Bifurcation Diagram for Saddle/Source Bimodal Linear Dynamical Systems

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Abstract: We continue the study of the structural stability and the bifurcations of planar bimodal linear dynamical systems (BLDS) (that is, systems consisting of two linear dynamics acting on each side of a straight line, assuming continuity along the separating line). Here, we enlarge the study of the bifurcation diagram of saddle/spiral BLDS to saddle/source BLDS and in particular we study the position of the homoclinic bifurcation with regard to the new improper node bifurcation.

Key–Words: Piecewise linear system, structural stability, bifurcation diagram

1 Introduction

Piecewise linear systems constitute a class of nonlinear systems which have attracted the interest of researchers because of their interesting properties and the wide range of applications from which they arise. Even the planar continuous BLDS (planar continuous bimodal linear dynamical systems, that is, two planar linear subsystems acting in complementary halfplanes, assuming continuity in the separating straight line) have complex dynamic behaviors as well as applications (see, for example, [1], [2], [3] and [5]).

Our aim is a full characterization of the planar continuous BLDS structurally stable and a systematic study of the bifurcations between them, both in terms of the coefficients of the matrices which define the system. The structural stability of a system guarantees that its qualitative behavior is preserved under small perturbations of their parameters, whereas qualitative changes occur at the bifurcation points. We point out that both concepts (structural stability and bifurcation) depend on the equivalence relation which precises the idea that two dynamical systems have the "same qualitative behavior". For example, for the equilibrium point of a single (non degenerate) planar linear system, those having positive trace and positive determinant form a unique (structurally stable) C^0-class (sources) whereas they are partitioned in four C^1-classes (spirals and nodes as structurally stable classes; improper nodes and starred nodes as bifurcations). Here we follow [11], where two planar continuous BLDS are equivalent if there is a homeomorphism of $\mathbf{R}^2$, preserving the separating line, which maps the orbits of a system into those of the other one and it is differentiable when restricted to finite periodic orbits (see Definition 3). We maintain also the nomenclature in [11].

Till now, several partial studies exist concerning equilibrium points, periodic orbits or homoclinic orbits (see [8], [9], [10], [12]). Our aim is a full integration of these and other behaviors in a complete bifurcation diagram, in particular analyzing their persistence under small perturbations. For example, in [6] it becomes clear that in general the periodic orbits are structurally stable and that two bifurcations are possible for disappearing: an ordinary homoclinic bifurcation and a special kind of Hopf bifurcation. Indeed, in an ordinary Hopf bifurcation the periodic orbit collapses to the equilibrium point inside it. Whereas in our case, the spiral inside the periodic orbit does not collapse but change from divergent to convergent, through a continuum of periodic orbits. In addition, in [7] we prove that beyond both bifurcations there is not a zone of structural stability, but a sequence of saddle/tangency or tangency/saddle bifurcations whose limit is the corresponding bifurcation.

For this global study, the starting point is the reduced form of the matrices representing a continuous BLDS obtained in [4]. Then, in [6] we have specialized the general criteria for structural stability in [11] and we have pointed out that additional specific studies (concerning periodic orbits, saddle-loop (or homoclinic) orbits, saddle/tangency orbits and tangency/saddle orbits) are needed when one of the subsystems is a spiral.

As a first goal, we focus our attention in the saddle/spiral case because it is the only one where all these elements can appear so that, more complex behaviors and applications are expected. Thus, also in [6], we have studied the periodic orbits and the saddle-loop orbits for the case of a saddle/spiral system. For this case, in [12] one specifies the conditions for the existence of saddle-loop orbits. In general, for planar continuous BLDS, in [8] it is proved that there exists at most one saddle-loop orbit or limit cycle, which then must be attracting or repelling. All these previous partial results are collected in Section 3. As referred, in [7] we complete the bifurcation diagram of a saddle/spiral system.

Here we enlarge this study to the transformation of the considered spiral into a node, through an improper node. More precisely, we consider a saddle (with negative trace T) as left subsystem and a source as right subsystem, that is, having positive both the determinant Δ and the trace τ. It is well-known that the right subsystem will be an improper divergent node for $\tau = \tau_0 \equiv 2\sqrt{\Delta}$, a divergent node for $\tau > \tau_0$ and a divergent spiral for $\tau < \tau_0$. We present figures of these transformation.

For the spiral case, Theorem 6 says that an homoclinic orbit appears for an unique value $0 < \tau_H < \tau_0$, and that there is a finite periodic hyperbolic orbit for each $0 < \tau < \tau_H$. Here we prove that $\tau_H \to 0^+$, when $T \to 0^-$ and that $\tau_H \to \tau_0^-$, when $T \to -\infty$. Thus the interval $\tau \in]0, \tau_H[$ of structurally stable systems having a finite periodic orbit increases to the whole $]0, \tau_0[$ when the trace of the spiral in the right decreases to $-\infty$. As a consequence, the interval $]\tau_H, \tau_0[$ containing the tangency/saddle bifurcation decreases to zero length.

Throughout the paper, $\mathbf{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbf{R})$ the set of matrices having n rows and m columns and entries in $\mathbf{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbf{R})$) and $Gl_n(\mathbf{R})$ the group of non-singular matrices in $M_n(\mathbf{R})$. Finally, we will denote by $e_1, \ldots, e_n$ the natural basis of the Euclidean space $\mathbf{R}^n$.

2 Structural stability of planar bimodal linear systems

Let us consider a bimodal linear dynamical system (BLDS) given by two subsystems each one acting in a halfspace:

$$\dot{x}(t) = A_1 x(t) + B_1 \qquad \text{if } Cx(t) \le 0,$$

$$\dot{x}(t) = A_2 x(t) + B_2 \qquad \text{if } Cx(t) \ge 0,$$

where $A_1, A_2 \in M_n(\mathbf{R})$; $B_1, B_2 \in M_{n\times 1}(\mathbf{R})$; $C \in M_{1\times n}(\mathbf{R})$. We assume that the dynamics is continuous along the separating hyperplane $H = \{x \in \mathbf{R}^n : Cx = 0\}$; namely, that both subsystems coincide for $Cx(t) = 0$.

By means of a linear change in the state variable $x(t)$, we can consider $C = (1\ 0 \ldots 0) \in M_{1\times n}(\mathbf{R})$. Hence $H = \{x \in \mathbf{R}^n : x_1 = 0\}$ and continuity along H is equivalent to:

$$B_2 = B_1, \qquad A_2 e_i = A_1 e_i, \quad 2 \le i \le n.$$

We will write from now on $B = B_1 = B_2$.

Definition 1 *In the above conditions, we say that the triple of matrices (A_1, A_2, B) defines a continuous* bimodal linear dynamical system. *(BLDS.)*

The placement of the equilibrium points will play a significative role in the dynamics of a BLDS. So, we define:

Definition 2 *Let us assume that a subsystem of a BLDS has a unique equilibrium point, not lying in the separating hyperplane. We say that this equilibrium point is* real *if it is located in the halfspace corresponding to the considered subsystem. Otherwise, we say that the equilibrium point is* virtual.

Our goal is to characterize the planar continuous BLDS which are structurally stable in the sense of [11] in terms of the coefficients A_1, A_2 and B, and to analyze the bifurcations appearing in the boundary values between them.

Definition 3 *A triple of matrices (A_1, A_2, B) defining a continuous BLDS is said to be (regularly)* structurally stable *if it has a neighborhood $V(A_1, A_2, B)$ such that for every triplet $(A_1', A_2', B') \in V(A_1, A_2, B)$ there is a homeomorphism of $\mathbf{R}^2$ preserving the hyperplane H which maps the oriented orbits of (A_1', A_2', B') into those of (A_1, A_2, B) and it is differentiable when restricted to finite periodic orbits.*

A natural tool in the study of BLDS is simplifying the matrices A_1, A_2, B by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system (in particular, the condition of structurally stability). So, we consider linear changes in the state variables space preserving the hyperplanes $x_1(t) = k$, which will be called *admissible basis changes*. Thus, they are basis changes given by a matrix $S \in Gl_n(\mathbf{R})$,

$$S = \begin{pmatrix} 1 & 0 \\ U & W \end{pmatrix}, W \in Gl_{n-1}(\mathbf{R}), U \in M_{n-1\times 1}(\mathbf{R}).$$

See [4] for the resulting reduced forms.

Also, translations parallel to the hyperplane H are allowed.

3 Preliminaries

By specializing to BLDS the general necessary and sufficient conditions in [11], in [6] one proves the following results.

Theorem 4 *[6] Let us consider planar continuous BLDS.*

(1) If such a BLDS is structurally stable, then the triples of matrices representing it can be reduced (by means of an admissible basis change and a translation parallel to the separating line) to the form:

$$A_1 = \begin{pmatrix} T & 1 \\ -D & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \tau & 1 \\ -\Delta & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ b \end{pmatrix}, b \neq 0$$

In particular, the only tangency point is $(0, 0)$.

(2) If one of the subsystems is a center, a degenerate node, an improper node or a starred node, then the BLDS is not structurally stable.

(3) For the remainder BLDS, if none subsystem is a real spiral then the BLDS is structurally stable. Explicitly (for $b > 0$; when $b < 0$, we obtain the symmetric ones) when:
the left subsystem is a real saddle, a virtual node or a virtual spiral
the right subsystem is a virtual saddle or a real node

(4) Additional conditions must be verified if one of the subsystems is a real spiral (in the right half-plane if $b > 0$):

(4.1) A BLDS real saddle/real spiral is structurally stable if and only if:

(a) the finite periodic orbits are hyperbolic

(b) there are not saddle-loop orbits

(c) there are not finite orbits connecting a saddle and a tangency point

(4.2) A BLDS virtual node/real spiral is structurally stable if and only if condition (a) holds

(4.3) A BLDS virtual spiral/real spiral is structurally stable if and only if condition (a) holds and also:

(a') the infinite periodic orbit at infinity is hyperbolic

Remark 5 *In (1) of the above Theorem one can take $b = 1$ (by means of a change of scale and a symmetry, if necessary), but we will consider general $b \neq 0$ because of the homogeneity in the obtained formulas.*

In [6] one focuses on conditions (a), (b) of case (4.1) for divergent spirals ($\tau > 0$). Thus, let us assume a BLDS as in (1) of Theorem 4, verifying:

- The left subsystem is a (real) saddle, i.e.: $D < 0, b > 0$. In particular, its equilibrium point is $(\frac{b}{D}, -T\frac{b}{D})$, and the invariant manifold cut the separating line at $(0, -\frac{b}{\lambda_2})$ and $(0, -\frac{b}{\lambda_1})$, where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of A_1 ($\lambda_1 + \lambda_2 = T$, $\lambda_1\lambda_2 = D$.)
- The right subsystem is a (real) spiral, i.e.: $\Delta > 0, \tau^2 < 4\Delta, b > 0$. In particular, its equilibrium point is $(\frac{b}{\Delta}, -\tau\frac{b}{\Delta})$. We write $\alpha \pm i\beta$, $\beta > 0$ the eigenvalues of A_2 ($2\alpha = \tau$, $\alpha^2 + \beta^2 = \Delta$.)

We summarize the results in [6] concerning this case (4.1). Moreover, we precise the uniqueness of the finite periodic orbit in (2.c) (see [8]).

Theorem 6 *[6] Let us assume the case (4.1) above, that is:*

$$b > 0, D < 0, \Delta > 0, \tau^2 < 4\Delta$$

in (1) of Theorem 4, and let be

$$\lambda_2 < 0 < \lambda_1 \quad \text{the eigenvalues of } A_1$$

$$\alpha \pm i\beta, \beta > 0, \quad \text{the eigenvalues of } A_2$$

Then, for $\tau > 0$:

(1) If $T \geq 0$, then there are not homoclinic orbits nor finite periodic orbits.

(2) If $T < 0$, the only homoclinic (i.e., saddle-loop) orbit appears for the value τ_H of τ verifying

$$t = \frac{1}{\tau}\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \tau\lambda_1 + \Delta}{\lambda_2^2 - \tau\lambda_2 + \Delta}\right)$$

being

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi$$

where $M > 0$ and $0 < \varphi < \pi$ are defined by

$$M\cos(\varphi) = \alpha - \frac{\Delta}{\lambda_2}, \quad M\sin(\varphi) = \beta.$$

Moreover, $\tau_H < \frac{T\Delta}{D}$.

(3) If $T < 0$, a unique finite periodic orbit exists for $0 < \tau < \tau_H$, being attractive hyperbolic (and transverse to the separating axis). No saddle/tangency orbits exist.

Corollary 7 *[6] The systems in case (4.1) with $T < 0$ and $0 < \tau < \tau_H$ are structurally stable.*

The following theorem specifies the value of τ_H.

Theorem 8 *[12] In the conditions of Theorem 6, τ_H is the value of τ verifying*

$$\frac{1}{2}\ln\left(\frac{\lambda_2^2}{\lambda_1^2}\frac{\lambda_1^2 - \tau\lambda_1 + \Delta}{\lambda_2^2 - \tau\lambda_2 + \Delta}\right) -$$

$$-\frac{\alpha}{\beta}\left(2\pi - \arctan\frac{\lambda_2\alpha - \Delta}{\lambda_2\beta} - \arctan\frac{\Delta - \lambda_1\alpha}{\lambda_1\beta}\right) = 0.$$

Remark 9 *For $\tau < 0$ one has symmetric results:*

(1') If $T \leq 0$, there are not homoclinic orbits nor finite periodic orbits.

(2') If $T > 0$, the only homoclinic orbit appears for a unique value $\tau_H < 0$ of τ verifying $\tau_H > \frac{T\Delta}{D}$.

(3') If $T > 0$, a unique finite periodic orbit exists for each $\tau_H < \tau < 0$, being hyperbolic (and transverse to the separating axis). Hence, the system is structurally stable.

4 The bifurcation at an improper node

Let us consider a BLDS (A_1, A_2, B) as in (1) of Theorem 4, with

$$T < 0, D < 0, \tau > 0, \Delta > 0, b > 0,$$

that is, the left subsystem is a real saddle and the right one is a real source. From Theorem 4, we have

Proposition 10 *In the above conditions, let $\tau_0 = 2\sqrt{\Delta}$. Then:*

(1) For $\tau > \tau_0$ one has a structurally stable system saddle/node.

(2) For $\tau = \tau_0$ one has a bifurcation saddle/improper node.

(3) For $\tau < \tau_0$ one has a system saddle/spiral, which is structurally stable if $\tau_H < \tau$ and no tangency/saddle orbit appears.

Remark 11 *When $\tau_H < \tau < \tau_0$ as in (3) above, in [7] one proves that tangency/saddle orbits appear just for a decreasing sequence $(\tau_1, \tau_2, ...) \to \tau_H^+$. Thus, in (3) above the system is structurally stable for all $\tau_1 < \tau < \tau_0$. In particular, the tangent orbit intersects the separating line in just one additional point.*

Example 12 *We include figures of the transition $\tau < \tau_0, \tau = \tau_0, \tau > \tau_0$ for $T = -1, D = -1, \Delta = 5$ and $b = 1$, so that $\tau_0 = 2\sqrt{5} \cong 4.4721$.*

More precisely, Figure 1 and Figure 2 correspond to $\tau = \tau_0 - 1$. Notice that the tangent orbit (as well as the ones under the spiral/saddle orbit) intersects the separating line in just an additional point. The remainder orbits intersect twice in the main quadrant of the saddle and a third time under it.

Figure 3 corresponds to $\tau = \tau_0$. Now the tangent orbit does not intersect the separating line because it cannot cross the new invariant line arising from the equilibrium point of the improper node.

In Figure 4, corresponding to $\tau = \tau_0 + 1$, this invariant line splits into two of them, giving an ordinary node.

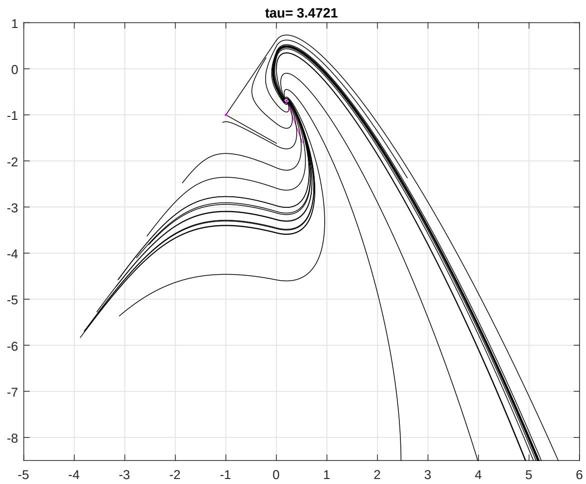

Figure 1: Fig. corresponding to Example 12, $\tau < \tau_0$

Our main goal is to determine the variation of τ_H with regard to τ_0 when $0 > T > -\infty$. In other words, the variation of the interval $]0, \tau_H[\subset]0, \tau_0[$ where the structurally stable finite periodic orbits appear. Or equivalently, the variation of the interval $]\tau_H, \tau_0[\subset]0, \tau_0[$ where the tangency/saddle bifurcations appear.

Theorem 13 *In the above conditions, we fix D, Δ and b. Then:*

(1) $\tau_H \to 0^+$, when $T \to 0^-$.

(2) $\tau_H \to \tau_0^-$, when $T \to -\infty$.

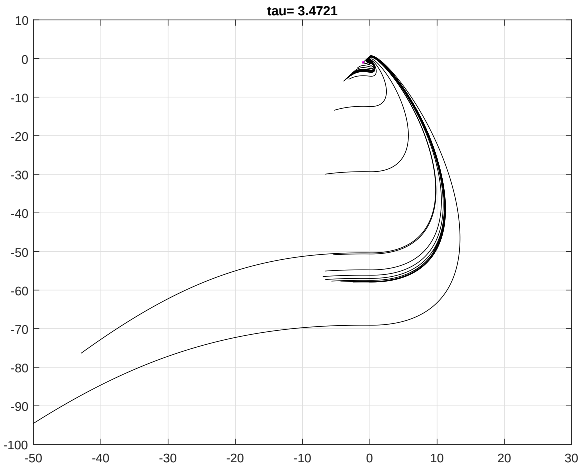

Figure 2: Fig. corresponding to Example 12, $\tau < \tau_0$

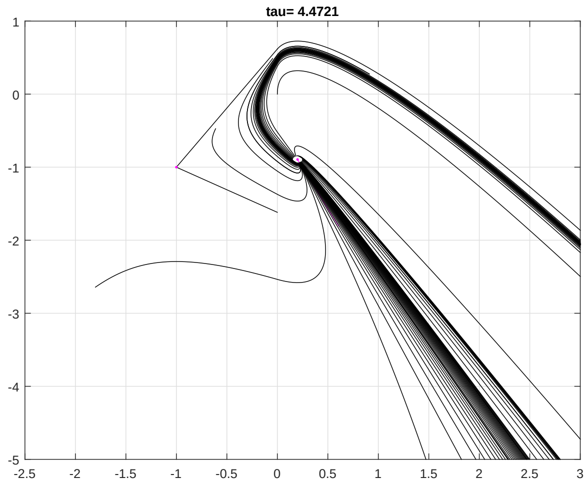

Figure 3: Fig. corresponding to Example 12, $\tau = \tau_0$

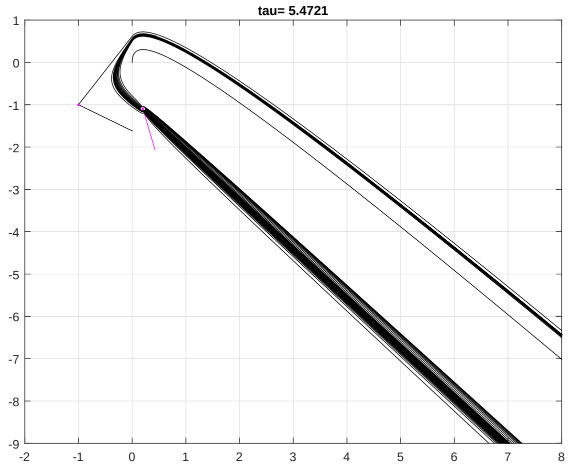

Figure 4: Fig. corresponding to Example 12, $\tau > \tau_0$

Proof: From Theorem 6, we know that, for each value of $T < 0$, τ_H is the unique value of $\tau > 0$ verifying the condition in (2) of Theorem 6, or equivalently, the equation in Theorem 8.

(1) Clearly, for $T = 0$ (that is, $\lambda_1 = \sqrt{-D}, \lambda_2 = -\sqrt{-D}$), the value $\tau = 0$ (that is, $\alpha = 0, \beta = \sqrt{\Delta}$) verifies the equation in Theorem 8.

(2) Now, we consider $T \to -\infty$. We must prove that, then, the equation in Theorem 8 is verified if and only if $\tau \to 2\sqrt{\Delta}$.

In order to that, we write

$$x = \frac{1}{\lambda_2}, \quad y = \sqrt{4\Delta - \tau^2}$$

so that

$$x < 0, \quad 0 \leq y < 2\Delta$$

As

$$T = \lambda_1 + \lambda_2, \quad \lambda_2 < 0 < \lambda_1$$

when $T \to -\infty$, necessarily $\lambda_2 \to -\infty$ and $x \to 0^-$. We must prove that, then, $y \to 0^+$.

With the above notation, we have

$$\lambda_1 = Dx, \beta = \frac{y}{2}, \alpha = \frac{\tau}{2}, \tau = \sqrt{4\Delta - y^2}$$

so that the equation in Theorem 8 results

$$\frac{1}{2}\ln\left(\frac{1}{D^2x^2}\frac{D^2x^2 - D\sqrt{4\Delta - y^2}x + \Delta}{1 - \sqrt{4\Delta - y^2}x + \Delta x^2}\right) =$$

$$= \frac{\sqrt{4\Delta - y^2}}{y}\left(2\pi - \arctan\frac{\sqrt{4\Delta - y^2} - \Delta x}{y} - \right.$$

$$\left. - \arctan\frac{2\Delta - D\sqrt{4\Delta - y^2}x}{Dxy}\right)$$

As

$$0 \leq \sqrt{4\Delta - y^2} < 2\sqrt{\Delta}$$

when $x \to 0^-$, for the left term we have

$$\frac{D^2x^2 - D\sqrt{4\Delta - y^2}x + \Delta}{1 - \sqrt{4\Delta - y^2}x + \Delta x^2} \to \Delta$$

$$\frac{1}{2}\ln\left(\frac{1}{D^2x^2}\frac{D^2x^2 - D\sqrt{4\Delta - y^2}x + \Delta}{1 - \sqrt{4\Delta - y^2}x + \Delta x^2}\right) \to +\infty$$

On the other hand, as

$$\pi \leq 2\pi - \arctan\frac{\sqrt{4\Delta - y^2} - \Delta x}{y} -$$

$$- \arctan\frac{2\Delta - D\sqrt{4\Delta - y^2}x}{Dxy} \leq 3\pi$$

the only possibility that the right term converges to $+\infty$ is that $y \to 0^+$, as desired.

$\square$

5 Conclusion

We enlarge the study of the bifurcation diagram of saddle/spiral BLDS to saddle/source BLDS. In particular, being τ the trace of the source, we precise the variation of the interval of values $0 < \tau < \tau_H$ for which structurally stable finite periodic orbits appear: it converges to 0 if $T \to 0$, and to $0 < \tau < 2\sqrt{\Delta}$ when $T \to -\infty$, where T is the trace of the saddle and Δ is the determinant of the source (assumed fixed).

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