Compact Hausdorff group topologies for the additive group of real numbers

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ABSTRACT

We deal with an example of a topology ν for the additive group of real numbers $\mathbb{R}$, which makes it a compact Hausdorff topological group. Further, $(\mathbb{R}, \nu)$ is connected, but neither arcwise connected, nor locally connected. Thus, it is neither a Lie group, nor a curve in the sense of H. Mazurkiewicz. The contribution of this short note is to provide an elementary proof of the fact that it is not arcwise connected.

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1. Introduction

The set $\mathbb{R}$ of real numbers is a corner stone in Mathematics. It supports all kind of structures: the simple operations of sum and multiplication make $\mathbb{R}$ the perfect paradigm of a group, a ring or a vector space. If it is endowed with the absolute value it becomes the model for metric spaces and consequently for topological spaces. We could point out so many features of the set $\mathbb{R}$ that make it a model, that maybe the following question is pertinent: is $\mathbb{R}$ a human creature, or is it just a gift of God to Mankind?

In this short note we do not pretend to answer that philosophical question, we only try to report on a concrete structure on $\mathbb{R}$ which is suprising according to our standard intuition. Namely, the additive group of real numbers $\mathbb{R}$ can be endowed with a topology ν such that $(\mathbb{R},\nu)$ is a compact, connected, Hausdorff topological group which is neither arcwise connected, nor locally connected. Thus $(\mathbb{R},\nu)$ is neither a Lie group, nor a curve in the sense of Hahn-Mazurkiewicz.

The mentioned topology ν called our attention from a one-page paper of Halmos, and we have denominated it Halmos topology. The route to define it in [7] is first to establish an algebraic isomorphism between $\mathbb{R}$ and the group of all characters on the rational numbers, say $\mathrm{Hom}(\mathbb{Q},\mathbb{T})$.Taking into account that $\mathrm{Hom}(\mathbb{Q},\mathbb{T})$ has a natural topology as a subspace of the product $\mathbb{T}^{\mathbb{Q}}$, a topology on $\mathbb{R}$ can be defined under the constrain of making topological (i.e. continuous and open) the algebraic isomorphism. This is precisely what we call the Halmos topology ν on $\mathbb{R}$. By its very definition, the properties of $(\mathbb{R},\nu)$ are precisely the same as those of the compact group $\mathrm{Hom}(\mathbb{Q},\mathbb{T})$, and for this reason we devote the first section of this paper to study the latter.

In [11] we dealt with the group $(\mathbb{R},\nu)$, giving there the details of the definition and a proof that it is not arcwise connected which seemed simple, but a deep artillery was hidden in it. In the present paper we provide an elementary proof of the fact that it is not arcwise connected. We will also give the reference to prove that it is not locally connected.

Notation. All groups considered in this note will be abelian. For a set A we denote by $\mathrm{Card}(A)$ or by $|A|$ the cardinality of A. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{T}$ and $\mathfrak{c}$ denote, respectively, the integers, the rational numbers, complex numbers of modulus one, and the cardinality of $\mathbb{R}$. We write $\mathbb{Q}_d$ or $\mathbb{Q}_u$ if $\mathbb{Q}$ must be considered with the discrete topology or with the induced from the euclidean topology of $\mathbb{R}$, and the same meaning have the subindex d and u in other contexts.

Let G, Y be groups. We denote by $\mathrm{Hom}(G,Y)$ the set of all group homomorphisms from G to Y: with operation defined pointwise it becomes a group. If G, Y are topological groups, $\mathrm{CHom}(G,Y)$ stands for the continuous elements of $\mathrm{Hom}(G,Y)$. If $Y=\mathbb{T}$, the elements of $\mathrm{Hom}(G,\mathbb{T})$ are called characters of G, and the group of continuous characters $G^\wedge := \mathrm{CHom}(G,\mathbb{T})$ is called the dual group of G, or the character group of G. Whenever $\mathbb{R}$ or $\mathbb{T}$ are target groups, say in $\mathrm{CHom(G,\mathbb{R})}$ or $\mathrm{CHom(G,\mathbb{T})}$, they are supposed to carry the corresponding euclidean topology. Observe that $\mathrm{Hom(G,\mathbb{T})}$ is a closed subgroup of $\mathbb{T}^G$, therefore it is compact and Hausdorff as a subspace of it. However, $\mathrm{CHom}(\mathrm{G}, \mathbb{T})$ may not be closed in $\mathbb{T}^G$.

All the dual groups considered are supposed to carry **the compact open topology**. This is in consonance with Pontryagin duality theory. Obviously if the original group G is discrete, then $G^\wedge = \mathrm{Hom}(G, \mathbb{T})$ and the compact open topology coincides with the pointwise convergence topology which is precisely the induced by the natural embedding of $G^\wedge$ in the product $\mathbb{T}^G$.

2. Groups of homomorphisms defined on $\mathbb{Q}$

We shall look at $\mathrm{Hom}(\mathbb{Q}, \mathbb{T})$ as the character group of $\mathbb{Q}_d$, the discrete group of rationals. Thus we write $\mathbb{Q}_d^\wedge := \mathrm{Hom}(\mathbb{Q}, \mathbb{T})$, and as said above, it carries the pointwise convergence topology. We also consider $\mathbb{Q}_u^\wedge := \mathrm{CHom}(\mathbb{Q}_\mathrm{u}, \mathbb{T})$ the character group of $\mathbb{Q}$ equipped with the topology induced from the euclidean topology of $\mathbb{R}$. If the target group $\mathbb{T}$ is substituted by $\mathbb{R}$, we have $\mathrm{Hom}(\mathbb{Q}, \mathbb{R})$, which happens to be the same as $\mathrm{CHom}(\mathbb{Q}_u, \mathbb{R})$, as proved below.

2.1. A few elementary facts about $\mathrm{Hom}(\mathbb{Q}, \mathbb{R})$

The set $\mathrm{Hom}(\mathbb{Q}, \mathbb{R})$ is quite simple, since every element $\chi \in \mathrm{Hom}(\mathbb{Q}, \mathbb{R})$ can be identified to the real number $\chi(1)$, as we claim in the following Lemma.

Lemma 2.1 *Every group homomorphism $\chi : \mathbb{Q} \to \mathbb{R}$ is defined by its value in $1 \in \mathbb{Q}$, being χ a linear mapping if $\mathbb{Q}$ and $\mathbb{R}$ are considered as vector spaces over $\mathbb{Q}$. Thus,* $\mathrm{CHom}(\mathbb{Q}_\mathrm{u}, \mathbb{R}) = \mathrm{Hom}(\mathbb{Q}, \mathbb{R})$.

The validity of Lemma 2.1 derives from the facts that $\mathbb{R}$ is divisible and χ an homomorphism: if $\chi(1) = r \in \mathbb{R}$, and m is any integer number, $\chi(1/m)$ must be r/m so that $\chi(1/m + \overset{m}{\ldots} + 1/m) = r$. Thus $\chi(n/m) = (n/m)\chi(1)$, and χ is in fact a linear mapping from the one-dimensional vector space $\mathbb{Q}$ to $\mathbb{R}$ (as a vector spaces over $\mathbb{Q}$). Consequently, $\chi : \mathbb{Q} \to \mathbb{R}$ is continuous if both $\mathbb{Q}$ and $\mathbb{R}$ are endowed with the euclidean topology.

However, if $\phi : \mathbb{Q} \to \mathbb{T}$ is a homomorphism and $\phi(1) \in \mathbb{T}$ is known, two possibilities arise for $\phi(1/2)$, the two square roots of $\phi(1)$ and roughly speaking, many more for $\phi(1/m)$ if $m > 2$. This explains the following negative claim:

Remark 2.1 *A homomorphism $\phi : \mathbb{Q} \to \mathbb{T}$ is not defined by its value in 1. Later on we shall prove that* $\mathrm{Hom}(\mathbb{Q}, \mathbb{T}) \neq \mathrm{CHom}(\mathbb{Q}_\mathrm{u}, \mathbb{T})$.

2.2. The group $\mathrm{Hom}(\mathbb{Q}, \mathbb{T})$.

In order to understand how the set $\mathrm{Hom}(\mathbb{Q}, \mathbb{T})$ looks like, we include the following description done in [8]. Let $\chi : \mathbb{Q} \to \mathbb{T}$ be a homomorphisms from $\mathbb{Q}$ to $\mathbb{T}$. It gives rise to a sequence $\{\alpha_n\}$ of elements of $\mathbb{T}$, namely $\alpha_n := \chi(1/n!)$. Clearly it holds:

$$(\alpha_n)^n = \alpha_{n-1}. \ (*)$$

On the other hand, every sequence $\{\beta_n\} \subset \mathbb{T}$ which satisfies the condition (*) gives rise to a character ϕ on $\mathbb{Q}$. Just define $\phi(1/n!) = \beta_n$, and take into account the expression of $\mathbb{Q}$ as a union of cyclic groups, $\mathbb{Q} = \bigcup_n \langle 1/n! \rangle$.

Thus, the character group of $\mathbb{Q}_d$ can be identified to a limit of an inverse sequence $\{\mathbb{T}, g_n\}$, where the linking mappings $g_n : \mathbb{T} \to \mathbb{T}$ are defined by $g_n(\gamma) = \gamma^n$.

2.3. $\mathrm{Hom}(\mathbb{Q}, \mathbb{T}) \neq \mathrm{CHom}(\mathbb{Q}_{\mathrm{u}}, \mathbb{T})$.

An example of a character χ defined on $\mathbb{Q}$ which is not continuous considered as a mapping from $\mathbb{Q}_u$ into $\mathbb{T}$ is provided in [5]. Following 2.2, it is achieved by means of a sequence $\{\alpha_n\}$ for which (*) holds but α_n does not converge to $1 \in \mathbb{T}$. The character χ defined through $\chi(1/n!) = \alpha_n$ for such an $\{\alpha_n\}$ cannot be continuous if $\mathbb{Q}$ carries the euclidean topology. In fact, from $1/n! \to 0$ in $\mathbb{Q}_u$ we would obtain $\alpha_n \to \chi(0) = 1$, which does not happen. We present the concrete example below.

Example 2.1 *Let $\alpha_n := e^{2\pi i \theta_n}$, being*

$$\theta_n = \frac{1}{2(n!)} + \frac{1}{n!}\left(\sum_{i=2}^{n} \left[\frac{i}{2}\right](i-1)!\right),$$

where the symbol $[.]$ stands for entire part. From the equality:

$$\theta_n = \frac{\theta_{n-1} + [n/2]}{n}, \ \textit{for all } n \in \mathbb{N} \setminus \{0, 1\},$$

it is straightforward to check that $\alpha_n^n = \alpha_{n-1}$, therefore χ is a character on $\mathbb{Q}$. On the other hand, it can be proved by induction that

$$1/4 \leq \theta_n \leq 3/4, \ \textit{for all } n \in \mathbb{N} \setminus \{0, 1\},$$

which implies $\alpha_n \not\to 1$ in $\mathbb{T}$.

3. Lifting continuous characters of a topological group G to real continuous characters

Taking into account that $\mathbb{R}_u$ is the universal cover of $\mathbb{T}$ by means of the exponential mapping ρ, it is natural to think about the possibility of lifting continuous characters defined on a topological group G, to continuous homomorphisms from G to $\mathbb{R}$. First a definition to make precise what we mean.

Definition 3.1 *Let G be a topological group. A continuous character $\phi : G \to \mathbb{T}$ is said to be liftable over the reals if there exists a continuous homomorphism $\tilde{\phi} : G \to \mathbb{R}$ such that $\rho\tilde{\phi} = \phi$. The term real character is used for a homomorphism from G to $\mathbb{R}$, and we will call $\tilde{\phi}$ a lift of ϕ.*

Let G be a topological group. Does the statement "every continuous character on G is liftable" hold?

Clearly for a simply connected group G every continuous character is liftable. (This follows from well known properties of the theory of coverings.) However, one of the most elementary connected groups, like $\mathbb{T}$ is not simply connected. Thus, the assumption of simple connectedness is too strong. Nevertheless, the statement will hold for any topological real vector space considered as a group.

Partial Answers 3.1 *1)* (Dixmier [6]) *If G is a locally compact group, and $G^\wedge$ is arcwise connected then every continuous character is liftable.*

2) If G is a k-space and $G^\wedge$ is arcwise connected, then every continuous character is liftable.

Obviously the result 1) is a particular case of 2) and 2) can be proved by means of the homotopy lifting property of a covering [14, Chapter 2, thm. 3]. Dixmier gives a proof of 1) ad hoc for locally compact groups, based upon the Structure Theorem for the latter, which is a delicate matter. The assertion 2) is from [12]. A detailed proof of it can be seen in [3, 5.2.1 and 5.2.2].

The possibility of lifting the characters of a topological group G to real characters by requiring conditions on the arcs of $G^\wedge$, rather than the global property of arcwise connectivity, is analyzed in a series of papers that have recently appeared. As defined in [2], a topological group G has the EAP (equicontinuous arc property) if every arc in $G^\wedge$ is equicontinuous with respect to the original topology of G. In the mentioned paper, the set of continuous characters of G that can be lifted is called $G^\wedge_{\text{lift}}$. It is a subgroup of $G^\wedge$ contained in the arc component, denominated $G^\wedge_a$. The main result of [2] asserts that $G^\wedge_{\text{lift}} = G^\wedge_a$ whenever G has the EAP property.

Since we have extensively dealt with the continuous convergence structure Λ defined on the dual of a topological group (see for instance [4] and [3]), it is clear for us that the EAP property has a natural setting in terms of it. Let us state here how to reformulate the result of [2] quoted above:

Theorem 3.2 *Let G be a topological abelian group. The following assertions are equivalent:*

1) Every character $\phi \in G^\wedge$ can be lifted over the reals.

2) The group $\mathrm{CHom}(\mathrm{G}, \mathbb{T})$ *endowed with the continuous convergence structure Λ is arcwise connected.*

4. The main results

In order to obtain a compact Hausdorff group topology on $\mathbb{R}$, we first state the following:

Proposition 4.1 *The group $\mathbb{Q}_d^\wedge$ is algebraically isomorphic to the group of real numbers $\mathbb{R}$.*

A detailed proof of this assertion can be seen in [11]. For the readers convenience we point out that it derives from the fact that $\mathbb{Q}_d^\wedge$ and $\mathbb{R}$ are vector spaces over the field of rationals $\mathbb{Q}$, such that $|\mathbb{Q}_d^\wedge| = |\mathbb{R}|$. Their respective Hamel bases have also the same cardinality. Thus, any fixed bijection from a Hamel basis of $\mathbb{R}$ to a Hamel basis of $\mathbb{Q}_d^\wedge$ can be extended to an algebraic isomorphism. For definiteness, let us call $\phi : \mathbb{R} \to \mathbb{Q}_d^\wedge$ such an isomorphism, and let us denote by ν the topology on $\mathbb{R}$ that makes ϕ a **topological** isomorphism. We call ν a **Halmos topology** for $\mathbb{R}$.

Since $\mathbb{Q}_d^\wedge$ and $(\mathbb{R}, \nu)$ are isomorphic topological groups, we prove now that the first one is not arcwise connected in order to have the same property for $(\mathbb{R}, \nu)$.

Theorem 4.2 *The group $\mathbb{Q}_d^\wedge$ is not arcwise connected.*

Proof. Assume by contradiction that $\mathbb{Q}_d^\wedge$ were arcwise connected. Since $\mathbb{Q}_d$ is locally compact by the assertion 1) in 3.1, every character on $\mathbb{Q}$ would be liftable over the reals. In particular χ, as described in the Example would be liftable to a real character $\tilde{\chi} : \mathbb{Q}_d \to \mathbb{R}$ such that $\rho\tilde{\chi} = \chi$. By the equality $\mathrm{CHom}(\mathbb{Q}_{\mathrm{u}}, \mathbb{R}) = \mathrm{Hom}(\mathbb{Q}, \mathbb{R})$ (see 2.2), we would have that $\tilde{\chi}$ is continuous with respect to the usual topology. But in that case $\rho\tilde{\chi} = \chi$ would be continuous with respect to the usual topology of $\mathbb{Q}$, in other words $\chi = \rho\tilde{\chi} \in \mathrm{CHom}(\mathbb{Q}_{\mathrm{u}}, \mathbb{T})$, which does not hold as proved in the Example. □

We provide now references which easily prove the facts that $(\mathbb{R}, \nu)$ is neither locally connected, nor a Lie group.

Proposition 4.3 [13, Theorem 42, pp. 169] *A compact locally connected and connected abelian group decomposes into the direct sum of a finite or countable number of subgroups, each isomorphic with the group $\mathbb{T}$.*

Clearly our group $(\mathbb{R}, \nu)$ cannot have any subgroup isomorphic to $\mathbb{T}$, since in $\mathbb{R}$ there are no torsion elements. Because of the definition of the topology ν, all the topological assertions done for $\mathbb{Q}_d^\wedge$ hold for $(\mathbb{R}, \nu)$, thus we already know that $(\mathbb{R}, \nu)$ is compact and connected. Therefore, according to Proposition 4.3, it cannot be locally connected. On the other hand a compact Lie group is necessarily locally connected (see, for instance [13, Remark H, p. 212]).

Finally we recall the following:

Proposition 4.4 [9, Theorem H. Mazurkiewicz] *A Hausdorff topological space is a curve if and only if it is a metrizable Peano continuum.*

By a Peano continuum it is meant a Hausdorff, compact, connected and locally connected topological space. We now have all the ingredients to formulate the Theorem:

Theorem 4.5 *The topological group $(\mathbb{R}, \nu)$ is compact, Hausdorff and connected, but neither arcwise connected nor locally connected. Therefore $(\mathbb{R}, \nu)$ is neither a Lie group, nor a curve in the sense of H. Mazurkiewicz.*

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