connes–Kreimer and other graphs.

**5.1.1. Leaf labelled and planar version.** First, we can look at the operad  $\mathcal{O}$  of leaf labelled rooted trees or planar planted trees. This gives a Feynman category by §4.7.1 and hence a bialgebra. Here  $\mathcal{O}(1)$  has two generators  $id_1$  which we denote by  $|$  and  $\blacklozenge$ , the rooted tree with one binary non–root vertex. Now composing  $\blacklozenge$  with itself will result in  $\blacklozenge n$ , the rooted tree with  $n$  binary non–root vertices. We also identify  $\blacklozenge 0 = |$ . Taking the dual, either as the free Abelian group of morphisms, or simply the dual as a cooperad, we obtain a cooperad and the multiplication is either  $\otimes$  from the Feynman category or  $\otimes$  from the free construction. That these two coincide follows from condition (ii) of a Feynman category.  $\eta$  is given by  $| = id_1$ . The Feynman category and the cooperad are almost connected, since  $\Delta(\blacklozenge n) = \sum_{(n_1, n_2): n_1, n_2 \geq 0, n_1 + n_2 = n} \blacklozenge n_1 \otimes \blacklozenge n_2$  and hence the reduced coproduct is given by  $\bar{\Delta}(\blacklozenge n) = \sum_{(n_1, n_2): n_1, n_2 \geq 1, n_1 + n_2 = n} \blacklozenge n_1 \otimes \blacklozenge n_2$  whence  $\check{\mathcal{O}}(1)$  is nilpotent.

If we take planar trees, there are no automorphisms and we obtain the first Hopf algebra of planted planar labelled forests. Notice that in the quotient  $[[ ] = [[ [ \dots ] ] = [1]$  which says that there is only one empty forest.

If we are in the non–planar case, we obtain a Hopf algebra of rooted forests, with labelled leaves. One uses  $\mathcal{V}$  as finite subsets of  $\mathbb{N}$  with isomorphisms.

These structures are also discussed in [Foi02b], [Foi02a] and [EFK05].

**5.1.2. Algebra description.** If one considers the algebras over the operad  $\mathcal{O}$ , then for a given algebra  $\rho, V, \rho(\bullet) \in \text{Hom}(V, V)$  is a “marked” endomorphism. This is the basis of the constructions of [Moe01]. One can also add more extra morphisms, say  $\bullet c$  for  $c \in C$  where  $C$  is some indexing set of colors. This was considered in [vdLM06b]. In general one can include such marked morphisms into Feynman categories (see [KW13][2.7]) as morphisms of  $\emptyset \rightarrow *_{[1]}$ .

**5.1.3. Unlabelled and symmetric version.** In the non-planar case, we have the action of the symmetric groups. In this case, we can use the symmetric construction or mod out by the automorphisms.

We then obtain the commutative Hopf algebra of rooted forests with non-labelled tails.

Alternatively, from the universal construction §4.7.3 on  $\mathfrak{F}_{operads}$  one directly obtains the structure of a Hopf algebra of non-labelled rooted forests with leaves. The action of the automorphisms is free and hence there is also the reduced version of the co- and Hopf algebras.

**5.1.4. No tail version.** For this particular operad, there is the construction of forgetting tails and we can use the construction of §2.11. In this case, we obtain the Hopf algebras of planted planar forests without tails or the commutative Hopf algebra of rooted forests, which is called  $\mathcal{H}_{CK}$ .

Finally, one can amputate the tails in the universal construction. One then obtains the cooperad dual to the pre-Lie operad. That is  $\mathcal{H}_{amp}$  is realized naturally from a weakly generating suboperad, in the nomenclature of [KW13].

**5.1.5. Graph version.** If we look at the Feynman category  $\mathfrak{G} = (\mathcal{Crl}, \text{Agg}, \iota)$  then, we obtain the Hopf algebra of graphs of Connes and Kreimer [CK98]. For this, we notice that the structure of composition in the Feynman category is given by grafting graphs into compatible vertices, i.e. those that have the correct structure of external legs; see Appendix A and [KW13]. Thus the coproduct gives a sum over subgraphs in a graph.

Taking the various quotients, we obtain the symmetric graph Hopf algebra, either with or without automorphism factors.

**5.1.6. 1-PI graph version.** It is easy to see that the property of being 1-PI is preserved under composition in  $\mathfrak{G}$  and hence, we obtain the Hopf algebra of 1-PI graphs.

Recall that a connected 1-PI graph is a connected graph that stays connected, when one severs any edge. A 1-PI graph is then a graph whose every component is 1-PI.

A nice way to write this is as follows [Bro15a]. Let  $b_1(\Gamma)$  be the first Betti number of the graph  $\Gamma$ . Then a graph is 1-PI if for any subgraph  $\gamma \subset \Gamma$ :  $b_1(\gamma) < b_1(\Gamma)$ . In this formulation the condition is also easily checked.

**5.1.7. Other graphs.** The constructions works for any of the Feynman categories built on graphs and their decorations mentioned in [KW13, KL13]. The key thing is that the extra structures respect the concatenation of morphisms, which boils down to plugging graphs into vertices. Examples of this type furnish bi- and Hopf algebras of modular graphs, non- $\Sigma$  modular graphs, trees, planar trees, etc..

**5.1.8. Brown’s motic Hopf algebras.** In [Bro15a] a generalization of 1–PI graphs is given. In this case there are the decorations of (ghost) edges of the morphisms by masses and the momenta; that is, maps  $m : E(\Gamma) \rightarrow \mathbb{R}$  and  $q : T(\Gamma) \rightarrow \mathbb{R}^d \cup \{\emptyset\}$ . Notice that these are decorations in the technical sense of [KL13] as well. The masses carry over onto the new edges upon insertion. For the tails the composition rule is as follows: the tails that are labelled by  $\emptyset$  become half of an edge on insertion and the tails that are labelled otherwise remain tails and keep their decoration. A subgraph  $\gamma$  of a graph  $\Gamma$  is called momentum and mass spanning (m.m.) if it contains all the tails and all the edges with non-zero mass.

A graph  $\Gamma$  is called motic if for any m.m. subgraph  $\gamma$ :  $b_1(\gamma) < b_1(\Gamma)$ .

This condition is again stable under composition, i.e. gluing graphs into vertices as shown in [Bro15a][Theorem 3.6]. After taking the quotient, we see that the one vertex ghost graph becomes identified with the empty graph and we obtain the Hopf algebra structure of [Bro15a][Theorem 4.2] from this Feynman category after amputating the tails marked by  $\emptyset$ .

## 5.2. (Semi)–Simplicial case.

**5.2.1. With  $\mathbb{Z}$  coefficients.** This is the case of a decorated Feynman category. Given a semi-simplicial set  $X_\bullet$  then  $C^*(X_\bullet)$  can be made into a functor from  $\mathfrak{F}_{Surj,<}$ . Namely, we assign to each  $n$  the set  $C^*(X_\bullet)^{\otimes n} \simeq C^*(X_\bullet^{\times n})$  and to the unique map  $n \rightarrow 1$  the iterated cup product  $\cup^{n-1}$ . This is just the fact that  $C^*(X_\bullet)$  is an algebra. In other words  $X_\bullet$  can be thought of as a functor  $\mathfrak{F}_{Surj,<} \rightarrow \mathcal{C}$  and we can decorate with it. After decorating, the objects become collections of cochains, and there is a unique map with source an  $n$ –collection of cochains and target a single cochain, which is the iterated cup product. Thus, one can identify the morphisms of this type with the objects. Furthermore, the set of morphisms then possesses a natural structure of Abelian group. Dualizing this Abelian group, we get the cooperad structure on  $C_*(X_\bullet)$  and the cooperad structure with multiplication on  $C_*(X_\bullet)^\otimes$  that coincides with the one considered in chapter §3.

The bialgebra is almost connected if the 1–skeleton of  $X_\bullet$  is connected. And after quotienting we obtain the same Hopf algebra structure from both constructions.

**5.2.2. Relation to  $\cup_i$  products.** It is here that we find the similarity to the  $\cup_i$  products also noticed by JDS Jones. Namely, in order to apply  $\cup^{n-1}$  to a simplex, we first use the Joyal dual map  $[1] \rightarrow [n]$  on the simplex. This is the map that is also used for the  $\cup_i$  product. The only difference is that instead of using  $n$  cochains, one only uses two. To formalize this one needs a surjection that is not in  $\Delta$ , but uses a permutation, and hence lives in  $S\Delta_+$ . Here the surjection  $Surj$  gives rise to what is alternatively called the sequence operad. Joyal duality is then the fact that one uses sequences and overlapping sequences in the language of [MS03]. The pictorial realizations and associated representations can be found in [Kau08] and [Kau09]. This is also related to the notion of discs in Joyal [Joy97]. This connection will be investigated in the future.

In the Hopf algebra situation, we see that the terms of the iterated  $\cup_1$  product coincide with the second factor of  $\Delta$ . Compare Figure 4.



**5.2.3. Over Set: Special case of the nerve of a category, colored operad structure.** In general there is no operad structure on  $X_\bullet$  itself. By the operad structure on simplices, we can try to put an operad structure on  $X_\bullet$  by composing an  $n$  simplex and an  $m$  simplex if the respective images of  $i$  and  $i + 1$  agree. This simplex need not exist, but it does if the simplicial set is the nerve of a category. In particular, if  $X_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_n} X_n$  is an  $n$  simplex and  $X_{i-1} = Y_0 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_m} Y_m = X_i$ , with  $\psi_m \circ \cdots \circ \psi_1 = \phi_i$ , then we can compose to

$$X_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{i-1}} X_{i-1} = Y_0 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_m} Y_m = X_{i+1} \xrightarrow{\phi_{i+1}} \cdots \xrightarrow{\phi_n} X_n$$

In the Feynman category language,  $\mathcal{V}$  is discrete, but not trivial, in particular  $\mathcal{V} = \{X_0 \rightarrow X_1\}$  is the set of one-simplices. The morphisms they yield a colored operad structure over the set  $Ob(\mathcal{V})$ . Each morphism/ $n$ -simplex  $X_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_n} X_n$  is a morphism from  $\phi_1 \otimes \cdots \otimes \phi_n \rightarrow \phi = \phi_n \circ \cdots \circ \phi_1$ .

**5.2.4. Over Set: Special case of the nerve of a complete groupoid.** If the underlying category is a complete groupoid, so that there is exactly one morphism per pair of objects, then any  $n$ -simplex can simply be replaced by the word  $X_0 \cdots X_n$  of its sources and targets. Notice that  $\mathcal{V} = \{X_0 X_1\}$  is the set of words of length 2 not 1. This explains the constructions of Goncharov involving MZVs, but also polylogs [Gon05], and the subsequent construction of Brown.

**5.3. Semi-simplicial objects and links to Chapter 3.** By definition a semi-simplicial object in  $\mathcal{C}$  is a functor  $X_\bullet : Surj_{<}^{op} \rightarrow \mathcal{C}$ , and rewriting this, we see that this is equivalent either to a functor  $Surj_{<} \rightarrow \mathcal{C}^{op}$  or to a functor  $Inj_{*,*} \rightarrow \mathcal{C}$ . Our constructions of §3 actually work with the last interpretation.

The second and third descriptions open this up for a description in terms of Feynman categories. Notice that in this interpretation  $X_\bullet$  is a functor from  $\mathfrak{F}_{Surj,<}$ , but it is not monoidal. In [KW13][Chapter 3.1], a free monoidal Feynman category  $\mathfrak{F}^\boxtimes$  is constructed, such that  $\mathfrak{F}^\boxtimes\text{-Ops}_{\mathcal{C}}$  is equal to  $Fun(\mathcal{F}, \mathcal{C})$ , that is all functors, not necessarily monoidal ones. So we could decorate  $\mathfrak{F}_{Surj,<}^\boxtimes$  with the semi-simplicial set  $X_\bullet$ , and then regard the decorated  $\mathfrak{F}_{Surj,<}^\boxtimes, dec X_\bullet$ .

What is more pertinent however, is that since there is the oplax monoidal structure §3.3, induced by  $X_{p+q} \rightarrow X_p \times X_q$  in the Feynman category language means that we get a morphism from the non-connected version  $\mathfrak{F}_{Surj,<}^{nc}$  of  $\mathfrak{F}_{Surj,<}$ . The cubical realization of this using the functors  $L$  of §3.3 in the more general context will be the subject of further investigation.

Another interesting fact is that  $Inj_{*,*}$  also forms a Feynman category. This is parallel to the discussion in [KW13, Chapter 2.9.3], although we need to tweak the construction slightly.  $\mathcal{V}$  is trivial and the underlying objects of  $\mathcal{F}$  are the natural numbers. To each  $n$  we associate  $[n + 1]$ . We take the identity in  $Hom(1, 1)$  and its tensor powers give the identities in  $Hom_{\mathcal{F}}(n, n)$ . Now we add one morphism in  $Hom_{\mathcal{F}}(\mathbb{1} = 0, 1)$  which we will call special. Any base point preserving injection from  $[n + 1]$  to  $[m + 1]$  is then represented by a tensor product of identities and special maps.



There is another way to see the constructions of chapter §3. For this note that we can consider functors  $\tilde{\mathcal{O}} : \mathfrak{F}_{Surj_{<}}^{hyp} = \mathfrak{F}_{operads,0} \rightarrow \mathcal{C}^{op}$  as almost pointed operads in  $\mathcal{C}^{op}$  or almost pointed cooperads in  $\mathcal{C}$ . Then enriching with  $\tilde{\mathcal{O}}$  gives us  $\mathfrak{F}_{Surj_{<},\tilde{\mathcal{O}}}$  which is enriched in  $\mathcal{C}^{op}$  now. Flipping to opposite categories, this is the same as considering  $\mathfrak{F}_{Inj_{*,*},\tilde{\mathcal{O}}}$ .

**5.3.1. Goncharov multiple zeta values and polylogarithms.** Taking the contractible groupoid on  $0, 1$  we obtain the construction of  $\mathcal{H}_{Gon}$  for the multi-zeta values. If we take that with objects  $z_i$ , we obtain Goncharov’s Hopf algebra for polylogarithms [Gon05].

**5.3.2. Baues.** This is the case of a general simplicial set, which however is 1-connected. We note that since we are dealing with graded objects, one has to specify that one is in the usual monoidal category of graded  $\mathbb{Z}$  modules whose tensor product is given by the Koszul or super sign.

**5.4. Boot–strap.** There is the following nice observation. The simplest Feynman category is given by  $\mathfrak{F}_{triv} = (\mathcal{V} = triv, \mathcal{F} = \mathcal{V}^{\otimes}, \iota)$  and  $\mathfrak{F}_{triv}^+ = \mathfrak{F}_{surj}$  [KW13, Example 3.6]. Going further,  $\mathfrak{F}_{surj}^+ = \mathfrak{F}_{May\ operads}$  [Example 3.7]. Adding units gives  $\mathfrak{F}_{operads}$  and then  $\mathfrak{F}_{\mathcal{V}}$  gives  $\mathfrak{F}_{surj,\mathcal{O}=leaf\ labelled\ trees}$ . Decorating by simplicial sets, we obtain the three original examples from these constructions.

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## APPENDIX

### APPENDIX A. GRAPH GLOSSARY

**A.1. The category of graphs.** Interesting examples of Feynman categories used in operad-like theories are indexed over a Feynman category built from graphs. It is important to note that although we will first introduce a category of graphs *Graphs*, the relevant Feynman category is given by a full subcategory *Agg* whose objects are disjoint unions or aggregates of corollas. The corollas themselves play the role of  $\mathcal{V}$ .

Before giving more examples in terms of graphs it will be useful to recall some terminology. A very useful presentation is given in [BM08] which we follow here.

**A.1.1. Abstract graphs.** An abstract graph  $\Gamma$  is a quadruple  $(V_\Gamma, F_\Gamma, i_\Gamma, \partial_\Gamma)$  of a finite set of vertices  $V_\Gamma$ , a finite set of half edges or flags  $F_\Gamma$ , an involution on flags  $i_\Gamma: F_\Gamma \rightarrow F_\Gamma; i_\Gamma^2 = id$  and a map  $\partial_\Gamma: F_\Gamma \rightarrow V_\Gamma$ . We will omit the subscript  $\Gamma$  if no confusion arises.

Since the map  $i$  is an involution, it has orbits of order one or two. We will call the flags in an orbit of order one *tails* and denote the set of tails by  $T_\Gamma$ . We will call an orbit of order two an *edge* and denote the set of edges by  $E_\Gamma$ . The flags of an edge are its elements. The function  $\partial$  gives the vertex a flag is incident to. It is clear that the set of vertices and edges form a 1-dimensional CW complex. The realization of a graph is the realization of this CW complex.

A graph is (simply) connected if and only if its realization is. Notice that the graphs do not need to be connected. Lone vertices, that is, vertices with no incident flags, are also possible.

We also allow the empty graph  $\mathbb{1}_\emptyset$ , that is, the unique graph with  $V = \emptyset$ . It will serve as the monoidal unit.

**Example A.1.** A graph with one vertex and no edges is called a *corolla*. Such a graph only has tails. For any set  $S$  the corolla  $*_{p,S}$  is the unique graph with  $V$  a singleton and  $F = S$ .

Given a vertex  $v$  of a graph, we set  $F_v = \partial^{-1}(v)$  and call it *the flags incident to  $v$* . This set naturally gives rise to a corolla. The *tails* at  $v$  is the subset of tails of  $F_v$ .

As remarked above,  $F_v$  defines a corolla  $*_v = *_{\{v\},F_v}$ .

**Remark A.2.** The way things are set up, we are talking about (finite) sets, so changing the sets even by bijection changes the graphs.

**Remark A.3.** As the graphs do not need to be connected, given two graphs  $\Gamma$  and  $\Gamma'$  we can form their disjoint union:

$$\Gamma \sqcup \Gamma' = (F_\Gamma \sqcup F_{\Gamma'}, V_\Gamma \sqcup V_{\Gamma'}, i_\Gamma \sqcup i_{\Gamma'}, \partial_\Gamma \sqcup \partial_{\Gamma'})$$

One actually needs to be a bit careful about how disjoint unions are defined. Although one tends to think that the disjoint union  $X \sqcup Y$  is strictly symmetric, this is not the case. This becomes apparent if  $X \cap Y \neq \emptyset$ . Of course there is a bijection  $X \sqcup Y \xrightarrow{1-1} Y \sqcup X$ . Thus the categories here are symmetric monoidal, but not strict symmetric monoidal. This is important, since we consider functors into other not necessarily strict monoidal categories.

Using MacLane's theorem it is however possible to make a technical construction that makes the monoidal structure (on both sides) into a strict symmetric monoidal structure

**Example A.4.** An *aggregate of corollas* or aggregate for short is a finite disjoint union of corollas, that is, a graph with no edges.

Notice that if one looks at  $X = \bigsqcup_{v \in I} *_{S_v}$  for some finite index set  $I$  and some finite sets of flags  $S_v$ , then the set of flags is automatically the disjoint union of the sets  $S_v$ . We will just say just say  $s \in F_X$  if  $s$  is in some  $S_v$ .

### A.1.2. Category structure; Morphisms of Graphs.

**Definition A.5.** [BM08] Given two graphs  $\Gamma$  and  $\Gamma'$ , consider a triple  $(\phi^F, \phi_V, i_\phi)$  where

- (i)  $\phi^F: F_{\Gamma'} \hookrightarrow F_\Gamma$  is an injection,
- (ii)  $\phi_V: V_\Gamma \twoheadrightarrow V_{\Gamma'}$  and  $i_\phi$  is a surjection and
- (iii)  $i_\phi$  is a fixed point free involution on the tails of  $\Gamma$  not in the image of  $\phi^F$ .

One calls the edges and flags that are not in the image of  $\phi$  the *contracted edges* and flags. The orbits of  $i_\phi$  are called *ghost edges* and denoted by  $E_{ghost}(\phi)$ .

Such a triple is a *morphism of graphs*  $\phi: \Gamma \rightarrow \Gamma'$  if

- (1) The involutions are compatible:
  - (a) An edge of  $\Gamma$  is either a subset of the image of  $\phi^F$  or not contained in it.
  - (b) If an edge is in the image of  $\phi^F$  then its pre-image is also an edge.
- (2)  $\phi^F$  and  $\phi_V$  are compatible with the maps  $\partial$ :
  - (a) Compatibility with  $\partial$  on the image of  $\phi^F$ :  
If  $f = \phi^F(f')$  then  $\phi_V(\partial f) = \partial f'$

(b) Compatibility with  $\partial$  on the complement of the image of  $\phi^F$ :

The two vertices of a ghost edge in  $\Gamma$  map to the same vertex in  $\Gamma'$  under  $\phi_V$ .

If the image of an edge under  $\phi^F$  is not an edge, we say that  $\phi$  grafts the two flags.

The composition  $\phi' \circ \phi: \Gamma \rightarrow \Gamma''$  of two morphisms  $\phi: \Gamma \rightarrow \Gamma'$  and  $\phi': \Gamma' \rightarrow \Gamma''$  is defined to be  $(\phi^F \circ \phi'^F, \phi'_V \circ \phi_V, i)$  where  $i$  is defined by its orbits viz. the ghost edges. Both maps  $\phi^F$  and  $\phi'^F$  are injective, so that the complement of their concatenation is in bijection with the disjoint union of the complements of the two maps. We take  $i$  to be the involution whose orbits are the union of the ghost edges of  $\phi$  and  $\phi'$  under this identification.

**Remark A.6.** A *naïve morphism* of graphs  $\psi: \Gamma \rightarrow \Gamma'$  is given by a pair of maps  $(\psi_F: F_\Gamma \rightarrow F_{\Gamma'}, \psi_V: V_\Gamma \rightarrow V_{\Gamma'})$  compatible with the maps  $i$  and  $\partial$  in the obvious fashion. This notion is good to define subgraphs and automorphisms.

It turns out that this data *is not enough* to capture all the needed aspects for composing along graphs. For instance it is not possible to contract edges with such a map or graft two flags into one edge. The basic operations of composition in an operad viewed in graphs is however exactly grafting two flags and then contracting.

For this and other more subtle aspects one needs the more involved definition above which we will use.

**Definition A.7.** We let *Graphs* be the category whose objects are abstract graphs and whose morphisms are the morphisms described in Definition A.5. We consider it to be a monoidal category with monoidal product  $\sqcup$  (see Remark A.3).

**A.1.3. Decomposition of morphisms.** Given a morphism  $\phi: X \rightarrow Y$  where  $X = \bigsqcup_{w \in V_X} *w$  and  $Y = \bigsqcup_{v \in V_Y} *v$  are two aggregates, we can decompose  $\phi = \bigsqcup \phi_v$  with  $\phi_v: X_v \rightarrow *v$  where  $X_v$  is the subaggregate  $\bigsqcup_{\phi_V(w)=v} *w$ , and  $\bigsqcup_v X_v = X$ . Here  $(\phi_v)_V$  is the restriction of  $\phi_V$  to  $V_{X_v}$ . Likewise  $\phi_v^F$  is the restriction of  $\phi^F$  to  $(\phi^F)^{-1}(F_{X_v} \cap \phi^F(F_Y))$ . This is still injective. Finally  $i_{\phi_v}$  is the restriction of  $i_\phi$  to  $F_{X_v} \setminus \phi^F(F_Y)$ . These restrictions are possible due to the condition (2) above.

**A.1.4. Ghost graph of a morphism.** The underlying ghost graph of a morphism of graphs  $\phi: \Gamma \rightarrow \Gamma'$  is the graph  $\Gamma(\phi) = (V(\Gamma), F_\Gamma, \hat{i}_\phi)$  where  $\hat{i}_\phi$  is  $i_\phi$  on the complement of  $\phi^F(\Gamma')$  and identity on the image of flags of  $\Gamma'$  under  $\phi^F$ . The edges of  $\Gamma(\phi)$  are called the ghost edges of  $\phi$ .

## A.2. Extra structures.

**A.2.1. Glossary.** This section is intended as a reference section. All the following definitions are standard.

Recall that an order of a finite set  $S$  is a bijection  $S \rightarrow \{1, \dots, |S|\}$ . Thus the group  $S_{|S|} = \text{Aut}\{1, \dots, n\}$  acts on all orders. An orientation of a finite set  $S$  is an equivalence class of orders, where two orders are equivalent if they are obtained from each other by an even permutation.

A tree	is a connected, simply connected graph.
A directed graph $\Gamma$	is a graph together with a map $F_\Gamma \rightarrow \{in, out\}$ such that the two flags of each edge are mapped to different values.
A rooted tree	is a directed tree such that each vertex has exactly one “out” flag.
A ribbon or fat graph	is a graph together with a cyclic order on each of the sets $F_v$ .
A planar graph	is a ribbon graph that can be embedded into the plane such that the induced cyclic orders of the sets $F_v$ from the orientation of the plane coincide with the chosen cyclic orders.
A planted planar tree	is a rooted planar tree together with a linear order on the set of flags incident to the root.
An oriented graph	is a graph with an orientation on the set of its edges.
An ordered graph	is a graph with an order on the set of its edges.
A $\gamma$ labelled graph	is a graph together with a map $\gamma : V_\Gamma \rightarrow \mathbb{N}_0$ .
A b/w graph	is a graph $\Gamma$ with a map $V_\Gamma \rightarrow \{black, white\}$ .
A bipartite graph	is a b/w graph whose edges connect only black to white vertices.
A $c$ colored graph	for a set $c$ is a graph $\Gamma$ together with a map $F_\Gamma \rightarrow c$ s.t. each edge has flags of the same color.

### A.2.2. Remarks and language.

- (1) In a directed graph one speaks about the “in” and the “out” edges, flags or tails at a vertex. For the edges this means the one flag of the edges is an “in” flag at the vertex. In pictorial versions the direction is indicated by an arrow. A flag is an “in” flag if the arrow points to the vertex.
- (2) As usual there are edge paths on a graph and the natural notion of an oriented edge path. An edge path is a (oriented) cycle if it starts and stops at the same vertex and all the edges are pairwise distinct. It is called simple if each vertex on the cycle has exactly one incoming flag and one outgoing flag belonging to the cycle. An oriented simple cycle will be called a *wheel*. An edge whose two vertices coincide is called a (*small*) *loop*.
- (3) There is a notion of the genus of a graph, which is the minimal dimension of the surface it can be embedded on. A ribbon graph is planar if this genus is 0.
- (4) For any graph, its Euler characteristic is given by

$$\chi(\Gamma) = b_0(\Gamma) - b_1(\Gamma) = |V_\Gamma| - |E_\Gamma|;$$

where  $b_0, b_1$  are the Betti numbers of the (realization of)  $\Gamma$ . Given a  $\gamma$  labelled graph, we define the total  $\gamma$  as

$$\gamma(\Gamma) = 1 - \chi(\Gamma) + \sum_{v \text{ vertex of } \Gamma} \gamma(v) \tag{A.1}$$

If  $\Gamma$  is *connected*, that is  $b_0(\Gamma) = 1$  then a  $\gamma$  labeled graph is traditionally called a genus labeled graph and

$$\gamma(\Gamma) = \sum_{v \in V_\Gamma} \gamma(v) + b_1(\Gamma) \quad (\text{A.2})$$

is called the genus of  $\Gamma$ . This is actually not the genus of the underlying graph, but the genus of a connected Riemann surface with possible double points whose dual graph is the genus labelled graph.

A genus labelled graph is called *stable* if each vertex with genus labeling 0 has at least 3 flags and each vertex with genus label 1 has at least one edge.

- (5) A planted planar tree induces a linear order on all sets  $F_v$ , by declaring the first flag to be the unique outgoing one. Moreover, there is a natural order on the edges, vertices and flags given by its planar embedding.
- (6) A rooted tree is usually taken to be a tree with a marked vertex. Note that necessarily a rooted tree as described above has exactly one “out” tail. The unique vertex whose “out” flag is not a part of an edge is the root vertex. The usual picture is obtained by deleting this unique “out” tail.

### A.2.3. Category of directed/ordered/oriented graphs.

- (1) Define the category of directed graphs  $\mathcal{G}raphs^{dir}$  to be the category whose objects are directed graphs. Morphisms are morphisms  $\phi$  of the underlying graphs, which additionally satisfy that  $\phi^F$  preserves orientation of the flags and the  $i_\phi$  also only has orbits consisting of one “in” and one “out” flag, that is the ghost graph is also directed.
- (2) The category of edge ordered graphs  $\mathcal{G}raphs^{or}$  has as objects graphs with an order on the edges. A morphism is a morphism together with an order  $ord$  on all of the edges of the ghost graph.

The composition of orders on the ghost edges is as follows.  $(\phi, ord) \circ \bigsqcup_{v \in V} (\phi_v, ord_v) := (\phi \circ \bigsqcup_{v \in V} \phi_v, ord \circ \bigsqcup_{v \in V} ord_v)$  where the order on the set of all ghost edges, that is  $E_{ghost}(\phi) \sqcup \bigsqcup_v E_{ghost}(\phi_v)$ , is given by first enumerating the elements of  $E_{ghost}(\phi_v)$  in the order  $ord_v$  where the order of the sets  $E(\phi_v)$  is given by the order on  $V$ , i.e. given by the explicit ordering of the tensor product in  $Y = \bigsqcup_v *_v$ .<sup>3</sup> and then enumerating the edges of  $E_{ghost}(\phi)$  in their order  $ord$ .

- (3) The oriented version  $\mathcal{G}raphs^{or}$  is then obtained by passing from orders to equivalence classes.

**A.2.4. Category of planar aggregates and tree morphisms.** Although it is hard to write down a consistent theory of planar graphs with planar morphisms, if not impossible, there does exist a planar version of special subcategory of  $\mathcal{G}raphs$ .

We let  $\mathcal{C}rl^{pl}$  have as objects planar corollas — which simply means that there is a cyclic order on the flags — and as morphisms isomorphisms of these, that is isomorphisms of

<sup>3</sup>Now we are working with ordered tensor products. Alternatively one can just index the outer order by the set  $V$  by using [Del90]



graphs, which preserve the cyclic order. The automorphisms of a corolla  $*_S$  are then isomorphic to  $C_{|S|}$ , the cyclic group of order  $|S|$ . Let  $\mathfrak{C}^{pl}$  be the full subcategory of aggregates of planar corollas whose morphisms are morphisms of the underlying corollas, for which the ghost graphs in their planar structure induced by the source is compatible with the planar structure on the target via  $\phi^F$ . For this we use the fact that the tails of a planar tree have a cyclic order.

Let  $\mathcal{C}rl^{pl,dir}$  be directed planar corollas with one output and let  $\mathfrak{D}^{pl}$  be the subcategory of  $\mathcal{A}gg^{pl,dir}$  of aggregates of corollas of the type just mentioned, whose morphisms are morphisms of the underlying directed corollas such that their associated ghost graphs are compatible with the planar structures as above.

### A.3. Flag killing and leaf operators; insertion operations.

**A.3.1. Killing tails.** We define the operator *trun*, which removes all tails from a graph. Technically,  $trun(\Gamma) = (V_\Gamma, F_\Gamma \setminus T_\Gamma, \partial_\Gamma|_{F_\Gamma \setminus T_\Gamma}, \iota_\Gamma|_{F_\Gamma \setminus T_\Gamma})$ .

**A.3.2. Adding tails.** Inversely, we define the formal expression *leaf* which associates to each  $\Gamma$  without tails the formal sum  $\sum_n \sum_{\Gamma': trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma) \sqcup \bar{n}} \Gamma'$ , that is all possible additions of tails where these tails are a standard set, to avoid isomorphic duplication. To make this well defined, we can consider the series as a power series in  $t$ :  $leaf(\Gamma) = \sum_n \sum_{\Gamma': trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma) \sqcup \bar{n}} \Gamma' t^n$

This is the foliage operator of [KS00, Kau07] which was rediscovered in [BBM13].

**A.3.3. Insertion.** Given graphs,  $\Gamma, \Gamma'$ , a vertex  $v \in V_\Gamma$  and an isomorphism  $\phi: F_v \mapsto T_{\Gamma'}$  we define  $\Gamma \circ_v \Gamma'$  to be the graph obtained by deleting  $v$  and identifying the flags of  $v$  with the tails of  $\Gamma'$  via  $\phi$ . Notice that if  $\Gamma$  and  $\Gamma'$  are ghost graphs of a morphism then it is just the composition of ghost graphs, with the morphisms at the other vertices being the identity.

**A.3.4. Unlabelled insertion.** If we are considering graphs with unlabelled tails, that is, classes  $[\Gamma]$  and  $[\Gamma']$  of coinvariants under the action of permutation of tails. The insertion naturally lifts as  $[\Gamma] \circ [\Gamma'] := [\sum_\phi \Gamma \circ_v \Gamma']$  where  $\phi$  runs through all the possible isomorphisms of two fixed lifts.

**A.3.5. No-tail insertion.** If  $\Gamma$  and  $\Gamma'$  are graphs without tails and  $v$  a vertex of  $v$ , then we define  $\Gamma \circ_v \Gamma' = \Gamma \circ_v \text{coeff}(leaf(\Gamma'), t^{|F_v|})$ , the (formal) sum of graphs where  $\phi$  is one fixed identification of  $F_v$  with  $\overline{|F_v|}$ . In other words one deletes  $v$  and grafts all the tails to all possible positions on  $\Gamma'$ . Alternatively one can sum over all  $\partial: F_\Gamma \sqcup F_{\Gamma'} \rightarrow V_\Gamma \setminus v \sqcup V_{\Gamma'}$  where  $\partial$  is  $\partial_G$  when restricted to  $F_w, w \in V_\Gamma$  and  $\partial_{\Gamma'}$  when restricted to  $F_{v'}, v' \in V_{\Gamma'}$ .

**A.3.6. Compatibility.** Let  $\Gamma$  and  $\Gamma'$  be two graphs without flags, then for any vertex  $v$  of  $\Gamma$   $leaf(\Gamma \circ_v \Gamma') = leaf(\Gamma) \circ_v leaf(\Gamma')$ .

**A.4. Graphs with tails and without tails.** There are two equivalent pictures one can use for the (co-)operad structure underlying the Connes–Kreimer Hopf algebra of rooted trees. One can either work with tails that are flags, or with tail vertices. These two concepts are of course equivalent in the setting where if one allows flag tails, disallows vertices with valence and vice-versa if one disallows tails, one allows one valenced vertices called tail vertices. In [CK98] graphs without tails are used. Here we collect some combinatorial facts which represent this equivalence as a useful dictionary.

There are the obvious two maps which either add a vertex at each the end of each tail, or, in the other direction, simply delete each valence one vertex and its unique incident flag, but what is relevant for the Connes–Kreimer example is another set of maps. The first takes a graph with no flag tails to the tree which to every vertex, we *add* a tail, we will denote this map by  $\sharp$  and we add one extra (outgoing) flag to the root, which will be called the root flag.

The second map  $\flat$  simply deletes all tails. We see that  $\flat \circ \sharp = id$ . But  $\flat$  is not the double sided inverse, since  $\sharp \circ \flat$  replaces any number of tails at a given vertex by one tail. It is the identity on the image of  $\sharp$ , which we call single tail graphs.

Notice that  $\sharp$  is well defined on leaf labelled trees by just transferring the labels as sets. Likewise  $\flat$  is well defined on single tail trees again by transferring the labels. This means that each vertex will be labelled.

There are the following degenerate graphs which are allowed in the two setups: the empty graph  $\emptyset$  and the graph with one flag and no vertices  $|$ . We declare that

$$\emptyset^\sharp = | \text{ and vice-versa } |\flat = \emptyset \tag{A.3}$$

**A.4.1. Planted vs. rooted.** A planted tree is a rooted tree whose root has valence 1. One can plant a rooted tree  $\tau$  to obtain a new planted rooted tree  $\tau^\downarrow$ , by adding a new vertex which will be the root of  $\tau^\downarrow$  and adding one edge between the new vertex and the old root. Vice-versa, given a planted rooted tree  $\tau$ , we let  $\tau^\uparrow$  be the uprooted tree that is obtained from  $\tau$  by deleting the root vertex and its unique incident edge, while declaring the other vertex of that edge to be the root.

**A.5. Operad structures on rooted/planted trees.** There are several operad structures on leaf-labelled trees which appear.

For rooted trees without tails and labelled vertices, we define

- (1)  $\tau \circ_i \tau'$  is the tree where the  $i$ -th vertex of  $\tau$  is identified with the root of  $\tau'$ . The root of the resulting tree being the image of the root of  $\tau$ .
- (2)  $\tau \circ_i^+ \tau'$  is the tree where the  $i$ -th vertex of  $\tau$  is joined to the root of  $\tau'$  by a new edge, with the root of the resulting tree is then the image of the root of  $\tau$ .

It is actually the second operad structure that underlies the Connes-Kreimer Hopf algebra. One can now easily check that

$$\tau \circ_i^+ \tau' = \tau \circ_i \tau'^\downarrow = (\tau^\downarrow \circ_i \tau'^\downarrow)^\uparrow \tag{A.4}$$

These constructions also allow us to relate the compositions of trees with and without tails as follows

$$(\tau^\# \circ_i \tau'^\#)^\flat = \tau \circ_i^+ \tau' \tag{A.5}$$

where the  $\circ_i$  operation on the left is the one connecting the  $i$ th flag to the root flag.

**A.5.1. Planar case: marking angles.** In the case of planar trees, we have to redefine  $\#$  by adding a flag to every *angle* of a planar tree. The labels are then not on the vertices, but rather the angles. The analogous equations hold as above. Notice that to give a root to a planar tree actually means to specify a vertex and an angle on it. Planting it connects a new vertex into that angle.

This angle marking is directly to the angle marking in Joyal duality, see below and Figures 3 and 6. This also explains the appearance of angle markings in [Gon05].

## APPENDIX B. COALGEBRAS AND HOPF ALGEBRAS

A good source for this material is [Car07].

**Definition B.1.** A coalgebra with a split counit is a triple  $(\mathcal{H}, \epsilon, \eta)$ , where  $(\mathcal{H}, \epsilon)$  is a cogebra and  $\eta : \mathbb{1} \rightarrow \mathcal{H}$  is a section of  $\eta$ , such that if  $| := \eta(1)$ ,  $\Delta(|) = | \otimes |$ .

Using  $\eta$ , we split  $\mathcal{H} = \mathbb{1} \oplus \bar{\mathcal{H}}$  where  $\bar{\mathcal{H}} := \ker(\epsilon)$

Following Quillen [Qui67], one defines  $\bar{\Delta}(a) := \Delta(a) - | \otimes a - a \otimes |$  where  $| := \eta(1)$

If  $(\mathcal{H}, \mu, \Delta, \eta, \epsilon)$  is a bialgebra then the restriction  $(\mathcal{H}, \Delta, \epsilon)$  is a coalgebra with split counit.

A coalgebra with split counit  $\mathcal{H}$  is said to be conilpotent if for all  $a \in \bar{\mathcal{H}}$  there is an  $n$  such that  $\bar{\Delta}^n(a) = 0$  or equivalently if for some  $m : a \in \ker(\text{pr}^{\otimes m+1} \circ \Delta^m)$ .

Quillen defined the following filtered object.

$$F^0 = \mathbb{1}; F^m = \{a : \bar{\Delta}a \in F^{m-1} \otimes F^{m-1}\} \tag{B.1}$$

$\mathcal{H}$  is said to be connected, if  $\mathcal{H} = \bigcup_m F^m$ . If  $\mathcal{H}$  is connected, then it is nilpotent, and conversely if taking kernels and the tensor product commute then conilpotent implies connected where  $F^m = \ker(\text{pr}^{\otimes m+1} \circ \Delta^m)$ .

For a conilpotent bialgebra algebra there is a unique formula for a possible antipode given by:

$$S(x) = \sum_{n \geq 0} (-1)^{n+1} \mu^n \circ \bar{\Delta}^n(x) \tag{B.2}$$

where  $\bar{\Delta}^n : \mathcal{H} \rightarrow \mathcal{H}^{\otimes n}$  is the  $n - 1$ -st iterate of  $\bar{\Delta}$  that is unique due to coassociativity and  $\mu^n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$  is the  $n - 1$ -st iterate of the multiplication  $\mu$  that is unique due to associativity.

## APPENDIX C. JOYAL DUALITY, SURJECTIONS, INJECTIONS AND LEAF VS. ANGLE MARKINGS

**C.1. Joyal duality.** There is a well known duality [Joy97] of two subcategories of  $\Delta_+$ . This history of this duality can be traced back to [Str80]. Here we review this operation

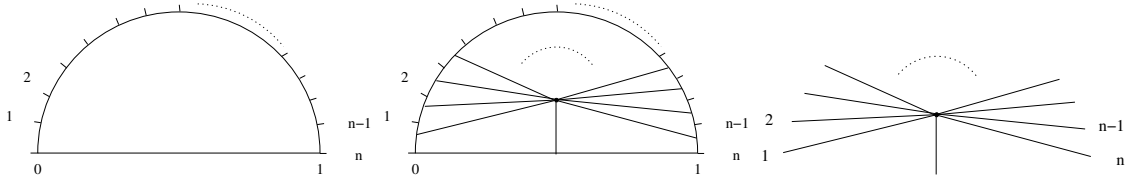


FIGURE 3. The interval injection  $[1] \rightarrow [n]$  on the left, the surjection  $\underline{n} \rightarrow \underline{1}$  on the right and Joyal duality in the middle. Here reading the morphism upwards yields the double base point preserving injection, while reading it downward the surjection.

and show how it can be graphically interpreted. The graphical notation we present in turn connects to the graphical notation in [Gon05] and [GGL09].

The first of the two subcategories of  $\Delta_+$  is  $\Delta$  and the second is the category of intervals. Since we will be dealing with both  $\Delta$  and  $\Delta_+$ , we will use the notation  $\underline{n}$  for the small category  $1 \rightarrow \cdots \rightarrow n$  in  $\Delta$  and  $[n]$  for  $0 \rightarrow 1 \rightarrow \cdots \rightarrow n$  in  $\Delta_+$ . The subcategory of intervals is then the wide subcategory of  $\Delta_+$  whose morphisms preserve both the beginning and the end point. We will denote these maps by  $Hom_{*,*}([m], [n])$ . Explicitly  $\phi \in Hom_{*,*}([m], [n])$  is  $\phi(0) = 0$  and  $\phi(m) = n$ .

The contravariant duality is then given by the association  $Hom_{*,*}([m], [n]) \simeq Hom(\underline{n}, \underline{m})$  defined by  $\phi \xleftrightarrow{1-1} \psi$  given by

$$\psi(i) = \min\{j : \phi(j) \geq i\} - 1, \quad \phi(j) = \max\{i : \psi(i) < j\} + 1.$$

This identification is contravariant.

**C.2. Semi-simplicial objects.** We will mostly be interested in the subcategory  $Surj_{<}^op$  of  $\Delta$  consisting of order preserving surjections. Notice that  $Fun(Surj_{<}^op, \mathcal{C})$  are the semi-simplicial objects in  $\mathcal{C}$ . The Joyal dual of  $Surj_{<}^op$  is the subcategory  $Inj_{*,*}$  of order preserving maps of intervals. In other words semi-simplicial objects are  $Fun(Inj_{*,*}, \mathcal{C})$

Just as the surjections are generated by the unique maps  $\underline{n} \rightarrow \underline{1}$  so dually are the injections by the unique maps  $[1] \rightarrow [n] \in Hom_{*,*}([1], [n])$ . Pictorially the surjection is naturally depicted by a corolla while the injection is nicely captured by drawing an injection as a half circle. The duality can then be seen by superimposing the two graphical images. This duality is also that of dual graphs on bordered surfaces. This is summarized in Figure 3. Notice that in this duality, the elements of  $[n]$  correspond to the angles of the corolla and the elements of  $\underline{n}$  label the leaves of the corolla.

This also explains the adding and subtraction of 1 in the correspondence (C.1).

For general surjections, the picture is the a forest of corollas and a collection of half circles. The composition then is given by composing corollas to corollas and by gluing on the half circles to the half circles by identifying the beginning and endpoints. This is exactly the map of combining simplicial strings. The prevalent picture for this in the literature on multi-zetas and polylogs is by adding line segments as the base for the arc segments. This is pictured in Figure 4. The composition is then given by contracting the internal edges or dually erasing the internal lines. This is depicted in Figure 5.

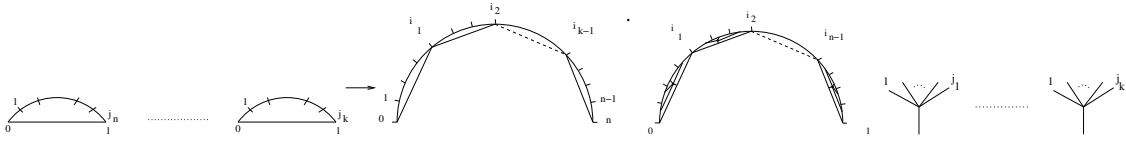


FIGURE 4. The first step of the composition is to assemble a forest or a collection of half discs into one morphism. On semicircles on the left corollas on the right and the duality in the middle. The  $j$  and  $i$  are related by  $i_l = j_1 + \dots + j_k$ . This also corresponds to an iterated cup product.

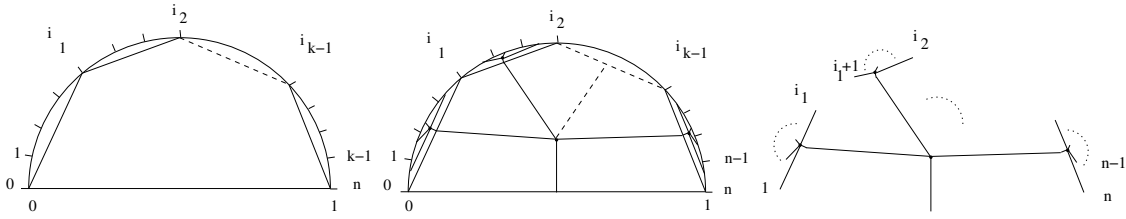


FIGURE 5. The second step of composition. For half circles on the left corollas on the right and the duality in the middle. The first step is to assemble a forest or a collection of half discs into one morphism, this is done in Figure 4. The result of the composition is in Figure 3.

We have chosen here the traditional way of using half circles. Another equivalent way would be to use polygons with a fixed base side. Finally, if one includes both the tree and the half circle, one can modify the picture into a more pleasing aesthetic by deforming the line segments into arcs as is done in §3, where also an explicit composition is given in all details, see Figure 2.

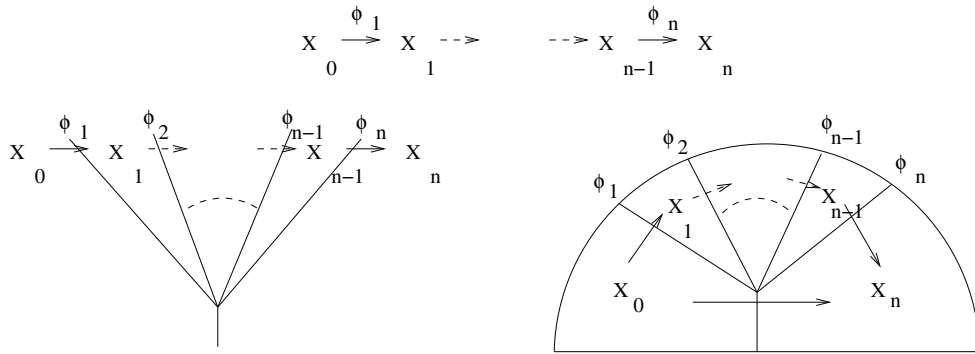
**C.2.1. Marking angles by morphisms.** A particularly nice example of the duality between marking angles vs. marking tails is given by considering the simplicial object given by the nerve of a category  $N_\bullet(\mathcal{C})$ . An  $n$ -simplex  $X_0 \xrightarrow{\phi_1} X_1 \cdots \xrightarrow{\phi_n} X_n$  naturally gives rise to a decorated corolla, where the leaves are decorated by the objects and the angles are decorated by the morphisms, see Figure 6. The operation that the corolla represents is the the composition of all of the morphisms to get a morphism  $\phi = \phi_n \circ \dots \circ \phi_0 : X_0 \rightarrow X_n$ .

If there is a single morphism between any two objects either one of the markings, tail or angle, will suffice to give a simplex. In the general case, one actually needs both the markings.

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FIGURE 6. Marking a corolla by a simplex in  $N_{\bullet}(\mathcal{C})$ 

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