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The group inverse of subdivision networks

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Abstract

In this paper, given a network and a subdivision of it, we show how the Group Inverse of the subdivision network can be related to the Group Inverse of initial given network. Our approach establishes a relationship between solutions of related Poisson problems on both networks and takes advantatge on the definition of the Group Inverse matrix.

Keywords: Network, Subdivision, Laplacian matrix, Group Inverse.

1 Introduction and preliminaries

In this paper $\Gamma = (V, E, c)$ denotes a *network;* that is, a finite, with no loops, nor multiple edges, connected graph, with n vertices that we can label $V = \{1, 2, \dots, n\}$ and m edges in E, in which each edge $\{i, j\}$ has been assigned a weight or *conductance* $c_{ij} > 0$. It is $c_{ij} = c_{ji}$ as defined on edges. In addition,

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when $\{i,j\} \notin E$ we define $c_{ij} = 0$ and, in particular, $c_{ii} = 0$ for any i. We define the *(weighted) degree of* i as $k_i = \sum_{j=1}^{n} c_{ij}$.

The *combinatorial Laplacian* of Γ is the $n \times n$ matrix L whose entries are $L_{ij} = -c_{ij}$ for all $i \neq j$ and $L_{ii} = k_i$. Therefore, for each vector $u \in \mathbb{R}^n$ and for each $i = 1, \dots, n$

$$[Lu]_i = k_i u_i - \sum_{j=1}^{n} c_{ij} u_j = \sum_{j=1}^{n} c_{ij}(u_i - u_j). \tag{1}$$

It is well-known that $Lu = 0$ iff $u = a\mathbf{1}$, $a \in \mathbb{R}$ and $\mathbf{1} \in \mathbb{R}^n$ the vector whose entries equal one. Moreover, given $f \in \mathbb{R}^n$, the Poisson problem, e.g. the linear system $Lu = f$, has solution iff $\sum_{i=1}^{n} f_i = 0$ and in this case two different solutions differ up to a constant. Therefore, there exists a unique solution orthogonal to $\mathbf{1}$ to every compatible linear system $Lu = f$ (*Fredholm's alternative*).

For a square matrix M, the *group inverse* of M, denoted as $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is very well known, see [3, and the references therein], that $M^{\#}$ exists if and only if $\operatorname{rank}(M) = \operatorname{rank}(M^2)$. Moreover, if M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$–inverse of M. As the combinatorial Laplacian of Γ matrix L, is a square, symmetric matrix that satisfies $\operatorname{rank}(M) = \operatorname{rank}(M^2)$ then $L^{\#}$ exists.

In particular, if for each $i = 1, \dots, n$, e_i denotes the i–th unit vector, with 1 in the ith position and 0 elsewhere, the linear system

$$Lu = \mathrm{e}_i - \frac{1}{n}\mathbf{1} \tag{2}$$

has a unique solution which is orthogonal to $\mathbf{1}$. This solution will be denoted by $L_i^{\#}$. We use this set of orthogonal to $\mathbf{1}$ solutions, varying the i vertices in V, to define an $n \times n$ matrix $L^{\#}$, the group inverse of L.

2 A Poisson Problem on a Subdivision Network

A *subdivision graph* (V^S, E^S), of a given graph (V, E), finite, with no loops, with no multiple edges and connected, is the one obtained by inserting a new vertex in every edge of E so that it is replaced by two new edges, let's say $\{i, v_{ij}\}$ and $\{j, v_{ij}\}$, once v_{ij} is the new inserted vertex in between $i, j \in V$ such that $\{i,j\} \in E$. In this way we obtain the new graph (V^S, E^S) is also finite, with no loops and no multiple edges, and connected. The set of vertices $V^S = V \cup V'$ is the union of $V = \{1, 2, \cdots n\}$, those in the base graph, and

$V' = \{v_{ij} \mid i,j \in V \text{ and } \{i,j\} \in E\}$, the set of the new generated vertices, so the total number of nodes of the subdivision graph is $n+m$ (because, in our notation, $v_{ij} = v_{ji}$). Moreover the set of edges E^S is composed by $2m$ new edges so that every one of them have exactly one node in V and one node in V'.

A *subdivision network* $\Gamma^S = (V^S, E^S, c^S)$ of a base network $\Gamma = (V, E, c)$ will be the subdivided graph of (V, E), provided with a positive symmetric *conductance* $c^S(i, v_{jk})$ that vanishes when the pair of vertices are not adjacent in E^S. Moreover, and in order to fulfill electrical conditions, we define conductances when non–null, such that

$$\frac{1}{c_{ij}} = \frac{1}{c^S(i, v_{ij})} + \frac{1}{c^S(j, v_{ij})}.$$

As all edges in E^S have both kind of vertices in V^S, the definition of c^S cannot be misunderstood so that, in the sequel, it will be denoted also as c.

Finally, a *Poisson problem in a subdivision network* is stated as

$$[L^S \overline{u}]_i = h_i, \text{ for every } i = 1, \dots, n+m, \tag{3}$$

where L^S is the combinatorial Laplacian of Γ^S, $h \in \mathbb{R}^{n+m}$ is a given vector, (e.g. a given data function), allegedly compatible, and $\overline{u} \in \mathbb{R}^{n+m}$ is a solution vector (e.g. a discrete function).

When labeling the vertices in V^S such that $V = \{x_1, x_2, \dots, x_n\}$ and $V' = \{x_{n+1}, \dots, x_{n+m}\}$, the Laplacian matrix can be written as a block matrix

$$L^S = \begin{bmatrix} H_1 & B \\ B^T & H_2 \end{bmatrix}$$

where $H_1 \in \mathcal{M}_{n\times n}$ and $H_2 \in \mathcal{M}_{m\times m}$ are diagonal, and $B \in \mathcal{M}_{n\times m}$ is just the weighted incidence matrix of Γ. In this context it is worthwhile to remark that the Laplacian matrix of Γ is the Schur complement of H_2, in other words $L = H_1 - BH_2^{-1}B^T$. Thus the group inverse of L^S depends on the group inverse of L.

The first main result we present in this work sets the precise relation between the solution of a Poisson problem in a subdivision network Γ^S and the solution of a conveniently stated Poisson problem on the base network Γ.

With the aim of usefulness we define for each pair $i, j \in V$ such that

$\{i,j\}\in E$, the coefficient

$$\alpha_{ij}=\frac{c(i,v_{ij})}{c(i,v_{ij})+c(j,v_{ij})}.$$

It could also be stated as $\alpha_{ij}=\frac{c(i,v_{ij})}{k(v_{ij})}$. Anyway, notice its value in $(0,1)$ and the relationship $\alpha_{ji}=1-\alpha_{ij}$. Also it is important for the sequel to consider, for each pair of functions (i.e vectors) $h\in\mathbb{R}^{n+m}$ and $u\in\mathbb{R}^n$, the two next definitions:

(i) The *contraction of* h *to* V (which is as $\mathbb{R}^n$) is

$$\underline{h}_i := h_i+\sum_{j\sim i}\alpha_{ij}h_{v_{ij}},\ \ i\in V,$$

(ii) The *extension of* u *to* V^S *(which is as* $\mathbb{R}^{n+m}$*), referred to* h*,* as

$$u_i^h=\begin{cases} u_i, & \text{for all } i\in V,\\ \frac{h_{v_{ij}}}{k(v_{ij})}+\alpha_{ij}u_i+\alpha_{ji}u_j, & \text{for all } v_{ij}\in V'.\end{cases}$$

With all these tools precisely defined, we now can establish our first theorem, (see [2] for a proof) which is

Theorem 2.1 *Given* $\Gamma=(V,E,c)$ *a network, let* $\Gamma^S=(V^S,E^S,c)$ *be a subdivision network generated with* Γ *as a base, let* $L^S(\overline{u})=h$ *be a Poisson problem on* Γ^S*, for* $h\in\mathbb{R}^{n+m}$ *a compatible data function and let* $L(u)=f$ *be a Poisson problem on* Γ*, then there is a relation between* $\overline{u}$ *and* u *if and only if*

$$f_i=h_i+\sum_{\substack{j\in V\\ \{i,j\}\in E}}\alpha_{ij}h_{v_{ij}},\ \ i=1,\cdots,n.$$

Moreover

(i) $\overline{u}_i=u_i$, *for all* $i\in V$,

(ii) $\overline{u}_{v_{ij}}=\frac{1}{k_{v_{iij}}}\left[h_{v_{ij}}+c(i,v_{ij})\overline{u}_i+c(j,v_{ij})\overline{u}_j\right]$, *for all* $v_{ij}\in V'$,

so $\overline{u}$ *is the extension of* u *to* V^S *referred by* h.

3 The Group Inverse of subdivision networks

In the next result we show how to obtain the solution of a Poisson problem on a subdivision network that is orthogonal to $\mathsf{1}_{V^S}$.

Corollary 3.1 *Given* $h \in \mathbb{R}^{n+m}$*, let* $\underline{h} \in \mathbb{R}^n$ *be its contraction to* V*, and* $u \in \mathbb{R}^n$ *be the unique solution to* $Lu = \underline{h}$ *that satisfies* $\langle u, 1_V \rangle = 0$*. Let*

$$\lambda = -\tfrac{1}{(n+m)} \sum_{i\sim j} \tfrac{h_{v_{ij}}}{k(v_{ij})} - \tfrac{1}{(n+m)} \sum_{i\sim j} (\alpha_{ij} u_i + \alpha_{ji} u_j)$$

be a constant, then $u^\perp = u^h + \lambda$ *is the unique solution to* $L^S u^\perp = h$ *that satisfies* $\langle u^\perp, 1_{V^S} \rangle = 0$*.*

Now we are ready to obtain the expression of the Group Inverse of a subdivision network Γ^S in terms of the Group Inverse of Γ, the base network. From equation 2, we are interested in solving the following Poisson problems on Γ^S. Those problems are not identical depending upon the pole vertex considered is in V or in V', as data functions for the subdivision network are to be contracted to pose a suitable Poisson problem on the base network.

$$[L^S (L^S)^\#_i] = \mathrm{e}_i - \frac{1}{n+m} \mathbf{1}, \qquad i = 1, 2, \dots, n;$$

$$[L^S (L^S)^\#_{v_{ij}}] = \mathrm{e}_{v_{ij}} - \frac{1}{n+m} \mathbf{1}, \quad v_{ij} = n+1, n+2, \dots, n+m;$$

Moreover the solution of the corresponding Poisson problems for Γ must be extended to the whole subdivision network. So we obtain the following result where $k_i^S = \displaystyle\sum_{j=1}^{n} \alpha_{ij}$, is a kind of a degree of vertex $i = 1, \dots, n$ once the subdivision operation has been performed.

Proposition 3.2 *Let* Γ^S *the subdivison network of* Γ*, then for any* $i, j =$

$1,\dots,n$ *and* $v_{ij}, v_{pq} = n+1,\dots,n+m$*, the Group Inverse for* Γ^S *is given by*

$$
\begin{aligned}
[(L^S)^{\#}_i]_j &= L^{\#}_{ij} - \frac{1}{n+m}\sum_{\ell\in V}\Big[L^{\#}_{i\ell}+L^{\#}_{j\ell}\Big]k^S_\ell \\
&\quad + \frac{1}{(n+m)^2}\sum_{r,s\in V} L^{\#}_{sr}k^S_r k^S_s + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})},\\
[(L^S)^{\#}_i]_{v_{pq}} &= \alpha_{pq}L^{\#}_{ip} + \alpha_{qp}L^{\#}_{iq} - \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha_{pq}L^{\#}_{p\ell}+\alpha_{qp}L^{\#}_{q\ell}+L^{\#}_{i\ell}\Big]k^S_\ell \\
&\quad + \frac{1}{(n+m)^2}\sum_{r,s} L^{\#}_{sr}k^S_r k^S_s + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})} - \frac{1}{(n+m)k(v_{pq})},\\
[(L^S)^{\#}_{v_{ij}}]_{v_{pq}} &= \alpha_{pq}\Big(\alpha_{ij}L^{\#}_{ip}+\alpha_{ji}L^{\#}_{jp}\Big) + \alpha_{qp}\Big(\alpha_{ij}L^{\#}_{iq}+\alpha_{ji}L^{\#}_{jq}\Big)\\
&\quad - \frac{1}{n+m}\sum_{\ell\in V}\Big[\alpha_{ij}L^{\#}_{i\ell}+\alpha_{ji}L^{\#}_{j\ell}+\alpha_{pq}L^{\#}_{p\ell}+\alpha_{qp}L^{\#}_{q\ell}\Big]k^S_\ell\\
&\quad + \frac{1}{(n+m)^2}\sum_{r,s\in V} L^{\#}_{sr}k^S_r k^S_s + \frac{1}{(n+m)^2}\sum_{r\sim s}\frac{1}{k(v_{rs})}\\
&\quad + \frac{[\mathrm{e}_{v_{zt}}]_{v_{ij}}}{k(v_{ij})} - \frac{1}{(n+m)k(v_{ij})} - \frac{1}{(n+m)k(v_{pq})}.
\end{aligned}
$$

4 The Group Inverse of a Star of length two

We adjoint to this work an easy work–out example of our results.

Let us consider S_n the $n-$Star network with $n+1$ vertices $\{x_0, x_1, \dots, x_n\}$ so that every $x_i,\ i = 1,\dots n$ is adjacent only to x_0, and equal conductances $a(x_0, x_i) = a > 0,\ i = 1,\dots n$.

The group inverse in this case is known, see [1, Corollary 6.1], and defined as

$$
\begin{aligned}
L^{\#}(x_0,x_0) &= \tfrac{n}{a(n+1)^2}, & L^{\#}(x_0,x_i) &= -\tfrac{1}{a(n+1)^2},\\
L^{\#}(x_i,x_i) &= \tfrac{(n-1)(n+2)+1}{a(n+1)^2}, & L^{\#}(x_i,x_j) &= -\tfrac{n+2}{a(n+1)^2},
\end{aligned}
$$

where $i, j = 1,\dots,n$ and $i \neq j$.

Then, S^S_n is an $n-$Star of length 2. We denote as $y_i,\ i = 1,\dots,n$ the new vertices generated by the subdivision process, adjacent to x_0 and to its corresponding x_i. Thus $V^S = \{x_0, x_1,\dots,x_n, y_1,\dots,y_n\}$ and conductances are to be determined. As the electrical condition that allow us to relate Poisson problems in S_n and S^S_n must be satisfied, we choose every conductance as $c(x_0, y_i) = 2a = c(x_i, y_i),\ i = 1,\dots,n$. Then

Proposition 4.1 *Given the Group inverse of S_n by the expressions just stated above then, with the above notation, the group inverse of S_n^S is defined by*

$$\begin{aligned}
(L^S)^{\#}(x_0,x_0) &= \tfrac{5}{2}\tfrac{n}{(2n+1)^2a}, & (L^S)^{\#}(x_0,x_i) &= -\tfrac{1}{2}\tfrac{n+3}{(2n+1)^2a},\\
(L^S)^{\#}(x_i,x_i) &= \tfrac{1}{2}\tfrac{8n^2+n-4}{(2n+1)^2a}, & (L^S)^{\#}(x_i,x_j) &= -\tfrac{1}{2}\tfrac{7n+6}{(2n+1)^2a},\\
(L^S)^{\#}(y_i,y_i) &= \tfrac{1}{2}\tfrac{(4n-3)(n+1)}{(2n+1)^2a}, & (L^S)^{\#}(y_i,y_j) &= -\tfrac{1}{2}\tfrac{3n+4}{(2n+1)^2a},\\
(L^S)^{\#}(x_0,y_i) &= \tfrac{1}{2}\tfrac{n-2}{(2n+1)^2a}, & (L^S)^{\#}(x_i,y_i) &= \tfrac{1}{2}\tfrac{4(n+1)(n-1)-n}{(2n+1)^2a},\\
(L^S)^{\#}(x_i,y_j) &= -\tfrac{5}{2}\tfrac{n+1}{(2n+1)^2a}.
\end{aligned}$$

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