MASTER'S DEGREE THESIS

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Title: A Liouville type result for fractional Schrödinger operators in 1D

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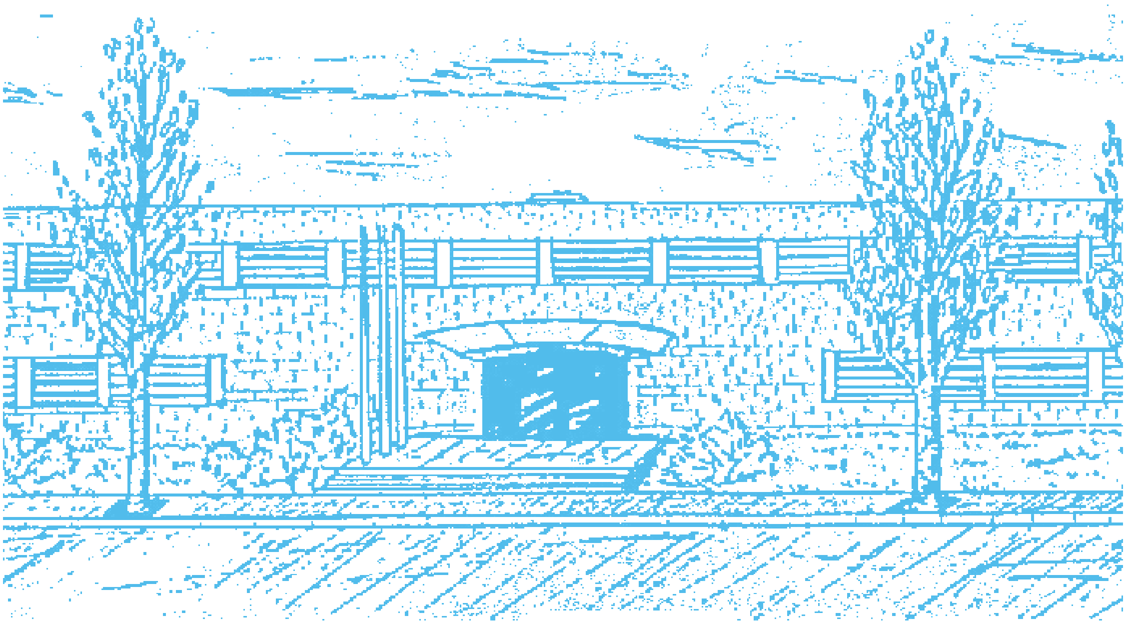

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En memoria de mi primo Paulino

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Abstract

Keywords: nonlocal equations, integral operators, Schrödinger operator, fractional Laplacian, Liouville type result, De Giorgi conjecture

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The aim of this master's thesis is to obtain an alternative and original proof of a Liouville type result for fractional Schrödinger operators, $\mathcal{L} = (-\Delta)^s - V$, in 1D without using a local extension problem, in the spirit of the recent work of Hamel et al. in [**20**]. Thanks to this new proof we can extend the Liouville theorem to other nonlocal operators that do not have a local extension problem, being the first time that a result of this kind is proven.

First, we introduce Schrödinger operators, the fractional Laplacian and its local extension problem. Then, we present a recent work about a nonlocal and nonlinear problem, where the prior study of fractional Schrödinger operators is needed. We also present the most important motivation for the study of Liouville type results: the conjecture of De Giorgi, and we review some Liouville type results both with local and nonlocal operators. Finally, we give the proof of the main theorems of the thesis.

Contents

Chapter 1
Introduction

This work is devoted to the study of bounded solutions to the linear problem

$$L\,u - V(x)u = 0, \quad \text{in } \mathbb{R}^n,$$

where the equation is driven by a nonlocal, integral operator of the form

$$L\,u(x) = \int_{\mathbb{R}^n} (u(x) - u(y))\,K(x-y)\,dy, \tag{1.0.1}$$

with K a measurable and nonnegative kernel such that $K(-z) = K(z)$ for a.e. $z \in \mathbb{R}^n$.

In particular we focus in the study of uniqueness (up to a multiplicative constant) of positive bounded solutions of this kind of equations, what is called a Liouville type result. As we shall see in chapters 3 and 4, these results are very useful when dealing with nonlinear problems.

This kind of results have their origin in Liouville's Theorem for harmonic functions, which has been generalized to a bigger class of equations replacing the Laplacian by classical or fractional Schrödinger operators, $\mathcal{L} = (-\Delta)^s - V$. In the same way as in the classical case the Laplacian is the main operator in the equation, in the fractional case this role belongs to the fractional Laplacian, which is the most basic elliptic integro-differential operator of order $2s$. It can be written as

$$(-\Delta)^s u = c_{n,s}\ P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

Nowadays, most of the nonlocal Liouville type results are proven just by using an extension problem and applying local properties. This type of arguments present an important limitation: most of the nonlocal operators do not have a local extension or its existence is a priori unclear. Recently, Hamel et al. have presented in [**20**] a result where they try to prove a Liouville type result (motivated by a conjecture of De Giorgi) for the fractional Laplacian in dimension 2 without using any local extension. Nevertheless, they do not get the desired result. They are only able to prove it with a different kind of nonlocal operator whose kernel has compact support.

The aim of this thesis is to apply the ideas of Hamel et al. together with other tools in order to prove a Liouville type result for the fractional Laplacian in the simplest case of dimension 1 without using local extension problems. The great advantage of this kind of proof is that we can extend the result to other nonlocal operators that are related to the fractional Laplacian. This methodology has also the potential to be extended to higher dimensions.

1.1. Main results

Now, we are going to present and comment the main results of the thesis.

Theorem 1.1.1. *Let $s \in [1/2, 1)$, and $\alpha > 2s$. Let $w, \tilde{w}$ be two solutions of the equation*

$$(-\Delta)^s u - V(x)u = 0 \quad \text{in } \mathbb{R},$$

with $w, \tilde{w} \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ (i.e. the L^∞-norm and the C^α-seminorm are bounded in all $\mathbb{R}$) and $w > 0$.

Assume that there exist two positive constants $m, b \in \mathbb{R}$ such that

$$V \leq -b < 0 \quad \text{in } \mathbb{R} \setminus [-m, m].$$

Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Note that this is not an original result of the thesis. It is a particular case of a result from Cabré and Sire in [**10**], that corresponds to Corollary 5.4.3 in this thesis. Nevertheless, the way in which we prove it, without using the local extension problem, is new.

Remark 1.1.2. (1) If we compare this theorem with Corollary 5.4.3, we note that the price we have to pay for not using the extension is adding some hypothesis in the potential function V. However, these hypothesis seem reasonable, at least in the kind of problems we have in mind. These are water waves problems, where the potential function is $V(x) = -1 + p\,u^{p-1}$, with u tending to 0 at infinity.

(2) We can see that we have not gotten the result for all possible values of the fractional power s. Nevertheless, we have the same rank of applicability that in Corollary 5.4.3, where we use the local extension to prove the result.

(3) Note that in order to apply the theorem we need certain Hölder regularity conditions for the solutions. We think that this is an unnecessary hypothesis that can be obtained from the L^∞ condition by making some regularity study, but it has not been possible in this thesis.

Theorem 1.1.3. *Let $\alpha > 2s_{max}$ and L be an integral operator of the form (1.0.1), with kernel K satisfying either*

(F1) $$K(z) \leq C\, K_s(z), \qquad \text{with } \; s \in [1/2, 1), \quad \text{and } C \in \mathbb{R}^+,$$

or

$$\text{(F2)} \qquad K(z) = \sum_{i=1}^{m} c_i\, K_{s_i}(z), \quad \text{with } s_i \in [1/2, 1), \text{ and } c_i \in \mathbb{R}^+,$$

where $K_s(z)$ is the kernel of the fractional Laplacian of order $2s$, and $s_{max} = s$ for the case $(F1)$ and $s_{max} = \max_i s_i$ for the case $(F2)$.

Let $w, \tilde{w}$ be two solutions of the equation

$$L\, u - V(x)u = 0 \quad \text{in } \mathbb{R},$$

with $w, \tilde{w} \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ (i.e. the L^∞-norm and the C^α-seminorm are bounded in all $\mathbb{R}$) and $w > 0$.

Assume that there exist two positive constants $m, b \in \mathbb{R}$ such that

$$V \leq -b < 0 \quad \text{in } \mathbb{R} \setminus [-m, m].$$

Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Note that Theorem 1.1.3, unlike Theorem 1.1.1, is a new and original result of the thesis. Apart from the work of Hamel et al. (see [**20**]), all the known nonlocal Liouville type results are proven just by using an extension problem. Therefore, since there is no extension problem for the nonlocal operators that appear in Theorem 1.1.3, a new methodology is needed to deal with them.

Due to the fact that the tools developed to prove Theorem 1.1.1 are quite general in terms of the nonlocal operator that drives the equation, the proofs of both Theorem 1.1.1 and 1.1.3 are essentially the same.

1.2. Outline of the thesis

The work is organized as follows:

- In chapter 2 we introduce nonlocal and integro-differential operators together with some examples and applications of them in science. Then we present the fractional Laplacian, one of the most important nonlocal operators, and fractional Schrödinger operators, which will play a principal role in this thesis. Finally, we have an interesting result about the fractional Laplacian, an extension problem that allows a better comprehension of this kind of operator, and is very useful when studying equations where it appears.

- Chapter 3 contains a recent result where the study of Schrödinger operators is crucial for obtaining interesting properties about nonlocal and nonlinear equations. In this case, we present the main ideas Frank and Lenzmann use in [**17**] to prove the uniqueness and nondegeneracy of ground states for the equation

$$(-\Delta)^s u + u - u^{\alpha+1} = 0 \quad \text{in } \mathbb{R}.$$

- In chapter 4 we introduce the conjecture of De Giorgi, which is the main motivation for the study of Liouville type results. We also present some of the most interesting ideas of the proof in dimension 3 and a list of results of some of the so-called fractional versions of the conjecture.

- Chapter 5 contains some of the most important Liouville type results. First we present the Classical Liouville theorem for harmonic functions, and we continue with a more general version, Theorem 5.2.2, where the Laplacian has been replaced by a Schrödinger operator with potential $V(x)$. Then we show the fractional version of this result. Although these results involve nonlocal operators, they are proven by using local arguments through the extension problem. Finally we present a recent Liouville type result involving nonlocal operators, a kind of integral operators with truncated kernels, that is proven without the extension problem.

- In chapter 6 we give the proof of Theorems 1.1.1 and 1.1.3. Although the statement of Theorem 1.1.1 was already known, we present here a different proof without using the extension problem of the fractional Laplacian. Thanks to this new proof we can extend Theorem 1.1.1 to other nonlocal operators that do not have a local extension problem, Theorem 1.1.3. This last result is a new and original result of the thesis.

1.3. Notation

Finally we want to present some remarks about the notation that we will use throughout the document.

Remark 1.3.1. Regarding half-spaces and half-balls, we use the notation

$$\mathbb{R}^n_+ = \{(x,y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, y > 0\},$$

$$B_R^+ = \{(x,y) \in \mathbb{R}^n_+ : |(x,y)| < R\},$$

$$\Gamma_R^0 = \{(x,0) \in \mathbb{R}^n_+ : |x| < R\},$$

$$\Gamma_R^+ = \{(x,y) \in \mathbb{R}^n_+ : |(x,y)| = R\}.$$

It is easy to see that $\partial B_R^+ = \Gamma_R^0 \cup \Gamma_R^+$.

Remark 1.3.2. When β is not an integer, we will denote by C^β the Hölder space of order β. That is

$$C^\beta = C^{\lfloor \beta \rfloor, \beta - \lfloor \beta \rfloor},$$

with $\lfloor \cdot \rfloor$ the integer part of a real number.

Remark 1.3.3. As usual, in the thesis the letter C stands for some constant which may change its value at different occurrences. Besides, a constant with subscripts usually means that its value only depends on the parameters appearing in the subscripts, but it also may change at different occurrences.

Chapter 2

Integro-differential operators. The fractional Laplacian

In this chapter we first introduce integro-differential operators together with some examples of application of them in science. Then, we present one of the most important nonlocal operator, the fractional Laplacian, as well as Schrödinger operators. They play a main role in this thesis. Finally we show an extension problem for the fractional Laplacian. This is a very interesting result that allows a better comprehension of this kind of operator, and is very useful when studying equations where it appears.

2.1. Integro-differential and Schrödinger operators

Partial differential equations are relations between an unknown function and its derivatives, and they have been used for the last centuries to model a great variety of problems. Nevertheless there are a lot of applications in which other interesting operators appear, most of them nonlocal operators. Integral operators and multiplier operators are two important families of operators that usually appear in models from different scientific disciplines. For example, the fractional Laplacian appears in mathematical finance, optimal search theory for marine predators (see [**5**] and [**21**]), or in fluid dynamics problems.

Definition 2.1.1 (Local and nonlocal operators)**.** A local operator is an operator whose value applied on a function u and evaluated at a point x_0 only depends on the value of the function u in a small neighborhood of the point x_0. A nonlocal operator is an operator which is not local.

Differential operators, which only involve a finite number of derivatives of the function, are the clearest example of local operators. The reason for this is that the derivatives of a function at a point only depend on the values of the function in a small neighborhood of the point, and it is possible for example to find the solution of a differential equation only in a neighborhood of a point of interest.

Nonlocal operators appear in modeling because of the necessity of studying natural phenomena which depend strongly on a long range iteration. That is, phenomena where it is not enough to know what happens in the neighborhood of a point, it is needed to know what happens in all the domain.

Definition 2.1.2 (Multiplier operator)**.** A multiplier operator L, is a linear operator that acts on a function multiplying its Fourier transform by a specified function m called multiplier or symbol. That is,

$$Lu(x) := \mathcal{F}^{-1}\left(m(\xi)\cdot\mathcal{F}(u)(\xi)\right).$$

This is a very important family of operators which includes some of the most used ones in both mathematics and physics. As we can see in the following examples, local and nonlocal operators are included in this family.

Example 2.1.3. Here we have a compilation of some of the most representative multiplier operators.

(1) Translation operator:

$$L_1u(x) = u(x-a) \Rightarrow \mathcal{F}(L_1u)(\xi) = \mathrm{e}^{ia\cdot\xi}\cdot\mathcal{F}(u)(\xi) \Rightarrow m_1(\xi) = \mathrm{e}^{ia\cdot\xi}.$$

(2) Partial derivative operator:

$$L_2u(x) = \frac{\partial u}{\partial x_j}(x) \Rightarrow \mathcal{F}(L_2u)(\xi) = i\xi_j\cdot\mathcal{F}(u)(\xi) \Rightarrow m_2(\xi) = i\xi_j.$$

(3) Convolution operator:

$$L_3u(x) = K(x)*u(x) \Rightarrow \mathcal{F}(L_3u)(\xi) = \mathcal{F}(K)(\xi)\cdot\mathcal{F}(u)(\xi) \Rightarrow m_3(\xi) = \mathcal{F}(K)(\xi).$$

Multiplier operators are very important in fluid dynamics, specially in the study of water waves, where the dispersion relation becomes the symbol of the operator that drives the dynamics of the problem (see [**1**] and [**7**]). The dispersion relation is the quotient between the velocity of propagation of waves at different wavelengths divided by the wavelength.

One interesting real example of these operators is the Intermediate Long-wave model (see [**2**] and [**22**]). It is used to study the interface between two fluids of different positive densities contained at rest in a long channel with a horizontal top and bottom. In this case, the dispersion relation is $\xi\coth(\xi)$. Therefore, the equation of the model is driven by a multiplier operator with this symbol. We note that this model generalizes the Korteweg-de Vries (KdV) model for shallow-water, with dispersion relation $|\xi|^2$ and the Benjamin-Ono (BO) model for deep-water, with dispersion relation $|\xi|$.

In this thesis we focus in the study of equations involving operators of the type (1.0.1). It is clear that this kind of operators are nonlocal because evaluating the operator at a point, the result is a pondered mean of the differences with all the points in the domain. These operators belong to a bigger family of operators called integro-differential operators.

Definition 2.1.4 (Integro-differential operator)**.** We say that the operator L is integro-differential if it is of the form

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - y \cdot \nabla u(x)\, \chi_{B_1}(y))\, K(x,y).$$

This operators are the generators of Lévy processes where we have removed the local parts. Lévy processes can be seen as the generalization of Brownian motion where arbitrarily long jumps are allowed.

Note that in the case of $K(x,y)$ being translation invariant (only depending on the variable y) and symmetric ($K(x,y) = K(x,-y)$), we recover the operators of type (1.0.1).

We can see that the operators of type (1.0.1) are multiplier operators.

Proposition 2.1.5 (Di Nezza, Palatucci, Valdinoci in [**16**])**.** *Let L be an operator of the form* (1.0.1)*. Then it is a multiplier operator, and its symbol can be computed from the kernel as*

$$m(\xi) = 2 \int_{\mathbb{R}^n} (1 - \cos(y \cdot \xi))\mathcal{K}(y)\ dy.$$

Proof. First we see that we can write the operator in an equivalent way. That is,

$$\begin{aligned} L\, u(x) &= \int_{\mathbb{R}^n} (u(x) - u(y))\, K(x-y)\, dy = \int_{\mathbb{R}^n} (u(x) - u(x+z))\, K(z)\, dz = \\ &= \frac{-1}{2} \int_{\mathbb{R}^n} (u(x-z) - 2\, u(x) + u(x+z))\, K(z)\, dz, \end{aligned}$$

where we only have to make a change of variables and use that the kernel is even.

Now we can obtain its symbol computing the Fourier transform:

$$\begin{aligned} \mathcal{F}(Lu)(\xi) &= \mathcal{F}\left(\int_{\mathbb{R}^n} \frac{-1}{2} \, (u(x+z) - 2u(x) + u(x-z))\, K(z)\ dz \right)(\xi) = \\ &= \frac{-1}{2} \int_{\mathbb{R}^n} \mathcal{F}\left(u(x-z) - 2u(x) + u(x+z)\right)(\xi) K(z)\ dz = \\ &= \frac{-1}{2} \int_{\mathbb{R}^n} (e^{iz\cdot\xi} + e^{-iz\cdot\xi} - 2)(\mathcal{F}u)(\xi) K(z)\ dz = \\ &= \frac{-1}{2} \int_{\mathbb{R}^n} (e^{iz\cdot\xi} + e^{-iz\cdot\xi} - 2) K(z)\ dz\ (\mathcal{F}u)(\xi) = \\ &= \int_{\mathbb{R}^n} (1 - \cos(z \cdot \xi))\mathcal{K}(z)\ dz\ (\mathcal{F}u)(\xi). \end{aligned}$$

□

Schrödinger operators are other of the fundamental families of operators we are going to deal with along this thesis.

Definition 2.1.6 (Schrödinger operator)**.** We say that $\mathcal{L}$ is a Schrödinger operator if it is of the form

$$\mathcal{L}u(x) = (-\Delta)u(x) - V(x)\,u(x),$$

for any function $V(x)$, which is called the potential of the operator.

These operators receive this name due to the fact that they are the time independent part of the Schrödinger equation

$$i\,\frac{\partial}{\partial t}\psi = \mathcal{L}\psi.$$

This equation describes the movement of a quantum mechanical particle under the effect of a magnetic field

$$F(x) = \nabla V(x).$$

Note that these operators are local ones, although we can generalize them by replacing the Laplacian by any other operator of the kind we have previously seen in this chapter.

2.2. The fractional Laplacian

One of the most important nonlocal operators is the fractional Laplacian, because it is the most basic linear integro-differential operator of order $2s$.

The Laplacian is a multiplier operator whose symbol can be computed easily as follows

$$\left.\begin{aligned} -\Delta u &= \sum_{j=1}^{n} -\partial_{jj}u \\ \mathcal{F}(\partial_j u) &= i\xi_j \mathcal{F}u \end{aligned}\right\} \Rightarrow \mathcal{F}(-\Delta u) = \sum_{j=1}^{n} \mathcal{F}(-\partial_{jj}u) = \sum_{j=1}^{n} -(i\xi_j)^2\mathcal{F}(u) = |\xi|^2\mathcal{F}(u).$$

Hence, the fractional Laplacian, $(-\Delta)^s u$ can be naturally obtained from the standard Laplacian via Fourier transform

$$(-\Delta)^s u := \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right), \quad s \in (0,1). \tag{2.2.1}$$

We say that this definition of the fractional Laplace operator comes natural because it carries the property that applying two fractional Laplacians to a function is equivalent to applying one fractional Laplacian whose index is the sum of them. That is,

$$(-\Delta)^{\alpha}(-\Delta)^{\beta}u = (-\Delta)^{\alpha+\beta}u. \tag{2.2.2}$$

Proposition 2.2.1 (Di Nezza, Palatucci, Valdinoci in [**16**])**.** *The fractional Laplacian defined in* (2.2.1) *is a nonlocal operator that can be written as*

$$(-\Delta)^s u = c_{n,s}\ P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\,dy, \tag{2.2.3}$$

where P.V. means that the integral is made in the principal value sense, and $c_{n,s}$ is a positive constant that depends on n and s and is going to be computed later.

Sometimes expression (2.2.3) is taken as the definition of the fractional Laplacian due to the equivalence between it and the definition (2.2.1). Next, we prove this.

PROOF. Let us call $\mathcal{L}(u)$ the right-hand side of expression (2.2.3). Then we have to prove that $\mathcal{L}(u) = (-\Delta)^s u$.

We are going to apply Proposition 2.1.5 with

$$\mathcal{K}(y) = \frac{c_{n,s}}{|y|^{n+2s}}.$$

Therefore, the symbol of $\mathcal{L}$ can be computed as

$$m(\xi) = \int_{\mathbb{R}^n} (1 - \cos(y \cdot \xi))\mathcal{K}(y)\, dy = c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot \xi)}{|y|^{n+2s}}\, dy.$$

To obtain a simpler expression of the symbol, first we show that $m(\xi)$ is rotationally invariant (i.e. $m(R(\xi)) = m(\xi)$ for all rotation R). That is,

$$\begin{aligned} m(R(\xi)) &= c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(y \cdot R(\xi))}{|y|^{n+2s}}\, dy = c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(R^T(y) \cdot \xi)}{|y|^{n+2s}}\, dy = \\ &= c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(\tilde{y} \cdot \xi)}{|\tilde{y}|^{n+2s}}\, d\tilde{y} = m(\xi), \end{aligned} \tag{2.2.4}$$

where we have made the change of variables $\tilde{y}$ by $R^T(y)$ and taken into account that the norm of a vector is invariant under all rotations.

Therefore, applying expression (2.2.4) to the rotation $R(\xi) = |\xi| e_1$, with e_1 the first direction vector, we obtain:

$$\begin{aligned} m(\xi) = m(|\xi|e_1) &= c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(|\xi| y_1)}{|y|^{n+2s}}\, dy = \frac{c_{n,s}}{|\xi|^n} \int_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x/\xi|^{n+2s}}\, dx = \\ &= c_{n,s} \left(\int_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x|^{n+2s}}\, dx \right) |\xi|^{2s} = c_{n,s} \tilde{c}_{n,s} |\xi|^{2s}. \end{aligned}$$

We need to prove that $\tilde{c}_{n,s}$ is finite in order to have $m(\xi)$ well defined. Indeed,

$$\begin{aligned} |\tilde{c}_{n,s}| &= \left| \int_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x|^{n+2s}}\, dx \right| \leq \int_{\mathbb{R}^n} \frac{|1 - \cos(x_1)|}{|x|^{n+2s}}\, dx = \\ &= \int_{\mathbb{R}^n / B_\epsilon} \frac{|1 - \cos(x_1)|}{|x|^{n+2s}} + \int_{B_\epsilon} \frac{|1 - \cos(x_1)|}{|x|^{n+2s}} \leq \\ &\leq \int_{\mathbb{R}^n / B_\epsilon} \frac{2}{|x|^{n+2s}} + \int_{B_\epsilon} \frac{|x_1|^2}{|x|^{n+2s}} \leq \\ &\leq \int_{\mathbb{R}^n / B_\epsilon} \frac{2}{|x|^{n+2s}} + \int_{B_\epsilon} \frac{1}{|x|^{n+2s-2}} < \infty. \end{aligned}$$

And these two integrals are convergent because $s \in (0,1)$. In the bounding of the central integral we have used that

$$\lim_{t\to 0} \frac{1-\cos(t)}{t^2} = \frac{1}{2} \Longrightarrow \exists\ \epsilon > 0 \text{ such that } \frac{1-\cos(t)}{t^2} < 1\ \forall t < \epsilon.$$

If we define

$$c_{n,s} = \tilde{c}_{n,s}^{-1} = \left(\int_{\mathbb{R}^n} \frac{1-\cos(x_1)}{|x|^{n+2s}}\ dx \right)^{-1}, \tag{2.2.5}$$

we finally have that

$$\mathcal{F}(\mathcal{L}(u)) = m(\xi)\mathcal{F}(u) = |\xi|^s \mathcal{F}(u) = \mathcal{F}((-\Delta)^s u \Longrightarrow \mathcal{L}u = (-\Delta)^s u.$$

□

We can see from the previous result that the fractional Laplacian corresponds to a translation invariant integro-differential operator for which $K(y)$ is radially symmetric and homogeneous.

So far, we have defined the fractional Laplacian but we have not worried about in which functions it can applied. Although in this thesis we are not working with regularity results, we present now a proposition that ensures the well-definition of the fractional Laplacian for a set of functions that are going to be of interest along the document.

Proposition 2.2.2. *Let $u \in L^\infty(\mathbb{R}) \cap C^\alpha_{loc}(\mathbb{R})$, with $\alpha > 2s$, then the fractional Laplacian $(-\Delta)^s u$ is well-defined pointwise.*

Proof. In order to prove the result we are going to distinguish two cases, if $\alpha \in (0,1]$ or $\alpha \in [1,2)$. If we are in the first case we have a condition about the function u, while in the second one we have a condition about the derivative u'. We begin with the first one, which is quite simpler.

Let us divide the integral in two parts, because at the origin we use that the function is Hölder, while at infinity we use that it is bounded. That is, given any δ we have

$$\begin{aligned}(-\Delta)^s u &= \int_{\mathbb{R}} \frac{u(x)-u(x+y)}{|y|^{1+2s}} dy = \\ &= \int_{-\infty}^{-\delta} \frac{u(x)-u(x+y)}{|y|^{1+2s}} dy + \int_{-\delta}^{\delta} \frac{u(x)-u(x+y)}{|y|^{1+2s}} dy + \\ &\qquad + \int_{\delta}^{\infty} \frac{u(x)-u(x+y)}{|y|^{1+2s}} dy.\end{aligned}$$

Now we bound each of the queue integrals using that $u \in L^\infty(\mathbb{R})$. That is,

$$\left| \int_{-\infty}^{-\delta} \frac{u(x) - u(x+y)}{|y|^{1+2s}} dy \right| \le \int_{-\infty}^{-\delta} \frac{|u(x) - u(x+y)|}{|y|^{1+2s}} dy \le C \int_{-\infty}^{-\delta} \frac{1}{|y|^{1+2s}} dy =$$

$$= C \int_{\delta}^{\infty} \frac{1}{y^{1+2s}} dy = C\,\delta^{-2s},$$

and analogous for the other queue integral.

In order to bound the central integral we are going to use the Hölder condition

$$|u(x) - u(x+y)| \le C\,|y|^\alpha.$$

Then we have

$$\left| \int_{-\delta}^{\delta} \frac{u(x) - u(x+y)}{|y|^{1+2s}} dy \right| \le \int_{-\delta}^{\delta} \frac{|u(x) - u(x+y)|}{|y|^{1+2s}} dy \le C \int_{-\delta}^{\delta} \frac{|y|^\alpha}{|y|^{1+2s}} dy =$$

$$= C \int_{0}^{\delta} \frac{1}{y^{1+2s-\alpha}} dy = C\,\delta^{\alpha-2s},$$

where it is crucial the fact that $\alpha > 2s$.

Therefore, if we put together the three parts, we have

$$|(-\Delta)^s\, u(x)| \le C\,(\delta^{-2s} + \delta^{\alpha-2s}) < \infty.$$

Now we are going to prove the second case in a similar way. For the part of the queue integrals we have to do exactly the same. The differences are when we have to apply the Hölder condition because, when $\alpha > 1$ we have that $u \in C^1(\mathbb{R})$ and $u' \in C^{\alpha-1}(\mathbb{R})$. Hence we are going to write the central integral in an equivalent way

$$\int_{-\delta}^{\delta} \frac{u(x) - u(x+y)}{|y|^{1+2s}} dy = \frac{1}{2} \int_{-\delta}^{\delta} \frac{(u(x) - u(x-y)) + (u(x) - u(x+y))}{|y|^{1+2s}}.$$

Since $u \in C^1(\mathbb{R})$ we can apply Taylor's theorem of order one. It says that

$$u(x+y) = u(x) + u'(\xi(y))\,y, \quad \text{with } \xi(y) \in (x, x+y).$$

On the other hand, as $u' \in C^{\alpha-1}(\mathbb{R})$ we have the condition

$$|u'(\xi(-y)) - u'(\xi(y))| \le C\,|\xi(-y) - \xi(y)|^{\alpha-1} \le C\,|y|^{\alpha-1}.$$

If we introduce these two expressions in the integral we get

$$\left| \int_{-\delta}^{\delta} \frac{u(x) - u(x+y)}{|y|^{1+2s}} dy \right| = \left| \frac{1}{2} \int_{-\delta}^{\delta} \frac{(u(x) - u(x-y)) + (u(x) - u(x+y))}{|y|^{1+2s}} \right| \le$$

$$\le \frac{1}{2} \int_{-\delta}^{\delta} \frac{|u'(\xi(-y))\,y - u'(\xi(y))\,y|}{|y|^{1+2s}} dy \le C \int_{-\delta}^{\delta} \frac{|y|^\alpha}{|y|^{1+2s}} dy =$$

$$= C \int_{0}^{\delta} \frac{1}{y^{1+2s-\alpha}} dy = C\,\delta^{\alpha-2s}.$$

Hence, as in the other case, if we put together the three parts of the integral we conclude that

$$|(-\Delta)^s\, u(x)| \le C\,(\delta^{-2s} + \delta^{\alpha-2s}) < \infty.$$

In this way we have completed the proof of the proposition. □

Finally we define the fractional Schrödinger operators, which are going to play a crucial role in several parts of the work.

Definition 2.2.3 (Fractional Schrödinger operator)**.** We say that L is a fractional Schrödinger operator with potential V if it is of the form

$$Lu(x) = (-\Delta)^s u(x) - V(x)\, u(x),$$

with $s \in (0, 1)$.

2.3. An extension problem for the fractional Laplacian

In this part of the work, we want to present a way to obtain any fractional power of the Laplacian from a local extension problem to the upper half space (see [**13**]). This is a very important characterization of these operators because it allows us to derive some properties by using purely local arguments. It is also important to understand how this kind of operator can appear in physics for modeling real phenomena.

Definition 2.3.1 (a-harmonic extension)**.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded function, and a a real parameter in $(-1, 1)$. We define its a-harmonic extension U as the solution of the problem

$$\begin{cases} div(y^a\, \nabla U) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ U(x, 0) = u(x), & \text{in } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases} \tag{2.3.1}$$

Remark 2.3.2. This terminology is motivated by the fact that when $a = 0$, then U is harmonic ($\Delta U = 0$) in $\mathbb{R}^{n+1}_+$.

Remark 2.3.3. By making the change of variables

$$z = \left(\frac{y}{1-a}\right)^{1-a}$$

we can rewrite equation (2.3.1) as

$$\begin{cases} \Delta_x U + z^{-\frac{a}{1-a}} U_{zz} = 0, & x \in \mathbb{R}^n, z > 0, \\ U(x, 0) = u(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.3.2}$$

where only second derivatives appear.

Once we have presented the a-harmonic extension we have to see if it is well defined. That is, we have to see if there exists a solution of this problem and if the solution is unique. We are going to prove the existence by writing an explicit solution via convolution with a Poisson kernel, and we are going to prove the uniqueness through a maximum principle.

Proposition 2.3.4 (Caffarelli-Silvestre in [**13**]). *The Poisson kernel*

$$P(x,y) = C_{n,a} \frac{y^{1-a}}{(|x|^2+|y|^2)^{\frac{n+1-\alpha}{2}}},$$

with $C_{n,a}$ a constant that only depends on n and a, is the fundamental solution of equation (2.3.1). *That is,*

$$\begin{cases} div(y^a \nabla P) = 0, & in \;\; \mathbb{R}^{n+1}_+, \\ P(x,0) = \delta(x), & in \;\; \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

In particular, it means that we can write explicitly the a-harmonic extension of a function u as

$$U(x,y) = u(x) * P(x,y) = \int_{\mathbb{R}^n} P(x-\xi, y)\, u(\xi)\, d\xi.$$

We can prove this result easily by direct computation. In [**13**] we can find the development of how this Poisson kernel is obtained.

Remark 2.3.5. The constant $C_{n,a}$ in the Poisson kernel expression is such that

$$\int_{\mathbb{R}^n} P(x,y)\, dx = 1, \;\; \forall\; y \in \mathbb{R}^+.$$

Now we are going to present a maximum principle for elliptic operators that will allow us to prove the uniqueness of the a-harmonic extension.

Proposition 2.3.6 (A maximum principle). *Let v be a function such that*

$$div(A(x)\,\nabla v) \geq 0, \qquad in \;\; \Omega, \tag{2.3.3}$$

with $A \geq 0$ in $\overline{\Omega}$. Then, the supremum of v is attained at the boundary. That is,

$$\sup_{\overline{\Omega}} v = \sup_{\partial\Omega} v.$$

Proof. Let $M = \sup_{\partial\Omega} v$, then $v - M \leq 0$ on $\partial\Omega$. Now we define the function

$$\tilde{v} = \max\{v - M, 0\}, \quad \text{in } \; \overline{\Omega}.$$

We know by definition that $\tilde{v} \equiv 0$ in $\partial\Omega$ and $\nabla\tilde{v} = \nabla v\, \chi_{v>M}$.

Since $\tilde{v}$ is non-negative, if we multiply it by equation (2.3.3) and integrate all over the domain we have

$$\int_\Omega \tilde{v}\, div(A(x)\,\nabla v) \geq 0.$$

On the other hand, by using the properties of $\tilde{v}$ that we have previously commented and integrating by parts we get

$$\int_\Omega \tilde{v}\, div(A(x)\,\nabla v) = \int_{\partial\Omega} \tilde{v}\, A(x)\,\nabla v \cdot n - \int_\Omega A(x)\,\nabla\tilde{v} \cdot \nabla v =$$
$$= - \int_{\{v>M\}} A(x)\, |\nabla v|^2 \leq 0.$$

Summarizing, we have

$$0 \le \int_\Omega \tilde{v}\, div(A(x)\, \nabla v) = -\int_{\{v>M\}} A(x)\, |\nabla v|^2 \le 0,$$

which finally implies that $v(x) \le M$ in Ω. Hence, we have completed the proof of the result. □

Corollary 2.3.7. *Let be $u : \mathbb{R}^n \to \mathbb{R}$ a smooth bounded function, a a real parameter in $(-1, 1)$, and U its a-harmonic extension, then we have*

$$\sup_{\mathbb{R}^{n+1}_+} U = \sup_{\mathbb{R}^n} u,$$

and

$$\inf_{\mathbb{R}^{n+1}_+} U = \inf_{\mathbb{R}^n} u.$$

Proof. We have to apply the maximum principle, Proposition 2.3.6, with $\Omega = \mathbb{R}^{n+1}_+$, $A(x) = y^a$, which is non-negative in the domain, $v = U$ in order to obtain the supremum equality and $v = -U$ to obtain the infimum one. □

Corollary 2.3.8. *Let be $u : \mathbb{R}^n \to \mathbb{R}$ a smooth bounded function, a a real parameter in $(-1, 1)$, then the a-harmonic extension is unique.*

Proof. Let us suppose that there are two a-harmonic extensions U_1 and U_2. Therefore, due to the linearity of equation (2.3.1), the difference of the two extensions is a solution of the problem

$$\begin{cases} div(y^a\, \nabla(U_1 - U_2)) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ U_1(x, 0) - U_2(x, 0) = 0, & \text{in } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

By applying Corollary 2.3.7 we have that $\sup(U_1 - U_2) = 0$ and $\inf(U_1 - U_2) = 0$, which implies that $U_1 \equiv U_2$, and we finally have a unique extension. □

It has been known for years that the half-Laplacian ($s = 1/2$) can be obtained from the harmonic extension problem to the upper half space as the operator that maps the Dirichlet boundary condition to the Neumann condition. That is,

$$-U_y(x, 0) = (-\Delta)^{1/2}\, u.$$

It is easy to show a heuristic argument which illustrates this relation. Let T be the operator such that $T\, u = -U_y(x, 0)$. We are going to see that applying twice T is the same as applying $-\Delta$.

Since $U(x, y)$ is the harmonic extension of $u(x)$, it is easy to prove that $U_y(x, y)$ is the harmonic extension of $U_y(x, 0)$. Therefore, we have

$$\begin{aligned} T^2(u) &= T(T(u)) = T(-U_y(x, 0)) = -(-U_{yy}(x, 0)) = U_{yy}(x, 0) = \\ &= -\Delta_x U(x, 0) = -\Delta u. \end{aligned}$$

Now we want to relate the fractional Laplacian of any order of a function with the extension problem in a similar way as the half-Laplacian.

Definition 2.3.9 (Conormal exterior derivative)**.** Let be $v : \mathbb{R}^{n+1}_+ \to \mathbb{R}$, we define its conormal derivative in the boundary as

$$\frac{\partial v}{\partial \nu^a} = -\lim_{y\to 0} y^a\, v_y(x,y),$$

with a a real parameter in $(-1,1)$.

Theorem 2.3.10 (Caffarelli-Silvestre in [**13**])**.** *Let be $u : \mathbb{R}^n \to \mathbb{R}$ a smooth bounded function, a a real parameter in $(-1,1)$, and U its a-harmonic extension, then we have*

$$\frac{\partial U}{\partial \nu^a} = -\lim_{y\to 0} y^a\, U_y(x,y) = C_{n,a} \int_{\mathbb{R}^n} \frac{u(\xi)-u(x)}{|\xi - x|^{n+2s}} = C_{n,a}\,(-\Delta)^s\, u,$$

with $a = 1-2s$, and $C_{n,a}$ a positive constant that only depends on n and a.

Proof. We only have to use the Poisson formula that gives the a-harmonic extension explicitly. That is,

$$\begin{aligned}
\lim_{y\to 0} y^a\, U_y(x,y) &= \lim_{y\to 0} \frac{U(x,y)-U(x,0)}{y^{1-a}} = \\
&= \lim_{y\to 0} \frac{1}{y^{1-a}} \left[\int_{\mathbb{R}^n} P(x-\xi,y)u(\xi)\; d\xi - u(x) \right] = \\
&= \lim_{y\to 0} \frac{1}{y^{1-a}} \left[\int_{\mathbb{R}^n} P(x-\xi,y)u(\xi)\; d\xi - u(x) \int_{\mathbb{R}^n} P(x-\xi,y)\; d\xi \right] = \\
&= \lim_{y\to 0} \frac{1}{y^{1-a}} \int_{\mathbb{R}^n} P(x-\xi,y)\,(u(\xi)-u(x))\; d\xi = \\
&= \lim_{y\to 0} \int_{\mathbb{R}^n} P(x-\xi,y) \frac{u(\xi)-u(x)}{y^{1-a}}\; d\xi = \\
&= \lim_{y\to 0} \int_{\mathbb{R}^n} C_{n,a} \frac{y^{1-a}}{(|x-\xi|^2+|y|^2)^{\frac{n+1-a}{2}}} \frac{u(\xi)-u(x)}{y^{1-a}}\; d\xi = \\
&= \int_{\mathbb{R}^n} C_{n,a} \frac{u(\xi)-u(x)}{|x-\xi|^{n+1-a}}\; d\xi = \int_{\mathbb{R}^n} C_{n,a} \frac{u(\xi)-u(x)}{|x-\xi|^{n+2s}}\; d\xi = \\
&= -C_{n,a}(-\Delta)^s u.
\end{aligned}$$

□

Remark 2.3.11. If we write the a-harmonic extension U in terms of the z variable (see remark 2.3.3). Then, we can rewrite Theorem 2.3.10 as

$$C_{n,a}\,(-\Delta)^s\, u = -U_z(x,0).$$

In conclusion, the Theorem of Caffarelli and Silvestre is a fundamental tool when we deal with the fractional Laplacian because it allows us to work with local equations although we have a nonlocal operator.

Since Theorem 2.3.10 appeared, some results in the spirit of it have been obtained. That is, local approaches to some interesting nonlocal problems. For example, in [**26**], *P.R. Stinga and J.L. Torrea* prove the same type of characterization for the fractional powers of second order partial differential operators in some class, as the fractional harmonic oscillator $H^s = (-\Delta + |x|^2)^s$. More recent is the extension problem of *X. Cabré and J. Serra* for sums of fractional Laplacians of different orders, $L = \sum_{i=1}^{k}(-\Delta)^{s_i}$ (see [**9**]) .

Chapter 3
Ground States of nonlinear fractional Schrödinger equations (in 1D)

Nonlocal problems have been intensively studied in recent years. Nevertheless, the available results are still far from the optimal. In this chapter we present the main ideas Frank and Lenzmann use in [**17**] to prove the nondegeneracy and uniqueness of ground states for the nonlinear equation

$$(-\Delta)^s Q + Q - Q^{\alpha+1} = 0 \quad \text{in } \mathbb{R}. \tag{3.0.4}$$

The proof of these results is based on the prior study of some properties of certain kind of fractional Schrödinger operator.

This is a recent and very important result in the study of nonlocal and nonlinear equations. In fact, this result plays a central role for the stability of solitary waves for nonlinear dispersive PDEs with fractional Laplacians, such as water waves.

3.1. Ground states

Definition 3.1.1 (Ground state)**.** A ground state is a nontrivial, nonnegative and radial function $Q = Q(|x|) \geq 0$ that vanishes at infinity and satisfies (in the distributional sense) an equation of the form

$$(-\Delta)^s Q - F(Q) = 0 \quad \text{in } \mathbb{R}^n, \tag{3.1.1}$$

where $F(Q)$ denotes some given nonlinearity.

We have the following result about existence of this kind of solutions in our case of interest $F(Q) = -Q + Q^{\alpha+1}$.

Proposition 3.1.2 (Frank and Lenzmann in [**17**])**.** *Let* $0 < s < 1$ *and* $0 < \alpha < \alpha_{max}(s)$, *with*

$$\alpha_{max}(s) = \begin{cases} \frac{4s}{1-2s} & \textit{for } 0 < s < \frac{1}{2}, \\ +\infty & \textit{for } \frac{1}{2} \leq s < 1. \end{cases}$$

Then the following holds.

(1) *There exists a solution* $Q \in H^s(\mathbb{R})$ *of equation* (3.0.4) *that is even, positive and strictly decreasing in* $|x|$.
(2) *If* $Q \in H^s(\mathbb{R})$ *is a nontrivial and nonnegative solution of equation* (3.0.4), *then there exists* $x_0 \in \mathbb{R}$ *such that* $Q(\cdot - x_0)$ *is even, positive and strictly decreasing.*

While the existence of this kind of solutions has been obtained in most of the examples of interest by using variational arguments, their uniqueness seems to be more difficult. Indeed, prior to this work of Frank and Lenzmann, uniqueness of ground states had been proven only in a few examples. One of them is the classical case with $s = 1$, where standard ODE methods are applicable, and the other is the Benjamin-Ono case ($s = 1/2$ and $F(Q) = -Q + Q^2$), where using complex analysis, the system is shown to be completely integrable. In this last case, the unique ground state is

$$Q(x) = \frac{2}{1 + x^2}.$$

In [**18**], Frank, Lenzmann and Silvestre generalize the study of equation (3.0.4) for arbitrary dimensions.

The importance of the study of ground state solutions for equation (3.0.4) lies in the fact that this solutions provide solitary wave solutions for three fundamental nonlinear dispersive models in dimension 1:

- The generalized Benjamin-Ono equation
$$u_t + u_x - ((-\Delta)^s u)_x + u^\alpha u_x = 0.$$
- The Benjamin-Bona-Mahony equation
$$u_t + u_x + ((-\Delta)^s u)_t + u^\alpha u_x = 0.$$
- The fractional nonlinear Schrödinger equation
$$iu_t - (-\Delta)^s u + |u|^\alpha u = 0.$$

The study of the nondegeneracy and uniqueness of the ground states is crucial for the stability analysis of the solitary waves.

In order to make a deep analysis of the problem, a more rigorous definition of the ground states is needed.

Definition 3.1.3 (Ground state)**.** Let $Q \in H^s(\mathbb{R})$ be an even and positive solution of equation (3.0.4) and consider the Weinstein functional

$$J^{s,\alpha}(u) := \frac{\left(\int_R |(-\Delta)^{s/2} u|^2\right)^{\frac{\alpha}{4s}} \left(\int_R |u|^2\right)^{\frac{\alpha}{4s}(2s-1)+1}}{\int_R |u|^{\alpha+2}}.$$

If

$$J^{s,\alpha}(Q) = \inf_{u \in H^s(\mathbb{R}) \setminus \{0\}} J^{s,\alpha}(u),$$

then we say that $Q \in H^s(\mathbb{R})$ is a ground state solution of equation (3.0.4).

3.2. Oscillation estimate for fractional Schrödinger operators

In this section we present the essential tool in the proof of the nondegeneracy of ground states. That is, we need a theory to estimate the number of sign changes for the first and second eigenfunctions for some fractional Schrödinger operators acting on $L^2(\mathbb{R})$ functions.

First we are going to define two important concepts that we are going to work with.

Definition 3.2.1 (Sign changes)**.** Let $\psi \in C^0(\mathbb{R})$ be real-valued and let $N \geq 1$ be an integer. We say that $\psi(x)$ changes its sign N times if there exist points $x_1 < ... < x_{N+1}$ such that $\psi(x_i) \neq 0$ for $i = 1, ..., N+1$ and $sign(\psi(x_i)) = -sign(\psi(x_{i+1}))$ for $i = 1, ..., N$.

Definition 3.2.2 (Nodal domains)**.** Let $\psi \in C^0(\Omega)$ be real-valued, with $\Omega \subset \mathbb{R}^n$ a connected open set. We can define the nodal domains of $\psi(x)$ as the connected components of the open set $\{x \in \Omega : \psi(x) \neq 0\}$.

Note that the maximal number of sign changes of $\psi(x)$ equals $K - 1$, where K is the number of nodal domains of ψ.

Now we introduce a suitable class of potentials V for the fractional Schrödinger operators.

Definition 3.2.3 (Kato class)**.** Let $0 < s < 1$. We say that the potential $V \in K_s$ if and only if $V : \mathbb{R} \to \mathbb{R}$ is measurable and satisfies

$$\lim_{E\to\infty} \big|\big|((-\Delta)^s + E)^{-1}|V|\big|\big|_{L^\infty \to L^\infty} = 0.$$

Remark 3.2.4. We list some comments about the previous definition.

(1) If $V \in K_s$, then $(-\Delta)^s - V$ defines a unique self-adjoint operator on $L^2(\mathbb{R})$ and the corresponding heat kernel $e^{-t((-\Delta)^s - V)}$ maps $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$. In particular, any L^2-eigenfunction of the fractional Schrödinger operator is continuous and bounded.
(2) In terms of L^p-spaces we have the following.
  - If $0 < s \leq \frac{1}{2}$ and $V \in L^p(\mathbb{R})$ for some $p > \frac{1}{2}s$, then $V \in K_s$.
  - If $\frac{1}{2} < s < 1$ and $V \in L^p(\mathbb{R})$ for some $p \geq 1$, then $V \in K_s$.

Note that the study of the first eigenfunction(s) is easier if we apply standard Perron-Frobenius type arguments. In fact we can obtain the following result:

Lemma 3.2.5 (Frank and Lenzmann in [**17**])**.** *Let $0 < s < 1$ and consider the fractional Schrödinger operator $\mathcal{L} = (-\Delta)^s - V$ acting on $L^2(\mathbb{R})$, where we assume that $V \in K_s$. Suppose that $e = \inf \sigma(\mathcal{L})$ is an eigenvalue. Then e is simple and its corresponding eigenfunction $\psi = \psi(x) > 0$ is positive (after replacing ψ by $-\psi$ if necessary).*

In order to study the second eigenfunction(s) we have to work with the a-harmonic extension problem.

Theorem 3.2.6 (Frank and Lenzmann in [**17**]). *Let $0 < s < 1$, $V \in K_s$, and define $a = 1 - 2s$. Suppose that $n \geq 1$ is an integer and assume that $\mathcal{L} = (-\Delta)^s - V$ has at least n eigenvalues*

$$\lambda_1 \leq ... \leq \lambda_n < 0.$$

If $\psi_n \in H^s(\mathbb{R}) \cap C^0(\mathbb{R})$ is a real eigenfunction of the Schrödinger operator $\mathcal{L}$ with eigenvalue λ_n, then its a-harmonic extension Ψ_n has at most n nodal domains in $\mathbb{R}^2_+$.

The proof of this result is based on a variational characterization of the eigenfunctions and eigenvalues of the Schrödinger operator together with a trace inequality for the extension problem. We can see it with all the details in the original paper of Frank and Lenzmann, [**17**].

Once we have this result, we can present and prove the main theorem of this section.

Theorem 3.2.7 (Frank and Lenzmann in [**17**]). *Let $0 < s < 1$, $V \in K_s$, and consider the fractional Schrödinger operator $\mathcal{L} = (-\Delta)^s - V$ acting on $L^2(\mathbb{R})$. Suppose that $\lambda_2 < \inf \sigma_{ess}(\mathcal{L})$ is the second eigenvalue of $\mathcal{L}$ and let $\psi_2 \in H^s(\mathbb{R}) \cap C^0(\mathbb{R})$ be a corresponding real-valued eigenfunction. Then ψ_2 changes its sign at most twice on $\mathbb{R}$.*

In particular, if ψ_2 is an even eigenfunction, then it changes its sign exactly once on the positive axis $\{x > 0\}$.

Note that since the operator is self-adjoint, and by Lemma 3.2.5 we have that $\psi_1(x) > 0$, the second eigenfunction ψ_2 changes its sign at least once in order to satisfy the orthogonality condition $\langle \psi_1, \psi_2 \rangle = 0$.

Proof. Let us prove the result by contradiction. Suppose that $\psi_2 : \mathbb{R} \to \mathbb{R}$ changes its sign at least three times on $\mathbb{R}$. Then, there exist points $x_1 < x_2 < x_3 < x_4$ such that without loss of generality

$$\psi(x_i) > 0 \quad \text{for } i = 1, 3,$$

and

$$\psi(x_i) < 0 \quad \text{for } i = 2, 4.$$

Now, we consider the a-harmonic extension Ψ_2 on $\mathbb{R}^2_+$. By the continuity of this function up to the boundary, we have that it has at least two nodal domains in $\mathbb{R}^2_+$. Then, by applying Theorem 3.2.6, we conclude that Ψ_2 has exactly two nodal domains, which we denote by Ω_+ and Ω_-.

Next, by the continuity of the a-harmonic extension up to the boundary, we have that there exists a sufficiently small constant $\epsilon_0 > 0$ such that

$$(x_i, \epsilon) \in \Omega_+ \quad \text{for } i = 1, 3,$$

and

$$(x_i, \epsilon) \in \Omega_- \quad \text{for } i = 2, 4,$$

for all $0 < \epsilon \leq \epsilon_0$.

Since the connected open sets $\Omega_\pm \subset \mathbb{R}^2_+$ must be arcwise connected, there exist two simple continuous curves $\gamma_+, \gamma_- \in C^0([0,1]; \overline{\mathbb{R}^2_+})$ satisfying

- $\gamma_+(0) = (x_1, 0)$, $\gamma_+(1) = (x_3, 0)$ and $\gamma_+(t) =\in \Omega_+$ for $t \in (0,1)$.
- $\gamma_-(0) = (x_2, 0)$, $\gamma_+(1) = (x_4, 0)$ and $\gamma_-(t) =\in \Omega_-$ for $t \in (0,1)$.

By the topology of $\overline{\mathbb{R}^2_+}$, we deduce that γ_+ and γ_- must intersect in $\mathbb{R}^2_+$. But this contradicts the fact that $\Omega_+ \cap \Omega_- = \varnothing$.

The second part of the theorem comes immediately. $\square$

Note that in this proof, the fact that we are working in $\mathbb{R}^2_+$ has a crucial role since we have to apply some topological arguments about this space.

3.3. Uniqueness and nondegeneracy of Ground States

First in this section we are going to show that nonnegative minimizers of the functional $J^{s,\alpha}(u)$ have a nondegenerate linearization. In fact we shall prove that ground states are nondegenerate.

Definition 3.3.1 (Nondegeneracy)**.** Let Q be a solution of the equation

$$(-\Delta)^s u - F(u) = 0, \tag{3.3.1}$$

with F any nonlinearity. We say that Q is nondegenerate if the linearized operator

$$\mathcal{L} = (-\Delta)^s - F'(Q)$$

satisfies that

$$ker\, \mathcal{L} = span\{Q'\}.$$

Note that differentiating the nonlinear equation (3.3.1) with respect to x, we have that $(-\Delta)^s Q' - F'(Q)Q' = 0$, and therefore

$$span\{Q'\} \subseteq ker\, \mathcal{L}.$$

Remark 3.3.2. Although we are interested in the study of the nonlinear equation

$$(-\Delta)^s Q + Q - Q^{\alpha+1} = 0,$$

for later purpose we are going to focus in the more general nonlinear equation

$$(-\Delta)^s Q + \lambda\, Q - Q^{\alpha+1} = 0, \tag{3.3.2}$$

with $\lambda > 0$.

In fact, it is easy to see that we can pass from the solutions to one of the equations to the other just by making a rescaling of the independent variable.

Associated to Q and equation (3.3.2) we have the linearized operator

$$L_+ = (-\Delta)^s + \lambda - (\alpha + 1)Q^\alpha.$$

It can be seen that the potential $V = -\lambda + (\alpha + 1)Q^\alpha$ belongs to the Kato class.

One important concept in the proof of the nondegeneracy is the definition of the Morse index of a linear operator.

Definition 3.3.3 (Morse index)**.** The Morse index of a linear operator $\mathcal{N}_-(\mathcal{L})$ is the number of strictly negative eigenvalues, i.e.,

$$\mathcal{N}_-(\mathcal{L}) = \#\{e < 0 : e \text{ is eigenvalue of } \mathcal{L} \text{ acting on } L^2(\mathbb{R})\}.$$

Theorem 3.3.4 (Frank and Lenzmann in [**17**])**.** *Let* $0 < s < 1$ *and* $0 < \alpha < \alpha_{max}(s)$*. Suppose that* $Q \in H^s(\mathbb{R})$ *is a positive solution of equation* (3.0.4) *and consider the linearized operator* L_+ *acting on* L^2*-functions. If* Q *is a local minimizer of the functional* $J^{s,\alpha}(u)$*, then* Q *is nondegenerate.*

We are not going to prove this result, but we note that once we have the oscillation estimate of the previous section, the problem turns the same as in the local case ($s = 1$). One of the key steps is the orthogonal decomposition $L^2(\mathbb{R}) = L^2_{even}(\mathbb{R}) \oplus L^2_{odd}(\mathbb{R})$.

Once we have a nondegeneracy result, we can use it to prove the uniqueness result.

Theorem 3.3.5 (Frank and Lenzmann in [**17**])**.** *Let* $0 < s < 1$ *and* $0 < \alpha < \alpha_{max}(s)$*. Then the ground state solution* $Q = Q(|x|)$ *for equation* (3.0.4) *is unique.*

Since the proof of this result is not easy and we can find it in [**17**], we are only going to briefly explain the strategy behind it.

Let us fix $0 < s_0 < 1$ and $0 < \alpha < \alpha_{max}(s_0)$, and suppose that $Q_0 = Q_0(|x|) > 0$ is a ground state solution. By using the nondegeneracy result we have that the operator L_+ is invertible on $L^2_{even}(\mathbb{R})$. Hence, we can use the implicit function theorem in order to construct a locally unique branch of solutions (Q_s, λ_s) around (Q_0, λ_0), which satisfies

$$(-\Delta^s)Q_s + \lambda_s Q_s - |Q_s|^\alpha Q_s = 0,$$

with $s \in [s_0, s_0 + \epsilon]$ and $\epsilon > 0$ small.

From some a priori estimates, we deduce that in fact the branch (Q_s, λ_s) can be extended to $s = 1$. Once we have established the global continuation, we conclude that $Q_s \to Q_*$ and $\lambda_s \to \lambda_*$ as $s \to 1$, with $Q_* = Q_*(|x|) > 0$ and $\lambda_* > 0$ satisfying

$$-\Delta Q_* + \lambda_* Q_* - Q_*^{\alpha+1} = 0.$$

From standard ODE techniques it is well-known the uniqueness and nondegeneracy of ground states for the local version. Furthermore, we can deduce that λ_* only depends on the starting fractional index s_0 and the parameter α. Hence, we can conclude that two different branches (Q_s, λ_s) and $(\tilde{Q}_s, \tilde{\lambda}_s)$ both starting from a ground state with $s = s_0$ must converge to the same limit (Q_*, λ_*). Therefore, if there exist these two different branches, they must intersect for some $s \in [s_0, 1)$, in contradiction to the local uniqueness of branches.

Chapter 4
A conjecture of De Giorgi for the Allen-Cahn equation

In this chapter we are going to present the conjecture of De Giorgi for the Allen-Cahn equation. First we introduce the conjecture, its motivation and the results that have been proven. Then, we show some of the most interesting ideas of the proof in dimension 3. We finally present the fractional version of this conjecture.

4.1. The conjecture

The following conjecture was raised by Ennio De Giorgi in 1978.

Conjecture 4.1.1 (E. De Giorgi in [**14**])**.** Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}^n \tag{4.1.1}$$

such that

$$|u| \leq 1 \ \text{ and } \ \partial_{x_n} > 0$$

in the whole $\mathbb{R}^n$. Then, all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \leq 8$. Equivalently, u is a function depending only on one Euclidean variable.

Equation (4.1.1) is the so-called Allen-Cahn equation, which models phase transitions. One example of phase transitions can be seen when we have two competing species with opposed interests in steady equilibrium. In this case, the function u would be the difference between the densities of the species, normalized in such way that they take values in the range $[-1, 1]$, where $u(x) = \pm 1$ means that one specie has exterminated the other at this point.

An heuristic motivation of the conjecture of De Giorgi is given now. Let u be a solution of (4.1.1), we consider the function $u_\epsilon(x) = u(x/\epsilon)$, which is a bounded solution of the re-scaled equation

$$-\Delta u_\epsilon = \epsilon^{-2}(u_\epsilon - u_\epsilon^3) \qquad \text{in } \mathbb{R}^n,$$

then, this new equation still models the normalized difference of densities of two species. Nevertheless, since $\epsilon \to 0$, the two species have stronger opposed interest, and in the limit case the two species are incompatible. That is, we will have the space divided in two zones, each one full of only one specie (i.e. $u_\epsilon \to \pm 1$). It is well known that the separating interphase between the two regions is a minimal surface. Hence, this fact together with other results about minimal surfaces, as Simons' classification of entire minimal graphs, motivated the conjecture of De Giorgi.

Now, we are going to present some results to see that the conjecture of De Giorgi is not empty. That is, that there are solutions of the Allen-Cahn equation satisfying the conditions stated.

Lemma 4.1.2. *The unique (up to translations) bounded and strictly increasing solution of equation*

$$u'' = u^3 - u \quad \textit{in } \mathbb{R} \tag{4.1.2}$$

is

$$u(x) = tanh\left(\frac{x}{\sqrt{2}}\right).$$

Proof. Since problem (4.1.2) is a conservative system with one degree of freedom and its associated potential $(1-u^2)^2/4$ has relative minima at $u = \pm 1$ and relative maximum at $u = 0$, then its phase portrait is very easy. The system has three types of solutions: unbounded solutions, periodic ones, and the heteroclinic connection. It is clear that the unique type of solution that satisfies the conditions of the Lemma is the heteroclinic connection.

Therefore, in order to find the heteroclinic we have to solve equation

$$(u')^2 = \frac{1}{2}\left(u^2 - 1\right)^2 \quad \text{in } \mathbb{R},$$

which is the expression that we obtain when multiplying equation (4.1.2) by u' and integrating. We also have to choose the constant of integration in order to impose that the solution is the connection between -1 and 1. In this case the constant is 1/2.

Since we want the solution to be strictly increasing, we finally have to solve

$$u' = \frac{\sqrt{2}}{2}(u^2 - 1) \quad \text{in } \mathbb{R},$$

which is a separable first order ODE.

If we solve this equation we obtain

$$u(x) = tanh\left(\frac{x+b}{\sqrt{2}}\right),$$

which concludes the proof of the result. □

Proposition 4.1.3. *The unique 1D solutions of the Allen-Cahn equation satisfying the conditions of the conjecture of De Giorgi are of the form*

$$u(x) = tanh\left(\frac{a \cdot x + b}{\sqrt{2}}\right),$$

for some $b \in \mathbb{R}$, and $a = (a_1, ..., a_n) \in \mathbb{R}^n$, with $|a| = 1$ and $a_n > 0$.

PROOF. Since $u(x)$ is a function depending only on one Euclidean variable, we can write it as

$$u(x) = U(a \cdot x + b),$$

with $|a| = 1$ and $a_n > 0$. It is easy to prove that

$$\Delta u(x) = U''(a \cdot x + b).$$

Therefore, in order to find u we have to solve equation

$$U'' = U^3 - U.$$

Finally, if we apply the result of Lemma 4.1.2 we obtain the desired result.

□

The conjecture has been proven to be true in dimension $n = 2$ by Ghoussoub and Gui in [**19**] and in dimension $n = 3$ by Ambrosio and Cabré in [**4**]. In [**24**], Savin proves the Conjecture for $4 \leq n \leq 8$ by assuming the additional hypothesis

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{n-1}.$$

In a more recent work, del Pino, Kowalczyk and Wei have presented in [**15**] a counterexample of the conjecture for dimensions $n \geq 9$.

Remark 4.1.4. The positive answers to the conjecture for $n = 2$ and 3 apply also to more general nonlinearities than the Allen-Cahn equation.

4.2. Proof of the conjecture in dimensions 2 and 3

In this section we are going to focus in the proof of Ambrosio and Cabré for dimension $n = 3$ (they also prove the case $n = 2$) since they use some techniques developed by Berestycki, Caffarelli and Nirenberg in [**6**]. This techniques are based on Liouville type results and energy estimates.

We are not going to show the detailed proofs of the results we present. In particular, we do not worry about regularity issues. They can be seen in the original paper, [**4**].

Lemma 4.2.1 (Modica in [**23**])**.** *If $F > 0$ in $\mathbb{R}$, then every bounded solution u of $\Delta u - F'(u) = 0$ in $\mathbb{R}^n$ satisfies the gradient bound*

$$\frac{1}{2}|\nabla u|^2 \leq F(u) \quad in \quad R^n.$$

Proposition 4.2.2 (Ambrosio-Cabré in [**4**])**.** *Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad in \quad \mathbb{R}^n,$$

where F is an arbitrary $C^2(\mathbb{R})$ function. Assume that

$$\partial_n u > 0 \quad in \quad \mathbb{R}^n \quad and \quad \lim_{x_n \to +\infty} u(x', x_n) = 1 \quad for\ all \quad x' \in \mathbb{R}^{n-1}.$$

For every $R > 1$, let $B_R = \{|x| < R\}$. Then,

$$\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + F(u) - F(1) \right\} \leq C\, R^{n-1}$$

for some constant C independent of R.

PROOF. We consider the family of functions

$$u^t(x) = u(x', x_n + t),$$

defined for $x = (x', x^n) \in \mathbb{R}^n$ and $t \in R$. For each t we have

$$\Delta u^t - F'(u^t) = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Since u is bounded and F is $C^2(\mathbb{R})$ and positive, we can apply Lemma 4.2.1 and we get

$$|u^t| + |\nabla u^t| \leq C \quad \text{in} \quad \mathbb{R}^n,$$

with C a positive constant independent of t.

We can also note that

$$\lim_{t \to +\infty} u^t(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

If we denote the derivative of $u^t(x)$ with respect to t by $\partial_t u^t(x)$, we have

$$\partial_t u^t(x) = \partial_n u(x', x_n + t) > 0 \qquad \text{for all} \;\; x \in \mathbb{R}^n.$$

Let us now define the energy of u^t in the ball B_R as

$$E_R(u^t) = \int_{B_R} \left\{ \frac{1}{2}|\nabla u^t|^2 + F(u^t) - F(1) \right\}.$$

We want to see that

$$\lim_{t \to +\infty} E_R(u^t) = 0.$$

Clearly, the term $\int_{B_R} \{F(u^t) - F(1)\}$ tends to zero as t tends to infinity by applying the dominated convergence theorem. On the other hand we have that

$$\int_{B_R} F'(u^t)(u^t - 1) = \int_{B_R} \Delta u^t (u^t - 1) = \int_{\partial B_R} \frac{\partial u^t}{\partial \nu}(u^t - 1) - \int_{B_R} |\nabla u^t|^2,$$

which is equivalent to

$$\int_{B_R} |\nabla u^t|^2 = \int_{\partial B_R} \frac{\partial u^t}{\partial \nu}(u^t - 1) - \int_{B_R} F'(u^t)(u^t - 1).$$

Clearly, the right-hand side converges to zero by the dominated convergence theorem and therefore we conclude that $E_R(u^t)$ goes to zero as we want.

Now we compute the derivative of $E_R(u^t)$ with respect to t. That is,

$$\begin{aligned}
\partial_t E_R(u^t) &= \int_{B_R} \nabla u^t \cdot \nabla(\partial_t u^t) + \int_{B_R} F'(u^t)\,\partial_t u^t = \\
&= -\int_{B_R} \Delta u^t\,\partial_t u^t + \int_{\partial B_R} \partial_t u^t \,\frac{\partial u^t}{\partial \nu} + \int_{B_R} F'(u^t)\,\partial_t u^t = \\
&= \int_{\partial B_R} \partial_t u^t \,\frac{\partial u^t}{\partial \nu} \geq -\int_{\partial B_R} |\partial_t u^t| \left|\frac{\partial u^t}{\partial \nu}\right| = \int_{\partial B_R} -\left|\frac{\partial u^t}{\partial \nu}\right| \partial_t u^t \geq \\
&\geq \int_{\partial B_R} -\left|\nabla u^t\right| \partial_t u^t \geq -C \int_{\partial B_R} \partial_t u^t,
\end{aligned}$$

where we have used the L^∞ bounds of ∇u^t and the fact that $\partial_t u^t > 0$.

Thus, for each $T > 0$, we have

$$\begin{aligned}
E_R(u) &= E_R(u^T) - \int_0^T dt\,\partial_t E_R(u^t) \leq \\
&\leq E_R(u^T) + C\int_0^T dt \int_{\partial B_R} d\sigma(x)\,\partial_t u^t(x) = \\
&= E_R(u^T) + C\int_{\partial B_R} d\sigma(x) \int_0^T dt\,\partial_t u^t(x) = \\
&= E_R(u^T) + C\int_{\partial B_R} d\sigma(x)(u^T - u)(x) \leq \\
&\leq E_R(u^T) + C|\partial B_R| = E_R(u^T) + CR^{n-1}.
\end{aligned}$$

Letting $T \to +\infty$ we finally prove the proposition. □

Theorem 4.2.3 (Ambrosio-Cabré in [**4**])**.** *Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \textit{in} \quad \mathbb{R}^3, \tag{4.2.1}$$

satisfying

$$\partial_3 u > 0 \quad \textit{in} \quad \mathbb{R}^3 \quad \textit{and} \quad \lim_{x_n \to \pm\infty} u(x', x_3) = \pm 1 \quad \textit{for all} \quad x' \in \mathbb{R}^2.$$

Assume that $F \in C^2(\mathbb{R})$ and that

$$F \geq \min\{F(-1), F(1)\} \quad \textit{in} \quad (-1, 1).$$

Then, the level sets of u are planes, i.e., there exists $a \in \mathbb{R}^3$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \quad \textit{for all} \quad x \in \mathbb{R}^3.$$

Proof. For each $i \in \{1, 2\}$, we consider the functions

$$\varphi = \partial_3 u \qquad \text{and} \qquad \sigma_i = \frac{\partial_i u}{\partial_3 u}.$$

We note that σ_i is well defined since $\partial_3 u > 0$. We can use that $\partial_i u$ and $\partial_3 u$ satisfy the same linearized equation $\Delta w - F''(u)w = 0$, which is just obtained by differentiating equation (4.2.1). Therefore we have that

$$\varphi^2 \nabla \sigma_i = \partial_3 \nabla \partial_i u - \partial_i u \, \nabla \partial_3 u,$$

and we can conclude that

$$\nabla \cdot (\varphi^2 \nabla \sigma_i) = 0$$

in the weak sense in $\mathbb{R}^3$. Now we want to apply Proposition 5.2.1. Then we have to show that, for each $R > 1$

$$\int_{B_R} (\varphi \sigma_i)^2 = \int_{B_R} (\partial_i u)^2 \leq C\, R^2,$$

which is equivalent to proving that

$$\int_{B_R} |\nabla u|^2 \leq C\, R^2,$$

for some constant independent of R.

By hypothesis we have that $F \geq \min\{F(-1), F(1)\}$ in $(-1, 1)$. Let us suppose that $\min\{F(-1), F(1)\} = F(1)$. In this case we have that $F(u) - F(1) \geq 0$ in $\mathbb{R}^3$. Therefore, we can apply Proposition 4.2.2 to obtain that

$$\frac{1}{2} \int_{B_R} |\nabla u|^2 \leq \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right\} \leq C\, R^2.$$

On the other hand, if $\min\{F(-1), F(1)\} = F(-1)$, we obtain the same result applying Proposition 4.2.2 to the function $-u(x', -x_3)$.

Finally, by proposition 5.2.1, we have that σ_i is constant, or equivalently that there exist some constant c_i such that

$$\partial_i u = c_i \, \partial_3 u.$$

Then, if we define $\omega = \frac{(c_1, c_2, 1)}{\sqrt{c_1^2 + c_2^2 + 1}}$ we have that

$$\nabla u(x) = (c_1, c_2, 1) \partial_3 u(x) = \sqrt{c_1^2 + c_2^2 + 1}\, \partial_3 u(x)\, \omega.$$

Thus, if $\omega \cdot y = 0$ then

$$u(x + y) - u(x) = \int_0^1 \nabla u(x + ty) \cdot y \, dt = \sqrt{c_1^2 + c_2^2 + 1}\, \partial_3 u(x + ty)\, \omega \cdot y \, dt = 0.$$

Therefore, if we set $u_*(t) := u(t\omega)$ for any $t \in \mathbb{R}$, and we write any $x \in \mathbb{R}^3$ as

$$x = (\omega \cdot x)\omega + y_x$$

with $\omega \cdot y_x = 0$, we conclude that

$$u(x) = u((\omega \cdot x)\omega + y_x) = u((\omega \cdot x)\omega) = u_*(\omega \cdot x).$$

This completes the proof of Theorem 4.2.3. □

Remark 4.2.4. In dimension $n = 2$ there is no need of the energy estimates of Proposition 4.2.2. That is because $|\nabla u| \leq C$ and balls have dimension of order R^2.

If we work a bit more with these tools, we find a more general result where we do not needed any condition about the nonlinearity F (apart from the regularity) and the convergence of the function at infinity.

Theorem 4.2.5 (Alberti-Ambrosio-Cabré in [**3**])**.** *Assume that $F \in C^2(\mathbb{R})$. Let u be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \textit{in} \;\; \mathbb{R}^3,$$

satisfying

$$\partial_3 u > 0 \quad \textit{in} \;\; \mathbb{R}^3.$$

If $n = 2$ or $n = 3$, then all level sets of u are hyperplanes, i.e., there exists $a \in \mathbb{R}^n$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \quad \textit{for all} \;\; x \in \mathbb{R}^n.$$

4.3. Fractional versions of the conjecture

We use the term fractional version of the conjecture of De Giorgi to refer to results that are inspired by the original Conjecture 4.1.1, but where the Laplace operator is replaced by the fractional one. That is, that bounded and monotone solutions to the equation

$$(-\Delta)^s u = f(u) \quad \text{in} \;\; \mathbb{R}^n$$

are one-dimensional.

Currently, the following cases have been proven:

- In dimension $n = 2$, for $s = 1/2$ by Cabré and Solà-Morales in [**12**].
- In dimension $n = 2$, for $0 < s < 1$ by Cabré and Sire in [**10**] and [**11**].
- In dimension $n = 3$, for $1/2 \leq s < 1$ by Cabré and Cinti in [**8**].
- In dimension $4 \leq n \leq 8$, for $1/2 < s < 1$ with the additional conditions

  $$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1,$$

  and f satisfying certain properties, by Savin in [**25**]. The Allen-Cahn nonlinearity satisfies the conditions on f, as it can be seen in the original paper of Savin.

As in the classical conjecture, the results for dimensions 2 and 3 are proven by using some Liouville type results together with energy estimates. On the other hand, the works of Savin are based on results about minimal surfaces. Then, we note that he can only prove the result for the case $1/2 < s < 1$ because when $0 < s < 1/2$, the role of minimal surfaces corresponds to nonlocal minimal surfaces, and there is now not much knowledge about the last ones.

Chapter 5
Several Liouville type results

In this chapter we present some of the most important Liouville type results. When we talk about a Liouville type result we mean a result that states that if we have two bounded solutions of a certain linear integro-differential equation with one of them always positive, then both solutions are the same up to a multiplicative constant.

First in the chapter we show the Classical Liouville theorem for harmonic functions, and we continue with a more general version, Theorem 5.2.2, which has also the Laplacian as main part of the equation but it includes a potential $V(x)$ as well. Then we present the fractional version of this results, in which the Laplacian is substitute by the fractional Laplacian. Although these results involve nonlocal operators they are proven by using local arguments through the extension problem. Finally we present a recent Liouville type result involving nonlocal operators, a kind of integral operators with truncated kernels, which is proven without the extension problem.

5.1. Classical Liouville's Theorem

Theorem 5.1.1. *(Liouville's Theorem) Any harmonic and bounded function in $\mathbb{R}^n$ is constant.*

This is a very well known result that is studied in any basic course of PDEs. Due to the close relationship between harmonic functions and holomorphic functions, this is also a result studied in any basic course of complex analysis.When proving it with arguments of complex analysis, Taylor representation and Cauchy's integral formula are the main tools used. On the other hand, when we use the PDEs' approach, the mean value property and integration by parts are used, as we will see in the following proof.

Proof. Since w is harmonic, w_i is also harmonic. Therefore we can use the mean value property on w_i. Using also Green's theorems

$$w_i(x) = \frac{1}{|B_R|}\int_{B_R(x)} w_i(y)dV = \frac{1}{|B_R|}\int_{\partial B_R(x)} w(y)\,\nu_i dS.$$

Now, if we compute its absolute value we have

$$|w_i(x)| = \frac{1}{|B_R|}\left|\int_{\partial B_R(x)} w(y)\,\nu_i dS\right| \leq \frac{1}{|B_R|}\int_{\partial B_R(x)} |w(y)|dS \leq$$
$$\leq ||u||_{L^\infty}\frac{|\partial B_R|}{|B_R|} = \frac{C}{R}||u||_{L^\infty},$$

with C only dependent on the dimension n. Since u is harmonic on $\mathbb{R}^n$, we can let R go to infinity and conclude that its derivatives are zero at any point, i.e.,

$$u_i(x) \equiv 0 \ \ \forall\, i,$$

and therefore $u(x) \equiv ctt.$ □

This is the most basic and known result we are going to present in this chapter, and is the one that gives the name to all this family of results that we are going to study, the Liouville type results. It seems that it is not a result of this type because we are only talking here about one solution of the equation instead of the two we have mentioned before. Nevertheless we can understand this theorem as a Liouville type result if we think that the second function that also solves the equation is the constant function $u \equiv 1$, which is the one that plays the role of positive function. Hence, in this case being both solutions equal up to a multiplicative constant is the same as being both solutions constant, which is what the theorem states.

5.2. A Liouville Theorem for classical Schrödinger operators

In this part of the chapter we have a generalization of the classical Liouville Theorem in which we modify the Laplace equation by adding a potential term $V(x)$, that is, we deal with a Schrödinger operator.

In order to obtain the Liouville result we first need to present a very important proposition by *L. Ambrosio and X. Cabré* from [**4**] that says that the unique supersolution of a certain kind of second order differential equation are the constant ones.

Proposition 5.2.1 (Ambrosio-Cabré in [**4**])**.** *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$\sigma\,\nabla\cdot(\varphi^2\nabla\sigma) \geq 0 \quad in\ \mathbb{R}^n$$

in the distributional sense. For every $R > 1$, let $B_R = \{|x| < R\}$ and assume that

$$\int_{B_R}(\varphi\sigma)^2 \leq CR^2,$$

for some constant C independent of R. Then σ is constant.

This result is very important because it is one of the main tools that were used to prove De Giorgi conjecture in dimensions 2 and 3, as we have seen in the Chapter 4 of this thesis.

PROOF. Let ζ be a C^∞ decreasing function on $\mathbb{R}^+$ such that $0 \le \zeta \le 1$ and

$$\zeta = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 2. \end{cases}$$

Since ζ' is a continuous function and it is zero if $t \ge 2$, then it is bounded. That is,

$$|\zeta'| \le C < +\infty.$$

For $R > 1$, let

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right) \qquad \text{for } x \in \mathbb{R}^n.$$

From the bound of $|\zeta'|$ we can obtain the following estimate for the gradient of ζ_R:

$$\nabla\zeta_R(x) = \frac{x}{R|x|}\zeta'\left(\frac{|x|}{R}\right) \Longrightarrow |\nabla\zeta_R(x)| = \frac{1}{R}\left|\zeta'\left(\frac{|x|}{R}\right)\right| \le \frac{C}{R}.$$

Note that ζ_R converges pointwise to the function 1 when R tends to infinity.

Multiplying $\varphi^2|\nabla\sigma|^2$ by ζ_R^2 and integrating in $\mathbb{R}^n$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta_R^2\varphi^2|\nabla\sigma|^2 &= \int_{B_{2R}} \zeta_R^2\varphi^2\nabla\sigma\cdot\nabla\sigma = \\
&= \int_{\partial B_{2R}} \zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} - \int_{B_{2R}} \sigma\nabla\left(\zeta_R^2\varphi^2\nabla\sigma\right) = \\
&= -\int_{B_{2R}} \zeta_R^2\,\sigma\nabla\left(\varphi^2\nabla\sigma\right) - 2\int_{B_{2R}} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \le \\
&\le -2\int_{B_{2R}} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma = -2\int_{\{R<|x|<2R\}} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \le \\
&\le 2\left[\int_{\{R<|x|<2R\}} \zeta_R^2\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{B_{2R}} \varphi^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \le \\
&\le 2\left[\int_{\{R<|x|<2R\}} \zeta_R^2\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\frac{C}{R^2}\int_{B_{2R}} (\varphi\sigma)^2\right]^{1/2},
\end{aligned}$$

where the constant C is the one that appears in the estimate of the gradient of ζ_R, which is independent of R. Using the hypothesis, we infer that

$$\int_{\mathbb{R}^n} \zeta_R^2\varphi^2|\nabla\sigma|^2 \le C\left[\int_{\{R<|x|<2R\}} \zeta_R^2\varphi^2|\nabla\sigma|^2\right]^{1/2} \le C\left[\int_{\mathbb{R}^n} \zeta_R^2\varphi^2|\nabla\sigma|^2\right]^{1/2},$$

again with C independent of R. This implies that

$$\int_{\mathbb{R}^n} \zeta_R^2\varphi^2|\nabla\sigma|^2 \le C.$$

Since $\zeta(t)$ is a decreasing function, the set of functions $\zeta_R(x)$ is a sequence of functions increasing pointwise, and therefore $\zeta_R^2\,\varphi^2|\nabla\sigma|^2$ has also the same property, and we can apply the monotone convergence theorem for the Lebesgue measure. That is,

$$\int_{\mathbb{R}^n}\varphi^2|\nabla\sigma|^2 = \int_{\mathbb{R}^n}\lim_{R\to\infty}\zeta_R^2\,\varphi^2|\nabla\sigma|^2 = \lim_{R\to\infty}\int_{\mathbb{R}^n}\zeta_R^2\,\varphi^2|\nabla\sigma|^2 \le C.$$

By applying the dominated convergence theorem to $\zeta_R^2\,\varphi^2|\nabla\sigma|^2\,\chi_{\{R<|x|<2R\}}$, it follows that

$$\int_{\mathbb{R}^n}\varphi^2|\nabla\sigma|^2 = 0.$$

We conclude that $|\nabla\sigma| = 0$, and then σ is constant. $\square$

We want to stress that the ideas of using cut-off functions, integration over all the domain and using convergence results with Lebesgue measure, which are applied in the proof of the proposition, are going to be used in most of the proofs of Liouville type results along this thesis.

Now we apply the previous proposition to obtain a generalization of the Liouville Theorem in low dimensions when we have a potential in the equation.

Corollary 5.2.2. *Let $w, \tilde{w}$ be two solutions of the linear equation*

$$-\Delta u - V(x)\,u = 0, \quad \text{in } \mathbb{R}^n,$$

with $w > 0$, $w, \tilde{w} \in L^\infty(\mathbb{R}^n)\cap C^1(\mathbb{R}^n)$ and $n \le 2$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Once we have this theorem we can see that Theorem 5.1.1 is the particular case when the potential is $V(x)\equiv 0$, and the dimension is 1 or 2.

Proof. Let

$$\sigma(x) := \frac{\tilde{w}(x)}{w(x)},$$

which is well defined and continuous due to the fact that w is always positive and both functions are continuous. Since the functions are of class C^1, so is σ, and therefore we conclude that $\sigma \in H^1_{loc}$. Then

$$\nabla\sigma = \nabla\left(\frac{\tilde{w}}{w}\right) = \frac{w\,\nabla\tilde{w} - \tilde{w}\,\nabla w}{w^2} \Longrightarrow w^2\,\nabla\sigma = w\,\nabla\tilde{w} - \tilde{w}\,\nabla w,$$

and if we compute the divergence of the last expression we obtain

$$\begin{aligned}\operatorname{div}(w^2\,\nabla\sigma) &= \operatorname{div}(w\,\nabla\tilde{w} - \tilde{w}\,\nabla w) = w\Delta\tilde{w} + \nabla w\cdot\nabla\tilde{w} - \tilde{w}\Delta w - \nabla\tilde{w}\cdot\nabla w = \\ &= w\Delta\tilde{w} - \tilde{w}\Delta w = -w\,V(x)\tilde{w} + \tilde{w}\,V(x)w = 0.\end{aligned}$$

Therefore, since $n \leq 2$,

$$\int_{B_R} \tilde{w}^2 \leq ||\tilde{w}||^2_{L^\infty} \int_{B_R} 1 \leq CR^2,$$

and we can apply Proposition 5.2.1 with $\sigma = \tilde{w}/w$ and $\varphi = w$ because they satisfy the necessary regularity conditions. This yields the desired result. $\square$

We can see that the hypothesis of having low dimensions is crucial, and the limitation comes from the fact that balls of radius R in dimension n have size proportional to R^n.

We can remark that in this proof we find another of the main ideas that will be used along this work, which is trying to find a simple equation, related to the original one, that is satisfied by the quotient, σ, of the two solutions involved in the theorem.

5.3. A Liouville Theorem for the Half-Laplacian

In this section we present the equivalent results of section 5.2 but replacing the classical Laplacian with the half-Laplacian.

The main difference of this problem with respect to the one in the previous section is the fact that instead of having a local operator we have now a nonlocal one. Nevertheless, as we have seen in Chapter 2, there exists a very useful relation of the fractional Laplacian with a local problem via the extension of Caffarelli and Silvestre.

Therefore, at the end, the only differences with respect to the previous sections are that we have to work in half-spaces and with Neumann boundary conditions, apart from keeping always in mind the extension problem.

Proposition 5.3.1 (Cabré, Solà-Morales in [**12**]). *Let $\varphi \in L^\infty_{loc}(\overline{\mathbb{R}^n_+})$ be a positive function, not necessarily bounded in all of $\mathbb{R}^n_+$. Suppose that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^n_+})$ satisfies*

$$\begin{cases} \sigma\, \nabla \cdot (\varphi^2 \nabla \sigma) \geq 0 & \text{in } \mathbb{R}^n_+, \\ \sigma \, \frac{\partial \sigma}{\partial \nu} \leq 0 & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

in the weak sense. Assume that, for every $R > 1$,

$$\int_{B_R^+} (\varphi\sigma)^2 \leq CR^2,$$

for some constant C independent of R. Then σ is constant.

Proof. Let ζ be a C^∞ decreasing function on $\mathbb{R}^+$ such that $0 \leq \zeta \leq 1$ and

$$\zeta = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

Since ζ' is a continuous function and it is zero if $t \geq 2$, then it is bounded. That is,

$$|\zeta'| \leq C < +\infty.$$

For $R > 1$, let

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right) \qquad \text{for } x \in \mathbb{R}^n.$$

From the bound of $|\zeta'|$ we can obtain the following estimate for the gradient of ζ_R,

$$\nabla\zeta_R(x) = \frac{x}{R|x|}\zeta'\left(\frac{|x|}{R}\right) \Longrightarrow |\nabla\zeta_R(x)| = \frac{1}{R}\left|\zeta'\left(\frac{|x|}{R}\right)\right| \leq \frac{C}{R}.$$

We can note that ζ_R converges pointwise to the function 1 when R tends to infinity. Now, multiplying $\varphi^2|\nabla\sigma|^2$ by ζ_R^2 and integrating in $\mathbb{R}^n_+$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \zeta_R^2\,\varphi^2|\nabla\sigma|^2 &= \int_{B^+_{2R}} \zeta_R^2\,\varphi^2\nabla\sigma\cdot\nabla\sigma = \\
&= \int_{\partial B^+_{2R}} \zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} - \int_{B^+_{2R}} \sigma\nabla\left(\zeta_R^2\,\varphi^2\nabla\sigma\right) = \\
&= \int_{\Gamma^0_{2R}} \zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} + \int_{\Gamma^+_{2R}} \zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} - \int_{B^+_{2R}} \sigma\nabla\left(\zeta_R^2\,\varphi^2\nabla\sigma\right) \leq \\
&\leq -\int_{B^+_{2R}} \zeta_R^2\,\sigma\nabla\left(\varphi^2\nabla\sigma\right) - 2\int_{B^+_{2R}} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \leq \\
&\leq -2\int_{B^+_{2R}} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma = -2\int_{B^+_{2R}\setminus B^+_R} \zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \leq \\
&\leq 2\left[\int_{B^+_{2R}\setminus B^+_R} \zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{B^+_{2R}} \varphi^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \leq \\
&\leq 2\left[\int_{B^+_{2R}\setminus B^+_R} \zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\frac{C}{R^2}\int_{B^+_{2R}} (\varphi\sigma)^2\right]^{1/2},
\end{aligned}$$

where the constant C is the one that appears in the estimate of the gradient of ζ_R, which is independent of R. Using the hypothesis, we infer that

$$\int_{\mathbb{R}^n_+} \zeta_R^2\,\varphi^2|\nabla\sigma|^2 \leq C\left[\int_{B^+_{2R}\setminus B^+_R} \zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2} \leq C\left[\int_{\mathbb{R}^n} \zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2},$$

again with C independent of R. This implies that

$$\int_{\mathbb{R}^n_+} \zeta_R^2\,\varphi^2|\nabla\sigma|^2 \leq C.$$

Since $\zeta(t)$ is a decreasing function, the set of functions $\zeta_R(x)$ is a sequence of functions increasing pointwise, and therefore $\zeta_R^2\,\varphi^2|\nabla\sigma|^2$ has also the same property.

Hence, we can apply the monotone convergence theorem for Lebesgue measure. That is,

$$\int_{\mathbb{R}^n_+} \varphi^2|\nabla\sigma|^2 = \int_{\mathbb{R}^n} \lim_{R\to\infty} \zeta_R^2\,\varphi^2|\nabla\sigma|^2 = \lim_{R\to\infty}\int_{\mathbb{R}^n} \zeta_R^2\,\varphi^2|\nabla\sigma|^2 \le C.$$

It follows by applying the dominated convergence theorem to $\zeta_R^2\,\varphi^2|\nabla\sigma|^2\,\chi_{\{R<|x|<2R\}}$ that

$$\int_{\mathbb{R}^n} \varphi^2|\nabla\sigma|^2 = 0.$$

We conclude that $|\nabla\sigma| = 0$, and then σ is constant. □

We remark that the proof of this proposition is essentially the same as for Proposition 5.2.1 but with half-balls instead of complete balls.

Now we present a new proposition, that can be considered as a Liouville type result for harmonic functions in a half-plane. This is of great significance because we will apply it to the extension function when we prove the Liouville type result for the half-Laplacian.

Proposition 5.3.2 (Cabré, Solà-Morales in [**12**]). *Let $w, \tilde{w}$ be two solutions of the linear problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ -u_y - V(x)\,u = 0 & \text{on } \{y=0\}, \end{cases}$$

with $w > 0$ in $\mathbb{R}^2_+$ and $w, \tilde{w} \in L^\infty(\overline{\mathbb{R}^2_+}) \cap C^1(\overline{\mathbb{R}^2_+})$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Proof. Let

$$\sigma(x) := \frac{\tilde{w}(x)}{w(x)},$$

which is well defined and continuous because w is always positive and both functions are continuous. Since the functions are of class C^1, σ is C^1 as well, and we can conclude that $\sigma \in H^1_{loc}$. Then

$$\nabla\sigma = \nabla\left(\frac{\tilde{w}}{w}\right) = \frac{w\,\nabla\tilde{w} - \tilde{w}\,\nabla w}{w^2} \Longrightarrow w^2\,\nabla\sigma = w\,\nabla\tilde{w} - \tilde{w}\,\nabla w,$$

and if we compute the divergence of the last expression we obtain

$$\begin{aligned}\operatorname{div}(w^2\,\nabla\sigma) &= \operatorname{div}(w\,\nabla\tilde{w} - \tilde{w}\,\nabla w) = w\Delta\tilde{w} + \nabla w\cdot\nabla\tilde{w} - \tilde{w}\Delta w - \nabla\tilde{w}\cdot\nabla w = \\ &= w\Delta\tilde{w} - \tilde{w}\Delta w = 0.\end{aligned}$$

On the other hand we have

$$\frac{\partial\sigma}{\partial\nu} = -\sigma_y = \left(\frac{\tilde{w}}{w}\right)_y = \frac{\tilde{w}_y\,w - \tilde{w}\,w_y}{w^2} = \frac{-V(x)\,\tilde{w}\,w + \tilde{w}\,V(x)\,w}{w^2} = 0.$$

Therefore, since $n = 2$,

$$\int_{B_R^+} \tilde{w}^2 \le ||\tilde{w}||_{L^\infty}^2 \int_{B_R^+} 1 = CR^2,$$

and we can apply Proposition 5.3.1 with $\sigma = \tilde{w}/w$ and $\varphi = w$ and we obtain the desired result. □

Finally, if we use all the results we have previously presented, we can obtain a Liouville type result for a nonlocal operator, the half-Laplacian, in dimension 1.

Corollary 5.3.3. *Let $w, \tilde{w}$ be two solutions of the linear nonlocal equation*

$$(-\Delta)^{1/2}\, u - V(x)\, u = 0 \quad \text{in } \mathbb{R}, \tag{5.3.1}$$

with $w > 0$ and $w, \tilde{w} \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Proof. We only have to use the extension problem of Caffarelli and Silvestre [**13**]. That is, we have seen in section 2.3 that given a function u defined in $\mathbb{R}^n$ we can construct its harmonic extension U in $\mathbb{R}^{n+1}_+$, which is the unique harmonic function in $\mathbb{R}^{n+1}_+$ such that $U|_{\partial\mathbb{R}^{n+1}_+} = u$.

Then, if $w, \tilde{w}$ are two solutions of the linear nonlocal equation (5.3.3), their harmonic extensions $W, \tilde{W}$ are solutions of the local problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+, \\ -U_y - V(x)\, U = 0 & \text{on } \{y = 0\}. \end{cases}$$

By the maximum (and minimum) principle for harmonic functions we can say that $W > 0$ in $\mathbb{R}^2_+$ and $W, \tilde{W} \in L^\infty(\overline{\mathbb{R}^2_+})$.

Finally we can apply Proposition 5.3.2 to conclude that the quotient between $\tilde{W}$ and W is constant, and therefore the quotient between $\tilde{w}$ and w is also constant. This completes the proof of Lemma 5.3.3. □

We remark that, unlike what happened in Corollary 5.2.2, where we obtain a theorem that is valid for dimensions 1 and 2, in the case of the half-Laplacian, Corollary 5.3.3, we obtain a result only valid for dimension 1. The reason of these differences lies in the fact that we are using ideas that are completely analogous, and that in the case of the half-Laplacian we need an extra dimension to work with the extension.

5.4. A Liouville Theorem for the Fractional Laplacian

This section is completely analogous to the previous one, but with the fractional Laplacian of any order. Therefore the results in section 5.3 can be considered as a particular case of the ones in this section. Nevertheless, we have made the distinction of sections due to historical reasons, that is, the results with the half-Laplacian were found prior to the general ones.

Proposition 5.4.1 (Cabré-Sire in [**10**]). *Let $\varphi \in L^\infty_{loc}(\overline{\mathbb{R}^n_+})$ be a positive function and $a \in (-1,1)$. Suppose that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^n_+}, y^a)$ is such that*

$$\begin{cases} \sigma\, \nabla \cdot (y^a\, \varphi^2 \nabla \sigma) \geq 0 & \text{in } \mathbb{R}^n_+, \\ \sigma\, \frac{\partial \sigma}{\partial \nu^a} \leq 0 & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

in the weak sense. Assume that, for every $R > 1$,

$$\int_{B_R^+} y^a\, (\varphi\sigma)^2 \leq CR^2,$$

for some constant C independent of R. Then σ is constant.

Proof. Let ζ be a C^∞ decreasing function on $\mathbb{R}^+$ such that $0 \leq \zeta \leq 1$ and

$$\zeta = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

Since ζ' is a continuous function and it is zero if $t \geq 2$, it is bounded. That is,

$$|\zeta'| \leq C < +\infty.$$

For $R > 1$, let

$$\zeta_R(x) = \zeta\left(\frac{|x|}{R}\right) \qquad \text{for } x \in \mathbb{R}^n_+.$$

From the bound of $|\zeta'|$ we can obtain the following estimate for the gradient of ζ_R,

$$\nabla \zeta_R(x) = \frac{x}{R|x|}\zeta'\left(\frac{|x|}{R}\right) \Longrightarrow |\nabla \zeta_R(x)| = \frac{1}{R}\left|\zeta'\left(\frac{|x|}{R}\right)\right| \leq \frac{C}{R}.$$

We can note that ζ_R converges pointwise to the function 1 when R tends to infinity.

Now, multiplying $y^a\,\varphi^2|\nabla\sigma|^2$ by ζ_R^2 and integrating in $\mathbb{R}^n_+$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n_+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2 &= \int_{B_{2R}^+} y^a\,\zeta_R^2\,\varphi^2\nabla\sigma\cdot\nabla\sigma = \\
&= \int_{\partial B_{2R}^+} y^a\,\zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} - \int_{B_{2R}^+}\sigma\nabla\left(y^a\,\zeta_R^2\,\varphi^2\nabla\sigma\right) = \\
&= \int_{\Gamma_{2R}^0} y^a\,\zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} + \int_{\Gamma_{2R}^+} y^a\,\zeta_R^2\,\sigma\,\varphi^2\frac{\partial\sigma}{\partial n} - \int_{B_{2R}^+}\sigma\nabla\left(y^a\,\zeta_R^2\,\varphi^2\nabla\sigma\right) \le \\
&\le -\int_{B_{2R}^+}\zeta_R^2\,\sigma\nabla\left(y^a\,\varphi^2\nabla\sigma\right) - 2\int_{B_{2R}^+} y^a\,\zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \le \\
&\le -2\int_{B_{2R}^+} y^a\,\zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma = -2\int_{B_{2R}^+\setminus B_R^+} y^a\,\zeta_R\,\varphi^2\sigma\,\nabla\zeta_R\cdot\nabla\sigma \le \\
&\le 2\left[\int_{B_{2R}^+\setminus B_R^+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{B_{2R}^+} y^a\,\varphi^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \le \\
&\le 2\left[\int_{B_{2R}^+\setminus B_R^+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2}\left[\frac{C}{R^2}\int_{B_{2R}^+} y^a\,(\varphi\sigma)^2\right]^{1/2},
\end{aligned}$$

where the constant C is the one that appears in the estimate of the gradient of ζ_R, which is independent of R. Using the hypothesis, we have that

$$\int_{\mathbb{R}^n_+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2 \le C\left[\int_{B_{2R}^+\setminus B_R^+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2} \le C\left[\int_{\mathbb{R}^n} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2\right]^{1/2},$$

again with C independent of R. This implies that

$$\int_{\mathbb{R}^n_+} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2 \le C.$$

Since $\zeta(t)$ is a decreasing function, then the set of functions $\zeta_R(x)$ is a sequence of functions increasing pointwise, and therefore $y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2$ has also the same property. Therefore, we can apply the monotone convergence theorem for Lebesgue measure. That is,

$$\int_{\mathbb{R}^n_+} y^a\,\varphi^2|\nabla\sigma|^2 = \int_{\mathbb{R}^n}\lim_{R\to\infty} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2 = \lim_{R\to\infty}\int_{\mathbb{R}^n} y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2 \le C.$$

It follows by applying the dominated convergence theorem to $y^a\,\zeta_R^2\,\varphi^2|\nabla\sigma|^2\,\chi_{\{R<|x|<2R\}}$ that

$$\int_{\mathbb{R}^n} y^a\,\varphi^2|\nabla\sigma|^2 = 0.$$

We finally conclude that $|\nabla\sigma| = 0$, and then σ is constant. □

Now, we continue with a Liouville type result in the half-plane.

Proposition 5.4.2 (Cabré-Sire in [**10**]). *Let $w, \tilde{w}$ be two solutions of the linear problem*

$$\begin{cases} \nabla \cdot (y^a \nabla u) = 0 & in\ \mathbb{R}^2_+, \\ (1+a)\,\frac{\partial u}{\partial \nu^a} - V(x)\,u = 0 & on\ \{y=0\}, \end{cases}$$

with $w > 0$ in $\mathbb{R}^2_+$,$w, \tilde{w} \in L^\infty(\overline{\mathbb{R}^2_+}) \cap C^1(\overline{\mathbb{R}^2_+})$ and $a \in (-1, 0]$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Proof. Let

$$\sigma(x) := \frac{\tilde{w}(x)}{w(x)},$$

which is well defined and continuous due to the fact that w is always positive and both functions are continuous. Since the functions are of class C^1, so is σ, and therefore we conclude that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^n_+}, y^a)$. Then

$$\nabla\sigma = \nabla\left(\frac{\tilde{w}}{w}\right) = \frac{w\,\nabla\tilde{w} - \tilde{w}\,\nabla w}{w^2} \Longrightarrow w^2\,\nabla\sigma = w\,\nabla\tilde{w} - \tilde{w}\,\nabla w,$$

and if we compute the divergence of the last expression multiplied by y^a we obtain

$$\begin{aligned} \operatorname{div}(y^a\,w^2\,\nabla\sigma) &= \operatorname{div}(y^a\,w\,\nabla\tilde{w} - y^a\,\tilde{w}\,\nabla w) = \\ &= w\,\nabla\cdot(y^a\,\nabla\tilde{w}) + y^a\,\nabla w\cdot\nabla\tilde{w} - \tilde{w}\,\nabla\cdot(y^a\,\nabla w) - y^a\,\nabla\tilde{w}\cdot\nabla w = \\ &= w\,\nabla\cdot(y^a\,\nabla\tilde{w}) - \tilde{w}\,\nabla\cdot(y^a\,\nabla w) = 0. \end{aligned}$$

On the other hand we have

$$\frac{\partial\sigma}{\partial\nu^a} = -y^a\,\sigma_y = -y^a\,\left(\frac{\tilde{w}}{w}\right)_y = -\frac{y^a\,\tilde{w}_y\,w - \tilde{w}\,y^a\,w_y}{w^2} = \frac{V(x)\,\tilde{w}\,w - \tilde{w}\,V(x)\,w}{(1+a)\,w^2} = 0.$$

Therefore, since $n = 2$ and $a \in (-1, 0]$,

$$\int_{B_R^+} y^a\,\tilde{w}^2 \le ||\tilde{w}||^2_{L^\infty} \int_{B_R^+} y^a = C\,R^{2+a} \le CR^2,$$

We can apply Proposition 5.4.1 with $\sigma = \tilde{w}/w$ and $\varphi = w$ and we have completed the proof. ☐

Finally we obtain a Liouville type result for the fractional Laplacian.

Corollary 5.4.3. *Let $w, \tilde{w}$ be two solutions of the linear nonlocal equation*

$$(-\Delta)^s\,u - V(x)\,u = 0 \quad in\ \mathbb{R}, \tag{5.4.1}$$

with $w > 0$, $w, \tilde{w} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and $s \in [1/2, 1)$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

Proof. We only have to use the extension problem of Caffarelli and Silvestre [**13**]. That is, we have seen in section 2.3 that given a function u defined in $\mathbb{R}^n$ we can construct its a-harmonic extension U in $\mathbb{R}^{n+1}_+$ that is the unique a-harmonic function in $\mathbb{R}^{n+1}_+$ such that $U|_{\partial\mathbb{R}^{n+1}_+} = u$.

Then, if $w, \tilde{w}$ are two solutions of the linear nonlocal equation (5.4.1), their extensions $W, \tilde{W}$ are solutions of the local problem

$$\begin{cases} \nabla \cdot (y^a \, \nabla U) = 0 & \text{in } \mathbb{R}^2_+, \\ c_a \, \frac{\partial U}{\partial \nu^a} - V(x) \, U = 0 & \text{on } \{y = 0\}, \end{cases}$$

with $a = 1 - 2s$.

By the maximum (and minimum) principle for a-harmonic functions (Corollary 2.3.7), we can say that $W > 0$ in $\mathbb{R}^2_+$ and $W, \tilde{W} \in L^\infty(\overline{\mathbb{R}^2_+})$.

Finally we can apply Proposition 5.4.2 to conclude that the quotient between $\tilde{W}$ and W is constant, and hence the quotient between $\tilde{w}$ and w is also constant. □

We note that in this section we obtain a Liouville type result involving the fractional Laplacian in dimension 1, but only when the order of the integro-differential operator is greater or equal to one. We can see that it has the same rank of validity in s as Theorem 1.1.1, although as we will see in chapter 6, the way we deal with the problem is totally different.

In this case, the limitations in the rank of s come from the integrability of the function y^{1-2s} in the half-balls of radius R, where we need the integrals to be of order 2 or less with respect to R.

5.5. A Liouville Theorem for a class of nonlocal equations in the plane

Finally, in this section of the thesis we present a very recent Liouville type result with nonlocal operators in dimension 2. Unlike what we did in the previous sections where, although the equations were nonlocal we worked with a local extension problem, in this case we work directly with the nonlocal equations.

Theorem 5.5.1 (Hamel, Ros-Oton, Sire, Valdinoci in [**20**]). *Let $n = 2$ and L an operator of the form* (1.0.1)*, with kernel K satisfying*

(H1) $K(z) \geq m_0 \, \chi_{B_{r_0}}(z)$ in $\mathbb{R}^2$ for some $m_0 > 0$ and $r_0 > 0$. Moreover, K has compact support in B_{R_0} for some $R_0 > 0$, that is

$$K \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R_0},$$

and

$$\int_{B_{R_0}} |z|^2 \, K(z) \, dz < \infty.$$

(H2) The operator L satisfies the following Harnack inequality: if φ is continuous and positive in $\mathbb{R}^2$ and is a weak solution to $L\,\varphi + c(x)\,\varphi = 0$ in B_R, with $c(x) \in L^\infty(B_R)$ and $||c||_{L^\infty} \leq b$, then

$$\sup_{B_{R/2}} \varphi \leq C \inf_{B_{R/2}} \varphi$$

for some constant C depending on L and b, but independent of φ.

Let $w, \tilde{w} \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be solutions of the equation

$$L\,u - V(x)\,u = 0 \quad in\ \mathbb{R}^2,$$

with $w > 0$ in $\mathbb{R}^2$ and $V(x) \in C(\mathbb{R}^2)$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

We are not going to prove this theorem because the proof is very similar to that of Theorem 1.1.1, which we are going to do with detail in Chapter 6. In fact, once you have read Chapter 6, the proof of this theorem comes immediately. The complete proof of the theorem can be found in [**20**].

Remark 5.5.2. Now we list some comments about this theorem.

(1) It is the first Liouville type result for nonlocal equations that is proven without using the extension and local arguments.
(2) One of the main steps in the proof of the theorem is finding an equation that is satisfied by the quotient, $\tilde{w}/w$. In the proof of theorem 1.1.1 we use an improvement of the same equation.
(3) Both hypothesis $(H1)$ and $(H2)$ are very restricting conditions and are far away from the desired results of this type. That is to say, in the case of the classical Laplacian we have seen a Liouville type result in dimension 2, therefore it is to be expected a similar result in the fractional case.

Chapter 6
A new Liouville type result for fractional Schrödinger operators in 1D

In this chapter we give the proof of Theorems 1.1.1 and 1.1.3. Although the statement of Theorem 1.1.1 was already known, we present here a different proof without using the extension problem of the fractional Laplacian. Thanks to this new proof we can extend Theorem 1.1.1 to other nonlocal operators that do not have a local extension problem, Theorem 1.1.3. This result is a new and original result of the thesis.

6.1. Preliminary Results

This section is devoted to presenting some results that are needed to finally prove Theorems 1.1.1 and 1.1.3.

Lemma 6.1.1. *Let L be an operator of the form* (1.0.1). *Then*

$$2\int_{\mathbb{R}} Lu(x)\, v(x)\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))\ (v(x)-v(y))\ K(x-y)\, dx\, dy.$$

Proof.

$$\begin{aligned}
2\int_{\mathbb{R}} Lu(x)\, v(x)\, dx &= 2\int_{\mathbb{R}}\left[\int_{\mathbb{R}} (u(x)-u(y))\ K(x-y)\, dy\right]\ v(x)\, dx = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))\ K(x-y)\, v(x)\, dy\, dx + \\
&\qquad + \int_{\mathbb{R}}\int_{\mathbb{R}} (u(y)-u(x))\ K(y-x)\, v(y)\, dy\, dx = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))\ v(x)\, K(x-y)\, dx\, dy - \\
&\qquad - \int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))\ v(y)\, K(x-y)\, dx\, dy = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (u(x)-u(y))\ (v(x)-v(y))\ K(x-y)\, dx\, dy.
\end{aligned}$$

$\square$

Note that the last expression is symmetric with respect to u and v.

Now we need a result about the integrability of the fractional Laplace's kernel in certain domains. This is one of the crucial results that is needed in the final proof. In fact, this result is the one that limits us to extend the rank of applicability of the Theorems to s in $(0, 1/2)$.

Lemma 6.1.2. *Let K_s be the kernel of the s-fractional Laplace operator in dimension one, that is*

$$K_s(z) = \frac{1}{|z|^{1+2s}},$$

and the sets

$$S_R = \{[-2R, 2R] \times \mathbb{R} \cup \mathbb{R} \times [-2R, 2R]\} \setminus [-R, R] \times [-R, R] \subset \mathbb{R}^2,$$

$$T_R(A) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x-y| \leq \frac{R}{A}\right\} \subset \mathbb{R}^2,$$

with A a positive constant.

Then

$$C_1\, R^{1-2s} \leq \int_{S_R \cap T_R} \frac{|x-y|^2}{R^2} K_s(x-y) dx\, dy \leq C_2\, R^{1-2s},$$

and

$$C_3\, R^{1-2s} \leq \int_{S_R \cap (T_R)^c} K_s(x-y) dx\, dy \leq C_4\, R^{1-2s},$$

with C_1, C_2, C_3 and C_4 positive constants that do not depend on R.

Proof. First, let us note that the lines with $|x-y| = ctt$ inside $S_R \cap T_R$ have length of order R, that is

$$c_1\, R \leq \mu(\{(x, y) \in S_R \cap T_R \text{ such that } |x-y| = z\}) \leq c_2\, R,$$

with c_1 and c_2 independent of R and z.

Therefore if we use this property together with the change of variable $z = x - y$ and Fubini's theorem we get on one hand

$$\int_{S_R \cap T_R} \frac{|x-y|^2}{R^2} K_s(x-y) dx\, dy = \frac{1}{R^2} \int_{S_R \cap T_R} |x-y|^{1-2s} dx\, dy \leq \frac{c_2\, R}{R^2} \int_0^{R/A} z^{1-2s}\, dz =$$
$$= \frac{C_2}{R} R^{2-2s} = C_2\, R^{1-2s},$$

and on the other hand

$$\int_{S_R \cap T_R} \frac{|x-y|^2}{R^2} K_s(x-y) dx\, dy = \frac{1}{R^2} \int_{S_R \cap T_R} |x-y|^{1-2s} dx\, dy \geq \frac{c_1\, R}{R^2} \int_0^{R/A} z^{1-2s}\, dz =$$
$$= \frac{C_1}{R} R^{2-2s} = C_1\, R^{1-2s},$$

Using a similar property about the length of the lines with $|x-y| = ctt$ inside $S_R \cap (T_R)^c$, we bound the other integral as

$$\int_{S_R\cap(T_R)^c} K_s(x-y)dx\,dy = \frac{1}{R^2}\int_{S_R\cap(T_R)^c} |x-y|^{-1-2s}dx\,dy \leq c_4\,R\int_{R/A}^{\infty} z^{-1-2s}\,dz = \\ = C_4\,R^{1-2s},$$

and

$$\int_{S_R\cap(T_R)^c} K_s(x-y)dx\,dy = \frac{1}{R^2}\int_{S_R\cap(T_R)^c} |x-y|^{-1-2s}dx\,dy \geq c_3\,R\int_{R/A}^{\infty} z^{-1-2s}\,dz = \\ = C_3\,R^{1-2s}.$$

□

Finally we also need a maximum principle for the type of operators we are dealing with in this work. This result is the one that includes the hypothesis on the potential function $V(x)$ that are also imposed in Theorems 1.1.1 and 1.1.3.

Lemma 6.1.3. *Let $\mathcal{L} = L - V(x)$ with L of one of the types $(F1)$ or $(F2)$ defined in the statement of Theorem 1.1.3. Let $\varphi \in C^{\alpha}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, with $\alpha > 2s_{max} \geq 1$, and s_{max} as in Theorem 1.1.3. Assume that there exist two positive constants $m, b \in \mathbb{R}$ such that*

$$V \leq -b < 0 \quad \text{in } \mathbb{R}\setminus[-m,m],$$
$$\mathcal{L}\varphi \geq 0 \quad \text{in } \mathbb{R}\setminus[-m,m],$$

and

$$\varphi \geq 0 \quad \text{in } [-m,m].$$

Then

$$\varphi \geq 0 \quad \text{in } \mathbb{R}.$$

Proof. We are only going to present the proof for the case $L = (-\Delta)^s$, which is a particular case. The other cases are essentially the same. Let us prove the result by contradiction. We suppose that there exists a non empty set $H \subset \mathbb{R}$ where $\varphi < 0$. Now we can distinguish two possibilities regarding the infimum of the function, depending on whether it is achieved or not.

If the infimum is achieved, it is in fact a minimum, and there exists a point where it is attained. That is,

$$\exists\, x_0 \in \mathbb{R} \text{ such that } \min \varphi = \varphi(x_0) < 0.$$

Since $\varphi \geq 0$ in $[-m,m]$, we have that $x_0 \in \mathbb{R}\setminus[-m,m]$.

Now, let us apply the operator $\mathcal{L}$ to φ in x_0. First we have that

$$L(\varphi)(x_0) = \int_R (\varphi(x_0) - \varphi(y))\; K(x_0 - y)\,dy \leq 0, \tag{6.1.1}$$

due to being x_0 a global minimum and K positive. On the other hand we have that

$$V(x_0)\,\varphi(x_0) > 0. \tag{6.1.2}$$

Summarizing we obtain a contradiction,

$$0 \le \mathcal{L}\phi(x_0) = L\phi(x_0) - V(x_0)\,\varphi(x_0) < 0, \tag{6.1.3}$$

where the first inequality comes from the hypothesis, while the second one comes from the two previous expressions, equations (6.1.1) and (6.1.2).

By the contrary, since the infimum is negative and it is not achieved, we can construct a sequence of points $(x_k)_{k\in\mathbb{Z}^+} \notin [-m, m]$ whose images are negative and tend to the infimum. That is,

$$\varphi(x_k) - \varphi(x) \le \varphi(x_k) - \inf\, \varphi \le \frac{1}{k} \qquad \forall\, x \in \mathbb{R}, \tag{6.1.4}$$

and

$$\varphi(x_k) < 0.$$

Now, we are going to evaluate the operator $\mathcal{L}$ in these points. In order to do this, we are going to divide the integral part of the fractional Laplacian in three terms, and we are going to bound each one separately:

$$\begin{aligned}(-\Delta)^s\,\varphi(x_k) &= \int_{-\infty}^{\infty} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy = \\ &= \int_{-\infty}^{x_k-\delta} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy + \int_{x_k-\delta}^{x_k+\delta} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy + \\ &\qquad + \int_{x_k+\delta}^{\infty} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy,\end{aligned}$$

where δ is a parameter to be determined later.

Let us begin the estimates with the tails of the integral, that are in fact the same. That is

$$\int_{x_k+\delta}^{\infty} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy \le \frac{1}{k}\int_0^{\infty} \frac{1}{z^{1+2s}}dz = \frac{1}{k}\,\frac{1}{2s}\,\frac{1}{\delta^{2s}},$$

where we have used equation (6.1.4) and the change of variables $y = z + x_k$. Similarly for the other tail,

$$\int_{-\infty}^{x_k-\delta} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy \le \frac{1}{k}\int_{-\infty}^{0} \frac{1}{|z|^{1+2s}}dz = \frac{1}{k}\int_0^{\infty} \frac{1}{z^{1+2s}}dz = \frac{1}{k}\,\frac{1}{2s}\,\frac{1}{\delta^{2s}}.$$

Since $\alpha > 2s \ge 1$, then $\varphi \in C^1(\mathbb{R})$ and we can apply Taylor's theorem of order one. It says that

$$\varphi(x_k + z) = \varphi(x_k) + \varphi'(\xi(z))\,z, \quad \text{with } \xi(z) \in (x_k, x_k + z).$$

On the other hand, as $\varphi' \in C^{\alpha-1}(\mathbb{R})$ we have the condition

$$|\varphi'(\xi(-z)) - \varphi'(\xi(z))| \le C\,|\xi(-z) - \xi(z)|^{\alpha-1} \le C\,|z|^{\alpha-1}.$$

If we introduce these two expressions in the integral we get

$$\begin{aligned}
\int_{x_k-\delta}^{x_k+\delta} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy &= \int_{-\delta}^{\delta} \frac{\varphi(x_k)-\varphi(x_k+z)}{|z|^{1+2s}}dz \leq \\
&\leq \left| \int_{-\delta}^{\delta} \frac{\varphi(x_k)-\varphi(x_k+z)}{|z|^{1+2s}}dz \right| = \\
&= \left| \frac{1}{2}\int_{-\delta}^{\delta} \frac{(\varphi(x_k)-\varphi(x_k-z))+(\varphi(x_k)-\varphi(x_k+z))}{|z|^{1+2s}} \right| \leq \\
&\leq \frac{1}{2}\int_{-\delta}^{\delta} \frac{|\varphi'(\xi(-z))\, z-\varphi'(\xi(z))\, y|}{|z|^{1+2s}}dz \leq C\int_{-\delta}^{\delta} \frac{|z|^{\alpha}}{|z|^{1+2s}}dz = \\
&= C\int_{0}^{\delta} \frac{1}{z^{1+2s-\alpha}}dz = C\,\delta^{\alpha-2s}.
\end{aligned}$$

Once we have all the parts bounded we can join them. That is

$$(-\Delta)^s\,\varphi(x_k) = \int_{-\infty}^{\infty} \frac{\varphi(x_k)-\varphi(y)}{|x_k-y|^{1+2s}}dy \leq \frac{1}{ks}\,\frac{1}{\delta^{2s}} + C\,\delta^{\alpha-2s}.$$

Set $\delta = k^{-\mu}$ with $0<\mu<1/(2s)$, then we can rewrite the previous expression as

$$(-\Delta)^s\,\varphi(x_k) \leq \frac{1}{s}\,k^{2s\mu-1} + C\,k^{(2s-\alpha)\mu}.$$

On the other hand we have that $\varphi(x_k)<0$ and $\varphi(x_k)\leq \frac{1}{k}+\inf\,\varphi$, which implies

$$-V(x_k)\,\varphi(x_k) \leq b\,\varphi(x_k) \leq \frac{b}{k} + b\,\inf\,\varphi.$$

Finally, summarizing, we have

$$\begin{aligned}
0 \leq \mathcal{L}\varphi(x_k) &= (-\Delta)^s\varphi(x_k) - \varphi(x_k)\,V(x_k) \leq \\
&\leq \frac{1}{s}\,k^{2s\mu-1} + C\,k^{(2s-\alpha)\mu} + \frac{b}{k} + b\,\inf\,\varphi \quad \forall\, k\in\mathbb{Z}^+.
\end{aligned}$$

Since the last expression holds for all k, we can compute the limit when k tends to infinity and the inequalities will continue being true. Therefore,

$$0 \leq b\,\inf\varphi < 0,$$

where we have used the fact that $2\mu\,s<1$, $2s-\alpha<0$ and that the infimum is strictly negative. At this point we arrive at a contradiction.

Both whether the infimum is achieved or not we obtain a contradiction. Thus, we conclude that the assumption of being φ negative in a nonempty set is not possible and we finish the proof. $\square$

6.2. Proof of the main theorems

The proof of these theorems follows the ideas of *F. Hamel, X. Ros-Oton, Y. Sire and E. Valdinoci* presented in [**20**]. While they work with a kind of operators with truncated kernels in dimension two, we work in dimension one but with more general operators.

We are going to divide the proof of the theorems in three parts. First we are going to find an equation where $\sigma = \tilde{w}/w$ appears (Lemma 6.2.1), in the spirit of what we do in chapter 5. Then we are going to bound σ (Proposition 6.2.2), and finally we are going to prove that in fact it is constant, concluding the proof.

Lemma 6.2.1. *Let be $w, \tilde{w} \in L^\infty(\mathbb{R})$, with $w > 0$ solutions of equation*

$$\mathcal{L}u := Lu - V(x)u = 0, \quad \text{in } \mathbb{R},$$

with L of the type (1.0.1) *and $V(x)$ any function. Also, let $\tau \in C_c^\infty(\mathbb{R})$, and $\sigma := \frac{\tilde{w}}{w}$. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x) - \sigma(y)]^2 \left[\tau^2(x) + \tau^2(y)\right] \, w(x)\, w(y)\, K(x-y)\, dx\, dy =$$
$$= -\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\sigma^2(x) - \sigma^2(y)\right] \left[\tau^2(x) - \tau^2(y)\right] w(x)\, w(y)\, K(x-y)\, dx\, dy.$$

PROOF. By hypothesis we have $L\,\tilde{w} = V(x)\,\tilde{w}$ and $L\,w = V(x)\,w$, therefore

$$(L\tilde{w})w = (V(x)\,\tilde{w})\; w = \tilde{w}\; (V(x)\,w) = \tilde{w}\; (Lw)\,.$$

If we write the last expression in terms of w and σ we get

$$0 = (L\tilde{w})w - \tilde{w}(Lw) = (L(\sigma w))\,w - \sigma w\,(Lw)\,. \tag{6.2.1}$$

Let us multiply equation (6.2.1) by $2\,\tau^2\,\sigma$, integrate over $\mathbb{R}$ and use the result in Lemma 6.1.1. That is

$$\begin{aligned}
0 &= 2\int_{\mathbb{R}} L(\sigma\, w)(x)\, \tau^2(x)\, \sigma(x)\, w(x)\, dx - 2\int_{\mathbb{R}} L(w)(x)\, \tau^2(x)\, \sigma^2(x)\, w(x)\, dx = \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)w(x) - \sigma(y)w(y)] \left[\tau^2(x)\sigma(x)w(x) - \tau^2(y)\sigma(y)w(y)\right] K(x-y)dx\, dy \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x) - w(y)] \left[\tau^2(x)\sigma^2(x)w(x) - \tau^2(y)\sigma^2(y)w(y)\right] K(x-y)dx\, dy \\
&=: I_1 - I_2.
\end{aligned}$$

Now we are going to work with I_1 and I_2 in order to obtain alternative expressions. Therefore let us rewrite some of the terms that appear in them,

a) $(w(x) - w(y))\;\sigma(x) + (\sigma(x) - \sigma(y))\,w(y) = \sigma(x)w(x) - \sigma(y)w(y).$

b) $\big(\tau^2(x)\sigma(x)w(x) - \tau^2(y)\sigma(y)w(y)\big)\,\sigma(x) + (\sigma(x) - \sigma(y))\,\tau^2(y)\sigma(y)w(y) =$
$$= \tau^2(x)\sigma^2(x)w(x) - \tau^2(y)\sigma^2(y)w(y).$$

Hence, if we substitute them in I_1 and I_2 we get

$$\begin{aligned} I_1 = & \int_{\mathbb{R}}\int_{\mathbb{R}} \left[(w(x)-w(y))\ \sigma(x) + (\sigma(x)-\sigma(y))\, w(y)\right]\ K(x-y)\, \cdot \\ & \qquad\qquad\qquad \cdot \left[\tau^2(x)\sigma(x)w(x) - \tau^2(y)\sigma(y)w(y)\right] dx\, dy = \\ = & \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x)-w(y)] \left[\tau^2(x)\sigma^2(x)w(x) - \tau^2(y)\sigma^2(y)w(y)\right] \sigma(x)\, K(x-y)\, dx\, dy + \\ & + \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)] \left[\tau^2(x)\sigma^2(x)w(x) - \tau^2(y)\sigma^2(y)w(y)\right] w(y)\, K(x-y)\, dx\, dy, \end{aligned}$$

and

$$\begin{aligned} I_2 = & \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x)-w(y)]\ K(x-y) \\ & \cdot \left[\left(\tau^2(x)\sigma(x)w(x) - \tau^2(y)\sigma(y)w(y)\right) \sigma(x) + (\sigma(x)-\sigma(y))\, \tau^2(y)\sigma(y)w(y)\right] dx\, dy = \\ & = \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x)-w(y)] \left[\tau^2(x)\sigma(x)w(x) - \tau^2(y)\sigma(y)w(y)\right] \sigma(x)\, K(x-y)\, dx\, dy \\ & + \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x)-w(y)]\, [\sigma(x)-\sigma(y)]\, \tau^2(y)\sigma(y)\, w(y)\, K(x-y)\, dx\, dy. \end{aligned}$$

Then, subtracting the new expressions of I_1 and I_2 we have

$$\begin{aligned} 0 & = I_1 - I_2 = \\ & = \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)] \left[\tau^2(x)\sigma^2(x)w(x) - \tau^2(y)\sigma^2(y)w(y)\right] w(y)\, K(x-y)\, dx\, dy \\ & \quad - \int_{\mathbb{R}}\int_{\mathbb{R}} [w(x)-w(y)]\, [\sigma(x)-\sigma(y)]\, \tau^2(y)\sigma(y)\, w(y)\, K(x-y)\, dx\, dy. \end{aligned}$$

By writing

$$\begin{aligned} [\sigma(x)-\sigma(y)]\, \tau^2(x)w(x) + \left[\tau^2(x)-\tau^2(y)\right] \sigma(y)\, w(x) + [w(x)-w(y)]\, \tau^2(y)\, \sigma(y) = \\ = \tau^2(x)\, \sigma(x)\, w(x) - \tau^2(y)\, \sigma(y)\, w(y), \end{aligned}$$

we can see that

$$\begin{aligned} 0 = I_1 - I_2 = & \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)]^2\, \tau^2(x)\, w(x)\, w(y)\, K(x-y)\, dx\, dy \\ & + \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)] \left[\tau^2(x)-\tau^2(y)\right] \sigma(y)\, w(x)\, w(y)\, K(x-y)\, dx\, dy. \end{aligned}$$

If we interchange now the variables x and y in the previous equality, we get

$$\begin{aligned} 0 = & \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)]^2\, \tau^2(y)\, w(x)\, w(y)\, K(x-y)\, dx\, dy \\ & + \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)] \left[\tau^2(x)-\tau^2(y)\right] \sigma(x)\, w(x)\, w(y)\, K(x-y)\, dx\, dy. \end{aligned}$$

Finally, we add the last two equalities

$$
\begin{aligned}
0 = &\int_{\mathbb{R}} \int_{\mathbb{R}} [\sigma(x) - \sigma(y)]^2 \left[\tau^2(x) + \tau^2(y)\right] \, w(x) \, w(y) \, K(x-y) \, dx \, dy \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\sigma^2(x) - \sigma^2(y)\right] \left[\tau^2(x) - \tau^2(y)\right] w(x) \, w(y) \, K(x-y) \, dx \, dy.
\end{aligned}
$$

□

We can see that this result is almost identical to the one in Lemma 2.1. from [**20**], although we have worked it a bit more in order to have a symmetric expression with respect to both variables x and y.

Proposition 6.2.2. *Let $\mathcal{L}$ be a linear operator and $s_{max} > 1/2$ both as in Lemma 6.1.3. Let $w, \tilde{w} \in C^\alpha(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $\alpha > 2s_{max}$ and $w > 0$. Assume that there exist two positive constants $m, b \in \mathbb{R}$ such that*

$$V \leq -b < 0 \quad in \ \mathbb{R} \setminus [-m, m],$$

$$\mathcal{L} w \geq 0 \quad in \ \mathbb{R},$$

and

$$\mathcal{L} \tilde{w} = 0 \quad in \ \mathbb{R}.$$

Then, there exists a constant $C \geq 0$ such that

$$|\sigma| = \left| \frac{\tilde{w}}{w} \right| \leq C \quad in \ \mathbb{R}.$$

Proof. Let $a > 0$ be such that

$$a \cdot \min_{x \in [-m,m]} w(x) \geq \max_{x \in [-m,m]} |\tilde{w}(x)|.$$

Note that this constant a exists due to the minimum and the maximum of continuous functions in a compact set (an interval in this case) are well-defined and $w > 0$.

Since we want to apply the result of Lemma 6.1.3 to the functions

$$\varphi_\pm = a \, w \pm \tilde{w},$$

we have to show that the hypothesis of Lemma 6.1.3 are satisfied for $\varphi_\pm$. It is obvious by definition of a that $\varphi_\pm \geq 0$ in $[-m, m]$.

Now we have to compute $\mathcal{L}\varphi_\pm$, that is

$$\mathcal{L}\varphi_\pm = a \, \mathcal{L} w \pm \mathcal{L}\tilde{w} = a \, \mathcal{L} w \geq 0.$$

Since both w and $\tilde{w}$ have the adequate regularity, $\varphi_\pm$ have also the same regularity because of being a linear combination. Therefore we can apply Lemma 6.1.3 to them and we have that

$$\varphi_\pm = a \, w \pm \tilde{w} \geq 0 \quad \text{ in } \mathbb{R},$$

which is equivalent to

$$|\sigma| = \left| \frac{\tilde{w}}{w} \right| \leq a \ \text{ in } \mathbb{R}.$$

Hence, the constant a that bounds σ in $[-m, m]$ is the same constant C that bounds it in all $\mathbb{R}$. $\square$

Proof of Theorem 1.1.1. Let τ be a C^∞ function on $\mathbb{R}^+$ such that $0 \leq \tau \leq 1$ and

$$\tau = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

Since τ' is a continuous function and it is zero if $t \geq 2$, then it is bounded. That is,

$$|\tau'| \leq ||\tau||_\infty < +\infty.$$

For $R > 1$, let

$$\tau_R(x) = \tau\left(\frac{|x|}{R}\right) \qquad \text{for } x \in \mathbb{R}.$$

From the bound of $|\tau'|$ we can obtain the following estimate for the derivative of τ_R,

$$\tau_R'(x) = \frac{x}{R|x|}\tau'\left(\frac{|x|}{R}\right) \Longrightarrow |\tau_R'(x)| = \frac{1}{R}\left|\tau'\left(\frac{|x|}{R}\right)\right| \leq \frac{||\tau||_\infty}{R}.$$

Then, on the one hand we have

$$|\tau_R(x) - \tau_R(y)| = \left|\int_x^y \tau_R'(t)\,dt\right| \leq \int_x^y |\tau_R'(t)|\,dt \leq \frac{||\tau||_\infty}{R}\int_x^y dt = \frac{||\tau||_\infty}{R}|x-y|$$

where we have assumed without loss of generality that $x \leq y$. On the other hand we have that by definition

$$|\tau_R(x) - \tau_R(y)| \leq 1.$$

Hence, we get

$$|\tau_R(x) - \tau_R(y)| \leq \min\left\{1, \frac{||\tau||_\infty}{R}|x-y|\right\}. \tag{6.2.2}$$

We can also note that τ_R converges pointwise to the function 1 when R tends to infinity.

Once we have seen some needed properties about the function τ_R we apply Lemma 6.2.1 with this function.

$$
\begin{aligned}
0 \leq J_1 :=& \int_{\mathbb{R}}\int_{\mathbb{R}} [\sigma(x)-\sigma(y)]^2 \left[\tau_R^2(x)+\tau_R^2(y)\right] \, w(x)\, w(y)\, K(x-y)\; dx\, dy = \\
=& -\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\sigma^2(x)-\sigma^2(y)\right] \left[\tau_R^2(x)-\tau_R^2(y)\right] w(x)\, w(y)\, K(x-y)\, dx\, dy \leq \\
\leq& \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(x)-\sigma(y)||\sigma(x)+\sigma(y)||\tau_R(x)-\tau_R(y)||\tau_R(x)+\tau_R(y)| \cdot \\
& \qquad\qquad\qquad\qquad \cdot w(x)\, w(y)\, K(x-y)\, dx\, dy = \\
=& \int_{S_R} |\sigma(x)-\sigma(y)||\sigma(x)+\sigma(y)||\tau_R(x)-\tau_R(y)||\tau_R(x)+\tau_R(y)| \cdot \\
& \qquad\qquad\qquad\qquad \cdot w(x)\, w(y)\, K(x-y)\, dx\, dy = \\
=:& \, J_2.
\end{aligned}
$$

The last equality in the previous expression comes from applying that the support of the function $|\tau_R(x)-\tau_R(y)|$ is just the set S_R defined in Lemma 6.1.2. That is, if both $|x|$ and $|y|$ are smaller than R, then $\tau_R(x)=\tau_R(y)=1$, and the function is zero, and on the other case if both $|x|$ and $|y|$ are greater than $2R$, then $\tau_R(x)=\tau_R(y)=0$, and the function is also zero.

Then by Cauchy-Schwartz inequality

$$
\begin{aligned}
J_2^2 \leq& \int_{S_R} [\sigma(x)-\sigma(y)]^2 \left[\tau_R(x)+\tau_R(y)\right]^2 \, w(x)\, w(y)\, K(x-y)\, dx\, dy \\
& \cdot \int_{S_R} [\sigma(x)+\sigma(y)]^2 \left[\tau_R(x)-\tau_R(y)\right]^2 \, w(x)\, w(y)\, K(x-y)\, dx\, dy \leq \\
& \leq 2\, J_1 \cdot \int_{S_R} [\sigma(x)+\sigma(y)]^2 \left[\tau_R(x)-\tau_R(y)\right]^2 \, w(x)\, w(y)\, K(x-y)\, dx\, dy.
\end{aligned}
$$

In the last inequality we have used that $\left[\tau_R(x)+\tau_R(y)\right]^2 \leq 2\left[\tau_R^2(x)+\tau_R^2(y)\right]$, and the fact that $S_R \subset \mathbb{R}^2$.

Now we apply the result of Proposition 6.2.2, which we have previously proven. Thus, since both σ and w are bounded we have

$$
\begin{aligned}
& \int_{S_R} [\sigma(x)+\sigma(y)]^2 \left[\tau_R(x)-\tau_R(y)\right]^2 w(x)\, w(y)\, K(x-y)\, dx\, dy \leq \\
& \qquad\qquad\qquad\qquad \leq C \int_{S_R} \left[\tau_R(x)-\tau_R(y)\right]^2 \, K(x-y)\, dx\, dy,
\end{aligned}
$$

and applying Lemma 6.1.2 and the bounds on $\tau_R(x)-\tau_R(y)$ we have previously computed, expression (6.2.2), we get

$$
\int_{S_R} [\sigma(x)+\sigma(y)]^2 \left[\tau_R(x)-\tau_R(y)\right]^2 \, w(x)\, w(y)\, K(x-y)\, dx\, dy \leq C\, R^{1-2s} \leq C,
$$

where the last inequality comes from having $R > 1$ and $s \in [1/2, 1)$.

Therefore
$$0 \leq J_1^2 \leq J_2^2 \leq C\, J_1,$$
implies that
$$J_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} [\sigma(x) - \sigma(y)]^2 \left[\tau_R^2(x) + \tau_R^2(y)\right]\, w(x)\, w(y)\, K(x-y)\, dx\, dy \leq C.$$
In particular, since $\tau_R = 1$ in B_R
$$\int_{B_R} \int_{B_R} [\sigma(x) - \sigma(y)]^2\, w(x)\, w(y)\, K(x-y)\, dx\, dy \leq C.$$
Seeing that C is independent of R, we can send $R \to +\infty$, and by the monotone convergence theorem we obtain that the map
$$\mathbb{R} \times \mathbb{R} \ni (x, y) \mapsto [\sigma(x) - \sigma(y)]^2\, w(x)\, w(y)\, K(x-y)$$
belongs to $L^1(\mathbb{R} \times \mathbb{R})$. Since χ_{S_R} tends pointwise to zero as $R \to \infty$, we conclude from the dominated convergence theorem that
$$\lim_{R\to\infty} \int_{S_R} [\sigma(x) - \sigma(y)]^2\, w(x)\, w(y)\, K(x-y) dx\, dy = 0.$$
Therefore, summarizing we have
$$\begin{aligned}
&\left[\int_{\mathbb{R}} \int_{\mathbb{R}} [\sigma(x) - \sigma(y)]^2\, w(x)\, w(y)\, K(x-y) dx\, dy\right]^2 = \\
&= \frac{1}{2} \lim_{R\to\infty} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} [\sigma(x) - \sigma(y)]^2 \left[\tau_R^2(x) + \tau_R^2(y)\right]\, w(x)\, w(y)\, K(x-y) dx\, dy\right]^2 \leq \\
&\leq C \lim_{R\to\infty} \int_{S_R} [\sigma(x) - \sigma(y)]^2 \left[\tau_R^2(x) + \tau_R^2(y)\right]\, w(x)\, w(y)\, K(x-y) dx\, dy \leq \\
&\leq C \lim_{R\to\infty} \int_{S_R} [\sigma(x) - \sigma(y)]^2\, w(x)\, w(y)\, K(x-y) dx\, dy = 0.
\end{aligned}$$
This together with the fact that $w > 0$ and $K > 0$ implies that $[\sigma(x) - \sigma(y)]^2 = 0$ for almost every $(x, y) \in \mathbb{R} \times \mathbb{R}$. Hence we have that $\sigma = ctt$ almost everywhere, and by its continuity we finally get
$$\sigma \equiv ctt.$$
□

We can see that the proof of Theorem 1.1.1 is based in the results presented in Lemmas 6.1.1, 6.1.2, 6.1.3, 6.2.1 and Proposition 6.2.2. Apart from Lemma 6.1.2, all the other are results for more general operators than the fractional Laplacian. Then they can be also applied to the two families of operators F1 and F2 that appear in the statement of Theorem 1.1.3. Therefore we only have to adapt Lemma 6.1.2 to be applicable to the operators of the type F1 and F2.

Corollary 6.2.3. *(of Lemma 6.1.2) Let K be the kernel of an integral operator in dimension one of the type F1 or F2, and the sets S_R and $T_R(A)$ as in Lemma 6.1.2, with $R > 1$. Then*

$$\int_{S_R\cap T_R} \frac{|x-y|^2}{R^2} K(x-y)dx\,dy \leq C_1,$$

and

$$\int_{S_R\cap (T_R)^c} K(x-y)dx\,dy \leq C_2,$$

with C_1, C_2 positive constants that do not depend on R.

Proof. Let us begin by proving the result for the case of $K(z)$ of type F1. That is,

$$\int_{S_R\cap T_R} \frac{|x-y|^2}{R^2} K(x-y)dx\,dy \leq C \int_{S_R\cap T_R} \frac{|x-y|^2}{R^2} K_s(x-y)dx\,dy \leq C_1\, R^{1-2s} \leq C_1,$$

and

$$\int_{S_R\cap (T_R)^c} K(x-y)dx\,dy \leq C \int_{S_R\cap (T_R)^c} K_s(x-y)dx\,dy \leq C_2\, R^{1-2s} \leq C_2.$$

And in a similar way for the case of type F2. That is,

$$\begin{aligned}\int_{S_R\cap T_R} \frac{|x-y|^2}{R^2} K(x-y)dx\,dy &= \sum_{i=1}^m c_i \int_{S_R\cap T_R} \frac{|x-y|^2}{R^2} K_{s_i}(x-y)dx\,dy \leq \\ &\leq \sum_{i=1}^m \overline{C}_i\, R^{1-2s_i} \leq \sum_{i=1}^m \overline{C}_i = C_1,\end{aligned}$$

and

$$\begin{aligned}\int_{S_R\cap (T_R)^c} K(x-y)dx\,dy &= \sum_{i=1}^m c_i \int_{S_R\cap (T_R)^c} K_{s_i}(x-y)dx\,dy \leq \\ &\leq \sum_{i=1}^m \overline{C}_i\, R^{1-2s_i} \leq \sum_{i=1}^m \overline{C}_i = C_2.\end{aligned}$$

In both cases we have used that $R > 1$, $s, s_i \in [1/2, 1)$ and the result for the fractional Laplacian from Lemma 6.1.2. □

Proof of Theorem 1.1.3. Since the tools developed to prove 1.1.1 are quite general in terms of the nonlocal operator that drives the equation, the proofs of both Theorem 1.1.1 and 1.1.3 are essentially the same. That is, we only have to repeat the proof of Theorem 1.1.1 but applying Corollary 6.2.3 instead of Lemma 6.1.2.

□

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