Consensus Stabilizability and Exact Consensus Controllability of Multi-agent Linear Systems

M. ISABEL GARCÍA-PLANAS
Universitat Politècnica de Catalunya
Departament de Matèmatiques
Minería 1, Esc. C, 1-3, 08038 Barcelona
SPAIN
maria.isabel.garcia@upc.edu

Abstract: A goal in engineering systems is to try to control them. Control theory offers mathematical tools for steering engineered systems towards a desired state. Stabilizability and controllability can be studied under different points of view, in particular, we focus on measure of controllability in the sense of the minimum set of controls that need for to steer the multiagent system toward any desired state. In this paper, we study the consensus stabilizability and exact consensus controllability of multi-agent linear systems, in which all agents have a same linear dynamic mode that can be in any order.

Key–Words: Multi-agent systems, consensus, stability, controllability, consensus stabilizability and exact consensus controllability.

1 Introduction

In the last years, the study of dynamic control multi-agents systems have attracted considerable interest, because they arise in a great number of engineering situations as for example in distributed control and coordination of networks consisting of multiple autonomous agents. There are many publications as for example ([5], [9], [13], [16], [18]). It is due to the multi-agents appear in different fields as for example in consensus problem of communication networks ([13]), or formation control of mobile robots ([3]).

The consensus problem has been studied under different points of view, for example Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [16], analyze the case of multiagent systems in which all agents have an identical stable linear dynamics system, M.I. García-Planas in [5], generalize this result to the case where the dynamic of the agents are controllable.

For the stability analysis problem, the first question is whether the multagent system is stable when there is no restriction on the topology. We will call this problem as consensus stability analysis.

Controllability is a fundamental topic in dynamic systems and it is studied under different approaches (see [1], [2], [4], [6], [8], for example). Given a linear system $\dot{x} = Ax$, there are many possible control matrices B making the system $\dot{x} = Ax + Bu$ controllable. The goal is to find the set of all possible matrices B, having the minimum number of columns corresponding to the minimum number $n_D(A)$ of independent controllers required to control the whole network. This minimum number is called exact controllability, that in a more formal manner is defined as follows.

Definition 1 *Let A be a matrix. The exact controllability $n_D(A)$ is the minimum of the rank of all possible matrices B making the system $\dot{x} = Ax + Bu$ controllable.*

$$n_D(A) = \min\{\operatorname{rank} B, \forall B \in M_{n\times i}\, 1 \leq i \leq n \mid (A,B) \text{ controllable}\}.$$

Z.Z. Yuan, C. Zhao, W.X.Wang, Z.R. Di, Y.C. Lai, in [19] and [20] introduce the concept of exact controllability for complex networks and give a characterization of systems verifying this condition, but they do not describe the possible matrices B making the system controllable. García-Planas in [10] characterize all possible matrices B for linear dynamical systems.

In this paper, we investigate the consensus stabilizability and exact controllability of a class of multi-agent systems consisting of k agents with dynamics

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= Ax^k + Bu^k \end{aligned}$$

where $A \in M_n(\mathbb{C})$, and B an unknown matrix having n rows and an indeterminate number $1 \leq \ell \leq n$ of columns.

The paper is structured as follows: in Section 2, we give the prelimnaries. In section 3, the consensus problem is introduced. In section 4, the concept of consensus stabilizable is introduced and a criterium that lets we know whether a multagent system is not consensus stabilizable. In section 5, exact Consensus Controllability is analyzed. Finally a list of references is presented.

2 Preliminaries

For this study, we need to introduce some basic concepts on Graph theory and matritial algebra.

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order k with the set of vertices $\mathcal{V} = \{1, \dots, k\}$ and the set of edges $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i, j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix $G = (g_{ij})$ called (unweighted) adjacency matrix defined as follows $g_{ii} = 0$, $g_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

In a more general case we can consider that a weighted adjacency matrix is $G = (g_{ij})$ with $g_{ii} = 0$, $g_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise).

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 2 *i) If the graph is undirected then the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t = \mathcal{D}$.*

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $\mathbf{1}_k = (1, \dots, 1)^t$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

For more details about graph theory see (D. West, 2007).

With respcet Kronecker product, remember that $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$ the Kronecker product is defined as follows.

Definition 3 *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A \otimes B$, is the matrix*

$$A\otimes B = \begin{pmatrix} a^1_1 B & a^1_2 B & \dots & a^1_m B \\ a^2_1 B & a^2_2 B & \dots & a^2_m B \\ \vdots & \vdots & & \vdots \\ a^n_1 B & a^n_2 B & \dots & a^n_m B \end{pmatrix} \in M_{np\times mq}(\mathbb{C})$$

Among the properties that verifies the product of Kronecker we will make use of the following

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
4) If $A \in Gl(n;\mathbb{C})$ and $B \in Gl(p;\mathbb{C})$), then $A \otimes B \in Gl(np;\mathbb{C})$) and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
5) If the products AC and BD are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

See [12] for more information and properties.

Given a square matrix $A \in M_n(\mathbb{C})$, it can be reduced to a canonical reduced form (Jordan form):

$$J = \begin{pmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_r) \end{pmatrix}, \; J(\lambda_i) = \begin{pmatrix} J_1(\lambda_i) & & \\ & \ddots & \\ & & J_{n_i}(\lambda_i) \end{pmatrix},$$

$$J_j(\lambda_i) = \begin{pmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_i \end{pmatrix}. \quad (1)$$

See [7] for more information and properties.

3 Consensus

The consensus problem can be introduced as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. It is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

The dynamic of each agent defining the system considered, is given by the following manner.

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= Ax^k + Bu^k \end{aligned} \tag{2}$$

$x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^\ell$, $1 \leq i \leq k$. Where matrices $A \in M_n(\mathbb{R})$ and $B \in M_{n\times\ell}(\mathbb{R})$, $1 \leq \ell \leq n$.

The communication topology among agents is defined by means the undirected graph $\mathcal{G}$ with

i) Vertex set: $\mathcal{V} = \{1, \dots, k\}$

ii) Edge set: $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

an in a more specific form, we have the following definition.

Definition 4 *Consider the multi-agent linear system 2. We say that the consensus is achieved using local information if there exists a state feedback*

$$u^i = K_i \sum_{j\in\mathcal{N}_i} (x^i - x^j),\ 1 \leq i \leq k$$

such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0,\ 1 \leq i, j \leq k.$$

$z^i = \sum_{j\in\mathcal{N}_i}(x^i - x^j),\ 1 \leq i \leq k.$

$$\begin{aligned} \dot{\mathcal{X}} &= (I_k \otimes A)\mathcal{X} + (I_k \otimes B)\mathcal{U} \\ \mathcal{Z} &= (\mathcal{L} \otimes I)\mathcal{X} \\ \mathcal{U} &= (I_k \otimes K)\mathcal{Z} \end{aligned}$$

Then, and taking into account that

$$\begin{aligned} &(I_k \otimes B)(I_k \otimes K)(\mathcal{L} \otimes I_n)\mathcal{X} = \\ &(\mathcal{L} \otimes BK)\mathcal{X} = (\mathcal{L} \otimes B)(I_k \otimes K)\mathcal{X} \end{aligned}$$

The system is equivalent to

$$\begin{aligned} \dot{\mathcal{X}} &= (I_k \otimes A)\mathcal{X} + (\mathcal{L} \otimes B)\bar{\mathcal{U}} \\ \bar{\mathcal{U}} &= (I_k \otimes K)\mathcal{X} \end{aligned} \tag{3}$$

4 Consensus Stabilizability

The system 2 is controllable if and only if the system

$$\dot{\mathcal{X}} = (I_k \otimes A)\mathcal{X} + (\mathcal{L} \otimes B)\bar{\mathcal{U}} \tag{4}$$

is controllable.

The controllability character can be analyzed using the Hautus criteria:

Proposition 5 *The system is controllable if and only if*

$$\text{rank}\begin{pmatrix} sI_{nk} - (I_k \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = kn$$

The system $\dot{\mathcal{X}} = (I_k \otimes A)\mathcal{X}$, is stable (or asymptotically stable) if all real parts of the eigenvalues of $(I_k \otimes A)$ are negative.

The system 4 is stabilizable if and only if there exist a feedback F such that the system

$$\dot{\mathcal{X}} = ((I_k \otimes A) + (\mathcal{L} \otimes B)F)\mathcal{X} \tag{5}$$

is stable

Using Hautus criteria:

Proposition 6 *The system 5 is stable if and only if*

$$\begin{aligned} &\text{rank}\begin{pmatrix} sI_{nk} - (I_k \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = kn, \\ &\forall s \in \mathbb{C}^+ = \{s \in \mathbb{C} \mid Re\,(s) \geq 0\},\ s \text{ finite} \end{aligned}$$

Definition 7 *The system is called consensus stabilizable if and only if the system 4 is stabilizable under feedback in the form*

$$F = (I_k \otimes K)$$

Suppose now, that the Laplacian matrix $\mathcal{L}$ is diagonalizable with $\lambda_1, \dots, \lambda_k$ as eigenvalues then, we have the following proposition.

Proposition 8 *A necessary and sufficient condition for existence of K in such a way the system 4 be stable is that the matrices $A + \lambda_i BK$ are Hurwitz.*

Proof.

Let $P \in Gl(n;\mathbb{C})$ such that $\mathcal{L} = P^{-1}DP$, with $D = \text{diag}\,(\lambda_1, \dots, \lambda_k)$.

$$\begin{aligned} &(I_k \otimes A) + (\mathcal{L} \otimes B)(I_k \otimes K) = \\ &(I_k \otimes A) + (P^{-1}DP \otimes B)(I_k \otimes K) = \\ &(P^{-1} \otimes I_n)(I_k \otimes A) + (D \otimes B)(I_k \otimes K))(P \otimes I_n). \end{aligned}$$

Then, the eigenvalues of

$$(I_k \otimes A) + (\mathcal{L} \otimes B)(I_k \otimes K)$$

are the same than

$$(I_k \otimes A) + (D \otimes B)(I_k \otimes K))$$

and

$$\begin{aligned} &(I_k \otimes A) + (D \otimes B)(I_k \otimes K)) = \\ &\text{diag}\,(A + \lambda_1 BK, \dots, A + \lambda_k BK), \end{aligned}$$

□

Proposition 9 *Suppose that the system $\dot{x}^i = Ax^i + Bu^i$, $i = 1, \dots, k$ is controllable and the eigenvalues of the Laplacian matrix $\mathcal{L}$ being positive $\lambda_j > 0$. Then, there exists the matrix K making the system consensus stable.*

Proof. It suffices to consider the following result. □

Lemma 10 ([16]) *Let (A, B) be a controllable pair of matrices and we consider the set of k-linear systems*

$$\dot{x}^i = Ax^i + \lambda_i Bu^i, \ 1 \le i \le k$$

with $\lambda_i > 0$. Then, there exist a feedback K which simultaneously assigns the eigenvalues of the systems as negative as possible.

More concretely, for any $M > 0$, there exist $u^i = Kx^i$ for $1 \le i \le k$ such that

$$Re\, \sigma(A + \lambda_i BK) < -M, \ 1 \le i \le k.$$

($\sigma(A+\lambda_i BK)$ denotes de spectrum of $A+\lambda_i BK$ for each $1 \le i \le k$).

Example We consider 3 identical agents with the following dynamics of each agent

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ \dot{x}^2 &= Ax^2 + Bu^2 \\ \dot{x}^3 &= Ax^3 + Bu^3 \end{aligned} \tag{6}$$

The communication topology is defined by the undirected graph $(\mathcal{V}, \mathcal{E})$:
$\mathcal{V} = \{1, 2, 3\}$
$\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1)\} \subset \mathcal{V} \times \mathcal{V}$
and the adjacency matrix:

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{1, 2\}, \mathcal{N}_2 = \{1, 3\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

a) with $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Clearly, the system

$$\dot{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \mathcal{X}$$

is not stable.

$$\text{rank} \begin{pmatrix} s-1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & s+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s-1 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & s+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & s+1 & 0 & 0 & 0 \end{pmatrix}$$

6 for all $s \in \mathbb{C} - \{-1\}$ then for all $s \in \mathbb{C}^+$.

Then, the system is stabilizable.

The matrix $(I_k \otimes A) + (\mathcal{L} \otimes B)(I_k \otimes K)$ with $K = \begin{pmatrix} a & b \end{pmatrix}$ is

$$\begin{pmatrix} 1+2a & 2b & -a & -b & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -a & -b & 1+2a & 2b & -a & -b \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -a & -b & 0 & 0 & 1+a & b \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

are and the eigenvalues are -1, -1, -1, $\frac{a(3-\sqrt{5})}{2} + 1$, $\frac{a(3+\sqrt{5})}{2} + 1$, $2a + 1$.

It suffices to consider $K = \begin{pmatrix} a & b \end{pmatrix}$ with $a < 2.6178$. So, the system is consensus stabilizable.

b) with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Clearly, the system

$$\dot{\mathcal{X}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{X}$$

is not stable.

$$\text{rank} \begin{pmatrix} s-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & s-1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s-1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & s-1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s-1 & -1 & 0 & 1 \end{pmatrix}$$

6 for all $s \in \mathbb{C}$ then for all $s \in \mathbb{C}^+$.

Then, the system is stabilizable.

Taking $F = (I_k \otimes K)$ with $K = \begin{pmatrix} -10.1 & -9.1 \end{pmatrix}$, the matrix $(I_k \otimes A) + (\mathcal{L} \otimes B)F$ is

$$\begin{pmatrix} 1.0000 & 1.0000 & 0 & 0 & 0 & 0 \\ -6.2000 & -5.2000 & 3.1000 & 3.1000 & 0 & 0 \\ 0 & 0 & 1.0000 & 1.0000 & 0 & 0 \\ 3.1000 & 3.1000 & -6.2000 & -5.2000 & 3.1000 & 3.1000 \\ 0 & 0 & 0 & 0 & 1.0000 & 1.0000 \\ 3.1000 & 3.1000 & 0 & 0 & -3.1000 & -3.1000 \end{pmatrix}$$

and the eigenvalues are -21.0405, -17.2539, $-0.9654 + 0.3111i$, $-0.9654 - 0.3111i$, -0.1490, -0.1258, having all negative real part. Then the system is consensus stabilizable.

5 Exact Consensus Controllability

We are interested in study the exact controllability of the obtained system 3. In our particular setup

Definition 11 *Let A be a matrix. The exact controllability $n_D(I_k \otimes A)$ is the minimum of the rank of all possible matrices B making the system 3 controllable.*

$$\begin{array}{l} n_D(I_k \otimes A) = \\ \min \{\operatorname{rank} B, \forall B \in M_{n\times i}\, 1 \leq i \leq n \mid \\ \qquad (I_k \otimes A, \mathcal{L} \otimes B) \text{ controllable}\}. \end{array}$$

The controllability condition depends directly on the structure of the matrix $\mathcal{L}$.

Proposition 12 *Let J be the Jordan reduced of the matrix $\mathcal{L}$ and P such that $\mathcal{L} = P^{-1}JP$. Then, the system 3 is controllable if and only if*

$$\operatorname{rank} \begin{pmatrix} sI_{nk} - (I_k \otimes A) & J \otimes B \end{pmatrix} = kn$$

Proof. Suppose that there exist S such that $P^{-1}JP = \mathcal{L}$ and

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} sI_{kn} - (I_k \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = \\ \operatorname{rank} (P^{-1} \otimes I_n) \cdot \begin{pmatrix} (sI_k \otimes I_n) - (I_k \otimes A) & J \otimes B \end{pmatrix} \\ \qquad \cdot \begin{pmatrix} P \otimes I_n & \\ & P \otimes I_n \end{pmatrix} = \\ \operatorname{rank} \begin{pmatrix} sI_{kn} - (I_k \otimes A) & J \otimes B \end{pmatrix} \end{array}$$

□

Corollary 13 *Suppose that the matrix $\mathcal{L}$ can be reduced to the Jordan form (1), with non-zero eigenvalues $\lambda_1, \dots, \lambda_r$. Then, the system 3 is controllable if and only if each agent is controllable.*

Proof. Let $\lambda_i \neq 0$, $i = 1, \dots r$ be the eigenvalues of $\mathcal{L}$.

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} s(I_{k_{ij}} \otimes I_n) - (I_{k_{ij}} \otimes A) & J_j(\lambda_i) \otimes B \end{pmatrix} = \\ \operatorname{rank} \begin{pmatrix} sI_n - A & & & & \lambda_i B & & & \\ & sI_n - A & & & B & \lambda_i B & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & sI_n - A & & & B & \lambda_i B \end{pmatrix} = \\ \operatorname{rank} \begin{pmatrix} sI_n - A & & & & B & & & \\ & sI_n - A & & & & B & & \\ & & \ddots & & & & \ddots & \\ & & & sI_n - A & & & & B \end{pmatrix} = \\ k \cdot \operatorname{rank} \begin{pmatrix} sI_n - A & B \end{pmatrix} \end{array}$$

with $k_1 + \ldots + k_r = k$, $k_{i_1} + \ldots k_{i_{n_i}} = k_i$.

□

Corollary 14 *A necessary condition for controllability of the system 3 is that the matrix $\mathcal{L}$ has full rank.*

Example We consider 3 identical agents with the following dynamics of each agent

$$\begin{array}{ll} \dot{x}^1 & = Ax^1 + Bu^1 \\ \dot{x}^2 & = Ax^2 + Bu^2 \\ \dot{x}^3 & = Ax^3 + Bu^3 \end{array} \tag{7}$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2\times\ell}(\mathbb{C})$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(\mathcal{V}, \mathcal{E})$:
$\mathcal{V} = \{1, 2, 3\}$
$\mathcal{E} = \{(i,j) \mid i, j \in \mathcal{V}\} = \{(1,2), (1,3)\} \subset \mathcal{V}\times\mathcal{V}$
and the adjacency matrix:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{2,3\}, \mathcal{N}_2 = \{1\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$.

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} sI_6 - (I \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = \\ \operatorname{rank} \begin{pmatrix} s & -1 & 0 & 0 & 0 & 0 & 2a & 2c & -a & -c & -a & -c \\ 0 & s & 0 & 0 & 0 & 0 & 2b & 2d & -b & -d & -b & -d \\ 0 & 0 & s & -1 & 0 & 0 & -a & -c & a & c & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 & -b & -d & b & d & 0 & 0 \\ 0 & 0 & 0 & 0 & s & -1 & -a & -c & 0 & 0 & a & c \\ 0 & 0 & 0 & 0 & 0 & s & -b & -d & 0 & 0 & b & d \end{pmatrix} \\ = \begin{cases} 6 \text{ for all } s \neq 0 \\ 5 \text{ for } s = 0 \end{cases} \end{array}$$

In fact, for all matrix $B \in M_{2\times\ell}(\mathbb{C})$ for all $\ell \geq 0$

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} sI_6 - (I \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = \\ \begin{cases} 6 \text{ for all } s \neq 0 \\ 5 \text{ for } s = 0 \end{cases} \end{array}$$

If the matrix $\mathcal{L}$ has full rank, then the number of columns for exact controllability of matrix $I_k \otimes A$ depends on the multiplicity of the eigenvalues of the matrix A and we have the following result.

Proposition 15 *Let $\mathcal{L}$ be the Laplacian matrix of a graph having full rank. Then, the exact controllability $n_D(I_k \otimes A)$ for the system $\dot{\mathcal{X}} = (I_k \otimes A)\mathrm{X} + (\mathcal{L} \otimes B)\bar{\mathcal{U}}$ coincides with the exact controllability $n_D(A)$ for the system $\dot{x} = Ax + Bu$.*

Example We consider 3 identical agents with the following dynamics of each agent

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ \dot{x}^2 &= Ax^2 + Bu^2 \\ \dot{x}^3 &= Ax^3 + Bu^3 \end{aligned} \tag{8}$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2\times\ell}(\mathbb{C})$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(\mathcal{V}, \mathcal{E})$:
$\mathcal{V} = \{1, 2, 3\}$
$\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} = \{(1,1),(1,2),(2,1),(2,3),(3,1)\} \subset \mathcal{V} \times \mathcal{V}$
and the adjacency matrix:

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{1,2\}, \mathcal{N}_2 = \{1,3\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0.3820$, $\lambda_2 = 2$, $\lambda_3 = 2.6180$.

$$\operatorname{rank} \begin{pmatrix} s & -1 & 0 & 0 & 0 & 0 & 2a & -a & 0 \\ 0 & s & 0 & 0 & 0 & 0 & 2b & -b & 0 \\ 0 & 0 & s & -1 & 0 & 0 & -a & 2a & -a \\ 0 & 0 & 0 & s & 0 & 0 & -b & 2b & -b \\ 0 & 0 & 0 & 0 & s & -1 & -a & 0 & a \\ 0 & 0 & 0 & 0 & 0 & s & -b & 0 & b \end{pmatrix}$$

6 for all s and $b \neq 0$.

Obviously the system $\dot{x} = Ax + Bu$ with $B = \begin{pmatrix} a \\ b \end{pmatrix}$ and $b \neq 0$ is controllable.

6 Conclusions

In this paper, the consensus stabilizability and exact consensus controllability for multi-agent systems where all agents have an identical linear dynamic mode, using linear algebra techniques, are analyzed.

The future work is focussed on description of possible controls making the system consensus controllable.

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