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General bounds on limited broadcast domination

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Abstract

Limited dominating broadcasts were proposed as a variant of dominating broadcasts, where the broadcast function is upper bounded by a constant k. The minimum cost of such a dominating broadcast is the k-broadcast dominating number. We present a unified upper bound on this parameter for any value of k in terms of both k and the order of the graph. For the specific case of the 2-broadcast dominating number, we show that this bound is tight for graphs as large as desired. We also study the family of caterpillars, providing a smaller upper bound, which is attained by a set of such graphs with unbounded order.

Keywords: Broadcast; Domination; Graph.

1 Introduction

Domination in graphs has been shown as an extremely fruitful concept, since it was originally defined in the late fifties [1] and named in the early sixties [14]. A dominating set of a graph G is a vertex set S such that any vertex not in S has at least one neighbor in S. Multiple variants of domination have been defined over the past fifty years, putting the focus on different aspects. One of them is the broadcast domination, firstly introduced in [12] and taken up more recently in [6]. This concept reflects the idea of several broadcast stations, with associated transmission powers, that can broadcast messages to places at distance greater than one. We recall the formal definition from [6]. For a graph G any function $f\colon V(G) \to \{0,1,\ldots,diam(G)\}$, where $diam(G)$ denotes the diameter of G, is called a *broadcast* on G. A vertex $v \in V(G)$ with $f(v) > 0$ is a *f-dominating vertex* and it is said to *f-dominate* every vertex u with $d(u,v) \leq f(v)$. A *dominating broadcast* on G is a broadcast f such that every

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vertex in G is f-dominated and the *cost* of f is $\omega(f) = \sum_{v \in V(G)} f(v)$. Finally, the *dominating broadcast number* is

$$\gamma_B(G) = \min\{\omega(f) \colon f \text{ is a dominating broadcast on } G\}.$$

A number of issues has been addressed regarding this variation of domination, for instance the role of the dominating broadcast number into the Domination Chain [5], the characterization of graphs where the dominating broadcast number reaches its natural upper bounds: radius and domination number [3, 9, 13] and some computational complexity aspects [4, 7].

In this paper, we follow the suggestion posed in [5] as an open problem of considering the broadcast dominating problem with limited broadcast power, say k. We formally define the concept of dominating k-broadcast in Section 2 and we present some basic properties, with the focus on the role of spanning trees regarding the associated parameter: the k-broadcast dominating number. We devote Section 3 to the study of the particular case in which the broadcast power is limited to $k = 2$ and we obtain tight upper bounds for the 2-broadcast dominating number in the family of caterpillars and also for general graph. Finally, in Section 4 we present a unified upper bound for the k-broadcast dominating number in terms of both k and the order of the graph.

All graphs considered in this paper are finite, undirected, simple and connected. The *open neighborhood* of a vertex v is $N(v)$ the set of its neighbors and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A *leaf* is a vertex of degree one and its unique neighbor is a *support vertex*. For a pair of vertices u, v the distance $d(u, v)$ is the length of a shortest path between them. For any graph G, the *eccentricity* of a vertex $u \in V(G)$ is $\max\{d(u, v) : v \in V(G)\}$ and is denoted by $ecc_G(u)$. The maximum (resp. minimum) of the eccentricities among all the vertices of G is the *diameter* (resp. *radius*) of G, denoted respectively by $diam(G)$ and $rad(G)$. Two vertices u and v are *antipodal* in G if $d(u, v) = diam(G)$. A *caterpillar* is a tree such that the set of vertices of degree greater than one induces a path. Given a tree T, a vertex in $u \in V(T)$ and an edge $e \in E(T)$, the tree $T(u, e)$ is the subtree containing u, obtained from T by deleting the edge e. For further undefined general concepts of graph theory see [2].

2 Dominating k-broadcast

We begin this section with the formal definition of dominating k-broadcast, following the ideas proposed in [5]. To our knowledge, this concept has not previously been studied, except in [10] where their authors focus on the case $k = 2$ and present some straightforward properties.

Let G be a graph and let $k \geq 1$ be an integer. For any function $f \colon V(G) \to \{0, 1, \ldots, k\}$, we define the sets $V_f^0 = \{u \in V(G) \colon f(u) = 0\}$ and $V_f^+ = \{v \in V(G) \colon f(v) \geq 1\}$. We say that f is a *dominating k-broadcast* if for every $u \in V(G)$ there exists $v \in V_f^+$ such that $d(u, v) \leq f(v)$. In such a case, we say that u *hears* v. The *cost* of a k-dominating broadcast f is $\omega(f) = \sum_{u \in V(G)} f(u) = \sum_{v \in V_f^+} f(v)$. Finally, the *$k$-broadcast dominating number* of G is

$$\gamma_{B_k}(G) = \min\{\omega(f) \colon f \text{ is a k-dominating broadcast on } G\}.$$

Moreover, a dominating k-broadcast with cost $\gamma_{B_k}(G)$ is called *optimal*.

It is clear from the definition that $\gamma(G) = \gamma_{B_1}(G)$ and $\gamma_B(G) \leq \gamma_{B_k}(G)$ for $1 \leq k \leq diam(G)$. For technical reasons, we consider any value of k in our definition instead of limiting it to the diameter of the graph (see Remark 2), although the parameter $\gamma_{B_k}(G)$ agrees with $\gamma_B(G)$ for large enough values of k.

If $r = rad(G)$, then the function $f\colon V(G) \to \{0, 1, \ldots, r\}$ satisfying $f(v) = r$ for a central vertex v and $f(u) = 0$ if $u \neq v$, is both a dominating broadcast and a dominating k-broadcast, for every $k \geq r$. Therefore, a dominating broadcast on minimum cost must be also a dominating r-broadcast so $\gamma_B(G) = \min\{\omega(f)\colon f \text{ is a r-dominating broadcast on } G\} = \gamma_{B_r}(G)$. Moreover, if $k \geq r$ then a dominating k-broadcast on minimum cost must be also a dominating r-broadcast, so $\gamma_{B_k}(G) = \min\{\omega(f)\colon f \text{ is a r-dominating broadcast on } G\} = \gamma_{B_r}(G)$.

Finally, it is clear that every dominating k-broadcast is also a dominating $(k+1)$-broadcast, for every $k \geq 1$, so $\gamma_{B_{k+1}}(G) \leq \gamma_{B_k}(G)$. All these relationships can be summarized in the following chain of inequalities

$$\gamma_B(G) = \gamma_{B_r}(G) \leq \gamma_{B_{r-1}}(G) \leq \cdots \leq \gamma_{B_2}(G) \leq \gamma_{B_1}(G) = \gamma(G)$$

We present next some general properties of these parameters.

Proposition 1. *Let G be a graph and let $k \geq 1$ be an integer.*

1. *If e is a cut-edge of G and G_1, G_2 are the connected components of $G - e$, then $\gamma_{B_k}(G) \leq \gamma_{B_k}(G_1) + \gamma_{B_k}(G_2)$.*

2. *There exists an optimal dominating k-broadcast f such that $f(u) = 0$, for every leaf u of G.*

Proof. 1. Let f_1, f_2 optimal dominating k-broadcast on G_1 and G_2 respectively. Then, the function $f\colon V(G) \to \{0, 1, \ldots, k\}$ such that $f(v) = f_i(v)$ for any $v \in V(G_i)$, is a dominating k-broadcast on G with cost $\omega(f) = \sum_{v \in V_f^+} f(v) = \sum_{v \in V_{f_1}^+} f_1(v) + \sum_{v \in V_{f_2}^+} f_2(v) = \gamma_{B_k}(G_1) + \gamma_{B_k}(G_2)$.

2. Suppose that f is an optimal dominating k-broadcast on G that assigns a positive value to r leaves. Suppose that $f(u) > 0$ for a leaf u with support vertex v. In such a case, the optimality of f implies $f(v) = 0$. Consider the function $g\colon V(G) \to \{0, 1, \ldots, k\}$ satisfying $g(u) = 0$, $g(v) = f(u)$, and $g(w) = f(w)$ if $w \neq u, v$. It is clear that g is a dominating k-broadcast with cost $\omega(g) = \omega(f)$, so g is also optimal and has $r - 1$ leaves with positive value. Repeating this procedure as many times as necessary, we obtain an optimal k-broadcast that assigns the value 0 to all leaves.

□

Remark 2. *Note that the definition of dominating k-broadcast as a function with fixed range set $\{0, 1, \ldots, k\}$, not depending on the diameter of the graph, ensures that the first property described in Proposition 1 makes sense even if the connected components have small diameters.*

Let T be a tree of order at least 3. We define the *twin-free tree associated* to T, and denote it by T^*, as the tree obtained from T by deleting all but one of the leaves of every maximal set of pairwise twin leaves. Observe that $rad(T) = rad(T^*)$.

Proposition 3. *Let T^* be the twin-free tree associated to a tree T. Then, for every integer $k \geq 1$, $\gamma_{B_k}(T) = \gamma_{B_k}(T^*)$.*

Proof. Let f be an optimal dominating k-broadcast on T that assigns the value 0 to all its leaves. Then, the restriction f^* of f to the set $V(T^*)$ is a dominating k-broadcast on T^* such that $\gamma_{B_k}(T^*) \leq \omega(f^*) = \omega(f) = \gamma_{B_k}(T)$.

Reciprocally, let f^* be an optimal dominating k-broadcast on T^* that assigns the value 0 to all its leaves and define $f\colon V(T) \to \{0, 1, \ldots, k\}$ such that $f(v) = f^*(v)$ if $v \in V(T^*)$ and $f(u) = 0$ if $u \in V(T) \setminus V(T^*)$. Then, f is a dominating k-broadcast on T satisfying $\gamma_{B_k}(T) \leq \omega(f) = \omega(f^*) = \gamma_{B_k}(T^*)$. □

Spanning trees play a central role in the problem of obtaining the dominating broadcast number of a graph, as it is shown in [8].

Theorem 4. *[8] Let G be a graph. Then,*

$$\gamma_B(G) = \min\{\gamma_B(T)\colon T \textit{ is a spanning tree of } G\}\,.$$

The proof of this result essentially uses the existence of an *efficient* optimal dominating broadcast [5] in every graph, that is, a dominating broadcast f with minimum cost such that any vertex u in G is f-dominated by exactly one vertex v with $f(v) > 0$. Nevertheless, there is no similar property for dominating k-broadcasts in general. For instance, the cycle C_7 satisfies $\gamma_{B_2}(C_7) = 3$ and however, it has no efficient optimal dominating 2-broadcast with cost equal to 3 (see Figure 1). Despite this fact, we can get a result similar to that in Theorem 4, by means of an specific construction.

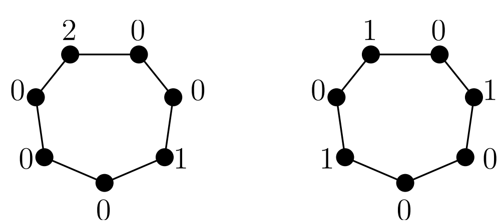


Figure 1: There are exactly two non-isomorphic optimal dominating 2-broadcasts on the cycle C_7.

Theorem 5. *Let G be a graph and let $k \geq 1$ be an integer. Then,*

$$\gamma_{B_k}(G) = \min\{\gamma_{B_k}(T) \colon T \textit{ is a spanning tree of } G\}.$$

Proof. Let T be a spanning tree of G such that

$$\gamma_{B_k}(T) = t = \min\{\gamma_{B_k}(T) \colon T \text{ is a spanning tree of } G\}$$

and let $f \colon V(T) \to \{0, 1, \ldots, k\}$ be an optimal dominating k-broadcast on T. Then, f is also a dominating k-broadcast on G and thus $\gamma_{B_k}(G) \leq t$.

Conversely, let $g \colon V(G) \to \{0, 1, \ldots, k\}$ be an optimal dominating k-broadcast on G. Let $V_g^+ = \{v_1, \ldots, v_m\}$ with the property $1 \leq g(v_1) \leq g(v_2) \leq \cdots \leq g(v_m)$. Consider, for every $i \in \{1, 2, \ldots, m\}$ and $j \in \{0, \ldots, g(v_i)\}$, the sets $L_j(v_i) = \{u \in V(G) \colon d(u, v_i) = j\}$ and $B(v_i) = \bigcup_{j=0}^{g(v_i)} L_j(v_i)$. Let T_i' be the tree rooted in v_i with vertex set $B(v_i)$, obtained by keeping a minimal set of edges of G ensuring that $d_{T_i'}(v_i, x) = d_G(v_i, x)$ and deleting the rest of edges. If $V(T_1'), V(T_2'), \ldots, V(T_m')$ are pairwise disjoint sets, then define $T_i = T_i'$. Otherwise, we modify the trees T_i' in the following way.

Firstly, suppose that $v_i \in V(T_\ell')$ with $i > \ell$, denote by $T_\ell'(v_i)$ the subtree of T_ℓ' rooted in v_i and let a be the distance from v_i to the farthest leaf of $T_\ell'(v_i)$. If $y \in V(T_\ell'(v_i))$, then $d_G(v_i, y) \leq d_{T_\ell'(v_i)}(v_i, y) \leq a \leq g(v_\ell) \leq g(v_i)$ so $y \in B(v_i)$. In this case, we modify the tree T_ℓ' by deleting the subtree $T_\ell'(v_i)$.

On the other hand, assume that $v_\ell \in V(T_i')$ with $\ell < i$, denote by $T_i'(v_\ell)$ the subtree of T_i' rooted in v_ℓ and let b be the distance from v_ℓ to the farthest leaf of $T_i'(v_\ell)$. Suppose that $g(v_\ell) \leq b$ and let $y \in V(T_\ell')$. Then $d_G(v_i, y) \leq d_G(v_i, v_\ell) + d_G(v_\ell, y) \leq d_{T_i'}(v_i, v_\ell) + d_{T_\ell'}(v_\ell, y) \leq d_{T_i'}(v_i, v_\ell) + g(v_\ell) \leq d_{T_i'}(v_i, v_\ell) + b \leq g(v_i)$, so $y \in B(v_i)$. But in this case, the function $h \colon V(G) \to \{0, 1, \ldots, k\}$ satisfying $h(v_\ell) = 0$ and $h(v) = g(v)$ if $v \neq v_\ell$ is a dominating k-broadcast with $\omega(h) < \omega(g)$ which is not possible because g is an optimal broadcast. Hence, $b < g(v_\ell)$ and if $y \in V(T_i'(v_\ell))$, then $d_G(v_\ell, y) \leq d_{T_i'(v_\ell)}(v_\ell, y) \leq b < g(v_\ell)$ so $y \in B(v_\ell)$. In this case, we modify the tree T_i' by deleting its subtree $T_i'(v_\ell)$. In the rest of the proof, we may assume that $v_i \in V(T_\ell')$ if and only if $i = \ell$.

Suppose now that, for $i \geq 2$, $V(T_i') \cap \bigcup_{r=1}^{i-1} V(T_r') \neq \emptyset$, being $V(T_1'), \ldots, V(T_{i-1}')$ pairwise disjoint and let $x \in L_j(v_i) \cap \bigcup_{r=1}^{i-1} V(T_r')$, where $j \in \{1, 2, \ldots, g(v_i)\}$ is the smallest index such that $L_j(v_i) \cap \bigcup_{r=1}^{i-1} V(T_r') \neq \emptyset$. Then, there exists a unique $r \in \{1, \ldots, i-1\}$ such that $x \in L_j(v_i) \cap V(T_r')$. Denote by $T_r'(x)$ the subtree of T_r' rooted in x and by d_r the distance from x to the farthest leaf of $T_r'(x)$. Similarly, denote by $T_i'(x)$ the subtree of T_i' rooted in x and by d_i the distance from x to the farthest leaf of $T_i'(x)$.

If $d_r \leq d_i$, then every vertex $y \in V(T_r'(x))$ satisfies $d_G(v_i, y) \leq d_G(v_i, x) + d_G(x, y) \leq d_{T_i'}(v_i, x) + d_{T_r'(x)}(x, y) \leq d_{T_i'}(v_i, x) + d_r \leq d_{T_i'}(v_i, x) + d_i \leq g(v_i)$ so $y \in V(T_i')$. In this case, we modify the tree T_r' by deleting its subtree $T_r'(x)$. If, to the contrary, $d_r > d_i$, then every vertex $y \in V(T_i'(x))$ satisfies $d_G(v_r, y) \leq d_G(v_r, x) + d_G(x, y) \leq d_{T_r'}(v_r, x) + d_{T_i'(x)}(x, y) \leq d_{T_r'}(v_r, x) + d_i < d_{T_r'}(v_r, x) + d_r \leq g(v_r)$ so $y \in V(T_r')$. In this case, we modify the tree T_i' by deleting its subtree $T_i'(x)$.

We proceed in the same way for every vertex in $L_j(v_i) \cap \bigcup_{r=1}^{i-1} V(T_r')$ and then we recursively repeat this process for $\ell \in \{j+1, \ldots g(v_i)\}$, which is the smallest index such that $L_\ell(v_i) \cap \bigcup_{r=1}^{i-1} V(T_r') \neq \emptyset$, until we obtain that $V(T_1'), V(T_2') \ldots V(T_i')$ are pairwise disjoint. We repeat this procedure as many times as is necessary until we obtain a family of trees $T_1, \ldots, T_m$ such that $V(T_1), \ldots V(T_m)$ provide a partition of $V(G)$ and, for every $i \in \{1, \ldots, m\}$ and for every $z \in V(T_i)$, $v_i \in V(T_i)$ and $d_{T_i}(v_i, z) \leq g(v_i)$.

Finally, it is possible to construct a spanning tree of G by adding some edges of G to $T_1, T_2, \ldots T_m$ in order to get a connected graph T with no cycles. The property $d_{T_i}(v_i, x) \leq g(v_i)$ for every $x \in V(T_i)$, ensures that $g\colon V(T) \to \{1, \ldots, k\}$ is a dominating k-broadcast on the spanning tree T, so $t \leq \gamma_{B_k}(T) \leq \omega(g) = \gamma_{B_k}(G)$. $\square$

3 Bounds on γ_{B_2}

In this section, an upper bound on γ_{B_2} is given. By Theorem 5, it is sufficient to provide an upper bound for trees. We begin by proving an upper bound for a concrete subfamily of trees, the caterpillars. First, we prove a technical lemma about the floor and the ceiling functions that will be used later.

Lemma 6. *If a, b, c, d are integers such that $a/b \leq c/d$, then $a + \lceil c(n-b)/d \rceil \leq \lceil cn/d \rceil$.*

Proof. Any pair of real numbers x and y satisfy $\lfloor x - y \rfloor \leq \lceil x \rceil - \lceil y \rceil$. Therefore, $\lfloor bc/d \rfloor = \lfloor cn/d - c(n-b)/d \rfloor \leq \lceil cn/d \rceil - \lceil c(n-b)/d \rceil$, so it is enough to prove that $a \leq \lfloor bc/d \rfloor$. We know that a is an integer such that $a \leq bc/d < \lfloor bc/d \rfloor + 1$. Hence, $a \leq \lfloor bc/d \rfloor$. $\square$

Remark 7. *If f is an optimal dominating 2-broadcast then we may assume that $f(u) \neq 2$ for any vertex u of degree 2, otherwise the function f' such that $f'(u) = 0$, $f'(x) = 1$ if $ux \in E(T)$ and $f'(x) = f(x)$ for $x \notin N[u]$ would be also an optimal dominating 2-broadcast.*

Proposition 8. *Let T be a caterpillar of order $n \geq 1$, then*

$$\gamma_{B_2}(T) \leq \lceil 2n/5 \rceil.$$

Moreover, there are caterpillars of order as large as desired attaining this bound.

Proof. We proceed by induction on the order of the caterpillar. By inspection of all cases, we know that the result is true for caterpillars of order at most 6 (see Figure 2). Now let T be a caterpillar of order $n \geq 7$ and assume that the statement is true for

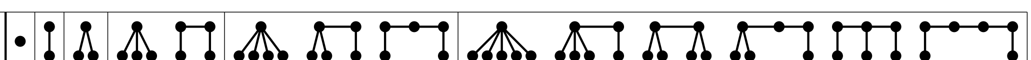
T														
γ_{B_2}	1	1	1	1	2	1	2	2	1	2	2	2	2	2
$\lceil \frac{2n}{5} \rceil$	1	1	1	2		2			3					
n	1	2	3	4		5			6					

Figure 2: All caterpillars T of order $n \leq 6$ satisfy $\gamma_{B_2}(T) \leq \lceil 2n/5 \rceil$.

caterpillars of order less than n. If T has twins, then the twin-free tree T^* associated to T has order n^* and $n^* < n$, and by inductive hypothesis

$$\gamma_{B_2}(T) = \gamma_{B_2}(T^*) \leq \lceil 2n^*/5 \rceil \leq \lceil 2n/5 \rceil.$$

If T has no twins, the path $u_1 ... u_r$ obtained by removing all leaves from T has order at least 4. Observe that there is a leaf hanging from u_1 and another leaf hanging from u_r. If there is no leaf hanging from u_2, then consider the trees $T_1 = T(u_2, u_2u_3)$, which has order 3, and $T_2 = T(u_3, u_2u_3)$. By inductive hypothesis and using Proposition 1 and Lemma 6, we have:

$$\gamma_{B_2}(T) \le \gamma_{B_2}(T_1) + \gamma_{B_2}(T_2) \le 1 + \lceil 2(n-3)/5 \rceil \le \lceil 2n/5 \rceil.$$

If there is a leaf hanging from u_2, then consider the trees $T_1 = T(u_3, u_3u_4)$, which has order 5 or 6, and $T_2 = T(u_4, u_3u_4)$. The function f such that $f(u_2) = 2$ and $f(x) = 0$, if $x \neq u_2$, is a dominating 2-broadcast on T_1. By inductive hypothesis and using Proposition 1 and Lemma 6, we have:

$$\gamma_{B_2}(T) \le \gamma_{B_2}(T_1) + \gamma_{B_2}(T_2) \le 2 + \lceil 2(n-5)/5 \rceil \le \lceil 2n/5 \rceil.$$

It remains to prove that there is a caterpillar attaining the bound with an arbitrarily large order. For each integer $h \ge 1$, consider the caterpillar T_n of order $n = 5h$ obtained from the path $u_1, u_2, \ldots, u_{3h}$ by hanging a leaf to each vertex u_i, where $i \equiv 1, 3 \mod 3$ (see Figure 3). By Proposition 1 and Remark 7, we know that

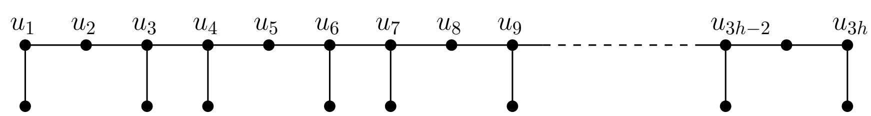


Figure 3: The caterpillar T_n of order $n = 5h$ satisfies $\gamma_{B_2}(T_n) = \lceil 2n/5 \rceil$.

T_n admits an optimal dominating 2-broadcast f such that $f(x) = 0$ if x is a leaf and $f(x) \neq 2$ if $x = u_i$, where $i \equiv 2 \mod 3$. Suppose that $f(u_j) = 1$, for some $j \equiv 2 \mod 3$, then u_j and the leaf hanging from u_{j-1} hears the same vertex $w \in V_f^+$. Therefore, the function $g\colon V(T_n) \to \{0, 1, 2\}$ defined as $g(x) = f(x)$ if $x \neq u_j$ and $g(u_j) = 0$ would be a dominating 2-broadcast with $\omega(g) < \omega(f)$, which is not possible. Thus, $f(x) = 0$ if x is a leaf or $x = u_i$, where $i \equiv 2 \mod 3$. Hence, $f(x) = 1$ for every support vertex. Therefore, $\gamma_{B_2}(T_n) = \omega(f) = 2h = \lceil 2n/5 \rceil$. □

The previous result provides an upper bound for a special class of graphs, the *caterpillar pure graphs*, introduced in [11]. A graph G is called *caterpillar pure* if every spanning tree of G is a caterpillar, and are characterized in the same paper as those graphs with no aster $A_{2,2,2}$ (that is, the graph obtained by subdividing once the three edges of $K_{1,3}$) as a subgraph, not necessarily induced.

Corollary 9. *If G is a caterpillar pure graph, then $\gamma_{B_2}(G) \le \lceil 2n/5 \rceil$.*

Next, we prove a general upper bound for trees.

Proposition 10. *Let T be a tree of order n, then*

$$\gamma_{B_2}(T) \le \lceil 4n/9 \rceil.$$

Moreover, there are trees of order as large as desired attaining this bound.

Proof. We proceed by induction on n, the order of T. It is straightforward to check that the statement is true if $1 \leq |V(T)| \leq 4$. Suppose that T is a tree of order n, $n \geq 5$, and the statement is true for trees of order less than n. If T has twins, then the twin-free tree T^* associated to T has order n^*, $n^* < n$, and by inductive hypothesis and Proposition 3

$$\gamma_{B_2}(T) = \gamma_{B_2}(T^*) \leq \lceil 4n^*/9 \rceil \leq \lceil 4n/9 \rceil.$$

Now, assume that T has no twins. If T is a path, then

$$\gamma_{B_2}(T) \leq \gamma(T) = \lceil n/3 \rceil \leq \lceil 4n/9 \rceil.$$

Now, suppose that T is not a path. Let u, u' be antipodal vertices in T and let $u, u_1, \ldots, u_{D-1}, u'$ be a shortest path from u to u', where $D = diam(T)$. Let v be the vertex of degree distinct from 2 closer to u lying on this path. Note that $v \neq u'$ and it is at distance at least 2 from u, since T has no twins. We distinguish two cases depending on the distance from u to v.

If $d(u,v) \geq 3$, then consider the trees $T_1 = T(u, u_2u_3)$ and $T_2 = T(u', u_2u_3)$. Since T_1 is a path of order 3, by inductive hypothesis and applying Proposition 1 and Lemma 6 we obtain

$$\gamma_{B_2}(T) \leq \gamma_{B_2}(T_1) + \gamma_{B_2}(T_2) \leq 1 + \lceil 4(n-3)/9 \rceil \leq \lceil 4n/9 \rceil.$$

If $d(u,v) = 2$, then $v = u_2$. Consider the trees $T_1 = T(u, u_2u_3)$, which has order at least 4, and $T_2 = T(u', u_2u_3)$. Notice that all the vertices of T_1 are at distance at most 2 from v, because u and u' are antipodal vertices in T. If $|V(T_1)| = n_1 \geq 5$, then consider the dominating 2-broadcast function on T_1 such that $f(v) = 2$ and $f(x) = 0$, if $x \neq v$. By inductive hypothesis and applying Proposition 1 and Lemma 6 we obtain

$$\gamma_{B_2}(T) \leq \gamma_{B_2}(T_1) + \gamma_{B_2}(T_2) \leq 2 + \lceil 4(n-n_1)/9 \rceil \leq 2 + \lceil 4(n-5)/9 \rceil \leq \lceil 4n/9 \rceil.$$

If $|V(T_1)| = 4$, then there is only a leaf w hanging from v. Consider the trees $T' = T(u, u_3u_4)$ and $T'' = T(u', u_3u_4)$. Consider the set $S = V(T') \setminus \{u, u_1, v, w\}$. Observe that the vertices of S are at distance at most 3 from u_3. We distinguish the following cases.

i) *Every vertex in S is at distance at most 1 from u_3.* In such a case, T' has 5 or 6 vertices and the function such that $f(v) = 2$ and $f(x) = 0$, if $x \neq v$, is a dominating 2-broadcast function on T'. By inductive hypothesis and using Proposition 1 and Lemma 6 we obtain:

$$\gamma_{B_2}(T) \leq \gamma_{B_2}(T') + \gamma_{B_2}(T'') \leq 2 + \lceil 4(n-5)/9 \rceil \leq \lceil 4n/9 \rceil.$$

ii) *There is at least one vertex in S at distance 2 from u_3, but there are no vertices at distance 3.* Then T' has at least 7 vertices and the function such that $f(u) = 1$, $f(u_3) = 2$ and $f(x) = 0$, if $x \neq u, u_3$, is a dominating 2-broadcast on T'. By inductive hypothesis and using Proposition 1 and Lemma 6 we obtain:

$$\gamma_{B_2}(T) \leq \gamma_{B_2}(T') + \gamma_{B_2}(T'') \leq 3 + \lceil 4(n-7)/9 \rceil \leq \lceil 4n/9 \rceil.$$

iii) *There is at least one vertex in S at distance 3 from u_3.* Consider a vertex t at distance 3 from u_3 and let u_3, t_1, t_2, t be the (u_3, t)-path in T'. Consider the connected component $C_t = T'(t, u_3t_1)$ of T_1. We may assume that C_t has exactly 4 vertices, otherwise we can proceed as in the preceding cases because t and u' are antipodal and we are done. Therefore, it only remains to prove that the result holds when, for every vertex t of S at distance 3 from u_3, C_t has exactly 4 vertices. In such a case, the tree induced by C_t must be a path t', t_1, t_2, t. Consider the function f defined on $V(T')$ such that $f(u_3) = 2$, $f(u) = 1$, $f(t) = 1$, if $d(t, u_3) = 3$, and $f(x) = 0$ otherwise. On one hand, f is a dominating 2-broadcast on T' with cost $r + 3$, where r is the number of vertices in S at distance 3 from u_3. On the other hand, T' has at least $4r + 5$ vertices (see Figure 4). It is

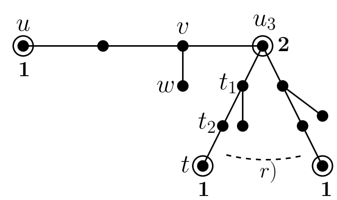


Figure 4: The tree T' of case iii) has order $n \geq 4r + 5$ and $\gamma_{B_2}(T') \leq 3 + r$.

straightforward to check that $r \geq 1$ implies $\dfrac{r+3}{4r+5} \leq \dfrac{4}{9}$. Therefore, by inductive hypothesis and using Proposition 1 and Lemma 6 we have:

$$\gamma_{B_2}(T) \leq \gamma_{B_2}(T') + \gamma_{B_2}(T'') \leq (r + 3) + \lceil 4(n - (4r + 5))/9 \rceil \leq \lceil 4n/9 \rceil.$$

It remains to prove that the given bound is tight. For this purpose, consider for every $m \geq 1$ the tree T_n of order $n = 9m$ obtained as follows. Let C be the caterpillar of order 9 obtained by hanging a leaf to the vertices u_3 and u_5 of the path $u_1 \dots u_7$ of order 7. Take m copies $C_1, \dots, C_m$ of C, and add $m - 1$ edges joining central vertices of consecutive copies (see Figure 5). Then, T_n has order $9m$ and

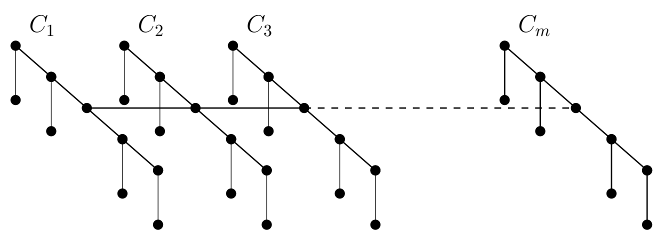


Figure 5: The tree T_n of order $n = 9m$ satisfies $\gamma_{B_2}(T_n) = \lceil 4n/9 \rceil$.

we claim that $\gamma_{B_2}(T_n) = 4m \leq \lceil 4n/9 \rceil$. Observe that the leaves of a copy of C do not hear any vertex of another copy, and thus, the sum of the values restricted to the vertices of a copy of C of any dominating 2-broadcast is at least 4. Therefore, $4m \leq \gamma_{B_2}(T_n) \leq \lceil 4n/9 \rceil = 4m$. $\square$

Corollary 11. *For every graph G of order n,*

$$\gamma_{B_2}(G) \leq \lceil 4n/9 \rceil.$$

4 Bounds on $\gamma_{B_k}(G)$

In this section, we give a general upper bound on $\gamma_{B_k}(G)$ for any graph G. By Theorem 5, it is sufficient to prove the bound for trees. Herke and Mynhardt (see [8, 9]) showed that $\gamma_B(T) \leq \lceil \frac{n}{3} \rceil$ for every tree of order n, and, as we have noticed before, we have $\gamma_{B_k}(T) = \gamma_{B_r}(T) = \gamma_B(T)$, whenever $k \geq r = rad(T)$. A consequence of these facts is the following result.

Proposition 12. *Let T be a tree of order n. If $k \geq rad(T)$, then $\gamma_{B_k}(T) \leq \lceil \frac{n}{3} \rceil$.*

Next, we give an upper bound on $\gamma_{B_k}(T)$ for $k < rad(T)$.

Theorem 13. *Let T be a tree of order n and let $k \geq 1$ be an integer such that $k < rad(T)$. Then,*

$$\gamma_{B_k}(T) \leq \left\lceil \frac{k+2}{k+1} \frac{n}{3} \right\rceil.$$

Proof. We know that $\gamma_{B_1}(T) = \gamma(T) \leq n/2$, and Proposition 10 states that the result is true for $k = 2$. Now set $k \geq 3$. Suppose to the contrary that the bound does not hold for some tree with radius greater than k. Let T be a tree of minimum order n and radius greater than k not satisfying the bound. On the one hand, observe that T is a tree of radius at least $k+1$ and any path of order n satisfies the upper bound, and thus $n \geq 2k+2$. On the other hand, observe that the given bound holds for every tree T' with radius $rad(T') \leq k$, since by Proposition 12,

$$\gamma_{B_k}(T') \leq \lceil n/3 \rceil \leq \left\lceil \frac{k+2}{k+1} \frac{n}{3} \right\rceil.$$

Consequently, the given bound is satisfied by every subtree T' of T, different from T. Next, we show some more properties of the tree T needed to prove the theorem.

Claim 1. *T has no twins.*

Proof. Suppose to the contrary that T has twins and let T^* be its associated twin-free tree of order n^*. Then, T^* satisfies the given bound, since it has less vertices than T. By Proposition 3,

$$\gamma_{B_k}(T) = \gamma_{B_k}(T^*) \leq \left\lceil \frac{k+2}{k+1} \frac{n^*}{3} \right\rceil \leq \left\lceil \frac{k+2}{k+1} \frac{n}{3} \right\rceil.$$

□

Now, consider a pair u and u' of antipodal vertices of T, and the (u, u')-path of length D, $u, u_1, \ldots, u_k, u_{k+1}, \ldots, u_{D-1}, u'$. Observe that $k < r < D = d(u, u')$ and $D \in \{2r-1, 2r\}$. For $i \geq 1$, let $T(u_i)$ be the connected component of $T - \{u_{i-1}u_i, u_iu_{i+1}\}$ containing u_i and let u_i' be an eccentric vertex of u_i in $T(u_i)$. Let $d_i = d(u_i, u_i') = ecc_{T(u_i)}(u_i)$ (see Figure 6). If $d_i = 0$, then $u_i' = u_i$. Note that $d_1 = 0$, since T has no twins, and $0 \leq d_i \leq i$ whenever $1 \leq i \leq k+1$, as u and u' are antipodal.

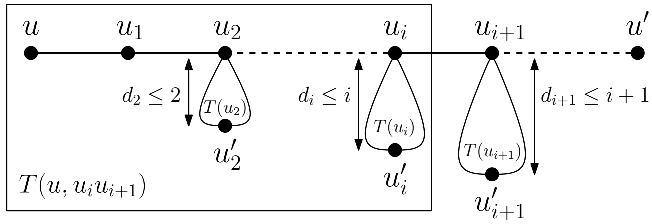


Figure 6: Vertex u'_i is such that $d(u_i, u'_i) = ecc_{T(u_i)}(u_i) = d_i$.

Claim 2. *The set of vertices belonging to the tree $T(u, u_{k+1}u_{k+2})$ does not contain two adjacent vertices x and y satisfying one of the following conditions: (i) $\deg_T(x) = \deg_T(y) = 2$; (ii) $\deg_T(x) = \deg_T(y) = 3$ and both have a leaf as a neighbor.*

Proof. First, suppose that x and y are adjacent vertices such that $\deg_T(x) = \deg_T(y) = 2$. If the edges incident to those vertices are $x'x$, xy and yy', then consider the trees $T(u, x'x)$, $T(u, xy)$ $T(u, yy')$, if x and y belong to the (u, u_{k+1})-path, and the trees $T(u'_j, x'x)$, $T(u'_j, xy)$ $T(u'_j, yy')$, if x and y belong to a tree $T(u_j)$, $2 \leq j \leq k+1$. In both cases, those trees have radius at most k and consecutive orders, so at least one of them, say T', has order multiple of 3. In a similar way, if $\deg(x) = \deg(y) = 3$ and the edges incident to those vertices are $x'x$, xy, yy', $x''x$, yy'', where x'' and y'' are leaves, then consider the trees $T(u, x'x)$, $T(u, xy)$ $T(u, yy')$, if x and y belong to the $u - u_{k+1}$ path, and the trees $T(u'_j, x'x)$, $T(u'_j, xy)$ $T(u'_j, yy')$, if x and y belong to a tree $T(u_j)$, $2 \leq j \leq k+1$. In both cases, those trees have radius at most k and order m, $m+2$, $m+4$ respectively, for some integer m. Thus, at least one of them, say T', has order a multiple of 3.

In both cases, we have a tree $T' = T(w, e)$ of radius at most k and order $n' = 3t$, $t \in \mathbb{Z}$, for some vertex w and some edge e. If T'' is the other connected component of $T - e$, then T'' has order less than n. Therefore, by Proposition 1 and Lemma 6,

$$\begin{aligned}\gamma_{B_k}(T) &\leq \gamma_{B_k}(T') + \gamma_{B_k}(T'') = \gamma_B(T') + \gamma_{B_k}(T'') \\ &\leq \lceil n'/3 \rceil + \left\lceil \frac{k+2}{3k+3}(n - n') \right\rceil = t + \left\lceil \frac{k+2}{3k+3}(n - 3t) \right\rceil \leq \left\lceil \frac{k+2}{3k+3}\, n \right\rceil,\end{aligned}$$

which is a contradiction. □

Claim 3. *If e is an edge of T, then any dominating k-broadcast f' on a connected component T' of $T - e$ satisfies*

$$\frac{\omega(f')}{n'} > \frac{k+2}{3k+3},$$

where n' is the order of T'.

Proof. Suppose to the contrary that $\frac{\omega(f')}{n'} \leq \frac{k+2}{3k+3}$ for some dominating k-broadcast function f' on T'. The other connected component of $T - e$, T'', satisfies Bound 13 because its order is less than n. Then, by Lemma 6 and Proposition 1 we have:

$$\begin{aligned}\gamma_{B_k}(T) &\leq \gamma_{B_k}(T') + \gamma_{B_k}(T'') \\ &\leq \omega(f') + \left\lceil \frac{k+2}{3k+3}(n-n') \right\rceil \leq \left\lceil \frac{k+2}{3k+3}\ n \right\rceil = \left\lceil \frac{k+2}{k+1}\ \frac{n}{3} \right\rceil.\end{aligned}$$

which is a contradiction. □

Claim 4. *There exists $i \in \{2, \dots, k\}$ such that $d_i \geq 2$. Moreover, if k is even, then $d_i \geq 2$ for some $i < k$.*

Proof. Suppose to the contrary that $d_i \in \{0, 1\}$ for every $i \in \{2, \dots, k\}$. By Claim 2, it is not possible that $d_i = d_{i+1} = 0$ or $d_i = d_{i+1} = 1$ for some $i \in \{2, \dots, k\}$. Therefore, by Claim 1, the vertices $u_2, u_3, \dots, u_k$ have degree $2, 3, 2, \dots$ respectively, and there is a leaf hanging from the vertices of degree 3.

For k odd, let $T' = T(u, u_k u_{k+1})$. Then, T' has order $n' = k + 1 + \lfloor k/2 \rfloor = (3k+1)/2$. If w is a center of the (u, u_k)-path, then we define the dominating k-broadcast function f' on T' such that $f'(w) = (k+1)/2$ and $f'(x) = 0$, otherwise. It is straightforward to check that for any odd integer $k \geq 1$,

$$\frac{\omega(f')}{n'} = \frac{k+1}{3k+1} \leq \frac{k+2}{3k+3}.$$

For k even, consider $T' = T(u, u_{k-1} u_k)$. The tree T' has order $n' = (k-1) + 1 + \lfloor (k-1)/2 \rfloor = (3k-2)/2$. Let w be a center of the (u, u_{k-1})-path. Consider the dominating k-broadcast function f' such that $f'(w) = k/2$ and $f'(x) = 0$, otherwise. It is straightforward to check that for any even integer $k \geq 4$,

$$\frac{\omega(f')}{n'} = \frac{k}{3k-2} \leq \frac{k+2}{3k+3}.$$

In both cases we get a contradiction by Claim 3. □

Now, let us calculate a lower bound on the order of some subtrees of T. For this purpose, let us define the function $B : \mathbb{N} \mapsto \mathbb{N}$, such that for every $i \in \mathbb{N}$,

$$B(i) = \begin{cases} \frac{3i+2}{2} & \text{if } i \text{ is even,} \\ \frac{3i+1}{2} & \text{if } i \text{ is odd.} \end{cases}$$

Claim 5.
1. *$|V(T(u, u_i u_{i+1}))| \geq B(i)$, for every $i \in \{1, \dots, k-1\}$.*
2. *$|V(T(u_i))| \geq B(d_i - 1) + 1$, for every $i \in \{1, \dots, k-1\}$.*
3. *$|V(T(u, u_{k-1} u_k))| \geq B(k-1) + 1 = \frac{3k}{2}$, for k even.*
4. *$|V(T(u, u_k u_{k+1}))| \geq B(k) + 1$.*

Proof. 1. Besides the $i+1$ vertices of the (u, u_i)-path, the tree $T' = T(u, u_i u_{i+1})$ contains at least $\lfloor i/2 \rfloor$ vertices adjacent to the (u, u_i)-path by Claim 2. Therefore, T' has at least $i + 1 + \lfloor i/2 \rfloor$ vertices, and from here the desired result follows.

2. To prove the bound on the order of $T(u_i)$, let T'' be the connected component of $T - u_i$ containing the furthest vertex u_i' from u_i in $V(T(u_i))$. Notice that the tree $T(u_i)$ contains vertex u_i and at least the vertices of T''. Moreover, for the lower bound on the order of T'' we get the same lower bound as for the order of $T(u, u_{d_i-1}u_{d_i})$ by using Claim 2. Hence, $|V(T(u_i))| \geq |V(T'')|+1 \geq B(d_i-1)+1$.

3. Proceed as in the proof of item 1 and take into account that Claim 4 ensures in such a case the existence of at least one more vertex.

4. As in the preceding item, Claim 4 ensures in such a case the existence of at least one more vertex.

□

Claim 6. *For every $i \in \{1, \dots, k\}$, we have $d_i < i$.*

Proof. We know that $d_1 = 0 < 1$. Suppose to the contrary that $d_i = i$, for some $i \in \{2, \dots, k\}$. We find an edge e such that one of the connected components of $T - e$, say T', has order n' and there is a dominating k-broadcast satisfying $\frac{\omega(f')}{n'} \leq \frac{k+2}{3k+3}$, contradicting Claim 3. Having this in mind, we distinguish the following cases. In all of them, we use Claim 5 to calculate a lower bound on the order of T'.

Case 1. *$d_i = i$ for some odd integer i.* Let $T' = T(u, u_iu_{i+1})$. The function f' such that $f'(u_i) = i$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast satisfying $\omega(f') = i$. The order of T' satisfies $n' \geq B(i-1) + B(d_i - 1) + 1 \geq 2B(i-1) + 1 = 2\,\frac{3(i-1)+2}{2} + 1 = 3i$. Therefore,

$$\frac{\omega(f')}{n'} \leq \frac{i}{3i} = \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

Case 2. *$d_i = i$ for some even integer i and $i < k$.* We may assume that $d_{i+1} \neq i+1$, otherwise we proceed as in Case 1. Let $T' = T(u, u_{i+1}u_{i+2})$. We distinguish two subcases.

2.1 If $d_{i+1} \leq i-1$, then the function f' such that $f'(u_i) = i$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast on T' satisfying $\omega(f') = i$. On the other hand, $n' \geq 2\,B(i-1) + 2 \geq 2\frac{3(i-1)+1}{2} + 2 = 3i$.

Then
$$\frac{\omega(f')}{n'} \leq \frac{i}{3i} = \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

2.2 If $d_{i+1} = i$ and $i \geq 4$, then the function f' such that $f'(u_i) = i+1$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast on T' satisfying $\omega(f') = i+1$. Moreover, $n' \geq 3\,B(i-1) + 2 \geq 3\frac{3(i-1)+1}{2} + 2 = \frac{9i-2}{2}$. Then,

$$\frac{\omega(f')}{n'} \leq \frac{2(i+1)}{9i-2} \leq \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

2.3 If $d_{i+1} = i$ and $i = 2$, then $d_2 = d_3 = 2$. Consider the tree $T' = T(u, u_4u_5)$ of order n'. If $d_4 \leq 2$, then $n' \geq 9$ and the function f' such that $f'(u_3) = 3$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast satisfying $\omega(f') = 3$. If

$d_4 \in \{3,4\}$, then it is easy to check that $n' \geq 13$ and the function f' such that $f'(u_4) = 4$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast for $k \geq 4$, satisfying $\omega(f') = 4$, whenever $k \geq 4$. In both cases,

$$\frac{\omega(f')}{n'} \leq \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

Finally, suppose that $k = 3$ and $d_4 \in \{3,4\}$. Note that for $k = 3$, $\frac{k+2}{3k+3} = \frac{5}{12}$. If $d_4 = 3$, then $n' \geq 13$ and the function f' such that $f'(u_4) = 3$, $f'(u_2) = 2$ and $f'(x) = 0$, otherwise, is a dominating 3-broadcast satisfying $\omega(f') = 5$. Hence,

$$\frac{\omega(f')}{n'} \leq \frac{5}{13} \leq \frac{5}{12}.$$

If $d_4 = 4$, then let $x_1, \ldots, x_r$ be the vertices at distance 4 from u_4, where $r \geq 1$. For every $i \in \{1, \ldots, r\}$, let z_i be the vertex adjacent to u_4 in the unique (x_i, u_4)-path. If for some $i \in \{1, \ldots, r\}$, then there are at least 2 vertices at distance 4 from u_4 belonging to $T'' = T(x_i, u_4 z_i)$, then the function f'' such that $f''(z_i) = 3$ and $f''(x) = 0$, otherwise, is a dominating 3-broadcast on the tree T'' of order $n'' \geq 9$ (see Figure 7a). Thus,

$$\frac{\omega(f'')}{n''} \leq \frac{3}{9} \leq \frac{5}{12}.$$

Now, assume that for every $i \in \{1, \ldots, r\}$, x_i is the only vertex at distance 4 from u_4 lying on $T(x_i, u_4 z_i)$. Consider the tree $T' = T(u, u_4 u_5)$ of order n'.

If $r \geq 2$, then the function f' such that $f'(u_4) = 3$, $f'(u_3) = 2$, $f'(x_i) = 1$, for $i \in \{1, \ldots, r\}$, and $f'(x) = 0$, otherwise, is a dominating 3-broadcast on T' and $n' \geq 9 + 5r$ (see Figure 7b).

Moreover, $\omega(f') = 5 + r$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{5+r}{9+5r} \leq \frac{5}{12}.$$

If $r = 1$ and $n' \geq 15$, then the function f' such that $f'(u_2) = 2$, $f'(u_4) = 3$, $f'(x_1) = 1$ and $f'(x) = 0$, otherwise, is a dominating 3-broadcast with $\omega(f') = 6$ (see Figure 7c) and if $n' = 14$, then the function f' such that $f'(u_3) = 3$, $f'(y) = 2$ and $f'(x) = 0$, otherwise, is a dominating 3-broadcast with $\omega(f') = 5$ (see Figure 7d). In both cases,

$$\frac{\omega(f')}{n'} \leq \frac{5}{12}.$$

Case 3. *$d_k = k$ and k is even.* Let $T' = T(u, u_k u_{k+1})$. Consider the dominating k-broadcast function such that $f'(u_k) = k$ and $f'(x) = 0$, otherwise. Then $\omega(f') = k$. Since $d_k = k$, then $n' \geq 2\,\frac{3k}{2} + 1 = 3k + 1$. It is straightforward to check that if $k \geq 4$, then

$$\frac{\omega(f')}{n'} \leq \frac{k}{3k+1} \leq \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

□

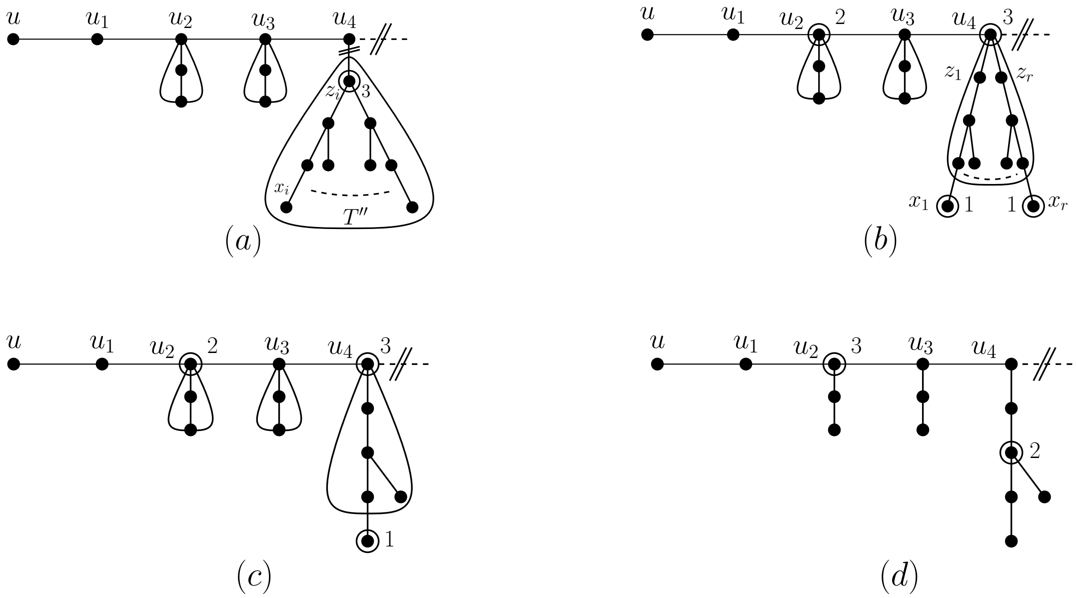


Figure 7: Dominating k-broadcasts illustrating case 2.3 of Claim 6.

Now, we proceed with the proof of the Theorem. Let i_0 be the minimum integer such that vertex u'_{i_0} is one of the furthest vertices from u in the tree $T(u, u_k u_{k+1})$, that is:

$$i_0 = \min\{i : 1 \le i \le k \text{ and } d(u, u'_i) \ge d(u, u'_j) \text{ for all } j \in \{1, \ldots, k\} \text{ such that } i \ne j\}.$$

(see Figure 8a).

Observe that, on the one hand, $i_0 > k/2$, otherwise $d(u, u'_{i_0}) < k = d(u, u_k) \le d(u, u'_k)$, which contradicts the definition of i_0. So, $i_0 \ge 2$. On the other hand, $k \le i_0 + d_{i_0} = d(u, u'_{i_0}) \le 2k$. Moreover, $d_{i_0} \ge 1$, as otherwise $d_{i_0 - 1}$ must be 0 by definition of i_0, contradicting Claim 2.

In all cases, we will find an edge e satisfying that the tree $T' = T(u, e)$ has order n' and admits a k-broadcast f' on T' such that $\omega(f')/n' \le (k+2)/(3k+3)$, leading

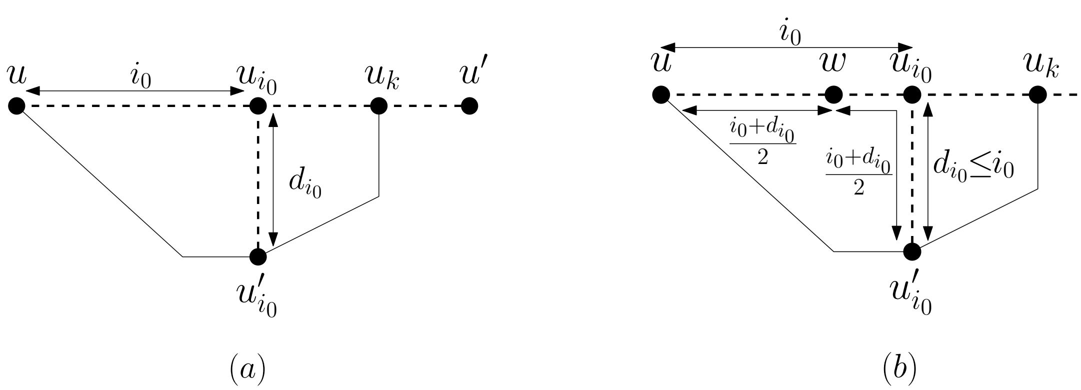


Figure 8: Vertex u'_{i_0} is one of the furthest vertices from u in the tree $T(u, u_k u_{k+1})$.

to a contradiction by Claim 3. A lower bound on the order of T' will be calculated using Claim 5.

Case 1. If $i_0 \leq k-1$, then let $T' = T(u, u_k u_{k+1})$. On the one hand, all the vertices of T' are at distance at most $\lceil d(u, u'_{i_0})/2 \rceil$ from a center w of the (u, u'_{i_0})-path (see Figure 8b). Thus, the function f' such that $f'(w) = \lceil d(u, u'_{i_0})/2 \rceil = \lceil (i_0 + d_{i_0})/2 \rceil$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast on T'. On the other hand, T' contains vertex u_{i_0+1} and the vertices of the trees $T(u, u_{i_0-1}u_{i_0})$ and $T(u_{i_0})$. Hence, $n' \geq B(i_0 - 1) + B(d_{i_0} - 1) + 2$. Now, we distinguish the following cases taking into account the parity of i_0 and d_{i_0}.

1.1 If i_0 and d_{i_0} are odd, then $n' \geq \frac{3(i_0+d_{i_0})+2}{2}$ and $\omega(f') = \frac{i_0+d_{i_0}}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{i_0 + d_{i_0}}{3(i_0 + d_{i_0}) + 2} \leq \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

1.2 If i_0 and d_{i_0} are even, then $n' \geq \frac{3(i_0+d_{i_0})}{2}$ and $\omega(f') = \frac{i_0+d_{i_0}}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{i_0 + d_{i_0}}{3(i_0 + d_{i_0})} = \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

1.3 If i_0 and d_{i_0} have distinct parity, then $n' \geq \frac{3(i_0+d_{i_0})+1}{2}$ and $\omega(f') = \frac{i_0+d_{i_0}+1}{2}$. Since $2k+1 \leq 2(i_0 + d_{i_0}) + 1 \leq 3(i_0 + d_{i_0})$, it can be easily checked that

$$\frac{\omega(f')}{n'} \leq \frac{i_0 + d_{i_0} + 1}{3(i_0 + d_{i_0}) + 1} \leq \frac{k+2}{3k+3}.$$

Case 2. If $i_0 = k$, then we distinguish the following cases.

2.1 $d_k > d_{k+1}$. Let $T' = T(u, u_{k+1}u_{k+2})$ and consider the dominating k-broadcast f' on T' such that $f'(w) = \left\lceil \frac{k+d_k}{2} \right\rceil$, where w is a center of the (u, u'_k)-path, and proceed analogously as in Case 1.

2.2 $d_k \leq d_{k+1}$. Recall that $d_{k+1} \geq d_k \geq 1$. Let $T' = T(u, u_{k+1}u_{k+2})$. The order n' of T' satisfies $n' \geq |V(T(u, u_k u_{k+1})| + |V(T(u_{k+1}))| \geq (B(k)+1) + B(d_{k+1}-1) + 1 = B(k) + B(d_{k+1} - 1) + 2$. We distinguish the following subcases:

(a) $d_{k+1} \leq k-1$. Since $\left\lceil \frac{(k+1)+d_{k+1}}{2} \right\rceil \leq k$, the function f' such that $f'(w) = \left\lceil \frac{(k+1)+d_{k+1}}{2} \right\rceil$ for a center w of the (u, u'_{k+1})-path, and $f'(x) = 0$, otherwise, is a dominating k-broadcast on T'

i) If $k+1$ and d_{k+1} are even, then $n' \geq \frac{3k+3d_{k+1}+3}{2}$ and $\omega(f') = \frac{(k+1)+d_{k+1}}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{k + d_{k+1} + 1}{3k + 3d_{k+1} + 3} = \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

ii) If $k+1$ and d_{k+1} are odd, then $n' \geq \frac{3k+3d_{k+1}+5}{2}$ and $\omega(f') = \frac{(k+1)+d_{k+1}}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{k + d_{k+1} + 1}{3k + 3d_{k+1} + 5} = \frac{1}{3} \leq \frac{k+2}{3k+3}.$$

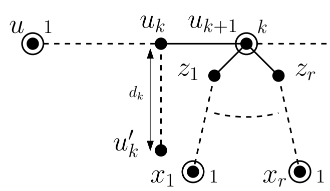


Figure 9: Vertex z_i is the only vertex adjacent to u_{k+1} on the (x_i, u_{k+1})-path, for every $i \in \{1, \dots, r\}$

iii) If $k+1$ and d_{k+1} have distinct parity, then $n' \geq \frac{3k+3d_{k+1}+4}{2}$ and $\omega(f') = \frac{(k+1)+d_{k+1}+1}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{k+d_{k+1}+2}{3k+3d_{k+1}+4} \leq \frac{k+2}{3k+3},$$

where it can be checked that the last inequality holds because $2k \leq 3(k+d_{k+1})+2$.

(b) $d_{k+1} = k$. By Claim 6, $d_i < i$ for $i \in \{1, \dots, k\}$, so the function f' such that $f'(u_{k+1}) = k$, $f'(u) = 1$ and $f'(x) = 0$, otherwise, is a dominating k-broadcast on T' satisfying $\omega(f') = k+1$. A lower bound on the order n' of T' is $B(k) + B(k-1) + 2 = \frac{6k+4}{2}$. Thus,

$$\frac{\omega(f')}{n'} \leq \frac{2(k+1)}{6k+4} \leq \frac{k+2}{3k+3}.$$

(c) $d_{k+1} = k+1$. Let $x_1, \dots, x_r$, $r \geq 1$, be the vertices of $T(u_{k+1})$ at distance $k+1$ from u_{k+1}. For each x_i, $i \in \{1, \dots, r\}$, let z_i be the vertex adjacent to u_{k+1} on the (x_i, u_{k+1})-path (see Figure 9). Since x_i and u' are antipodal, the preceding claims apply also by interchanging u and x_i. Thus, by Claim 6, x_i is the only vertex at distance $k+1$ from u_{k+1} in $T(x_i, u_{k+1}z_i)$. Consider the function f' such that $f'(u_{k+1}) = k$, $f'(u) = f'(x_1) = \cdots = f'(x_r) = 1$ and $f'(x) = 0$, otherwise. By Claim 6, $d_i < i$ for $i \in \{1, \dots, k\}$, and thus f' is a dominating k-broadcast function that satisfies $\omega(f') = k+1+r$.

Let us now calculate a lower bound on the order n' of $T' = T(u, u_{k+1}u_{k+2})$. Notice that, $n' = |V(T(u, u_k u_{k+1}))| + \sum_{j=1}^{r} |V(x_j, u_{k+1}z_j)| + 1 \geq (r+1)\,(B(k)+1)+1$.

If k is odd, then $n' \geq (r+1)\,(\frac{3k+1}{2}+1)+1 = \frac{(r+1)(3k+3)+2}{2}$, implying that

$$\frac{\omega(f')}{n'} \leq \frac{2(k+1+r)}{(r+1)(3k+3)+2}.$$

If the inequality

$$\frac{2(k+1+r)}{(r+1)(3k+3)+2} \leq \frac{k+2}{3k+3},$$

holds, we are done, but this inequality is equivalent to $3\,k^2\,(r-1) + 3\,k\,(r-1) + 4 \geq 0$ and it is easy to check that it is true for $r \geq 1$ and $k \geq 3$.

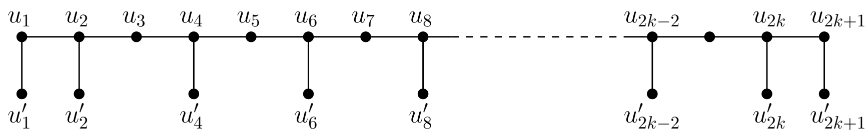


Figure 10: The tree T_k of order $n = 3k + 3$ satisfies $\gamma_{B_k}(T_k) = \lceil \frac{k+2}{k+1} \frac{n}{3} \rceil$.

Finally, if k is even, then $n' \geq (r + 1)\,(\frac{3k+2}{2} + 1) + 1 = \frac{(r+1)(3k+4)+2}{2}$. Thus, taking into account the calculations of the preceding case,

$$\frac{\omega(f')}{n'} \leq \frac{2(k + 1 + r)}{(r + 1)(3k + 4) + 2} \leq \frac{2(k + 1 + r)}{(r + 1)(3k + 3) + 2} \leq \frac{k + 2}{3k + 3}.$$

This concludes the proof of the upper bound on γ_{B_k} for any $k \geq 3$. □

In the following propositions we give some trees attaining the upper bound on γ_{B_k}.

Proposition 14. *For every $k \geq 3$ and $r \leq k$, the path P_n, where $n = 2r+1$, satisfies $rad(P_n) = r \leq k$ and $\gamma_{B_k}(P_n) = \lceil n/3 \rceil$.*

Proof. It is known that a path of order n has radius $\lfloor n/2 \rfloor$ and broadcast number $\lceil n/3 \rceil$ (see [8, 9]). Therefore, for $n = 2r + 1$ we have $rad(P_n) = r \leq k$, and thus $\gamma_{B_k}(P_n) = \gamma_B(P_n) = \lceil n/3 \rceil$, since we have $rad(P_n) \leq k$. □

Proposition 15. *For every $k \geq 3$, there exists a tree T such that $rad(T) > k$ and $\gamma_{B_k}(T) = \left\lceil \dfrac{k+2}{k+1} \dfrac{n}{3} \right\rceil$.*

Proof. Let $k \geq 3$ and consider the tree T_k of order $3k+3$ and radius $r+1$ obtained from a path $P = u_1 u_2 \ldots u_{2k+1}$ by hanging a leaf u'_i to each vertex u_i, for $i = 1$, $i = 2k+1$ and for i even with $2 \leq i \leq 2k$ (see Figure 10). We claim that $\gamma_{B_k}(T_k) = k+2$. First observe that the set formed by the $k + 2$ support vertices is a dominating set, and thus $\gamma_{B_k}(T_k) \leq \gamma(T_k) = k + 2$ for every $k \geq 3$.

Next, we show that $\gamma_{B_k}(T_k) = k + 2$ by induction on $k \geq 3$. It is easy to check that $\gamma_{B_3}(T_3) = 5$.

Now, suppose that $k \geq 4$. We want to prove that $\gamma_{B_k}(T_{k-1}) = k + 1$ implies $\gamma_{B_k}(T_k) = k + 2$.

Suppose to the contrary that $\gamma_{B_k}(T_k) \leq k + 1$. Let f be an optimal dominating k-broadcast on T_k. On the one hand, by Proposition 1, we may assume that $f(u) = 0$ for every leaf u. On the other hand, f is a dominating $(k - 1)$-broadcast on T_k. Indeed, suppose to the contrary that there is some vertex x with $f(x) = k$, then there must be another vertex y such that $f(y) = 1$ and $f(z) = 0$, for $z \neq x, y$. But in such a case, it is not possible that all vertices hear a vertex of V_f^+ because there are two vertices at distance $2k + 2$ in T_k.

Notice that T_{k-1} is isomorphic to the tree induced by $V(T_k) \setminus \{u_1, u'_1, u'_2\}$. Suppose that u'_1 hears vertex $x \in V_f^+$. If $f(x) = 1$, then $x = u_1$ and u'_2 hears a vertex $y \neq u_1$. Hence, u_2 hears y and the restriction of f to the vertices of T_{k-1} is a dominating $(k - 1)$-broadcast with cost k, which contradicts that $\gamma_{B_k} mu(T_{k-1}) = k + 1$. If

$f(x) = 2$, we can assume that $x = u_2$. Then, the function g defined on the set of vertices of T_{k-1} such that $g(u) = f(u)$ if $u \neq u_2, u_3$, $g(u_2) = 0$ and $g(u_3) = 1$ is a dominating $(k-1)$-broadcast on T_{k-1} with $\omega(g) = k$, which is again a contradiction. Finally, if u'_1 hears a vertex $x = u_j$ with $f(u_j) = h \geq 3$, then $d(u_2, u_j) \leq h-2$ and $d(u_2, u_{j+1}) \leq h-1$. In such a case, the function g on $V(T_{k-1})$ such that $g(u) = f(u)$ if $u \neq u_j, u_{j+1}$, $g(u_j) = 0$ and $g(u_{j+1}) = h-1$ is a dominating $(k-1)$-broadcast on T_{k-1} with $\omega(g) = k$, a contradiction. This completes the proof. □

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