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Tesi Doctoral

**ASYMPTOTIC TRACKING WITH DC-TO-DC  
BILINEAR POWER CONVERTERS**

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*A l'Àngels, per tot.*

# Agraïments

A l'Enric Fossas, director de la tesi. Sempre has tingut una estona per dedicar-me quan ha calgut, una idea per sortir d'un impasse o per encetar un nou camí i, sobretot, paciència suficient per aguantar el meu ritme de treball sense fer mai cap comentari al respecte, fins i tot quan la velocitat tendia a zero i el que em mereixia era que m'engeguessis a pasturar d'una vegada. Espero que, després de tot, consideris que ha pagat la pena. Gràcies.

Als que heu perdut més o menys estones amb mí -no pas jo amb vosaltres!- parlant d'aspectes diversos relacionats amb la tesi. Ara mateix recordo el Domingo Biel, el Xavier Cabré, l'Antonio Capella, l'Amadeu Delshams, el Jaume Franch, el Joaquim Font, l'Armengol Gasull, el Robert Griñó i el Joan C. de Solà-Morales. També agraeixo a l'Àngel Raluy la feinada de correcció de l'anglès de mitja tesi. Demano disculpes de debò a tots els que em deixo.

Als meus pares, per tot allò que ha d'agrair un fill als seus pares en un moment com aquest. I, òbviament, per haver-me apuntat de petit a fer anglès.

A l'Àngels, pel que et dic a la dedicatòria.

Al Bob Dylan per la pregunta i als Stones per la resposta.

Barcelona, novembre de 2003

Josep M. Olm.

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# Chapter 1

## Introduction

Power electronic systems consist of one or more electronic power converters that use power semiconductor devices controlled by integrated circuits. Roughly speaking, a power converter transforms an input voltage  $V_i$  into an output voltage  $V_0$ .

There are different patterns to classify the converters used in power electronics: type of device, function, connexions between the different parts of the converter, etc. According to the last pattern there are the switched converters: they have switches that, adequately commanded (at higher frequencies compared with the line), produce DC or AC output voltage with line-close frequency.

Nowadays, DC-to-AC conversion has an important practical application in the field of uninterruptible power systems, currently known as UPS. Basic DC-to-DC switch mode power converters possess a very simple structure. During the last fifteen years, considerable research effort has been addressed to study the possibility of using them in DC-to-AC conversion schemes.

There are many commercially available DC-to-DC switch mode power converters, but only two topologies can be considered basic. These are the (step down) buck converter, which produces an output voltage lower than the input, and the (step-up) boost converter, provider of an output greater than the input. Both are mainly used in DC power supplies and DC motors velocity control. Variations of the two basic topologies constitute the rest of the set.

However, attention should be paid to another important structure despite being derived from the buck and the boost. The (up-down) buck-boost results after a cascade connection of the buck and the boost; it may provide an output voltage lower or greater than the input and it is used in power supplies.

It is worth mentioning that the space state model of the buck converter is linear, while the boost and buck-boost converter show nonlinear representations that increase the difficulty of the study.

The aim of this thesis is to achieve that the output voltage of the DC-to-DC buck, boost and buck-boost power converters can track periodic references. Robust schemes to eliminate disturbance effects in the tracking task will also be developed. Sliding modes will be used as control technique, and the obtained results will be validated by numeric simulation.

## **Distribution of the Contents**

The thesis is organized in chapters according to the following distribution:

Chapter 2 reviews the results reported in specialized literature that have a direct impact on the topics of the thesis. Chapter 3 deals with the exact and asymptotic tracking of a time varying reference by the load voltage of a step-down DC-to-DC power converter, indirectly controlled through the input current. Restrictions on reference signals, due to the fixed control gains of the device, are obtained. It also contains a brief survey of sliding mode control. Chapter 4 is devoted to achieve exact tracking of a periodic reference in a nonminimum phase, nonlinear control system by means of an inversion-based procedure. Restrictions on the signals to be followed are also derived. The plant parameters are assumed to be known. Chapter 5 studies



the use of the Galerkin method to approximately solve the inverse problem which appeared in the previous chapter, as well as the effect of its use in the tracking control of the system. Convergence and error analysis of the Galerkin sequences are performed. Chapter 6 considers the robustness of the nonlinear boost and buck-boost converters in the presence of load perturbations, and introduces an adaptive scheme that identifies the disturbed parameter and allows the asymptotic tracking of periodic signals at the output resistances. Chapter 7 proposes the use of bidirectional boost and buck-boost converters to perform a direct control of the output voltage, thus taking advantage of the insensitiveness to external disturbances offered by this type of control actuation. Finally, the main contributions of this thesis and a set of suggestions for further research are listed in chapter 8. Appendix A provides a summary of results in module theory related to linear control systems.

# Chapter 2

## Literature Review

This chapter contains a commented list of the relevant studies which have appeared in the specialized literature and are closely related to the thesis.

### 2.1 Introduction

The literature is organized into sections. Section 2.2 considers noteworthy texts developing variable structure theory. Section 2.3 quotes important contributions in the field of regulation of switched converters, which precedes a discussion of the asymptotic tracking problem in section 2.4. The studies carried on under the hypothesis of a locally constant reference hypothesis may be found in section 2.5. In section 2.6 attention is paid to the presence of the Galerkin method and the mapping degree theory in control literature.

### 2.2 Variable structure systems

Among the excellent books and tutorials on Variable Structure Systems (VSS), [Utk77] and [Utk92] are of particular interest; both of them are usually referenced in studies

that develop and apply such theory. The tutorial [DZM88] is also an outstanding piece of material. An interesting differential geometric exposition of certain aspects of sliding mode control may be found in [Sir88]. Finally, [HGH89] gathers a complete state of the art in VSS till 1989, with near 200 references.

Special attention will be given to Filippov's paper [Fil64], which furnishes a definition of the solution for differential equations with discontinuous right hand side. Other proposals also exist, but none is so related to the intuitive idea of the operation of switched systems.

## 2.3 Regulation of switching converters with sliding mode control

Switched mode DC-to-DC power converters [MUR89], [SB85], constitute a natural field of application of VSS techniques due to the abrupt topological changes that occur during operation. The first reports trying sliding mode control date from the beginning of the eighties. The different strategies used in the sliding mode regulation of switched converters have two common aspects. Firstly, a sliding surface that provides the desired asymptotic behavior when the converter dynamics are constrained to evolve on it. Secondly, the feedback control circuit that takes the system to the mentioned surface is designed.

The first device to be regulated with such technique is the buck converter [BMS83a], and an application to nonlinear converters is described in [VŠČ85]. The sliding surfaces are straight lines in the state space.

According to the mathematical model of [Bro72], a general treatment of the problem

for bilinear networks is in [Sir87], where the use of  $x_j - k$ -type sliding surfaces is reported to offer good robustness results. However, the basic DC-to-DC power converter structures are shown to need indirect control. Since the control action does not appear explicitly in the equation of the output voltage, the buck must be indirectly controlled through the input current (affine combinations of input current and output voltage are also allowed, whenever the input current has nonzero component). The nonlinear boost and buck-boost show nonminimum phase characteristics (i.e., unbounded internal dynamics) when the capacitor voltage is taken as the system's output. The drawback of indirect control is that system parameters appear in the switching surface equations, entailing undesirable sensitivity to disturbances.

A sliding mode control design based on the Lyapunov function approach was designed in [NFC95] for the regulation of a buck converter with input filter. The control strategy takes into account the filter oscillations and results in stable system behavior and good dynamic performance.

Sliding modes plus other control methods have also been used to regulate switched converters. State feedback linearization [Isi89], [NS90] is a good example. In [SVC86], [San89] and [SI89], linearizing transformations and posterior sliding control are performed in buck, boost and buck-boost systems. Rapid transients and robustness to certain parameters are observed.

A combination of dynamical input-output linearization and a backstepping controller design method results in an adaptive regulation of PWM<sup>1</sup> controlled nonlinear power supplies [SGZ96]. The devices demonstrate robustness to bounded and external disturbances.

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<sup>1</sup>A geometric equivalence between PWM and sliding mode control is proven in [Sir89].

Passivity based control techniques have been successfully tested for robust control of switched converters. In [SOE96a], the energy dissipation and passivity properties of the boost converter are incorporated into a sliding mode controller design for regulation tasks. Having in mind a dynamical model of the converters obtained with the Euler-Lagrange formulation [SOE96b], [Gar00] proposes a piecewise unstable, dynamical, adaptive feedback regulation combined with a suitable controller resetting strategy. The resetting is performed on the unstable duty ratio function, used as stabilizing controller, when a certain threshold is exceeded. Therefore, the duty ratio is constrained in a small vicinity of the required constant equilibrium value, evolving into a pseudo-sliding regime. Robust regulation of the capacitor voltage is reported. The Ph. D. Thesis [Esc99] uses port controlled hamiltonian modelling in the regulation of DC-to-DC converters and proposes several passivity based techniques, one of them combined with sliding mode control. This last strategy was experimentally proved to be very robust to source disturbances, but was highly sensitive to parameter uncertainties. However, using the energy-balancing method proposed in [OSME02], regulation of the output voltage and insensitivity to load resistance uncertainty with partial state measurement is achieved in [ROE00].

In [UGS99], an observer-based sliding mode control incorporates an ideal model simulated in the controller in parallel with the real plant. For the sliding mode control itself, real state measures are substituted by observer states which may converge to the real state. Reduction in the number of plant measurements is a proven fact, but high complexity in the controller design is a handicap.

Regulation of an uncertain boost converter is achieved in [SFRF03] by means of an algebraic, on-line parameter estimation algorithm, together with a particular certainty

equivalent linearizing state feedback controller.

## 2.4 Asymptotic tracking in DC-to-DC power converters

Time varying sliding surfaces were used twenty years ago in path control of robotic manipulators [Utk92]. The dynamic stabilization problem was reduced to a static one by means of an error variable that had to be taken to zero [SL91]. Sliding surfaces were defined using the error and its derivative in such a way that, when the system reaches it, the dynamics takes the error to zero exponentially. Following this idea, a sliding mode control scheme for both power conditioning or UPS systems was proposed in [CMOP88] and [CM96]. The equivalent circuit was modelled as a switched, linear, bidimensional system with state variables proportional to capacitor current and voltage. The sliding surface, built as an affine relation between state variables, allows tracking of a sinusoidal wave with the output voltage to be achieved. Insensitivity to parameter variations and robustness to disturbances were reported. Similar sliding surfaces have been successfully employed in [CFT93] and [JRM93] with a three phase inverter and a step-up converter. In the latter, the nonlinear character of the boost device results in the control action appearing inside the sliding surface equation.

Full bridge buck converters were used to track sinusoidal references by means of sliding mode and PWM control schemes in [BBBMVA89], via full state feedback and pole assignment.

A dynamical PWM feedback control scheme accomplishing indirect asymptotic output tracking, again in a full bridge power converter, is proposed in [SP93]. The approach uses Fliess' generalized observer canonical form (GOCF) [Fli90b] and partial inversion techniques. GOCF is also at the heart of a dynamic feedback strategy of sliding mode type for a chattering free, robust asymptotic output tracking suitable for some electromechanical systems [Sir93].

The study of a single phase inverter working under sliding mode control is performed in [PMP94]. The dynamic behavior of the inverter is governed by a linear system for which a Carpita-type [CMOP88] sliding surface is tested. The system shows overload and short circuit protection, while the use of a reduced observer eliminates load current measurements and improves noise insensitivity.

Constant switching frequency is required in [NL95] for the tracking control of a buck converter. Restrictions on the time varying output reference coincide with the ones obtained in [FO94a].

An interesting idea for robust generation is the concept of equivalent perturbations [FB96], [Bie99]. The authors prove that load disturbances may be considered equivalent to input voltage perturbations, being thus counteracted with external voltage injections. Knowledge of the load perturbations is assumed.

AC signal tracking task is achieved with a sliding feedback control scheme for a boost-buck converter [Bie99], [BFGR99]. Different sliding surfaces are proposed for the boost and buck stages. Experimental results validate the design for both linear and nonlinear loads.

The necessity of indirect control of the output voltage through the input current changes from regulation to tracking tasks. The obtention of the necessary current

for a given output voltage reference is easy in the step-down converter, but not so much for the nonlinear ones: a highly unstable nonlinear differential relation with no analytic solution appears. Different approaches have been proposed to solve the problem.

In [Sir99a], [Sir99b] and from a differential flatness [FLMR94] point of view, an approximate solution is obtained with a rapidly convergent iterative procedure. This line of research has produced no other results.

Other contributions [OFB96], [Bie99], [ZFSB98a], [ZFSB98b] suggest the calculation of an approximation to a periodic solution through the Harmonic Balance method. The robustness of the procedure is later studied in [FZ01], where an adaptation scheme for the identification of unknown parameters using FFT is suggested. Also, a transient optimization algorithm is provided.

In the communication [ACA00], the current indirect reference is obtained with a time reversal in the differential equation that describes the internal dynamics. However, the method is not applicable in the presence of perturbations, because a time inversion may produce a system response before the arrival of the disturbance. Extreme sensitivity to perturbations is expected. It also contains alternative proofs of analogous results to theorem 4.4.2 and corollary 4.4.3 of this thesis.

An extension of the algebraic on-line parameter identification approach, proposed in [SFRF03] for the regulation of a step-up converter, based on input and output measurements as commented at the end of section 2.3, is performed in [SFF02] for trajectory tracking in an uncertain double bridge buck converter. A controller of the generalized proportional integral type is proposed for the stabilization of the minimum phase output to the desired reference. Good performance is shown, even in



the presence of unmodelled stochastic disturbance inputs.

A discrete time sliding mode controller is suggested in [MVLLC00] for output voltage tracking in a boost converter. The switching surface is obtained by imposing a desired dynamic behavior to the system. The use of an adaptive law for the estimation of perturbed parameters results in fast transient response, absence of steady state errors and robust performance to input voltage and load disturbances.

We may finish the section with an overall vision of general results about tracking in nonminimum phase systems. As an introduction, we should say that the problem of asymptotic tracking in minimum phase SISO systems is solvable under the assumptions of relative degree  $r$  for the system, and boundedness for the reference and its first  $r$  derivatives [Isi89], [Sas99]. A sliding mode control law that also achieves asymptotic tracking in the presence of perturbation vector fields satisfying a (generalized) matching condition may be found in [Sas99].

Exact tracking of a known output reference for nonminimum phase systems is developed in [DCP96] and [DP98] both for the time invariant and time varying cases. The method tries to determine a bounded input-state trajectory that achieves a desired output behavior with an inversion-based procedure. If this is possible, a composite control law is used in such a way that its first component produces exact tracking and the second one stabilizes the overall system once linearized about the nominal trajectory. An interesting modified version of this work is contained in [Sas99].

However, plant uncertainties may negatively impact on output tracking performance in inversion based controllers. [Dev00] contains acceptance bounds on the size of the uncertainties under which is advantageous to use inverse feedforward for linear, time-invariant systems.

The robust asymptotic tracking problem, reduced to output regulation when working with tracking error variables, is considered in [IB90], [Sas99] for nonlinear systems. The trajectory to be followed and disturbances to be rejected are not known, but may range over the set of all possible trajectories of a given autonomous system, so-called exosystem. With a controller that incorporates an internal model of the exosystem, robust asymptotic stabilization to zero is guaranteed for every possible exogenous input in the class of signals generated by the exosystem. The main result states that if we stabilize the closed loop system, there will exist a center manifold that can be chosen so as to have zero output error on it. Such a manifold may be obtained solving a partial differential equation. In the paper [BI00] one may find an excellent overview of the output regulation problem for nonlinear systems, with comments on further improvements of the method and current research lines.

Recent results in [SS01] and [SS02] report both approximate and asymptotic output tracking in sliding modes for certain classes of nonminimum phase and uncertain nonlinear systems. The key is in the definition of a proper output reference profile to be followed by the system that avoids unstable internal states. It is known as the stable system center design because it is based on the center manifold theory [Isi89], [Sas99]. Application to boost and buck-boost converters [SZS02a] and to systems with output delay [SZS02b] has already been developed.

A dynamical sliding manifold is used in [SZS02c] to address asymptotic tracking in nonlinear boost and buck-boost devices. The classical sliding mode robustness to matched disturbances is enhanced with the accommodation to unmatched disturbances inherent to conventional dynamic compensators.

## 2.5 Locally constant reference hypothesis

An approximate treatment of the tracking problem in the fourth order Čuk converter is given in the series of articles [FMO92a], [FM93] and [FMO92b].

Assume that to regulate the state variable  $x_j$  of the single input system

$$x' = f(x) + g(x)u$$

to a level  $k_d$ , a sliding regime over the surface  $s(x, k_d)$  is created such that the ideal sliding dynamics possesses an asymptotically stable equilibrium point  $x^* = x^*(k_d)$ , being  $x_j^* = k_d$ . Hence, in case that the control target consists of the tracking of a time dependent but hypothetic locally constant reference  $k_d(t)$  by  $x_j$ , the new time varying sliding surface  $s[x, k_d(t)]$  may induce a sliding regime with approximately bounded stationary dynamics  $x^*(t) \approx x^*(k_d(t))$ , being also  $x_j^* \approx k_d(t)$ .

The technique shows good results and a considerable attenuating effect of load and input voltage perturbations. In practical applications, however, particular analysis is needed for each system in order to determine the validity of the approximation. Some restrictions to be satisfied by  $k_d(t)$  may be taken into account [OFB96].

Studies on the boost converter have been reported in [CB99] and [CFA01], where bounds on the steady state error are obtained.

## 2.6 Galerkin method and mapping degree in control systems

The Harmonic Balance [Kha92a], [Sas99] is a general method of prediction and approximation of limit cycles in nonlinear control systems. Essentially, the idea is to

represent the supposed periodic solution by a Fourier series, truncated at the  $n$ -th harmonic, and to look for a frequency and a set of Fourier coefficients which satisfy the system's dynamic equations. This search results in an overdetermined, infinite nonlinear system of equations involving frequency and Fourier coefficients which is hardly solvable. A simplification consists of reducing the problem to a finite dimensional one just by considering the first  $2n + 1$  equations, which are termed the *determining equations* of the nonlinear system.

For large  $n$ , the finite system may still be difficult to solve. A further simplification is performed if  $n = 1$  is taken, giving rise to the well known (first order) harmonic balance equation. Conditions under which the existence or inexistence of solution for the determining equations guarantee the existence or inexistence of a limit cycle in our system can be derived. This identifies the classical Describing Function method. In Functional Analysis, with a more general setting of operators in Hilbert spaces and complete orthonormal systems, the Harmonic Balance is a special case of the Galerkin method. The discussion about the conditions to be satisfied by the vector field  $f(x, t)$  to ensure the existence of a periodic solution for an  $x' = f(x, t)$ -type ODE if there is solution for the determining equations is presented in [Ces63], [Kno63] and [Ces64]. Success in that task, however, left two questions not completely solved: restrictions for the existence of solution for the determining equations and the validity of the approximation of the initial periodic solution given by them. In fact, this is a generalization of the describing function problem.

Later, the use of the mapping degree theory allowed an elegant mathematical justification for the describing function method [BF71] (a summary may be found in [Sas99]). Moreover, it gave the key to answer in [Maw71] and [GM77] the above

mentioned questions. The excellent textbooks [Zei93], [Zei90a], [Zei90b] and [Zei97] supply a complete idea of the importance of the Galerkin method and the mapping degree theory in Functional Analysis, with updated results.

Apart from that, the mapping degree theory is included as mathematical background material for the analysis of control systems in [Sas99].

In [Tad02], a Functional Analysis approach to approximate dynamic phasor models in bilinear dissipative systems with nonlinear lossless dynamics is performed. It studies the existence of periodic steady states, as well as the existence and convergence of approximate stationary solutions. Although the problem is not the same, the structure of the paper and the tools therein used are close to the posing and development of chapter 5 of the thesis.

# Chapter 3

## Tracking Control of the Buck Converter

In this chapter we will deal with the tracking of a time varying reference by the output voltage of a buck converter.

### 3.1 Introduction

The tracking problem in single input controllable linear systems was early faced by Slotine and Sastry in [SS83]. They derived a generalized, linear, time varying sliding surface and a control law such that the first component of the vector state of an  $n$ -dimensional system in Brunovsky canonical form could asymptotically track any signal with bounded  $n$ -th derivative, stated in advance.

However, when fixed gain systems are considered, as the buck converter, the restrictions to be satisfied by the desired output harden. The reason being that the sliding domain of the switching surface cannot be fitted to every reference. In other words, the fixed gains define a region in the phase plane out of which the control saturates. The ideal buck converter can be mathematically modelled as a linear, single input,

variable structure system, thus allowing it to be controlled by means of sliding mode control. A Carpita-type [CMOP88] time dependent sliding surface, obtained from an autonomous one that behaves appropriately in regulation tasks, is the key to make the output voltage of the converter follow a certain periodic signal. Restrictions for candidate references are derived. Also, a full bridge buck converter is used to track non offset signals. The results we are presenting here were partially reported in [FO94a], [FO94b] and [Olm94], with a slight difference: we are performing a change of state variable following again [CMOP88] to enhance the robustness of the device. The chapter is organized as follows. Section 3.2 contains a short survey on sliding mode control. Section 3.3 studies the ideal dynamic response of a general, variable structure control system with fixed gains, to a certain non autonomous switching surface. The description and solution of our problem in a class of linear systems that includes the buck converter is performed in section 3.4. As one of the main results of that section is expressed from a module theoretic approach, the reader is referred to appendix A for a brief review of its basic concepts. The buck converter is presented in section 3.5, where it is also studied as a regulator. The general methodology for tracking purposes developed in the former sections is applied to the converter in section 3.6; a sinusoidal signal is taken to exemplify the technique. Section 3.7 contains a robustness study and simulations are found in section 3.8.

## 3.2 Sliding regimes in switched systems

Essentially, a variable structure control uses a switching control law to direct the system trajectories towards a certain manifold of the phase plane, called the sliding or switching surface and, once there, maintains the evolution of the system constrained

to the manifold, giving rise to what is known as sliding mode or sliding regime.

This section contains a summary of the main results in the theory of variable structure systems and its associated sliding modes. The reader is referred to the excellent book of Utkin [Utk92] and the tutorials [DZM88], [Sir88], for background material.

### 3.2.1 Definitions

Consider the single-input, nonlinear dynamical system

$$x' = f(x) + g(x)u \quad (3.2.1)$$

where the state vector  $x$  belongs to an open subset  $D$  of  $\mathbb{R}^n$ . Moreover,  $f, g : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  are smooth vector fields with  $g(x) \neq 0, \forall x \in D$ . The switched control  $u : D \rightarrow \mathbb{R}$  acts as

$$u = \begin{cases} u^+(x) & \text{if } s(x, t) > 0 \\ u^-(x) & \text{if } s(x, t) < 0, \end{cases} \quad (3.2.2)$$

with  $u^+, u^-$  smooth, real scalar fields in  $D$ ; without loss of generality we let them satisfy  $u^+(x) > u^-(x)$  locally in  $D$ . The also scalar field  $s : D \times \mathbb{R} \rightarrow \mathbb{R}$  stands for a smooth function with non zero gradient on  $D$ . The set

$$S = \{(x, t) \in D \times \mathbb{R}, \quad s(x, t) = 0\}$$

defines a  $n - 1$  dimensional manifold in  $D \times R$  called the *switching surface*. The switching surfaces are designed in such a way that the motion of the system restricted to  $S$  exhibits a desired behavior, as regulation or tracking.

### 3.2.2 Existence of sliding modes

Suppose that, in a neighbourhood of the switching surface, the tangent vectors of the state trajectory always point towards  $S$ , thus resulting in the system evolving in an



immediate vicinity of  $S$ . This ensures the crossing of the surface from each side of it.

**Definition 3.2.1.** It is said that a *sliding mode* or a *sliding regime* locally exists on  $S$  iff there exists an open neighbourhood  $N$  of  $S$ ,  $N \cap S \neq \emptyset$ , such that

$$\frac{d}{dt}s^2(x, t) < 0, \quad \forall (x, t) \in N \setminus S,$$

where the derivative is evaluated along the trajectories of  $f + gu^+$  when  $s > 0$  and  $f + gu^-$  when  $s < 0$ .  $S$  is then called a *sliding surface*.

Ideal sliding modes appear only when  $x \in S, \forall t \geq t_0$ , which demands an infinitely fast switching frequency not attainable in real systems. This results in a motion within a neighbourhood of the switching surface, the so-called *chattering* phenomenon. A first approach, however, neglects the chattering effects and considers the dynamics ideally restricted to the sliding surface.

*Remark 3.2.1.* The definition of sliding modes in SISO systems given by definition 3.2.1 is quite intuitive and actually useful. However, its extension to the multi-input case must be carefully performed; see, for example, chapter 4 of [Utk92], which contains clarifying examples. An alternative definition may be found in [DZM88].

### 3.2.3 Dynamics on a sliding surface

The Filippov's [Fil64] method to determine the ideal dynamics of system (3.2.1) in sliding regime, i.e., over the surface  $s(x, t) = 0$ , known as *ideal sliding dynamics* (ISD), is computed as an average of the system's dynamics on both sides of the surface.

Let

$$h_{\pm}(x, u) = f(x) + g(x)u^{\pm}$$

denote the velocity vectors where  $s > 0$  and  $s < 0$ , respectively. Assume that a sliding regime has been created for system (3.2.1) with the control policy (3.2.2). The average field on the sliding surface  $h_0(x, u)$  is proposed to be found as a convex combination of the field values on each side:

$$h_0(x, u) = \alpha h_+(x, u) + (1 - \alpha)h_-(x, u).$$

The parameter  $\alpha$  is easily obtainable, because we may require the system trajectories to be tangent to the surface  $s(x, t) = 0$ . Therefore, solving the equation

$$\partial_x s \cdot h_0 + \partial_t s = 0$$

for  $\alpha$  yields:

$$\alpha = \frac{\partial_x s \cdot h_- + \partial_t s}{(u^- - u^+) \partial_x s \cdot g}.$$

The dynamics  $x' = h_0(x, u)$  over the sliding surface then becomes

$$x' = f(x) - g(x) \frac{\partial_x s \cdot f + \partial_t s}{\partial_x s \cdot g}. \quad (3.2.3)$$

*Remark 3.2.2.* An elegant introduction to discontinuous differential equations and sliding mode control using the notion of differential inclusion is provided in [Zol99].

### 3.2.4 The equivalent control method

The equivalent control method constitutes an alternative to Filippov's technique, easily applicable to the multi input case and with coincident results for single input systems.

The ideal dynamics of a system in sliding mode implies the existence of a continuous control  $u_{eq}$ , called the *equivalent control*, that maintains the system on the sliding

surface once it has been reached. Formally, the demands may be  $s(x, t) = 0$  and  $s'(x, t) = 0$ . From the last,

$$0 = \partial_x s \cdot x' + \partial_t s = \partial_x s \cdot (f + gu_{eq}) + \partial_t s,$$

which gives

$$u_{eq} = -\frac{\partial_x s \cdot f + \partial_t s}{\partial_x s \cdot g}, \quad \forall (x, t) \in S. \quad (3.2.4)$$

The dynamical system

$$x' = f(x) + g(x)u_{eq}, \quad (x, t) \in S, \quad (3.2.5)$$

describes the ISD, and the substitution of  $u_{eq}$  by its expression (3.2.4) leads straightforwardly to Filippov's motion model over the sliding surface (3.2.3). This last may be written

$$x' = F(f) - g(\partial_x s \cdot g)^{-1} \partial_t s,$$

with

$$F = [\mathbb{I} - g(\partial_x s \cdot g)^{-1} \partial_x s]. \quad (3.2.6)$$

It is trivially verifiable that  $F$  is a projection operator [Sir88] that maps  $\mathbb{R}^n$  vectors into the tangent space to the switching surface  $S$  in the direction of  $\text{span}\{g(x)\}$ , denoted  $T_x S$ . Therefore, the component of the vector field  $f(x)$  on such a direction has no influence on the ISD.

**Theorem 3.2.1.** *Let system (3.2.1) be controlled with the variable structure control law (3.2.2),  $S$  being the switching surface. Then,*

(i) *The equivalent control is well defined iff the transversality condition*

$$\partial_x s \cdot g \neq 0$$

is satisfied locally on  $S$ .

(ii) If the control gains  $u^+, u^-$  may possess arbitrary value, it is a sufficient condition for the existence of sliding regime in  $S$  that

$$\partial_x s \cdot g < 0.$$

(iii) If the control gains  $u^+, u^-$  cannot take arbitrary values, it is sufficient for the local existence of a sliding motion on  $S$  that the transversality condition is verified and

$$u^-(x) < u_{eq}(x, t) < u^+(x), \quad (3.2.7)$$

locally  $\forall(x, t) \in S$ . ■

*Remark 3.2.3.* Notice from theorem 3.2.1 that:

- (i) Statement (i) requires the vector field  $g$  is not tangential to the sliding manifold  $S$ .
- (ii) The sign of the transversality condition, taken as negative in statement (ii), is quite arbitrary and depends on the orientation given to  $S$  and on the sign of  $u^+ - u^-$ .
- (iii) If the assumption  $u^+ < u^-$  had been made in the definition of the control policy, the inequalities of (3.2.7) may exhibit opposite orientation. This is the reason why such a condition is generally expressed as [Utk92]

$$\inf\{u^+, u^-\} < u_{eq} < \sup\{u^+, u^-\}.$$

- (iv) In the case that (3.2.7) does not hold  $\forall(x, t) \in S$  but in a certain  $S_d \subset S$ , a sliding regime in a subset, known as *sliding domain* or *sliding zone*.

The following result establishes that, if a sliding regime exists on  $S$ , it is always possible to design particular control gain values and a control policy that drives the system to  $S$ .

**Corollary 3.2.2.** *If a sliding regime locally exists on  $S$ , the switching logic*

$$u = k|u_{eq}(x, t)|\text{sign}[s(x, t)], \quad k > 1,$$

*allows the system to achieve the sliding regime.* ■

*Remark 3.2.4.* The former corollary is not always applicable when the system has fixed gain values.

### 3.2.5 The switching frequency

As it has been previously mentioned in subsection 3.2.2, the infinitely fast velocity of the switch that is necessary for the sliding modes to exist is not attainable in real physical systems, inducing the appearance of the chattering phenomenon. The modelling of such effect is commonly done through the use of a comparator with a hysteresis band of amplitude  $2\Delta s_h$ .

[NFC95] provides a calculation of the maximum switching frequency that can be achieved in a (3.2.1)-type system commanded with a control law as (3.2.2), assuming the existence of a hysteresis band in the comparator and supposing for the sliding surface a linear behavior around switching states, as depicted in figure 3.1.

Notice that

$$s'(x, t; u) = \partial_x s \cdot x' + \partial_t s = \partial_x s \cdot f + \partial_t s + u \partial_x s \cdot g;$$

when the system evolutions in sliding mode, with infinite switching frequency, it happens that  $s'(x, t; u_{eq}) = 0$  (see subsection 3.2.4). Then, from equality (3.2.4),

$$\partial_x s \cdot f + \partial_t s = -u_{eq} \partial_x s \cdot g.$$

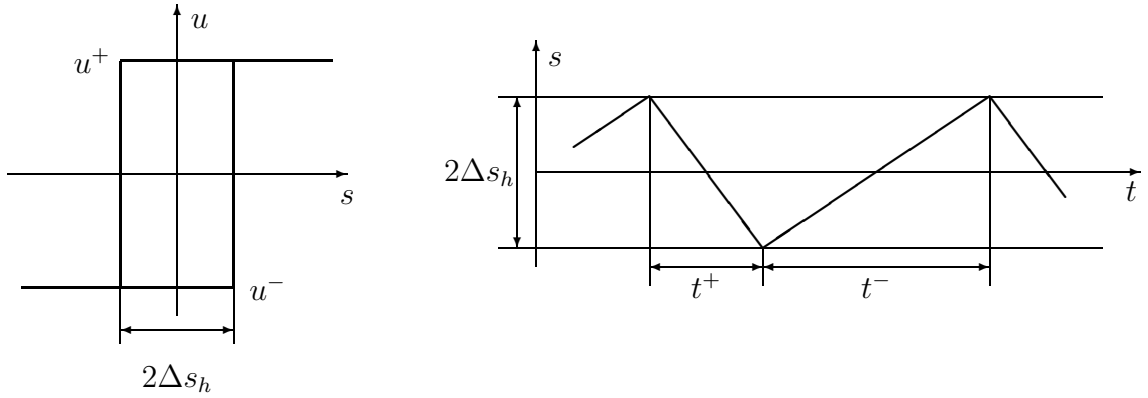


Figure 3.1: Switching surface hysteresis

This allows to write

$$s'(x, t; u) = (u - u_{eq})\partial_x s \cdot g.$$

Assuming for the sliding surface a linear evolution around  $s(x, t) = 0$ , it results

$$s'(x, t; u^+) = -\frac{2\Delta s_h}{t^+}, \quad s'(x, t; u^-) = \frac{2\Delta s_h}{t^-}.$$

Then, the switching frequency  $\nu_s$  is

$$\nu_s = \frac{1}{t^+ + t^-} = \frac{(u^+ - u_{eq})(u_{eq} - u^-)|\partial_x s \cdot g|}{2\Delta s_h(u^+ - u^-)},$$

where the usual hypothesis about the negativity of  $\partial_x s \cdot g$  and  $u^- - u^+$  has been assumed.

[Bie99] contains a detailed study of the maximum and minimum values of  $\nu_s$ , particularized for the tracking of a sinusoidal wave by the output voltage of a buck converter.

A simpler analysis, however, assuming fixed gains and  $|\partial_x s \cdot g| = 1$ , reduces the problem to the optimization of

$$\nu_s = \nu_s(u_{eq}) = \frac{(u^+ - u_{eq})(u_{eq} - u^-)}{2\Delta s_h(u^+ - u^-)}$$

for  $\nu_s \in [0, +\infty)$ . It is straightforward that  $\nu_s(u_{eq})$  is an inverted parabola with  $\nu_s(u^+) = \nu_s(u^-) = 0$ , having its maximum at the vertex

$$u_{eq} = \frac{u^+ + u^-}{2},$$

where it reaches the value

$$\nu_{sM} = \frac{u^+ - u^-}{8\Delta s_h}. \quad (3.2.8)$$

### 3.2.6 Rejection of disturbances

$F$  governs the ISD of a sliding mode controlled system, acts as a projection operator and is responsible of its invariance to additive disturbances  $p(x, t)$  that are in the control vector direction.

Let system (3.2.1) suffer a field perturbation  $p(x, t)$  such that its dynamics become

$$x' = f(x) + g(x)u + p(x, t).$$

**Definition 3.2.2.** The ideal sliding mode is said to show a *strong invariance property* with respect to  $p(x, t)$  iff the ISD is independent of  $p(x, t)$ .

**Theorem 3.2.3.** *The ISD satisfies a strong invariance property with respect to  $p(x, t)$  iff*

$$p(x, t) \in \text{span}\{g(x)\},$$

*which is known as the matching condition.* ■

**Definition 3.2.3.** When a disturbance  $p(x, t)$  does not verify the matching condition, the respective ISD is said to exhibit a *weak invariance property*.

The analysis of the effect produced by a disturbance that does not verify the matching condition, also known as *Drazenovic's condition*, is straightforwardly performed. Consider the unique decomposition of  $p(x, t)$  in two vectors permitted by the operator  $F$ : one along the direction of  $g(x)$  and another along the tangent space to the sliding manifold  $S$ , namely,

$$p(x, t) = \alpha(x, t)g(x) + n(x, t),$$

where  $\alpha(x, t)$  is a scalar field and  $n(x, t)$  is a vector field.

Notice now that, from (3.2.6),

$$F[p(x, t)] = F[n(x, t)] = n(x, t).$$

Moreover, denoting  $u_{eq}$  and  $u_{eq}^p$  the equivalent controls of the unperturbed and the perturbed system, respectively, it results from (3.2.4) that

$$u_{eq}^p = u_{eq} - \frac{\partial_x s \cdot p(x, t)}{\partial_x s \cdot g} = u_{eq} - \alpha(x, t). \quad (3.2.9)$$

This entails the following statement:

**Proposition 3.2.4.**  *$\alpha(x, t)$  does not affect the ISD and affects the existence of sliding regime. In its turn,  $n(x, t)$  affects the ISD and does not affect the existence of sliding regime.* ■

*Remark 3.2.5.* In case of additive disturbances  $g_p(x, t)$  in the system's input channel, i.e., such that  $g(x) \rightarrow g(x) + g_p(x, t)$ , definition 3.2.2 and theorem 3.2.3 are entirely applicable; to apply this last result, we may consider  $p(x, t) = g_p(x, t)u$ . About the ISD and the existence of sliding regime, departing from the decomposition provided by  $F$ , that is,

$$g_p(x, t) = \beta(x, t)g(x) + q(x, t),$$



with  $\beta(x, t)$  and  $q(x, t)$  scalar and vector fields, respectively, straightforward calculations leads to

$$\begin{aligned} u_{eq}^p &= \frac{1}{1 + \beta(x, t)} u_{eq}, \\ x' &= f(x) + g(x)u_{eq} + \frac{q(x, t)}{1 + \beta(x, t)} u_{eq}. \end{aligned}$$

This allows to conclude that  $\beta(x, t)$  affects both the existence of sliding regime and, also, the ISD when  $q(x, t) \neq 0$ , while  $q(x, t)$  just affects the ISD.

*Remark 3.2.6.* Different strategies to achieve a robust performance in sliding mode controlled systems have been developed for situations where the matching condition is not satisfied. Some of them have already been commented in chapter 2.

### 3.3 Time dependent sliding surfaces and tracking control in systems with fixed gains

This section is devoted to a generic study of the ideal dynamic response of a system on a certain time dependent sliding surface, derived from an autonomous one that provides good performance in regulation tasks.

Let system (3.2.1) be commanded by a control law as (3.2.2), the control gains being fixed:  $u^+, u^- \in \mathbb{R}$ ,  $u^- < u^+$ . Consider also that  $s$  stands for the autonomous switching surface

$$s(x) := v(x) - w^*,$$

being  $w^*$  a constant real value and  $v(x)$  a smooth real function of the state such that  $\partial_x v \cdot g < 0$ . Assume that a sliding regime exists for system (3.2.1), and that its ISD shows an asymptotically stable equilibrium point.

The behavior of the ISD on a time-dependent switching surface

$$\hat{s}(x, t) := v(x) - w(t),$$

obtained by substitution of the constant term  $w^*$  of  $s$  for a smooth real scalar function  $w = w(t)$  is studied below.

**Proposition 3.3.1.** *The transversality condition is satisfied on  $\hat{s}$ .*

**Proof.** Notice that  $\partial_x \hat{s} \cdot g = \partial_x(v - w) \cdot g = \partial_x v \cdot g \neq 0$ , because the transversality condition is satisfied on  $s$  by hypothesis. ■

**Proposition 3.3.2.** *The control law*

$$u = \begin{cases} u^+ & \text{if } \hat{s} > 0 \\ u^- & \text{if } \hat{s} < 0 \end{cases}$$

*directs system (3.2.1) towards the switching surface  $\hat{s}(x, t) = 0$ .*

**Proof.**  $V(x, t) = \frac{1}{2} \hat{s}^2(x, t)$  satisfies the first and second Lyapunov conditions. Moreover,

$$V' = \hat{s} \hat{s}' = \hat{s}(\partial_x v \cdot x' - w') = -|\partial_x v \cdot g| \hat{s}(u - u_{eq}) < 0, \quad \forall(x, t), \quad \hat{s}(x, t) \neq 0,$$

with the proposed control law, aided by the hypothesis  $\partial_x v \cdot g < 0$ . ■

**Proposition 3.3.3.** *A necessary and sufficient condition for system (3.2.1) to exhibit sliding mode on  $\hat{s}$  is*

$$u^- < \frac{\partial_x v \cdot f - w'}{|\partial_x v \cdot g|} < u^+.$$

**Proof.** Let  $\hat{u}_{eq}$  be the equivalent control that governs the ISD of system (3.2.1) over  $\hat{s}$ . From equation (3.2.7) in theorem 3.2.1 (iii), the fulfillment of  $u^- < \hat{u}_{eq} < u^+$  is a necessary and sufficient condition for system (3.2.1) to exhibit sliding regime on  $\hat{s}$ .

Trivially,  $\partial_x \hat{s} = \partial_x v$  and  $\partial_t \hat{s} = -w'$ . Therefore,

$$\hat{u}_{eq} = -\frac{\partial_x v \cdot f - w'}{\partial_x v \cdot g} = \frac{\partial_x v \cdot f - w'}{|\partial_x v \cdot g|},$$

and the result follows immediately. ■

We have already seen that the ISD of a system on a sliding surface describes the motions that take place about the sliding surface assuming an infinitely fast switching velocity. This, of course, forces the system to be always on the surface. The differential equation that satisfies the ISD of a system when it slides on a surface comes from the substitution of the discontinuous control  $u$  for the corresponding equivalent control.

**Proposition 3.3.4.** *The ISD derived for system (3.2.1) when it slides over  $s$  and  $\hat{s}$  are, respectively,*

$$x' = f(x) + \frac{\partial_x v \cdot f}{|\partial_x v \cdot g|} g(x), \quad (3.3.1)$$

$$x' = f(x) + \frac{\partial_x v \cdot f - w'}{|\partial_x v \cdot g|} g(x). \quad (3.3.2)$$

**Proof.** Follows straightforward by simple calculation of the respective ISD equations according to (3.2.5). ■

### 3.4 Linear systems with fixed gains

Consider a controllable, time invariant, single input linear system, commanded by a function  $u$  with fixed control gains  $u^+$ ,  $u^-$ :

$$x' = Ax + bu, \quad (3.4.1)$$

where  $x, b \in \mathbb{R}^n$ ,  $u^+, u^- \in \mathbb{R}$  with  $u^- < u^+$  and  $A \in M_n(\mathbb{R})$ . We are interested in the tracking of a certain time varying reference  $f(t)$  by a state vector component, i.e.  $x_i$ , and the question is how to design a switching surface and a control law in order to asymptotically reach a steady state ISD where our control purpose may be accomplished. It will also be taken into account that, the gains being fix, they cannot be tuned attending to the new target (see remark 3.2.4).

The following result shows that the imposition of a reference for a single state component determines the behavior of all the vector state:

**Proposition 3.4.1.** *Consider system (3.4.1); if we fix  $x_i = f(t)$ , then it exist a vector  $\Phi(t) = (\Phi_1(t), \dots, \Phi_n(t))$  and a scalar function  $\bar{u}(t)$  such that  $(x, u) = (\Phi(t), \bar{u}(t))$  is a solution of (3.4.1), with  $\Phi_i(t) = f(t)$ .*

**Proof.** Let system (3.4.1) be considered as the finitely generated left  $\mathbb{R} \left[ \frac{d}{dt} \right]$ -module

$$\Lambda = \frac{\mathbb{R} \left[ \frac{d}{dt} \right] \langle x, u \rangle}{\mathbb{R} \left[ \frac{d}{dt} \right] \langle x' - Ax - bu \rangle}.$$

The controllability assumption implies that the module is free. As the system is single input, it allows one generator, say  $y$ . Thus, the state variables and the control input can be expressed in terms of  $y$  by:

$$\begin{aligned} x_j(y) &= \left( a_0^j I + a_1^j \frac{d}{dt} + \dots + a_{n_j}^j \frac{d^{n_j}}{dt^{n_j}} \right) y, \\ u(y) &= \left( u_0 I + u_1 \frac{d}{dt} + \dots + u_r \frac{d^r}{dt^r} \right) y, \end{aligned}$$

with  $a_i^j, u_k \in \mathbb{R}$ . Hence, the demand

$$x_i(y) = f(t)$$

defines an ODE in  $y$ . Let  $Y(t)$  be a solution; then,

$$\Phi_j(t) = x_j(Y(t)) \quad \text{and} \quad \bar{u}(t) = u(Y(t)),$$

$(\Phi, \bar{u})$  being a solution of system (3.4.1). ■

*Remark 3.4.1.*  $y - Y(t)$  could very well be used as switching surface. However, this surface does not necessarily satisfy the transversality condition as happens with the buck converter.

Assume that the generator of system (3.4.1), formerly denoted by  $y$ , is such that the sliding surface  $y - Y(t)$  does not satisfy the transversality condition. Assume that for the regulation purpose  $f(t) = x_i^*$  a sliding motion is created for system (3.4.1) on the switching surface

$$s = c^\top \cdot (x - \Phi^*) = 0, \quad (3.4.2)$$

where  $c \in \mathbb{R}^n$  and  $\Phi_j^* = x_j(Y^*)$  with  $x_i(Y^*) = x_i^*$ , the control law being

$$u = \begin{cases} u^+ & \text{if } s > 0 \\ u^- & \text{if } s < 0. \end{cases}$$

Without loss of generality assume that  $c^\top b < 0$ . Finally suppose that the behavior on  $s$  is such that  $\Phi^*$  is an asymptotically stable equilibrium point. Proposition 3.4.1 and section 3.3 suggest how to modify the switching surface  $s$  of (3.4.2) in order to ideally obtain  $x_i = f(t)$  in steady state.

Let the new switching surface be

$$\hat{s} = c^\top \cdot (x - \Phi(t)), \quad (3.4.3)$$

with  $\Phi(t) = (\Phi_1(t), \dots, \Phi_{i-1}(t), f(t), \Phi_{i+1}(t), \dots, \Phi_n(t))^\top$ . Next proposition describes the sliding mode properties of  $\hat{s}$  related to those of the original autonomous  $s$  and, at the same time, guarantees for the ISD the asymptotic stability of the solution  $\Phi = \Phi(t)$ .

**Proposition 3.4.2.** *Let us take  $\hat{s}$  as the time dependent hyperplane over which system (3.4.1) switches. Therefore,*

(i)  $\hat{s}$  satisfies the transversality condition.

(ii) The control law

$$u = \begin{cases} u^+ & \text{if } \hat{s} > 0 \\ u^- & \text{if } \hat{s} < 0 \end{cases}$$

locates system (3.4.1) on the switching surface  $\hat{s}(x, t) = 0$ .

(iii) The sliding domain for  $\hat{s}$  is given by

$$u^- < \frac{c^\top Ax - c^\top \Phi'}{|c^\top b|} < u^+. \quad (3.4.4)$$

(iv) The restrictions to be satisfied by the reference signal  $f(t)$  in order to have the system inside the sliding domain when it reaches the stationary state are:

$$|c^\top b|u^- < \inf_t \{c^\top [A\Phi - \Phi']\} < \sup_t \{c^\top [A\Phi - \Phi']\} < |c^\top b|u^+. \quad (3.4.5)$$

(v) The ISD of (3.4.1) restricted to  $\hat{s}$  possesses  $x = \Phi(t)$  as an asymptotically stable steady state solution.

**Proof.** (i), (ii) and (iii) follow directly from propositions 3.3.1, 3.3.2 and 3.3.3, respectively. About (iv), the inequalities of (3.4.5) are due to the substitution  $x = \Phi$  in (3.4.4).

(v) The ISD of the system over the surfaces  $s$  and  $\hat{s}$  are calculated in proposition 3.3.4. Since  $x = \Phi^*$  and  $x = \Phi(t)$  are solutions of both systems (3.3.1) and (3.3.2), the changes of variable  $e = x - \Phi^*$  and  $e = x - \Phi$  reduce both of them to

$$e' = Ae + \frac{c^\top \cdot Ae}{|c^\top \cdot b|} b.$$

Hence, the asymptotic stability of the equilibrium point  $\Phi^*$  over  $s$  forces the same behavior for  $x = \Phi(t)$  over its corresponding  $\hat{s}$ . ■

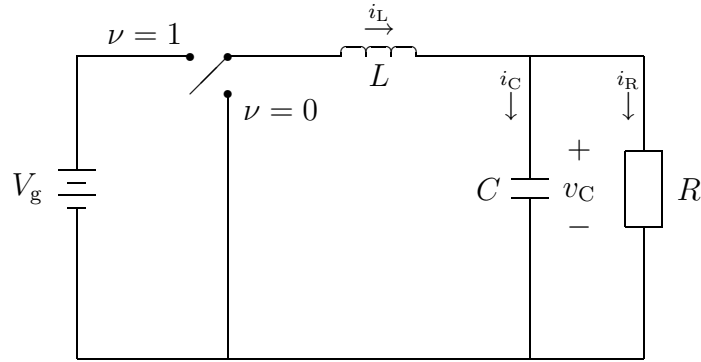


Figure 3.2: Buck converter circuit

*Remark 3.4.2.* Notice from the former proposition that:

(ii) The restrictions (3.4.5) should be taken into account when designing the plant parameters, thus starting from the reference to be tracked.

(ii) Instability of  $\Phi^*$  produces equal effect for  $\Phi(t)$ . Therefore, linear, autonomous sliding surfaces that induce unstable ISD in regulation tasks are unuseful when re-converted in the time dependent (3.4.3) for tracking purposes.

### 3.5 The buck converter

The Kirchhoff equations of the ideal buck converter, depicted in figure 3.2, are

$$\begin{aligned} i_L &= i_R + i_C \\ V_g \nu &= L \frac{di_L}{d\tau} + v_C, \end{aligned}$$

where

$$i_R = \frac{v_C}{R}, \quad i_C = C \frac{dv_C}{d\tau}.$$

$i_L$ ,  $i_R$  and  $i_C$  stand for input, capacitor and load currents, respectively. The communications [FO94a], [FO94b] and [Olm94], as in [Sir87], use  $i_L$  and  $v_C$  as state variables.

However, this formalism leaves load current disturbances as unmatched perturbations. To counteract these perturbations it is necessary to use an appropriate time varying sliding surface that incorporates on-line updating of the perturbed parameter through indirect observation.

In the series of papers [CMOP88], [CFT93] and [CM96], the use of the capacitor voltage  $v_C$  and its derivative  $\dot{v}_C$  for the state space representation of a certain switched network results in load perturbations being matched disturbances. The same state variable choice has identical effect in the buck system.

Hence, consider the phase space dynamic behavior of the ideal buck converter described by

$$LC \frac{d}{d\tau} \left( \frac{dv_C}{d\tau} \right) = -L \frac{d}{d\tau} \left( \frac{v_C}{R} \right) - v_C + V_g \nu, \quad (3.5.1)$$

$$\frac{d}{d\tau} (v_C) = \frac{dv_C}{d\tau}. \quad (3.5.2)$$

The control gain  $\nu$  takes values in the discrete set  $\{0, 1\}$ . The change of variables

$$x_1 = \frac{dx_2}{dt}, \quad x_2 = \frac{1}{|V_g|} v_C, \quad t = \frac{1}{\sqrt{LC}} \tau,$$

and the introduction of  $\lambda = R^{-1} \sqrt{LC}^{-1}$  and  $u = \nu \text{sign}(V_g)$ , adimensionalizes the model and minimizes the number of parameters, leaving it as

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} -\lambda' x_2 \\ 0 \end{pmatrix}, \quad (3.5.3)$$

where the possibility of appearance of a load disturbance  $R = R(t)$  has been taken into account. The value of the control gains  $u^-, u^+$  depends on the sense of the voltage source  $V_g$ . Thus, if its position coincides with that of figure 3.2, they take values in  $\{0, 1\}$ ; if the source has opposite sense, the set is  $\{-1, 0\}$ . Furthermore,



the full bridge buck converter also answers to the dynamical model (3.5.3), but with gains in  $\{-1, 0, 1\}$  or  $\{-1, +1\}$ , depending on the switching logic. From now on we will refer to them as  $\{u^-, u^+\}$  when we have no interest in distinguishing between them.

The basic reductor, non perturbed ( $R = R_N \in \mathbb{R}^+$ ) converter is a variable structure system described by (3.4.1), with controllability guaranteed by its Kalman matrix

$$(b, Ab) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}$$

having maximum rank. Notice also that, as the ideal buck model is controllable and single input, the module

$$\Lambda = \frac{\mathbb{R} \left[ \frac{d}{dt} \right] \langle x_1, x_2, u \rangle}{\mathbb{R} \left[ \frac{d}{dt} \right] \langle x_1' + \lambda x_1 + x_2 - u, x_2' - x_1 \rangle}$$

possesses  $x_2$  as an unique generator, because it is straightforward from (3.5.3) that

$$x_1 = x_2', \tag{3.5.4}$$

$$u = x_2'' + \lambda x_2' + x_2. \tag{3.5.5}$$

We are interested in the regulation of the state variable  $x_2$  (proportional to the output voltage) to a reference level  $x_2^*$ , via the creation of sliding modes over a certain switching surface. Taking advantage of the fact that  $x_2$  is the generator of the module, remark 3.4.1 suggests to exert a direct control over the state variable  $x_2$  by using the hyperplane  $s = x_2 - x_2^*$  as the switching surface. However, as the quoted remark points out as a possibility and is later assumed in section 3.4, this natural surface does not satisfy the transversality condition:

$$\partial_x s \cdot g = (0, 1) \cdot (1, 0)^\top = 0.$$

Therefore, no sliding motion can be induced over it. An alternative surface will be used in such a way that the ISD forces indirectly the control goal.

Consider an autonomous linear surface as proposed in section 3.4. Then,

**Proposition 3.5.1.** *The buck converter system (3.5.3), forced to switch over*

$$s = -(1, k) \cdot (x_1, x_2 - x_2^*)^\top, \quad k > 0, \quad (3.5.6)$$

according to the control law

$$u = \begin{cases} u^+ & \text{if } s > 0 \\ u^- & \text{if } s < 0, \end{cases}$$

shows sliding motion on  $s$  with its ISD exhibiting  $\Phi^* = (0, x_2^*)^\top$  as an asymptotically stable equilibrium point.

**Proof.** Firstly, we will observe that the control law locates the system over  $s$ ; in a second stage we will verify the asymptotic stability of the equilibrium point.

(i) The scalar function  $V(x) = \frac{1}{2}s^2(x)$  is such that its time derivative along the system trajectories is

$$V'(x) = ss' = -s(u - u_{eq}),$$

with

$$u_{eq} = (\lambda - k)x_1 + x_2.$$

The proposed control law guarantees  $V'(x) < 0$  for every point situated in a neighbourhood of the sliding domain of  $s$ .

(ii) The ISD satisfies

$$\begin{aligned} 0 &= x_1 + k(x_2 - x_2^*), \\ x_2' &= x_1; \end{aligned}$$

its solutions are

$$\begin{aligned}x_1 &= -k [x_2(0) - x_2^*] \exp\{-kt\}, \\x_2 &= [x_2(0) - x_2^*] \exp\{-kt\} + x_2^*.\end{aligned}$$

Notice that the steady state is given by  $\Phi^* = (0, x_2^*)^\top$  iff  $k > 0$ . ■

*Remark 3.5.1.* The sliding domain, i.e., the phase plane region where the equivalent control remains between the values of the control gains, is

$$u^- < (\lambda - k)x_1 + x_2 < u^+.$$

If the system is wanted to be regulated at  $x_2 = x_2^*$ , the conditions to be satisfied by such reference are

$$u^- < x_2^* < u^+, \tag{3.5.7}$$

and they do not depend on the parameter  $k$ . Notice that, for  $V_g > 0$ , (3.5.7) may be written

$$0 < v_C < V_g,$$

thus explaining the reductor character of the basic buck converter (the behavior is analogous in case that  $V_g < 0$ ).

## 3.6 Tracking signals with the buck converter

The preceding section has shown the buck converter fits the model (3.2.1) and, particularly, (3.4.1), for the sliding mode regulation of a system with an autonomous switching surface as (3.5.6). Hence, the theory developed in sections 3.3 and 3.4 is ready to be used in the solution of the tracking problem  $x_2 = f(t)$ .

Equations (3.5.4) and (3.5.5) indicate that  $x_2$  is a generator of the module defined by the buck system. Therefore, the function  $\Phi(t)$  introduced in proposition 3.4.1 is now

$$\Phi(t) = (f', f)^\top,$$

and the switching surface candidate (3.4.3) takes the form:

$$\hat{s}(x, t) = -(1, k) \cdot (x_1 - f', x_2 - f)^\top, \quad k > 0. \quad (3.6.1)$$

Hence, proposition 3.4.2 shows that:

(i)  $\hat{s}(x, t)$  satisfies the transversality condition.

(ii) The control law

$$u = \begin{cases} u^+ & \text{if } \hat{s} > 0 \\ u^- & \text{if } \hat{s} < 0 \end{cases} \quad (3.6.2)$$

puts the system on the hyperplane  $\hat{s}$ .

(iii) The sliding domain associated with  $\hat{s}$  becomes

$$u^- < f'' + \lambda x_1 + x_2 - k(x_1 - f') < u^+.$$

(iv) Denoting

$$M(t) = f''(t) + \lambda f'(t) + f(t), \quad (3.6.3)$$

the restrictions to be fulfilled by the reference signal are

$$u^- < M(t) < u^+, \quad \text{or, equivalently,} \quad u^- < \inf_t \{M(t)\} < \sup_t \{M(t)\} < u^+. \quad (3.6.4)$$

Notice that  $f \in C^2$ , at least.

(v) In such a situation, the ideal equilibrium solution is

$$(x_1, x_2) = (f', f),$$

with asymptotic stability guaranteed by the fact that  $k > 0$ .

*Remark 3.6.1.* (i) Notice that the sliding surface  $\hat{s}$  in (3.6.1) may be written as

$$\hat{s}(x, t) := -(1, k) \cdot (x'_2 - f', x_2 - f)^\top;$$

defining  $e_2 = x_2 - f$ , in ISD one has

$$e'_2 + ke_2 = 0,$$

being thus guaranteed the asymptotic tendency of  $x_2$  to  $f$ . An analogous reasoning is applicable to the autonomous surface  $s$  of (3.5.6).

(ii) It was mentioned in section 3.5 that the basic buck converter, depicted in figure 3.2, is such that the cases  $V_g < 0$  and  $V_g > 0$  define different sliding regions for the corresponding device and, therefore, different restrictions for the reference  $f(t)$  are obtainable from (3.6.4). These are

$$-1 < M(t) < 0, \quad 0 < M(t) < 1.$$

A simple glance at them allows us to notice that they have 0 as common border and, therefore, the possibility of weaken the restrictions for  $f$  exists by using the two sliding domains. The system should be controlled by a control law appropriate to its situation in the phase plane, changing it when necessary. Physically, it entails the device being capable of inverting the polarity of the voltage source at will. This is possible with the bidirectional buck converter, which performs this task by means of a full bridge of switches. The mathematical model may then consider the control action  $u$  taking values in the three level control set  $\{-1, 0, 1\}$ , the new restrictions for the references being

$$-1 < M(t) < 1 \quad \text{or} \quad |M(t)| < 1.$$

The full bridge of switches, however, also allows the two level switch  $u \in \{-1, 1\}$  ([Bie99] contains the switching logics that define the three level and the two level buck converters) and, consequently, it also leads to  $|M(t)| < 1$ .

One of the advantages of the three level buck is the possibility of halving the frequency of the switches (and, indirectly, the switching losses) while the output voltage frequency remains constant [CFT93]. Nevertheless, it has two main disadvantages: a more complex control law, and a lack of robustness in front of load disturbances [Bie99], because the change of polarity of the source is driven by the sign of  $M(t)$  which, in turn, depends on the load resistance value (3.6.3).

### 3.6.1 Offsetted sinusoidal waves

Consider at this point the problem of tracking the reference

$$f(t) = A + B \sin \omega t$$

by the state variable  $x_2$ , proportional to the output voltage of the buck converter.

According to the previously developed theory, the control policy that allows the achievement of the target is based on the switching surface (3.6.1), particularized now as

$$\hat{s}(x, t) := -(x_1 - B\omega \cos \omega t) - k(x_2 - A - B \sin \omega t),$$

and the control law (3.6.2). The restrictions to be satisfied by the reference  $f(t)$  are obtained from (3.6.4): denoting as (3.6.3)

$$M(t) = A + B\sqrt{\lambda^2\omega^2 + (1 - \omega^2)^2} \sin \left( \omega t + \arctan \frac{\lambda\omega}{1 - \omega^2} \right),$$

they become

$$u^- < M(t) < u^+, \quad (3.6.5)$$

and its concretion comes through the calculation of the extreme values of the function  $M(t)$ . These are

$$A \pm B\sqrt{\lambda^2\omega^2 + (1 - \omega^2)^2}, \quad (3.6.6)$$

where the signs  $+$ ,  $-$  will be taken considering the signs of  $A$  and  $B$ .

*Remark 3.6.2.* [Bie99] contains a featuring of the reference signals using Bode plots.

### 3.6.2 Pure sinusoidal waves

From (3.6.5) we observe that a necessary condition for a certain  $f$  to be tracked by  $x_2$  with a basic buck converter ( $u \in \{0, 1\}$ ) is that the corresponding  $M(t)$  conserves its sign. This is not possible for non offset sinusoidal references as

$$f(t) = B \sin \omega t.$$

However, a solution may be found in remark 3.6.1 (ii), which referred to the possibility of enhancing the sliding domain offered by the three level and the two level buck devices.

A three level control strategy for non offset tracking is provided in [FO94a]. The problem is solved here by using the two level buck converter.

Hence, the restrictions become

$$|B|\sqrt{\lambda^2\omega^2 + (1 - \omega^2)^2} < 1, \quad (3.6.7)$$

while the control law is that of (3.6.2), with  $u^- = -1$ ,  $u^+ = 1$ .

### 3.7 Robustness

In system (3.5.1, 3.5.2) disturbances of the voltage source constitute a perturbation of the input channel that satisfies the matching condition of theorem 3.2.3, thus exhibiting the strong invariance property of the corresponding ISD. Then, following remark 3.2.5 we can explore its effect on the sliding domain and calculate the new zone.

Let us now consider a buck converter that suffers a load perturbation, changing from its nominal value  $R_N$  to  $R_N + R_p(t)$ ; the parameter  $\lambda_N$  of the dimensionless system changes to

$$\lambda(t) = \lambda_N + \lambda_p(t) = \lambda_N - \frac{\lambda_N R_p(t)}{R_N + R_p(t)}. \quad (3.7.1)$$

The perturbed (3.5.3) is, therefore

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -\lambda_N & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} -\lambda_p(t)x_1 - \lambda'_p(t)x_2 \\ 0 \end{pmatrix}. \quad (3.7.2)$$

The field perturbation vector, denoted  $p(x, t)$ , is such that  $p(x, t) \in \text{span}\{(1, 0)\}$ , i.e., the corresponding ISD satisfies a strong invariance property. Hence, its effect is limited to the domain where sliding regime exists (see proposition 3.2.4), because the new equivalent control  $u_{eq}^p$  (3.2.9) for the system sliding over the surface  $\hat{s}$  defined in (3.6.1) must satisfy

$$u^- < f'' + \lambda(t)x_1 + [1 + \lambda'(t)]x_2 - k(x_1 - f') < u^+.$$

The restrictions for the references are now

$$u^- < f'' + \lambda(t)f' + [1 + \lambda'(t)]f < u^+.$$



### 3.8 Simulation results

Figure 3.3 contains a SIMULINK model of an ideal buck converter that allows the introduction of a load perturbation. The converter parameters are set to  $V_g = 200V$ ,  $R_N = 30\Omega$ ,  $L = 0.007H$  and  $C = 0.00033F$ , providing a nominal  $\lambda_N = 0.1535$ . The load variations are introduced as a train of pulses when necessary. The sliding surface is (3.6.1)

$$\hat{s}(x, t) = -(x_1 - f') - k(x_2 - f),$$

with  $k = 1.2$ . Non-ideal effects due to finite switching velocity can be modelled through a relay with hysteresis band of amplitude  $\Delta S_h$  following section 3.2.5. A fixed step fifth order Runge-Kutta algorithm is used to integrate the differential system, with an integration step of 0.000658 units in the dimensionless time variable, corresponding to  $10^{-6}s$ . It is worth mentioning that the integration step has been chosen so that its reduction does not affect the relative output error ( $erx_2$ ) values (except for the ideal, non-perturbed case). The simulations are carried on during 50 new time units, corresponding to 0.076s.

The output voltage reference signal

$$v_{Cr} = 100 + 20 \sin 2\pi\nu\tau V$$

turns out to be

$$x_{2r} = f(t) = 0.5 + 0.1 \sin \omega t \tag{3.8.1}$$

in our state-space formalism. Setting  $\nu = 50Hz$ , corresponding to  $\omega = 0.4775$ , ideal ( $\Delta S_h = 0$ ), unperturbed ( $R_p = 0$ ) response details are further discussed. It is worth mentioning that the fulfillment of (3.6.5) is guaranteed, because the calculation of (3.6.6) provides  $\{0.42, 0.58\}$ . The system, with initial condition  $(x_{1ic}, x_{2ic})^\top = (0, 0)$ ,

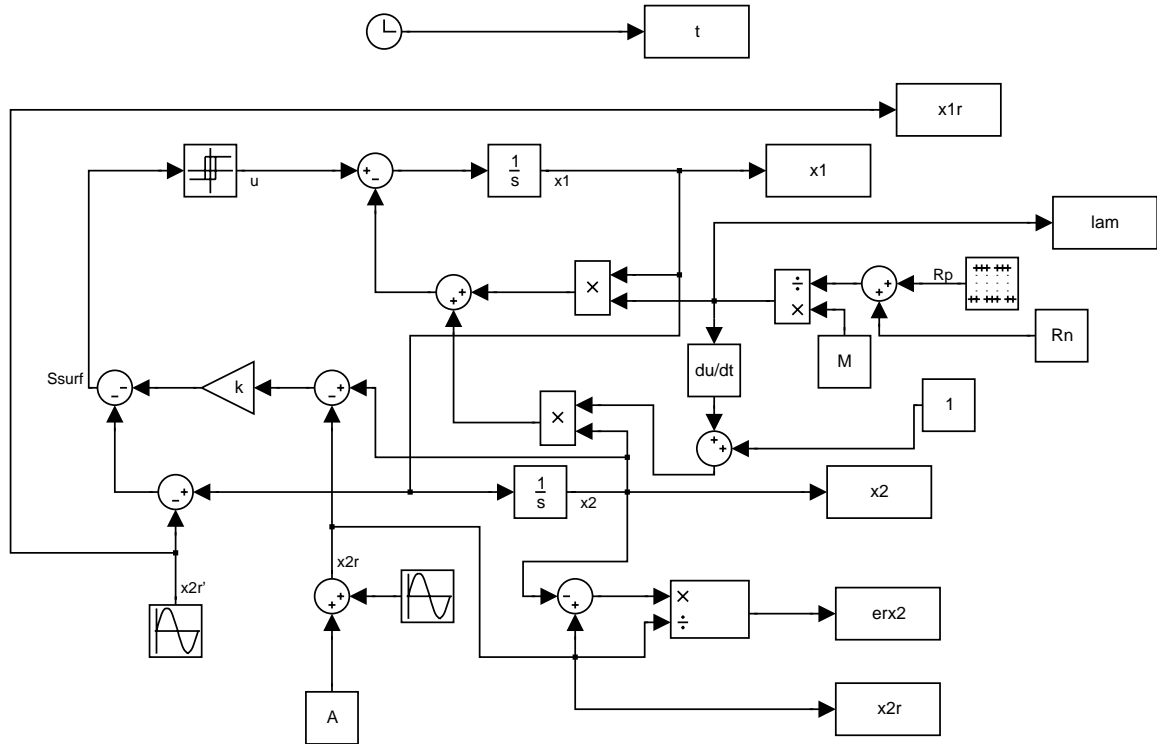


Figure 3.3: Buck converter model.

reaches asymptotically the reference and exhibits the expected behavior, as shown in figures 3.4 and 3.5, which depict details of the state variables  $x_1$ ,  $x_2$  and the output relative error  $erx_2$ .

The following set of simulation results have been obtained under the presence of an additive load resistance disturbance that causes a 100% variation of its nominal value ( $R_p = 30\Omega$ ), with a frequency of  $200Hz$  (corresponding to  $\omega_p = 1.9099$  in the new variables), represented in figure 3.6. See (3.7.1) to observe its effect on  $\lambda$ . Figures 3.7 and 3.8 depict the behavior of  $x_1$ ,  $x_2$  and  $erx_2$ . Notice that the error amount is similar to the non perturbed situation, that is, close to the 0.01%.

The simulation results obtained with a non ideal buck converter operating under a

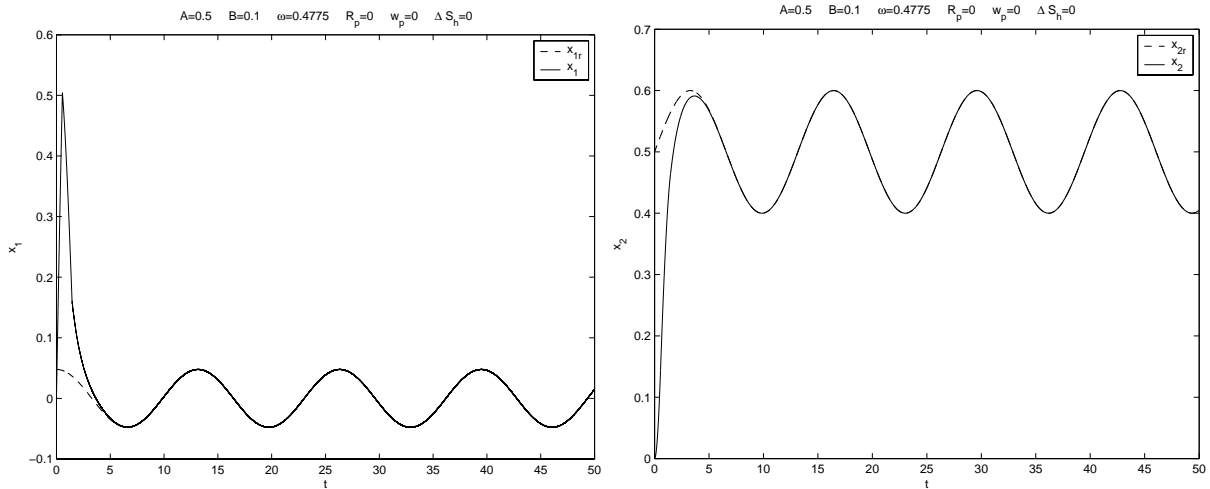


Figure 3.4: Details of  $x_1$  and  $x_2$  in the ideal, non-perturbed case.

maximum switching frequency of  $20\text{ kHz}$  are portrayed in figures 3.9 and 3.10. The effect is modelled through a relay with a hysteresis amplitude of  $\Delta S_h = 0.00411$ , calculated using (3.2.8). The chattering phenomenon inherent to real sliding modes is seen in the  $x_1$  behavior. The relative error  $erx_2$  is around the 0.015%.

Non offset signals have been obtained with a non ideal, perturbed, two level, full bridge buck converter. The hysteresis amplitude that is necessary to maintain a maximum switching frequency of  $20\text{ kHz}$  is  $\Delta S_h = 0.00822$ , because the control gains are now  $\pm 1$ . The load variation is still a 100% jump of the nominal value with frequency  $200\text{ Hz}$ . The output reference is the same as (3.8.1), but with null offset:

$$x_{2r} = f(t) = 0.1 \sin \omega t.$$

Restriction (3.6.7) is fulfilled, because it leads to  $0.08 < 1$ . The SIMULINK model for this purpose coincides exactly with that of figure 3.3 except for the fact that the relative error is now calculated dividing by the amplitude  $B$  of the reference, instead of using the reference itself, because it takes null values periodically. The corresponding figures are 3.11 and 3.12. The relative error of the output is again around the 0.015%.

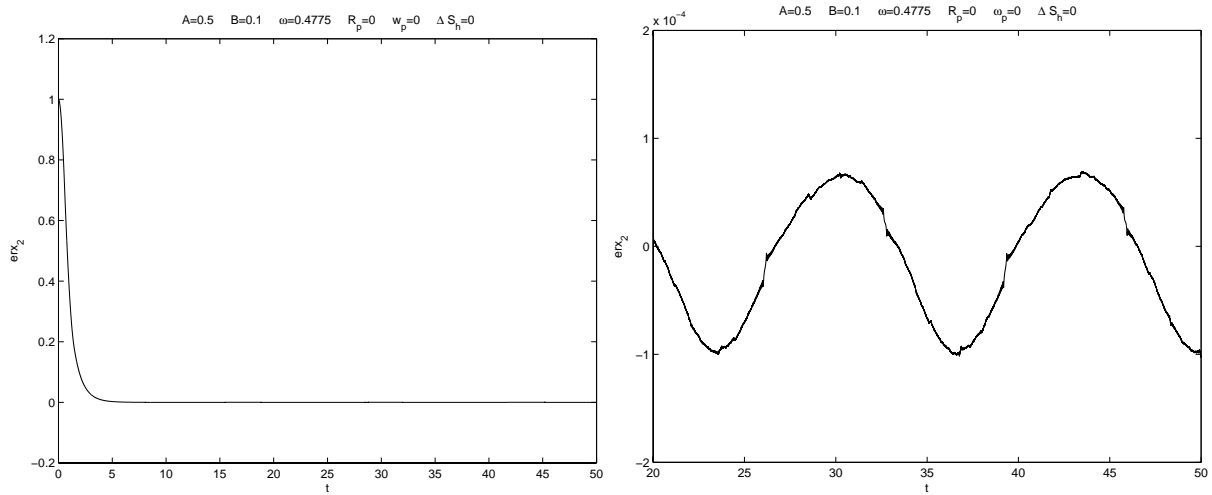
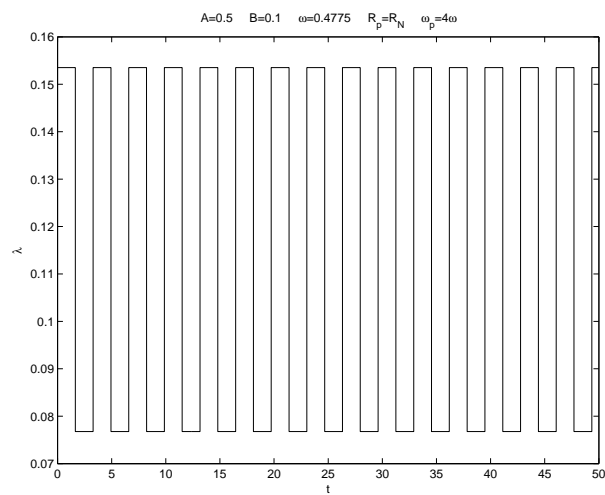
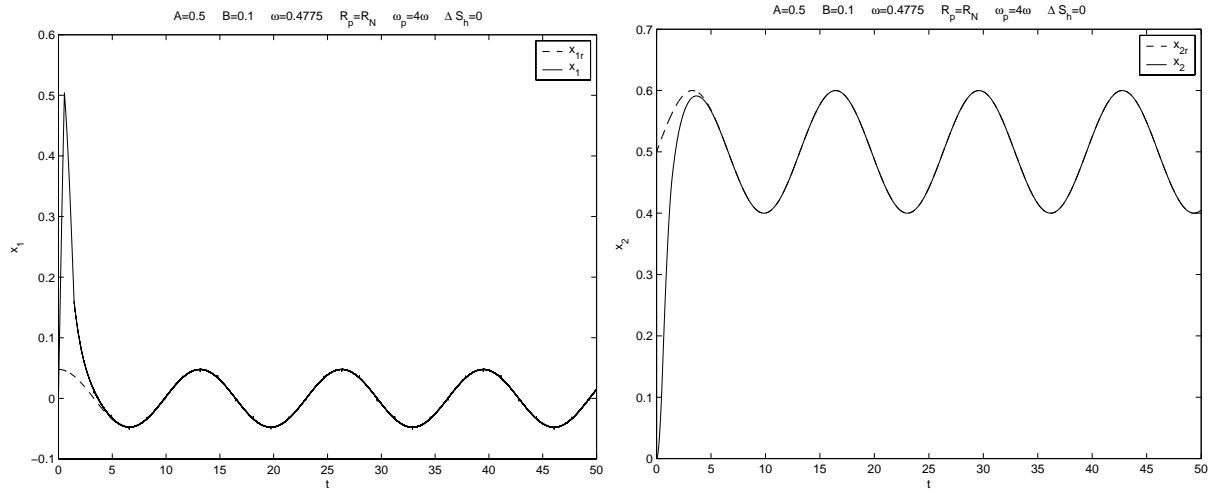
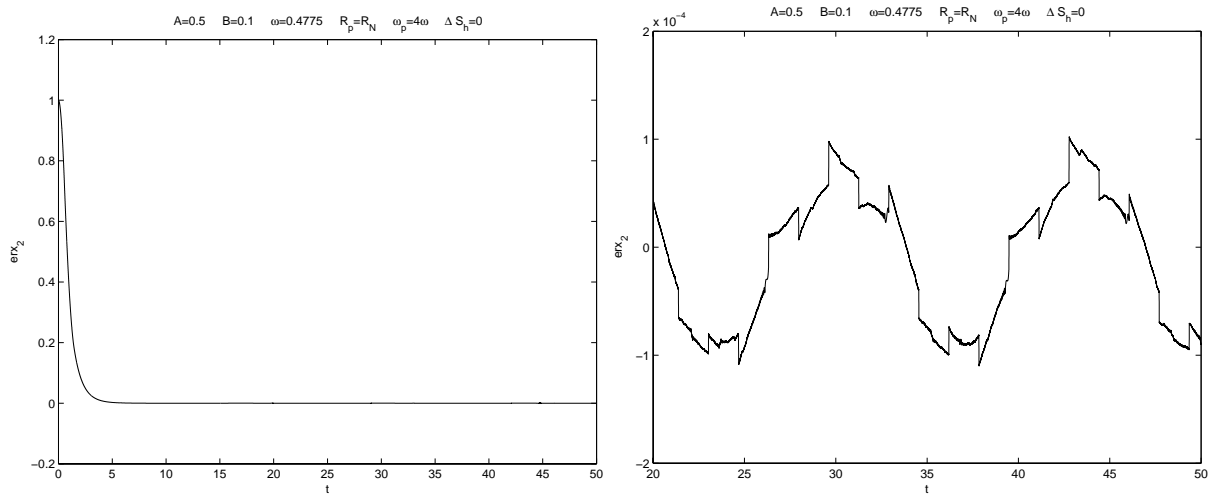
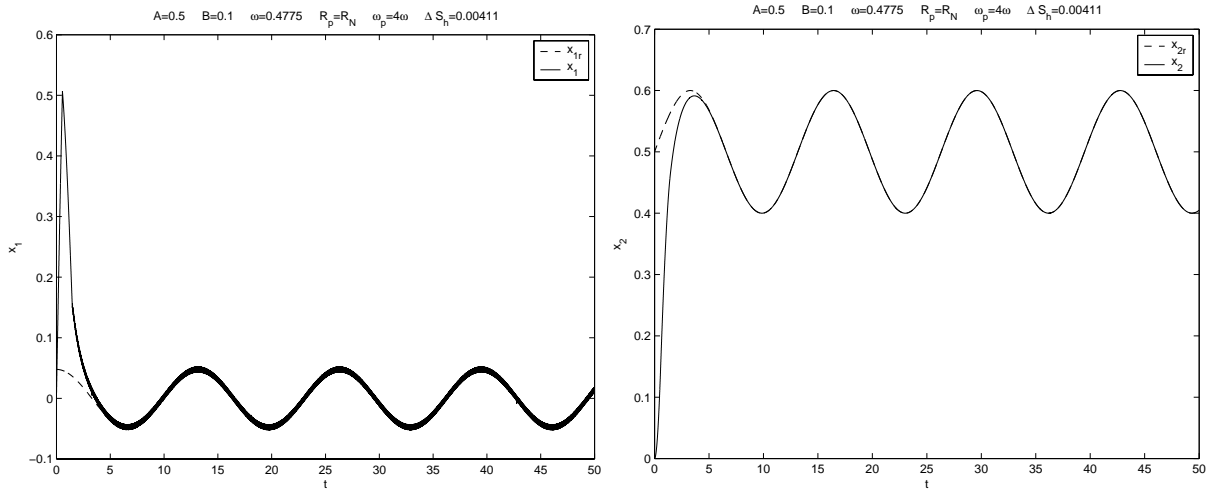
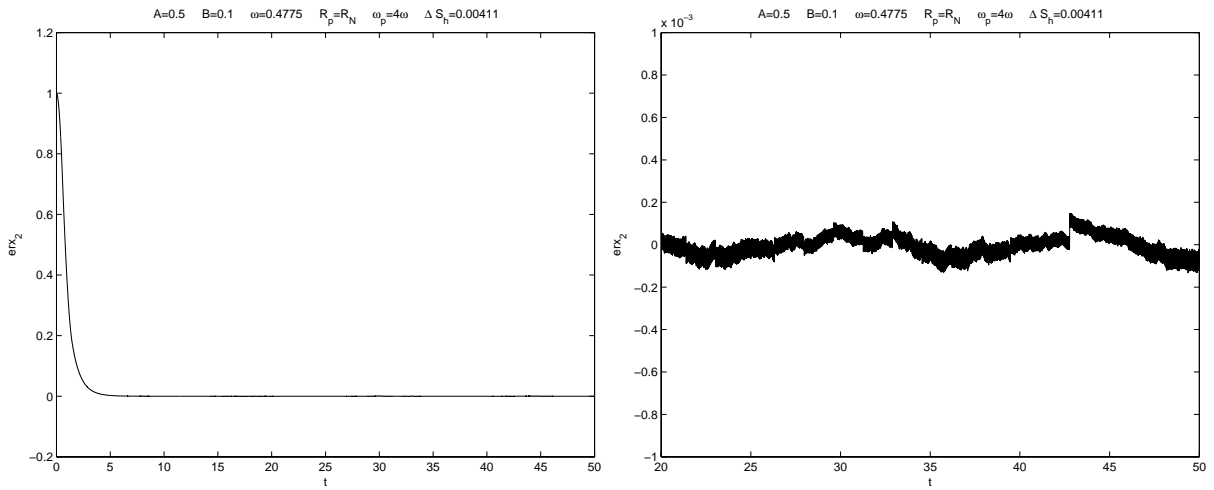
Figure 3.5: Details of  $erx_2$  in the ideal, non-perturbed case.

Figure 3.6: Perturbed parameter.

Figure 3.7: Details of  $x_1$  and  $x_2$  in the ideal, perturbed case.Figure 3.8: Details of  $err_2$  in the ideal, perturbed case.

Figure 3.9: Details of  $x_1$  and  $x_2$  in the non ideal, perturbed case.Figure 3.10: Details of  $erx_2$  in the non ideal, perturbed case.

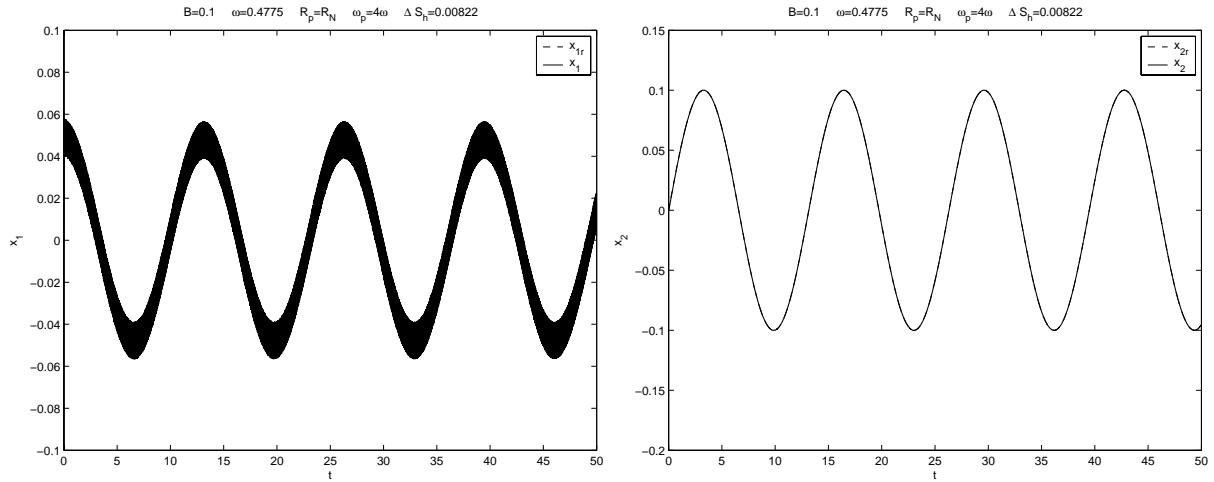


Figure 3.11: Details of  $x_1$  and  $x_2$  in the non offset, non ideal, perturbed case

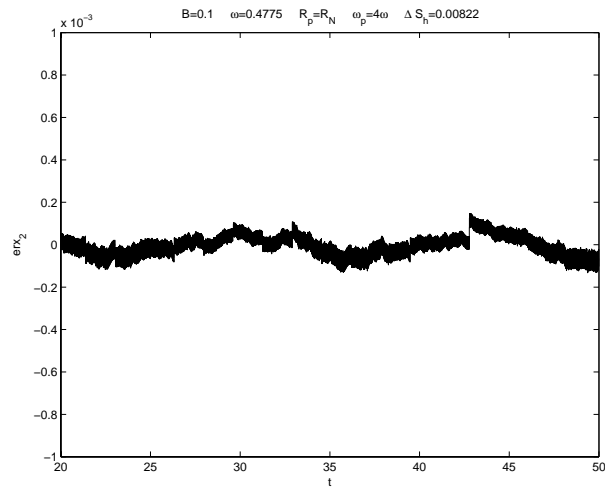


Figure 3.12: Detail of  $erx_2$  in the non offset, non ideal, perturbed case.

# Chapter 4

## The Tracking Problem in Nonlinear Converters

The aim of this chapter is to solve a tracking problem in a particular nonminimum phase, second order, nonlinear control system by means of indirect control. A complete knowledge of the plant parameters is assumed.

### 4.1 Introduction

In the design of controllers for a real physical problem, it is not rare to find systems with internal dynamics that become unstable when the output tracks a given reference; it is then said that they have unstable tracking dynamics and they are referred to as *nonminimum phase systems* [MT95]. The technique known as indirect control changes the output to obtain an asymptotically stable tracking dynamics, which forces the initial output to track the desired reference. At a certain stage of the procedure an inverse problem arises from the computation of the target for the new output as a function of the initial target and needs to be solved. This scenario is often found in the field of nonlinear power converters.



Roughly speaking, a power converter is supposed to be able to regulate and/or modulate an input voltage. This would suggest output voltage control; however, a direct action taking into account just this variable is not appropriate. Although the target may be asymptotically reached, the internal dynamics results in an unstable behavior for the inductor current. Nevertheless, when a current mode control is implemented the internal dynamics is not only bounded input - bounded state but also forces the output voltage to perform asymptotically stable tracking. The problem then needs an indirect treatment and focuses on finding the appropriate signal, i.e. bounded and preferable periodic, to be followed by such a current in order to produce, in steady state, an output coincident with the desired reference.

In the case under study, the solution of the inverse problem is given by an ordinary differential equation. Under certain restrictions it is shown here that this equation has a bounded, unstable periodic solution for a given periodic target. Unfortunately, such a solution may not be analytically obtained; thus, a numerical technique will be used to manage it when the control is implemented. The procedure is similar to the method in [DCP96] (see also section 2.4) for inversion based exact output tracking. The communication [FO94b] contains a first approach to the indirect tracking control of nonlinear converters. A complete study including the results obtained in this chapter is reported in [FO02].

The chapter is structured as follows. Section 4.2 establishes the mathematical model of the ideal nonlinear converters boost and buck-boost and the equation that governs the dynamic behavior of the inductor current. Section 4.3 introduces the return map, a key tool used in section 4.4 to prove the existence of a periodic solution for the ODE. Section 4.5 deals with the questions inherent to the tracking control and

justifies the need for current mode regulation. Section 4.6 exemplifies the technique through the tracking of a sinusoidal reference. Section 4.7 is devoted to generalize the methodology to a certain class of bilinear systems, taking advantage of a change of variables that converts the ODE in one of Abel type. Finally, simulation results are found in section 4.8.

## 4.2 Mathematical model

Basic nonlinear switched boost and buck-boost devices, ideally represented in figures 4.1 and 4.2, admit a general state-space representation in terms of a two dimensional bilinear system with the inductor current and the capacitor voltage as state variables and a control action  $\nu$  taking its values in the discrete set  $\{0, 1\}$ . Namely,

$$L \frac{di_L}{d\tau} = -v_C + \nu v_C + V_g [1 + k(\nu - 1)] \quad (4.2.1)$$

$$C \frac{dv_C}{d\tau} = i_L - \frac{v_C}{R} - \nu i_L, \quad (4.2.2)$$

where  $k = 0$  for the boost converter and  $k = 1$  for the buck-boost converter.

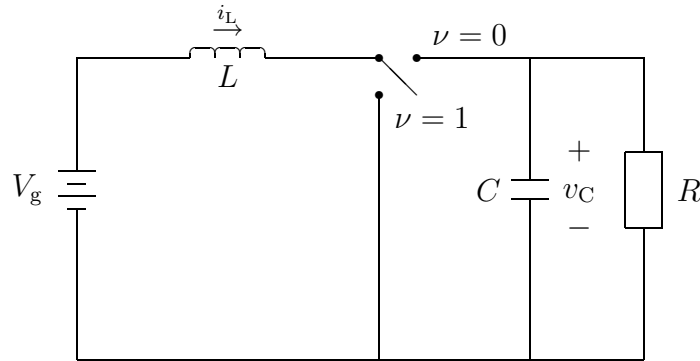


Figure 4.1: Boost converter

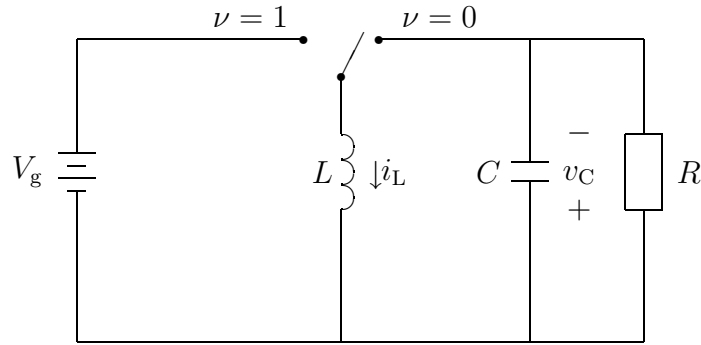


Figure 4.2: Buck-Boost converter

For a systematic study it is advisable to consider a dimensionless model obtained by a change of variables similar to that already used in section 3.5 with the buck converter, namely

$$x_1 = \frac{1}{V_g} \sqrt{\frac{L}{C}} i_L, \quad x_2 = \frac{1}{V_g} v_C, \quad t = \frac{1}{\sqrt{LC}} \tau,$$

and the introduction of  $\lambda = R^{-1} \sqrt{LC^{-1}}$  and  $u = 1 - \nu$ . The equations then become

$$x_1' = 1 - u(k + x_2) \quad (4.2.3)$$

$$x_2' = -\lambda x_2 + u x_1. \quad (4.2.4)$$

Since  $L$ ,  $R$  and  $C$  are positive constants,  $\lambda$  is positive.  $L$  and  $C$  are usually considered well known parameters, while perturbations may affect  $R$  and  $V_g$ . The chapter is devoted to the non perturbed case.

*Remark 4.2.1.* The coupled-inductor Čuk converter, depicted in figure 4.3, can also be described by (4.2.3, 4.2.4). In fact the system's dynamical model, expressed with the current state variables, is

$$x_1' = 1 - u x_2,$$

$$x_2' = -\lambda(x_2 - 1) + u x_1.$$

The additional assignment  $\hat{x}_2 = x_2 - 1$  leads directly to the buck-boost case in equations (4.2.3, 4.2.4).

The equation resulting after the elimination of the control action  $u$  in equations (4.2.3, 4.2.4) contains a differential relation between the state variables which does not depend on any input, as it can be directly verified

$$x_1(1 - x_1') = (k + x_2)(x_2' + \lambda x_2). \quad (4.2.5)$$

This last ODE plays a key role in the inverse problem that will be later solved.

*Remark 4.2.2.* In the case of regulation,  $x_1' = x_2' = 0$ , (4.2.5) provides the parabola of equilibrium points

$$x_1 = \lambda x_2(k + x_2).$$

### 4.3 The return map

Consider the Cauchy problem

$$x' = S(x, t), \quad x(0) = z, \quad (4.3.1)$$

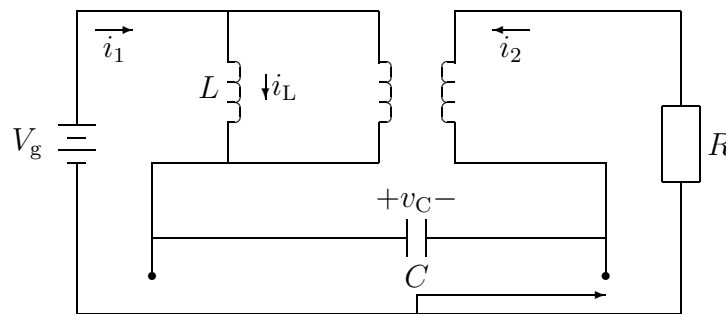


Figure 4.3: Coupled-inductor Ćuk converter

with  $S(x, t + T) = S(x, t)$ ,  $\forall t$ , and let  $x(t, z)$  be a solution. The map defined as

$$\begin{aligned} h : I \subseteq \mathbb{R} &\longrightarrow \mathbb{R} \\ z &\longmapsto h(z) = x(T, z), \end{aligned}$$

is known as the *return map* (see [Llo79] and [GL90]). Notice that any  $T$ -periodic solution of the equation produces a fixed point of  $h$  and, conversely, every fixed point of  $h$  is the initial value of a periodic solution (see [Ver90] for a proof). Hence, it is useful to consider the map  $H(z) = h(z) - z$  and relate the periodic solutions of (4.3.1) to the zeros of  $H(z)$ .

The expression of  $h(z)$  is difficult to obtain because it implies the analytic solution of the ODE. Nevertheless, information related to the graph of  $h(z)$  will be provided through the behavior of its derivatives, available by means of implicit derivation. Actually, when  $S(x, t)$  is smooth, standard results on the dependence of solutions on initial conditions [Sot82] guarantee that  $h$  is continuously differentiable.

**Proposition 4.3.1.** (See [GL90], and [Llo79])

Let  $h(z)$  be the return map associated to equation (4.3.1). Then,

$$h'(z) = \exp \left\{ \int_0^T \frac{\partial S}{\partial x}(x(t, z), t) dt \right\}.$$

■

## 4.4 Periodic solutions for the internal dynamics

The following lemma will be used in the proof of the main result of this section:

**Lemma 4.4.1.** Let  $x(t, x_0)$  be the solution of the Cauchy problem

$$x(1 - x') = c, \quad x(0) = x_0, \quad (4.4.1)$$

with  $c > 0$ . Then,

(i)  $x = c$  is an equilibrium solution.

(ii) For any initial condition  $x_0$ ,  $0 < c < x_0$ ,  $x(t, x_0)$  is increasing and it is defined  $\forall t \geq 0$ .

(iii) For any initial condition  $x_0$ ,  $0 < x_0 < c$ ,  $x(t, x_0)$  is decreasing and it is defined in the set  $[0, t^*)$ , where

$$t^* = c \log \frac{c}{c - x_0} - x_0$$

**Proof.** The first statement is trivial. It is also obvious that  $x(t, x_0)$  never takes null values, because this would mean  $0 = c$ . Hence, (4.4.1) can be written as

$$x' = 1 - \frac{c}{x}.$$

Notice that  $x' > 0$  for every  $x$  such that  $x > c > 0$ , and  $x' < 0$  for every  $x$  such that  $c > x > 0$ , thus indicating the increasing and decreasing character of the solutions in the corresponding domains. Moreover, the general solution of the ODE is implicitly given by

$$x - x_0 + c \log \frac{x - c}{x_0 - c} = t(x, x_0);$$

then item (ii) comes from

$$\lim_{x \rightarrow +\infty} t(x, x_0) = +\infty, \quad \forall x_0 > c.$$

As for item (iii), if  $0 < x_0 < c$ ,

$$\lim_{x \rightarrow 0} t(x, x_0) = c \log \frac{c}{c - x_0} - x_0,$$

which yields the result. ■

Let us consider again system (4.2.3, 4.2.4) and the differential relation (4.2.5). When  $x_2$  tracks a  $T$ -periodic reference  $f(t)$ , the internal dynamics is given by

$$x_1(1 - x_1') = (k + f)(f' + \lambda f), \quad x_1(0) = x_{10}. \quad (4.4.2)$$

Setting  $x = x_1$ , equation (4.4.2) can be rewritten as

$$x(1 - x') = g(t), \quad x(0) = x_0, \quad (4.4.3)$$

where

$$g(t) = (k + f)(f' + \lambda f).$$

Assume from now on the smoothness and  $T$ -periodicity of  $f$ , and therefore of  $g$ .

**Theorem 4.4.2.** *If  $g(t) > 0$ , then equation (4.4.3) has one and only one periodic solution in  $\mathbb{R}^+$ , which is hyperbolic<sup>1</sup> and unstable.*

**Proof.** EXISTENCE. The solutions of (4.4.3) are different from 0 everywhere because, otherwise, this would imply  $0 = g(t)$ . Hence, the ODE can be written as

$$x' = S(x, t) = 1 - \frac{g(t)}{x}, \quad x(0) = x_0. \quad (4.4.4)$$

Let  $h : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be the return map of (4.4.4), with its associated function  $H$ . The smoothness,  $T$ -periodicity and positivity of  $g(t)$  ensure the existence of  $a, b \in \mathbb{R}^+$  such that  $a \leq g(t) \leq b$ .

Consider then the differential equations

$$x' = S_a(x) = 1 - \frac{a}{x}, \quad (4.4.5)$$

$$x' = S_b(x) = 1 - \frac{b}{x}, \quad (4.4.6)$$

---

<sup>1</sup>Let  $\Phi(t, z_0)$  be such a periodic solution; it is hyperbolic iff  $H'(z_0) \neq 0$ ,  $H := h - \mathbb{I}$  being associated to the return map  $h$  of (4.4.3).

with return maps  $h_a : I_a \subseteq \mathbb{R} \rightarrow \mathbb{R}$ ,  $h_b : I_b \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , and associated functions  $H_a, H_b$ . It is straightforward that

$$S_b(x) \leq S(x, t) \leq S_a(x), \quad \forall x > 0, \forall t \geq 0. \quad (4.4.7)$$

Now let  $x(t, z)$ ,  $x_a(t, z)$ ,  $x_b(t, z)$  be the solutions of (4.4.4, 4.4.5, 4.4.6), respectively, with initial conditions  $x(0, z) = x_a(0, z) = x_b(0, z) = z$ . Taking into account (4.4.7) and the mean value theorem it turns out that

$$x_b(t, z) \leq x(t, z) \leq x_a(t, z), \quad \forall z \in \mathbb{R}^+ \quad (4.4.8)$$

and for all  $t$  in the common domain. Then  $H_b(z) \leq H(z) \leq H_a(z) \forall z > 0, z \in I \cap I_a \cap I_b$ . Moreover,  $S_a(a) = S_b(b) = 0$  which, in case that  $a, b \in I$ , leads to  $H(a) \leq H_a(a) = 0$  and  $0 = H_b(b) \leq H(b)$ . Thus, the existence of  $z_0 \in [a, b]$  with  $H(z_0) = 0$  is ensured, which results in a periodic solution of equation (4.4.3).

Lemma 4.4.1 and equation (4.4.8) allow us to state that, while it is true that  $b \in I$ ,  $a$  does not necessarily belong to  $I$ . When it happens, the continuous dependence of the solutions of (4.4.3) on initial conditions and the impossibility of having crossing trajectories in the  $(x, t)$  plane guarantee the existence of  $\xi > a$  such that

$$\lim_{t \rightarrow T} x(t, \xi) = 0.$$

Therefore,  $\exists \varepsilon, 0 < \varepsilon < a$ , and  $\exists \xi', \xi' > \xi > a$ , such that  $h(\xi') = \varepsilon$ , then  $H(\xi') < 0$ . Since  $H(b)$  is null or positive, a zero of  $H$  may be found in  $(\xi, b]$ .

INSTABILITY. As

$$\frac{\partial}{\partial x} S(x, t) = \frac{g(t)}{x^2} > 0, \quad \forall x \neq 0, \quad (4.4.9)$$

proposition 4.3.1 yields  $H'(z) > 0, \forall z \in I$ . Hence, any periodic solution of (4.4.3) is unstable.



UNIQUENESS. Again from  $H'(z) > 0, \forall z \in I$ ,  $H$  has at most one zero in each connected component of  $I = (-\infty, \mu_-) \cup (\mu_+, +\infty)$ . Moreover,  $\forall z \in (-\infty, \mu_-)$ ,

$$x'(t, z) = 1 - \frac{g(t)}{x(t, z)} = 1 + \left| \frac{g(t)}{x(t, z)} \right| > 0,$$

which indicates the strictly increasing character of these solutions. Therefore,  $H(z) > 0, \forall z \in (-\infty, \mu_-)$ . ■

**Corollary 4.4.3.** *If  $0 > g(t)$ , equation (4.4.3) has one and only one asymptotically stable, periodic solution in  $\mathbb{R}^-$ .*

**Proof.** The change of variables  $(x, t) \longrightarrow (-x, -t)$  in (4.4.3) gives

$$\frac{dx}{dt} = 1 - \frac{|g(t)|}{x}.$$

Then, the proof of the preceding theorem yields the result. Concerning asymptotic stability, note that instability in backward time means stability in forward time. ■

Alternative proofs of theorem 4.4.2 and corollary 4.5.2 are provided in [ACA00].

## 4.5 Tracking control

This section deals with the output voltage tracking problem for elementary nonlinear converters modelled by equations (4.2.3, 4.2.4). The need for indirect control through the inductor current will be substantiated.

The signals  $f(t)$  to be tracked are supposed to be  $T$ -periodic, smooth and such that

$$g(t) = [k + f(t)] [f'(t) + \lambda f(t)] \neq 0, \quad \forall t. \quad (4.5.1)$$

**Lemma 4.5.1.**  $\forall t, g(t) \neq 0 \Rightarrow \forall t, f(t) \neq 0$ .

**Proof.** Two situations can be distinguished:

(i) In the case  $k = 0$ , if  $\bar{t}$  exists such that  $f(\bar{t}) = 0$ , then  $g(\bar{t}) = 0$ .

(i) In the case  $k = 1$ , let  $t_n, t_{n+1}$  be two consecutive simple zeros of  $f$ . Then it is possible to find two different neighbourhoods for them where, while  $1 + f(t)$  remains positive,  $f' + \lambda f$  changes sign. The continuity of  $g$  and Bolzano theorem ensure the existence of  $\bar{t} \in [t_n, t_{n+1}]$  with  $g(\bar{t}) = 0$ . For a higher order zero of  $f$ , say  $\tilde{t}$ ,  $g(\tilde{t}) = 0$  trivially. ■

It is also worth mentioning that the study of the regulation problem for basic DC-to-DC power converters is also included when (4.5.1) is taken as hypothesis. Then, the following results are applicable to the case  $f(t) = c \in \mathbb{R}$  except for  $c = -k$ ,  $k = 0, 1$  because, otherwise,  $g(t) = 0, \forall t$ .

Assume now an ideal steady state where  $x_1$  and  $x_2$  track  $\phi(t)$  and  $f(t)$ , respectively,  $\phi(t)$  being a  $T$ -periodic solution of (4.4.2) and  $\bar{u}(t)$  being the ideal continuous control that allows it. From equations (4.2.3, 4.2.4) we have

$$\phi' = 1 - \bar{u}(k + f)$$

$$f' = -\lambda f + \bar{u}\phi.$$

Furthermore, the characteristic restrictions on switched converters result in the input satisfying

$$0 \leq \bar{u}(t) \leq 1,$$

so

$$0 \leq \frac{1 - \phi'}{k + f} \leq 1 \quad \text{or, equivalently,} \quad 0 \leq \frac{f' + \lambda f}{\phi} \leq 1,$$

thus entailing constrains on  $f(t)$ . These inequalities can also be written as

$$0 < f' + \lambda f \leq \phi \quad \text{or} \quad \phi \leq f' + \lambda f < 0, \quad (4.5.2)$$

where  $g(t) \neq 0$  has been taken into account. At this stage, the attempt to reconcile the restrictions over  $g(t)$  and  $\bar{u}(t)$  leads to the following results: proposition 4.5.2, where a necessary condition over  $f(t)$  for the satisfaction of (4.5.2) is stated, and proposition 4.5.3, which contains a sufficient condition for the same target.

**Proposition 4.5.2.** *The fulfillment of (4.5.2) demands  $g(t) > 0, \forall t$ .*

**Proof.** By corollary 4.4.3,  $g(t) < 0$  entails  $\phi(t) < 0$ ; then, from (4.5.1) and (4.5.2),

$$g < 0 \Rightarrow f' + \lambda f < 0 \text{ and } k + f > 0$$

and, from lemma 4.5.1,  $f(t) \neq 0$ . But

1.  $f' + \lambda f < 0$  and  $f > 0$ , then  $0 < f(t) \leq f(0)e^{-\lambda t} \forall t$ , which is incompatible with the periodicity of  $f$ .
2.  $f' + \lambda f < 0$  and  $f < 0$ , then  $k = 1$  and  $-1 < f(t) \leq f(0)e^{-\lambda t} < 0, \forall t$ . In addition,  $\phi$  is periodic and smooth; hence, from Rolle theorem,  $\exists t_0$  such that  $\phi'(t_0) = 0$ . Then, taking into account equations (4.4.2) and (4.5.2)

$$(1 + f(t_0))(f'(t_0) + \lambda f(t_0)) = \phi(t_0) \leq f'(t_0) + \lambda f(t_0).$$

Therefore  $1 + f(t_0) \geq 1$ , which is in contradiction to  $f < 0$ .

■

**Proposition 4.5.3.** *If  $\inf\{g(t)\} \geq \sup\{f'(t) + \lambda f(t)\}, \forall t \in [0, +\infty)$ , then*

$$\phi(t) \geq f'(t) + \lambda f(t), \quad \forall t \quad \text{and} \quad [f > 0 \Leftrightarrow f' + \lambda f > 0].$$

**Proof.** Let  $t_m$  be the minimum of  $\phi$  in  $[0, T]$ . Since  $\phi$  satisfies the equation

$$\phi(t)(1 - \phi'(t)) = (k + f(t))(f'(t) + \lambda f(t)) = g(t)$$

and is periodic,  $\phi(t_m) = g(t_m)$  as  $\phi'(t_m) = 0$ . Finally,

$$\phi(t) \geq \phi(t_m) = g(t_m) \geq \sup\{f'(t) + \lambda f(t)\}, \quad \forall t \in [0, +\infty).$$

Let us consider the equivalence. Assume  $f > 0$ ; then  $f' + \lambda f > 0$  since  $f' + \lambda f$  is continuous, different from zero everywhere and in  $t = t_0$ , where  $f$  reaches a minimum, one has  $f'(t_0) + \lambda f(t_0) = \lambda f(t_0) > 0$ .

Moreover, since  $f \neq 0$  everywhere,  $f > 0$  or  $f < 0$ . But both  $f' + \lambda f > 0$  and  $f < 0$  result in

$$0 > f(t) > f(0)e^{-\lambda t},$$

which is not possible for a periodic  $f$ . Thus,

$$f' + \lambda f > 0 \Rightarrow f > 0.$$

■

The next step is to consider the tracking problem  $x_2 = f(t)$  for the system described in equations (4.2.3, 4.2.4). The internal dynamics are given by

$$x_1' = 1 - u_1(k + f) \tag{4.5.3}$$

$$u_1 = (f' + \lambda f)x_1^{-1}. \tag{4.5.4}$$

This control policy would be implementable if the internal dynamics were stable, i.e., the behavior of  $(x_1, u_1)$  were bounded. Nevertheless, (4.5.3, 4.5.4) yield equation (4.4.3). As it has been proven in the previous section, it has one and only one unstable periodic solution. The other solutions are unbounded.

**Proposition 4.5.4.** *System (4.2.3, 4.2.4) has unstable tracking dynamics when the  $x_2$  state variable is taken as the output.* ■

Once direct control has been discarded, the alternative is a current mode indirect control action. [BMS83a], [Sir87] and [VŠČ85] used this methodology in regulation tasks. In [FO94b] and [OFB96] the authors expanded it to the tracking problem.

The internal dynamics are now defined by

$$\begin{aligned}x_2' &= -\lambda x_2 + \phi u_2 \\u_2 &= (1 - \phi')(k + x_2)^{-1},\end{aligned}$$

which yield

$$(k + x_2)(x_2' + \lambda x_2) = \phi(1 - \phi'). \quad (4.5.5)$$

**Proposition 4.5.5.** *System (4.2.3, 4.2.4) has asymptotically stable tracking dynamics when  $x_1$  is taken as the output and the restrictions stated in proposition 4.5.3 are fulfilled.*

**Proof.** Since  $\phi(t, x_{10})$  is a  $T$ -periodic solution of (4.4.2),

$$\phi(1 - \phi') = (k + f)(f' + \lambda f);$$

then (4.5.5) can be written as

$$x_2' = N(x_2, t) = -\lambda x_2 + \frac{(k + f)(f' + \lambda f)}{k + x_2}. \quad (4.5.6)$$

Take now  $H_2(z) = h_2(z) - z$ ,  $h_2(z)$  being the return map associated to equation (4.5.6). As  $x_2 = f(t)$  is a  $T$ -periodic solution of (4.5.6),  $H_2(f(0)) = 0$ . Moreover, following proposition 4.3.1 it is straightforward that

$$H_2'(f(0)) = \exp \left\{ - \int_0^T \left( \lambda + \frac{f' + \lambda f}{k + f} \right) dt \right\} - 1 < 0,$$

because  $\lambda > 0$  and  $g(t) = (f' + \lambda f)(k + f) > 0$ . The asymptotic stability of  $f(t)$  is then ensured. ■

Although the periodic solution of (4.4.3) is not analytically obtainable, it could be numerically approximated by the integration of the ODE with the appropriate initial condition which, in turn, may be numerically computed solving the equation  $H(z) = 0$ . However, highly unstable solutions demand a high precision calculation of the zero, which may imply great computational effort.

Another possibility is to consider the integration of (4.4.3) in backward time, which allows the obtention of the periodic solution with the desired error due to its asymptotic stability in reverse time. Less computational effort is then needed, because we just need to let the simulation run enough (backward) time.

In both cases the idea is to obtain one period of the solution and use its periodic extension throughout time. Nevertheless, as the control technique is clearly sensitive to external perturbations, alternative methodologies will be explored in the next chapter, searching for an easier handling of such a solution and the possibility of introducing a robust scheme.

The section will close with the establishment of a sliding control law that satisfies the control goal, that is, the tracking of  $\phi(t)$  by  $x_1$ , which will induce internal dynamics that lead  $x_2$  to track the output voltage reference  $f(t)$ . It is worth mentioning that other control techniques may be also applicable in this case to achieve  $x_1 = \phi(t)$ . In fact, the observer based methodology developed in [UGS99] and commented in section 2.3 for regulation purposes can be easily extended to the tracking situation.

**Proposition 4.5.6.** *Let  $s(x_1, t) = x_1 - \phi(t)$  be the switching surface. The control law*

$$u = \begin{cases} 0 & \text{if } (k + x_2)s < 0 \\ 1 & \text{if } (k + x_2)s > 0, \end{cases}$$

*produces in system (4.2.3, 4.2.4) an asymptotic tendency to  $s(x_1, t) = 0$ .*

**Proof.** Let  $u_{eq}$  be the continuous control that ideally maintains the system on the switching surface in the case it initially starts there. Therefore,  $s'(x_1(t, u_{eq}), t) = 0$  defines the equivalent control, which in our case satisfies

$$x_1' - \phi'(t) = 0 \quad \Longrightarrow \quad \phi'(t) = 1 - u_{eq}(k + x_2).$$

The system trajectories will be directed towards the switching surface when  $ss' < 0$ . Then,

$$ss' = s(x_1' - \phi') = s[1 - u(k + x_2) - 1 + u_{eq}(k + x_2)] = -s(k + x_2)(u - u_{eq}) < 0$$

if the control switches according to the law proposed in the hypothesis. ■

The sliding domain is trivially given by

$$0 < \frac{1 - \phi'}{k + x_2} < 1.$$

## 4.6 Tracking a sinusoidal reference

The general process described at the end of the last section can be directly applied to the tracking control of a sinusoidal wave without further detail. However, it might be interesting to obtain the analytic restrictions for these type of references.

Hence, let

$$f(t) = A + B \sin \omega t$$

be a candidate to be tracked by the state variable  $x_2$  of system (4.2.3, 4.2.4), where  $A > 0$ ,  $B > 0$  is supposed. Therefore,

$$A > B \Longrightarrow f > 0 \Longrightarrow k + f > 0.$$

In turn,

$$\begin{aligned} f' + \lambda f &= A\lambda + B\omega \cos \omega t + B\lambda \sin \omega t = \\ &= A\lambda + B\sqrt{\lambda^2 + \omega^2} \sin \left( \omega t + \arctan \frac{\omega}{\lambda} \right); \end{aligned}$$

then, the hypothesis on the signs of  $A$  and  $B$  make that

$$A > B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} \implies f' + \lambda f > 0,$$

which ensures  $g(t) > 0$ .

On the other hand, proposition 4.5.3 asks for

$$\inf \{g\} \geq \sup \{f' + \lambda f\},$$

which may be written

$$\inf \{(k + f)(f' + \lambda f)\} \geq \sup \{f' + \lambda f\}. \quad (4.6.1)$$

Trivially, the satisfaction of

$$\inf \{k + f\} \geq \frac{\sup \{f' + \lambda f\}}{\inf \{f' + \lambda f\}} \quad (4.6.2)$$

is sufficient to guarantee (4.6.1). Taking into account the expression for  $f' + \lambda f$  derived above, equation (4.6.2) becomes

$$k + A - B \geq \frac{A\lambda + B\sqrt{\lambda^2 + \omega^2}}{A\lambda - B\sqrt{\lambda^2 + \omega^2}}.$$

In short, the restrictions to be fulfilled by  $f(t)$  are:

$$A > B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} > 0, \quad (4.6.3)$$

$$k + A \geq B + \frac{A + B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}{A - B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}. \quad (4.6.4)$$



## 4.7 Abel equations in bilinear systems

The differential equation (4.4.3) for which theorem 4.4.2 has proven the existence of an unstable, periodic solution in  $\mathbb{R}^+$  takes the form of an Abel ODE [Zwi93] under the change of variable  $x = y^{-1}$ . In fact, these type of ODE's arise naturally in two dimensional bilinear systems when indirect control, derived from an inverse problem, is performed. Let

$$x' = (Ax + \delta) + (Bx + \gamma)u \quad (4.7.1)$$

be a two dimensional bilinear system.  $x$ ,  $\delta$ ,  $\gamma$  are vectors in  $\mathbb{R}^2$ ,  $A$  and  $B$  are square matrices and  $u$  is a single input.  $u$  can be eliminated from equation (4.7.1) by a scalar product, this leading to

$$(x' - Ax - \delta) \cdot (Bx + \gamma)^\perp = 0, \quad (4.7.2)$$

where  $w^\perp = (-w_2, w_1)$  for any given vector  $w = (w_1, w_2)^\top \in \mathbb{R}^2$ . Equation (4.7.2) results in

$$x'_1 p_1(x_1, x_2) + p_2(x_1, x_2) = x'_2 q_1(x_1, x_2) + q_2(x_1, x_2),$$

where  $p_i$ ,  $q_i$  are polynomials of degree  $i$  in the variables  $x_1$ ,  $x_2$ .

Let us assume  $x_2$  to be a nonminimum phase output which should track a certain periodic reference  $f(t)$ , and let  $x_1$  be a minimum phase output indirectly used to achieve the tracking purpose. For  $x_2 = f(t)$  to hold, the variable  $x_1$  must be forced to be a solution of the ordinary differential equation

$$x'_1 p_1(x_1, f(t)) + p_2(x_1, f(t)) = f'(t) q_1(x_1, f(t)) + q_2(x_1, f(t)). \quad (4.7.3)$$

As in the problem we have solved, the interest is in finding bounded or, rather, periodic solutions for (4.7.3), which would allow a physical implementation of the

control. Equation (4.7.3) is of the Abel type provided that

$$\frac{\partial p_1(x_1, f(t))}{\partial x_1} \neq 0;$$

then, the change of variable

$$\frac{1}{y} = p_1(x_1, f(t))$$

allows the standard polynomial description

$$y' = M(t)y + N(t)y^2 + P(t)y^3.$$

At this point, general results on the maximum number of limit cycles for this type of equations reported in [GL90] may be used in the investigation.

In order to illustrate the above exposed discussion, notice that the Abel ODE that appears after performing the change of variable mentioned at the beginning of the section to (4.4.3) is

$$y' = -y^2 + g(t)y^3. \quad (4.7.4)$$

It is worth saying that an equivalent result to theorem 4.4.2 is readily obtainable for (4.7.4) with the same tools.

## 4.8 Simulation results

A buck-boost converter with parameters  $V_g = 50V$ ,  $R = 10\Omega$ ,  $L = 0.018H$  and  $C = 0.00022F$  has been used in the simulations. These values make  $\lambda = 0.9045$ . The output voltage reference is

$$v_{Cr} = 135 + 15 \sin 2\pi\nu\tau \text{ V},$$

which becomes

$$x_{2r} = f(t) = 2.7 + 0.3 \sin \omega t$$

in the dimensionless variables. Fixing  $\nu = 50Hz$  results in  $w = 0.6252$ . With these settings, the fulfillment of (4.6.3) and (4.6.4) is guaranteed:

$$2.7 > \sup \left\{ B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}, B + \frac{A + B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}{A - B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}} - k \right\} = \sup \{0.36, 0.61\}.$$

Figure 4.4 depicts function  $H(z)$  with its unique zero. If the change of variables  $x = y^{-1}$  proposed in section 4.7 is performed, the function  $H(z)$  associated to the ODE (4.4.3) in polynomial form may be seen in figure 4.5.

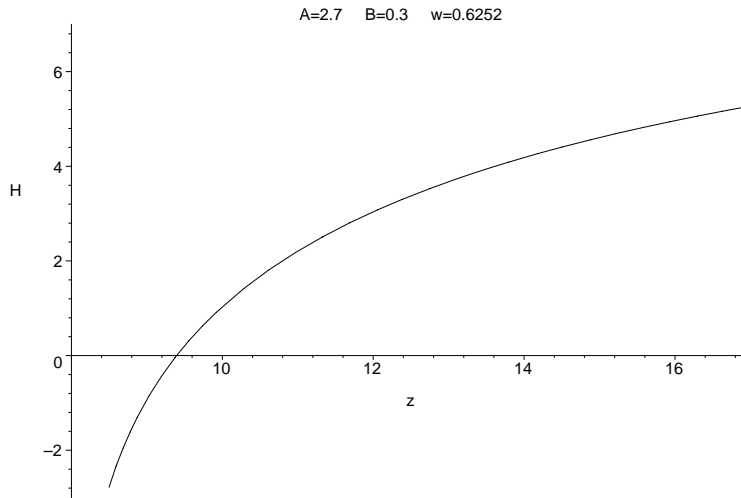


Figure 4.4: Detail of  $H(z)$  associated to (4.4.3).

The periodic solution  $\phi(t, z_0)$  of (4.4.3),  $z_0 = 9.3941719902$ , located between other two solutions  $y(t, z_1)$  and  $y(t, z_2)$ , with initial conditions above and below  $z_0$ , that is,  $z_1 = 9.39$  and  $z_2 = 9.40$ , is portrayed in figure 4.6. The instability is evident.

Figure 4.7 contains the SIMULINK model of the buck-boost converter described at the beginning of the section and used to simulate the tracking of the previously mentioned

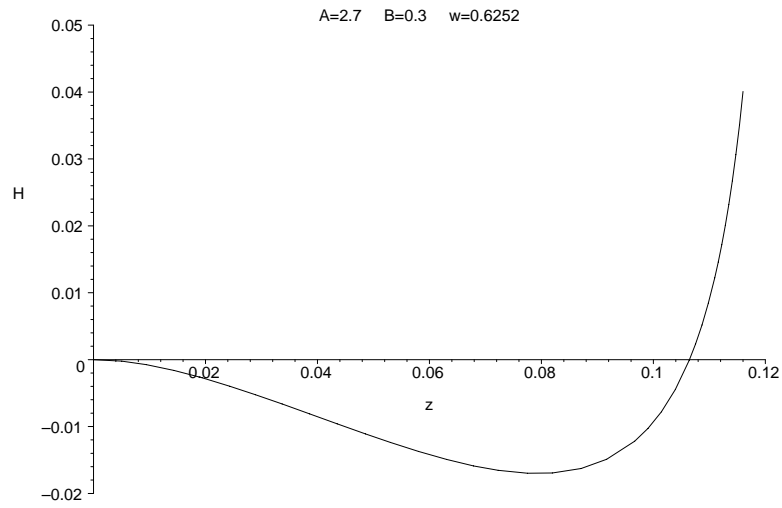


Figure 4.5: Detail of  $H(z)$  associated to (4.4.3) in polynomial form.

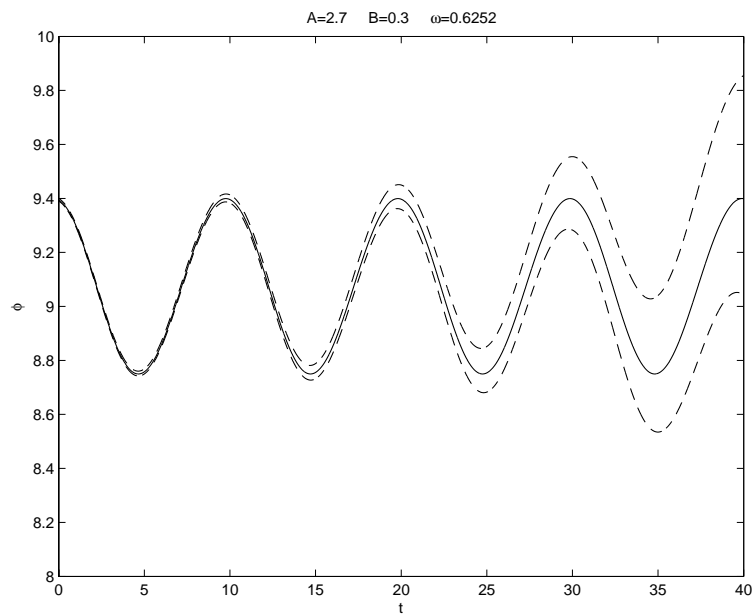


Figure 4.6: The periodic solution  $\phi(t, z_0)$  between neighbouring solutions.



to  $10^{-6}$  s) is lowered.

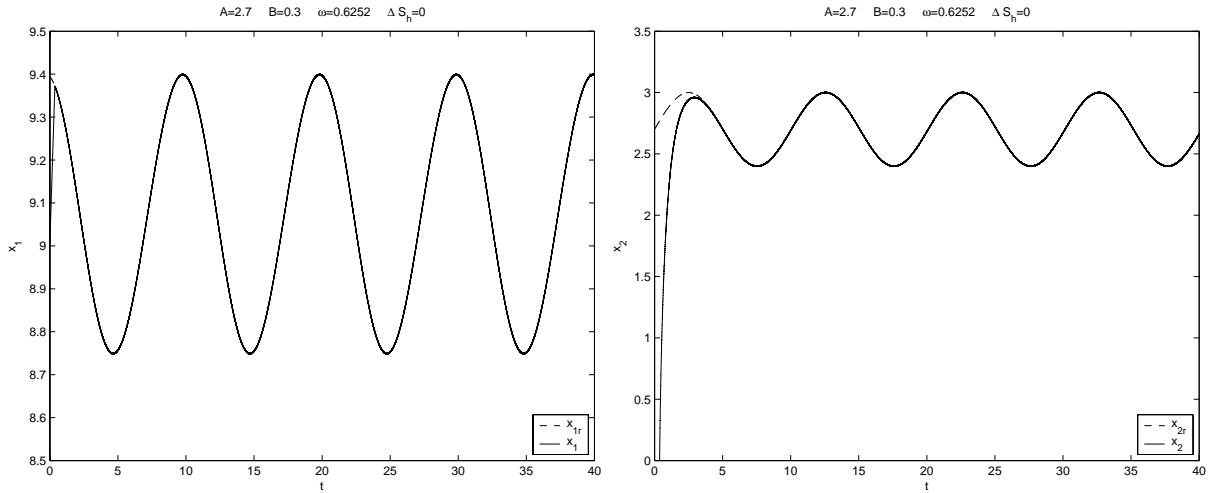


Figure 4.8: Details of  $x_1$  and  $x_2$  in the ideal case.

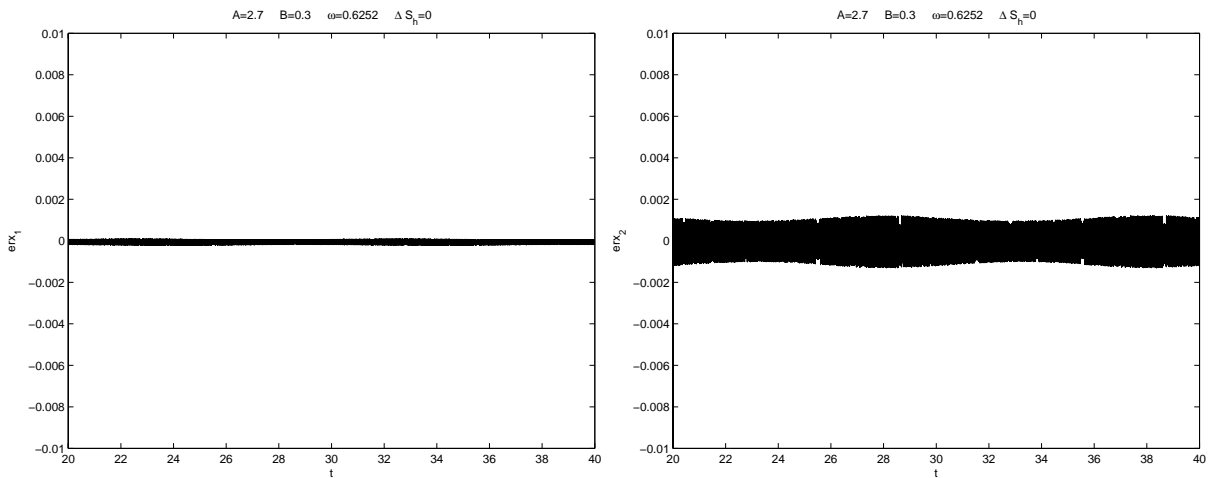
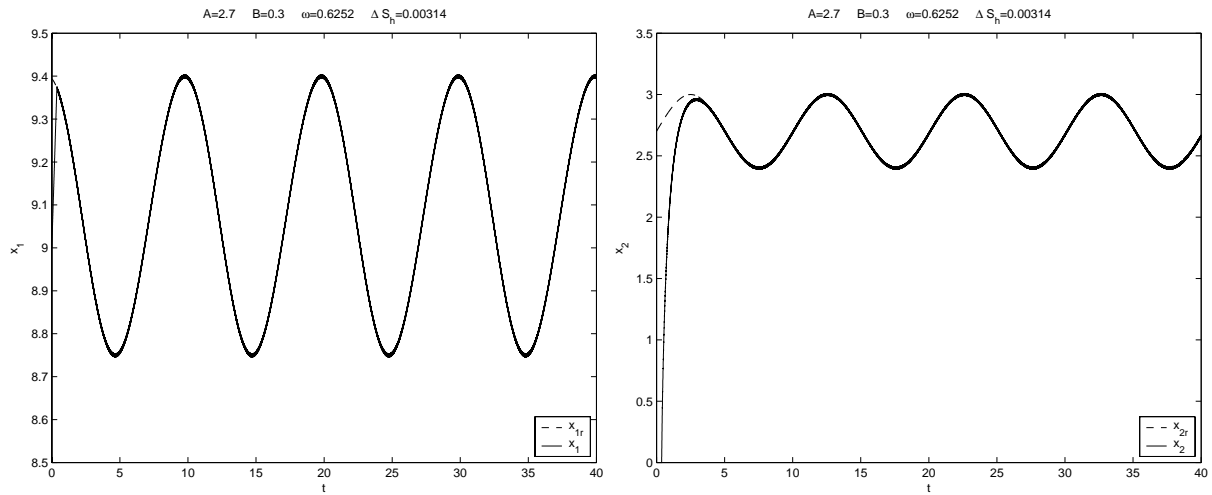
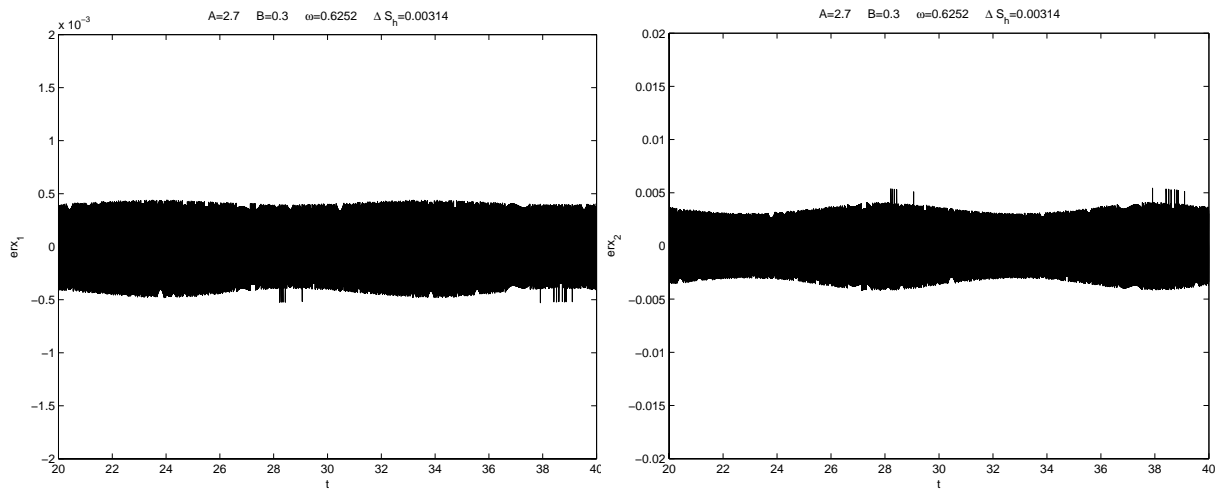


Figure 4.9: Details of  $x_1$  and  $x_2$  relative errors in the ideal case.

A more realistic simulation, achieved by means of a hysteresis amplitude band of  $\Delta S_h = 0.00314$ , associated to a maximum switching frequency of  $20 \text{ kHz}$ , is presented in figures 4.10 and 4.11. Good performance is again a fact, with relative output errors that do not reach the 0.5% in steady state.

Figure 4.10: Details of  $x_1$  and  $x_2$  in the non ideal case.Figure 4.11: Details of  $x_1$  and  $x_2$  relative errors in the non ideal case.

# Chapter 5

## Galerkin Method and Approximate Tracking

In this chapter we study the tracking problem solved in the previous chapter with an approximate technique, improving the results of [FO02].

### 5.1 Introduction

Chapter 4 developed the tracking of signals with the output voltage in basic, nonlinear DC-to-DC power converters. The problem is proved to be solvable via indirect control of such a voltage through the input current. That is, when the current is forced to follow a certain signal, the internal dynamics leads the output to track the desired reference. The signal to be followed by the input current is a periodic and unstable solution of an ordinary differential equation that depends on the output reference.

Despite the numerical solution, it is often interesting to consider an analytical approximation. The paper [FO02] proposes the Harmonic Balance method to obtain an approximate periodic solution of the ODE. When the current followed by the system is an approximation of the current actually needed, the output voltage is indirectly



affected and an output error appears. The quoted article gives the ODE satisfied by the output voltage, analyzes the boost converter and concludes that the obtained signal tends to the desired one when the input error tends to zero; for the buck-boost converter, the paper refers to simulation results to state that it exhibits the same behavior. This approximate methodology has been used by one of the authors of [FO02] in [ZFSB98a] and [ZFSB98b]. In [Sir99a], the same problem is treated using an iterative process instead of the harmonic balance method. Both techniques receive here a theoretical basis.

The Harmonic Balance method will be identified as a particular case of the Galerkin method [Zei90a], [Zei90b], widely used in Functional Analysis. The Galerkin method provides an algorithm that finds sets of equations whose solutions are used to build a sequence of approximate periodic solutions to the periodic input current reference. However, several questions not completely answered in [FO02] arise naturally:

(QA) Do all the Galerkin equations have a solution; that is, does such a sequence really exist?

(QB) Does the Galerkin sequence exhibit any type of convergence to the input current reference?

(QC) Which type of response can be expected from the output voltage when approximate inputs are used?

(QD) What can be stated about the influence of intrinsic system restrictions on the signals to be followed when approximate tracking is performed?

(QE) Is it possible to evaluate the input and output errors?

The chapter begins with four sections of mathematical background. The Galerkin method and the Leray-Schauder fixed point index, the main tools to answer QA,

are introduced in sections 5.2 and 5.3. Section 5.4 contains the basic ideas about weak convergence. The definition and main properties of Sobolev spaces are found in section 5.5. The statement of the problem is contained in section 5.6. In section 5.7, question QA is affirmatively answered, while the input error evaluation demanded in QE is in section 5.8. QB is treated in section 5.9. Section 5.10 develops QC, rebuilding section 6 of [FO02] and introducing a result that guarantees the existence of a periodic output even though an approximate current is used in the indirect control. Restrictions on the signals to be tracked are contained in section 5.11, and the evaluation of the output error is in section 5.12, answering questions QD and QE, respectively. In section 5.13, the convergence of the output to the desired reference is studied. A sliding mode control strategy for the devices is shown in section 5.14. Sections 5.15 and 5.16 end the chapter exemplifying the technique with the tracking of a sinusoidal wave and providing simulation results.

## 5.2 Galerkin method

The material of this section has been mainly extracted from [Zei90a], with some ideas from [Zei90b] and [Die76]. Elementary properties of Banach and Hilbert spaces have been reproduced to help to introduce some concepts.

**Definition 5.2.1.** Let  $X$  be a linear space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ .

(i) The mapping  $\|\cdot\| : X \rightarrow [0, \infty)$  is called a *norm* iff,  $\forall x, y \in X, \forall \lambda \in \mathbb{K}$ ,

$$(a) \|\lambda x\| = |\lambda| \|x\|, \quad (b) \|x + y\| \leq \|x\| + \|y\| \quad \text{and} \quad (c) \|x\| = 0 \iff x = 0.$$

(ii)  $(X, \|\cdot\|)$  is a *normed space* over  $\mathbb{K}$ .

(iii) The *distance*  $d(x, y) := \|x - y\|$  makes  $(X, \|\cdot\|)$  a *metric space*.

**Definition 5.2.2.** A normed space  $(X, \|\cdot\|)$  which is complete<sup>1</sup> as a metric space is called a *Banach space* (B-space).

**Definition 5.2.3.** Let  $X$  be a B-space. Then,

(i) A subset  $M$  of  $X$  is called *dense* in  $X$  iff for every  $x \in X$ , there exists a sequence  $(x_n) \in M$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . This is equivalent to the condition that  $\forall x \in X, \forall \epsilon > 0, \exists y \in M$  such that  $\|x - y\| < \epsilon$ .

(ii)  $X$  is *separable* iff it contains a numerable, dense subset.

(iii) A *basis* of the B-space  $X$  is a numerable sequence  $\{w_n\}_n$  of elements  $w_n \in X$ ,  $\forall n$ , such that a finite collection  $w_1, \dots, w_n$  is always linearly independent and

$$X = \overline{\cup_n X_n},$$

with  $X_n = \text{span}\{w_1, \dots, w_n\}$ .

**Definition 5.2.4.** Let  $X$  be a B-space, and  $\{Y_n\}_n$  a sequence of subspaces of  $X$ , with  $Y_n \neq \emptyset$  and  $\dim Y_n < \infty, \forall n$ . The sequence  $\{Y_n\}_n$  is a *Galerkin scheme* in  $X$  iff

$$\lim_{n \rightarrow \infty} \text{dist}_X(u, Y_n) = \lim_{n \rightarrow \infty} \inf_{v \in Y_n} \|u - v\|_X = 0, \quad \forall u \in X,$$

$\|\cdot\|_X$  being the norm defined in the B-space  $X$ .

**Proposition 5.2.1.** Let  $X$  be a separable B-space. Then,

(i)  $X$  has a basis.

(ii) If  $\{w_n\}_n$  is a basis of  $X$ , the sequence of subspaces  $\{X_n\}_n$ ,  $X_n = \{w_1, \dots, w_n\}$ , is a Galerkin scheme in  $X$ .

(iii) If  $\{Y_n\}_n$  is a Galerkin scheme in  $X$ , a basis in  $X$  can be obtained with  $\{Y_n\}_n$ . ■

---

<sup>1</sup>Every Cauchy sequence converges.

**Definition 5.2.5.** Let  $X$  be a linear space over  $\mathbb{K}$ . A *scalar product*  $(\cdot|\cdot) : X \times X \longrightarrow \mathbb{K}$  is a mapping such that  $\forall x, y, z \in X, \forall \lambda, \mu \in \mathbb{K}$ ,

$$(a) (x|\lambda y + \mu z) = \lambda(x|y) + \mu(x|z), \quad (b) (x|y) = \overline{(y|x)} \quad \text{and} \quad (c) (x|x) > 0 \text{ iff } x \neq 0,$$

where the bar denotes complex conjugation.

**Definition 5.2.6.** Let  $X$  be a linear space over  $\mathbb{K}$ , and let  $(\cdot|\cdot)$  be a scalar product in  $X$ .  $X$  becomes a normed space through  $\|x\| := \sqrt{(x|x)}$ . If this normed space is a B-space, then  $X$  is called a *Hilbert space* (H-space).

**Definition 5.2.7.** Let  $X$  be an H-space over  $\mathbb{R}$  with scalar product  $(\cdot|\cdot)$ , and let  $\{w_n\}_n$  be a numerable, orthonormal system in  $X$ . The numerable system  $\{w_n\}_n$  is said to be *complete* in  $X$  iff we can write

$$x = \sum_{n=1}^{\infty} (x|w_n)w_n, \quad \forall x \in X. \quad (5.2.1)$$

**Proposition 5.2.2.** Let  $X$  be a real H-space and  $\{w_n\}_n$  a numerable, orthonormal system in  $X$ . Then, the following statements are equivalent:

- (i)  $\{w_n\}_n$  is complete.
- (ii) The set  $\text{span}\{w_n\}$  is dense in  $X$ .
- (iii) The Parseval relation

$$\|x\|^2 = \sum_n |(x|w_n)|^2, \quad \forall x \in X$$

is satisfied. ■

**Proposition 5.2.3.** Let  $\{w_n\}_n$  be a complete, orthonormal system in the separable H-space  $X$ . Then,

- (i)  $\{w_n\}_n$  is a basis in  $X$ .

(ii) Given  $x \in X$ , we assign

$$P_n x = \sum_{j=1}^n (x|w_j) w_j.$$

The mapping  $P_n : X \rightarrow X_n$  is an orthogonal projection operator in the set  $X_n = \text{span}\{w_1, \dots, w_n\}$ . ■

Let now  $F : X \rightarrow X$  be an operator in the B-space  $X$ , and consider the problem

$$Fx = 0. \quad (5.2.2)$$

Being  $\{w_1, w_2, \dots\}$  a basis in  $X$ , the *Galerkin method* propounds to approximate the solution of (5.2.2) replacing  $x \in X$  by  $x_n \in X_n$ :

$$x_n = \sum_{j=1}^n c_{nj} w_j,$$

and searching for the coefficients  $\{c_{nj}\}_j$  that satisfy the system

$$Fx_n = 0$$

restricted to  $X_n$ , which is known as the *Galerkin equations*.

The Galerkin method may also be seen as a *projection method*. This is possible when we consider  $X$  an H-space, because the Galerkin equations can then be written (see proposition 5.2.3 (ii))

$$P_n F x_n = 0$$

or, equivalently,

$$(F x_n | w_j) = 0, \quad j = 1, \dots, n.$$

This may be better understood with the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \text{inj} \uparrow & & \downarrow P_n \\ X_n & \longrightarrow & X_n \end{array}$$

In our case, solutions of the Galerkin equations will be guaranteed via the *fixed point index* theory.

*Remark 5.2.1.* When a Galerkin approximation is used instead of the exact solution, an error appears due to the fact that, in general,  $Fx_n \neq 0$ . The properties of projection operators lead to

$$Fx_n = P_n Fx_n + (I - P_n)Fx_n = (I - P_n)Fx_n. \quad (5.2.3)$$

Consider now the set of square integrable functions in  $(0, T)$ , denoted  $L_2(0, T)$ , provided with the scalar product

$$(x|y) = \int_0^T xy. \quad (5.2.4)$$

We may find in it the so called *trigonometric system*  $\{w_n\}_n$ ,  $w_n \in L_2 \forall n \geq 0$ , with

$$w_0 = \frac{1}{\sqrt{T}}, \quad w_{2k-1} = \sqrt{\frac{2}{T}} \cos \frac{2\pi kt}{T}, \quad w_{2k} = \sqrt{\frac{2}{T}} \sin \frac{2\pi kt}{T}, \quad k \geq 1. \quad (5.2.5)$$

**Proposition 5.2.4.** *Let  $\{w_n\}_n$  stand for the trigonometric system. Then,*

(i)  $L_2(0, T)$ , together with the scalar product defined in (5.2.4), is a real, separable  $H$ -space for which the trigonometric system is a complete, orthonormal system.

(ii)  $\{w_n\}_n$  is a basis of  $L_2(0, T)$ .

(iii) The sequence of subspaces  $\{X_n\}_n$ , with

$$X_0 = \{w_0\}, X_1 = \{w_0, w_1, w_2\}, \dots, X_n = \{w_0, w_1, w_2, \dots, w_{2n-1}, w_{2n}\}, \dots$$

is a Galerkin scheme in  $L_2$ , each of the  $X_n$  being a Banach subspace. ■

*Remark 5.2.2.* Equality (5.2.1) must be understood in the sense of the norm defined in  $X$ ; for example, in  $L_2(0, T)$  and for a complete, orthonormal system  $\{w_n\}_n$ , it means that

$$\lim_{n \rightarrow \infty} \int_0^T \left[ x - \sum_{j=1}^n (x|w_j)w_j \right]^2 = 0.$$

Furthermore, for the trigonometric system, the convergence becomes uniform when  $x$  belongs to the subset of continuous,  $T$ -periodic functions with piecewise continuous derivative.

For a better identification of the Galerkin method and the Harmonic Balance method, we redefine the projection operator  $P_n$  for the Hilbert space  $L_2(0, T)$  and the trigonometric system  $\{w_n\}_n$  as

$$P_n x = \sum_{j=0}^{2n} (x|w_j) w_j. \quad (5.2.6)$$

Hence, the  $n$ -th Galerkin approximation  $x_n$  contains harmonics up to  $n$ -th order and may be written as

$$x_n = \sum_{j=0}^{2n} c_{nj} w_j.$$

Finally, the Galerkin equations  $P_n F x_n = 0$  become

$$(F x_n | w_j) = 0, \quad j = 0, \dots, 2n.$$

### 5.3 Fixed point index and mapping degree

The fixed point index and the mapping degree give an answer to a generalization of the so called *index theory*, which allows the existence of equilibrium points in planar, real systems to be predicted with few calculations. The material has been borrowed from [Zei93].

We first of all introduce the concept of compact operator. These kinds of operators play a main role in nonlinear functional analysis. Their importance derives from the fact that many results on continuous operators in  $\mathbb{R}^n$  remain valid in Banach spaces for compact operators. Afterwards, we define the mapping degree via the fixed point

index, because in Banach spaces it is possible to work with either, depending on the particular problem.

**Definition 5.3.1.** Let  $X$  be a B-space. A subset  $M \subseteq X$  is *relatively compact* (resp. *compact*) iff every sequence in  $M$  contains a convergent subsequence (resp. the limit of which also belongs to  $M$ ).

**Definition 5.3.2.** Let  $X$  and  $Y$  be two B-spaces,  $D(T) \subseteq X$  a subset of  $X$  and  $T : D(T) \rightarrow Y$  an operator. Then,

- (i)  $T$  is *bounded* iff it maps bounded sets into bounded sets.
- (ii)  $T$  is *continuous* iff  $x_n \rightarrow x$  when  $n \rightarrow \infty$  implies  $T(x_n) \rightarrow T(x)$ .
- (iii)  $T$  is *compact* iff it is continuous and it maps bounded sets into relatively compact sets.

The next property identifies continuous operators with compact operators in finite dimensional situations.

**Proposition 5.3.1.** *In finite dimensional B-spaces, continuous mappings and compact mappings are the same whenever the domain  $D(T)$  is closed.* ■

**Definition 5.3.3.** Let  $X$  be a B-space and  $G \subset X$  an open, bounded subset of  $X$ , and denote  $V(G, X)$  the set of compact mappings  $f : \overline{G} \rightarrow X$  with no fixed points in  $\partial G$ . Then, two mappings  $f, g \in V(G, X)$  are said to be *homotopically compact* in  $\partial G$  iff there exists a mapping  $H$  with the following properties:

- (P1)  $H : \overline{G} \times [0, 1] \rightarrow X$  is compact;
- (P2)  $H(x, \lambda) \neq x, \forall (x, \lambda) \in \partial G \times [0, 1]$ ;
- (P3)  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  in  $\overline{G}$ .



In this case, we write  $\partial G : f \cong g$ . The mapping  $H$  is called *compact homotopy* or, simply, *homotopy*.

**Proposition 5.3.2.** *The homotopy  $\partial G : f \cong g$  holds true iff there is a compact mapping  $H : \overline{G} \times [0, 1] \longrightarrow X$ , with*

$$H(x, \lambda) \neq x, \forall (x, \lambda) \in \partial G \times [0, 1], \text{ and } H(x, 0) = f(x), H(x, 1) = g(x) \text{ on } \partial G.$$

■

*Remark 5.3.1.* The characterization of homotopies given by proposition 5.3.2 constitutes a condition weaker than the one established in definition 5.3.3: notice that to obtain  $\partial G : f \cong g$  it suffices to know the value of  $f$  and  $g$  on  $\partial G$ ; namely,

$$\partial G : f = g \text{ always implies } \partial G : f \cong g.$$

The system of axioms that define the fixed point index is:

**Definition 5.3.4.** To every  $f \in V(G, X)$  let there be assigned an integer  $i(f, G)$  called the *fixed point index* of  $f$  on  $G$  so that it satisfies the axioms:

(A1) (Normalization). If  $f(x) = x_0, \forall x \in \overline{G}$  and some fixed  $x_0 \in G$ , then  $i(f, G) = 1$ .

(A2) (Kronecker existence principle). If  $i(f, G) \neq 0, \exists x \in G$  such that  $f(x) = x$ .

(A3) (Additivity). We have

$$i(f, G) = \sum_{j=1}^n i(f, G_j)$$

whenever  $f \in V(G, X)$  and  $f \in V(G_j, X) \forall j$ , where  $\{G_j\}_j$  is a partition of  $G$ .

(A4) (Homotopy invariance). If  $\partial G : f \cong g$ , then  $i(f, G) = i(g, G)$ .

This is completed with the following uniqueness principle:

**Proposition 5.3.3.** (Leray-Schauder). For every mapping  $f \in V(G, X)$  and every  $V(G, X)$ ,  $X$  being an arbitrary  $B$ -space, there is exactly one fixed point index that satisfies axioms (A1)-(A4) of definition 5.3.4. ■

*Remark 5.3.2.* With this tool, the strategy of proving the existence of a fixed point for a certain mapping  $f$  consists in relating it by homotopy with a simpler mapping  $g$  for which it happens  $i(g, G) \neq 0$ . Hence, (A4) and (A2) entail the desired result.

Let us finally introduce an alternative way of calculating the fixed point index in  $\mathbb{R}^n$ :

**Definition 5.3.5.** Let  $f : G \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$ -mapping. The point  $x \in G$  is called a *regular value* of  $f$  iff  $\det(f'(x)) \neq 0$ .

**Definition 5.3.6.** Let  $G$  be an open bounded set in  $\mathbb{R}^n$ . Then  $V_0(G, \mathbb{R}^n)$  denotes the set of all maps  $f$  with the following properties:

- (i) The map  $f : \overline{G} \rightarrow \mathbb{R}^n$  is continuous and  $C^1$  in  $G$ .
- (ii) The map  $f$  has finitely many fixed points, if any, all of which are regular and none of which lies on the boundary  $\partial G$ .

**Proposition 5.3.4.** For every  $f \in V_0(G, \mathbb{R}^n)$ ,

$$i(f, G) = \sum_{j=1}^m \operatorname{sgn}\{\det[F'(x_j)]\},$$

where  $F(x) = x - f(x)$  and  $x_1, \dots, x_m$  are all the fixed points of  $f$  in  $G$ . If  $f$  has no fixed points in  $G$ , then we set  $i(f, G) = 0$ . ■

Consider now the equation

$$F(x) = y;$$

its equivalence with the fixed point problem

$$x - F(x) + y = x$$

is evident. This leads us to the mapping degree.

**Definition 5.3.7.** Let  $F : \overline{G} \subset X \longrightarrow X$  be a compact perturbation of the identity; that is, it can be written as the identity plus a compact mapping, and let  $G$  be an open bounded subset of the B-space  $X$ . Let also  $y \in X$  be such that  $y \notin F(\partial G)$ . The *mapping degree* is defined as

$$\deg(F, G, y) = i(\mathbb{I} - F + y, G).$$

Notice that the mapping degree is well defined: if  $F$  is a compact perturbation of the identity,  $g = \mathbb{I} - F + y$  is also compact; furthermore, as  $y \notin F(\partial G)$ ,  $g$  cannot possess any fixed point on  $\partial G$ .

Notice also that

$$\deg(F, G, y) = \deg(F - y, G, 0),$$

and, by convention, we write

$$\deg(F, G, 0) = \deg(F, G).$$

The first properties of the mapping degree gather the equivalent to axioms (A1)-(A4) of the fixed point index:

**Proposition 5.3.5.** *Let  $X$  be a B-space,  $G \subset X$  an open, bounded subset of  $X$  and  $y \in X$  an element of  $X$ . Then,*

(i) *For the identity mapping  $\mathbb{I}$  it results that*

$$\deg(\mathbb{I}, G, y) = \begin{cases} 1 & \text{if } y \in G \\ 0 & \text{if } y \notin \overline{G}. \end{cases}$$

(ii) *If  $\deg(F, G, y) \neq 0$ ,  $\exists x \in G$  such that  $F(x) = y$ .*

(iii) Let  $H(x, \lambda) = x - h(x, \lambda)$ , with  $h : \overline{G} \times [0, 1] \rightarrow X$  compact and with  $y \notin H(\partial G \times [0, 1])$ . Then,  $\deg[H(\cdot, \lambda), G, y]$  is constant  $\forall \lambda \in [0, 1]$ . ■

*Remark 5.3.3.* An important consequence of proposition 5.3.2 and remark 5.3.1 is that the fixed point index and the mapping degree depend on boundary values only.

*Remark 5.3.4.* The relation between the index theory and the mapping degree may be clarified as follows. Let  $\overline{U}(0, R) \subset \mathbb{R}^2$  be the closed disk of radius  $R$  centered at the origin, and let  $F : \overline{U} \rightarrow \mathbb{R}^2$  be a continuous mapping. As a point  $x$  travels around the border of the disk  $\partial U$  in a positive sense, the vector field  $F(x)$  traverses a closed, oriented contour  $J$ . Assume that  $0 \notin J$ . The *index of  $\overline{U}$  with respect to  $F$* ,  $i_F(\overline{U})$ , is defined as the net change in the direction of  $F(x)$  as  $x$  completes a traverse along  $\partial U$ , divided by  $2\pi$  (also called *rotation of the vector field  $F$* ). Hence, denoting  $w_+$ ,  $w_-$  the number of windings of  $J$  about the origin in a positive and negative sense, respectively, we have that

$$i_F(\overline{U}) = w_+ - w_-.$$

Finally, it is evident that this definition satisfies the following well known properties:

- (i) If  $i_F(\overline{U}) \neq 0$ , there is a point  $x_0 \in U$ , such that  $F(x_0) = 0$  (*Kronecker existence principle*).
- (ii) If  $F$  is continuously deformed in such a way that the corresponding contours  $J$  do not touch the origin,  $i_F(\overline{U})$  remains constant (*homotopy invariance*).

## 5.4 Reflexivity and weak convergence

The main use of the reflexivity concept comes down specifically to the following fact. In infinite dimensional B-spaces it is not necessarily true that a bounded sequence

contains a convergent subsequence. However, this becomes true in reflexive B-spaces when weak convergence is used.

### 5.4.1 Duality in Banach spaces and reflexivity

**Definition 5.4.1.** Let  $X$  be a B-space over  $\mathbb{K}$ . A *continuous linear functional* in  $X$  is a continuous linear mapping  $f : X \rightarrow \mathbb{K}$ .

**Proposition 5.4.1.** *The set of all linear continuous functionals in  $X$ , denoted  $X^*$  and called dual space of  $X$ , exhibits a B-space structure over  $\mathbb{K}$  with the norm*

$$\|f\| = \sup_{\|x\| \leq 1} |\langle f, x \rangle|,$$

where  $\langle f, x \rangle := f(x)$ ,  $\forall x \in X$ . ■

**Proposition 5.4.2.** *For  $\mathbb{K} = \mathbb{R}$  and  $X \neq \{0\}$ ,*

(i)  $|\langle f, x \rangle| \leq \|f\| \|x\|$ .

(ii)  $\|f\| = \sup_{\|x\|=1} |\langle f, x \rangle|$ .

(iii)  $\|x\| = \sup_{\|f\|=1} |\langle f, x \rangle|$ . ■

Let us now denote  $X^{**} = (X^*)^*$  and consider  $x \in X$ ; we define the continuous linear functional  $U_x : X^* \rightarrow \mathbb{K}$ , assigning to each  $f \in X^*$

$$U_x(f) = \langle f, x \rangle.$$

Hence,  $U_x \in X^{**}$  and

$$\|U_x\| = \sup_{\|f\|=1} |\langle U_x, f \rangle| = \sup_{\|f\|=1} |\langle f, x \rangle| = \|x\|.$$

We then see the existence of a natural injection

$$a : X \rightarrow X^{**} \quad \text{such that} \quad a(x) = U_x,$$

which allows us to write  $X \subseteq X^{**}$ .

**Definition 5.4.2.** The B-space  $X$  is said to be *reflexive* iff the mapping  $a$  is surjective.

Therefore, if  $X$  is reflexive, the mapping  $a$  is a *norm isomorphism*<sup>2</sup> between  $X$  and  $X^{**}$ . It is in this sense that we can identify  $X = X^{**}$  and  $x$  with  $U_x$ ; hence,

$$\langle f, x \rangle = \langle x, f \rangle, \quad \forall x \in X, \forall f \in X^*.$$

*Remark 5.4.1.* The following are examples of reflexive B-spaces:

- (i) Every finite dimensional B-space.
- (ii) The B-space  $L_p(G)$ ,  $\forall p \in \mathbb{N}$ ,  $G$  being an open, bounded set.

## 5.4.2 Duality in Hilbert spaces

We will see here the existence of a close relation between  $X$  and  $X^*$ ,  $X$  being a real H-space. Hence, let  $J : X \rightarrow X^*$ , denote  $J_x \in X^*$  the image of  $x \in X$  by  $J$ , that is,  $J_x = J(x)$ , and establish

$$\langle J_x, y \rangle = (x|y), \quad \forall x, y \in X. \quad (5.4.1)$$

In fact, the Riesz representation theorem states that for every bounded, linear functional  $f$  over the H-space  $X$ , there exists a unique  $y \in X$  such that  $f(x) = (x|y)$ ,  $\forall x \in X$ . Thus,

$$X^* = \{f_y; f_y(x) = (x|y), \quad \forall x, y \in X\}.$$

**Proposition 5.4.3.** *Let  $X$  be a real H-space. Then,*

---

<sup>2</sup>Also known as isometry, it entails the existence of a linear bijection  $f : X \rightarrow X^{**}$  such that  $\|f(x)\| = \|x\|$ ,  $\forall x \in X$ .

(i) For every  $x \in X$  there exists a single linear continuous functional  $J_x \in X^*$  that satisfies (5.4.1). The operator  $J : X \rightarrow X^*$ , called dual map of  $X$ , is linear, bijective and such that  $\|J_x\| = \|x\|$ ,  $\forall x \in X$ .

(ii) As the dual map  $J : X \rightarrow X^*$  is a norm isomorphism, we can identify  $J_x$  with  $x$ , and it is in this sense that we have  $X^* = X$ . Hence,

$$\langle x, y \rangle = (x|y), \quad \forall x, y \in X.$$

(iii) Every H-space is reflexive. ■

*Remark 5.4.2.* Remember that  $L_p(0, T)^* = L_q(0, T)$ , with  $p^{-1} + q^{-1} = 1$ .

### 5.4.3 Weak convergence

**Definition 5.4.3.** Let  $X$  be a B-space and  $(x_n)$  a sequence in  $X$ . Then,

(i)  $(x_n)$  is said to *converge strongly* to  $x \in X$  when the norm of the difference  $x_n - x$  tends to zero. We write this as

$$x_n \rightarrow x \iff \|x_n - x\| \rightarrow 0.$$

Strong convergence is also designed as *norm convergence*.

(ii)  $(x_n)$  is said to *converge weakly* to  $x \in X$  when  $\langle f, x_n \rangle \rightarrow \langle f, x \rangle$ ,  $\forall f \in X^*$ . We write this as

$$x_n \rightharpoonup x \iff \langle f, x_n \rangle \rightarrow \langle f, x \rangle, \quad \forall f \in X^*.$$

(iii) If  $X$  is an H-space,

$$x_n \rightharpoonup x \iff (x_n|y) \rightarrow (x|y), \quad \forall y \in X.$$

**Theorem 5.4.4.** *Every bounded sequence in a reflexive B-space has a weakly convergent subsequence.* ■

**Definition 5.4.4.** Let  $X$  be a B-space. Then,

(i)  $X$  is *strictly convex* iff

$$\|\lambda x + (1 - \lambda)y\| < 1,$$

$\forall x, y \in X$  satisfying  $\|x\| = \|y\| = 1$ ,  $x \neq y$  and  $0 < \lambda < 1$ .

(ii)  $X$  is *locally uniformly convex* iff  $\forall \epsilon$ ,  $0 < \epsilon \leq 2$ , and  $\forall x \in X$ ,  $\|x\| = 1$ ,  $\exists \delta(\epsilon, x) > 0$  such that  $\forall x, y \in X$ ,

$$\|x - y\| \geq \epsilon, \quad \|x\| = \|y\| = 1, \implies \frac{1}{2}\|x - y\| \leq 1 - \delta(\epsilon, x).$$

(iii)  $X$  is *uniformly convex* iff it is locally uniformly convex and  $\delta$  can be chosen independently of  $x$ .

Geometrically, the uniform convexity for  $X$  means that the unit sphere is round; that is, for every two points of its border, the midpoint of the rectilinear segment that connects them is inside the sphere.

**Proposition 5.4.5.** *Every H-space is uniformly convex.* ■

The most important properties related to the weak convergence are:

**Proposition 5.4.6.** *Let  $(x_n)$  be a sequence in a B-space  $X$ , real or complex. Then,*

(i) *The strong convergence implies the weak convergence:*

$$x_n \rightarrow x \implies x_n \rightharpoonup x.$$

(ii) *If  $\dim X < \infty$ , the weak convergence implies the strong convergence.*



(iii) If  $x_n \rightharpoonup x$ ,  $(x_n)$  is bounded and

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

(iv) If  $X$  is locally uniformly convex,  $x_n \rightharpoonup x$  and  $\|x_n\| \rightarrow \|x\| \implies x_n \rightarrow x$ .

(v) Let  $(x_n)$  be bounded. If every convergent subsequence of  $(x_n)$  has limit  $x$ , then  $x_n \rightharpoonup x$ .

(vi) If  $(x_n)$  is bounded in  $X$  and there exist  $x \in X$  and a subset  $D \subset X^*$ , dense in  $X^*$  and such that

$$\langle f, x_n \rangle \rightarrow \langle f, x \rangle, \quad \forall f \in D,$$

then  $x_n \rightharpoonup x$ .

■

## 5.5 Sobolev spaces

Let  $I = (a, b)$  be an open interval in  $\mathbb{R}$ .

**Definition 5.5.1.**  $C_0^\infty(I)$  is the space of all real functions  $x \in C^\infty(I)$  with compact support in  $I$ , i.e. such that they take null value everywhere except in a compact subset  $K \subset I$  that depends on  $x$ .

**Definition 5.5.2.** Let  $x, y \in L_1(I)$ ;  $y$  is said to be the  $n$ -th generalized derivative of  $x$  in  $I$  iff

$$\int_I \varphi^{(n)}(t)x(t)dt = (-1)^n \int_I \varphi(t)y(t)dt, \quad \forall \varphi \in C_0^\infty(I).$$

Therefore, we write  $y = D^n x$ .

*Remark 5.5.1.* (i) Let  $x \in C^n(\bar{I})$ . Then, the continuous  $n$ -th derivative  $x^{(n)} : \bar{I} \rightarrow \mathbb{R}$  is also the generalized  $n$ -th derivative of  $x$  in  $\bar{I}$ . This follows immediately from the classical integration by parts formula.

(ii) The generalized derivative of  $x$  is unique in  $L_1(I)$ . In fact, from  $D^n x = y_1 = y_2$  we obtain  $y_1 = y_2$  almost everywhere in  $I$ .

**Definition 5.5.3.** The *Sobolev space*  $W_2^1(I)$  is the set of all functions  $x \in L_2(I)$  which have first generalized derivative in  $L_2(I)$ .

The norm in  $W_2^1$  is set to

$$\|x\|_{W_2^1} = \|x\|_{L_2} + \|Dx\|_{L_2}.$$

Let us now define a scalar product in  $W_2^1$ :

$$(x|y)_{W_2^1} = (x|y)_{L_2} + (Dx|Dy)_{L_2}. \quad (5.5.1)$$

**Proposition 5.5.1.** *The Sobolev space  $W_2^1(I)$  is a reflexive, separable B-space. Moreover,  $W_2^1(I)$  is a separable H-space with the scalar product defined in (5.5.1). In the latter case we denote  $H^1(I) = W_2^1(I)$ . ■*

**Definition 5.5.4.** Let  $X, Y$  be B-spaces over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ , with  $X \subseteq Y$ . The embedding operator  $j : X \rightarrow Y$  is defined by  $j(x) = x, \forall x \in X$ .

(i) The embedding  $X \subseteq Y$  is called *continuous* iff  $j$  is continuous, i.e.,

$$\|x\|_Y \leq \|x\|_X, \quad \forall x \in X. \quad (5.5.2)$$

(ii) The embedding  $X \subseteq Y$  is called *compact* iff  $j$  is compact, i.e., (5.5.2) holds and each bounded sequence  $\{x_n\}_n$  in  $X$  has a subsequence  $\{x_{n'}\}_{n'}$  which is convergent in  $Y$ .

**Proposition 5.5.2.** *Let  $X, Y, Z$  be  $B$ -spaces over  $\mathbb{K}$ .*

(i) *If the embeddings  $X \subseteq Y$  and  $Y \subseteq Z$  are continuous, then so is  $X \subseteq Z$ . If, in addition, one of the embeddings  $X \subseteq Y$  or  $Y \subseteq Z$  is compact, then so is  $X \subseteq Z$ .*

(ii) *If the embedding  $X \subseteq Y$  is continuous, then, as  $n \rightarrow \infty$ ,*

$$x_n \rightarrow x \text{ in } X \implies x_n \rightarrow x \text{ in } Y$$

and

$$x_n \rightharpoonup x \text{ in } X \implies x_n \rightharpoonup x \text{ in } Y.$$

(iii) *If the embedding  $X \subseteq Y$  is compact, then, as  $n \rightarrow \infty$ ,*

$$x_n \rightharpoonup x \text{ in } X \implies x_n \rightarrow x \text{ in } Y.$$

■

**Theorem 5.5.3.** *(Rellich-Kondratjev, [Bre83]) Let  $I$  be a bounded open interval in  $\mathbb{R}$ . Then, the following embeddings are compact:*

$$W_2^1(I) \subseteq L_2(I), \quad W_2^1 \subseteq C(\bar{I}).$$

■

*Remark 5.5.2.*  $C(\bar{I})$  stands for the space of continuous functions from  $\bar{I}$  into  $\mathbb{R}$ , with norm

$$\|x\|_{C(\bar{I})} = \|x\|_{L_\infty} = \sup_{t \in \bar{I}} \{|x(t)|\}.$$

**Proposition 5.5.4.** *Let  $\{x_n y_n\}$  be a bounded sequence in  $W_2^1(I)$ , and suppose that*

$$x_n \rightarrow x \text{ in } C(\bar{I}), \quad y_n \rightharpoonup y \text{ in } L_2(I).$$

Then,

$$x_n y_n \rightharpoonup xy \quad \text{in } W_2^1(I).$$

■

## 5.6 Statement of the problem

We have already seen in chapter 4 that the state variables proportional to the input current and the output voltage, respectively  $x$  and  $y$ , of the nonlinear converters which have been studied, are related by the ODE

$$x(1 - x') = (y' + \lambda y)(k + y).$$

If we succeed in forcing the current to follow a  $T$ -periodic reference  $x = \phi(t)$  such that

$$\phi(1 - \phi') = (f' + \lambda)(k + f), \quad (5.6.1)$$

$f(t)$  being a certain  $T$ -periodic output reference, the internal dynamics of the system will lead the output to asymptotically track  $f(t)$ . The reason lies in the fact that

$$(y' + \lambda y)(k + y) = (f' + \lambda)(k + f) \quad (5.6.2)$$

admits  $y = f(t)$  as an asymptotically stable solution provided that (see proposition 4.5.5)

$$g = (f' + \lambda)(k + f) > 0. \quad (5.6.3)$$

However, the high instability of  $\phi(t)$  makes us look for an analytical approximation via the Galerkin method. We therefore consider the equation

$$x(1 - x') = g(t), \quad (5.6.4)$$

also written  $Fx = 0$  following (5.2.2), with  $g(t) \in C^\infty$  defined in (5.6.3), positive and  $T$ -periodic. We know from theorem 4.4.2 that (5.6.4) has a positive,  $T$ -periodic, unstable solution  $x(t, x_0)$ , with  $x(0, x_0) = x_0$ , denoted  $x(t, x_0) = \phi(t)$  from now on. At this point we wonder whether a sequence  $\{\phi_n\}_n$  of solutions of the Galerkin equations associated to (5.6.4) exists and converges.

Additionally, it is obvious that the use of an  $n$ -th Galerkin approximation  $\phi_n(t)$  instead of  $\phi$  will affect the output  $y$ , converting it into a  $y_n$  that satisfies

$$(y'_n + \lambda y_n)(k + y_n) = \phi_n(1 - \phi'_n).$$

According to section 5.2, the right hand side of this equation can be written as

$$\phi_n(1 - \phi'_n) = (f' + \lambda)(k + f) + F\phi_n, \quad (5.6.5)$$

which is the equivalent to (5.6.1). The error term  $F\phi_n$  is given by (5.2.3) and is, trivially, also  $T$ -periodic. Equation (5.6.2) therefore becomes

$$(y'_n + \lambda y_n)(k + y_n) = g + F\phi_n, \quad (5.6.6)$$

or using  $G_n(t) = g(t) + F\phi_n(t)$ ,

$$(y'_n + \lambda y_n)(k + y_n) = G_n. \quad (5.6.7)$$

The questions that arise at this point are very evident. The first one is whether there still exists an asymptotically stable,  $T$ -periodic solution for (5.6.7); that is, whether we can obtain a sequence  $\{y_n\}_n$  with such a feature. This is very important, because a negative answer would imply much difficulty or even impossibility of using this technique. The second one is about the convergence of  $\{y_n\}_n$  to  $y = f(t)$ , which, if true, definitively validates the method from a mathematical viewpoint. Complementary subjects are a study of the restrictions to avoid saturation problems and an evaluation of input and output errors.

## 5.7 Solution of the Galerkin equations

Let us take up problem (5.6.4), which we now write as

$$Fx = 0, \quad (5.7.1)$$

with  $F : CP_1([0, T]) \subset L_2(0, T) \longrightarrow L_2(0, T)$ , where  $CP_1([0, T])$  stands for the set of continuous,  $T$ -periodic functions with continuous first derivative.  $F$  is defined as

$$Fx = x - xx' - g,$$

with  $g(\cdot) \in C^\infty$ , positive and  $T$ -periodic.

Notice that it is  $F(CP_1([0, T])) \subset L_2(0, T)$  because any function continuous almost everywhere and bounded is in  $L_2(0, T)$ , which is fulfilled by  $Fx$  when  $x \in CP_1([0, T])$  and  $g$  belongs to the above mentioned set.

It can be deduced from section 5.6 that the mapping  $F$  has a zero in  $CP_1([0, T])$  because the solution  $\phi(t)$  of (5.6.4) is positive,  $T$ -periodic and satisfies

$$\phi' = 1 - \frac{g}{\phi},$$

the continuity of  $\phi$  and  $\phi'$  being thus guaranteed.

Given the trigonometric<sup>3</sup> system  $\{w_n\}_n$  and an element  $x_n$  from the subspace  $X_n = \text{span}\{w_0, \dots, w_{2n}\}$ , the Galerkin equations associated to (5.6.4) or (5.7.1) in  $X_n$  are

$$(Fx_n|w_j) = 0, \quad j = 0, \dots, 2n. \quad (5.7.2)$$

Moreover, letting  $P_n : L_2(0, T) \longrightarrow X_n$  be a projection operator defined as in (5.2.6), its equivalent form

$$P_n Fx_n = 0, \quad (5.7.3)$$

---

<sup>3</sup>Its choice is justified by remark 5.2.2.

can also be written, recalling that  $P_n x_n = x_n, \forall x_n \in X_n$ , as

$$x_n - P_n(x_n x'_n + g) = 0. \quad (5.7.4)$$

Let us denote by  $\tilde{X}_n$  the Banach subspace of the functions with zero mean value:

$$\tilde{X}_n = \text{span} \{w_1, w_2, \dots, w_{2n-1}, w_{2n}\}.$$

Then, the decomposition

$$X_n = X_0 \oplus \tilde{X}_n \quad (5.7.5)$$

allows us to write

$$x_n = x_{n0} + \bar{x}_n, \quad g = g_0 + \bar{g},$$

$x_{n0}, \bar{x}_n, g_0$  and  $\bar{g}$  being unique and such that  $x_{n0}, g_0 \in X_0, \bar{x}_n \in \tilde{X}_n$  and  $\bar{g} \in \cup_{n \geq 1} \tilde{X}_n$ .

Using these expressions in (5.7.4) and observing that

- (i)  $x'_n = \bar{x}'_n$ ,
- (ii)  $P_n(\bar{x}_n \bar{x}'_n), P_n \bar{g} \in \tilde{X}_n$ ,

the system can be decomposed into

$$x_{n0} = g_0, \quad (5.7.6)$$

$$\bar{x}_n = x_{n0} \bar{x}'_n + P_n(\bar{x}_n \bar{x}'_n + \bar{g}), \quad (5.7.7)$$

which are problems in  $X_0$  and  $\tilde{X}_n$ , respectively. Then, for  $n = 0$  there is a single solution  $x_{00} = g_0$ , while for  $n \geq 1$  the 0-th component is  $x_{n0} = g_0$ . In this case, equation (5.7.7) may be read as the fixed point problem

$$g_0 \bar{x}'_n + P_n(\bar{x}_n \bar{x}'_n + \bar{g}) = \bar{x}_n. \quad (5.7.8)$$

The proof of the existence of a solution of (5.7.8) is based on the strategy already mentioned in remark 5.3.2.

Let us set  $R > 0$  and define  $U_n \subset \tilde{X}_n$  as

$$U_n = \{\bar{x}_n \in \tilde{X}_n; \|\bar{x}'_n\| < R\}.$$

Notice that  $U_n$  is bounded because

$$\bar{x}_n = \sum_{j=1}^{2n} c_{nj} w_j, \quad \text{and} \quad \bar{x}'_n = \omega \sum_{k=1}^n k(-c_{n,2k-1} w_{2k} + c_{n,2k} w_{2k-1}),$$

where  $\omega = 2\pi T^{-1}$ . Hence,

$$\|\bar{x}'_n\| = \omega \sqrt{\sum_{k=1}^n k^2 (c_{n,2k-1}^2 + c_{n,2k}^2)} \geq \omega \sqrt{\sum_{j=1}^{2n} c_{nj}^2} = \omega \|\bar{x}_n\|.$$

This immediately leads to

$$\|\bar{x}_n\| \leq \frac{R}{\omega}, \quad \forall \bar{x}_n \in U_n. \quad (5.7.9)$$

With  $g(t) \in C^\infty([0, T])$  also fixed, consider the restrictions

$$g_0 \omega > 1, \quad (g_0 \omega - 1)^2 \geq 4\omega \|\bar{g}\|. \quad (5.7.10)$$

Then, we construct the mapping  $H_n : \bar{U}_n \times [0, 1] \rightarrow \tilde{X}_n$  with

$$H_n(\bar{x}_n, \lambda) = g_0 \bar{x}'_n + \lambda P_n(\bar{x}_n \bar{x}'_n + \bar{g}).$$

**Proposition 5.7.1.** *The mapping  $H_n(x_n, \lambda)$  is such that:*

- (i) *It is compact in  $\bar{U}_n \times [0, 1]$ ,  $\forall n \geq 1$ .*
- (ii) *If (5.7.10) are fulfilled, there exists  $R > 0$  such that  $H_n(\bar{x}_n, \lambda)$  has no fixed points on  $\partial U_n$ ,  $\forall n \geq 1$  and  $\forall \lambda \in [0, 1]$ .*

**Proof.** (i) By proposition 5.3.1, the compactness of  $H_n$  is ensured by its continuity.

(ii) The second statement follows because

$$\|H_n(\bar{x}_n, \lambda) - \bar{x}_n\| \neq 0, \quad \forall \bar{x}_n \in \partial U_n, \forall \lambda \in [0, 1], \forall n \geq 1.$$



First note that

$$\|P_n x\| \leq \|x\|, \quad \forall n \geq 0, \forall x \in L_2(0, T).$$

Therefore, using this relation, the Schwarz inequality, the positivity of  $g_0$  (arising from  $g > 0$ ) and (5.7.9), we have that

$$\begin{aligned} \|H_n(\bar{x}_n, \lambda) - \bar{x}_n\| &= \|g_0 \bar{x}'_n + \lambda P_n(\bar{x}_n \bar{x}'_n + \bar{g}) - \bar{x}_n\| \geq \\ &\geq g_0 \|\bar{x}'_n\| - \lambda \|P_n(\bar{x}_n \bar{x}'_n + \bar{g})\| - \|\bar{x}_n\| \geq \\ &\geq g_0 \|\bar{x}'_n\| - \|\bar{x}_n \bar{x}'_n + \bar{g}\| - \|\bar{x}_n\| \geq \\ &\geq g_0 \|\bar{x}'_n\| - \|\bar{x}_n\| \|\bar{x}'_n\| - \|\bar{g}\| - \|\bar{x}_n\| \geq \\ &\geq g_0 R - \frac{R^2}{\omega} - \|\bar{g}\| - \frac{R}{\omega} = -\frac{R^2}{\omega} + \left(g_0 - \frac{1}{\omega}\right) R - \|\bar{g}\| = p(R) \end{aligned}$$

The vertex of the inverted parabola  $p(R)$  has coordinates  $(R_v, p(R_v))^\top$ :

$$\begin{aligned} R_v &= -\frac{g_0 - \frac{1}{\omega}}{-\frac{2}{\omega}} = \frac{g_0 \omega - 1}{2}, \\ p(R_v) &= -\frac{\left(g_0 - \frac{1}{\omega}\right)^2}{-\frac{4}{\omega}} - \|\bar{g}\| = \frac{(g_0 \omega - 1)^2}{4\omega} - \|\bar{g}\|. \end{aligned}$$

Consequently, it is easy to check that the fulfillment of (5.7.10) ensures the location of  $(R_v, p(R_v))^\top$  in the first quadrant of  $\mathbb{R}^2$ . The existence of  $R > 0$  is therefore guaranteed:

$$R \in (R_m, R_M), \quad \text{with} \quad R_{M,m} = \frac{g_0 \omega - 1 \pm \sqrt{(g_0 \omega - 1)^2 - 4\omega \|\bar{g}\|}}{2}$$

and such that  $H_n$  has no fixed points on  $\partial U_n$ ,  $\forall n \geq 1, \forall \lambda \in [0, 1]$ . ■

**Proposition 5.7.2.** *If (5.7.10) are fulfilled,*

$$i(H_n(\bar{x}_n, 0), U_n) = i(H_n(\bar{x}_n, 1), U_n) = 1.$$

**Proof.** When such conditions are satisfied, proposition 5.7.1 ensures that  $H_n(\bar{x}_n, 0)$  and  $H_n(\bar{x}_n, 1)$  are homotopically compact (see definition 5.3.3). Therefore, axiom (A4) of definition 5.3.4 guarantees the equality of their fixed point index. It remains to prove that

$$i(H_n(\bar{x}_n, 0), U_n) = i(g_0\bar{x}'_n, U_n) = 1.$$

The fixed point problem

$$g_0\bar{x}'_n = \bar{x}_n, \quad \bar{x}_n \in U_n,$$

is equivalent to the following problem in  $\mathbb{R}^{2n}$ : let  $R, \omega \in \mathbb{R}^+$  and let  $W_n \subset \mathbb{R}^{2n}$  be defined as

$$W_n = \left\{ z_n = (c_{n,1}, \dots, c_{n,2n})^\top \in \mathbb{R}^{2n}; \sqrt{\sum_{k=1}^n k^2 (c_{n,2k-1}^2 + c_{n,2k}^2)} < \frac{R}{\omega} \right\}.$$

Notice that  $W_n$  is open and bounded because the euclidean norm  $\|\cdot\|$  of its elements is bounded:

$$\forall z_n \in W_n, \frac{R}{\omega} > \sqrt{\sum_{k=1}^n k^2 (c_{n,2k-1}^2 + c_{n,2k}^2)} > \sqrt{\sum_{j=1}^{2n} c_{nj}^2} = \|z_n\|.$$

Let  $f : W_n \rightarrow \mathbb{R}^{2n}$  be the mapping such that

$$f(z_n) = g_0\omega(c_{n2}, -c_{n1}, 2c_{n4}, -2c_{n3}, \dots, nc_{n,2n}, -nc_{n,2n-1})^\top,$$

which is trivially continuous and  $C^1$  in  $\mathbb{R}^{2n}$ . The fixed points of  $f$  are the solutions of  $f(z_n) = z_n$ , which can be written as

$$\begin{cases} c_{n,2k-1} &= k\omega g_0 c_{n,2k} \\ c_{n,2k} &= -k\omega g_0 c_{n,2k-1}, \end{cases} \quad k = 1, \dots, n,$$

leading to

$$c_{nj} = -j^2\omega^2 g_0^2 c_{nj} \implies (1 + j^2\omega^2 g_0^2) c_{nj} = 0 \implies c_{nj} = 0, \quad \forall j, j = 1, \dots, 2n.$$

Hence,  $z_n = 0$  is the only fixed point of  $f$ , with  $0 \in W_n$  and regular:

$$f'(z_n) = \left[ \text{diag} \left( \begin{array}{cc} 0 & k\omega g_0 \\ -k\omega g_0 & 0 \end{array} \right)_{k=1, \dots, n} \right],$$

and

$$\det [f'(0)] = \prod_{k=1}^n \det \left( \begin{array}{cc} 0 & k\omega g_0 \\ -k\omega g_0 & 0 \end{array} \right) = (\omega g_0)^{2n} \prod_{k=1}^n k^2 = (n! \omega^n g_0^n)^2 \neq 0.$$

Let now be  $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ , with

$$F(z_n) = z_n - f(z_n) = (c_{n,2k-1} - k\omega g_0 c_{n,2k}, c_{n,2k} + k\omega g_0 c_{n,2k-1})_{k=1, \dots, n}^\top.$$

We have that

$$F'(z_n) = \left[ \text{diag} \left( \begin{array}{cc} 1 & -k\omega g_0 \\ k\omega g_0 & 1 \end{array} \right)_{k=1, \dots, n} \right],$$

and

$$\det [F'(0)] = \prod_{k=1}^n \det \left( \begin{array}{cc} 1 & -k\omega g_0 \\ k\omega g_0 & 1 \end{array} \right) = \prod_{k=1}^n (1 + k^2 \omega^2 g_0^2) > 0.$$

According to proposition 5.3.4,  $i(f, W_n) = 1$ ,  $\forall n \geq 1$ , which immediately implies

$$i(H_n(\bar{x}_n, 0), U_n) = 1, \quad \forall n \geq 1.$$

■

We are now ready to state and prove the main result of the section.

**Theorem 5.7.3.** *Let us assume that (5.7.10) are satisfied. Then, the Galerkin equations (5.7.4) associated with the ODE defined in (5.6.4) have solution  $\phi_n$ ,  $\forall n \geq 0$ .*

**Proof.** For the case  $n = 0$ , equation (5.7.6) leads to  $\phi_0 = g_0$ . For  $n \geq 1$ , proposition 5.7.2 and axiom (A2) of definition 5.3.4 ensure the existence of a solution  $\bar{\phi}_n \in U_n$  for the fixed point problem (5.7.8). Therefore,  $\phi_n = g_0 + \bar{\phi}_n$  is a solution of the Galerkin equations (5.7.4). ■

## 5.8 Input error evaluation

We will evaluate the error with the  $L_\infty$  norm.

Let  $e_{nx}(t) = \phi_n(t) - \phi(t)$  be the error between an  $n$ -th Galerkin approximation and the exact input, with  $\phi$  and  $\phi_n$  satisfying (5.6.1) and (5.6.5), respectively. Denote

$$\delta = \frac{\|g\|_\infty}{\inf_{t \in [0, T]} \{g(t)\}}. \quad (5.8.1)$$

We know that  $g > 0$ ; then, its periodicity ensures it possesses a non zero infimum which, in turn, guarantees  $\delta \in \mathbb{R}^+$ .

**Theorem 5.8.1.** *The error  $e_{nx}$  satisfies the following inequality:*

$$\|e_{nx}\|_\infty \leq \delta \|F\phi_n\|_\infty. \quad (5.8.2)$$

**Proof.** From the definition,  $e_{nx}$  is continuous,  $T$ -periodic and has continuous first derivative, thus exhibiting maximum and minimum values in each closed interval.

Then, when we replace  $\phi_n$  by  $e_{nx} + \phi$  in (5.6.5), we obtain

$$-(\phi + e_{nx})e'_{nx} + (1 - \phi')e_{nx} = F\phi_n.$$

Therefore, at any instant  $\hat{t}$  where  $e_{nx}$  has an extreme,  $e'_{nx}(\hat{t}) = 0$ , and the use of (5.6.1) yields

$$\frac{\inf\{\phi\} \inf\{F\phi_n\}}{\sup\{g\}} \leq e_{nx} \leq \frac{\sup\{\phi\} \sup\{F\phi_n\}}{\inf\{g\}},$$

where the infimums and supremes are searched on  $[0, T]$ . With analogous reasoning using again (5.6.1) we arrive at  $\inf\{g\} \leq \phi \leq \sup\{g\}$ , which leads the previous relation to

$$\frac{\inf\{g\} \inf\{F\phi_n\}}{\sup\{g\}} \leq e_{nx} \leq \frac{\sup\{g\} \sup\{F\phi_n\}}{\inf\{g\}}.$$

Hence,

$$|e_{nx}| \leq \sup \{ \delta^{-1} | \inf \{ F\phi_n \} |, \delta | \sup \{ F\phi_n \} | \} \implies \|e_{nx}\|_\infty \leq \delta \|F\phi_n\|_\infty,$$

where  $\delta \geq 1$  has been used. ■

## 5.9 Convergence of the Galerkin approximation

In this section we will see that the sequence of solutions of the Galerkin equations (5.7.2), or their equivalent forms (5.7.3) and (5.7.4), denoted from now on as  $\{\phi_n\}_n$ , exhibits uniform convergence to the periodic solution of the ODE defined in (5.6.4).

**Lemma 5.9.1.** *The error sequence  $\{F\phi_n\}_n$  converges weakly to 0 in  $L_2(0, T)$ .*

**Proof.** From decomposition (5.7.5) and equation (5.7.6) it follows that

$$\phi_n = g_0 + \bar{\phi}_n, \quad \bar{\phi}_n \in \tilde{X}_n.$$

Then, taking into account the periodicity of  $g$  and the fact that  $\bar{\phi}_n \in U_n$ ,

$$\begin{aligned} \|F\phi_n\| &= \|\phi_n - \phi_n\phi_n' - g\| = \|\bar{\phi}_n - g_0\bar{\phi}_n' - \bar{\phi}_n\bar{\phi}_n' - \bar{g}\| \leq \\ &\leq \|\bar{\phi}_n\| + g_0\|\bar{\phi}_n'\| + \|\bar{\phi}_n\|\|\bar{\phi}_n'\| + \|\bar{g}\| < \\ &< \frac{R}{\omega} + g_0R + \frac{R^2}{\omega} + \|\bar{g}\| = \frac{R^2}{\omega} + \left(g_0 + \frac{1}{\omega}\right)R + \|\bar{g}\| < \infty \quad \forall n \geq 0, \end{aligned}$$

which indicates that the sequence  $\{F\phi_n\}_n$  is bounded. Moreover, (5.7.2) yields

$$\lim_{n \rightarrow \infty} (F\phi_n|w_j) = 0, \quad \forall w_j \in \{w_n\}_n.$$

As the trigonometric system  $\{w_n\}_n$  is dense in  $L_2(0, T)$ , proposition 5.4.6 (vi) entails the result. ■

**Lemma 5.9.2.** *The sequence  $\{\phi_n\}_n$  is such that:*

(i)  $\{\phi_n\}_n$  belongs to the Sobolev space  $H^1(0, T)$ .

(ii)  $\{\phi_n\}_n$  possesses a weakly convergent subsequence in  $H^1(0, T)$ .

**Proof.** (i) As  $\phi_n$  is trivially a  $C^1(0, T)$  function  $\forall n \geq 0$ , remark 5.5.1 ensures that its first generalized derivative coincides with the first classical derivative. Moreover, the  $L_2$ -norms of  $\phi_n$  and  $\phi'_n$  are bounded  $\forall n \geq 0$ :

$$\|\phi_n\| = \|g_0 + \bar{\phi}_n\| \leq \|g_0\| + \|\bar{\phi}_n\| < \|g_0\| + \frac{R}{\omega}, \quad (5.9.1)$$

$$\|\phi'_n\| = \|(g_0 + \bar{\phi}_n)'\| = \|\bar{\phi}'_n\| < R \quad (5.9.2)$$

because  $\bar{\phi}_n \in U_n$ ,  $\forall n \geq 0$ .

(ii) We have just observed the bounded character of the  $L_2$ -norms of  $\{\phi_n\}_n$  and  $\{\phi'_n\}$ . Therefore, the result is a direct consequence of proposition 5.4.3 (iii) and theorem 5.4.4. ■

**Lemma 5.9.3.** *Let  $\{\hat{\phi}_n\}_n$  be a weakly convergent subsequence of  $\{\phi_n\}_n$ , and let  $\hat{\phi}$  be its weak limit. Then,*

(i)  $\{\hat{\phi}_n\}_n$  converges uniformly to  $\hat{\phi}$  in  $C([0, T])$ .

(ii)  $\hat{\phi}_n^2 \rightharpoonup \hat{\phi}^2$  in  $H^1(0, T)$ .

**Proof.** (i) Starting from lemma 5.9.2, the result follows from the Rellich-Kondratjev theorem 5.5.3 and from proposition 5.5.2 (iii).

(ii) Immediate from proposition 5.5.4 once the boundedness of  $\{\hat{\phi}_n^2\}_n$  in  $H^1(0, T)$  is observed. Due to the trivial fact that  $\phi_n^2 \in C^1(0, T)$ ,  $\forall n \geq 0$ , remark 5.5.1 reminds us of the equivalence of the first generalized derivative and the classical derivative of

$\phi_n$ . Hence,

$$\begin{aligned} \left\| \hat{\phi}_n^2 \right\|_{H^1} &= \left\| \hat{\phi}_n^2 \right\|_{L_2} + \left\| (\hat{\phi}_n^2)' \right\|_{L_2} \leq \left\| \hat{\phi}_n \right\|_{L_2}^2 + 2 \left\| \hat{\phi}_n \right\|_{L_2} \left\| \hat{\phi}_n' \right\|_{L_2} < \\ &< \left( \|g_0\|_{L_2} + \frac{R}{\omega} \right) \left( \|g_0\|_{L_2} + 2R + \frac{R}{\omega} \right), \end{aligned}$$

where (5.9.1) and (5.9.2) have been used. ■

Let us now establish the weak problem associated to the periodic solutions of the ODE (5.6.4) in  $L_2(0, T)$ . Performing a scalar product on both sides of the equation with any function  $\varphi \in CP_1([0, T])$ ,

$$(x(1-x')|\varphi) = (g|\varphi) \iff (x|\varphi) - \frac{1}{2}((x^2)'|\varphi) = (g|\varphi).$$

Integrating by parts while taking into account the  $T$ -periodicity of  $x$  and  $\varphi$  yields

$$(x|\varphi) + \frac{1}{2}(x^2|\varphi') = (g|\varphi), \quad \forall \varphi \in CP_1([0, T]). \quad (5.9.3)$$

**Lemma 5.9.4.** *The classical, positive and  $T$ -periodic solution  $\phi$  of (5.6.4) and the weak limit of every weakly convergent subsequence of  $\{\phi_n\}$  are weak  $T$ -periodic solutions of (5.6.2).*

**Proof.** The statement is obvious for  $\phi$ . Then, denote  $\{\hat{\phi}_n\}_n$  a subsequence of  $\{\phi_n\}_n$ , weakly convergent to a certain  $\hat{\phi}$  by lemma 5.9.2 (ii). Every element of the subsequence satisfies the ODE (5.6.5), now written

$$\hat{\phi}_n(1 - \hat{\phi}_n') = g + F\hat{\phi}_n.$$

Therefore,

$$(\hat{\phi}_n(1 - \hat{\phi}_n')|\varphi) = (g|\varphi) + (F\hat{\phi}_n|\varphi), \quad \forall \varphi \in CP_1([0, T]).$$

The scalar product may be expressed as

$$(\hat{\phi}_n|\varphi) - \frac{1}{2}((\hat{\phi}_n^2)'|\varphi) = (g|\varphi) + (F\hat{\phi}_n|\varphi)$$

and, integrating by parts, we easily arrive at

$$(\hat{\phi}_n|\varphi) + \frac{1}{2}(\hat{\phi}_n^2|\varphi') = (g|\varphi) + (F\hat{\phi}_n|\varphi).$$

Moreover, for  $n \rightarrow \infty$ , the weak convergences  $F\hat{\phi}_n \rightharpoonup 0$ ,  $\hat{\phi}_n \rightharpoonup \hat{\phi}$  and  $\hat{\phi}_n^2 \rightharpoonup \hat{\phi}^2$ , ensured by lemmas 5.9.1 and 5.9.2 (i) and (ii), lead to

$$(\hat{\phi}|\varphi) + \frac{1}{2}(\hat{\phi}^2|\varphi') = (g|\varphi).$$

■

**Lemma 5.9.5.** *Let  $p(t), q(t) \in C([0, T])$  be nonsingular and  $T$ -periodic. Then, the set of functions  $V_{p,q} \subset L_2$  defined as*

$$V_{p,q} = \{v_{p,q} \in L_2(0, T) / v_{p,q} = p(t)\varphi' + q(t)\varphi, \forall \varphi \in CP_1([0, T])\},$$

*is dense in  $L_2(0, T)$ .*

**Proof.** We will prove the lemma by observing that the trigonometric system belongs to  $V_{pq}$ , i.e., for every element  $w_n$  of the trigonometric system, there exists  $\varphi_n$  in  $CP_1([0, T])$  such that

$$p(t)\varphi_n' + q(t)\varphi_n = w_n.$$

Therefore, let us write the preceding ODE as

$$\varphi_n' = -\frac{q(t)}{p(t)}\varphi_n + \frac{w_n}{p(t)}. \quad (5.9.4)$$

The terms in the equation are dominated by the linear part when  $t \rightarrow \infty$ . Even more, the ODE

$$z' = -\frac{q(t)}{p(t)}z$$



has no  $T$ -periodic solutions except  $z = 0$ : its general solution is

$$z(t) = K \exp \left\{ - \int_0^t \frac{q(s)}{p(s)} ds \right\}$$

and, as  $q(t)p^{-1}(t)$  has definite sign,  $z(t)$  has also a definite sign and is strictly decreasing or strictly increasing  $\forall K \neq 0$ . Following [Ver90], we can state that equation (5.9.4) has at least one  $T$ -periodic solution. ■

Assume now that the following hypothesis is satisfied for the rest of the section:

**H0.**  $G_n = g + F\phi_n > 0, \forall n \geq 0$ .

**Lemma 5.9.6.** *The sequence  $\{\phi_n\}_n$  is such that  $\phi_n > 0, \forall n \geq 0$ .*

**Proof.** As  $g = g_0 + \bar{g} > 0$  and  $\bar{g} \in \cup_{n \geq 1} \tilde{X}_n, g_0 > 0$ . We also know that  $\phi_n = g_0 + \bar{\phi}_n, \bar{\phi}_n \in \tilde{X}_n$ ; then,  $\phi_n > 0$  at least in an open interval  $I \subset (0, T)$ . Suppose that  $\phi_n$  takes negative values. As it is continuous and  $T$ -periodic, Rolle's theorem ensure the existence of a minimum (at  $t = \bar{t}$ , for example) where it happens  $\phi(\bar{t}) < 0$  and  $\phi'_n(\bar{t}) = 0$ ; therefore, from (5.6.5) we find that

$$G_n(\bar{t}) = \phi_n(\bar{t}) [1 - \phi'_n(\bar{t})] = \phi_n(\bar{t}) < 0,$$

contradicting the hypothesis. ■

**Lemma 5.9.7.** *Every weakly convergent subsequence of  $\{\phi_n\}_n$  has a weak limit  $\phi$ , the classical, positive and  $T$ -periodic solution of (5.6.4).*

**Proof.** Consider  $\{\hat{\phi}_n\}_n$  a subsequence of  $\{\phi_n\}_n$  weakly convergent to a certain  $\hat{\phi}$ . Lemma 5.9.4 ensures that both  $\phi$  and  $\hat{\phi}$  satisfy (5.9.3). We may then write

$$(\hat{\phi}|\varphi) + \frac{1}{2}(\hat{\phi}^2|\varphi') = (\phi|\varphi) + \frac{1}{2}(\phi^2|\varphi').$$

The previous equation can be re-written as

$$(\hat{\phi} - \phi | \left[ \frac{1}{2}(\hat{\phi} + \phi)\varphi' + \varphi \right]) = 0, \quad \forall \varphi \in CP_1([0, T])$$

or, alternatively

$$(\hat{\phi} - \phi | v_{p_x, q_x}) = 0, \quad \forall v_{p_x, q_x} \in V_{p_x, q_x},$$

where now

$$p_x(t) = \frac{\hat{\phi}(t) + \phi(t)}{2}, \quad q_x(t) = 1.$$

The positivity of  $p_x(t)$  is guaranteed by the fact that  $\phi$  is positive and  $\hat{\phi}$  is, at least, non negative by lemma 5.9.6. Therefore, as  $V_{p_x, q_x}$  is dense in  $L_2$  by lemma 5.9.5,  $\hat{\phi} = \phi$  almost for all  $t$  in  $[0, T]$ . The continuity of both  $\hat{\phi}$  and  $\phi$  entails

$$\hat{\phi}(t) = \phi(t), \quad \forall t \in [0, T].$$

■

**Theorem 5.9.8.** *The sequence  $\{\phi_n\}_n$  of solutions of the Galerkin equations (5.7.2) converges uniformly to the periodic solution  $\phi$  of the ODE (5.6.4).*

**Proof.** Lemma 5.9.7 guarantees that every weakly convergent subsequence of  $\{\phi_n\}_n$  has weak a limit  $\phi$ . Item (v) of proposition 5.4.6 leads to  $\phi_n \rightharpoonup \phi$  and, finally, theorem 5.5.3 and proposition 5.5.2 (iii) yield the result. ■

## 5.10 System output

In this section we will try to answer the questions stated in section 5.6 about the type of output we can expect when an approximate indirect control is induced in the system. Specifically, theorem 5.10.2 gives sufficient conditions to guarantee the desired output behavior. Such a result is supported by the following lemma.

**Lemma 5.10.1.** *Consider the Cauchy problem*

$$(y' + \lambda y)(k + y) = c, \quad y(0) = y_0, \quad (5.10.1)$$

with  $c \in \mathbb{R}^+$  and  $y_0 \neq -k$ , and denote  $y(t, y_0)$  its solution. Then,

(i) *Asymptotically stable equilibrium solutions of (5.10.1) are*

$$y_+ = -\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 + \frac{c}{\lambda}}, \quad y_- = -\frac{k}{2} - \sqrt{\left(\frac{k}{2}\right)^2 + \frac{c}{\lambda}}.$$

(ii)  $\forall y_0, -k < y_0 < y_+$  and,  $\forall y_0, y_0 < y_-, y(t, y_0)$  is  $C^\infty$ , increasing and defined  $\forall t \geq 0$ .

(iii)  $\forall y_0, y_+ < y_0$  and,  $\forall y_0, y_- < y_0 < -k, y(t, y_0)$  is  $C^\infty$ , decreasing and defined  $\forall t \geq 0$ .

**Proof.** (i) It can be tested by simple calculations that  $y_+, y_-$  are the two solutions of  $\lambda y(k + y) = c$ , with  $y_- < -k \leq 0 < y_+$ . Moreover,  $y(t, y_0) \neq -k \forall t$  because the opposite entails  $0 = c$ . Therefore, (5.10.1) can be written

$$y' = \Omega_c(y) = -\lambda y + \frac{c}{k + y} \quad (5.10.2)$$

and, as the derivative of the field  $\Omega_c(y)$  over the system trajectories is

$$\frac{\partial \Omega_c(y)}{\partial y} = -\lambda - \frac{c}{(k + y)^2} < 0, \quad \forall y \neq -k,$$

the asymptotic stability of  $y_+$  and  $y_-$  follows.

(ii) It is obvious that the solutions with  $y_0 \neq -k$  are  $C^\infty$ . The increasing and decreasing character according to the situation of  $y_0$ , established in statements (ii) and (iii), can be seen studying the sign of  $\Omega_c(y)$ . Thus, denoting

$$\Omega_c(y) = \frac{-\lambda y^2 - k\lambda y + c}{k + y} = \frac{-\lambda(y - y_-)(y - y_+)}{k + y},$$

it follows immediately that

$$\begin{aligned} y' = \Omega_c(y) < 0 &\iff (y_+ < y) \vee (y_- < y < -k), \\ y' = \Omega_c(y) > 0 &\iff (y < y_-) \vee (-k < y < y_+). \end{aligned}$$

(iii) Finally, the existence of  $y(t, y_0)$ ,  $\forall t \geq 0$ , is observed: when (5.10.2) is integrated we obtain

$$t(y, y_0) = \frac{1}{2\lambda} \left[ \log \frac{(y_0 - y_+)(y_0 - y_-)}{(y - y_+)(y - y_-)} + \frac{k}{\Delta y_+} \log \frac{(y - y_-)(y_0 - y_+)}{(y - y_+)(y_0 - y_-)} \right],$$

with

$$\Delta y_+ = \sqrt{k^2 + \frac{4c}{\lambda}} > 0.$$

In this way,

$$\begin{aligned} \lim_{y \rightarrow y_+^+} t(y, y_0) &= +\infty \quad \forall y_0, \quad y_+ < y_0, \\ \lim_{y \rightarrow y_+^-} t(y, y_0) &= +\infty \quad \forall y_0, \quad -k < y_0 < y_+. \end{aligned}$$

Analogously,

$$\begin{aligned} \lim_{y \rightarrow y_-^+} t(y, y_0) &= +\infty \quad \forall y_0, \quad y_- < y_0 < -k, \\ \lim_{y \rightarrow y_-^-} t(y, y_0) &= +\infty \quad \forall y_0, \quad y_0 < y_- \end{aligned}$$

because for these two last cases we have

$$\frac{k}{\Delta y_+} = \begin{cases} 0 & k = 0 \\ \frac{1}{\sqrt{1 + \frac{4c}{\lambda}}} < 1 & k = 1. \end{cases}$$

■

Let us now return to our problem and consider the following hypothesis:

**H1.** The output reference  $f(t)$  is  $C^\infty$  and  $T$ -periodic;  $f(t) > 0$ ,  $g(t) > 0$ ,  $\forall t \geq 0$ ; and there exists an  $n$ -th Galerkin approximation  $\{\phi_n\}_n$  such that we can construct a sequence  $\{G_n\}_n$  that satisfies hypothesis H0.

**Theorem 5.10.2.** *Consider equation (5.6.7) as a Cauchy problem with  $y_n(0) = y_0$  and  $G_n$  being any function of the sequence  $\{G_n\}_n$ . If H1 is verified, equation (5.6.7) has one and only one periodic solution in  $\mathbb{R}^+$ , hyperbolic and asymptotically stable.*

**Proof.** EXISTENCE. The solutions of (5.6.7) do not take the value  $-k$  anywhere because, otherwise, we would have  $0 = G_n(t)$ . Therefore, the ODE can be written

$$y'_n = \Omega_n(t, y_n) = -\lambda y_n + \frac{G_n(t)}{k + y_n}, \quad y_n(0) = y_0. \quad (5.10.3)$$

Denote as  $y_n(t, z)$  the solution of (5.10.3) such that  $y_n(0, z) = z$ , trivially it is  $C^\infty$  for  $z \neq -k$ . Let  $h_n(z) = y_n(T, z)$  be the return map associated to (5.10.3), with associated function  $H_n = h_n - \mathbb{I}$ . It is clear that  $z_0$  is a zero of  $H_n$  iff  $y_n(t, z_0)$  is  $T$ -periodic (see section 4.3).

The smoothness,  $T$ -periodicity and positivity of  $G_n$  ensure the existence of  $A, B \in \mathbb{R}^+$  such that  $A \leq G_n(t) \leq B$ . Consider then the ODE's

$$y'_n = \Omega_A(y_n) = -\lambda y_n + \frac{A}{k + y_n}, \quad (5.10.4)$$

$$y'_n = \Omega_B(y_n) = -\lambda y_n + \frac{B}{k + y_n}, \quad (5.10.5)$$

with return maps  $h_A, h_B$  and associated functions  $H_A, H_B$ . It is clear that

$$\Omega_A(y_n) \leq \Omega_n(t, y_n) \leq \Omega_B(y_n), \quad \forall y_n > -k, \forall t \geq 0. \quad (5.10.6)$$

Let now  $y_n(t, z)$ ,  $y_A(t, z)$ ,  $y_B(t, z)$  be solutions of (5.10.3), (5.10.4), (5.10.5), respectively, with initial conditions  $y_n(0, z) = y_A(0, z) = y_B(0, z) = z$ . Taking into account (5.10.6) and the mean value theorem,

$$y_A(t, z) \leq y_n(t, z) \leq y_B(t, z), \quad \forall z > -k, \forall t \geq 0. \quad (5.10.7)$$

Therefore,  $H_A(z) \leq H_n(z) \leq H_B(z)$ ,  $\forall z > -k$  because, in fact, lemma 5.10.1 and (5.10.7) guarantee that  $H_A$ ,  $H_n$  and  $H_B$  are defined in  $\mathbb{R} \setminus \{-k\}$ . Furthermore, if

$$y_{A+} = -\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 + \frac{A}{\lambda}}, \quad y_{B+} = -\frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 + \frac{B}{\lambda}},$$

from lemma 5.10.1,  $H_A(y_{A+}) = H_B(y_{B+}) = 0$ . Therefore,

$$0 = H_A(y_{A+}) \leq H_n(y_{A+}) \quad \wedge \quad H_n(y_{B+}) \leq H_B(y_{B+}) = 0.$$

The existence of  $z_{n0} \in [y_{A+}, y_{B+}]$  with  $H_n(z_{n0}) = 0$  is then ensured and, consequently, of a solution  $y_n(t, z_{n0}) \in S$  for (5.10.3) with initial value  $z_{n0} \in \mathbb{R}^+$ .

POSITIVITY. Let  $t_- < T$  be such that  $-k < y_n(t_-, z_{n0}) < 0$ ; then, the periodicity and smooth character of the solution should also demand the existence of a minimum  $(t_m, y_n(t_m, z_{n0}))$  with  $-k < y_n(t_m, z_{n0}) < 0$  and  $y'_n(t_m, z_{n0}) = 0$ . This, taken with (5.10.3), means  $0 > \lambda y_n(t_m, z_{n0})(k + y_n(t_m, z_{n0})) = G_n(t_m)$ , contradicting the hypothesis. Thus,  $y_n(t, z_{n0}) > 0$ ,  $\forall t \geq 0$ .

ASYMPTOTIC STABILITY. Notice that

$$\frac{\partial \Omega_n(y_n, t)}{\partial y_n} = -\lambda - \frac{G_n(t)}{(k + y)^2} > 0, \quad \forall y_n \neq -k.$$

From proposition 4.3.1

$$h'(z) = \exp \left\{ - \int_0^T \left[ \lambda + \frac{G_n(t)}{(k + y(t, z))^2} \right] dt \right\},$$

and since all the elements in the integral are positive, it follows that  $0 < h'(z) < 1$ ,  $\forall z \neq -k$ . This implies  $H'(z) = h'(z) - 1 < 0$ ,  $\forall z \neq -k$ , which leads to the asymptotic stability of any periodic solution of (5.10.3).

UNIQUENESS. As we have just observed,  $H'(z) < 0 \forall z \neq -k$ ; then,  $H(z)$  is strictly decreasing in  $(-k, +\infty)$  and, therefore, it can only possess one zero. As  $H(z_{n0}) = 0$ , there can be no other zero. ■

Let us recall now that lemma 5.10.1 proved the existence of two asymptotically stable solutions of the ODE (5.10.1), one in  $\mathbb{R}^+$  and another in  $\mathbb{R}^-$ . Theorem 5.10.2 has just worked in  $\mathbb{R}^+$ , but a corollary can be established with a parallel result in  $\mathbb{R}^-$ .

**Corollary 5.10.3.** *If H1 is satisfied, equation (5.6.7) has  $y_n(t) = \bar{y}_n(t)$  as a single periodic solution in  $(-\infty, -k)$ , hyperbolic and asymptotically stable. Moreover, in the exact problem ( $G_n(t) = g(t)$ ) and for  $k = 0$  it is  $\bar{y}(t) = -f(t)$ .*

**Proof.** (i) It just suffices to follow step by step the proof of theorem 5.10.2. The changes to keep in mind are

$$(5.10.6) \longrightarrow \Omega_B(y_n) \leq \Omega_n(t, y_n) \leq \Omega_A(y_n), \quad \forall y_n < -k, \quad \forall t \geq 0,$$

$$(5.10.7) \longrightarrow y_B(t, z) \leq y_n(t, z) \leq y_A(t, z), \quad \forall z < -k, \quad \forall t \geq 0.$$

With this and lemma 5.10.1 we deduce that  $H_B(z) \leq H_n(z) \leq H_A(z)$ ,  $\forall z < -k$ . If we denote

$$y_{A-} = -\sqrt{\left(\frac{k}{2}\right)^2 + \frac{A}{\lambda} - \frac{k}{2}}, \quad y_{B-} = -\sqrt{\left(\frac{k}{2}\right)^2 + \frac{B}{\lambda} - \frac{k}{2}},$$

with  $y_{A-}, y_{B-} < -k$ , again by lemma 5.10.1,  $H_A(y_{A-}) = H_B(y_{B-}) = 0$ . Therefore,

$$0 = H_B(y_{B-}) \leq H_n(y_{B-}) \quad \wedge \quad H_n(y_{A-}) \leq H_A(y_{A-}) = 0.$$

The existence of  $\bar{z}_{n0} \in [y_{B-}, y_{A-}] \in (-\infty, -k)$  is then guaranteed with  $H_n(z_{n0}) = 0$  and, consequently, a periodic solution  $\bar{y}_n(t, \bar{z}_{n0})$  of the ODE (5.6.7) with  $\bar{y}_n(t, \bar{z}_{n0}) < -k$ ,  $\forall t \geq 0$ .

(ii) It is straightforward that  $\bar{y}(t) = -f(t)$  is a solution of (5.6.7) for the case  $G_n = g$ ,  $k = 0$ . Its unicity, periodicity, hyperbolicity and asymptotic stability follow from the above general proof. ■

*Remark 5.10.1.* Proposition 4.5.5 is a particular case of theorem 5.10.2 if we consider  $G_n = g$ . In this situation,  $y(t) = f(t)$  is a solution of the ODE with the same features as the functions  $y_n$ .

## 5.11 Restrictions on the signals to be tracked

As seen in chapter 4, the basic restriction suffered by our DC-to-DC switched converters is due to the fact that their performance is located in a certain region of the phase plane, the so called *insaturation zone*. This means that two inequalities where we can find inputs and outputs must be satisfied, which will lead to conditions on the output reference. In section 4.5 of chapter 4 we studied the situation for an exact input current. Here we will try to establish parallel results for the approximate case. In steady state, with  $x$  and  $y$  tracking  $\phi_n(t)$  and  $y_n$ , system (4.2.3, 4.2.4) becomes

$$\begin{aligned}\phi_n' &= 1 - \bar{u}_n(k + y_n) \\ y_n' &= -\lambda y_n + \bar{u}_n \phi_n,\end{aligned}$$

where  $\bar{u}_n$  is the control action. The insaturation region, defined by  $0 \leq \bar{u}_n \leq 1$ , is

$$0 \leq \frac{1 - \phi_n'}{k + y_n} \leq 1 \quad \text{or, equivalently,} \quad 0 \leq \frac{y_n' + \lambda y_n}{\phi_n} \leq 1.$$

Let us state the following hypothesis:

**H2.** H1 is verified and  $f(t)$  is such that the system is in the insaturation zone with exact approach; that is, the conditions of proposition 4.5.3 are satisfied.

**Proposition 5.11.1.** *If H2 is fulfilled, the insaturation region is defined by*

$$0 < 1 - \phi_n' \leq k + y_n \quad \text{or, equivalently,} \quad 0 < y_n' + \lambda y_n \leq \phi_n. \quad (5.11.1)$$



**Proof.**  $G_n(t) > 0$  demands  $\phi_n > 0$  by lemma 5.9.6. In turn, from (5.6.7) it is straightforward to show that

$$G_n > 0 \implies \text{sign}(y'_n + \lambda y_n) = \text{sign}(k + y_n).$$

It is then necessary to use the region

$$0 < y'_n + \lambda y_n \leq \phi_n.$$

■

A sufficient condition for insaturation is:

**Proposition 5.11.2.** *If H2 is fulfilled and  $\inf_{t \in [0, T]} \{G_n(t)\} \geq \|y'_n + \lambda y_n\|_\infty$ , the system is in insaturation zone.*

**Proof.** Immediate following the proof of proposition 4.5.3. ■

## 5.12 Output error evaluation

We will again evaluate with the  $L_\infty$  norm. Therefore, denote  $e_{ny}(t) = y_n(t) - f(t)$  the error between an  $n$ -th approximation and the exact output, with  $y_n$  satisfying (5.6.7). Suppose also that the system is in the insaturation region if it works with an exact input current as well as if it does so with an approximate one, that is, H2 and (5.11.1) hold.

**Theorem 5.12.1.** *The error  $e_{ny}$  is such that*

$$\|e_{ny}\|_\infty \leq \frac{\sqrt{\|F\phi_n\|_\infty}}{\lambda}. \quad (5.12.1)$$

**Proof.** Since  $e_{ny}$  is continuous,  $T$ -periodic and with continuous first derivative, the same process as that in section 5.8 for  $e_{nx}$  will be used to compute bounds. When we substitute  $y_n$  by  $e_{ny} + f$  in (5.6.7), we obtain

$$(k + f + e_{ny})e'_{ny} + \lambda e_y^2 + (f' + 2\lambda f + \lambda k)e_{ny} = F\phi_n.$$

At any instant  $\bar{t}$  where  $e_{ny}$  exhibits an extreme, we will have  $e'_{ny}(\bar{t}) = 0$ , and this makes

$$e_{ny}(\bar{t}) = \frac{-p(\bar{t}) \pm \sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{2\lambda}, \quad (5.12.2)$$

with

$$p(t) = f'(t) + 2\lambda f(t) + \lambda k = [f'(t) + \lambda f(t)] + [k + f(t)] \geq 0$$

by hypothesis. The negative option in (5.12.2) is incompatible with the fact that we work with  $y_n > 0$ , because when we consider it, we find that

$$y_n(\bar{t}) - f(\bar{t}) = \frac{-p(\bar{t}) - \sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{2\lambda},$$

which yields

$$\begin{aligned} y_n(\bar{t}) &= \frac{2\lambda f(\bar{t}) - p(\bar{t}) - \sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{2\lambda} \\ &= -\frac{f'(\bar{t})}{2\lambda} - \frac{\lambda k + \sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{2\lambda}. \end{aligned}$$

But as we are in an extreme, we may deduct from  $e'_{ny}(\bar{t}) = 0$  that  $y'_n(\bar{t}) = f'(\bar{t})$ . Using (5.6.7) and (5.6.5) it is possible to find an expression for  $y'_n(\bar{t})$  that, once taken to the above equality, results in

$$y_n(\bar{t}) = \frac{y_n(\bar{t})}{2} - \frac{1}{2\lambda} \cdot \frac{\phi_n(\bar{t}) [1 - \phi_n(\bar{t})]}{k + y_n(\bar{t})} - \frac{\lambda k + \sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{2\lambda}.$$

Assigning

$$q = \frac{\phi_n(\bar{t}) [1 - \phi_n(\bar{t})]}{\lambda}, \quad r = k + \frac{\sqrt{p^2(\bar{t}) + 4F\phi_n(\bar{t})}}{\lambda},$$

where  $q > 0$  by (5.11.1) and  $r \geq 0$ ,  $k \in \{0, 1\}$ , the second order equation that gives  $y_n(\bar{t})$  is

$$y_n^2(\bar{t}) + (k + r)y_n(\bar{t}) + (q + kr) = 0.$$

The fact that the coefficient of the first order term is positive or null and the independent term is strictly positive prevents the possibility of a positive solution. We must therefore take the positive solution of (5.12.2), which leads to

$$\frac{-p(\bar{t}) + \sqrt{p^2(\bar{t}) - 4|F\phi_n(\bar{t})|}}{2\lambda} \leq e_{ny}(\bar{t}) \leq \frac{-p(\bar{t}) + \sqrt{p^2(\bar{t}) + 4|F\phi_n(\bar{t})|}}{2\lambda}. \quad (5.12.3)$$

As

$$a - \sqrt{|b|} \leq \sqrt{a^2 + b} \leq a + \sqrt{|b|}, \quad a \geq \sqrt{|b|} \geq 0,$$

(5.12.3) becomes

$$-\sqrt{|F\phi_n(\bar{t})|} \leq \lambda e_{ny}(\bar{t}) \leq \sqrt{|F\phi_n(\bar{t})|},$$

and the result follows immediately. ■

## 5.13 Convergence of the system output

Theorem 5.10.2, and a glance at equation (5.6.7), allow an output sequence  $\{y_n\}_n$  of continuous,  $T$ -periodic, positive solutions of such an ODE to be constructed. Uniform convergence of the output sequence is now studied. The structure of this section is very close to that of section 5.9.

Suppose the next hypothesis is fulfilled:

**H3.** Consider that the following conditions are verified:

(i) Hypothesis H2 is satisfied. Let  $\{y_n\}_n$  be the sequence of positive and periodic solutions of equation (5.6.7), with their existence guaranteed by theorem 5.10.2.

(ii) The system is in an insaturation zone when it undergoes approximate indirect control, i.e., (5.11.1) is fulfilled  $\forall n \geq 0$ .

Also take into account that  $\|\cdot\| = \|\cdot\|_{L_2}$  will be used throughout the present section.

Notice, moreover, that  $y_n$ ,  $y'_n$  and  $y''_n$  are continuous. Then, the Fourier series of both  $y_n$  and  $y'_n$  are

$$y_n = y_{0n}w_0 + \sum_{k \geq 1} y_{2k-1,n}w_{2k-1} + y_{2k,n}w_{2k}, \quad (5.13.1)$$

$$y'_n = \omega \sum_{k \geq 1} k(-y_{2k-1,n}w_{2k} + y_{2k,n}w_{2k-1}), \quad (5.13.2)$$

where  $\{w_j\}_j$  again stands for the trigonometric system (5.2.5). From (5.13.2),

$$\|y'_n\|^2 = \omega^2 \sum_{k \geq 1} k^2 (y_{2k-1,n}^2 + y_{2k,n}^2) = \omega^2 (\|y_n^2\| - y_{0n}^2),$$

and this leads to

$$\|y_n\|^2 \leq y_{0n}^2 + \frac{\|y'_n\|^2}{\omega^2},$$

which means

$$\|y_n\| \leq y_{0n} + \frac{\|y'_n\|}{\omega}, \quad (5.13.3)$$

with  $y_{0n} > 0$  from hypothesis H3 (i).

**Lemma 5.13.1.** *Consider the projection operator  $P_0$  (see proposition 5.2.3 (ii)).*

*Then,*

$$(i) P_0 y'_n = P_0 (y_n y'_n) = 0,$$

$$(ii) P_0 y_n^2 = T^{-\frac{1}{2}} \|y_n\|^2.$$

**Proof.** (i) The first relation,  $P_0 y'_n = 0$ , is obvious. For the second one,

$$P_0 (y_n y'_n) = \frac{1}{2} P_0 (y_n^2)' = 0,$$

because the derivative operation eliminates the  $w_0$ -component.

(ii) Notice that

$$y_n^2 = \left( \sum_{j \geq 0} y_{jn} w_j \right)^2 = \sum_{j \geq 0} y_{jn}^2 w_j^2 + 2 \sum_{i \neq j} y_{in} y_{jn} w_i w_j.$$

The product of two different elements of the trigonometric system is proportional to the product of two different trigonometric functions, which has no component in the  $w_0$  direction. Let us now look at the quadratic terms:

$$w_0 = \frac{1}{\sqrt{T}} \implies w_0^2 = \frac{1}{\sqrt{T}} w_0,$$

$$\begin{aligned} w_{2k-1} &= \sqrt{\frac{2}{T}} \cos \frac{2\pi kt}{T}, \\ w_{2k-1}^2 &= \frac{2}{T} \cos^2 \frac{2\pi kt}{T} = \frac{2}{T} \cdot \frac{1}{2} \left( 1 + \cos \frac{4\pi kt}{T} \right) = \frac{1}{\sqrt{T}} w_0 + \frac{1}{\sqrt{2T}} w_{4k-1}, \end{aligned}$$

$$\begin{aligned} w_{2k} &= \sqrt{\frac{2}{T}} \sin \frac{2\pi kt}{T}, \\ w_{2k}^2 &= \frac{2}{T} \sin^2 \frac{2\pi kt}{T} = \frac{2}{T} \left( 1 - \cos^2 \frac{2\pi kt}{T} \right) = \frac{1}{\sqrt{T}} w_0 - \frac{1}{\sqrt{2T}} w_{4k-1}. \end{aligned}$$

The result follows immediately. ■

**Lemma 5.13.2.** *The sequence  $\{y_n\}_n$  is such that:*

(i)  $\{y_n\}_n$  belongs to the Sobolev space  $H^1(0, T)$ .

(ii)  $\{y_n\}_n$  possesses a weakly convergent subsequence in  $H^1(0, T)$ .

**Proof.** (i)  $y_n$  is a periodic and  $C^1(0, T)$  function  $\forall n \geq 0$ . Hence, remark 5.5.1 guarantees that its first generalized derivative coincides with the first classical derivative.

As the system is in an insaturation zone by hypothesis H3 (ii), it follows that

$$0 < y_n' + \lambda y_n \leq \phi_n;$$

therefore,

$$\|\phi_n\| \geq \|y'_n + \lambda y_n\| \geq \|y'_n\| - \lambda \|y_n\|,$$

which yields

$$\|y'_n\| \leq \|\phi_n\| + \lambda \|y_n\|. \quad (5.13.4)$$

Hence, as  $\|\phi_n\| < \infty$  by (5.9.1), it suffices to prove the boundedness of  $\|y_n\|$ .

Let us now rewrite the ODE (5.6.7) as:

$$(y'_n + \lambda y_n)(k + y_n) = G_n.$$

Since  $G_n \in L_2(0, T)$ , so is the left hand term. Then, projecting over the first element of the trigonometric system we have that

$$P_0(y'_n + \lambda y_n)(k + y_n) = P_0 G_n;$$

using (5.13.1) and the properties of  $P_0$ , it follows that

$$kP_0 y'_n + P_0 y_n y'_n + \lambda k P_0 y_n + \lambda P_0 y_n^2 = P_0 G_n,$$

and with the aid of lemma 5.13.1, we have

$$\lambda k y_{0n} + \lambda T^{-\frac{1}{2}} \|y_n\|^2 = P_0 G_n \leq \|G_n\| \quad (5.13.5)$$

by hypothesis H1.

For the case  $k = 0$ , (5.13.5) shows that

$$\|y_n\| \leq \sqrt{\frac{T^{\frac{1}{2}}}{\lambda} \|G_n\|}. \quad (5.13.6)$$

In case that  $k = 1$ , (5.13.5) reads as

$$y_{0n} \leq \frac{\|G_n\|}{\lambda} - \frac{\|y_n\|^2}{\sqrt{T}}.$$

Using this inequality and (5.13.4) in (5.13.3) it follows that

$$\|y_n\| \leq \frac{\|\phi_n\|}{\omega} + \frac{\lambda}{\omega} \|y_n\| + \frac{\|G_n\|}{\lambda} - \frac{\|y_n\|^2}{\sqrt{T}},$$

which becomes

$$\|y_n\|^2 + \sqrt{T} \left(1 - \frac{\lambda}{\omega}\right) \|y_n\| \leq \sqrt{T} \left(\frac{\|\phi_n\|}{\omega} + \frac{\|G_n\|}{\lambda}\right).$$

Then,

$$\|y_n\| \leq \sqrt{\frac{T}{4} \left(1 - \frac{\lambda}{\omega}\right)^2 + T^{\frac{1}{2}} \left(\frac{\|\phi_n\|}{\omega} + \frac{\|G_n\|}{\lambda}\right)} - \frac{\sqrt{T}}{2} \left(1 - \frac{\lambda}{\omega}\right). \quad (5.13.7)$$

Since

$$\|G_n\| = \|g + F\phi_n\| \leq \|g\| + \|F\phi_n\|,$$

the bounded characters of  $g$  and of the sequences  $\{\phi_n\}$  and  $\{F\phi_n\}$  detailed in (5.9.1) and in lemma 5.9.1 ensure  $\|y_n\| < \infty$  in both (5.13.6) and (5.13.7).

(ii) The result is a direct consequence of  $\|y_n\| < \infty$  and  $\|y'_n\| < \infty, \forall n \geq 0$ , proposition 5.4.3 (iii) and theorem 5.4.4. ■

**Lemma 5.13.3.** *Let  $\{\hat{y}_n\}_n$  be a weakly convergent subsequence of  $\{y_n\}_n$ , and let  $\hat{f}$  be its weak limit. Then,*

(i)  $\{\hat{y}_n\}_n$  converges uniformly to  $\hat{f}$  in  $C([0, T])$ .

(ii)  $\hat{y}_n^2 \rightharpoonup \hat{f}^2$  in  $H^1(0, T)$ .

**Proof.** (i) The result follows taking into account lemma 5.13.2, theorem 5.5.3 and proposition 5.5.2 (iii).

(ii) Immediate from proposition 5.5.4 once the boundedness of  $\{\hat{y}_n^2\}_n$  in  $H^1(0, T)$  is observed. Due to the trivial fact that  $y_n^2 \in C^1(0, T), \forall n \geq 0$ , remark 5.5.1 reminds

us of the equivalence of the first generalized derivative and the classical derivative of  $y_n$ . Hence,

$$\|\hat{y}_n^2\|_{H^1} = \|\hat{y}_n^2\|_{L_2} + \|(\hat{y}_n^2)'\|_{L_2} \leq \|\hat{y}_n\|_{L_2}^2 + 2\|\hat{y}_n\|_{L_2}\|\hat{y}_n'\|_{L_2} < \infty$$

by lemma 5.13.2 (i). ■

Let us consider the weak problem associated with the periodic solutions of the ODE (5.6.2) in  $L_2(0, T)$ . Performing a scalar product on both sides of the equation with any function  $\varphi \in CP_1([0, T])$  and denoting the right hand side as  $g$  we have that

$$((y' + \lambda y)(k + y)|\varphi) = (g|\varphi) \iff k(y'|\varphi) + \lambda k(y|\varphi) + \lambda(y^2|\varphi) + \frac{1}{2}((y^2)'|\varphi) = (g|\varphi).$$

Integrating by parts while taking into account the  $T$ -periodicity of  $y$  and  $\varphi$  entails

$$-k(y|\varphi') + \lambda k(y|\varphi) + \lambda(y^2|\varphi) - \frac{1}{2}(y^2|\varphi') = (g|\varphi), \quad \forall \varphi \in CP_1([0, T]). \quad (5.13.8)$$

**Lemma 5.13.4.** *The classical, positive and  $T$ -periodic solution  $f$  of (5.6.2) and the weak limit of every weakly convergent subsequence of  $\{y_n\}$  are weak  $T$ -periodic solutions of (5.6.4).*

**Proof.** The statement is trivial for  $f$ . Then, denote  $\{\hat{y}_n\}_n$  as a subsequence of  $\{y_n\}_n$ , weakly convergent to a certain  $\hat{f}$  by lemma 5.13.2 (ii). Every element of the subsequence satisfies the ODE (5.6.6), written as

$$(\hat{y}_n' + \lambda \hat{y}_n)(k + \hat{y}_n) = g + F\hat{\phi}_n.$$

Therefore,

$$((\hat{y}_n' + \lambda \hat{y}_n)(k + \hat{y}_n)|\varphi) = (g + F\hat{\phi}_n|\varphi), \quad \forall \varphi \in CP_1([0, T]).$$



The scalar product may be expressed as

$$k(\hat{y}'_n|\varphi) + \lambda k(\hat{y}_n|\varphi) + \lambda(\hat{y}_n^2|\varphi) + \frac{1}{2}((\hat{y}'_n)^2|\varphi) = (g|\varphi) + (F\hat{\phi}_n|\varphi)$$

and, integrating by parts,

$$-k(\hat{y}_n|\varphi') + \lambda k(\hat{y}_n|\varphi) + \lambda(\hat{y}_n^2|\varphi) - \frac{1}{2}(\hat{y}_n^2|\varphi') = (g|\varphi) + (F\hat{\phi}_n|\varphi).$$

For  $n \rightarrow \infty$ , the weak convergences  $F\hat{\phi}_n \rightarrow 0$ ,  $\hat{y}_n \rightarrow \hat{f}$  and  $\hat{y}_n^2 \rightarrow \hat{f}^2$ , ensured by lemmas 5.9.1 and 5.13.3 (i) and (ii), leads to

$$-k(\hat{f}|\varphi') + \lambda k(\hat{f}|\varphi) + \lambda(\hat{f}^2|\varphi) - \frac{1}{2}(\hat{f}^2|\varphi') = (g|\varphi).$$

■

**Lemma 5.13.5.** *Every weakly convergent subsequence of  $\{y_n\}_n$  possesses  $f$  as a weak limit.*

**Proof.** Let  $\{\hat{y}_n\}_n$  be a subsequence of  $\{y_n\}_n$  weakly convergent to a certain  $\hat{f}$ . Lemma 5.13.4 guarantees that both  $f$  and  $\hat{f}$  fulfill (5.13.8). We then write

$$-k(\hat{f}|\varphi') + \lambda k(\hat{f}|\varphi) + \lambda(\hat{f}^2|\varphi) - \frac{1}{2}(\hat{f}^2|\varphi') = -k(f|\varphi') + \lambda k(f|\varphi) + \lambda(f^2|\varphi) - \frac{1}{2}(f^2|\varphi').$$

After some algebraic manipulation we arrive at

$$(\hat{f} - f | \left[ \left( k + \frac{\hat{f} + f}{2} \right) \varphi' - \lambda(k + \hat{f} + f)\varphi \right]) = 0, \quad \forall \varphi \in CP_1([0, T]),$$

which can be alternatively written as

$$(\hat{f} - f | v_{p_y, q_y}) = 0, \quad \forall v_{p_y, q_y} \in V_{p_y, q_y},$$

with

$$p_y(t) = k + \frac{\hat{f}(t) + f(t)}{2}, \quad q_y(t) = -\lambda \left[ k + \hat{f}(t) + f(t) \right].$$

As  $f$  is positive and  $\hat{f}$  is, at least, nonnegative by lemma hypothesis H3 (i),  $p_y(t)$  and  $q_y(t)$  have definite sign. Hence, the density of  $V_{p_y, q_y}$  in  $L_2$  is ensured by lemma 5.9.5 and  $\hat{f} = f$  almost for all  $t$  in  $[0, T]$ . The continuity of both  $\hat{f}$  and  $f$  entails

$$\hat{f}(t) = f(t), \quad \forall t \in [0, T].$$

■

**Theorem 5.13.6.** *The sequence  $\{y_n\}_n$  of solutions of the ODE (5.6.7) converges uniformly to the periodic reference  $f$ .*

**Proof.** Lemma 5.13.5 guarantees the weak limit of every weakly convergent subsequence of  $\{y_n\}_n$  coincides with  $f$ . Item (v) of proposition 5.4.6 leads to  $y_n \rightharpoonup f$ , while theorem 5.5.3 and proposition 5.5.2 (iii) yield the result. ■

## 5.14 Sliding control of the device

The control law that produces  $x = \phi_n(t)$  through the creation of a sliding regime in the phase plane is analogous to that developed at the end of section 4.5. Hence, let  $s_n(x, t) = x - \phi_n(t)$  be the switching surface.

**Proposition 5.14.1.** *The control law*

$$u = \begin{cases} 0 & \text{if } (k + y)s_n < 0 \\ 1 & \text{if } (k + y)s_n > 0 \end{cases}$$

*induces a sliding motion of system (4.2.3, 4.2.4) over the surface  $s_n(y, t) = 0$ .*

**Proof.** Firstly, take into account that we continue denoting the state variables with  $x$  and  $y$ , instead of the notation  $x_1, x_2$  used in (4.2.3, 4.2.4). The equivalent control

$u_{neq}$ , defined via  $s'_n(x, t) = 0$ , leads to

$$x' - \phi'_n(t) = 0 \quad \Longrightarrow \quad \phi'_n(t) = 1 - u_{neq}(k + y).$$

The system trajectories will be directed towards the switching surface when  $s_n s'_n < 0$ .

Then,

$$s_n s'_n = s_n(x' - \phi'_n) = s_n[1 - u(k + y) - 1 + u_{neq}(k + y)] = -s_n(k + y)(u - u_{neq}),$$

and the result follows. ■

The sliding domain is given by

$$0 < \frac{1 - \phi'_n}{k + y} < 1 \quad \text{or its equivalent} \quad 0 < \frac{y' + \lambda y}{\phi_n} < 1.$$

## 5.15 Approximate tracking of a sinusoidal wave

The target deals with the approximate indirect achievement of

$$y \approx f(t) = A + B \sin \omega t$$

in steady state. We will work with the first Galerkin approximation  $\phi_1$  of  $\phi(t)$ , the periodic solution of (5.6.4). The procedure follows section 5.6 and uses the general results obtained in the foregoing ones.

Thus,  $\phi_1(t)$  is a solution of

$$P_1 F x_1 = 0, \tag{5.15.1}$$

where

$$F x_1 = x_1 - x_1 x'_1 - g,$$

with  $x_1$  of type

$$x_1(t) = \lambda\alpha_0 + \alpha_1 \cos \omega t + \beta_1 \sin \omega t.$$

The function  $g(t)$ , calculated with (5.6.3), is

$$\begin{aligned} g(t) = & \lambda \left( A^2 + kA + \frac{B^2}{2} \right) + (k + A)B\omega \cos \omega t + (k + 2A)B\lambda \sin \omega t + \\ & - \frac{\lambda B^2}{2} \cos 2\omega t + \frac{B^2\omega}{2} \sin 2\omega t. \end{aligned}$$

Therefore,

$$\begin{aligned} Fx_1 = & \left[ \lambda\alpha_0 - \lambda \left( A^2 + kA + \frac{B^2}{2} \right) \right] + [\alpha_1 - \lambda\omega\alpha_0\beta_1 - \omega B(k + A)] \cos \omega t + \\ & + [\lambda\omega\alpha_0\alpha_1 + \beta_1 - \lambda B(k + 2A)] \sin \omega t + \left( \frac{\lambda B^2}{2} - \omega\alpha_1\beta_1 \right) \cos 2\omega t + \\ & + \frac{\omega}{2} (\alpha_1^2 - \beta_1^2 - B^2) \sin 2\omega t. \end{aligned}$$

The Galerkin equations (5.15.1), written as (5.7.2), result in

$$\begin{aligned} \alpha_0 - \left( A^2 + kA + \frac{B^2}{2} \right) & = 0 \\ \alpha_1 - \lambda\omega\alpha_0\beta_1 - \omega B(k + A) & = 0 \\ \lambda\omega\alpha_0\alpha_1 + \beta_1 - \lambda B(k + 2A) & = 0, \end{aligned}$$

and their only solution is

$$\begin{aligned} \alpha_0 & = A^2 + kA + \frac{B^2}{2} \\ \alpha_1 & = \frac{\lambda^2\alpha_0(k + 2A) + k + A}{1 + \lambda^2\omega^2\alpha_0^2} B\omega \\ \beta_1 & = \frac{k + 2A - \omega^2\alpha_0(k + A)}{1 + \lambda^2\omega^2\alpha_0^2} B\lambda, \end{aligned}$$

where  $\alpha_1, \beta_1$  are rational functions of  $\lambda$ . Then,  $\phi_1(t)$  takes the form

$$\phi_1(t) = \lambda\alpha_0 + \frac{\lambda^2(k + 2A)\alpha_0 + k + A}{1 + \lambda^2\omega^2\alpha_0^2} B\omega \cos \omega t + \frac{k + 2A - \omega^2\alpha_0(k + A)}{1 + \lambda^2\omega^2\alpha_0^2} B\lambda \sin \omega t =$$

$$= \lambda\alpha_0 + B \sqrt{\frac{\lambda^2(k+2A)^2 + \omega^2(k+A)^2}{1 + \lambda^2\omega^2\alpha_0^2}} \sin \left[ \omega t + \arctan \frac{\lambda^2\omega\alpha_0(k+2A) + \omega(k+A)}{\lambda(k+2A) - \lambda\omega^2\alpha_0(k+A)} \right]. \quad (5.15.2)$$

The error term is

$$\begin{aligned} F\phi_1(t) &= (\mathbb{I} - P_1) F\phi_1 = \left( \frac{\lambda B^2}{2} - \alpha_1\beta_1\omega \right) \cos 2\omega t + \frac{\omega}{2} (\alpha_1^2 - \beta_1^2 - B^2) \sin 2\omega t = \\ &= \sqrt{\left( \frac{\lambda B^2}{2} - \alpha_1\beta_1\omega \right)^2 + \frac{\omega^2}{4} (\alpha_1^2 - \beta_1^2 - B^2)^2} \sin(2\omega t + \arctan \gamma), \end{aligned}$$

$\gamma$  being

$$\gamma = \frac{\lambda B^2 - 2\alpha_1\beta_1\omega}{\omega(\alpha_1^2 - \beta_1^2 - B^2)}.$$

Care must be taken with the tangent inversions since the corresponding arguments may not belong to  $[-\pi/2, \pi/2]$ .

The evaluation of the error, performed through proposition 5.8.1, is

$$\|e_{1x}\|_\infty = \|\phi_1 - \phi\|_\infty \leq \delta \sqrt{\left( \frac{\lambda B^2}{2} - \alpha_1\beta_1\omega \right)^2 + \frac{\omega^2}{4} (\alpha_1^2 - \beta_1^2 - B^2)^2};$$

the parameter  $\delta$  can be obtained from (5.8.1).

*Remark 5.15.1.* It is important to realize that  $\phi_1(t)$  always exists for our system whatever the function  $f(t)$  may be, its analytical expression being easily obtainable and reasonably manageable. Namely, writing  $g(t)$  as

$$g(t) = (k+f)f' + \lambda f(k+f),$$

we see that its Fourier development

$$g(t) = C_0 + C_1 \cos \omega t + D_1 \sin \omega t + \sum_{n \geq 2} C_n \cos n\omega t + D_n \sin n\omega t$$

is such that the coefficients  $C_i, D_i$ , are affine functions of  $\lambda$ . Therefore,

$$\begin{aligned}\phi_1(t) &= E_0 + E_1 \cos \omega t + F_1 \sin \omega t = \\ &= C_0 + \frac{C_1 + C_0 D_1 \omega}{1 + C_0^2 \omega^2} \cos \omega t + \frac{D_1 - C_0 C_1 \omega}{1 + C_0^2 \omega^2} \sin \omega t = \\ &= C_0 + \sqrt{\frac{C_1^2 + D_1^2}{1 + C_0^2 \omega^2}} \sin \left( \omega t + \arctan \frac{C_1 + C_0 D_1 \omega}{D_1 - C_0 C_1 \omega} \right),\end{aligned}$$

and

$$\begin{aligned}F\phi_1(t) &= (-E_1 F_1 \omega - C_2) \cos 2\omega t + \left( \frac{E_1^2 - F_1^2}{2} \omega - D_2 \right) \sin 2\omega t + \\ &\quad - \sum_{n \geq 3} C_n \cos n\omega t + D_n \sin n\omega t.\end{aligned}$$

Once more, the tangent inversion must be carefully performed.

According to theorem 5.10.2, it is sufficient to satisfy hypothesis H1 for the existence of a positive output  $y_1(t)$  that approximates the tracking target  $f(t)$ . As the restrictions corresponding to  $f$  and  $g$  have been considered at the beginning of the section, it remains to check that

$$\begin{aligned}G_1(t) = g(t) + F\phi_1(t) &= \lambda \left( A^2 + kA + \frac{B^2}{2} \right) + (k + A)Bw \cos \omega t + \\ &\quad + (k + 2A)B\lambda \sin \omega t - \alpha_1 \beta_1 \omega \cos 2\omega t + \frac{\omega(\alpha_1^2 - \beta_1^2)}{2} \sin \omega t > 0;\end{aligned}$$

a sufficient condition for  $G_1 > 0$  to be fulfilled may be

$$\inf \{g\} > \|F\phi_1\|_\infty. \quad (5.15.3)$$

Theorem 5.12.1 allows an easy evaluation of the output error: with (5.12.1) we obtain

$$\|e_{1y}\|_\infty = \|y_1 - f\|_\infty \leq \frac{1}{\lambda} \sqrt{\|F\phi_1\|_\infty} = \frac{1}{\lambda} \sqrt{\left( \frac{\lambda B^2}{2} - \alpha_1 \beta_1 \omega \right)^2 + \frac{\omega^2}{4} (\alpha_1^2 - \beta_1^2 - B^2)^2}.$$

Finally, the location of the indirectly controlled approximate system in a non saturated region of the state space may be accomplished following section 5.11, which also allows the corresponding restrictions over the output reference signal parameters to be derived. Hence, according to proposition 5.11.2, hypothesis H2 and

$$\inf_{t \in [0, T]} \{G_1(t)\} \geq \|y_1' + \lambda y_1\|_\infty \quad (5.15.4)$$

are to be fulfilled. A further approximation to (5.15.4) may be given by the substitution of  $f(t)$  instead of  $y_1(t)$ . Even more, a first Galerkin approximation of  $y_1$  may be obtained and its analytic expression also used in (5.15.4).

Finally, the procedure suggests the use of the sliding control law given in section 5.14 to achieve the control target.

## 5.16 Simulation results

The technique developed through the chapter is applied to the buck-boost converter already used in the simulations of chapter 4. The Galerkin approximations are obtained for an output reference equal to the one considered there.

Thus, let us recall that the converter parameters are  $V_g = 50V$ ,  $R = 10\Omega$ ,  $L = 0.018H$  and  $C = 0.00022F$ , which make  $\lambda = 0.9045$ . The output voltage reference is

$$v_{Cr} = 135 + 15 \sin 2\pi\nu\tau \text{ V},$$

becoming

$$y_r = f(t) = 2.7 + 0.3 \sin \omega t$$

in the dimensionless variables. A frequency of  $\nu = 50Hz$  results in  $w = 0.6252$ . These settings guarantee the fulfillment of (4.6.3) and (4.6.4), as seen in section 4.8.

Moreover, they allow the satisfaction of (5.7.10), which may be written

$$g_0\omega = 5.67 > 4.24 = 1 + 2\sqrt{\omega\|\bar{g}\|}.$$

The Galerkin equations (5.7.2) have been solved with MAPLE for the cases  $n = 1$ ,  $n = 2$ ,  $n = 3$ ,  $n = 4$  and  $n = 5$ ; that is, Galerkin approximations to the periodic solution of (5.6.4) with one, two, three, four and five harmonics have been obtained.

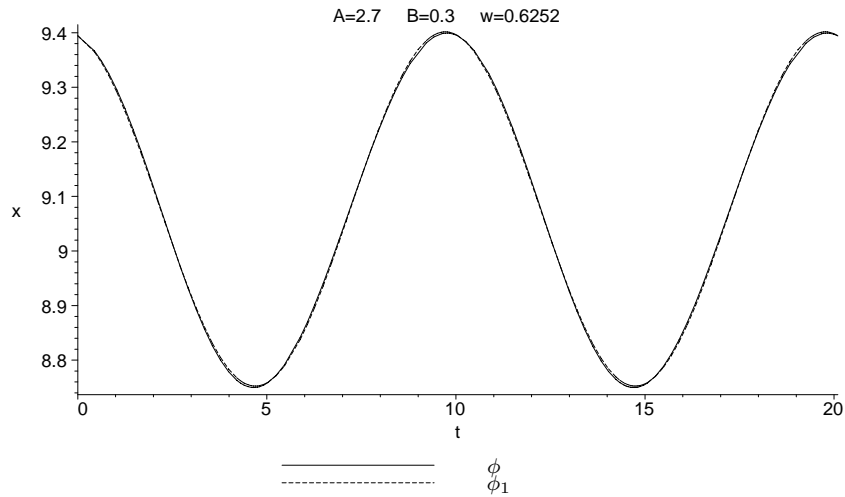


Figure 5.1:  $\phi(t)$  and  $\phi_1(t)$ .

Figure 5.1 depicts the periodic solution  $\phi(t)$  of (5.6.4), already portrayed in figure 4.6, together with  $\phi_1$ . When  $\phi_2$ ,  $\phi_3$ ,  $\phi_4$  or  $\phi_5$  are plotted with  $\phi$ , they are indistinguishable from it. Table 5.1 indicates the closeness of the approximations to the exact solution providing the absolute and relative errors of  $e_{nx} = \phi_n - \phi$ , measured with the  $L_2$  and  $L_\infty$  norms.

The errors  $F\phi_n$ ,  $n = 1, \dots, 5$  exhibit a clear tendency to decrease to 0 in table 5.2, which contains their  $L_2$  and  $L_\infty$  norms. This table also allows the fulfillment of the input error bound (5.8.2) to be verified:

$$\|e_{nx}\|_\infty < \delta \|F\phi_n\|_\infty,$$



Table 5.1: Absolute and relative errors of the Galerkin approximations measured with the  $L_2$  and the  $L_\infty$  norms.

	$\phi_1$	$\phi_2$	$\phi_3$	$\phi_4$	$\phi_5$
$\ e_{nx}\ _{L_2}$	$1.06 \cdot 10^{-2}$	$1.90 \cdot 10^{-4}$	$4.21 \cdot 10^{-6}$	$1.25 \cdot 10^{-7}$	$7.43 \cdot 10^{-9}$
$\frac{\ e_{nx}\ _{L_2}}{\ \phi\ _{L_2}} \cdot 100$ (%)	$3.69 \cdot 10^{-4}$	$6.59 \cdot 10^{-6}$	$1.46 \cdot 10^{-7}$	$4.36 \cdot 10^{-9}$	$2.58 \cdot 10^{-10}$
$\ e_{nx}\ _\infty$	$4.89 \cdot 10^{-3}$	$8.80 \cdot 10^{-5}$	$1.97 \cdot 10^{-6}$	$5.61 \cdot 10^{-8}$	$4.30 \cdot 10^{-9}$
$\frac{\ e_{nx}\ _\infty}{\ \phi\ _\infty} \cdot 100$ (%)	$5.20 \cdot 10^{-4}$	$9.36 \cdot 10^{-6}$	$2.09 \cdot 10^{-7}$	$5.98 \cdot 10^{-9}$	$4.58 \cdot 10^{-10}$

because  $\delta \geq 1$  for every admissible<sup>4</sup> function  $g(t)$ , and a straightforward comparison with the row  $\|e_{nx}\|_\infty$  of table 5.1 shows  $\|e_{nx}\|_\infty < \|F\phi_n\|_\infty$ .

Table 5.2:  $L_2$  and  $L_\infty$  norms of the Galerkin errors  $F\phi_n$ .

	$\phi_1$	$\phi_2$	$\phi_3$	$\phi_4$	$\phi_5$
$\ F\phi_n\ _{L_2}$	$1.21 \cdot 10^{-1}$	$3.23 \cdot 10^{-3}$	$9.56 \cdot 10^{-5}$	$3.28 \cdot 10^{-6}$	$1.16 \cdot 10^{-7}$
$\ F\phi_n\ _\infty$	$5.39 \cdot 10^{-2}$	$1.45 \cdot 10^{-3}$	$4.32 \cdot 10^{-5}$	$1.49 \cdot 10^{-6}$	$5.70 \cdot 10^{-8}$

The existence of positive, asymptotically stable periodic output when a Galerkin approximation is used in equation (5.6.6) is guaranteed by theorem 5.10.2. The positivity of  $G_n$ ,  $n = 1, \dots, 5$ , demanded by hypothesis H1, follows from the fulfillment of the sufficient condition (5.15.3):

$$7.26 = \inf \{g\} > \|F\phi_1\|_\infty = 0.05,$$

and table 5.2 shows that  $\|F\phi_1\|_\infty > \|F\phi_n\|_\infty$ ,  $n = 2, \dots, 5$ .

Figure 5.2 depicts a detail of the functions  $H_n(z)$ ,  $n = 1, 2, 3$  ( $H_4$  and  $H_5$  are visually indistinguishable from  $H_3$ ). All of them, including  $H_4$  and  $H_5$ , are shown to have

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<sup>4</sup>Consider  $g > 0$  in (5.8.1).

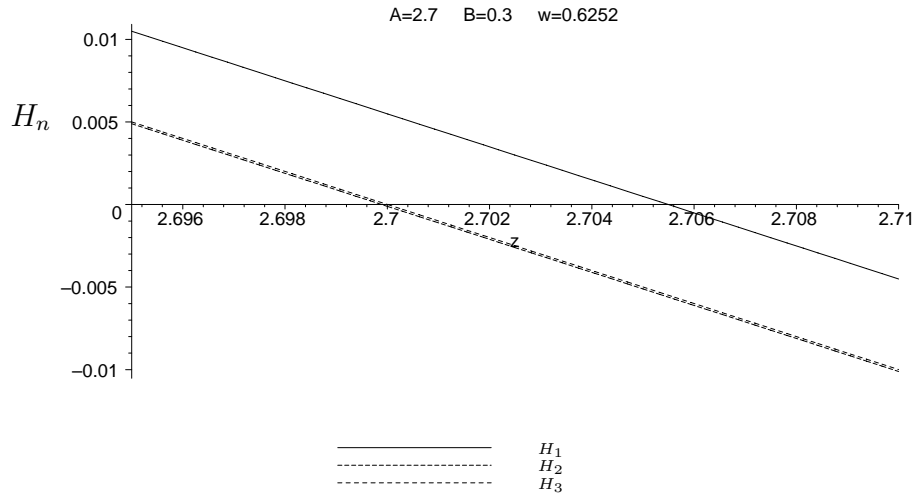


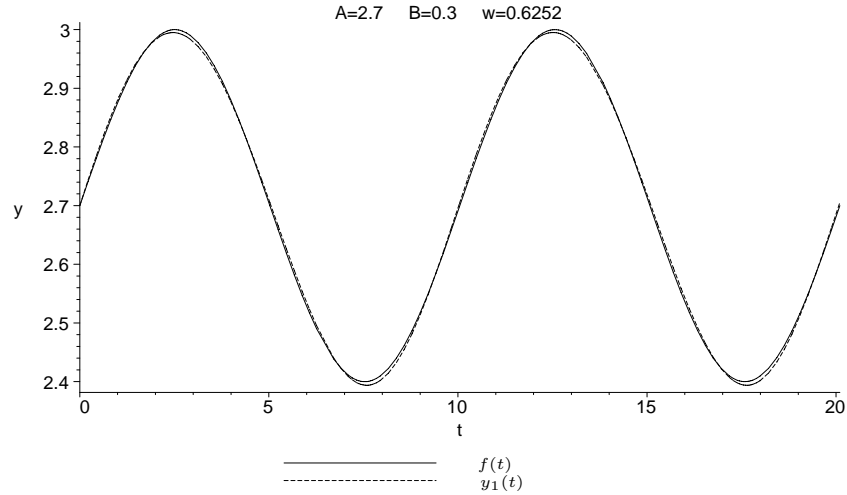
Figure 5.2: Detail of  $H_n(z)$ ,  $n = 1, 2, 3$ , crossing the  $z$ -axis.

a positive, decreasing solution close to  $f(0) = 2.7$ . Table 5.3 contains the solution points, which show a tendency to  $f(0)$ .

Table 5.3:  $L_2$  and  $L_\infty$  norms of the Galerkin errors  $F\phi_n$ .

	$H_n^{-1}(0)$
$n = 1$	2.705492596
$n = 2$	2.699889915
$n = 3$	2.700003399
$n = 4$	2.699999889
$n = 5$	2.700000003
$f(0)$	2.7

The ideal output behavior, also studied with the aid of MAPLE software, is observable in figure 5.3, where  $y_1$ , corresponding to the use of  $\phi_1$  in equation (5.6.6), is plotted together with the reference  $f(t)$ . Functions  $y_2$ ,  $y_3$ ,  $y_4$  and  $y_5$  cannot be distinguished from  $f$  in a plot. Table 5.4 contains the  $L_2$  and  $L_\infty$  norms of the output error  $e_{ny} = y_n - f(t)$  in absolute and relative form. Again, the tendency of  $y_n$  to  $f$  is

Figure 5.3: The output reference  $f(t)$  and the approximation  $y_1$ .

evident.

Table 5.4: Absolute and relative errors of the output measured with the  $L_2$  and  $L_\infty$  norms.

	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$\ e_{ny}\ _{L_2}$	$1.55 \cdot 10^{-2}$	$3.45 \cdot 10^{-4}$	$8.56 \cdot 10^{-6}$	$3.51 \cdot 10^{-7}$	$1.14 \cdot 10^{-8}$
$\frac{\ e_{ny}\ _{L_2}}{\ f\ _{L_2}} \cdot 100$ (%)	$1.81 \cdot 10^{-3}$	$4.02 \cdot 10^{-5}$	$9.96 \cdot 10^{-7}$	$4.09 \cdot 10^{-8}$	$1.33 \cdot 10^{-9}$
$\ e_{ny}\ _\infty$	$7.95 \cdot 10^{-3}$	$1.73 \cdot 10^{-4}$	$4.22 \cdot 10^{-6}$	$1.29 \cdot 10^{-7}$	$4.60 \cdot 10^{-9}$
$\frac{\ e_{ny}\ _\infty}{\ f\ _\infty} \cdot 100$ (%)	$2.65 \cdot 10^{-3}$	$5.77 \cdot 10^{-5}$	$1.41 \cdot 10^{-6}$	$4.31 \cdot 10^{-8}$	$1.53 \cdot 10^{-9}$

The fulfillment of the output error bound determined in equation (5.12.1), that is

$$\|e_{ny}\|_\infty \leq \frac{\sqrt{\|F\phi_n\|_\infty}}{\lambda},$$

can be observed in table 5.5 with the positivity of the differences

$$\frac{\sqrt{\|F\phi_n\|_\infty}}{\lambda} - \|e_{ny}\|_\infty.$$

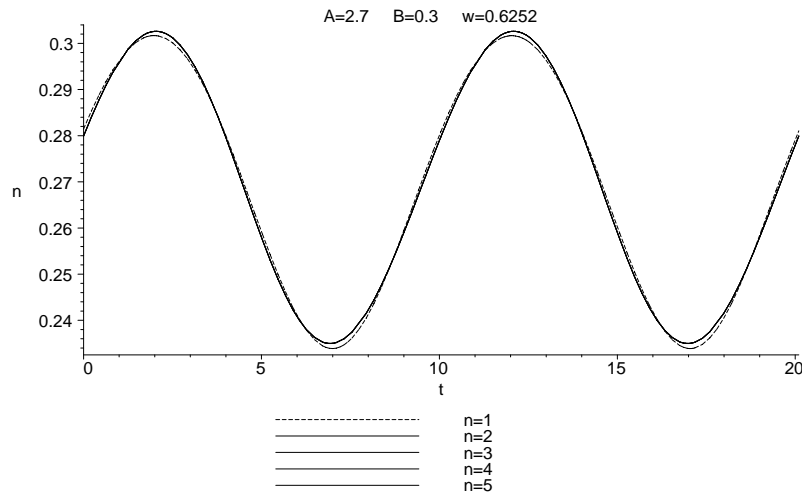
Table 5.5: Fulfillment of the output error bound.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\frac{\ F\phi_n\ _{\infty}^{\frac{1}{2}}}{\lambda} - \ e_{ny}\ _{\infty}$	$2.49 \cdot 10^{-1}$	$4.20 \cdot 10^{-2}$	$7.27 \cdot 10^{-3}$	$1.35 \cdot 10^{-3}$	$2.64 \cdot 10^{-4}$

The presence of the approximately controlled systems in the insaturation zone is verifiable in figure 5.4, where the plot of

$$\frac{1 - \phi'_n}{k + y_n}, \quad n = 1, \dots, 5$$

is shown to lie between 0 and 1, as required in section 5.11.

Figure 5.4: Verification of the insaturation condition in cases  $n = 1, \dots, 5$ .

The following pictures consider the non ideal case, here treated with SIMULINK (see section 3.8 for details about the numerical integration procedure). As has been done in the preceding chapters, the non ideal system is modelled with an hysteretic control of bandwidth  $\Delta S_h = 0.00314$ , which limits the switching frequency to a maximum of 20 kHz (see section 3.2.5).

Notice that the chattering is the reason why the use of higher order Galerkin approximations does not improve the response of the system. The lowest possible values of the errors, which equal those resulting from an exact treatment of the problem in section 4.8, are reached with the 2nd approximation.

Figure 5.5 contains the input currents  $x_1$  and  $x_2$ , obtained by the Galerkin approximations  $\phi_1$  and  $\phi_2$  in the switching surfaces, plotted against  $x_r = \phi$ . Figure 5.6 portrays the corresponding outputs  $y_1$  and  $y_2$  tracking the voltage reference  $y_r = f(t)$ .

The relative errors of such input currents and output voltages are depicted in figure 5.7. A simple comparison with figure 4.11 shows the coincidence of error values for the exact treatment and for the situation corresponding to the use of a 2nd Galerkin approximation. Naturally, no improvement can be expected with higher harmonics.

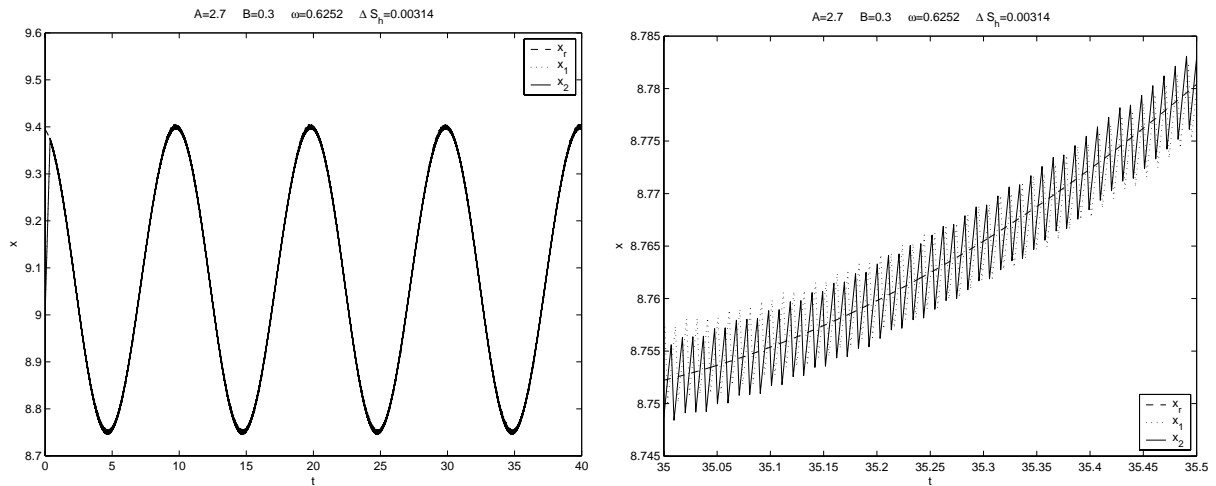


Figure 5.5: Details of  $x_r$ ,  $x_1$  and  $x_2$  in the non ideal case.

Finally, a SIMULINK model of a buck-boost converter controlled with a 1st Galerkin approximation is shown in figure 5.8. An extension of it, consisting in the introduction of equivalent systems controlled with 2nd, 3rd, 4th and 5th Galerkin approximations, has been used to obtain the non ideal case plots.

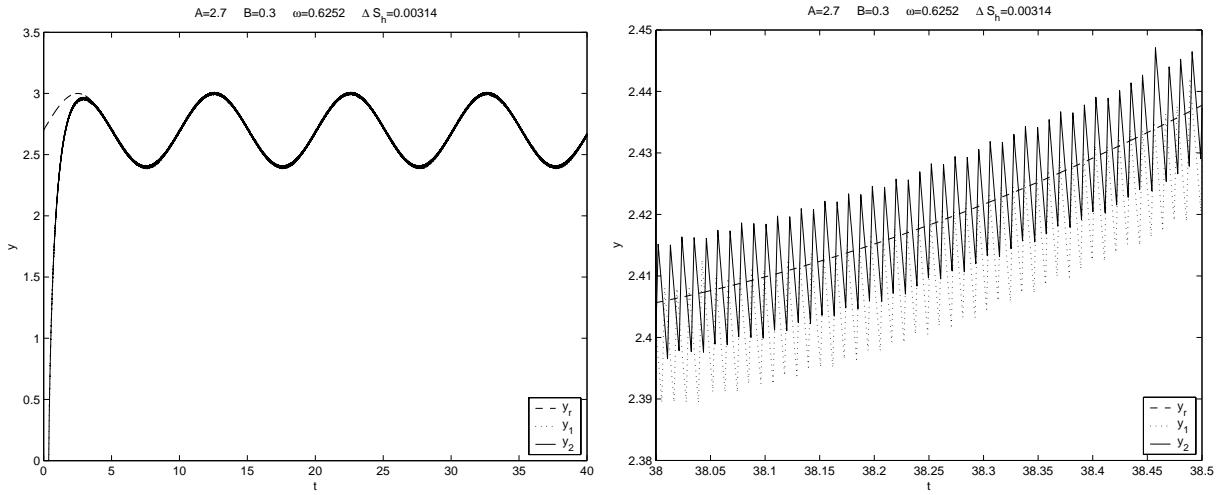


Figure 5.6: Details of  $y_r$ ,  $y_1$  and  $y_2$  in the non ideal case.

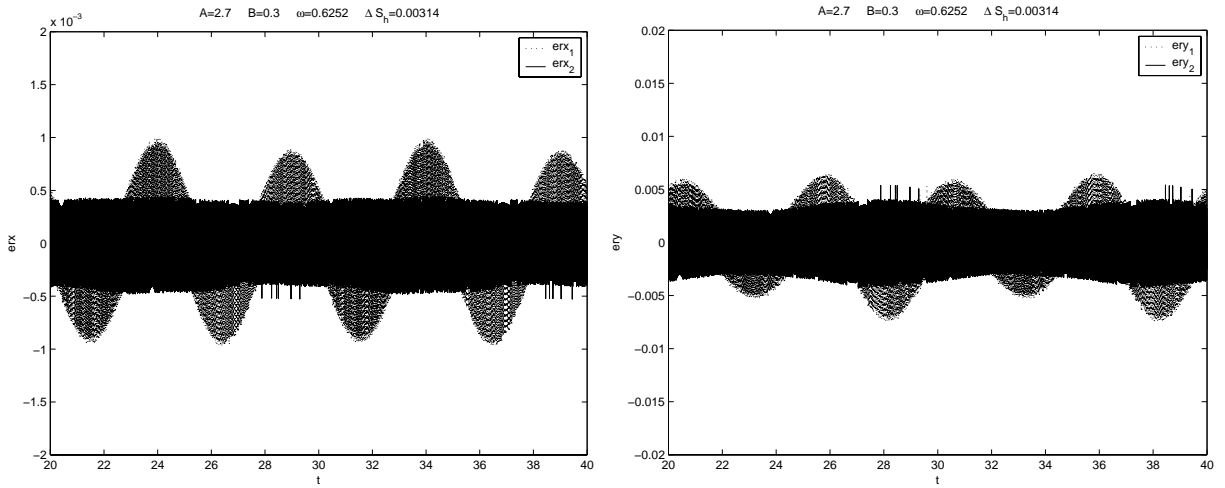


Figure 5.7: Details of  $erx_1$ ,  $erx_2$  and  $ery_1$ ,  $ery_2$ , in the non ideal case.

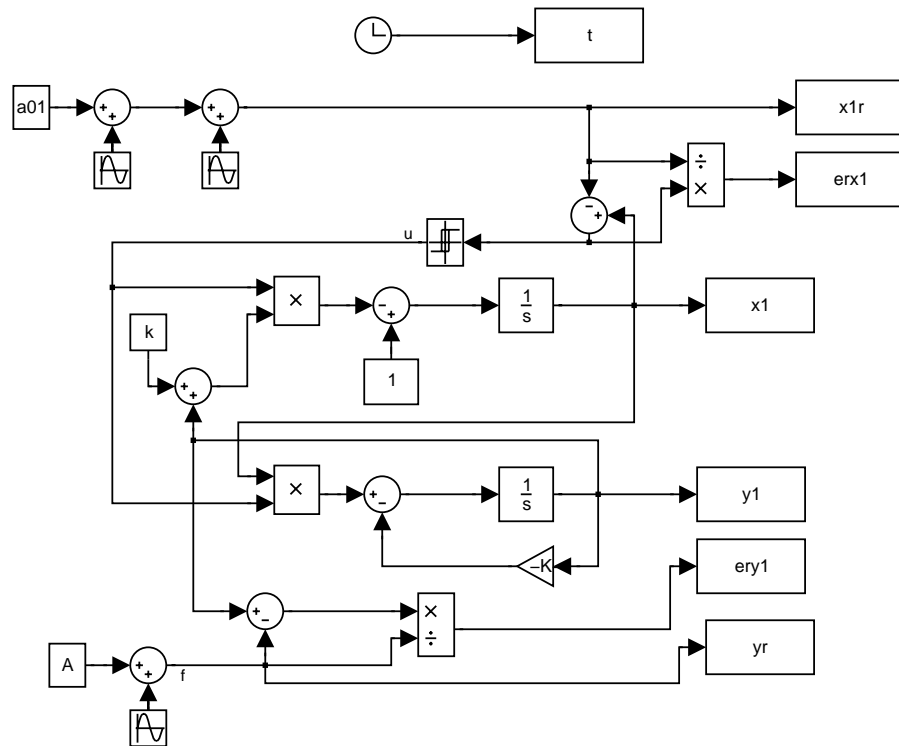


Figure 5.8: Buck-boost converter controlled with a 1st Galerkin approximation.

# Chapter 6

## Robust Tracking Control of Nonlinear Converters

Output voltage control of boost and buck-boost converters must be performed indirectly, through the input current, due to the non minimum phase character shown by the systems. The devices then have sensitivity to load perturbations, which can be removed with the introduction of an observer that identifies the disturbed parameter and allows the tracking of periodic signals at the output resistances.

### 6.1 Introduction

The need for indirect control for the output voltage of the nonlinear switched converters that have been dealt with has been shown in chapter 4. Exact and approximate control techniques that allow the tracking of periodic references have been developed in that chapter and in the previous one. However, perfect tracking control schemes are usually very sensitive to external perturbations and parameter uncertainties [BI00], [Dev00], and even more in inversion-based control schemes.

The next goal is to design robust control strategies to prevent undesirable effects



of load perturbations. We propose the introduction of a disturbance observer with dynamics proportional to the tracking error of the output voltage which shows a reasonably rapid speed of identification and good simulation results. A similar law has been used in [Esc99] for regulation tasks. Its design complexity is considerably lower than the proposal in [FZ01], as already commented in section 2.4. The on-line updating of the input current is performed, as in [FZ01], through a first harmonic Galerkin approximation of the solution of the ordinary differential equation (5.6.4). This chapter is structured as follows. In sections 6.2 and 6.3 we develop a control strategy that furnishes the performance of boost and buck-boost converters with load perturbation robustness. Section 6.4 exemplifies the robust tracking of a sinusoidal signal with both converters. Finally, the simulation results are presented in section 6.5.

## 6.2 Adaptive control of the boost converter

The devices that work under indirect control are especially sensitive to disturbances. This is an easily observable fact in our system because the input-output differential relation (5.6.1) depends on the system parameter  $\lambda$ , which is a strong candidate to suffer perturbations. Here we will try to eliminate the undesirable effects that load change induces on the converter dynamics by means of adaptive control.

We may assume an unknown value  $R$  for the load resistance, due to the addition of a constant disturbance term  $R_p$  to its nominal value  $R_N$ ; that is,  $R = R_N + R_p$ , where  $R > 0$ . Consequently, the parameter  $\lambda$  may be written as  $\lambda = \lambda_N + \lambda_p$  with

$$\lambda_p = -\frac{\lambda_N R_p}{R_N + R_p}, \quad (6.2.1)$$

where  $\lambda > 0$ . Therefore, system (4.2.3, 4.2.4) can be described as

$$x' = 1 - uy \quad (6.2.2)$$

$$y' = -(\lambda_N + \lambda_p)y + ux, \quad (6.2.3)$$

where the notation  $x, y$ , instead of  $x_1, x_2$  introduced in section 5.6, is already used.

Let  $f(t) > 0, \forall t$ , be the  $C^\infty$  and  $T$ -periodic reference to be tracked by  $y$ , while  $\hat{\lambda}_p$  represents an estimation of the additive disturbance  $\lambda_p$  detailed below. Assume likewise that a certain control forces the state variable  $x$  to follow a certain signal  $\phi(t; \lambda_N + \hat{\lambda}_p)$  such that

$$\phi(t; \lambda_N + \hat{\lambda}_p) \left[ 1 - \phi'(t; \lambda_N + \hat{\lambda}_p) \right] = f(t) \left[ f'(t) + (\lambda_N + \hat{\lambda}_p)f(t) \right]; \quad (6.2.4)$$

its existence is guaranteed by standard results of ODE theory. The quoted law can be, for example,

$$\bar{u} = \frac{1}{y}(1 - \phi')$$

which, when taken to (6.2.2), produces  $x(t) = \phi(t, \lambda_N + \hat{\lambda}_p)$  in steady state with appropriated initial conditions, and the behavior of the state variable  $y$  is given by (5.6.2):

$$y[y' + (\lambda_N + \lambda_p)y] = \phi(t; \lambda_N + \hat{\lambda}_p) \left[ 1 - \phi'(t; \lambda_N + \hat{\lambda}_p) \right]$$

which, using (6.2.4), will be expressed as

$$y[y' + (\lambda_N + \lambda_p)y] = f(t) \left[ f'(t) + (\lambda_N + \hat{\lambda}_p)f(t) \right].$$

With the change of variable

$$e_y = \frac{1}{2}(y^2 - f^2)$$

the former relation becomes

$$e'_y = -2(\lambda_N + \lambda_p)e_y + f^2(t)(\hat{\lambda}_p - \lambda_p).$$

The estimator dynamics is defined as

$$\hat{\lambda}'_p = -\beta f^2(t)e_y,$$

where  $\beta$  stands for a positive gain. Note that the observation error may be written  $e_\lambda = \hat{\lambda}_p - \lambda_p$  and, the perturbation being constant, it yields  $e'_\lambda = \hat{\lambda}'_p$ . Then, the adaptive system allows the description

$$e'_y = -2\lambda e_y + f^2(t)e_\lambda \quad (6.2.5)$$

$$e'_\lambda = -\beta f^2(t)e_y, \quad (6.2.6)$$

which admits  $(e_y, e_\lambda)^\top = (0, 0)$  as an equilibrium point. Its asymptotic stability must now be demonstrated.

Let us take

$$V(e_y, e_\lambda) = \frac{1}{2} \left( e_y^2 + \frac{e_\lambda^2}{\beta} \right);$$

it is easy to check that this is a positive definite, radially unbounded, decrescent function. The derivative of  $V$  over the system trajectories is

$$V'(e_y, e_\lambda) = -2\lambda e_y^2,$$

and is negative semidefinite. Moreover, the subset  $S$  of the phase plane points where the former derivative vanishes is  $S = \{(0, e_\lambda)\}$ , and the greatest invariant set inside  $S$  is  $\{(0, 0)\}$ :

$$e_y = 0 \implies e'_\lambda = -\beta \cdot f^2(t) \cdot 0 \implies e_\lambda = K, \quad K \in \mathbb{R},$$

and, as  $e'_y = 0$  (otherwise, the system would abandon  $S$  immediately), it entails

$$e'_y = 0 = -2 \cdot \lambda \cdot 0 + f^2(t) \cdot K \implies 0 = f^2(t) \cdot K.$$

Furthermore, by hypothesis it is  $f(t) \neq 0, \forall t$ , this leading to  $K = 0$ . Taking into account the invariance principle of LaSalle [Sas99], it can be established that:

**Proposition 6.2.1.** *The origin  $(0, 0)$  of system (6.2.5, 6.2.6) is globally uniformly asymptotically stable. ■*

Notice that when such an equilibrium point is reached, the output  $y$  coincides with the desired reference  $f$  and, moreover, the signal  $\phi(t, \lambda_N + \hat{\lambda}_p)$  tracked by the state variable  $x$  can be periodic. This follows from the application of theorem 4.4.2 to equation (6.2.4), where  $\hat{\lambda}_p$  is now constant.

The on-line updating of the current reference  $\phi(t; \lambda_N + \hat{\lambda}_p)$  according to the instant variations suffered by  $\hat{\lambda}_p$  is almost impossible in real problems. In fact, we have already commented the difficulties that arise if we intend to work with  $\phi(t; \lambda_N)$ ,  $\lambda_N$  being constant, which has justified the introduction of the Galerkin method in chapter 5. Our proposal considers the use of the first harmonic Galerkin approximation of  $\phi$ , rapidly obtainable for our system from the study in section 5.15.

Let then

$$\phi_1(t; \lambda_N) = \alpha_0(\lambda_N) + \alpha_1(\lambda_N) \cos(\omega t) + \beta_1(\lambda_N) \sin(\omega t)$$

be the first Galerkin approximation of  $\phi(t; \lambda_N)$ . Notice that its coefficients are  $\lambda_N$ -dependent (see remark 5.15.1). To work the perturbed case we may consider the expression of  $\phi_1(t; \lambda_N)$  and substitute  $\lambda_N$  by  $\lambda_N + \hat{\lambda}_p(t)$ . We will therefore make use of

$$\phi_1(t; \lambda_N + \hat{\lambda}_p) \approx \phi(t; \lambda_N + \hat{\lambda}_p).$$

*Remark 6.2.1.* (i) The sliding control laws proposed in sections 4.5 and 5.14 for the exact and approximate treatment will induce the desired behavior for the state variable

$x$ . Notice that in such a context the load perturbations of the ideal boost converter are additive field disturbances that do not satisfy the matching condition (see theorem 3.2.3), as is the case of the buck converter:

$$(0, -\lambda_p y)^\top \notin \text{span} \{(-y, x)^\top\}.$$

Hence, the theory for perturbations that hold a weak invariance property developed in subsection 3.2.6 is applicable to this situation.

(ii) The observer-based control purpose developed in [UGS99] and quoted in section 4.5 loses its advantage of eliminating the need to sense the output variable when a perturbation observer is introduced.

(iii) When the system is physically implemented, the fulfillment of the restrictions established in sections 4.5 for the exact case and in section 5.11 and 5.14 for the approximate case must be guaranteed. If they do not hold for the whole state space, the global character of proposition 6.2.1 would become local.

### 6.3 Adaptive control of the buck-boost converter

The objectives and structure of the present section are those of the preceding one. It must be noticed, however, that the buck-boost system appears to be slightly more complicated than the boost system. Hence, small differences may be observed in some proofs.

Consider the description of system (4.2.3, 4.2.4) for the buck-boost converter under the effect of a constant load resistance disturbance:

$$x' = 1 - u(1 + y) \tag{6.3.1}$$

$$y' = -(\lambda_N + \lambda_p)y + ux. \tag{6.3.2}$$

Assume that there exists a certain control strategy that induces  $x$  to follow a signal  $\phi(t; \lambda_N + \hat{\lambda}_p)$  such that

$$\phi(t; \lambda_N + \hat{\lambda}_p) \left[ 1 - \phi'(t; \lambda_N + \hat{\lambda}_p) \right] = [1 + f(t)] \left[ f'(t) + (\lambda_N + \hat{\lambda}_p)f(t) \right], \quad (6.3.3)$$

where  $f(t) > 0, \forall t$ , is the  $C^\infty$  and  $T$ -periodic reference for  $y$ , and  $\hat{\lambda}_p$  stands for an estimation of the disturbance  $\lambda_p$ . Basic results on ODE solutions ensure the existence of  $\phi$ . Such a control law can be

$$\bar{u} = \frac{1}{1+y}(1-\phi')$$

because, taking into account (6.3.1), it determines  $x = \phi(t, \lambda_N + \hat{\lambda}_p)$  with well chosen initial conditions. The dynamic behavior of the state variable  $y$  will be given by (5.6.2):

$$(1+y)[y' + (\lambda_N + \lambda_p)y] = \phi(t; \lambda_N + \hat{\lambda}_p) \left[ 1 - \phi'(t; \lambda_N + \hat{\lambda}_p) \right]$$

which, using (6.3.3), can be expressed as

$$(1+y)[y' + (\lambda_N + \lambda_p)y] = [1 + f(t)] \left[ f'(t) + (\lambda_N + \hat{\lambda}_p)f(t) \right].$$

Introducing the error  $e_y = y - f$ , the former equation becomes

$$e'_y = - \left( \lambda + \frac{f' + \lambda f}{1 + f + e_y} \right) e_y + \frac{f(1+f)}{1 + f + e_y} (\hat{\lambda}_p - \lambda_p).$$

The estimator dynamics is defined as

$$\hat{\lambda}'_p = -\beta f(t) e_y,$$

where  $\beta$  is a positive gain. The observation error can be written  $e_\lambda = \hat{\lambda}_p - \lambda_p$  and, since the perturbation is constant,  $e'_\lambda = \hat{\lambda}'_p$ . Thus, the adaptive system is

$$e'_y = - \left( \lambda + \frac{f' + \lambda f}{1 + f + e_y} \right) e_y + \frac{f(1+f)}{1 + f + e_y} e_\lambda \quad (6.3.4)$$

$$e'_\lambda = -\beta f(t) e_y, \quad (6.3.5)$$

showing  $(e_y, e_\lambda)^\top = (0, 0)$  to be an equilibrium point.

Let us now prove its asymptotic stability. Assigning  $\xi = (e_y, e_\lambda)^\top$  and

$$\Psi(\xi, t) = \left( - \left[ \lambda + \frac{f' + \lambda f}{1 + f + e_y} \right] e_y + \frac{f(1 + f)}{1 + f + e_y} e_\lambda, -\beta f(t) e_y \right)^\top,$$

(6.3.4, 6.3.5) can be written as the  $T$ -periodic differential system

$$\xi' = \Psi(\xi, t). \quad (6.3.6)$$

The matrix

$$A(t) = \left( \frac{\partial \Psi}{\partial \xi} \right)_{\xi=0} = \begin{pmatrix} -\lambda - \frac{f'(t) + \lambda f(t)}{1 + f(t)} & f(t) \\ -\beta f(t) & 0 \end{pmatrix}$$

is bounded if the hypotheses about  $f(t)$  stated at the beginning of the subsection are fulfilled; moreover, let

$$\Psi_1(\xi, t) = \Psi(\xi, t) - A(t)\xi = \left( \frac{f' + \lambda f}{(1 + f)(1 + f + e_y)} e_y^2 - \frac{f}{1 + f + e_y} e_y e_\lambda, 0 \right)^\top.$$

Thus, one may write (6.3.6) detailing its linear part in a neighbourhood of the origin:

$$\dot{\xi} = A(t)\xi + \Psi_1(\xi, t). \quad (6.3.7)$$

Notice that  $\Psi_1(0, t) = 0$  and that

$$\lim_{\|\xi\| \rightarrow 0} \frac{\|\Psi_1(\xi, t)\|}{\|\xi\|} = \lim_{\|\xi\| \rightarrow 0} \frac{1}{\sqrt{e_y^2 + e_\lambda^2}} \left| \frac{(f' + \lambda f)e_y^2 - f(1 + f)e_y e_\lambda}{(1 + f)(1 + f + e_y)} \right| = 0$$

uniformly in  $t$ , because the two terms of the numerator of the limit have degree two, while the smallest degree of the denominator is 1.

Take also

$$V(\xi) = \frac{1}{2} \left( e_y^2 + \frac{e_\lambda^2}{\beta} \right);$$

it is straightforward that this is a positive definite, radially unbounded, decrescent function. The derivative of  $V$  calculated over the linearized system (6.3.7) trajectories, that is, over  $\xi' = A(t)\xi$ , is

$$V'(\xi) = - \left[ \lambda + \frac{f'(t) + \lambda f(t)}{1 + f(t)} \right] e_y^2.$$

Reasons identical to the ones that allow the boundedness of  $A(t)$  to be established, together with the additional restriction  $(1 + f)^{-1}(f' + \lambda f) > 0$  (equivalent to the demand  $g(t) > 0$  in section 4.5), entail the existence of  $\rho \in \mathbb{R}$  such that  $(1 + f)^{-1}(f' + \lambda f) \geq \rho > 0$ . Therefore,

$$V'(\xi) \leq -(\lambda + \rho)e_y^2,$$

thus being negative semidefinite. It is straightforward to notice that the subset of the phase plane where the former derivative vanishes is  $S = \{(0, e_\lambda)\}$ , and the greatest invariant set inside  $S$  is  $\{(0, 0)\}$  since

$$e_y = 0 \implies e'_\lambda = -\beta \cdot f(t) \cdot 0 \implies e_\lambda = K, \quad K \in \mathbb{R}$$

and, as  $e'_y = 0$  (otherwise, the system would abandon  $S$  immediately), we have

$$e'_y = 0 = - \left[ \lambda + \frac{f'(t) + \lambda f(t)}{1 + f(t)} \right] \cdot 0 + f(t) \cdot K \implies 0 = f(t) \cdot K.$$

By hypothesis it is  $f(t) \neq 0, \forall t$ , which leads to  $K = 0$ . Hence, applying the invariance principle of LaSalle and the indirect method of Lyapunov for non autonomous systems [SL91], we can establish the next result:

**Proposition 6.3.1.** *The origin  $(0, 0)$  of system (6.3.4, 6.3.5) is locally uniformly asymptotically stable.* ■



As happened with the boost converter, when the system is at the equilibrium point, the output  $y$  tracks  $f(t)$ , while  $\phi(t, \lambda_N + \hat{\lambda}_p)$  may be chosen periodic.

Analogously to the boost case developed in the preceding section, the first harmonic Galerkin approximation of the input current reference is used to update its value according to the instant variation of the perturbation observer.

*Remark 6.3.1.* The notes of remark 6.2.1 for the boost converter are also valid for the present situation.

## 6.4 Robust tracking of a sinusoidal reference

The interest is focused on obtaining

$$y = A + B \sin \omega t, \quad A, B > 0$$

at the output resistance of a nonlinear boost or buck-boost converter under load disturbances. The restrictions established in sections 6.2 and 6.3 act on the signal parameters as follows:

- (i) The existence of a periodic indirect current reference  $\phi$  for the non perturbed exact problem, studied in chapter 4, is conditioned to (4.6.3, 4.6.4).
- (ii) The approximate approach needs the Galerkin approximation  $\phi_1(t; \lambda_N)$ , which always exists (see remark 5.15.1). Hence, no additional sufficient conditions for the existence of the whole Galerkin sequence  $\{\phi_n(t; \lambda_N)\}_n$  are needed. The expression of  $\phi_1(t; \lambda_N)$  may be found in (5.15.2).
- (iii) Under the current indirect control achieved through  $\phi_1(t; \lambda_N)$ , theorem 5.10.2 ensures the existence of a periodic output  $y \approx f(t)$  if

$$G_1(t; \lambda_N) = g(t; \lambda_N) + F\phi_1(t; \lambda_N) > 0.$$

A sufficient condition for the former relation to be fulfilled, involving the signal parameters, is

$$\inf \{g\} > \|F\phi_1\|_\infty.$$

The expressions of  $G_1(t; \lambda_N)$ ,  $F\phi_1(t; \lambda_N)$  and the former sufficient condition are calculated in section 5.15.

(iv) Equation (5.11.1) in proposition 5.11.1 provides the restrictions that guarantee the presence of the system in an insaturation region at steady state. A sufficient condition for the fulfillment of (5.11.1) may be found in proposition 5.11.2.

(v) The construction of  $\phi_1(t; \lambda_N + \hat{\lambda}_p)$  must be made from  $\phi_1(t; \lambda_N)$  in (ii). Care must be taken in the verification of (5.11.1),  $\forall t$ .

(vi) Once the system is (approximately) stabilized at  $\hat{\lambda}_p = \lambda_p$ , (iii) must be observed.

(vii) The sliding surface is

$$s(x, t) = x - \phi_1(t; \lambda_N + \hat{\lambda}_p),$$

while the control law is given in proposition 5.14.1.

## 6.5 Simulation results

The robust procedure is tested in a boost converter and a buck-boost converter with parameters  $V_g = 50V$ ,  $R_N = 10\Omega$ ,  $L = 0.018H$  and  $C = 0.00022F$ . The output voltage reference is

$$v_{Cr} = 135 + 15 \sin 2\pi\nu\tau V,$$

with  $\nu = 50Hz$ . In dimensionless variables this corresponds to  $\lambda_N = 0.9045$ ,  $w = 0.6252$  and

$$y_r = f(t) = 2.7 + 0.3 \sin \omega t.$$

The existence of a periodic current reference  $\phi$  is ensured for both converters in the non perturbed case as well as their location in an insaturated phase plane zone for that situation -(4.6.3), (4.6.4)- and the fulfillment of (5.7.10) and (5.15.3). This has been observed for the buck-boost converter in the simulation section of the preceding chapters. For the boost system the calculations are, respectively:

$$2.7 = A > \sup \left\{ B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}, B + \frac{A + B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}{A - B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}} \right\} = \sup \{0.365, 1.612\};$$

$$4.15 = g_0\omega > 1 + 2\sqrt{\omega \|\bar{g}\|} = 3.95; \quad 5.1 = \inf \{g\} > \|F\phi_1\|_\infty = 0.50.$$

Figure 6.1 contains the SIMULINK model used in the simulations (see section 3.8 for details about the numerical integration algorithm). The control actuator switches at a maximum frequency of 20 *kHz*, resulting in a relay hysteresis bandwidth of  $\Delta S_h = 0.00314$  (see relation (3.2.8) with  $\nu_{sM}$  in the dimensionless variables). The initial conditions of the state variables are close to the non perturbed current reference for  $x_1$  and to the output reference for  $y_1$ .

The results show good identification of the perturbations, with values up to 100% of the load resistance level. The optimal gains of the observer dynamics are around  $\beta = 0.007$  for the boost converter and  $\beta = 0.0125$  for the buck-boost, and obviously depend on the disturbance. For these gain values, figure 6.2 plots the observer behavior in the presence of initial perturbations such as  $R_p = 2\Omega$ ,  $R_p = 5\Omega$  and  $R_p = 10\Omega$ , that is, 20%, 50% and 100% of the nominal value. The overshoot increases as the disturbance does.

Figure 6.3 portrays the input current and the output voltage tracking their respective references. It can be seen that, while the input current never loses its reference, the output voltage needs two periods to correct the disturbance. This phenomenon is

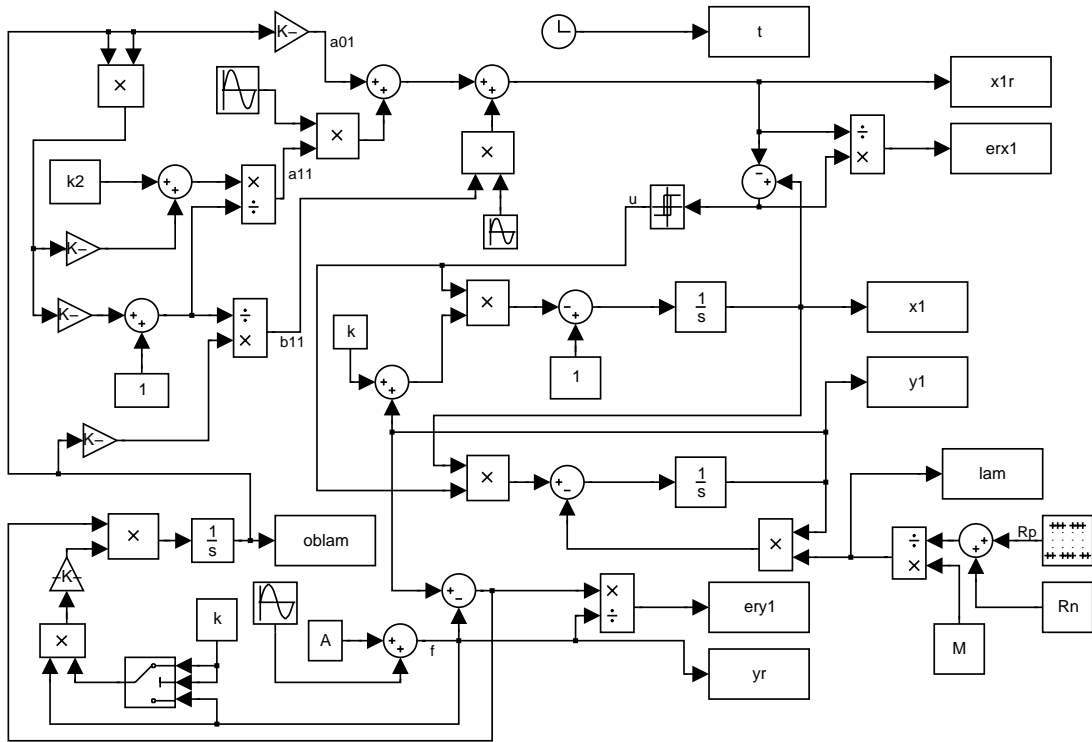


Figure 6.1: Robust boost and buck-boost converters model.

due to the indirect control that this variable undergoes. For a certain value of  $\lambda$  a concrete current reference is needed so that the internal dynamics make the output track the desired reference. If  $\lambda$  changes, a new current reference is needed to maintain the output tracking. Although the system is always under control, the unavoidable transition of the current from one reference to another is responsible for the transient tracking error at the output. This is more clearly observed in figure 6.4, where the perturbation occurs at instant  $t = 10$ . Notice that  $x_1$  never leaves its own reference, from which it is not distinguishable.

It may also be pointed out that greater gain values of the observer dynamics result in faster identification velocities and greater overshoots which, in turn, increase the

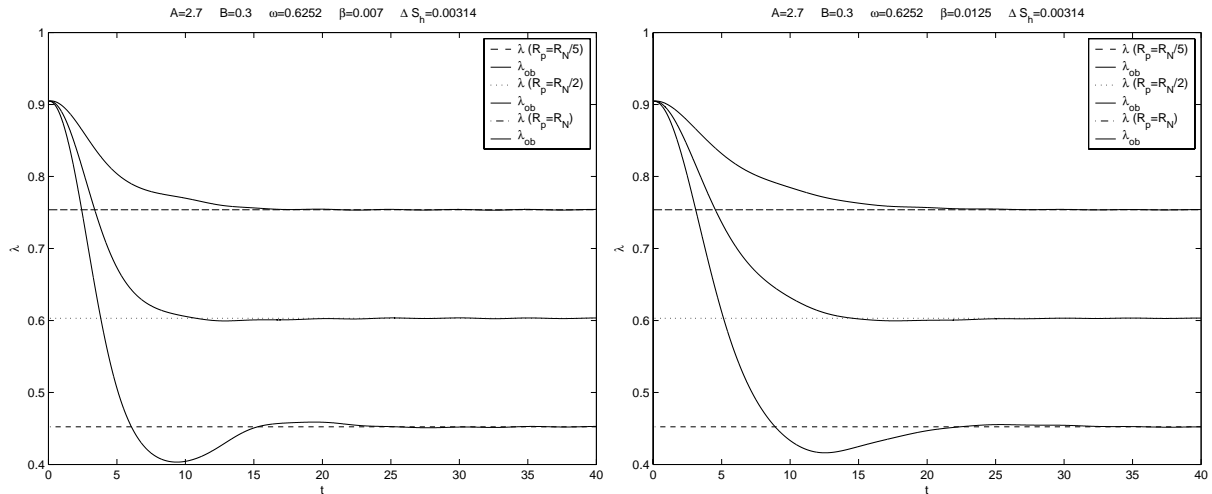


Figure 6.2: Boost and buck-boost: estimation of several disturbances.

duration of the transient. Hence, a compromise between these two features is to be taken into account when choosing  $\beta$ . The boost estimation algorithm seems to show a better relationship between velocity and overshoot. Moreover, from a certain threshold for  $\beta$  the observer does not identify the disturbance and the system unbalances. Such aspects can be observed in figures 6.5 and 6.6, where the behavior of the observer and of the state variables are plotted for several values of  $\beta$ .

Finally, the observer performance and behavior in steady state are detailed in figure 6.7. The steady state relative errors in the input and output tracking are depicted in figures 6.8 and 6.9. Notice that the output relative error does not exceed the 0.7% for both converters. For the buck-boost, this amount is almost equal to the reported in figure 5.7, where no adaptive control is implemented.

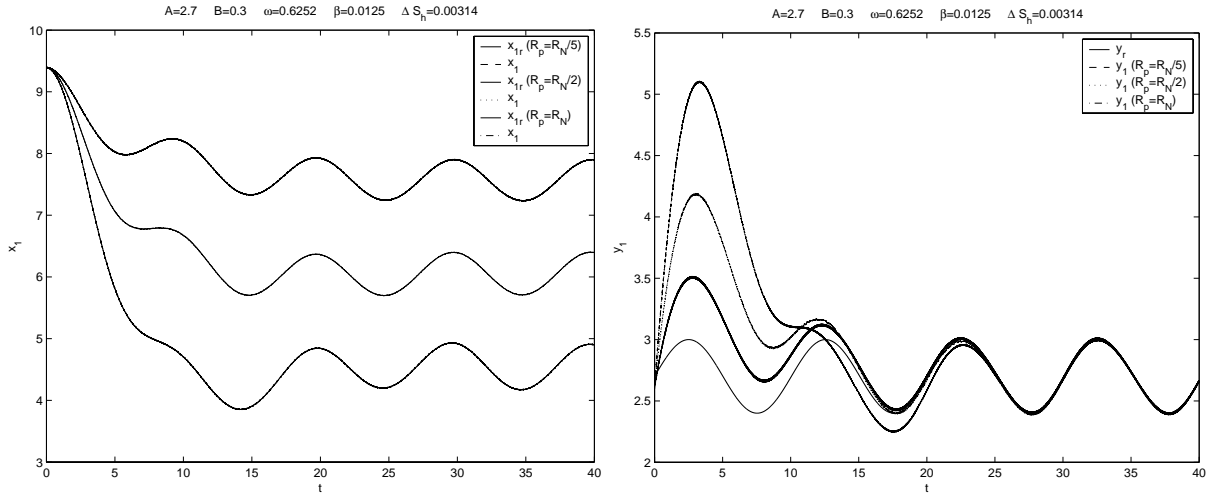


Figure 6.3: Buck-boost: state variables behavior for several disturbances, the gain  $\beta$  being constant.

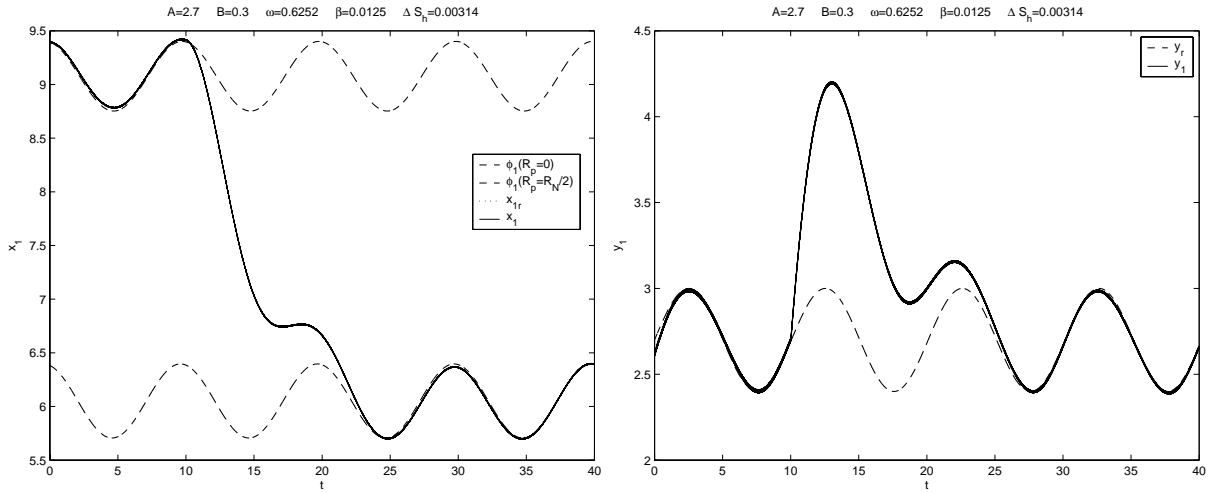


Figure 6.4: Buck-boost: current and voltage variables responding to a disturbance occurring at  $t = 10$ .

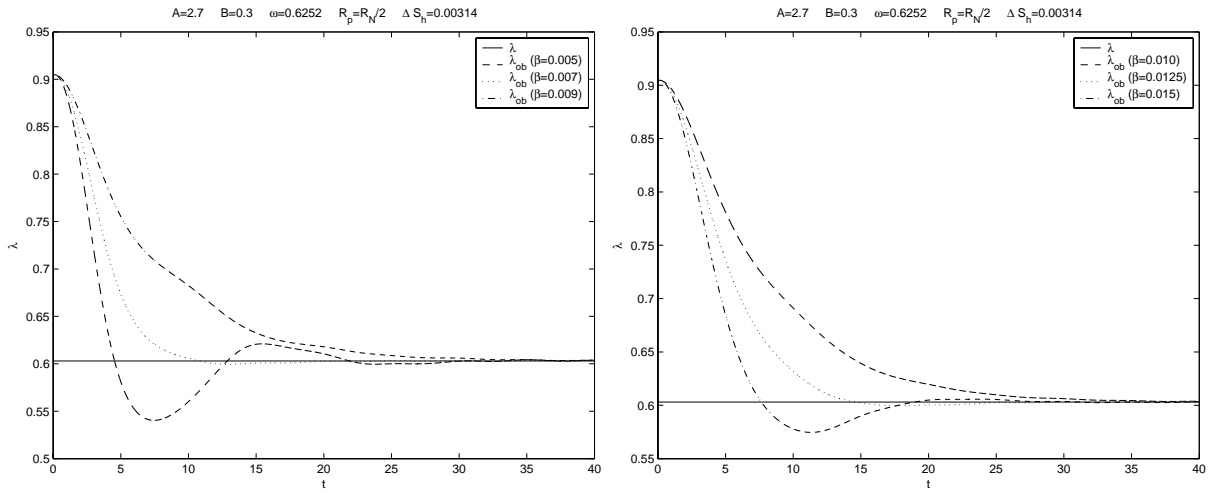


Figure 6.5: Boost and buck-boost: estimation of  $\lambda$  for several gain values of the observer.

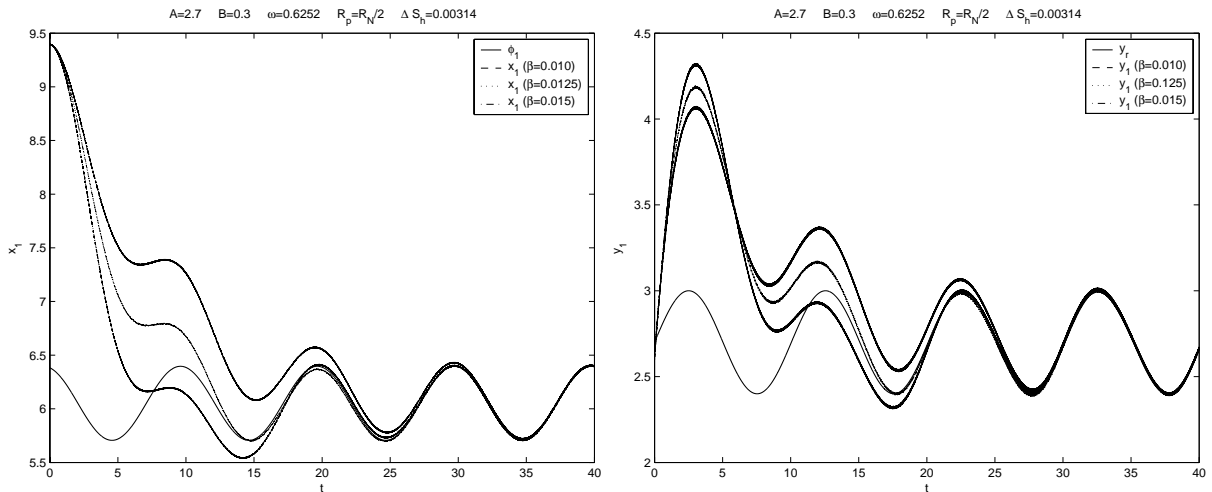


Figure 6.6: Buck-boost: details of  $x_1$  and  $y_1$  for several gain values of the observer.

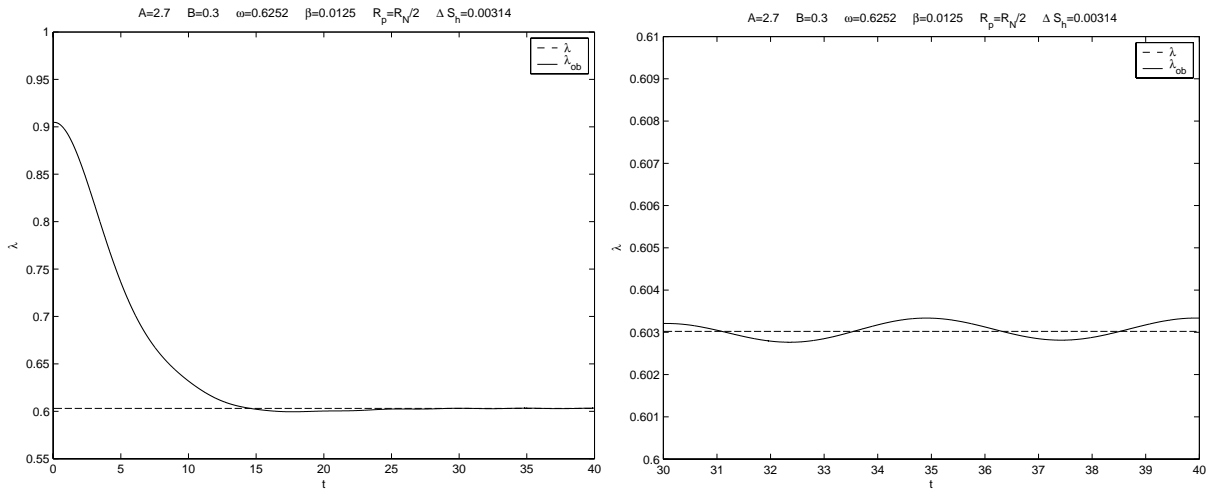


Figure 6.7: Buck-boost: details of the observer behavior.

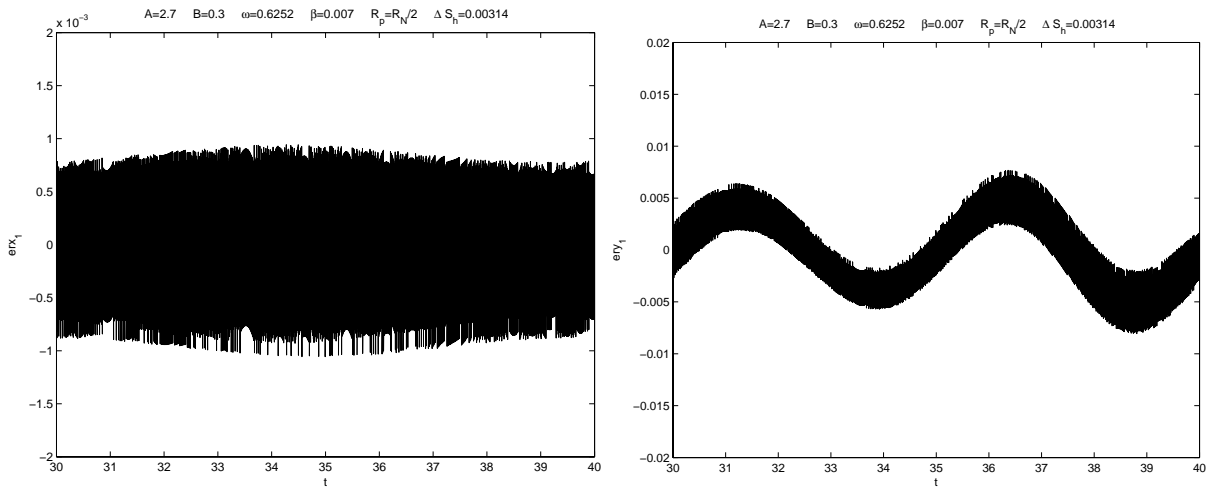


Figure 6.8: Boost: relative errors of the state variables in stationary state.



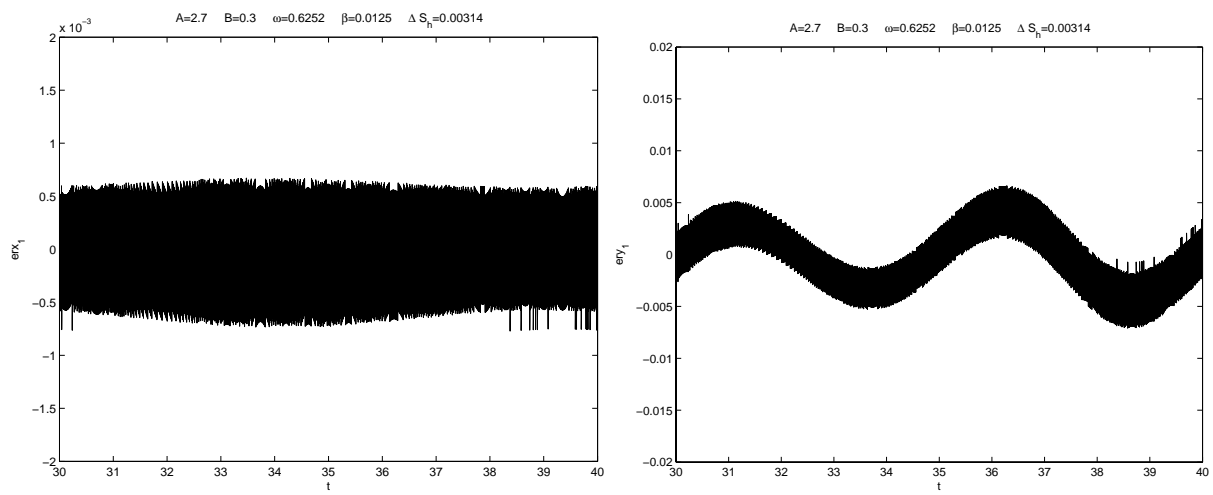


Figure 6.9: Buck-boost: relative errors of the state variables in stationary state.

# Chapter 7

## Direct control of the output voltage in nonlinear converters

In this chapter we propose a control scheme to exert direct control of the output voltage in bidirectional boost and buck-boost converters.

### 7.1 Introduction

Direct control of the output voltage was shown in chapter 4 (see proposition 4.5.4) to induce unstable tracking dynamics for our converters; that is, the state variable proportional to the input current exhibited unstable behavior while the output voltage tracked the reference. A physical implementation of such a control policy is impossible.

However, direct control is especially attractive in the sense that the robustness of the devices would be enhanced. We suggest here the use of bidirectional boost and buck-boost, as in the case of the buck system in section 3.6 for the tracking of a pure sinusoidal signal, to overcome the above mentioned difficulties.

Following an idea suggested by professor H. Sira-Ramírez, a direct tracking control

of the output voltage was performed in a bidirectional nonlinear converter [OFB96] while the input current was kept within a tolerance bandwidth through appropriate inversions of the source polarity. A similar structure has been used in [Gar00] to regulate boost and buck-boost converters (see section 2.3).

This scheme may be improved by allowing the new actuator to maintain the input current not within an interval but following a convenient signal. Meanwhile, the output voltage would be tracking the desired reference owing to the action of the original switch.

Section 7.2 establishes the theoretical basis that supports section 7.3, which contains the features associated with the robust tracking of an offset signal. The example with a sinusoidal reference and the corresponding simulation results are presented in sections 7.4 and 7.5.

## 7.2 Mathematical model

The dynamical system (4.2.1, 4.2.2) governs the behavior of the ideal, nonlinear boost and buck-boost converters. As pointed out in section 7.1, we use a bidirectional source that provides  $V_g = -|V_g|$  or  $V_g = |V_g|$  at will. This is equivalent to performing a substitution of  $V_g$  by  $|V_g|u_1$ ,  $u_1$  being a control action that takes values in the discrete set  $\{-1, 1\}$ . With the change of variables

$$x_1 = \frac{1}{|V_g|} \sqrt{\frac{L}{C}} i_L \quad x_2 = \frac{1}{|V_g|} v_C \quad t = \frac{1}{\sqrt{LC}} \tau$$

and the introduction of  $\lambda = \frac{1}{R} \sqrt{\frac{L}{C}}$  and  $u_2 = 1 - \nu$ , the system becomes dimensionless:

$$x_1' = u_1 - (ku_1 + x_2)u_2 \tag{7.2.1}$$

$$x_2' = -\lambda x_2 + x_1 u_2. \tag{7.2.2}$$

Consider the ideal steady state situation in which the state variable  $x = (x_1, x_2)^\top$  tracks the  $T$ -periodic reference  $x_r(t) = (f_1(t), f_2(t))^\top$ . The evolution of the system is given by

$$f_1' = \bar{u}_1 - (k\bar{u}_1 + f_2)\bar{u}_2$$

$$f_2' = -\lambda f_2 + \bar{u}_2 f_1,$$

$\bar{u} = (\bar{u}_1, \bar{u}_2)^\top$  being the also ideal, continuous control that allows the above situation:

$$\bar{u}_1 = \frac{f_1 f_1' + f_2 (f_2' + \lambda f_2)}{f_1 - k(f_2' + \lambda f_2)} \quad (7.2.3)$$

$$\bar{u}_2 = \frac{f_2' + \lambda f_2}{f_1}. \quad (7.2.4)$$

Notice that (7.2.3, 7.2.4) expressions make sense under the hypothesis  $f_1(t) \neq 0$  and, at the same time,  $f_1(t) \neq k[f_2'(t) + \lambda f_2(t)]$ ,  $\forall t$ . Additional conditions for the reference to allow the physical implementation of the device are  $f_1(t), f_2(t)$  bounded and  $C^1$ . The reference  $x_r(t)$  must also satisfy the control insaturation restrictions that require it to be within the area  $(-1, 1) \times (0, 1)$ . Explicitly,

$$-1 < \frac{f_1 f_1' + f_2 (f_2' + \lambda f_2)}{f_1 - k(f_2' + \lambda f_2)} < 1, \quad 0 < \frac{f_2' + \lambda f_2}{f_1} < 1. \quad (7.2.5)$$

The development of appropriate control laws implies the design of a switching logic for  $u_1$  and  $u_2$  in such a way that their mean behavior is that of  $\bar{u}_1$  and  $\bar{u}_2$ . In the case of the converters we deal with, their intrinsically variable nature makes us choose again sliding modes to achieve the control goal.

Let us define the switching surface  $s(x, t) = x(t) - x_r(t)$ , whose components are the errors associated to each state variable, that is,  $x_1 - f_1 = e_1$  and  $x_2 - f_2 = e_2$ .

**Proposition 7.2.1.** *The control law*

$$u_1 = \begin{cases} -1 & \text{if } e_1 > 0 \\ 1 & \text{if } e_1 < 0 \end{cases}, \quad u_2 = \begin{cases} 0 & \text{if } e_2x_1 - e_1(x_2 + ku_1) > 0 \\ 1 & \text{if } e_2x_1 - e_1(x_2 + ku_1) < 0 \end{cases},$$

*makes the system (7.2.1, 7.2.2) tend asymptotically to  $s(x, t) = 0$ .*

**Proof.** Consider the positive definite and continuously differentiable function

$$V(s) = \frac{1}{2}s^\top s.$$

Let us denote system (7.2.1, 7.2.2) as

$$x' = F(x, u),$$

and let  $u_{eq}^\top = (u_{1eq}, u_{2eq})$  be the equivalent control, which fulfills

$$x' - x'_r = 0 \quad \implies \quad x'_r = F(x, u_{eq}).$$

Keeping in mind that

$$s' = x' - x'_r = F(x, u) - F(x, u_{eq}),$$

a slight manipulation allows

$$\dot{V} = s^\top s' = (1 - ku_{2eq})(u_1 - u_{1eq})e_1 + [e_2x_1 - e_1(x_2 + ku_1)](u_2 - u_{2eq}),$$

which is maintained negative everywhere except on the switching surface by the proposed control law. Then, a stable sliding mode motion along the intersection of the discontinuity surfaces  $s(x, t) = 0$  occurs [Utk92]. ■

*Remark 7.2.1.* The requirement of continuous differentiability for the Lyapunov function is essential for the existence of sliding mode (see remark 3.2.1).

### 7.3 Robust tracking

A glance at the second inequation of (7.2.5) tells us that the reference  $x_r(t)$  must satisfy

$$\text{sign} \{f'_2 + \lambda f_2\} = \text{sign} \{f_1\}$$

to be a tracking candidate. Thus, in accordance with the hypothesis on the output reference which we worked with in section 4.5, we will now ask as a hypothesis:

**H.**  $f_2(t) > 0$ ,  $f(t+T) = f(t)$  and  $g_2(t) = (k + f_2)(f'_2 + \lambda f_2) > 0$ ,  $\forall t \geq 0$ .

In this situation, the intention is to force the state variable  $x_1$  to follow a constant, positive reference  $f_1(t) = c_1$ . Then, restrictions (7.2.5) become

$$-1 < \frac{f_2(f'_2 + \lambda f_2)}{c_1 - k(f'_2 + \lambda f_2)} < 1, \quad 0 < \frac{f'_2 + \lambda f_2}{c_1} < 1.$$

The following result gives sufficient conditions over  $c_1$  and  $f_2$  to have the system in the insaturation zone.

**Proposition 7.3.1.** *If hypothesis **H** is satisfied,  $c_1 > \|g_2\|_\infty$  and  $f_2 > 1 - k$ ,  $\forall t \geq 0$ , then the system is in insaturation zone.*

**Proof.** First notice that  $f'_2 + \lambda f_2 > 0$  follows from hypothesis **H**, as seen in proposition 4.5.3. Moreover,  $c_1 > \|g_2\|_\infty$  entails, together with **H**,  $c_1 > (k + f_2)(f'_2 + \lambda f_2) > 0$ ; as  $k + f_2 > 1$ , the second inequality is fulfilled. Furthermore,

$$c_1 - k(f'_2 + \lambda f_2) > f_2(f'_2 + \lambda f_2),$$

and the first restriction also holds. ■

When the possibility of having a load disturbance is considered, the robustness is guaranteed as parameter  $\lambda$  does not appear in the sliding surface equation and in the

control law. This makes the system trajectories continue pointing towards the sliding variety. However,  $\lambda$  is contained in the control insaturation conditions, which will naturally be affected. Thus, it is advisable to consider choosing a value for  $c_1$  such that these restrictions may be fulfilled in case that the load varies within an expected interval.

## 7.4 Tracking a sinusoidal reference

Let us assume that we are interested in the tracking of the sinusoidal signal

$$f_2(t) = A + B \sin \omega t, \quad A, B > 0,$$

by the state variable  $x_2$ .

From section 4.6,

$$A > B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} \implies \left\{ \begin{array}{l} f_2' + \lambda f_2 > 0 \\ A > B \implies f_2 > 0 \implies k + f_2 > 0 \end{array} \right\} \implies g_2(t) > 0,$$

thus guaranteeing the fulfillment of hypothesis **H**. A constant reference  $c_1$  to be tracked by  $x_1$  is also needed. Proposition 7.3.1 requires:

(i)  $c_1 > \|g_2\|_\infty$ . This may be accomplished with

$$c_1 > \left[ A + B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} \right] (k + A + B) = \|f_2' + \lambda f_2\|_\infty \|k + f_2\|_\infty > \|g_2\|_\infty.$$

(ii)  $f_2 > 1 - k \implies A > 1 + B - k$ .

In summary, the restrictions over  $A$ ,  $B$  and  $c_1$  are

$$A > \sup \left\{ 1 + B - k, B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} \right\}, \quad c_1 > \left[ A + B \sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} \right] (k + A + B). \quad (7.4.1)$$

In the presence of perturbations, the system is guaranteed to work in the insaturation zone if (7.4.1) is fulfilled for all  $\lambda$  within an expected interval of variation  $[\lambda_{min}, \lambda_{max}]$ .

## 7.5 Simulation results

Boost and buck-boost converters with parameters  $V_g = 50V$ ,  $R_N = 10\Omega$ ,  $L = 0.018H$  and  $C = 0.00022F$  have been chosen to test the theory developed above. The output voltage reference for our tracking purpose is

$$v_{Cr} = 135 + 15 \sin 2\pi\nu\tau \text{ V},$$

with  $\nu = 50Hz$ . The values of the corresponding dimensionless variables are  $\lambda_N = 0.9045$ ,  $\omega = 0.6252$  and

$$x_{2r} = f(t) = 2.7 + 0.3 \sin \omega t.$$

Choosing  $c_1 = 12$  for the boost converter ( $k = 0$ ) and  $c_1 = 15$  for the buck-boost case ( $k = 1$ ), both systems satisfy the restrictions of (7.4.1):

$$2.7 > \sup \{1.3, 0.36\} = 1.3, \quad c_1(k = 0) = 12 > 9.19, \quad c_1(k = 1) = 15 > 12.26.$$

The SIMULINK model depicted in figure 7.1 has been used in the simulations. Section 3.8 details the numerical integration algorithm parameters. The control actuators  $u_1$  and  $u_2$  switch at a maximum frequency of  $\nu_{sM} = 20 \text{ kHz}$ , resulting in relay hysteresis bandwidths of  $\Delta S_{h1} = 0.00628$  and  $\Delta S_{h2} = 0.00314$ . These values follow from (3.2.8), once  $\nu_{sM}$  is converted into the new variables.

The simulation results show the systems to quickly reach the reference and exhibit robust performance under load perturbations up to 200% of the nominal value  $R_N$ , that is,  $R_p = 20\Omega$ , and a frequency of  $\omega_p = 4\omega$ . Figure 7.2 portrays the effect of the disturbance on the parameter  $\lambda$ . Take into account that  $\lambda = \lambda_N + \lambda_p$ ,  $\lambda_p$  being provided by (6.2.1).

Figures 7.3 and 7.4 contain the response of the input and output state variables. No perturbation effect is observed in their behavior.



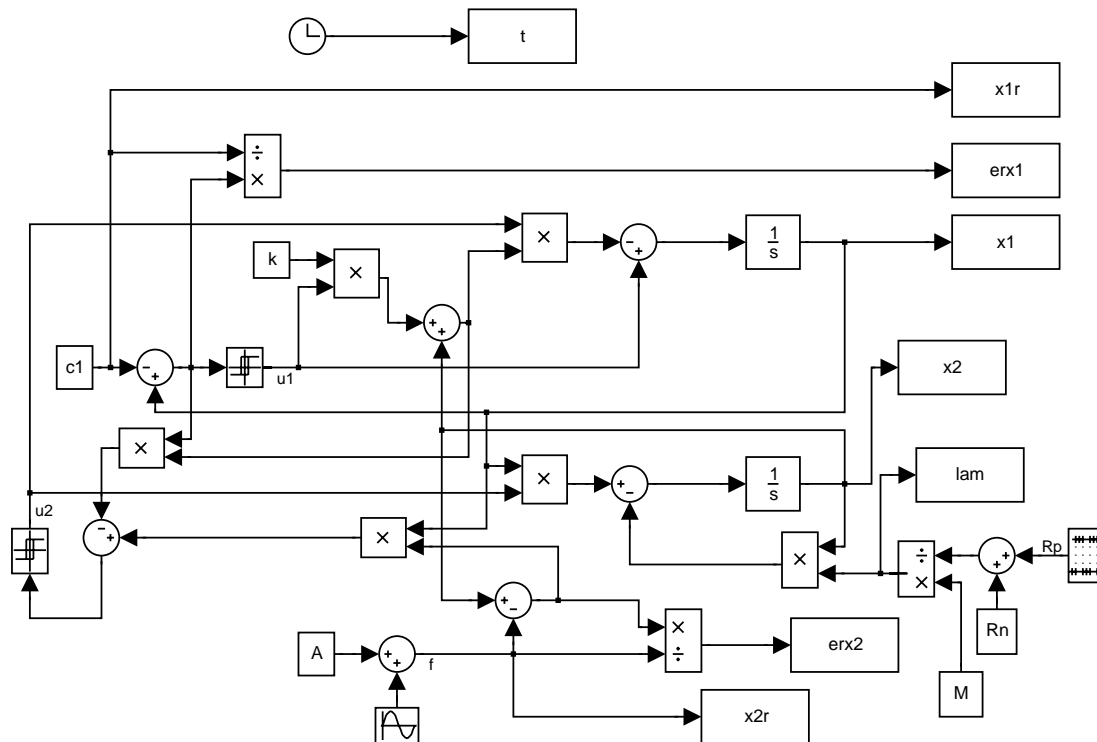
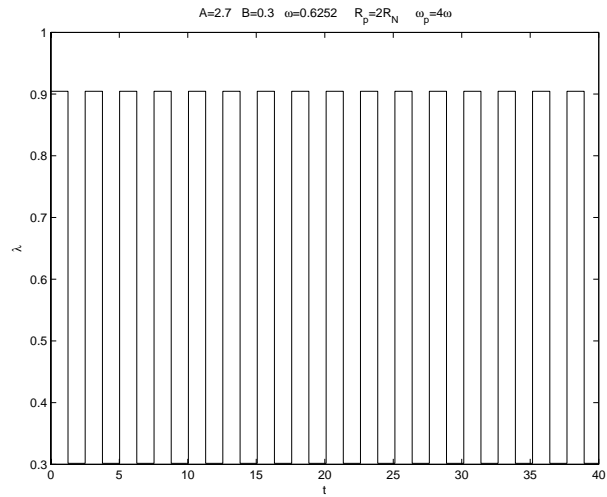
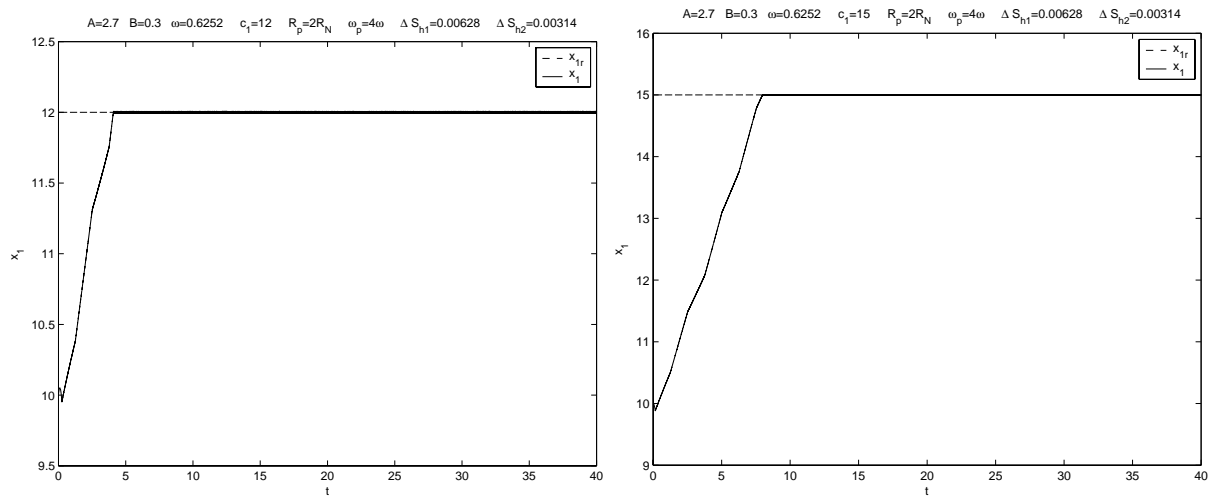
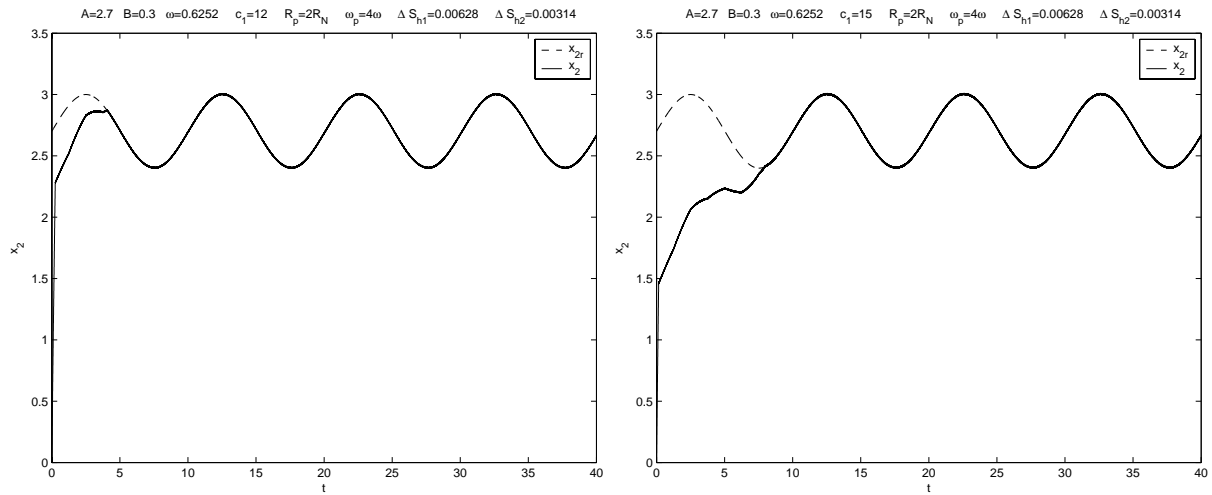
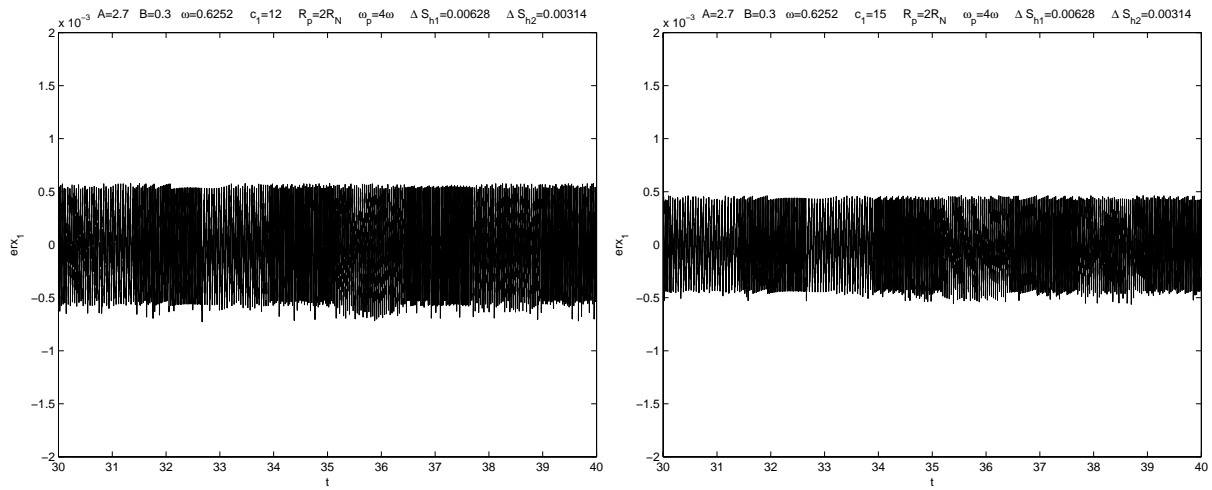
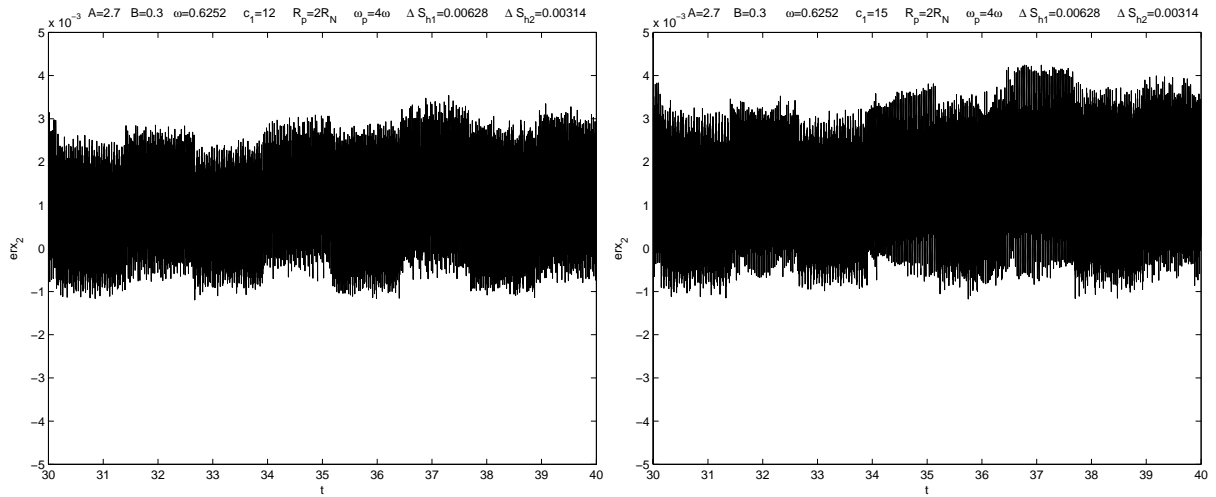
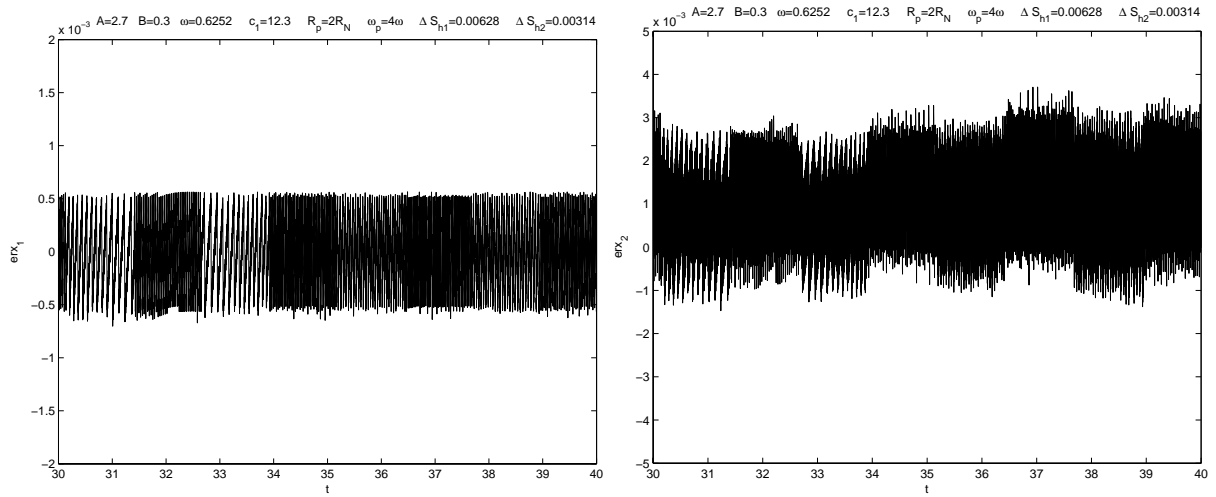


Figure 7.1: Robust full bridge boost and buck-boost converters model.

The relative errors in steady state are plotted in figures 7.5 and 7.6. Their values oscillate between 0.05% of the input variable and 0.4% of the output variable. In figure 7.7 it can be noted that when the input reference  $c_1$  is lowered the  $x_1$  relative error shows a small growth, while  $erx_2$  seems to correct slightly the steady state error observed in figure 7.6.

Figure 7.2: The perturbed parameter  $\lambda$ .Figure 7.3: Robust full bridge boost and buck-boost: the state variable  $x_1$ .

Figure 7.4: Robust full bridge boost and buck-boost: the state variable  $x_2$ .Figure 7.5: Robust full bridge boost and buck-boost: detail of the relative error  $erx_1$ .

Figure 7.6: Robust full bridge boost and buck-boost: detail of the relative error  $erx_2$ .Figure 7.7: Robust full bridge buck-boost: detail of the relative errors  $erx_1$  and  $erx_2$ .

# Chapter 8

## Contributions of this Thesis and Suggestions for Further Research

This chapter summarizes the main contributions of the thesis and outlines some ideas for future research.

### 8.1 Contributions of this thesis

The objectives of the thesis have been successfully fulfilled, and the following contributions may be pointed out.

1. The tracking problem in single input, linear systems with fixed gains has been studied in detail with the aid of module theory. Restrictions on the signals to be followed have been derived. A sliding mode strategy to achieve the control target, consisting of a procedure to modify a switching surface initially good for regulation tasks and a control law, is provided.
2. The aforementioned theory has been applied to the asymptotic tracking of signals by the output resistance of the buck converter under load disturbances.

This has been achieved by means of an appropriate choice of state variables that allows the perturbation to satisfy the matching condition.

3. Inversion-based indirect control is used to obtain exact tracking of periodic references by the load resistance of nonminimum phase, nonlinear boost and buck-boost converters. The return map allows the existence of a continuous and periodic (thus bounded) solution for the internal dynamics equation to be proved. Sufficient conditions for candidate references have also been obtained. A sliding mode control scheme has been chosen to implement the technique.
4. A general framework for an inversion-based treatment of the perfect tracking problem in a certain class of nonminimum phase, second order bilinear systems is proposed. The approach may be applicable to the situations in which the inverse problem gives rise to a differential equation of the Abel type.
5. The Harmonic Balance method has been identified as a particular case of the Galerkin method, widely used in Functional Analysis. Leray-Schauder fixed point index theory has been used to prove the existence of a sequence of approximate solutions for the internal dynamics equation. This sequence is proved to converge uniformly to the periodic solution of the ODE, and an error bound has been derived.
6. The system output exhibits a periodic and asymptotically stable behavior when indirect control using the sequence of Galerkin approximations is performed. In turn, the sequence of periodic outputs is shown to exhibit uniform convergence to the original target function under a reasonable hypothesis. Error bounds have also been obtained.

7. Approximate tracking of periodic references using a Galerkin approximation of the inverse problem solution has been successfully applied to nonlinear boost and buck-boost converters. Plant parameters are assumed to be known. A sliding mode methodology has been used as the control strategy.
8. Approximate asymptotic tracking has been achieved for load perturbed, basic, nonlinear power converters by means of an adaptive control that estimates the perturbation parameter and a first order Galerkin approximation that incorporates the on-line update into an appropriate sliding surface.
9. Sliding mode direct control of the output voltage has been performed in bidirectional boost and buck-boost converters. Periodic references have been followed, while the unstable inductor current has been independently regulated at a prescribed level. Robustness to external disturbances is a fact.
10. Simulation results validate the proposals.

## 8.2 Suggestions for further research

The knowledge about the tracking possibilities of DC-to-DC switched power converters may advance in the following directions:

- The hypothesis under which the uniform convergence of  $\{\phi_n\}$  and  $\{y_n\}$  has been established may be weakened.
- Efforts may also be directed to studying in depth the restrictions for the signals to be followed when a Galerkin approximation method is employed. This should help to clarify the possibilities of the technique.

- Further analysis of the strategy proposed in chapter 7 is necessary to achieve robust tracking results of non offset periodic signals with nonlinear converters.
- The methodologies developed in chapters 6 and 7 should be applied to other nonlinear networks such as the fourth order Čuk converter.
- Also in the field of signal generation (see [Bie99]), advantage may be taken of the contributions here presented.
- Optimization of the transient response for the asymptotic tracking control proposed in chapter 6, following the idea developed in [FZ01], is also left for further research.
- The finiteness of the switching frequency is responsible for the appearance of a steady state tracking error. Integral control has been proved to reduce its effect [BGFR00] (and before [VD90]). Its use in our tracking schemes remains to be tested.
- A universal output feedback integral controller that asymptotically regulates the output of a minimum phase, nonlinear system to a bounded time-varying reference signal with constant limit is reported in [Kha92b], using only knowledge of the relative degree and the sign of the high frequency gain. Moreover, the introduction of the integral of the output in a linear switching surface ensures robust regulation in a (non minimum phase) boost converter [BFGMR99]. The extension to non minimum phase tracking problems is currently under study.
- Passivity-based control, at the moment used for regulation tasks, might be investigated as a control tool for tracking purposes [OSME02].



- Extension to the nonlinear case of the algebraic approach to the identification of uncertain linear systems developed in [FS03] and further application to DC-to-DC power converters according to [SFF02] is our last advice.

# Appendix A

## Basic results in linear control systems from module theory

We present here an outline of the main results in linear systems from a module theoretic point of view. The material follows [Fli90a], [Fli92] and [FS93].

### A.1 Introduction

The use of module theory tools in the field of linear systems, introduced by Fliess in [Fli90a], [Fli92], provides a general algebraic setting that allows the improvement of their description and study. Moreover, it is the key to the understanding of the differentially flat systems concept and its use in the treatment of nonlinear control problems [FLMR94]. Background material may be found in [Fra99].

### A.2 Preliminaries

**Definition A.2.1.** Let  $R$  be a ring with identity and  $M$  an abelian group. Let also  $(\cdot)$  be an external product  $\cdot : R \times M \longrightarrow M$  satisfying,  $\forall a, b \in R, \forall m, n \in M$ , the

properties:

$$(i) \ a \cdot (m + n) = a \cdot m + a \cdot n,$$

$$(ii) \ (a + b) \cdot m = a \cdot m + b \cdot m,$$

$$(iii) \ (ab) \cdot m = a \cdot (b \cdot m),$$

$$(iv) \ 1 \cdot m = m.$$

Then,  $M$  is a *left  $R$ -module*.

**Definition A.2.2.** Any subgroup  $N \subset M$  is a *submodule* of  $M$  iff  $N$  is a left  $R$ -module.

**Proposition A.2.1.** *The intersection of submodules is also a submodule.* ■

**Definition A.2.3.** Let  $S \subset M$  and denote  $\{S_i\}_i$  the family of all submodules of  $M$  containing  $S$ . Then,

(i)  $[S]$  is the *submodule of  $M$  generated by  $S$* .

(ii)  $M$  is *finitely generated* iff  $S$  is finite and  $M = [S]$ . The elements of  $S$  are the *generators* of  $M$ .

### A.3 $k \left[ \frac{d}{dt} \right]$ -modules

Let  $k$  be a commutative field with a derivation  $\frac{d}{dt} = ' such that  $(a + b)' = a' + b'$  and  $(ab)' = a'b + ab'$ ,  $\forall a, b \in k$ . Consider then the ring of linear differential operators of the form$

$$k \left[ \frac{d}{dt} \right] = \left\{ \sum_{finite} a_i \frac{d^i}{dt^i}, \ a_i \in k \right\}.$$

**Proposition A.3.1.**  $k \left[ \frac{d}{dt} \right]$  is commutative iff  $k$  is a field of constants. ■

Let  $M$  be a left  $k\left[\frac{d}{dt}\right]$ -module.

**Definition A.3.1.** (i) A finite set of elements in  $M$  is a *basis* iff every element in  $M$  may be uniquely expressed as a  $k\left[\frac{d}{dt}\right]$  linear combination of such elements.

(ii)  $M$  is said to be *free* iff it has a basis.

**Definition A.3.2.** (i)  $m \in M$  is said to be a *torsion* iff  $\exists p \in k\left[\frac{d}{dt}\right]$  such that  $p(m) = 0$ .

(ii)  $T$  is a *torsion module* iff all its elements are torsions.

(iii) The subset  $T \subset M$  such that  $T = \{t \in M, t \text{ is a torsion}\}$  is a submodule called *torsion submodule* of  $M$ .

Notice that a torsion element satisfies a linear ODE with coefficients in  $k$ .

**Proposition A.3.2.** *Let  $M$  be finitely generated. Then, the following conditions are equivalent:*

(i)  $M$  is a torsion.

(ii) The dimension of  $M$  as a  $k$ -vector space is finite. ■

**Proposition A.3.3.**  $M$  is free iff its torsion submodule is trivial. ■

**Theorem A.3.4.** *Any finitely generated left  $k\left[\frac{d}{dt}\right]$ -module can be decomposed into a direct sum of its torsion submodule  $T$  and a free module  $F$ , i.e.,  $M = T \oplus F$ . ■*

## A.4 Quotient modules

Let  $M$  be an  $R$ -module and  $N \subset M$  a submodule of  $M$ . As  $N$  is, therefore, a subgroup, we can construct the quotient group

$$M/N = \{m + N, m \in M\}$$

and endow it with an  $R$ -module structure as we did with  $M$ :

$$a \cdot (m + N) = a \cdot m + N, \quad m \in M.$$

Then,

**Proposition A.4.1.** *The quotient group  $M/N$  is an  $R$ -module.* ■

**Definition A.4.1.** Let  $M$  be an  $R$ -module and  $N \subset M$  a submodule.

- (i) The elements  $m' = m \pmod{N}$  are the *residues* of  $M$  in  $M/N$ .
- (ii) The mapping  $M \rightarrow M/N$  that assigns  $m \rightarrow m' = m + N$  is the *canonical projection*.

## A.5 Linear systems and left $k \left[ \frac{d}{dt} \right]$ -modules

From now on, all modules will be left  $k \left[ \frac{d}{dt} \right]$ -modules.

**Definition A.5.1.** (i) A *linear system* is defined by a finitely generated left  $k \left[ \frac{d}{dt} \right]$ -module  $\Lambda$ .

(ii) A *linear dynamics* is a linear system  $\Lambda$  with a finite set of *inputs*  $u = (u_1, \dots, u_m)$  and such that the module  $\Lambda/[u]$  is a torsion.

(iii) The set of inputs  $u$  are said to be *independent* iff  $[u]$  is a free module.

(iv) An output  $y = (y_1, \dots, y_p)$  is a finite set of elements of the system.

**Definition A.5.2.** (i) A linear system  $\Lambda$  is *controllable* iff it is a free module.

(ii) A linear dynamics  $\Lambda$  with input  $u$  is *controllable* iff its associated linear system is controllable.

Let  $\Lambda$  be a linear dynamics with control input  $u = (u_1, \dots, u_m)$ , such that the module  $[u]$  is free. Consider also that  $\Lambda$  is affected by a perturbation  $p = (p_1, \dots, p_q)$  accomplishing  $[u] \cap [p] = \{0\}$ .

**Definition A.5.3.** Let  $\Lambda/[u, p]$  be a torsion. Then,  $\Lambda$  is a *linear perturbed dynamics*.

Let  $\bar{\Lambda} = \Lambda/[p]$ , and consider the canonical epimorphism  $i : \Lambda \longrightarrow \bar{\Lambda}$ . As  $[u] \cap [p] = \{0\}$  by hypothesis, the restriction of  $i$  to  $[u]$ , denoted  $i_{[u]}$ , is an isomorphism between  $[u]$  and  $[\bar{u}] = [i_{[u]}] = [i_{u_1}, \dots, i_{u_m}]$ . As  $\bar{\Lambda}/[\bar{u}]$  is a torsion,  $\bar{\Lambda}$  is known as the *unperturbed linear dynamics*,  $\bar{u}$  being the *unperturbed control*.

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