Solitary waves due to $\chi^{(2)}:\chi^{(2)}$ cascading

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Solitary waves in materials with a cascaded $\chi^{(2)}:\chi^{(2)}$ nonlinearity are investigated, and the implications of the robustness hypothesis for these solitary waves are discussed. Both temporal and spatial solitary waves are studied. First, the basic equations that describe the $\chi^{(2)}:\chi^{(2)}$ nonlinearity in the presence of dispersion or diffraction are derived in the plane-wave approximation, and we show that these equations reduce to the nonlinear Schrödinger equation in the limit of large phase mismatch and can be considered a Hamiltonian deformation of the nonlinear Schrödinger equation. We then proceed to a comprehensive description of all the solitary-wave solutions of the basic equations that can be expressed as a simple sum of a constant term, a term proportional to a power of the hyperbolic secant, and a term proportional to a power of the hyperbolic secant multiplied by the hyperbolic tangent. This formulation includes all the previously known solitary-wave solutions and some exotic new ones as well. Our solutions are derived in the presence of an arbitrary group-velocity difference between the two harmonics, but a transformation that relates our solutions to zero-velocity solutions is derived. We find that all the solitary-wave solutions are zero-parameter and one-parameter families, as opposed to nonlinear-Schrödinger-equation solitons, which are a two-parameter family of solutions. Finally, we discuss the prediction of the robustness hypothesis that there should be a two-parameter family of solutions with solitonlike behavior, and we discuss the experimental requirements for observation of solitonlike behavior.

1. INTRODUCTION

Stegeman *et al.*[1] pointed out recently that it is possible to obtain large nonlinear phase shifts by using a cascaded $\chi^{(2)}:\chi^{(2)}$ nonlinearity. While it has long been known that the product of second-order nonlinearities can lead to effective third-order nonlinearities,[2] this is only now becoming widely appreciated because of recent experimental results that can be explained only by this fact.[3,4]

Wave propagation in materials with substantial dispersion or diffraction and a significant $\chi^{(3)}$ nonlinearity can be described by the nonlinear Schrödinger equation and its variants.[5] The nonlinear Schrödinger equation has exact soliton solutions that correspond to a balance between nonlinearity and dispersion in the case of temporal solitons or between nonlinearity and diffraction in the case of spatial solitons. Since the nonlinear Schrödinger equation belongs to that very small and very special class of equations that can be solved by the inverse scattering method,[6] one might naïvely anticipate that the balance between nonlinearity and dispersion is delicate and that any small perturbations that are due to real-world effects added to the nonlinear Schrödinger equation will destroy the solitons. From a strict mathematical standpoint this expectation is, in fact, true. Nevertheless, there is extensive theoretical and experimental evidence that solitonlike behavior persists even in the presence of changes in the nonlinear Schrödinger equation that are so large that they are not properly referred to as perturbations and are called deformations, as long as these deformations are Hamiltonian.[7] For example, one finds in optical fibers that solitons are robust in the presence of third-order dispersion,[8] birefringence,[9] and the real part of the Raman susceptibility,[10] all of which are Hamiltonian, while attentuation[11] and the imaginary part of the Raman susceptibility,[12] both of which are non-Hamiltonian, lead to soliton destruction.

Although this robustness hypothesis has been studied extensively in the context of optical fibers, its applicability is not limited to optical fibers but is expected to apply in a broad range of physical contexts.[7,13] In particular, a number of authors[14–18] showed that the equations that describe the $\chi^{(2)}:\chi^{(2)}$ cascaded nonlinearity reduce to the nonlinear Schrödinger equation when the phase mismatch is large and the amplitudes of the fundamental and the second-harmonic waves are correspondingly large. Since these equations are Hamiltonian, they can be viewed as a Hamiltonian deformation of the nonlinear Schrödinger equation, and it follows from the robustness hypothesis that solitonlike behavior should be visible in this system under appropriate circumstances.[7] By solitonlike behavior, we mean that an object numerically and experimentally indistinguishable from a solitary wave will ultimately emerge from a wide range of initial conditions. Previous research has shown that it is not generally necessary for exact solitary-wave solutions to exist for solitonlike behavior to be observed[7–9]; however, it is already known that the equations that describe the $\chi^{(2)}:\chi^{(2)}$ cascaded nonlinearity in the presence of dispersion or diffraction have a variety of exact solitary-wave solutions.[16,17]

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In this paper we obtain all the solitary-wave solutions that are expressible as a simple sum of a constant term, a term proportional to a power of the hyperbolic secant, and a term proportional to a power of the hyperbolic secant multiplied by the hyperbolic tangent. This formulation includes all the solitary-wave solutions that are currently known as well as some exotic new ones, where by exotic we mean that the solutions are neither bright nor dark but have different, somewhat complicated phase and intensity variations. We also discuss the physical meaning of these solutions to the extent possible, their relationship to the robustness hypothesis, and the requirements for experimentally observing them. These solutions are intrinsically interesting, but they are not expected to exhaust the solitonlike behavior, since the robustness hypothesis predicts that there is a larger class of solitonlike solutions that may have no exact analytical form and may not even be solitary waves. Numerical simulations to date, while they have been carried out over a limited range in parameter space, support the contentions that solitonlike behavior is robust and that a larger class of solitonlike solutions exists.[15–17]

Most of the solitary-wave solutions that we describe were previously discovered by Werner and Drummond.[16,17] Our investigation adds to theirs in the following respects: first, we show that the set of solitary-wave solutions that we present is comprehensive, as mentioned in the previous paragraph, and included among them are some novel solutions that have not been presented previously and do not have the typical form of either bright or dark solitons. Second, Werner and Drummond find solutions only in a frame in which there is no group-velocity mismatch between the two harmonics, whereas we find the solutions in all possible frames. While it is possible for most parameter values to transform into a frame in which the group velocities match, there are special parameter values for which this transformation is not possible, and these parameter values have a special physical meaning that we describe. Third, the principal interest of Werner and Drummond was in degenerate parametric oscillators, which obey the same equations that describe the $\chi^{(2)}:\chi^{(2)}$ cascaded nonlinearity but which are physically quite distinct.[19] In this investigation we discuss the physical requirements for observing solitary waves by means of the $\chi^{(2)}:\chi^{(2)}$ nonlinearity. Fourth, we discuss the robustness of the solitonlike behavior.

The remainder of this manuscript is organized as follows: in Section 2 we present a derivation of the basic equations, using the plane-wave approximation, and we outline the relationship of these equations to the nonlinear Schrödinger equation. In Section 3 we present the solitary-wave solutions and discuss their physical significance. Finally, in Section 4 we discuss the robustness of solitonlike behavior, and we outline the conditions that are required for observing this behavior experimentally.

2. BASIC EQUATIONS

A. Temporal Solitons

We begin our discussion of the $\chi^{(2)}:\chi^{(2)}$ solitary waves by deriving the equations that describe temporal solitary waves; we move on from there to a derivation of the equations that describe spatial solitary waves. The equations are identical in structure, but the coefficients have different physical meaning. We carry out the derivation in the plane-wave approximation, and it is similar in style to an earlier derivation of the coupled nonlinear Schrödinger equation in Kerr media.[20] These equations have been known for a long time, and alternative discussions may be found, for example, in Refs. 21 and 22.

We begin by assuming that the fundamental and the second-harmonic waves are propagating in the z direction and are tightly confined in the transverse directions. Assuming that the variation along the waveguide occurs on a length scale that is long compared with the transverse dimensions, we may separate the evolution in the transverse directions so that $\mathcal{E}_1(x, y, z, t) = E_1(z, t)\Psi_1(x, y)$ and $\mathcal{E}_2(x, y, z, t) = E_2(z, t)\Psi_2(x, y)$ are the electric fields for the fundamental and the second-harmonic waves. The functional forms of Ψ_1 and Ψ_2 are given by the normal modes of the waveguide. We may ignore the transverse variation of $\mathcal{E}_1$ and $\mathcal{E}_2$ and use the plane-wave approximation in deriving the wave equations if we take into account the following two consequences of the transverse variation. First, the dispersion relations $k_1(\omega)$ and $k_2(\omega)$ for the fundamental and the second-harmonic waves will be affected not just by the material properties but also by the transverse structure of the normal modes. Second, the effect of the nonlinearity is determined by an average over the transverse profile of the modes; the coefficient of the bulk nonlinearity is multiplied by a coefficient somewhat less than one.[5,15] These issues affect only the coefficients of the resulting equations and not their structure. These coefficients will depend significantly on the bulk properties as well as waveguide effects.

We make the slowly varying envelope approximation, in which we assume that the spectral widths of the fundamental and the second-harmonic waves are small compared with the spectral widths over which $k_1(\omega)$ and $k_2(\omega)$ vary significantly, so it is legitimate to use a truncated Taylor expansion of $k_1(\omega)$ and $k_2(\omega)$. When this assumption fails, we must use a complete frequency-domain analysis whose properties depend on the details of the material and the geometry being considered.[15] Thus we write

$$\begin{aligned} E_1(z, t) &= A_1(z, t)\exp[ik_1(\omega_0)z - i\omega_0 t], \\ E_2(z, t) &= A_2(z, t)\exp[ik_2(2\omega_0)z - 2i\omega_0 t], \end{aligned} \tag{1}$$

where ω_0 is the central frequency of the fundamental wave. We assume that the z variation of A_1 and A_2 is slow compared with k_1 and k_2 and that the t variation is slow compared with ω_0. We use two dispersion relations, indicated with the subscripts 1 and 2, to allow for the possibility that the fundamental and the second-harmonic waves are on different branches of the complete dispersion relation.

Starting with Maxwell's equation in the plane-wave approximation, we write

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = 0, \tag{2}$$

where c is the speed of light,

$$\begin{aligned} \mathbf{E}(z, t) = {} & E_1(z, t)\hat{\mathbf{e}}_1 + E_1^*(z, t)\hat{\mathbf{e}}_1^* \\ & + E_2(z, t)\hat{\mathbf{e}}_2 + E_2^*(z, t)\hat{\mathbf{e}}_2^* \end{aligned} \tag{3}$$

is the total field, and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are the possibly complex unit vectors that describe the polarization of the modes at ω_0 and $2\omega_0$; we are assuming that these unit vectors have no spectral variation over the frequency range of interest. Assuming that the electric displacement $\mathbf{D}$ has contributions that are linear and quadratically nonlinear and that are nonlocal in time, we may set

$$\mathbf{D}(z, t) = \mathbf{E}(z, t) + 4\pi \int_{-\infty}^{t} dt' \mathsf{X}^{(1)}(t - t') \cdot \mathbf{E}(z, t) + 4\pi \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \mathsf{X}^{(2)}(t - t', t - t'') \cdot \mathbf{E}(z, t')\mathbf{E}(z, t''), \tag{4}$$

where $\mathsf{X}^{(1)}$ is a second-rank tensor and $\mathsf{X}^{(2)}$ is a third-rank tensor. The first term on the right-hand side of Eq. (4) is the vacuum contribution, the second term is the material's linear contribution, and the third term is the material's nonlinear contribution. Writing by analogy to Eqs. (1) and (3) that

$$\mathbf{D}(z, t) = D_1(z, t)\hat{\mathbf{e}}_1 + D_1{}^*(z, t)\hat{\mathbf{e}}_1{}^* + D_2(z, t)\hat{\mathbf{e}}_2 + D_2{}^*(z, t)\hat{\mathbf{e}}_2{}^*, \tag{5}$$

where

$$D_1(z, t) = U_1(z, t)\exp[ik_1(\omega_0)z - i\omega_0 t],$$
$$D_2(z, t) = U_2(z, t)\exp[ik_2(2\omega_0)z - 2i\omega_0 t], \tag{6}$$

we find that

$$\begin{aligned} U_1(z, t) = {} & A_1(z, t) + 4\pi \int_{-\infty}^{t} dt' [\hat{\mathbf{e}}_1{}^* \cdot \mathsf{X}^{(1)}(t - t') \cdot \hat{\mathbf{e}}_1] \\ & \times \exp[i\omega_0(t - t')]A_1(z, t') \\ & + 8\pi \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' [\hat{\mathbf{e}}_1{}^* \cdot \mathsf{X}^{(2)}(t - t', t - t'') \\ & \cdot \hat{\mathbf{e}}_1{}^*\hat{\mathbf{e}}_2] \exp[2i\omega_0(t - t'') - i\omega_0(t - t')] \\ & \times A_1{}^*(z, t')A_2(z, t'') \\ & \times \exp\{-i[2k_1(\omega_0) - k_2(2\omega_0)]z\}, \\ U_2(z, t) = {} & A_2(z, t) + 4\pi \int_{-\infty}^{t} dt' [\hat{\mathbf{e}}_2{}^* \cdot \mathsf{X}^{(1)}(t - t') \cdot \hat{\mathbf{e}}_2] \\ & \times \exp[2i\omega_0(t - t')]A_2(z, t') \\ & + 4\pi \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' [\hat{\mathbf{e}}_2{}^* \cdot \mathsf{X}^{(2)}(t - t', t - t'') \\ & \cdot \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1] \exp[i\omega_0(t - t'') + i\omega_0(t - t')] \\ & \times A_1(z, t')A_2(z, t'')\exp\{i[2k_1(\omega_0) - k_2(2\omega_0)]z\}. \end{aligned} \tag{7}$$

We now define

$$\tilde{\chi}_1^{(1)}(\omega) = \int_{-\infty}^{\infty} dt [\hat{\mathbf{e}}_1{}^* \cdot \mathsf{X}^{(1)} \cdot \hat{\mathbf{e}}_1]\exp(i\omega t),$$
$$\tilde{\chi}_2^{(1)}(\omega) = \int_{-\infty}^{\infty} dt [\hat{\mathbf{e}}_2{}^* \cdot \mathsf{X}^{(1)} \cdot \hat{\mathbf{e}}_2]\exp(i\omega t), \tag{8}$$

and we set $\mathsf{X}^{(1)}(t) = 0$ if $t < 0$. We also define

$$\begin{aligned} \tilde{\chi}_1^{(2)}(2\omega, -\omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [\hat{\mathbf{e}}_1{}^* \cdot \mathsf{X}^{(2)}(t, t') \cdot \hat{\mathbf{e}}_1{}^*\hat{\mathbf{e}}_2] \\ & \quad \times \exp(2i\omega t' - i\omega t), \\ \tilde{\chi}_2^{(2)}(\omega, \omega) &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [\hat{\mathbf{e}}_2{}^* \cdot \mathsf{X}^{(2)}(t, t') \cdot \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1] \\ & \quad \times \exp(i\omega t' + i\omega t), \end{aligned} \tag{9}$$

where we set $\mathsf{X}^{(2)}(t, t') = 0$ if either $t < 0$ or $t' < 0$. We use the slowly varying envelope approximation and keep terms through second order in the linear contribution and zero order in the nonlinear contribution, so that

$$\begin{aligned} \tilde{\chi}_1^{(1)}(\omega) &\simeq \tilde{\chi}_1^{(1)}(\omega_0) + \left.\frac{\partial \tilde{\chi}_1^{(1)}}{\partial \omega}\right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left.\frac{\partial^2 \tilde{\chi}_1^{(1)}}{\partial \omega^2}\right|_{\omega_0} (\omega - \omega_0)^2, \\ \tilde{\chi}_2^{(1)}(\omega) &\simeq \tilde{\chi}_2^{(1)}(2\omega_0) + \left.\frac{\partial \tilde{\chi}_2^{(1)}}{\partial \omega}\right|_{2\omega_0} (\omega - 2\omega_0) + \frac{1}{2} \left.\frac{\partial^2 \tilde{\chi}_2^{(1)}}{\partial \omega^2}\right|_{2\omega_0} (\omega - 2\omega_0)^2, \end{aligned} \tag{10}$$

$$\tilde{\chi}_1^{(2)}(2\omega, -\omega) \simeq \tilde{\chi}_1^{(2)}(2\omega_0, -\omega_0),$$
$$\tilde{\chi}_1^{(2)}(\omega, \omega) \simeq \tilde{\chi}_1^{(2)}(\omega_0, \omega_0). \tag{11}$$

When $\Delta k = 2k_1(\omega_0) - k_2(2\omega_0)$, Eqs. (7) become

$$\begin{aligned} U_1(z, t) = {} & \tilde{\epsilon}_1 A_1(z, t) + i\tilde{\epsilon}_1{}' \frac{\partial A_1(z, t)}{\partial t} - \frac{1}{2} \tilde{\epsilon}_1{}'' \frac{\partial^2 A_1(z, t)}{\partial t^2} \\ & + 2\tilde{\epsilon}_1^{(2)} A_1{}^*(z, t)A_2(z, t)\exp(-i\Delta kz), \\ U_2(z, t) = {} & \tilde{\epsilon}_2 A_2(z, t) + i\tilde{\epsilon}_2{}' \frac{\partial A_2(z, t)}{\partial t} \\ & - \frac{1}{2} \tilde{\epsilon}_2{}'' \frac{\partial^2 A_2(z, t)}{\partial t^2} + \tilde{\epsilon}_2^{(2)} A_1{}^2(z, t)\exp(i\Delta kz), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \tilde{\epsilon}_1 &= 1 + 4\pi\tilde{\chi}_1^{(1)}(\omega_0), \qquad \tilde{\epsilon}_1{}' = 4\pi \left.\frac{\partial \tilde{\chi}_1^{(1)}}{\partial \omega}\right|_{\omega_0}, \\ \tilde{\epsilon}_1{}'' &= 4\pi \left.\frac{\partial^2 \tilde{\chi}_1^{(1)}}{\partial \omega^2}\right|_{\omega_0}, \\ \tilde{\epsilon}_2 &= 1 + 4\pi\tilde{\chi}_2^{(1)}(2\omega_0), \qquad \tilde{\epsilon}_2{}' = 4\pi \left.\frac{\partial \tilde{\chi}_2^{(1)}}{\partial \omega}\right|_{2\omega_0}, \\ \tilde{\epsilon}_2{}'' &= 4\pi \left.\frac{\partial^2 \tilde{\chi}_2^{(1)}}{\partial \omega^2}\right|_{2\omega_0}, \end{aligned} \tag{13}$$

$$\tilde{\epsilon}_1^{(2)} = 4\pi\tilde{\chi}_1^{(2)}(2\omega_0, -\omega_0),$$
$$\tilde{\epsilon}_2^{(2)} = 4\pi\tilde{\chi}_2^{(2)}(\omega_0, \omega_0). \tag{14}$$

Returning now to Maxwell's equation, Eq. (2), we first isolate the contribution that is due to E_1 and D_1. Explicitly, we find

$$-k_1{}^2 A_1 + 2ik_1 \frac{\partial A_1}{\partial z} + \frac{\partial^2 A_1}{\partial z^2} + \frac{\omega_0{}^2}{c^2} \tilde{\epsilon}_1 A_1 + i\left(\frac{\omega_0{}^2}{c^2} \tilde{\epsilon}_1{}' + \frac{2\omega_0}{c^2} \tilde{\epsilon}_1\right) \frac{\partial A_1}{\partial t} - \left(\frac{\omega_0{}^2}{2c^2} \tilde{\epsilon}_1{}'' + \frac{2\omega_0}{c^2} \tilde{\epsilon}_1{}' + \frac{1}{c^2} \tilde{\epsilon}_1\right) \frac{\partial^2 A_1}{\partial t^2} + \frac{2\omega_0{}^2}{c^2} \tilde{\epsilon}_1^{(2)} A_1{}^* A_2 \exp(-i\Delta kz) = 0. \quad (15)$$

It follows from the slowly varying envelope approximation that

$$|k_1{}^2 A_1| \gg \left| k_1 \frac{\partial A_1}{\partial z} \right| \gg \left| \frac{\partial^2 A_1}{\partial z^2} \right| \quad (16)$$

and similarly that

$$|\omega_0{}^2 A_1| \gg \left| \omega_0 \frac{\partial A_1}{\partial t} \right| \gg \left| \frac{\partial^2 A_1}{\partial t^2} \right| . \quad (17)$$

We may thus expand the linear portion of Eq. (15) in order of the number of derivatives. We assume that the nonlinear contribution is of the same order as the second derivatives, since that is the physical requirement for obtaining solitons. At lowest order we obtain

$$k_1{}^2 - \frac{\omega_0{}^2}{c^2} \tilde{\epsilon}_1 = 0, \quad (18)$$

which in essence fixes the linear dispersion relation. At the next order we find

$$i \frac{\partial A_1}{\partial z} + i\left(\frac{\omega_0{}^2}{2k_1 c^2} \tilde{\epsilon}_1{}' + \frac{\omega_0}{k_1 c^2} \tilde{\epsilon}_1\right) \frac{\partial A_1}{\partial t} = i \frac{\partial A_1}{\partial z} + ik_1{}' \frac{\partial A_1}{\partial t} = 0, \quad (19)$$

where $k_1{}' = \mathrm{d}k_1/\mathrm{d}\omega$ is evaluated from the dispersion relation, Eq. (18), at $\omega = \omega_0$. From Eq. (19) it follows through second order that

$$\frac{\partial^2 A_1}{\partial z^2} = (k_1{}')^2 \frac{\partial^2 A_1}{\partial t^2}, \quad (20)$$

which allows us to eliminate $\partial^2 A_1/\partial z^2$ in Eq. (15). We finally obtain

$$i \frac{\partial A_1}{\partial z} + ik_1{}' \frac{\partial A_1}{\partial t} - \frac{1}{2} k_1{}'' \frac{\partial^2 A_1}{\partial t^2} + K_1 A_1{}^* A_2 \exp(-i\Delta kz) = 0, \quad (21)$$

where

$$k_1{}'' = \frac{\omega_0{}^2}{2k_1 c^2} \tilde{\epsilon}_1{}'' + \frac{2\omega_0}{k_1 c^2} \tilde{\epsilon}_1{}' + \frac{1}{k_1 c^2} \tilde{\epsilon}_1 - \frac{1}{k_1} (k_1{}')^2 = \left.\frac{\mathrm{d}^2 k_1}{\mathrm{d}\omega^2}\right|_{\omega_0} \quad (22)$$

and $K_1 = \omega_0{}^2 \tilde{\epsilon}_1^{(2)}/k_1 c^2$. We find in a completely analogous fashion that

$$i \frac{\partial A_2}{\partial z} + ik_2{}' \frac{\partial A_2}{\partial t} - \frac{1}{2} k_2{}'' \frac{\partial^2 A^2}{\partial t^2} + K_2 A_1{}^2 \exp(i\Delta kz) = 0, \quad (23)$$

where

$$k_2{}' = \left.\frac{\mathrm{d}k_2}{\mathrm{d}\omega}\right|_{2\omega_0}, \qquad k_2{}'' = \left.\frac{\mathrm{d}^2 k_2}{\mathrm{d}\omega^2}\right|_{2\omega_0}, \qquad K_2 = \frac{2\omega_0{}^2}{k_2 c^2} \tilde{\epsilon}_2^{(2)}. \quad (24)$$

We finally reduce the equations to normalized form. To do so, we make the following variable transformations:

$$\xi = \frac{|k_1{}''|}{\tau^2} z, \qquad s = \frac{t}{\tau} - \frac{k_1{}'}{\tau} z,$$
$$\delta = \frac{(k_1{}' - k_2{}')\tau}{|k_1{}''|}, \qquad \alpha = \frac{k_2{}''}{|k_1{}''|}, \qquad \beta = \frac{\Delta k \tau^2}{|k_1{}''|},$$
$$a_1 = \frac{|K_1 K_2|^{1/2} \tau^2}{|k_1{}''|} A_1, \qquad a_2 = \frac{K_1 \tau^2}{|k_1{}''|} A_2,$$
$$r = \mathrm{sgn}(k_1{}''), \quad (25)$$

where we assume that $k_1{}'' \neq 0$ and that τ is an arbitrary time parameter that is usefully chosen to equal roughly the duration over which the amplitude of the solitary wave varies. We now obtain the reduced equations

$$i \frac{\partial a_1}{\partial \xi} - \frac{r}{2} \frac{\partial^2 a_1}{\partial s^2} + a_1{}^* a_2 \exp(-i\beta\xi) = 0,$$
$$i \frac{\partial a_2}{\partial \xi} - i\delta \frac{\partial a_2}{\partial s^2} - \frac{\alpha}{2} \frac{\partial^2 a_2}{\partial s^2} + a_1{}^2 \exp(i\beta\xi) = 0. \quad (26)$$

We have implicitly assumed that K_1 and K_2 are purely real. If they contain imaginary components, as happens when the system being modeled operates near a resonance,[1] then the coefficients of the nonlinear terms in Eqs. (26) become complex. We do not consider this possibility in this paper. We note also that, owing to the Kleinman symmetry, $\mathrm{sgn}(K_1 K_2) = 1$.

B. Spatial Solitons

We now show that Eq. (26) also holds in a physical context in which spatial solitary waves can appear. The appropriate physical context is the introduction of a continuous-wave input into a slab waveguide. The wave is confined in one direction, which we choose to be the x direction. We treat the x coordinate as ignorable on the basis of the same argument that we used to treat the transverse coordinates as ignorable when we derived the temporal equations. Decomposing the electric field, we assume that the fundamental and the second-harmonic fields have the forms

$$E_1(y, z, t) = A_1(y - \rho_1 z, z) \exp(ik_1 z + ik_y y - i\omega_0 t),$$
$$E_2(y, z, t) = A_2(y - \rho_2 z, z) \exp(ik_2 z + 2ik_y y - 2i\omega_0 t), \quad (27)$$

where ρ_1 and ρ_2 equal the angles between the Poynting vectors of the fundamental and the second-harmonic waves and the z axis. Equations (27) allow for the possibility that the waves do not propagate along one of the principal axes of the $\chi^{(2)}$ crystal, in which case the energy propagation directions are different from the phase propagation direction and different from each other. The wave numbers at both frequencies are fixed by the dispersion relations, i.e.,

$$k_1{}^2 = \frac{\omega_0{}^2}{c^2}\,\tilde{\epsilon}_1(\omega_0)\,, \qquad k_2{}^2 = \frac{4\omega_0{}^2}{c^2}\,\tilde{\epsilon}_2(2\omega_0)\,. \tag{28}$$

Since we are dealing with continuous-wave sources, Maxwell's equation takes on the simple form

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\omega_0{}^2}{c^2}\,\mathbf{D} = 0\,. \tag{29}$$

Focusing on the E_1 contribution, we find that

$$U_1 = \tilde{\epsilon}_1 A_1 + 2\tilde{\epsilon}_1^{(2)} A_1{}^* A_2 \exp(-i\Delta k z)\,, \tag{30}$$

where $\Delta k = 2k_1 - k_2$, and Maxwell's equation yields

$$2ik_1 \frac{\partial A_1}{\partial z} - 2ik_1\rho_1 \frac{\partial A_1}{\partial y} + \frac{\partial^2 A_1}{\partial y^2} + \frac{2\omega_0{}^2}{c^2}\,\tilde{\epsilon}_1^{(2)} A_1{}^* A_2 \exp(-i\Delta k z) = 0\,, \tag{31}$$

where we have invoked slowly varying envelope approximation (16) and dispersion relations (28). Setting

$$k_1{}' = \rho_1\,, \qquad k_1{}'' = -1/k_1\,, \tag{32}$$

one can write this equation in the form

$$i\frac{\partial A_1}{\partial z} - ik_1{}'\frac{\partial A_1}{\partial y} - \frac{1}{2}k_1{}''\frac{\partial^2 A_1}{\partial y^2} + K_1 A_1{}^* A_2 \exp(-i\Delta k z) = 0\,. \tag{33}$$

By analogy, we find that

$$i\frac{\partial A_2}{\partial z} - ik_2{}'\frac{\partial A_2}{\partial y} - \frac{1}{2}k_2{}''\frac{\partial^2 A_2}{\partial y^2} + K_2 A_1{}^2 \exp(i\Delta k z) = 0\,, \tag{34}$$

where $k_2{}' = \rho_2$ and $k_2{}'' = -1/k_2$. These equations are formally identical to Eqs. (21) and (23), so that, noting that $k_1{}'' < 0$ and $k_2{}'' < 0$, we once again obtain Eqs. (26) if we set

$$\xi = \frac{|k_1{}''|}{\eta^2}\,z, \qquad s = \frac{y}{\eta} + \frac{k_1{}'}{\eta}\,z\,,$$
$$\delta = -\frac{(k_1{}' - k_2{}')\eta}{|k_1{}''|}\,, \qquad \alpha = \frac{k_2{}''}{|k_1{}''|}\,, \qquad \beta = \frac{\Delta k \eta^2}{|k_1{}''|}\,,$$
$$a_1 = \frac{|K_1K_2|^{1/2}\eta^2}{|k_1{}''|}\,A_1\,, \qquad a_2 = \frac{K_1\eta^2}{|k_1{}''|}\,A_2\,,$$
$$r = -1\,, \tag{35}$$

where η is an arbitrary parameter that may be conveniently chosen to be of the order of the width of the solitary wave's variation in the y dimension. Note the change of sign in the definition of δ.

C. Reduction to the Nonlinear Schrödinger Equation

It is useful in this discussion to redefine a_1 and a_2, setting $\hat{a}_1 = a_1/|\beta|^{1/2}$ and $\hat{a}_2 = a_2 \exp(-i\beta\xi)$, so that Eqs. (26) become

$$i\frac{\partial \hat{a}_1}{\partial \xi} - \frac{r}{2}\frac{\partial^2 \hat{a}_1}{\partial s^2} + \hat{a}_1{}^* \hat{a}_2 = 0\,, \tag{36a}$$

$$i\frac{\partial \hat{a}_2}{\partial \xi} - \beta \hat{a}_2 - i\delta\frac{\partial \hat{a}_2}{\partial s} - \frac{\alpha}{2}\frac{\partial^2 \hat{a}_2}{\partial s^2} + |\beta|\hat{a}_1{}^2 = 0\,. \tag{36b}$$

If we assume that $|\beta|$ is large, then we may use Eqs. (36) to eliminate $\hat{a}_2$ in favor of $\hat{a}_1$ by repeated substitution. At lowest order, we obtain

$$\hat{a}_2 = \frac{\beta}{|\beta|}\,\hat{a}_1{}^2\,. \tag{37}$$

At the next order we obtain

$$\hat{a}_2 = \frac{\beta}{|\beta|}\,\hat{a}_1{}^2 + \frac{1}{|\beta|}\left(-2i\delta\hat{a}_1\frac{\partial \hat{a}_1}{\partial s} + r\hat{a}_1\frac{\partial^2 \hat{a}_1}{\partial s^2} - \frac{\alpha}{2}\frac{\partial^2 \hat{a}_1{}^2}{\partial s^2} - \frac{2\beta}{|\beta|}\,|\hat{a}_1|^2\hat{a}_1{}^2\right). \tag{38}$$

This procedure can be continued indefinitely, yielding increasingly high powers of $1/|\beta|$. Substitution of Eq. (38) into Eq. (36a) yields

$$i\frac{\partial \hat{a}_1}{\partial \xi} - \frac{r}{2}\frac{\partial^2 \hat{a}_1}{\partial s^2} + \frac{\beta}{|\beta|}\,|\hat{a}_1|^2\hat{a}_1 = O(1/|\beta|)\,, \tag{39}$$

where $O(1/|\beta|)$ indicates terms that are of order $1/|\beta|$ and vanish as $|\beta| \to \infty$. Thus we have demonstrated that our basic equations, Eqs. (26), reduce to the nonlinear Schrödinger equation in the limit $|\beta| \to \infty$, corresponding physically to a large phase mismatch. We note, however, that the effective strength of the nonlinearity is inversely proportional to $|\beta|$, and hence the intensity of the electric field must increase proportionally to $|\beta|$ if the nonlinearity is to balance the dispersion or the diffraction.

Thus our basic equations, Eqs. (26), may be viewed as a deformation of the nonlinear Schrödinger equation. To demonstrate that it is a Hamiltonian deformation, we must show that Eq. (26) has both a Hamiltonian and a Poisson bracket. It is a bit easier to work directly with Eq. (36), rather than Eqs. (26). In this case, the Hamiltonian H is given by

$$H = \int_{-\infty}^{\infty} \mathrm{d}s\left(-\frac{\beta}{2|\beta|}\,\hat{a}_2\hat{a}_2{}^* - i\frac{\delta}{4|\beta|}\,\hat{a}_2{}^*\frac{\partial \hat{a}_2}{\partial s} + i\frac{\delta}{4|\beta|}\,\hat{a}_2\frac{\partial \hat{a}_2{}^*}{\partial s} + \frac{r}{2}\frac{\partial \hat{a}_1}{\partial s}\frac{\partial \hat{a}_1{}^*}{\partial s} + \frac{\alpha}{4|\beta|}\frac{\partial \hat{a}_2}{\partial s}\frac{\partial \hat{a}_2{}^*}{\partial s} + \frac{1}{2}\,\hat{a}_1{}^2\hat{a}_2{}^* + \frac{1}{2}\,\hat{a}_1{}^{*2}\hat{a}_2\right), \tag{40}$$

and the Poisson bracket $[F, G]$ is given by

$$[F, G] = i\int_{-\infty}^{\infty} \mathrm{d}s\left(\frac{\delta F}{\delta \hat{a}_1}\frac{\delta G}{\delta \hat{a}_1{}^*} - \frac{\delta F}{\delta \hat{a}_1{}^*}\frac{\delta G}{\delta \hat{a}_1} + 2|\beta|\frac{\delta F}{\delta \hat{a}_2}\frac{\delta G}{\delta \hat{a}_2{}^*} - 2|\beta|\frac{\delta F}{\delta \hat{a}_2{}^*}\frac{\delta G}{\delta \hat{a}_2}\right). \tag{41}$$

Here F and G are functionals of $\hat{a}_1$, $\hat{a}_1{}^*$, $\hat{a}_2$, and $\hat{a}_2{}^*$, while $\delta X/\delta x$ indicates the functional derivative. One can verify Eqs. (40) and (41) by showing that $\partial\hat{a}_1/\partial\xi = [\hat{a}_1, H]$, $\partial\hat{a}_1{}^*/\partial\xi = [\hat{a}_1{}^*, H]$, $\partial\hat{a}_2/\partial\xi = [\hat{a}_2, H]$, and $\partial\hat{a}_2{}^*/\partial\xi =$

$[\hat{a}_2^*, H]$. The reader who is unfamiliar with functional calculus may find a brief description of the functional derivative and its application to infinite-dimensional Hamiltonian systems in the appendix of Ref. 7.

The nonlinear Schrödinger equation as given in Eq. (39) will have bright-soliton solutions if $r\beta < 0$ and dark-soliton solutions if $r\beta > 0$. The signs of two different material properties determine whether bright or dark solitons appear, which gives the experimentalist considerable flexibility in tailoring a system that can produce either bright or dark solitons.

The single bright-soliton solution to the nonlinear Schrödinger equation, Eq. (39), can be written in the form

$$\hat{a}_1 = A\,\mathrm{sech}[A(s - r\omega\xi - s_0)] \times \exp[-i(r/2)(A^2 - \omega^2)\xi - i\omega s + i\phi_0]. \tag{42}$$

This solution has four arbitrary parameters: A, ω, s_0, and ϕ_0. Two of the parameters, s_0 and ϕ_0, are arbitrary as a result of the general symmetry properties of the nonlinear Schrödinger equation; if $f(s, \xi)$ is any solution of this equation, then so is $f(s - s_0, \xi)\exp(i\phi_0)$. Our basic equations, Eqs. (26), have an analogous symmetry property. If $a_1 = f_1(s, \xi)$ and $a_2 = f_2(s, \xi)$ is a solution of the basic equations, then so is $a_1 = f_1(s - s_0, \xi)\exp(i\phi_0)$ and $a_2 = f_2(s - s_0, \xi)\exp(i\phi_0)$. Physically these two parameters correspond to the arbitrariness of the time origin and the complex phase origin. The two other parameters A and ω correspond physically to the amplitude/width of and the central frequency of the soliton and are properties of the soliton solution alone. When we study the solitary-wave solutions of our basic equations, we find that they have no arbitrary parameters or at most one rather than two. This result is significant because the robustness hypothesis, together with the fact that our basic equations are a Hamiltonian deformation of the nonlinear Schrödinger equation, implies that there should be a two-parameter family of solitonlike solutions, at least when $|\beta|$ is large. Hence, according to this hypothesis, the family of solitary-wave solutions is not broad enough to encompass the totality of solitonlike solutions. We return to this point in Section 4.

3. SOLITARY-WAVE SOLUTIONS

We will be presenting solutions of our basic equations, Eqs. (26), which have the form

$$\begin{aligned} a_1 &= [A_1 + iB_1\,\mathrm{sech}^{l_1}(ws - v\xi)\tanh(ws - v\xi) \\ &\quad + C_1\,\mathrm{sech}^{m_1}(ws - v\xi)]\exp(i\kappa_1\xi - i\omega_1 s), \\ a_2 &= [A_2 + iB_2\,\mathrm{sech}^{l_2}(ws - v\xi)\tanh(ws - v\xi) \\ &\quad + C_2\,\mathrm{sech}^{m_2}(ws - v\xi)]\exp(i\kappa_2\xi - i\omega_2 s). \end{aligned} \tag{43}$$

The quantities $A_1, A_2, B_1, B_2, C_1, C_2, \kappa_1, \kappa_2, \omega_1, \omega_2, w$ and v are all parameters whose allowed values are determined by substitution into the basic equations. The exponents l_1 and l_2 are nonnegative integers, and m_1 and m_2 are positive integers that must also be determined by substitution into Eqs. (26). This ansatz is sufficiently general to include all the previously discovered solitary-wave solutions as well as some new ones; however, it does not necessarily exhaust all solitary-wave solutions or, indeed, even all solitary-wave solutions that are composed of the hyperbolic secant and tangent. A more general ansatz that could be considered is

$$\begin{aligned} a_1 &= \sum_{j=0}^{m_1} [A_{1,j}\,\mathrm{sech}^j(ws - v\xi) + B_{1,j}\,\mathrm{sech}^j(ws - v\xi) \\ &\quad \times \tanh(ws - v\xi)]\exp(i\kappa_1\xi - i\omega_1 s), \\ a_2 &= \sum_{j=0}^{m_2} [A_{2,j}\,\mathrm{sech}^j(ws - v\xi) + B_{2,j}\,\mathrm{sech}^j(ws - v\xi) \\ &\quad \times \tanh(ws - v\xi)]\exp(i\kappa_2\xi - i\omega_2 s). \end{aligned} \tag{44}$$

However, there are so many different cases that must be considered separately with this ansatz that we were not able to explore it completely.

We now turn to considering specific cases of the ansatz given in Eqs. (43).

A. $r = 0, a \neq 0$

In our definition of r in Section 2 we set $r = \pm 1$. Here we temporarily set $r = 0$ to explore the case of no dispersion or diffraction in the first harmonic when there is dispersion or diffraction in the second harmonic. The highest powers of the hyperbolic functions yielded by the linear and the nonlinear contributions must balance. There are two possibilities to consider: If $B_1B_2 \neq 0$, we find that $l_1 \geq l_1 + m_2$, and if $B_1B_2 = 0$, we find that $m_1 \geq m_1 + m_2$. In either case there is no solution in which $m_2 \geq 1$, so that a solution of the form of Eqs. (43) is not possible. This result justifies our use of the dispersion or the diffraction of the first harmonic in defining a normalized length.

B. $r \neq 0, a = 0$

There are several different cases to be considered. In all cases but one, we find that

$$\omega_1 = -r\delta, \qquad \omega_2 = -2r\delta, \qquad v = -\delta w, \tag{45}$$

so we do not repeat these results. In our analysis we first set

$$l_1 = l_2 = 0 \tag{46}$$

and then return below to the general case.

Matching once again the highest powers, we find that

$$m_1 + 2 = m_1 + m_2, \qquad 2 = 2m_1, \tag{47}$$

which yields the solution

$$m_1 = 1, \qquad m_2 = 2. \tag{48}$$

Our ansatz thus reduces to the form

$$\begin{aligned} a_1 &= [A_1 + iB_1\tanh(ws - v\xi) + C_1\,\mathrm{sech}(ws - v\xi)] \\ &\quad \times \exp(i\kappa_1\xi - i\omega_1 s), \\ a_2 &= [A_2 + iB_2\tanh(ws - v\xi) + C_2\,\mathrm{sech}^2(ws - v\xi)] \\ &\quad \times \exp(i\kappa_2\xi - i\omega_2 s). \end{aligned} \tag{49}$$

Using this ansatz, we first find after equating the arguments of the exponential function that

$$\omega_2 = 2\omega_1, \qquad \kappa_2 = 2\kappa_1 + \beta. \tag{50}$$

All solutions of the form given by Eqs. (43) must satisfy this relationship, so we do not repeat it below. Equating equal powers of the hyperbolic functions, we immediately arrive at the following relations:

$$A_1C_1 = 0, \qquad B_1C_1 = 0; \tag{51}$$

hence the solutions that we are considering immediately subdivide into two cases: $A_1 = B_1 = 0$ and $C_1 = 0$. After some further algebraic manipulation to be described shortly, we find that the second case further subdivides into two cases: $A_2 = 0$ and $B_2 = 0$. We now treat these three cases in turn.

1. $l_1 = l_2 = 0$, $A_1 = B_1 = 0$

After substitution into the basic equations, we obtain the relations $A_2 = B_2 = 0$, $C_2 = -rw^2$, and $C_1 = [-rw^2(k_2 + \delta\omega_2)]^{1/2}$, where $-r(\kappa_2 + \delta\omega_2) > 0$ is required for a nontrivial solution. Both signs of the square root are allowed—a result that we do not mark explicitly here or hereafter. All the quantities can be expressed in terms of one arbitrary parameter, which we may choose to be w. Explicitly, we find

$$C_1 = [w^2(\delta^2 + w^2 - r\beta)]^{1/2}, \qquad C_2 = -rw^2,$$
$$\kappa_1 = (r/2)(\delta^2 - w^2). \tag{52}$$

The requirement for solutions to exist is

$$r\beta < \delta^2 + w^2, \tag{53}$$

which in the limit of large $|\beta|$ becomes the requirement $r\beta < 0$, which we previously found in the reduction to the nonlinear Schrödinger equation. These solutions correspond physically to bright solitary waves.

2. $l_1 = l_2 = 0$, $B_2 = C_1 = 0$

Our ansatz becomes

$$a_1 = [A_1 + iB_1 \tanh(ws - v\xi)]\exp(i\kappa_1\xi - i\omega_1 s),$$
$$a_2 = [A_2 + iB_2 \tanh(ws - v\xi) + C_2 \operatorname{sech}^2(ws - v\xi)] \times \exp(i\kappa_2\xi - i\omega_2 s); \tag{54}$$

we impose the condition $B_2 = 0$ below. After substitution into the basic equations, we find $C_2 = rw^2$ and $B_1 = [rw^2(\kappa_2 + \delta\omega_2)]^{1/2}$. We further obtain the relations

$$A_2 = \frac{A_1{}^2 - B_1{}^2}{\kappa_2 + \delta\omega_2}, \qquad B_2 = \frac{2A_1B_1}{\kappa_2 + \delta\omega_2}, \tag{55}$$

which in turn imply that

$$\left[\left(\frac{r}{2}\omega_1{}^2 - \kappa_1\right)(\delta\omega_2 + \kappa_2) + 2B_1{}^2\right]B_1 = 0,$$
$$\left[\left(\frac{r}{2}\omega_1{}^2 - \kappa_1\right)(\delta\omega_2 + \kappa_2) + 2A_1{}^2\right]A_1 = 0. \tag{56}$$

Now, if $A_1 \neq 0$, then $A_1 = \pm B_1$ and

$$B_2 = \pm\frac{2B_1{}^2}{\kappa_2 + \delta\omega_2} = \pm 2C_2, \tag{57}$$

but if $A_1 = 0$, then it follows from Eqs. (55) that $B_2 = 0$ and $A_2 = -C_2$. As above, all quantities can be expressed in terms of one parameter, which we choose to be w. Explicitly, we find

$$B_1 = [w^2(2w^2 - \delta^2 + r\beta)]^{1/2}, \qquad A_2 = -C_2 = -rw^2,$$
$$\kappa_1 = (r/2)(\delta^2 + 2w^2). \tag{58}$$

The requirement for solutions to exist is

$$r\beta > 2w^2 - \delta^2. \tag{59}$$

These solutions correspond physically to dark solitary waves, and they exist in the limit $r\beta \to \infty$, rather than in the limit $r\beta \to -\infty$ as is the case for bright solitary waves.

3. $l_1 = l_2 = 0$, $A_2 = C_1 = 0$

The third case corresponds to the condition give in Eq. (57). As before, we may write all quantities in terms of w. We find as in the previous cases that $v = -\delta w$, but instead of obtaining $\omega_1 = -r\delta$ we obtain $\omega_1 = -r\delta \mp w$. The other quantities are given by

$$B_2 = \pm 2C_2 = \pm 2rw^2,$$
$$A_1 = \pm B_1 = \pm[w^2(5w^2 - \delta^2 + r\beta)]^{1/2},$$
$$\kappa_1 = \frac{r}{2}\delta^2 \pm \delta w + \frac{5r}{2}w^2, \tag{60}$$

with the condition $5w^2 - \delta^2 + r\beta > 0$ for a solution to exist. Physically this solution corresponds to neither dark nor bright solitons but instead is a novel phase wave solution. The intensity profile of the first harmonic dips at the origin of the argument of the hyperbolic functions, and its phase undergoes a 90° phase rotation. The second harmonic goes through a somewhat complex intensity variation, and its phase undergoes a 180° rotation.

4. l_1 and l_2 Arbitrary

We now turn to the general case, in which we make no assumptions initially about l_1 and l_2. We obtain the relations

$$l_1 + 2 = \max(l_1 + m_2, l_2 + m_1), \qquad l_2 = l_1 + m_1,$$
$$m_1 + 2 = \max(m_1 + m_2, l_1 + l_2 + 2),$$
$$m_2 = \max(2m_1, 2l_1 + 2), \tag{61}$$

which hold when $B_1C_1 \neq 0$. If $B_1C_1 = 0$, then this case reduces to one of the three cases already considered. Equations (61) can be satisfied only when $m_1 = l_2 = 1$ and $m_2 = 2$. In this case we find immediately that we must set $A_1 = 0$ to have a nontrivial solution. After some further algebra, we can show that $B_2{}^2 = -C_2{}^2$, so that there is no solution with both B_2 and C_2 real. Investigating the possibility that either B_2 or C_2 is imaginary, we find that the equations become overdetermined and, once again, there is no solution.

C. $r \neq 0$, $a \neq 0$

We find the following three possibilities after equating the highest powers of the hyperbolic functions:

$$m_1 = m_2 = 2, \qquad l_1 = l_2 = 0;$$
$$m_1 = m_2 = 2, \qquad l_1 = l_2 = 1;$$
$$m_1 = 1, m_2 = 2, \qquad l_1 = 1, l_2 = 0. \tag{62}$$

Concentrating on the first possibility, we find that unless $B_1 = B_2 = 0$ our equations are overdetermined. We first set $A_1 = A_2 = 0$ and consider the solutions in detail. There are two special values of α that must be considered separately: $\alpha = r/2$ and $\alpha = 2r$.

1. $m_1 = m_2 = 2$, $l_1 = l_2 = 0$, $A_1 = A_2 = 0$

In this case our ansatz reduces to

$$a_1 = C_1 \operatorname{sech}^2(ws - v\xi)\exp(i\kappa_1\xi - i\omega_1 s),$$
$$a_2 = C_2 \operatorname{sech}^2(ws - v\xi)\exp(i\kappa_2\xi - i\omega_2 s), \qquad (63)$$

which is a bright solitary-wave solution.

a. $\alpha \neq r/2$, $\alpha \neq 2r$: With $\alpha \neq r/2$ and $\alpha \neq 2r$, we find for our parameters

$$\omega_1 = \frac{\delta}{2\alpha - r}, \qquad v = \frac{r\delta w}{2\alpha - r},$$
$$C_1 = 3(\alpha r)^{1/2} w^2, \qquad C_2 = -3rw^2,$$
$$\kappa_1 = (r/2)\omega_1{}^2 - 2rw^2, \qquad (64)$$

with

$$w^2 = \frac{1}{4r - 2\alpha}\left(\beta + \frac{\delta^2}{2\alpha - r}\right). \qquad (65)$$

The solution constraints are that $\alpha r > 0$ and $w^2 > 0$. When $\alpha < 2r$, the allowed values of β extend to $+\infty$, whereas when $\alpha > 2r$, the allowed values of β extend to $-\infty$. All the solitary-wave parameters, w, v, κ_1, κ_2, ω_1, and ω_2, are fixed once the equation parameters δ, β, r, and α are known, in contrast to the solitary-wave parameters when $\alpha = 0$, in which case we find one free parameter, which we take to be w.

b. $\alpha = r/2$: In this case, we must have $\delta = 0$ to have a solution. We then find

$$v = r\omega_1(r\beta/3)^{1/2}, \qquad C_1 = r\beta(1/2)^{1/2}, \qquad C_2 = -\beta,$$
$$w^2 = r\beta/3, \qquad \kappa_1 = r\omega_1{}^2/2 - 2\beta/3, \qquad (66)$$

where ω_1 is arbitrary. From a physical standpoint, we find that when $\alpha \neq r/2$, ω_1 and ω_2 are set by the physical requirement that the group velocities of the two harmonics must be the same. The frequencies shift so as to equalize the group velocities. When $\alpha = r/2$, it is no longer possible to shift the relative group velocities by changing the frequencies. Hence the group velocities must be the same in the frame of reference of the basic equations, so that $\delta = 0$ is required but at the same time the frequencies are arbitrary. This case is of importance because, as follows from Eqs. (35), $\alpha \simeq r/2$ for spatial solitary waves.

c. $\alpha = 2r$: In this case, we must have $\beta = -r\delta^2/3$ to have a solution. With this condition fulfilled, we find that

$$\omega_1 = r\delta/3, \qquad v = \delta w/3,$$
$$C_1 = 3\sqrt{2}\,w^2, \qquad C_2 = -3rw^2,$$
$$\kappa_1 = \frac{r}{18}\delta^2 - 2rw^2, \qquad (67)$$

with w arbitrary. The solution constraint is $w^2 > 0$. From a physical standpoint we find that when $\alpha \neq 2r$ the width w is set by the balance among nonlinearity, dispersion or diffraction, and phase mismatch. When $\alpha = 2r$, this balance is unaffected by the width. Thus, β must be chosen so that this balance is achieved, but w then becomes arbitrary.

We now turn to consideration of other cases in which $\alpha \neq 0$ and $r \neq 0$. We do not specifically consider special values of α in these other cases, but it should be borne in mind that they exist for the same physical reasons that they exist for bright solitary waves.

2. $m_1 = m_2 = 2$, $l_1 = l_2 = 0$, $A_1 \neq 0$, $A_2 \neq 0$

In this case we obtain relations

$$\omega_1 = \frac{\delta}{2\alpha - r}, \qquad v = \frac{r\delta w}{2\alpha - r},$$
$$C_1 = 3(\alpha r)^{1/2} w^2, \qquad C_2 = -3rw^2,$$
$$A_1 = -2(\alpha r)^{1/2} w^2, \qquad A_2 = 2rw^2,$$
$$\kappa_1 = \frac{r}{2}\frac{\delta^2}{(2\alpha - r)^2} + 2rw^2, \qquad (68)$$

with

$$w^2 = \frac{1}{2\alpha - 4r}\left(\beta + \frac{\delta^2}{2\alpha - r}\right). \qquad (69)$$

The solution constraints are $\alpha r > 0$ and $w^2 > 0$. We find that when $\alpha < 2r$ the allowed values of β extend to $-\infty$, whereas when $\alpha > 2r$ the allowed values of β extend to $+\infty$, which is the opposite of bright solitary waves. These solutions are neither bright nor dark. Like bright solitary waves they have a fixed phase, and like dark solitary waves they extend to $s = \pm\infty$. Their intensity variation is somewhat complicated.

3. $m_1 = m_2 = 2$, $l_1 = l_2 = 1$

In this case, if we let $B_1 = B_2 = 0$, we find the two solutions previously reported. Our equations are overdetermined if we let $B_1 \neq 0$ and $B_2 \neq 0$. If we let $B_1 = 0$ and $B_2 \neq 0$, we find that no solution exists. Finally, if we let $B_1 \neq 0$ and $B_2 = 0$, we find one new solution. This last choice entrains $A_1 = A_2 = 0$, and the solution that we finally obtain has the form

$$\omega_1 = \frac{\delta}{2\alpha - r}, \qquad v = \frac{r\delta w}{2\alpha - r},$$
$$B_1 = 3(-\alpha r)^{1/2} w^2, \qquad C_2 = 3rw^2,$$
$$\kappa_1 = \frac{r}{2}\omega_1{}^2 - \frac{r}{2}w^2, \qquad (70)$$

with

$$w^2 = \frac{1}{\alpha + r}\left(\beta + \frac{\delta^2}{2\alpha - r}\right) \qquad (71)$$

and $C_1 = 0$. The solution constraints are $\alpha r < 0$ and $w^2 > 0$. In contrast to the previous two solutions considered, a special value of β is required when $\alpha = -r$, not when $\alpha = 2r$, for this solution to exist. In common with that for dark solitary waves, this solution has a phase flip, and the first harmonic has a zero crossing at the origin of the arguments of the hyperbolic functions. In common with that for bright solitary waves, this solution tends to zero as $s \to \pm\infty$. Thus the features that it shares with bright and dark solitary waves are complementary to the features that the solution of Subsection 3.C.2 shared.

4. $m_1 = 1$, $m_2 = 2$, $l_1 = 1$, $l_2 = 0$

We find that there is no solution in this case unless $B_2 = C_1 = 0$, and then this case reduces to the case just considered in Subsection 3.C.3.

D. Transformation to Zero-Velocity Solutions

When $\alpha \neq r/2$, it is possible to transform our equations and solutions so that they correspond to zero-velocity solutions. We then find that the solutions described in this paper (Subsections 3.B.1, 3.B.2, 3.C.1., and 3.C.3), which correspond to the solutions found earlier by Werner and Drummond,[16,17] reduce to exactly the solutions of Werner and Drummond. Writing

$$a_1 = f_1(ws - v\xi)\exp(i\kappa_1\xi - i\omega_1 s),$$
$$a_2 = f_2(ws - v\xi)\exp(i\kappa_2\xi - i\omega_2 s), \tag{72}$$

we have found that all our solutions except one (Subsection 3.B.3) obey the relations

$$\omega_1 = \frac{\delta}{2\alpha - r}, \qquad \omega_2 = \frac{\delta}{2\alpha - r}, \qquad v = \frac{r\delta w}{2\alpha - r}, \tag{73}$$

even when $\alpha = 0$. This result suggests that we transform our equations so that

$$s' = s - \frac{r\delta}{2\alpha - r}\xi, \qquad \xi' = \xi,$$
$$\beta' = \beta + \frac{\delta^2}{2\alpha - r}, \qquad \delta' = 0, \tag{74}$$

and our solutions so that

$$\kappa_1' = \kappa_1 - \frac{r}{2}\frac{\delta^2}{(2\alpha - r)^2}, \qquad \kappa_2' = \kappa_2 + \frac{2(\alpha - r)\delta^2}{(2\alpha - r)^2}, \tag{75}$$

with

$$a_1' = f_1(ws')\exp(i\kappa_1'\xi'), \qquad a_2' = f_2(ws')\exp(i\kappa_2'\xi'). \tag{76}$$

We then find that if a_1 and a_2 satisfy the basic equations, Eqs. (26), then a_1' and a_2' satisfy the transformed equations. These equations are identical in form to the basic equations except that $\delta \to 0$. The solution given in Subsection 3.B.3 can be treated similarly.

4. DISCUSSION

In this paper we have discussed solutions to the basic equations that describe the $\chi^{(2)}:\chi^{(2)}$ cascaded nonlinearity in the presence of dispersion or diffraction. We began by deriving the basic equations in the plane-wave approximation. We then searched for all the solitary-wave solutions that have the form given in Eq. (43). This form includes all the previously discovered solitary-wave solutions discussed by Werner and Drummond[16,17] (our Subsections 3.B.1, 3.B.2, 3.C.1, and 3.C.3) as well as new ones (Subsections 3.B.3 and 3.C.2). We found these solutions by assuming an arbitrary group-velocity difference between the first and the second harmonics, but we also showed that one can transform these solutions into zero-group-velocity solutions by making an appropriate transformation unless $\alpha = r/2$. This exception is important because to good approximation it applies to spatial solitary waves.

We found a rich structure of solutions. In addition to bright solitary-wave solutions, which appear at all values of α except $\alpha = \infty$, we found dark solitary-wave solutions at $\alpha = 0$ and other exotic solutions with a variety of phase and intensity variations. Despite this richness, all these solutions are zero-parameter families; i.e, the width and the frequency of the solitary wave are fixed except at certain special values of α, at which these solutions become one-parameter families. By contrast, the soliton solutions to the nonlinear Schrödinger equation are a two-parameter family in which both the width and the frequency can vary. It has been shown by ourselves and others[14–18] that when $|\beta|$ is large our basic equations tend toward the nonlinear Schrödinger equation. It follows from the robustness hypothesis[7] that when $|\beta|$ is large there should be a two-parameter family of solitonlike solutions, at least in the case of bright solitons. While we have not proved that no two-parameter family of exact solitary-wave solutions exists, we have proved that any such family cannot have the form given in Eqs. (43). This family would have to have the more complicated form given in Eqs. (44) or a different functional form altogether and at the same time conform to the usual inverse hyperbolic secant solutions in the large-β limit. These solutions would be exotic, and their existence seems improbable. Thus we anticipate that the exact solitary-wave solutions capture only an infinitesimal fraction of the solitonlike behavior that is inherent in our basic equations. A detailed numerical investigation of this issue is being carried out.

To observe bright solitary waves either temporally or spatially, one must first have materials with sufficiently strong nonlinearity so that light waves undergo a 2π phase rotation after propagating a few millimeters at reasonable power levels, since reasonable lengths for typical waveguides are of the order of a few centimeters. There are several materials that fulfill this condition for the cascaded $\chi^{(2)}:\chi^{(2)}$ nonlinearity, which has led to its observation in contexts in which the dispersion or diffraction are unimportant.[1–4] A more serious issue is obtaining sufficient dispersion or diffraction to generate solitons. The most highly dispersive optical materials are organics that can have dispersions as high as 1 ps^2/m,[23] implying that a 100-fs pulse has a dispersive scale length of the order of 1 cm. This case is marginal at best. The dispersive scale length is barely long enough to permit experiments to be done in waveguides of reasonable length. At the same time, with such a short pulse duration the slowly varying envelope approximation may fail, and other nonlinear effects, such as the Raman effect, may overwhelm the soliton effects that we wish to observe. Moreover, losses in these materials and fabrication difficulties make this experiment difficult at present.[23] Less dispersive materials lead to even longer scale lengths and similar difficulties. A more promising avenue for the near future is to study spatial solitons in slab waveguides. The diffractive scale length L is of the order of ny^2/λ_0, where n is the waveguide's index of refraction, λ_0 is the external wavelength, and y is the spot size. Assuming that $n \simeq 1.5$ and $\lambda_0 \simeq 1.5\ \mu\text{m}$, we find that $y \simeq 0.03$ mm implies that $L \simeq 1$ mm, and this spot size is easily achievable. As long as the directions of propagation of both beams are sufficiently close, it should be possible to generate spatial solitary waves.

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