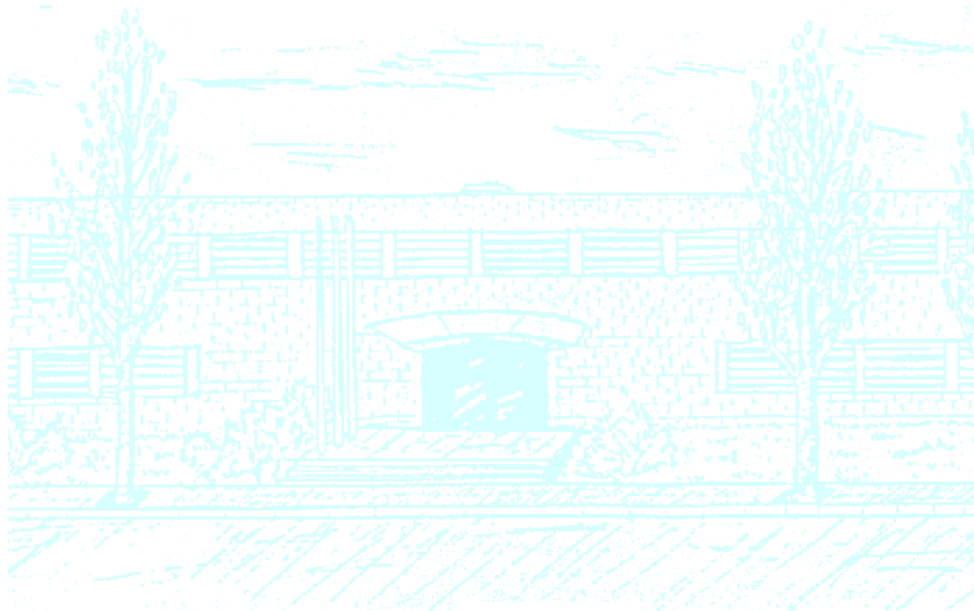# BACHELOR'S THESIS

Degree in Mathematics

Title: Singularities and qualitative study in LQC

Author: Llibert Aresté Saló

Advisor: Jaume Amorós Torrent, Jaume Haro Cases

Department: Departament de Matemàtiques

Academic year: 2017

Universitat Politècnica de Catalunya

Facultat de Matemàtiques i Estadística

Degree in Mathematics

Bachelor's Degree Thesis

Singularities and qualitative study in LQC

Llibert Aresté Saló

Supervised by: Jaume Amorós Torrent, Jaume Haro Cases

January, 2017

Abstract

We will perform a detailed analysis of singularities in Einstein Cosmology and in LQC (Loop Quantum Cosmology). We will obtain explicit analytical expressions for the energy density and the Hubble constant for a given set of possible Equations of State. We will also consider the case when the background is driven by a single scalar field, obtaining analytical expressions for the corresponding potential. And, in a given particular case, we will perform a qualitative study of the orbits in the associated phase space of the scalar field.

Keywords

Contents

1. Introduction

In 1915, Albert Einstein published his field equations of General Relativity [1], giving birth to a new conception of gravity, which would now be understood as the curvature of space-time. Among the different exact solutions that would be found throughout the following years, Alexander Friedmann [2] and Georges Lemaître [3] independently described a homogeneous, isotropic expanding or contracting universe. Howard P. Robertson [4] and Arthur Geoffrey Walker [5] proved that this metric is the only one on a spacetime that is spatially homogeneous and isotropic. Thus, it was named after Friedmann-Lemaître-Robertson-Walker. The so-called Friedmann equations, which come from this metric, are the ones that describe the Big Bang model.

In the second half of the 20th century, physicists started to realize that the recent Big Bang model led to some problems that needed to be solved. The "horizon problem" [6] appeared since, although the Big Bang theory predicted that certain regions of the universe were not causally connected, they actually had the same temperature, as shown by the Cosmic Microwave Background Radiation (CMB). The flatness problem [7] emerged because of the need of extreme fine-tuning of the density of matter and energy in the universe. The magnetic monopole problem [8] arose because the high temperature of the early universe predicted by the Big Bang implied the existence of magnetic monopoles, which would have persisted until nowadays.

As a solution to these problems of the Big Bang cosmology, Alan Guth introduced in 1981 [9] the cosmic inflation, which predicted an exponential expansion of space. This model was improved by Linde [10] and by Albrecht and Steindhardt [11], proposing the so-called new inflation or slow-roll inflation. Later on, it was understood that this inflation could be used to explain the inhomogeneities of the universe from the quantum vacuum fluctuations. These fluctuations were first calculated by Mukhanov and Chibisov [12,13] and then also by Hawking [14], Starobinsky [15], Guth and S-Young Pi [16], and Bardeen, Steinhardt and Turner [17].

Throughout this period, a quantum theory of gravity was also being built. We know that General Relativity is incomplete because it is a classical theory that ignores quantum effects. It is thought that these effects should become really dominant when, going backwards in time, we find huge matter densities and curvatures. Hence, the existence of a Big Bang should not be taken for granted because it is a prediction of a theory that might fail at high scales.

The so-called Wheeler-DeWitt equation was the first step towards quantum gravity in the 1960s, when Wheeler, Misner and DeWitt [18–20] used quantum geometrodynamics with the aim to resolve classical singularities through the quantum fluctuations of geometry. Later on, in 1986 Abbay Ashtekar developed the connection formulation of General Relativity [21], known as the "new variables" formulation, which laid as the basis for Loop Quantum Gravity (LQG). Thanks to this new formulation, Ted Jacobson and Lee Smolin found loop-like solutions to the Wheeler-DeWitt equation [22], thus giving birth to loop quantum gravity.

The application of Loop Quantum Gravity to cosmology is known as Loop Quantum Cosmology (LQC), which arose with the aim to solve the problems of the Big Bang model already stated. Abbay Ashtekar and Martin Bojowald [23–27] proved the avoidance in LQC of the future singularities classified in [28]. Parampreet Singh studied the holonomy correction in LQC [29] that leads to ρ^2 correction in the Friedmann equation at high energies. Thus, the Big Bang is replaced by a Big Bounce and the Friedmann equation describes an ellipse in the plane (ρ, H). Hence, the values of H and ρ are bounded [30–32].

The aim of this thesis is to study singularities both in Einstein Cosmology (EC) and in Loop Quantum Cosmology (LQC) when the universe is filled with a fluid with Equation of State $P = -\rho - A\rho^\alpha$. And we will also qualitatively study when we have a scalar field φ which mimicks this fluid. Among the infinite solutions that arise from the conservation equation, which can be plotted in the phase space $(\varphi, \dot{\varphi})$, there is only one which gives the same background as the fluid. We are going to study whether this solution is an attractor or a repeller of the dynamical system that comes from the conservation equation.

The thesis will be organised as follows:

In section 2, we are going to derive the Friedmann equations for a flat homogeneous and isotropic space-time from Einstein's field equations of General Relativity, analysing its singularities for an Equation of State $P = -\rho - A\rho^\alpha$, as done in [33]. Then, we are going to reconstruct the scalar field that fills the space-time with its corresponding potential. Finally, we will perform a qualitative study for a linear Equation of State, as in [34], obtaining some phase portraits.

In section 3, we are going to proceed analogously as with Einstein Cosmology, but now including in the Friedmann equations holonomic corrections that come from LQC. Since the equations become more complicated, when analysing singularities and reconstructing the scalar field, we will not be able to obtain analytical results for every value of α in the Equation of State, but only for certain particular values, treating some further cases than the ones in [34] and [35].

However, we will be able to do a detailed classification of the singularities, both stating which types can be avoided in comparison to Einstein Cosmology and which values of α lead to the types that persist. We will verify that the ρ^2 correction origins a bounce, preventing the existence of Big Rip singularities, that will be explained later in a more detailed way. Finally, we are going to do as well a qualitative study for a linear Equation of State, obtaining similar phase portraits as the one shown in [36] for a particular case of the ones we are going to study.

A version in form of paper of this thesis has been submitted in arXiv and can be found at arXiv:1612.05480. It has been sent to the journal *Physics Letter B* so as to be considered for publication.

We will use natural units ($\hbar = 8\pi G = c = 1$) and the Latin indices will refer to spatial coordinates $\{x, y, z\} \equiv \{1, 2, 3\}$ while Greek indices will be used when we include as well the temporal coordinate $t \equiv \{0\}$.

2. Einstein Cosmology

2.1 Friedmann Equations

Assuming homogeneity and isotropy of the universe, which is valid at sufficiently large scales, it can be shown that a suitable change of coordinates leads to the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric [37]:

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) \tag{1}$$

where $a(t)$ is a scale factor that parametrizes the relative expansion of the universe and the curvature κ is -1,0 or 1 if we are dealing respectively with an open, flat or closed universe.

We will perform all our calculations [38] in the flat FLRW space-time, in which the metric in Cartesian coordinates becomes:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{2}$$

Now, we want to obtain the relations that will eventually follow from Einstein equation ($R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}$) for this particular metric [39]. First, we are going to calculate the Christoffel symbols with Levi-Civita connection ($\Gamma^k_{ij} = \frac{1}{2}g^{kr}(g_{ri,j} + g_{rj,i} - g_{ij,r})$). Since we are in a synchronous frame (i.e. $g_{00} = 1$ and $g_{0i} = 0$), with a diagonal metric such that its components only depend on time, the only non-null coefficients are:

$$\begin{aligned} \Gamma^0_{ii} &= -\frac{1}{2}g^{00}g_{ii,0} = a\dot{a} \\ \Gamma^i_{i0} &= \Gamma^i_{0i} = \frac{1}{2}g^{ii}g_{ii,0} = H \end{aligned} \tag{3}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble constant.

Let us proceed to compute the Ricci tensor, $R_{\mu\nu} = \Gamma^\rho_{\nu\mu,\rho} - \Gamma^\rho_{\rho\mu,\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$:

$$R_{00} = \Gamma^\rho_{00,\rho} - \Gamma^\rho_{\rho0,0} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{00} - \Gamma^\rho_{0\lambda}\Gamma^\lambda_{\rho0} = -\Gamma^\rho_{\rho0,0} - \Gamma^\rho_{0\rho}\Gamma^\rho_{\rho0} = -3(\dot{H} + H^2) = -3\frac{\ddot{a}}{a} \tag{4}$$

$$\begin{aligned} R_{ii} &= \Gamma^\rho_{ii,\rho} - \Gamma^\rho_{\rho i,i} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{ii} - \Gamma^\rho_{i\lambda}\Gamma^\lambda_{\rho i} = \Gamma^0_{ii,0} + \Gamma^\rho_{\rho 0}\Gamma^0_{ii} - \Gamma^\rho_{i0}\Gamma^0_{\rho i} - \Gamma^0_{i\lambda}\Gamma^\lambda_{0i} = \\ &\partial_t(a\dot{a}) + 3Ha\dot{a} - Ha\dot{a} - Ha\dot{a} = 2\dot{a}^2 + a\ddot{a} \end{aligned} \tag{5}$$

And the scalar curvature is:

$$R = g^{\mu\nu}R_{\mu\nu} = -\frac{6}{a^2}(\dot{a}^2 + a\ddot{a}) = -6\left(H^2 + \frac{\ddot{a}}{a} \right) \tag{6}$$

We consider that the FLRW space-time is filled with a perfect fluid, only characterized by its pressure P and energy density ρ, with no shear stresses, viscosity or heat conduction. It fulfills the equation of state $P = -\rho - f(\rho)$. Therefore, the stress-energy-momentum tensor of this perfect fluid is $T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - pg^{\mu\nu}$, with $u^{\mu} = (1, 0, 0, 0)$ being the quadrivelocity in a reference system at rest relative to matter [40]. Hence, calculating the components of $T^{\mu\nu}$, we obtain:

$$T^{00} = \rho , \quad T^{ii} = \frac{P}{a^2} \tag{7}$$

Thus, using Einstein equation for both the R_{00} component and the scalar curvature ($R = -T$), we obtain Friedmann equations:

$$\begin{cases} H^2 = \frac{\rho}{3} \\ -6\frac{\ddot{a}}{a} = \rho + 3P \end{cases} \tag{8}$$

So, it is straightforward to compute Raychaudhuri equation:

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{-\rho - 3P}{6} - \frac{\rho}{3} = -\frac{\rho + P}{2} = \frac{f(\rho)}{2} \tag{9}$$

Finally, conservation equation comes from differentiating Friedmann equation:

$$\dot{\rho} = -3H(\rho + P) = 3Hf(\rho) \tag{10}$$

In next section we will particularize with the EoS given by $f(\rho) = A\rho^{\alpha}$ [33]. Firstly we analyze the singularities and then we will introduce a scalar field with its corresponding potential. Finally we will qualitatively study the dynamical system for the linear case ($\alpha = 1$).

2.2 Analysis of the singularities

Solving equation (10) for $f(\rho) = A\rho^{\alpha}$, we have to differ whether we are in the expanding phase ($H = H_+ > 0$) or contracting phase ($H = H_- < 0$):

$$\dot{\rho}_{\pm} = \pm\sqrt{3}A\rho_{\pm}^{\alpha+\frac{1}{2}}$$

$$\rho_{\pm}(t) = \begin{cases} \left(\rho_0^{1/2-\alpha} \pm \sqrt{3}\left(\frac{1}{2}-\alpha\right)At\right)^{1/(1-2\alpha)} & \alpha \neq \frac{1}{2} \\ \rho_0 e^{\pm A\sqrt{3}t} & \alpha = \frac{1}{2} \end{cases} \tag{11}$$

where ρ_0 is the energy density at $t = 0$.

The case $\alpha = 1/2$, corresponding to $H = H_0 e^{\pm A\sqrt{3}t/2}$ leads, in the expanding phase, to the so-called "Little Rip" singularity [41] for a phantom fluid and a "Little Bang" [33] -there are not singularities at finite cosmic time and the effective EoS parameter tends asymptotically to -1 at early times- for a standard fluid. Thus, we suppose $\alpha \neq \frac{1}{2}$. In this case,

$$\begin{cases} H_+(t) = \left(H_0^{1-2\alpha} + \frac{3^{\alpha}}{2}(1-2\alpha)At\right)^{1/(1-2\alpha)} \\ H_-(t) = -\left((-H_0)^{1-2\alpha} - \frac{3^{\alpha}}{2}(1-2\alpha)At\right)^{1/(1-2\alpha)} \end{cases} \tag{12}$$

If $A = 0$, $(\rho(t), H(t)) = (\rho_0, H_0)$ and $a(t) = a_0 e^{H_0(t-t_0)}$. Therefore, from now on, we will consider the more interesting case $A \neq 0$. If we define $t_s^{\pm} = \pm\frac{2|H_0|^{1-2\alpha}}{3^{\alpha}(2\alpha-1)A}$, that for $A < 0$ is positive for $\alpha < 1/2$ and negative for $\alpha > 1/2$ in the expanding phase and vice-versa in the contracting phase. Then, the Hubble parameter is given by:

$$H_{\pm}(t) = \pm\left(\pm\frac{3^{\alpha}}{2}(1-2\alpha)A(t-t_s^{\pm})\right)^{1/(1-2\alpha)} := \pm(k_{\pm}(t-t_s^{\pm}))^{1/(1-2\alpha)} \tag{13}$$

and the scale factor by

$$\begin{cases} \ln\left(\frac{a_{\pm}(t)}{a_s}\right) = \frac{1}{(1-\alpha)\cdot3^{\alpha}A}(k_{\pm}(t-t_s^{\pm}))^{2(1-\alpha)/(1-2\alpha)} & \alpha \neq 1 \\ a_{\pm}(t) = a_s|t-t_s^{\pm}|^{\pm1/k_{\pm}} & \alpha = 1 \end{cases} \tag{14}$$

From these expressions it is rather easy to analyze all the singularities, but first of all, we will classify them in the following types, as done in [28]:

FUTURE SINGULARITIES:

- Type I (Big Rip): $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|P| \to \infty$.

- Type II (Sudden): $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|P| \to \infty$.

- Type III (Big Freeze): $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|P| \to \infty$.

- Type IV (Generalized Sudden): $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|P| \to 0$ and derivatives of H diverge.

PAST SINGULARITIES (defined analogously as the future ones, as in [33]):

- Type I (Big Bang): $t \to t_s$, $a \to 0$, $\rho \to \infty$ and $|P| \to \infty$.

- Type II (Past Sudden): $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|P| \to \infty$.

- Type III (Big Hottest): $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|P| \to \infty$.

- Type IV (Generalized Past Sudden): $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|P| \to 0$ and derivatives of H diverge.

According to the different possible values of α, we are going to particularize for the expanding phase and $A < 0$, using equations (13) and (14):

- If $\alpha > 1/2$, H is defined for $t > t_s$ and $t = t_s$ is a past singularity where $H \to \infty$, $\dot{H} \to -\infty$ and

$$
a \to \begin{cases} a_s, & \alpha > 1 \\ 0(\text{polynomially}), & \alpha = 1 \\ 0(\text{exponentially}), & \frac{1}{2} < \alpha < 1 \end{cases} \tag{15}
$$

  Thus, $\alpha > 1$ corresponds to a Type III singularity, while $\frac{1}{2} < \alpha \leq 1$ is a Type I singularity.

- If $\alpha < 1/2$, H is defined for $t < t_s$ and, in $t \to t_s$, $H \to 0$ and, hence, $a \to a_s$. From $P = -\rho - A\rho^\alpha$, it is trivial to see that, when $t \to t_s$, for $\alpha < 0$, $P \to -\infty$ (corresponding to a Type II future singularity). For $\alpha = 0$, when $t \to t_s$, $\dot{H} = \frac{A\rho^\alpha}{2}$ is constant, and so for this case we have no singularities. Regarding the values $0 < \alpha < 1/2$, we see that, given $k \in \mathbb{N}$, $\frac{dH^k}{dt^k} = C_k \cdot \rho^{\alpha k - \frac{1}{2}(k-1)}$, where C_k is independent on ρ. Then, we can easily verify that when $\rho \to 0$,

$$
\frac{dH^k}{dt^k} \to \begin{cases} \text{constant,} & \text{if } \alpha = \frac{k-1}{2k} \\ \pm\infty, & \text{if } \alpha < \frac{k-1}{2k} \text{ and } \alpha \neq \frac{r-1}{2r} \ \forall r \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \tag{16}
$$

  Hence, in the first and third case there are no singularities, while the second case corresponds to Type IV future singularities.

Finally, we would lite to remark that, for the cases not treated, the analysis would be analogous and yield the same types of singularities.

2.3 Reconstruction method

The perfect fluid which fills FLRW space-time could be mimicked by a scalar field defined in all space-time with a kinetic and a potential term. By reconstruction we mean that we are going to obtain the potential to which this scalar field is submitted as a function of the parameters that define the Equation of State.

We will start treating the case $A < 0$. With the corresponding scalar canonical field φ subject to a potential $V(\varphi)$, the associated Lagrangian and stress-energy-momentum tensor are [42]:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\varphi - V(\varphi) \qquad T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L} \tag{17}$$

Thus, assuming the scalar field is spatially homogeneous, the energy density and the pressure have the following expressions:

$$\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi) \;, \quad P = \frac{\dot{\varphi}^2}{2} - V(\varphi) \tag{18}$$

From conservation equation $\dot{\rho} = -3H(\rho + P) = -3H\dot{\varphi}^2$, differentiating $\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi)$ we obtain

$$\ddot{\varphi} + 3H_\pm(\varphi, \dot{\varphi})\dot{\varphi} + V_\varphi = 0 \tag{19}$$

where $H_\pm(\varphi, \dot{\varphi}) = \pm\frac{1}{\sqrt{3}}\sqrt{\frac{\dot{\varphi}^2}{2} + V(\varphi)}$ with H_+ (resp. H_-) referring to the expanding (resp. contracting) phase.

Now, from equation (13) and Raychaudhuri equation $\dot{H} = -\frac{\rho+P}{2} = -\frac{\dot{\varphi}^2}{2}$, choosing the scalar field φ to be an increasing function, we obtain for $\alpha \neq 1/2$

$$\dot{\varphi}_\pm(t) = \sqrt{-3^\alpha A}\left(k_\pm(t - t_s^\pm)\right)^{\alpha/(1-2\alpha)} \tag{20}$$

So, if $\alpha \neq 1$:

$$\varphi_\pm(t) = \varphi_0 \pm \frac{2}{\sqrt{-A3^\alpha}}\frac{1}{\alpha - 1}\left(k_\pm(t - t_s^\pm)\right)^{(1-\alpha)/(1-2\alpha)} \tag{21}$$

Hence, from this relation between φ and t and using equation (13), we obtain:

$$H_\pm(\varphi) = \pm\left(\pm\frac{\sqrt{-A3^\alpha}}{2}(\alpha - 1)(\varphi - \varphi_0)\right)^{1/(1-\alpha)} \tag{22}$$

So as to compute the potential, we use that

$$\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi) = -\frac{f(\rho)}{2} + V(\varphi) = -\frac{A}{2}\rho^\alpha + V(\varphi)$$

obtaining

$$V_\pm(\varphi) = 3\left(\pm\frac{\sqrt{-A3^\alpha}}{2}(\alpha-1)(\varphi-\varphi_0)\right)^{2/(1-\alpha)} + \frac{A3^\alpha}{2}\left(\pm\frac{\sqrt{-A3^\alpha}}{2}(\alpha-1)(\varphi-\varphi_0)\right)^{2\alpha/(1-\alpha)} \quad (23)$$

We consider now the case $\alpha = 1$, which corresponds to the Equation of State $P = \omega\rho$, that is, $f(\rho) = -(\omega+1)\rho$. Thus, if $A < 0$ we are considering $\omega > -1$. The scalar field will have the following expression:

$$\dot{\varphi}_\pm(t) = \frac{2}{\sqrt{3(1+\omega)}}\frac{1}{|t - t_s^\pm|}$$

which can be integrated as:

$$\varphi_\pm(t) = \pm\frac{1}{\sqrt{3(1+\omega)}}\ln\left(\left(\frac{t - t_s^\pm}{t_0}\right)^2\right) \quad (24)$$

And the potential will be such that:

$$(\omega+1)\left(\frac{\dot{\varphi}^2}{2} + V(\varphi)\right) = \dot{\varphi}^2 \implies V(\varphi) = \frac{1-\omega}{1+\omega}\frac{\dot{\varphi}^2}{2}$$

Hence, from relation (24), its expression turns out to be:

$$V_\pm(t) = \frac{2(1-\omega)}{3(1+\omega)^2}\frac{1}{(t - t_s^\pm)^2}$$

$$V_\pm(\varphi) = \frac{2(1-\omega)}{3(1+\omega)^2}\frac{1}{t_0^2}e^{\mp\sqrt{3(1+\omega)}\varphi} \equiv V_0 e^{\mp\sqrt{3(1+\omega)}\varphi} \quad (25)$$

where we have taken $t_0^2 = \frac{2(1-\omega)}{3(1+\omega)^2 V_0}$.

Finally, we will analyse the case $\alpha = 1/2$. From the expression obtained in equation (12), we see that there is a Little Bang singularity. In this case the scalar field results being:

$$\dot{\varphi}_\pm(t) = \sqrt{-\sqrt{3}A|H_0|}e^{\pm\frac{\sqrt{3}}{4}At} \quad (26)$$

Thus,

$$\varphi_\pm(t) = \varphi_0 \pm 4\frac{\sqrt{-\sqrt{3}A|H_0|}}{\sqrt{3}A}e^{\pm\frac{\sqrt{3}}{4}At} \tag{27}$$

From this relation and using the case $\alpha = 1/2$ in equation (11), we obtain:

$$V(\varphi) = \frac{A}{2}\rho^{1/2} + \rho = \frac{-3A^2}{32}(\varphi - \varphi_0)^2\left(1 - \frac{3}{8}(\varphi - \varphi_0)^2\right) \tag{28}$$

Regarding the case $A > 0$, all the results are very similar. The only difference is that we need to consider a phantom scalar field, i.e, such that:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\partial^\mu\varphi - V(\varphi) \qquad T^{\mu\nu} = -\partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L} \tag{29}$$

Therefore, again under the assumption of spatial homogeneity, energy density and pressure are:

$$\rho = -\frac{\dot{\varphi}^2}{2} + V(\varphi) , \quad P = -\frac{\dot{\varphi}^2}{2} - V(\varphi) \tag{30}$$

2.4 Dynamics for the linear case

In the case of the Equation of State $P = \omega\rho$ for $\omega > -1$, we want to study the behaviour of the analytical solution, finding out if it is either an attractor or a repeller [34] and comparing its behaviour with that of other solutions of the system. We are going to analyse the dynamics in the contracting phase from this dynamical system:

$$\ddot{\varphi} + 3H_-(\varphi, \dot{\varphi}) + V_\varphi = 0 \tag{31}$$

$$\text{where} \quad H_-(\varphi, \dot{\varphi}) = -\sqrt{\frac{\dot{\varphi}^2}{2} + V(\varphi)} \quad, \quad V(\varphi) = V_0 e^{\sqrt{3(1+\omega)}\varphi}$$

With the following change of variable

$$\varphi = \frac{-2}{\sqrt{3(1+\omega)}} \ln \psi \tag{32}$$

we can obtain this system:

$$\frac{d\dot{\psi}}{d\varphi} = F_-(\dot{\psi}) := -\frac{3}{2}\sqrt{1+\omega}\left(\sqrt{\frac{2\dot{\psi}^2}{3(1+\omega)} + V_0} + \frac{\sqrt{3}(1+\omega)}{2\dot{\psi}}\left(\frac{2\dot{\psi}^2}{3(1+\omega)} + V_0\right)\right) \tag{33}$$

The different cases to distinguish are the following ones:

- $\omega = 1$: This case is known as a kination (or deflationary) phase [43, 44].

$$\frac{d\dot{\psi}}{d\varphi} = -\sqrt{\frac{3}{2}}(|\dot{\psi}| + \dot{\psi}) \tag{34}$$

In the semiplane $\dot{\psi} > 0$ ($\dot{\varphi} < 0$), the solution is given by:

$$\psi(t) = -|C|(t - t_s) \quad t < t_s$$

$$(\varphi(t), \dot{\varphi}(t)) = \left(-\sqrt{\frac{2}{3}} \ln(-|C|(t - t_s)), -\sqrt{\frac{2}{3}}\frac{1}{t - t_s}\right) \quad t < t_s \tag{35}$$

which is a stable orbit coinciding with the obtained result in equation (24), with the sign corresponding to the contracting phase. Therefore, this solution corresponds all the time to a universe with Equation of State $P = \omega\rho$ in the contracting phase, with $H(t) = \frac{1}{3(t-t_s)}$.

Regarding the semiplane $\dot{\psi} > 0$ ($\dot{\varphi} < 0$), the solution becomes:

$$(\varphi(t), \dot{\varphi}(t)) = \left(\sqrt{\frac{2}{3}} \ln(-|C|(t - t_s)), \sqrt{\frac{2}{3}}\frac{1}{t - t_s}\right) \quad t < t_s \tag{36}$$

which is stable analogously to the former case.

Finally, the case $\dot{\psi} = 0$, i.e. $(\varphi(t), \dot{\varphi}(t)) = (C, 0)$, corresponds to $H = 0$.

In the following figure, we have represented two possible orbits for $\dot{\varphi}(t) > 0$ and $\dot{\varphi}(t) < 0$, having taken in both cases $|C| = 1$.
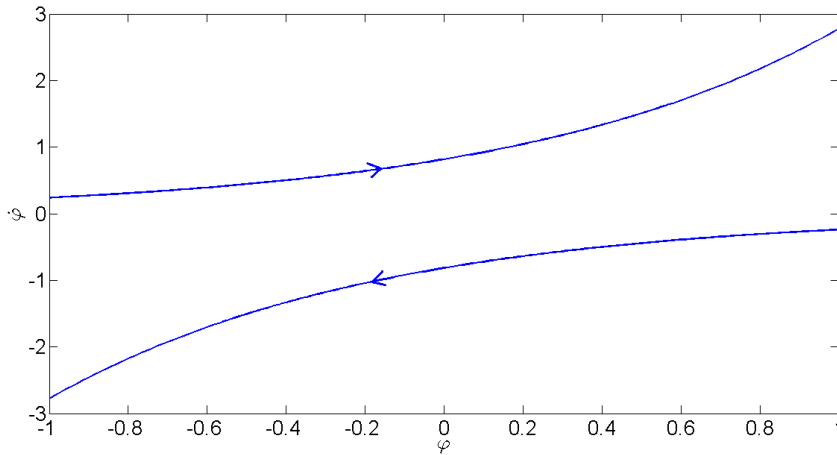

Figure 1: Orbits for $\omega = 1$.

- $-1 < \omega < 1$: Given that $F_-(\dot{\psi})$ can only vanish for $\dot{\psi} < 0$, we have a single critical point for $\dot{\psi}$:

$$\dot{\psi}_- = -(1 + \omega)\sqrt{\frac{3V_0}{2(1 - \omega)}} \tag{37}$$

which is a global repeller, given that $F_-(\dot{\psi}) > 0 \ \forall \dot{\psi}_- < \dot{\psi} < 0$ and $F_-(\dot{\psi}) < 0 \ \forall \dot{\psi} < \dot{\psi}_-$, as we can see in the phase portrait in Figure 2.

We point out that the blue horizontal line in Figure 2 corresponds to:

$$\varphi(t) = -\frac{1}{\sqrt{3(1 + \omega)}} \ln\left(\frac{3V_0(1 + \omega^2)}{2(1 - \omega)}(t - t_s)^2\right) \tag{38}$$

which again coincides with equation (24), depicting a universe with Equation of State $P = \omega\rho$.
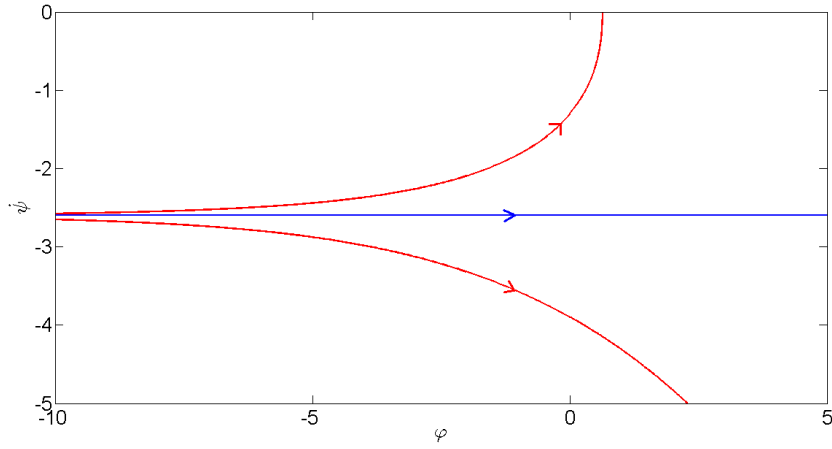
Figure 2: Phase portrait for $\omega = 1/2$ and $V_0 = 1$. We note that for $\dot{\psi} < 0$, $\dot{\varphi} > 0$ and, thus, throughout the evolution of time φ increases.

- $\omega > 1$: This case is known as an ekpyrotic phase or regime [45] and the system is only defined for $|\dot{\psi}| \geq \sqrt{\frac{3(1+\omega)|V_0|}{2}}$. We have three critical points:

$$\dot{\psi}_- = -(1+\omega)\sqrt{\frac{3|V_0|}{2|1-\omega|}} \qquad \dot{\psi}_0^{\pm} = \pm\sqrt{\frac{3(1+\omega)|V_0|}{2}} \qquad (39)$$

where $\dot{\psi}_0^{\pm}$ are repellers corresponding to $H = 0$. On the other hand, $\dot{\psi}_-$ is an attractor for $\dot{\psi} < \dot{\psi}_0^-$, solution that leads to a universe that all the time behaves as $P = \omega\rho$ in the contracting phase.
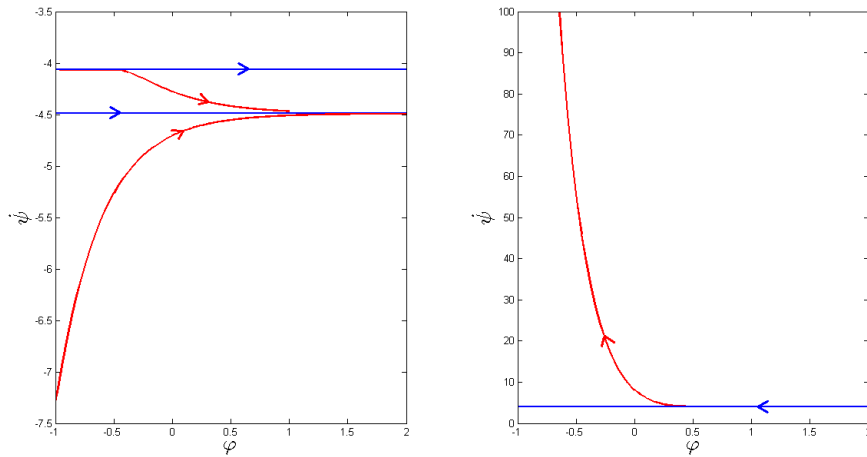

Figure 3: Phase portrait for $\omega = 10$ and $V_0 = -1$.

In the plot on the left-hand side, the upper horizontal line represents $\dot{\psi}_0^-$, corresponding to $H = 0$, while the lower horizontal line is the analytical solution. On the right-hand side, for the semiplane $\dot{\psi} > 0$, the blue horizontal line is $\dot{\psi}_0^+$, which corresponds to $H = 0$. In both plots, we appreciate that the orbits behave as we have already explained.

An analogous analysis for the expanding phase [34] would show that for $\omega = 1$ there are as well solutions coinciding with (24) and others with $H = 0$. The solution that depicts a fluid with EoS $P = \omega\rho$ is an attractor for $|\omega| < 1$ and a repeller for $\omega > 1$, which also coincides with equation (24). And for $\omega > 1$, we would also find critical points corresponding to $H = 0$, as the ones found in the contracting phase, that are in this case attractors.

3. Loop Quantum Cosmology

3.1 Modified Friedmann equations

The general formula of loop gravity, which takes into account the discrete nature of space-time, expresses the Hamiltonian in terms of holonomies $h_j(\lambda) \equiv e^{-i\frac{\lambda\beta}{2}\sigma_j}$, where σ_j are the Pauli matrices [29, 35]:

$$\mathcal{H}_{\mathsf{LQC}} = -\frac{2V}{\gamma^3\lambda^3} \sum_{i,j,k} \epsilon^{ijk} \mathsf{Tr}[h_i(\lambda)h_j(\lambda)h_i^{-1}(\lambda)h_j^{-1}(\lambda)\{h_k^{-1}(\lambda), V\} + \rho V \tag{40}$$

where $\gamma \approx 0.2375$ is the Barbero-Immirzi parameter and $\lambda = \sqrt{\frac{\sqrt{3}}{4}\gamma}$ is a parameter with the dimension of length, which is determined by invoking the quantum nature of the geometry.

The hamiltonian expression in (40) leads to [46, 47]

$$\mathcal{H}_{\mathsf{LQC}} = -3V\frac{\sin^2(\lambda\beta)}{\gamma^2\lambda^2} + \rho V \tag{41}$$

Using Hamiltonian equation

$$\dot{V} = \{V, \mathcal{H}_{\mathsf{LQC}}\} = -\frac{\gamma}{2}\frac{\partial\mathcal{H}_{\mathsf{LQC}}}{\partial\beta} \tag{42}$$

by imposing $\mathcal{H}_{\mathsf{LQC}} = 0$, i.e., $\rho = \frac{3\sin^2(\lambda\beta)}{\gamma^2\lambda^2}$, we obtain that

$$H^2 = \frac{\sin^2(2\lambda\beta)}{4\gamma^2\lambda^2} = \frac{\sin^2(\lambda\beta)}{\gamma^2\lambda^2}(1 - \sin^2(\lambda\beta)) = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_c}\right) \tag{43}$$

where $\rho_c = \frac{3}{\gamma^2\lambda^2}$ is the so-called critical energy density (the maximum value that reaches the energy density).

Equation (43) corresponds to an ellipse in the plane (ρ, H), that we can parametrize in the following form:

$$\begin{cases} H = \sqrt{\frac{\rho_c}{12}}\sin\eta \\ \rho = \rho_c\cos^2\frac{\eta}{2} \end{cases} \tag{44}$$

The conservation equation does not differ from standard GR, i.e., since the fluid fulfills the relation $d(\rho V) = -PdV$, where $V = a^3$, again with Equation of State $P = -\rho - f(\rho)$. Thus,

$$\dot{\rho} = \frac{1}{V}(-P - \rho)\dot{V} = 3Hf(\rho)$$

So as to obtain Raychadhuri equation, we have to differentiate Eq.(43), using conservation equation:

$$2H\dot{H} = \frac{\dot{\rho}}{3}\left(1 - \frac{2\rho}{\rho_c}\right) \implies \dot{H} = \frac{f(\rho)}{2}\left(1 - \frac{2\rho}{\rho_c}\right)$$

Again using conservation equation with the parametrization proposed in (44), it is straightforward that:

$$-\frac{\rho_c}{2}\sin\eta \cdot \dot{\eta} = 3\sqrt{\frac{\rho_c}{12}}\sin\eta \cdot A\rho_c^{\alpha}cos^{2\alpha}\frac{\eta}{2}$$

$$\int_{\eta_0}^{\eta}\frac{dx}{\cos^{2\alpha}\frac{x}{2}} = -A\sqrt{3}\rho_c^{\alpha-\frac{1}{2}}t := \theta t \tag{45}$$

where $\eta_0 = \arcsin\left(H_0\sqrt{\frac{12}{\rho_c}}\right)$.

In the expansive phase, $0 < \eta_0 < \pi$, while in the contracting phase $\pi < \eta_0 < 2\pi$.

Therefore, in the cases that it was possible to solve analytically this integral one will obtain an analytic expression of the evolution of the universe. We will see some examples in next subsection.

3.2 Analysis of the singularities

In LQC, one never finds singularities of Type I and III because the ellipse is a bounded subset, meaning that the energy density and the Hubble parameter are always finite quantities. Thus, the only singularities that we can find in LQC are for finite ρ and H. For our case, i.e. $f(\rho) = A\rho^{\alpha}$, we find:

- Type II singularity: It appears for $\alpha < 0$, since in this case $|P| \to \infty$ when $\rho \to 0$.

- Type IV singularity: They can be found with the same analysis that was done in Einstein cosmology, since the behaviour of modified Friedmann Equation near $\rho = 0$ is asymptotically the same as without the holonomic correction. Hence, near $\rho = 0$ it is satisfied that $\frac{dH^k}{dt^k} = C_k \cdot \rho^{\alpha k - \frac{1}{2}(k-1)}$, where C_k is independent on ρ. Therefore, for values $0 < \alpha < 1/2$ such that $\alpha \neq \frac{r-1}{2r} \; \forall r \in \mathbb{N}$, there exists $m \in \mathbb{N}$ for which $\frac{dH^m}{dt^m}$ diverges when $\rho \to 0$, being therefore Type IV singularities.

Analysing the dynamical system coming from Friedmann and Raychaudhuri equations

$$\begin{cases} \dot{\rho} = 3HA\rho^{\alpha} \\ \dot{H} = \frac{A\rho^{\alpha}}{2}\left(1 - \frac{2\rho}{\rho_c}\right) \end{cases} \tag{46}$$

we see that for $\alpha > 0$ the point $(0,0)$ will be a fixed point, concretely a saddle point. We observe that the dynamical system is not $\mathcal{C}^k$ for $\alpha < k$, with $k \in \mathbb{N}$. We will consider that $A \neq 0$, which corresponds as in GR to $(\rho(t), H(t)) = (\rho_0, H_0)$ and $a(t) = a_0 e^{H_0(t-t_0)}$. In the plane (ρ, H), the evolution in the ellipse determined by equation (43) will be anticlockwise for $A < 0$ (non-phantom fluid) and clockwise for $A > 0$ (phantom fluid). We see, as well, that the time spent to do a complete round in the ellipse starting from $(\epsilon, \text{sgn}(A)\sqrt{\frac{\epsilon}{3}\left(1 - \frac{\epsilon}{\rho_c}\right)})$ with $\epsilon \to 0$ is:

$$t = 2\int_0^{\rho_c} \frac{d\rho}{\sqrt{3\rho}\sqrt{1 - \frac{\rho}{\rho_c}}|A|\rho^{\alpha}} \tag{47}$$

which diverges for $\alpha \geq 1/2$.

We also point out that in all cases there will be a bounce, since the time that the universe lasts to bounce when it has a energy density $\rho_0 \neq 0$ is $\int_{-\rho_c/2}^{\rho_c/2} \frac{d\rho}{\sqrt{3\rho}\sqrt{1-\frac{\rho}{\rho_c}}|A|\rho^{\alpha}}$ converges $\forall \alpha$.

Now, having already done the classification of singularities of LQC, we can proceed to find analytical expressions of (ρ, H) for some values of α, in all cases solving equation (45) and using the parametrization in (44). We will compute, as well, $a(t)$ from $H = \frac{\dot{a}}{a}$.

- $\alpha = 1$: It corresponds to the linear equation of state $P = \omega\rho$ [34]. So

$$2\tan\frac{\eta}{2} = \theta(t - t_0)$$

where $t_0 = -\frac{2}{\theta}\tan\frac{\eta_0}{2}$.

Thus,

$$\begin{cases} H(t) = \frac{1+\omega}{2}\frac{\rho_c(t-t_0)}{\frac{3}{4}(1+\omega)^2\rho_c(t-t_0)^2+1} \\ \rho(t) = \frac{\rho_c}{\frac{3}{4}(1+\omega)^2\rho_c(t-t_0)^2+1} \end{cases} \qquad (48)$$

Therefore,

$$a(t) = a_0\left(1 + \frac{3}{4}(1+\omega)^2\rho_c(t-t_0)^2\right)^{\frac{1}{3(1+\omega)}} \qquad (49)$$

The bounce takes place at $t = t_0$.

- $\alpha = 0$:

$$\begin{cases} H(t) = \sqrt{\frac{\rho_c}{12}}\sin(\theta(t-t_0)) \\ \rho(t) = \rho_c\cos^2\left(\frac{\theta}{2}(t-t_0)\right) \end{cases} \qquad (50)$$

where $t_0 = -\frac{\eta_0}{\theta}$.

$$\ln\frac{a(t)}{a_0} = -\sqrt{\frac{\rho_c}{12}}\frac{1}{\theta}(\cos(\theta(t-t_0)) - 1) \qquad (51)$$

The bounce takes place at $t = t_0$.

- $\alpha = \frac{1}{2}$: Solving equation (45), we have:

$$2\ln\left(\tan\frac{\eta}{2} + \sec\frac{\eta}{2}\right) = \theta(t - t_0)$$

where $t_0 = -\frac{2}{\theta}\ln\left(\tan\frac{\eta_0}{2} + \sec\frac{\eta_0}{2}\right)$. So, we obtain:

$$\cos\frac{\eta}{2} = \frac{2e^{\theta(t-t_0)/2}}{1 + e^{\theta(t-t_0)}}$$

$$\begin{cases} H(t) = \frac{2\sqrt{3\rho_c}}{3}\frac{e^{\theta(t-t_0)/2}(1-e^{\theta(t-t_0)})}{(1+e^{\theta(t-t_0)})^2} = -\frac{\sqrt{3\rho_c}}{3}\frac{\sinh\frac{\theta(t-t_0)}{2}}{\cosh^2\frac{\theta(t-t_0)}{2}} \\ \rho(t) = 4\rho_c\frac{e^{\theta(t-t_0)}}{(1+e^{\theta(t-t_0)})^2} = \frac{\rho_c}{\cosh^2\frac{\theta(t-t_0)}{2}} \end{cases} \qquad (52)$$

$$\ln \frac{a(t)}{a_0} = -\frac{2\sqrt{3\rho_c}}{3\theta} \left(\frac{1}{\cosh \frac{\theta(t-t_0)}{2}} - 1 \right) \tag{53}$$

The bounce takes place at $t = t_0$.

- $\alpha = -\frac{1}{2}$: In this case we obtain the following expressions:

$$\begin{cases} H = \sqrt{\frac{\rho_c}{12}} \theta(t - t_0) \sqrt{1 - \frac{(\theta(t-t_0))^2}{4}} \\ \rho = \rho_c \left(1 - \frac{(\theta(t-t_0))^2}{4} \right) \end{cases} \tag{54}$$

defined for $|t - t_0| \leq \frac{2\rho_c}{|A|\sqrt{3}}$, where $t_0 = -\frac{2}{\theta} \cos \frac{\eta_0}{2}$.

$$\ln \frac{a(t)}{a_0} = \frac{2\sqrt{3\rho_c}}{9} \left[1 - \left(1 - \frac{\theta^2(t - t_0)^2}{4} \right)^{3/2} \right] \tag{55}$$

The bounce takes place at $t = t_0$. And we have a Type II singularity at $t_s^{\pm} = t_0 \pm \frac{2\rho_c}{A\sqrt{3}}$.

3.3 Reconstruction method

Now we will proceed, as done with General Relativity, to build the potential from the scalar field φ. We will work in the case $A < 0$, so the scalar field will be canonical and will satisfy relations in (18). We analyse separately the different cases that we have studied in the former section, in which we managed to obtain analytical values of (ρ, H):

- $\alpha = 1$. It corresponds to Equation of State $P = \omega\rho$, with $\omega > -1$. Thus, reminding (18), we obtain:

$$\dot{\varphi}^2 = P + \rho = (1 + \omega)\rho = \frac{(1 + \omega)\rho_c}{\frac{3}{4}(1 + \omega)^2 \rho_c (t - t_0)^2 + 1} \tag{56}$$

Hence,

$$\varphi(t) = \varphi_0 + \frac{2}{\sqrt{3(1+\omega)}} \text{arcsinh}\left(\frac{\sqrt{3\rho_c}}{2}(1+\omega)(t - t_0)\right) =$$

$$\frac{2}{\sqrt{3(1+\omega)}} \ln\left(\frac{\frac{\sqrt{3\rho_c}}{2}(1+\omega)(t-t_0) + \sqrt{\frac{3\rho_c}{4}(1+\omega)^2\rho_c(t-t_0)^2 + 1}}{|\tilde{\varphi}_0|}\right) \tag{57}$$

where $\varphi_0 = -\frac{2}{\sqrt{3(1+\omega)}} \ln|\tilde{\varphi}_0|$.

So, now we are able to compute the potential as a function of the scalar field. Taking $\tilde{\varphi}_0^2 = \frac{V_0}{2\rho_c(1-\omega)}$, this is done by expressing ρ in equation (48) as a function of φ using the relation found in (57):

$$V(\varphi) = \frac{\rho - P}{2} = \frac{1 - \omega}{2}\rho = \frac{(1 - \omega^2)\rho_c}{2\cosh^2\left((\varphi - \varphi_0)\frac{\sqrt{3(1+\omega)}}{2}\right)} = V_0 \frac{e^{\sqrt{3(1+\omega)}\varphi}}{\left(1 + \frac{V_0}{2\rho_c(1-\omega)}e^{\sqrt{3(1+\omega)}\varphi}\right)^2} \tag{58}$$

- $\alpha = 0$: In this case the scalar field is:

$$\dot{\varphi} = \sqrt{-A} \implies \varphi_\pm(t) = \varphi_0 + \sqrt{-A}(t - t_0)$$

Thus, the corresponding potential is, using (50):

$$V(\varphi) = \rho + \frac{A}{2} = \rho_c \cos^2\left(\sqrt{\frac{-3A}{\rho_c}}\frac{\varphi - \varphi_0}{2}\right) + \frac{A}{2} \tag{59}$$

Regarding the two left cases, it is not possible to integrate analytically the scalar field φ. However, we can express the potential in the following form:

- $\alpha = \frac{1}{2}$: From equation (52),

$$V = \rho + \frac{A\rho^{1/2}}{2} = \dot{\varphi}^2 \left(\frac{\dot{\varphi}^2}{A^2} - \frac{1}{2} \right) \tag{60}$$

where $\dot{\varphi}^2 = -A\rho^{1/2} = -\dfrac{A\rho_c^{1/2}}{\cosh\left(\frac{\theta(t-t_0)}{2}\right)}$

- $\alpha = -\frac{1}{2}$: From equation (54):

$$V = \rho + \frac{A\rho^{1/2}}{2} = \frac{A^2}{\dot{\varphi}^4} - \frac{\dot{\varphi}^2}{2} \tag{61}$$

where $\dot{\varphi}^2 = -A\rho^{-1/2} = -\dfrac{A}{\rho_c^{1/2}\sqrt{1 - \frac{\theta^2(t-t_0)^2}{4}}}$

3.4 Dynamics for the linear case

We want to analyse the behaviour of the analytical solution corresponding to the Equation of State $P = \omega\rho$ with $\omega > -1$. Using the potential found in (58), we are going to study the dynamics of the equation

$$\ddot{\varphi} + 3H_{\pm}(\varphi, \dot{\varphi})\dot{\varphi} + V_{\varphi} = 0 \tag{62}$$

where $H_{\pm}(\varphi, \dot{\varphi}) = \pm\sqrt{\frac{\rho(\varphi,\dot{\varphi})}{3}\left(1 - \frac{\rho(\varphi,\dot{\varphi})}{\rho_c}\right)}$, with $\rho(\varphi, \dot{\varphi}) = \frac{\dot{\varphi}^2}{2} + V(\varphi)$

Firstly, we note that this implies that:

$$\dot{\rho} = -3H_{\pm}(\varphi, \dot{\varphi})\dot{\varphi}^2 \tag{63}$$

Therefore, the evolution in time will take place in an anticlockwise sense throughout the ellipse, being $(0, 0)$ a fixed point.

Before proceeding to the analysis of the different cases, we are going to take a glance at the geometry of the phase space $(\varphi, \dot{\varphi})$.

Since we are dealing with a bi-valued dynamical system, we need a cover 2:1 (of two sheets) of the allowed region in the plane of the phase space $(\varphi, \dot{\varphi})$, which is ramified in the curves $H(\varphi, \dot{\varphi}) = 0$. Hence, for the case $|\omega| < 1$, this is a cylinder being projected in the plane, whereas for $\omega > 1$ we have two cylinders, as we can see in Figure 4. This explains why in the phase portrait that we will later obtain we can have intersecting orbits, which happens always between an orbit in the expanding phase and another one in the contracting phase.

Therefore, when solving the dynamical system one option would be to use local coordinates in a cylinder. However, this appears to be somehow cumbersome and, thus, we have opted for integrating the solution taking into account whether we are in the expanding or contracting phase, so that we change sign of H when reaching the curve $H = 0$. For the numerical results, we will use an RK78 method, that holds in memory the sign of H and changes it when we switch from the contracting to the expanding phase.
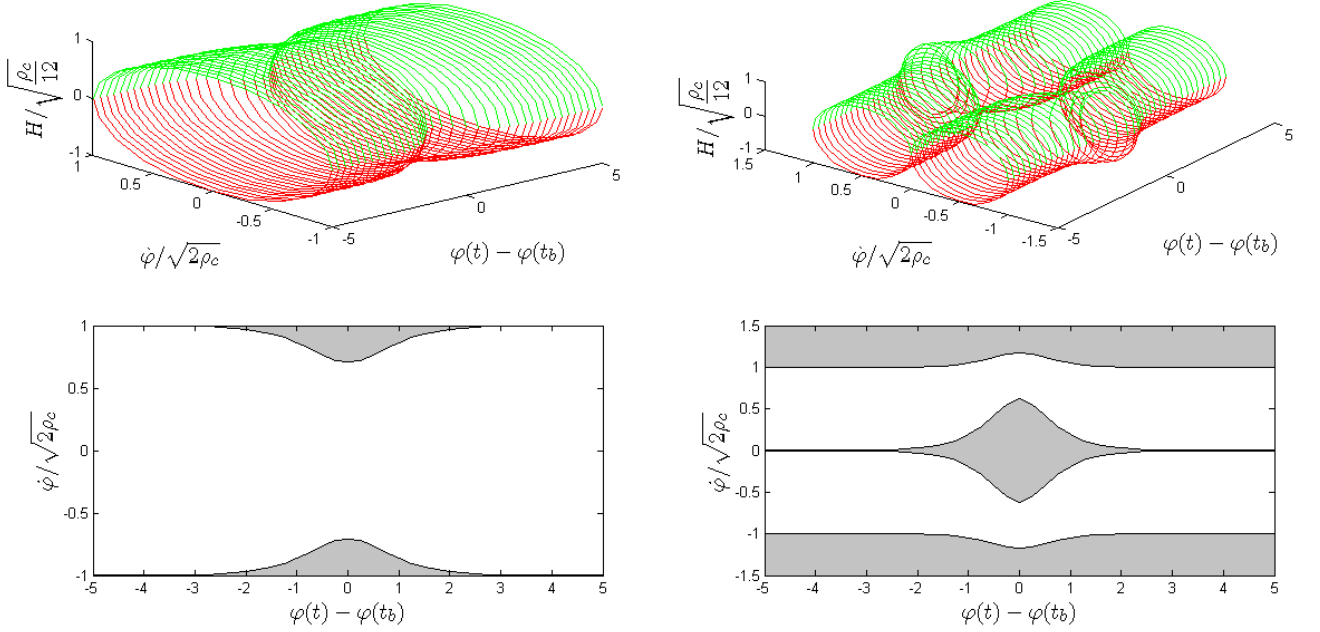
Figure 4: Top row: there is the phase space $(\varphi, \dot{\varphi}, H)$ for $|\omega| < 1$ (left) and $\omega > 1$ (right). The green (resp. red) lines recover all the region corresponding to the expanding (resp. contracting) phase. Bottom row: there is its projection in the plane $(\varphi, \dot{\varphi})$, in which the white region delimited by the curves $H(\varphi, \dot{\varphi}) = 0$ is the allowed one.

Now, we are going to treat separately the following cases:

- $\omega = 1$: In this case, the potential is zero. Therefore, (62) becomes:

$$\ddot{\varphi} = \mp \sqrt{\frac{3}{2}} |\dot{\varphi}| \dot{\varphi} \sqrt{1 - \frac{\dot{\varphi}^2}{2\rho_c}} \qquad (64)$$

Since $\rho \leq \rho_c$, $|\dot{\varphi}| \leq \sqrt{2\rho_c}$. So, we can use the change of variables $\dot{\varphi} = \sqrt{2\rho_c} \cos(\xi)$. Assuming that at t=0 we are in the expanding phase of the semi-plane $\dot{\varphi}_0 > 0$, we will have

$$\tan \xi - \tan \xi_0 = \sqrt{3\rho_c} t \qquad (65)$$

Hence,

$$\dot{\varphi}(t) = \sqrt{\frac{2\rho_c}{1 + (\sqrt{3\rho_c} t + C)^2}} \ , \quad t > \frac{-C}{\sqrt{3\rho_c}} \qquad (66)$$

$$\text{where} \quad C = \sqrt{\frac{2\rho_c}{\dot{\varphi}_0^2} - 1}$$

It is easy to see that for the contracting phase ($t < \frac{-C}{\sqrt{3\rho_c}}$), the expression of $\dot{\varphi}(t)$ would be exactly analogous. Moreover, if t=0 takes place during the contracting phase, the constant C would be defined as $C = -\sqrt{\frac{2\rho_c}{\dot{\varphi}_0^2} - 1}$. Therefore, the orbit in the phase portrait is:

$$(\varphi(t), \dot{\varphi}(t)) = \left(\varphi_0 + \sqrt{\frac{2}{3}} \ln\left(\sqrt{3\rho_c}t + C + \sqrt{1 + (\sqrt{3\rho_c}t + C)^2} \right), \sqrt{\frac{2\rho_c}{1 + (\sqrt{3\rho_c}t + C)^2}} \right) \tag{67}$$

and, in the semi-plane $\dot{\varphi}_0 < 0$, it is given by

$$(\varphi(t), \dot{\varphi}(t)) = \left(\varphi_0 - \sqrt{\frac{2}{3}} \ln\left(\sqrt{3\rho_c}t + C + \sqrt{1 + (\sqrt{3\rho_c}t + C)^2} \right), -\sqrt{\frac{2\rho_c}{1 + (\sqrt{3\rho_c}t + C)^2}} \right) \tag{68}$$

Hence, we see that all these are stable orbits that foliate all the space $0 < |\dot{\varphi}| \leq \sqrt{2\rho_c}$, corresponding to the analytical solution found in (57), with the bounce taking place in $t_b = -\frac{C}{\sqrt{3\rho_c}}$ and such that $H(t) = (t - t_b)\rho(t) = \frac{\rho_c(t-t_b)}{1+3\rho_c(t-t_b)^2}$.

On the other hand, if $\dot{\varphi}_0 = 0$, the correspondent orbit $(\varphi, \dot{\varphi}) = (\varphi_0, 0)$ would correspond to $\rho(t) = H(t) = 0$.

In the following figure, we have represented two possible orbits for $\dot{\varphi}(t) > 0$ and $\dot{\varphi}(t) < 0$, having taken in both cases $\varphi_0 = 0$.
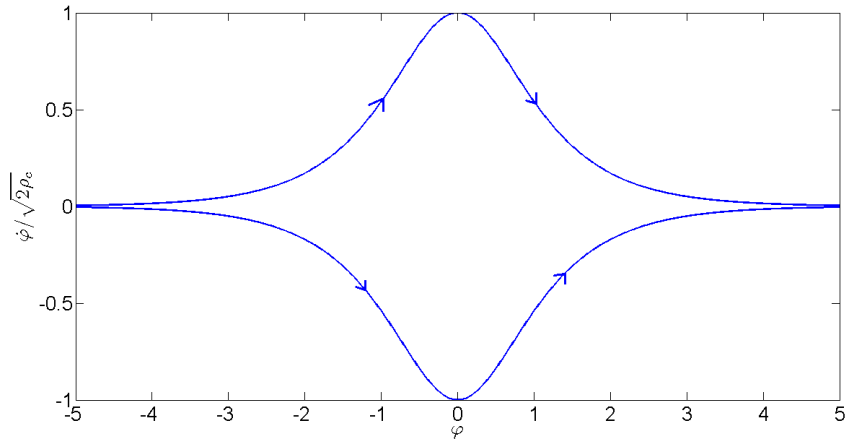

Figure 5: Orbits for $\omega = 1$.

Before analyzing the other cases, we are going to introduce the following change of variables, motivated by the solution (57):

$$\psi = \sinh\left(\frac{\varphi\sqrt{3(1+\omega)}}{2} + \frac{1}{2}\ln\left(\frac{V_0}{2\rho_c(1-\omega)}\right)\right) \tag{69}$$

where we have taken, as we already did previously, $|\tilde{\varphi}_0|^2 = \frac{V_0}{2\rho_c(1-\omega)}$.

Now, we need to express the equation (62), using the potential found in (58):

$$\dot{\psi} = \dot{\varphi}\sqrt{1+\psi^2}\frac{\sqrt{3(1+\omega)}}{2}$$

$$\ddot{\varphi} = \frac{2}{\sqrt{3(1+\omega)}}\frac{\ddot{\psi}\sqrt{1+\psi^2} - \dot{\psi}\frac{\psi\dot{\psi}}{\sqrt{1+\psi^2}}}{1+\psi^2} = \frac{2}{\sqrt{3(1+\omega)}}\frac{1}{\sqrt{1+\psi^2}}\left(\ddot{\psi} - \frac{\psi\dot{\psi}^2}{1+\psi^2}\right)$$

$$V(\psi) = \frac{1-\omega}{2}\frac{\rho_c}{1+\psi^2}$$

$$V_\varphi = \frac{dV(\psi)}{d\psi}\frac{d\psi}{d\varphi} = -\frac{1-\omega}{2}\sqrt{3(1+\omega)}\rho_c\frac{\psi}{(1+\psi^2)^{3/2}}$$

$$\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi) = \frac{2}{3(1+\omega)(1+\psi^2)}\left(\dot{\psi}^2 + \frac{3(1-\omega^2)}{4}\rho_c\right)$$

So,

$$\frac{2}{3(1+\omega)}\frac{1}{\sqrt{1+\psi^2}}\left(\ddot{\psi} - \frac{\psi\dot{\psi}^2}{1+\psi^2}\right) + 3H(\psi,\dot{\psi})\frac{2}{\sqrt{3(1+\omega)}}\frac{\dot{\psi}}{1+\psi^2} - \frac{1-\omega}{2}\sqrt{3(1+\omega)}\rho_c\frac{\psi}{(1+\psi^2)^{3/2}} = 0$$

which becomes in a more compact form:

$$\ddot{\psi} = -3H_\pm(\psi,\dot{\psi})\dot{\psi} + \rho(\psi,\dot{\psi})\psi\frac{3(1+\omega)}{2} \tag{70}$$

where $H_\pm(\psi,\dot{\psi}) = \pm\sqrt{\frac{\rho(\psi,\dot{\psi})}{3}\left(1 - \frac{\rho(\psi,\dot{\psi})}{\rho_c}\right)}$

So as to analyse this dynamical system, it is useful to see in which set of the plane $(\psi,\dot{\psi})$ $\ddot{\psi}$ vanishes.

$$\frac{\rho^2\psi^2(1+\omega)^2}{4} = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_c}\right)\dot{\psi}^2$$

If $\rho \neq 0$:

$$\rho = \frac{\dot{\psi}^2/3}{\frac{\psi^2(1+\omega)^2}{4} + \frac{\dot{\psi}^2}{3\rho_c}} = \frac{2}{3(1+\omega)(1+\psi^2)}\left(\dot{\psi}^2 + \frac{3(1-\omega^2)}{4}\rho_c\right)$$

Hence, if $|\dot{\psi}| \neq \frac{\sqrt{3\rho_c}}{2}(1+\omega)$,

$$\psi^2 = \frac{4}{3\rho_c(1-\omega^2)}\dot{\psi}^2 \tag{71}$$

Therefore, $\ddot{\psi}$ vanishes if

$$\rho = 0 \quad \text{or} \quad \left\{ \text{sgn}(H\dot{\psi}) = \text{sgn}(\psi) \text{ and } \left(|\dot{\psi}| = \frac{\sqrt{3\rho_c}}{2}(1+\omega) \text{ or } \psi^2 = \frac{4}{3\rho_c(1-\omega^2)}\dot{\psi}^2\right)\right\} \tag{72}$$

Now we can proceed to analyse the rest of cases that are left:

- $|\omega| < 1$: We can distinguish two types of orbits: those that cross the axis $\psi = 0$ (Type I) and those that cross the axis $\dot{\psi} = 0$ (Type II).

Regarding Type I orbits, we are going to consider that at the initial point $t = 0$ we are at $(\psi, \dot{\psi}) = (0, \dot{\psi}_0)$, where $0 < \dot{\psi}_0 \leq \frac{\sqrt{3\rho_c}}{2}(1+\omega)$, which comes from the restriction $0 < \rho_0 \leq \rho_c$. If $\rho_0 = \rho_c$, at t=0 we are at the bounce and, by (72), the value of $\dot{\psi}$ will be the same throughout all the contracting and expanding phase, coinciding with the analytical solution found in (57). With respect to Type II orbits, the initial point $t = 0$ will be at $(\psi, \dot{\psi}) = (\psi_0, 0)$ where $\psi_0 \neq 0$. This is the corresponding phase portrait:
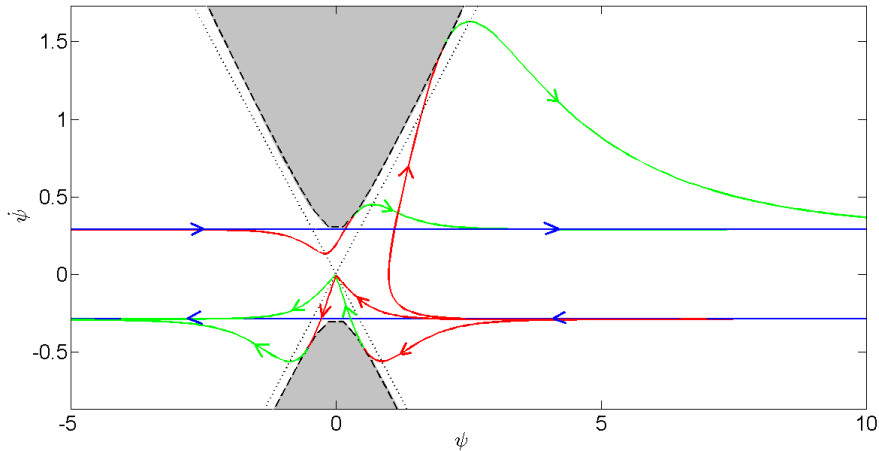

Figure 6: Phase portrait for $w = -2/3$ and $\rho_c = 1$.

In Fig 6, we have represented the set $\rho = \rho_c$, which is the discontinuous black line corresponding to $\dot{\psi} = \pm\sqrt{\frac{3\rho_c}{2}(1+\omega)\left(\psi^2 + \frac{1+\omega}{2}\right)}$. The pointed diagonal lines refer to the set where $\ddot{\psi} = 0$, as seen in (71). The blue horizontal lines are the orbits corresponding to the analytical solution.

With respect to the rest of the curves, we have used the following colour notation: red for the contracting phase and green for the expanding phase. We have plotted one possible Type I orbit and one possible Type II orbit. In both we have considered that either $\dot{\psi}_0$ or ψ_0 are in the positive axis and that $t = 0$ takes place during the contracting phase. We note that applying the symmetry with respect to the axis $\dot{\psi} = 0$ and/or $\psi = 0$ we would obtain the other possibilities for these orbits, considering that the initial point is in the negative axis and/or $t = 0$ takes place during the expanding phase.

Finally, in Fig 6 we have drawn as well the invariant curves that come in and out from the saddle point (0,0), the only critical point of the dynamical system. For clarity, we have only plotted the invariant curves for $\dot{\psi} < 0$. The others could be obtained with the symmetry respect to the axis $\dot{\psi} = 0$.

So, we clearly see in each orbit the bounce at the time in which it touches the curve $\rho = \rho_c$. We observe that $\rho = 0$ takes place for $\psi \to \infty$. The points in which the orbits intersect with the diagonal lines are where they change the sign of their slope. And finally the horizontal lines corresponding to the analytical solution are attractors for the expanding phase and repellers for the contracting phase.

We can also characterize orbits with the following value:

$$\omega_{eff}(t) := \frac{P(t)}{\rho(t)} = \frac{\dot{\psi}(t)^2 - \frac{3}{4}(1-\omega^2)\rho_c}{\dot{\psi}(t)^2 + \frac{3}{4}(1-\omega^2)\rho_c} \tag{73}$$
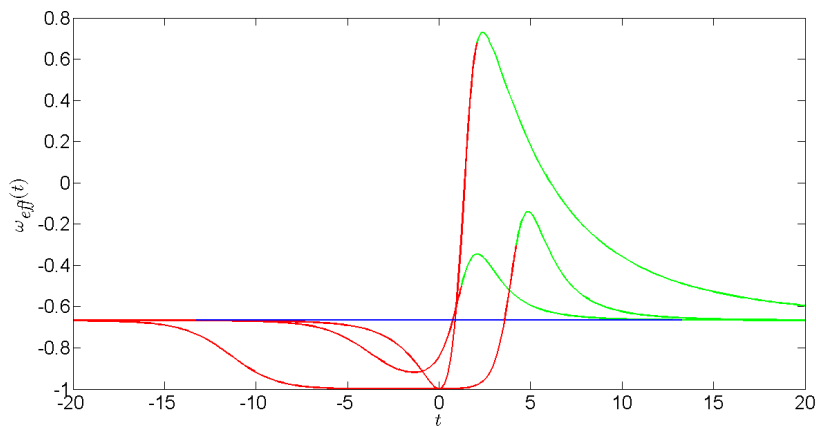

Figure 7: Evolution of $\omega_{eff}(t)$ for the orbits represented in the phase portrait for $\omega = -2/3$ and $\rho_c = 1$.

We observe that $-1 \leq \omega_{eff}(t) < 1$ and that for the analytical value $|\dot{\psi}| = \frac{\sqrt{3\rho_c}}{2}(1+\omega)$, $\omega_{eff}(t) = \omega$. Regarding the other orbits, the bounce takes place at $\omega_{eff}(t_b) > \omega$ and, when $\rho(t) \to 0$, it is verified that $\omega_{eff}(t) \to \omega$.

Finally, we also show how the phase portrait would be in the original phase space $(\psi, \dot{\psi}, H)$ with the same colour notation. Its projection leads to Figure 6.
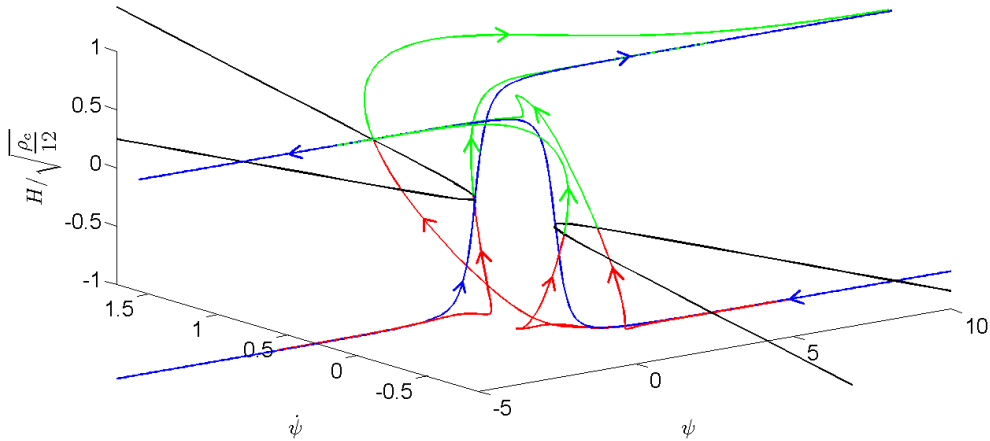

Figure 8: Phase portrait in $(\psi, \dot{\psi}, H)$ for $\omega = -2/3$ and $\rho_c = 1$

- $\omega > 1$: In this case, since the potential is negative we have the lower bound of $|\dot{\psi}| \geq \frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$. Therefore, we only have Type I orbits.

If $\rho_0 = 0$, we are stuck in this value of ρ during all the orbit $|\dot{\psi}| = \frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$. If $\rho_0 = \rho_c$, analogously as in the $|\omega| < 1$ case, we stay throughout all the contracting and expanding phase in the analytical solution. This is the phase portrait obtained:
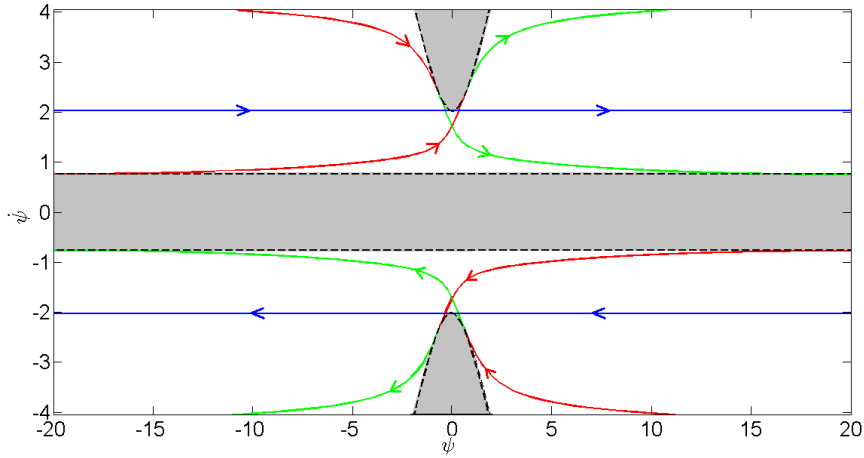
Figure 9: Phase portrait for $w = 4/3$ and $\rho_c = 1$.

We have represented as a discontinuous black line the curve corresponding to $\rho = \rho_c$. The other two discontinuous horizontal lines that delimit the forbidden region refer to an orbit with $\rho = 0$. The blue horizontal lines correspond to the orbits coming from the analytical solution.

Regarding the other orbits, we have used the same colour notation as before. It is important to note that, since $\omega > 1$, (71) is never fulfilled. Hence, equation (72) implies that the sign of the slope of the orbit can never change, i.e., ψ will always be convex or concave. So, we see that the curve reaches $\rho = 0$ at $\psi \to \pm\infty$ with $\dot{\psi}$ either converging to $\pm\frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$ or diverging such that $\frac{\dot{\psi}}{\psi} \to 0$. We also observe a bounce for $\rho = \rho_c$. Thus, the analytical solution is a repeller in the expanding phase and an attractor (though not global) in the contracting phase.
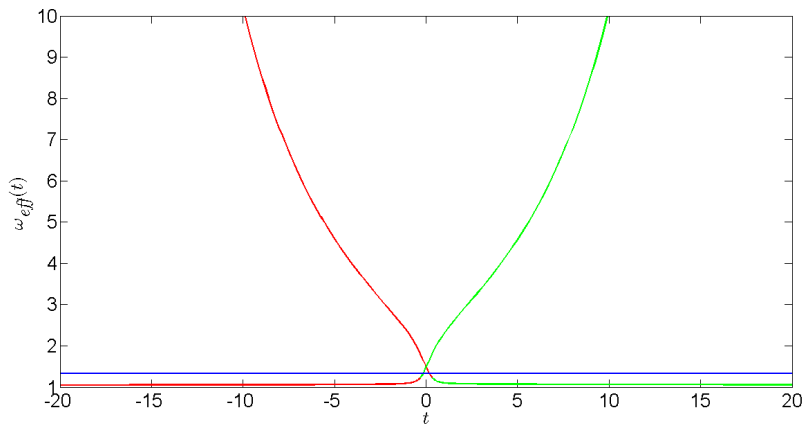

Figure 10: Evolution of $\omega_{eff}(t)$ for the orbits represented in the phase portrait for $w = 4/3$ and $\rho_c = 1$.

In this case, the same equation (73) is valid. We observe that $\omega_{eff} > 1$ and that $\omega_{eff} = \omega$ for the analytical orbit. In the other orbits, the bounce takes place at $\omega_{eff}(t_b) < \omega$ and, when $\rho(t) \to 0$, it is verified that $\omega_{eff}(t) \to \infty$ when the orbits converge to $\pm\frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$ or $\omega_{eff}(t) \to 1$ when $|\dot{\psi}| \to \infty$.

In figure 10, we clearly appreciate the two types of orbits:

- Type A: During the beginning of the contracting phase the orbit comes asymptotically from $|\dot{\psi}| = \frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$ ($\omega_{eff} \to \infty$), such that the value of $|\dot{\psi}|$ is below the one of the orbit of the analytic solution. Then it crosses this orbit, bounces and in the expanding phase $|\dot{\psi}| \to \infty$, i.e. $\omega_{eff} \to 1$.
- Type B: During the contracting phase the orbit comes asymptotically from $|\dot{\psi}| \to \infty$ ($\omega_{eff} \to 1$), such that the value of $|\dot{\psi}|$ is above the one of the orbit of the analytic solution. Then it bounces, crosses this orbit, and in the expanding phase $|\dot{\psi}| \to \frac{\sqrt{3\rho_c}}{2}\sqrt{\omega^2 - 1}$, i.e. $\omega_{eff} \to \infty$.

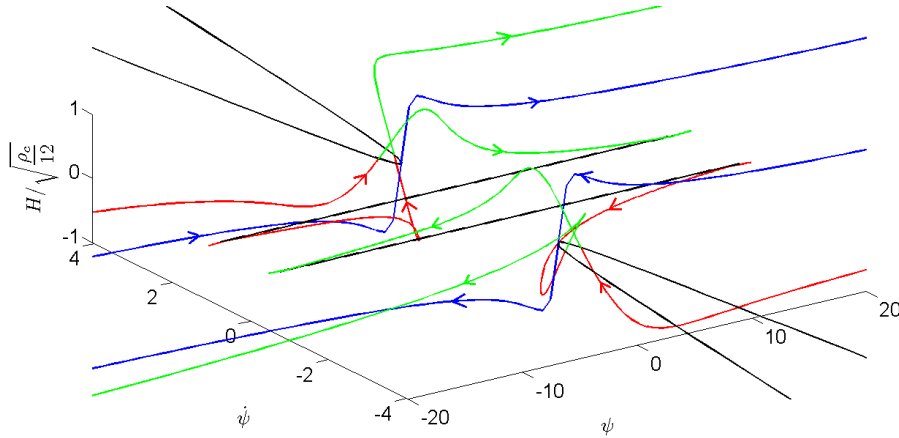Finally, we also show the phase portrait in the phase space $(\psi, \dot{\psi}, H)$.

Figure 11: Phase portrait in $(\psi, \dot{\psi}, H)$ for $\omega = 4/3$ and $\rho_c = 1$

4. Conclusions

We have explored singularities in General Relativity and in Loop Quantum Cosmology (LQC) with the Equation of State $P = -\rho - A\rho^\alpha$. We have observed that in LQC we only have Type II and Type IV singularities and, in particular, for $\alpha \geq 1/2$ there are no singularities. Unlike in GR, in LQC we have only been able to calculate analytically $H(t)$ and $\rho(t)$ for certain values of α.

We have introduced as well a scalar field φ that accounts for the perfect fluid that fills the spacetime. We have computed its corresponding potential $V(\varphi)$ both in GR and for some values of α in LQC. Moreover, we have observed that for $A < 0$ in the Equation of State, we need a canonical scalar field, while for $A > 0$ it must be a so-called phantom scalar field.

Finally, for the linear equation of state (i.e., $\alpha = 1$), we have made both in GR and in LQC a qualitative study of the orbits in the phase space $(\varphi, \dot{\varphi})$, concluding that, for a canonical scalar field (i.e., $\omega > -1$), in the expanding (resp. contracting) phase, the analytical solution is an attractor (resp. repeller) for $|\omega| < 1$ both in GR and LQC. For $\omega > 1$, both in GR and in LQC it is a repeller (resp. attractor) in the expanding (resp. contracting) phase. However, whereas in GR the analytical solution is a global attractor in the contracting phase and a repeller in the expanding one, in LQC the other solutions do not catch (do not converge asymptotically) the analytical one because of the bounce, and when they enter in the expanding phase they move away from the analytical orbit depecting at late times a universe with an effective EoS parameter equal to 1 or ∞.

References

[1] A. Einstein, "Die Feldgleichungen der Gravitation," *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 844-847.*, 1915.

[2] A. Friedmann, "Über die krümmung des raumes," *Zeitschrift für Physik*, vol. 10, no. 1, pp. 377–386, 1922.

[3] G. Lemaître, "Un Univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques," *Annales de la Société Scientifique de Bruxelles*, vol. 47, pp. 49–59, 1927.

[4] H. P. Robertson, "Kinematics and World-Structure," *ApJ*, vol. 82, p. 284, Nov. 1935.

[5] A. Walker, "On Milne's theory of world-structure.," *Proc. Lond. Math. Soc. (2)*, vol. 42, pp. 90–127, 1936.

[6] C. W. Misner, "The Isotropy of the universe," *Astrophys. J.*, vol. 151, pp. 431–457, 1968.

[7] R. H. Dicke and P. J. E. Peebles, "The big bang cosmology: Enigmas and nostrums," in *General Relativity: An Einstein Centenary Survey*, 1979.

[8] J. Preskill, "Cosmological Production of Superheavy Magnetic Monopoles," *Phys. Rev. Lett.*, vol. 43, p. 1365, 1979.

[9] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," *Phys. Rev.*, vol. D23, pp. 347–356, 1981.

[10] A. D. Linde, "A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems," *Phys. Lett.*, vol. B108, pp. 389–393, 1982.

[11] A. Albrecht and P. J. Steinhardt, "Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking," *Phys. Rev. Lett.*, vol. 48, pp. 1220–1223, 1982.

[12] V. F. Mukhanov and G. V. Chibisov, "The Vacuum energy and large scale structure of the universe," *Sov. Phys. JETP*, vol. 56, pp. 258–265, 1982. [Zh. Eksp. Teor. Fiz.83,475(1982)].

[13] V. F. Mukhanov and G. V. Chibisov, "Quantum Fluctuations and a Nonsingular Universe," *JETP Lett.*, vol. 33, pp. 532–535, 1981. [Pisma Zh. Eksp. Teor. Fiz.33,549(1981)].

[14] S. W. Hawking, "The Development of Irregularities in a Single Bubble Inflationary Universe," *Phys. Lett.*, vol. B115, p. 295, 1982.

[15] A. A. Starobinsky, "Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations," *Phys. Lett.*, vol. B117, pp. 175–178, 1982.

[16] A. H. Guth and S. Y. Pi, "Fluctuations in the New Inflationary Universe," *Phys. Rev. Lett.*, vol. 49, pp. 1110–1113, 1982.

[17] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, "Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe," *Phys. Rev.*, vol. D28, p. 679, 1983.

[18] B. S. DeWitt, "Quantum Theory of Gravity. 1. The Canonical Theory," *Phys. Rev.*, vol. 160, pp. 1113–1148, 1967.

[19] B. S. DeWitt, "Quantum Theory of Gravity. 2. The Manifestly Covariant Theory," *Phys. Rev.*, vol. 162, pp. 1195–1239, 1967.

[20] B. S. DeWitt, "Quantum Theory of Gravity. 3. Applications of the Covariant Theory," *Phys. Rev.*, vol. 162, pp. 1239–1256, 1967.

[21] A. Ashtekar, "New Variables for Classical and Quantum Gravity," *Phys. Rev. Lett.*, vol. 57, pp. 2244–2247, 1986.

[22] T. Jacobson and L. Smolin, "Nonperturbative Quantum Geometries," *Nucl. Phys.*, vol. B299, pp. 295–345, 1988.

[23] M. Bojowald, "Absence of singularity in loop quantum cosmology," *Phys. Rev. Lett.*, vol. 86, pp. 5227–5230, 2001.

[24] M. Bojowald, "Inflation from quantum geometry," *Phys. Rev. Lett.*, vol. 89, p. 261301, 2002.

[25] A. Ashtekar, "An Introduction to Loop Quantum Gravity Through Cosmology," *Nuovo Cim.*, vol. B122, pp. 135–155, 2007.

[26] A. Ashtekar and M. Bojowald, "Quantum geometry and the Schwarzschild singularity," *Class. Quant. Grav.*, vol. 23, pp. 391–411, 2006.

[27] A. Ashtekar, A. Corichi, and P. Singh, "Robustness of key features of loop quantum cosmology," *Phys. Rev.*, vol. D77, p. 024046, 2008.

[28] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, "Properties of singularities in (phantom) dark energy universe," *Phys. Rev.*, vol. D71, p. 063004, 2005.

[29] P. Singh, "Loop cosmological dynamics and dualities with Randall-Sundrum braneworlds," *Phys. Rev.*, vol. D73, p. 063508, 2006.

[30] J. Haro, "Future singularity avoidance in semiclassical cosmology," *Gen. Rel. Grav.*, vol. 44, pp. 2215–2238, 2012.

[31] J. de Haro, "Future singularity avoidance in phantom dark energy models," *JCAP*, vol. 1207, p. 007, 2012.

[32] J. de Haro, "Does loop quantum cosmology replace the big rip singularity by a non-singular bounce?," *JCAP*, vol. 1211, p. 037, 2012.

[33] S. Hidalgo, "Study of the Singularities in the flat Friedmann-Lemaître-Robertson-Walker Universe," Master's thesis, FME (Universitat Politècnica de Catalunya), Barcelona, 2015.

[34] J. Haro and J. Amorós, "The matter-ekpyrotic bounce scenario in Loop Quantum Cosmology," To appear.

[35] K. Bamba, J. de Haro, and S. D. Odintsov, "Future Singularities and Teleparallelism in Loop Quantum Cosmology," *JCAP*, vol. 1302, p. 008, 2013.

[36] J. de Haro and J. Amorós, "Viability of the matter bounce scenario in Loop Quantum Cosmology from BICEP2 last data," *JCAP*, vol. 1408, p. 025, 2014.

[37] H. Kurki-Suonio, "Cosmology I, University of Helsinki lecture notes," 2005.

[38] L. Amendola, "Introduction to cosmology." http://www.thphys.uni-heidelberg.de/~amendola/teaching/introduction.pdf. Accessed: 2016-09-24.

[39] M. P. Do Carmo, *Differential geometry of curves and surfaces*, vol. 2. Prentice-hall Englewood Cliffs, 1976.

[40] L. D. Landau and E. M. Lifshitz, "The classical theory of fields," *Course of theoretical physics*, vol. 2, 1971.

[41] P. H. Frampton, K. J. Ludwick, and R. J. Scherrer, "The Little Rip," *Phys. Rev.*, vol. D84, p. 063003, 2011.

[42] S. Watson, "An Exposition on inflationary cosmology," 2000.

[43] M. Joyce, "Electroweak baryogenesis and the expansion rate of the universe," *Phys. Rev. D*, vol. 55, pp. 1875–1878, Feb 1997.

[44] B. Spokoiny, "Deflationary universe scenario," *Physics Letters B*, vol. 315, no. 1, pp. 40 – 45, 1993.

[45] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, "Ekpyrotic universe: Colliding branes and the origin of the hot big bang," *Phys. Rev. D*, vol. 64, p. 123522, Nov 2001.

[46] P. Dzierżak, P. Małkiewicz, and W. Piechocki, "Turning big bang into big bounce. I. Classical dynamics," *Phys. Rev. D*, vol. 80, p. 104001, Nov 2009.

[47] J. Haro and E. Elizalde, "Loop cosmology: Regularization vs. quantization," *EPL (Europhysics Letters)*, vol. 89, no. 6, p. 69001, 2010.