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Positioning Systems on Light Coordinates

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Abstract

Keywords: GPS, Relativistic Mechanics, Reference Systems, Light Coordinates, Positioning

GPS Technologies are widely spread and growing everyday. They are based on multilateration techniques, assuming Newtonian mechanics for the satellite constellation. Corrections have been made to their models to reduce error from unaccounted noise sources. Most relativistic effects are seen as mere noises and taken into account as corrections to the classical model.

Instead, we propose building a new model embedded in a more complete mathematical framework considering relativistic effects from the basis.

We suggest properties for a future usable relativistic positioning system, we have checked its existence and studied different builds. We have also worked on a two-dimensional approach for a theoretical construction of a Light Coordinate Positioning System.

We have found that the improvement of accuracy on Earth positioning is negligible due to the level of experimental corrections on classical models. Also, the proposed model is clearly harder to solve and could even be unsolvable for some metrics. On the other hand, if one wants positioning systems not bound to Earth and avoid years of experimental corrections, Relativistic Positioning Systems are needed. Another advantage is that they avoid some actual technical issues such as clock synchronicity and provide information about the metric we are immersed in.

Resum

Keywords: GPS, Mecànica Relativista, Sistemes de Referència, Coordenades Llum, Posicionament

Les tecnologies GPS són esteses arreu i en constant creixement. Basades en tècniques de multilateració, assumint mecàniques newtonianes per a la constel·lació de satèl·lits.

S'han aplicat correccions als seus models per a reduir errors de diferents fonts de soroll. La majoria d'efectes relativistes són tractats com a soroll i considerats correccions al model clàssic.

En comptes d'això, proposem construir un nou model immers en un marc matemàtic més complet, considerant efectes relativistes des de l'inici.

Suggerim propietats per a futurs sistemes de posicionament relativistes usables, en comprovem l'existència i estudiem diferents construccions. Hem treballat en una aproximació bidimensional de construcció teòrica d'un Sistema de Posicionament en Coordenades Llum.

Hem trobat que la millora de precisió en posicionament terrestre és negligible donat el nivell de correccions experimentals sobre el model clàssic. A més, el model proposat és clarament més difícil de resoldre i podria ser irresoluble per algunes mètriques. Per altra banda, per poder construir sistemes de posicionament no vinculats a la Terra evitant anys de correccions experimentals, els Sistemes de Posicionament Relativistes són necessaris. Un altre avantatge és que eviten problemes tècnics actuals com la sincronització de rellotges i proporcionen informació de la mètrica en la qual són immersos.

Resumen

Keywords: GPS, Mecánica Relativista, Sistemas de Referencia, Coordenadas Luz, Posicionamiento

Las tecnologías GPS están extendidas por todo el mundo y en auge permanente. Basadas en técnicas de multilateración, asumiendo mecánicas newtonianas para la constelación de satélites.

Se han aplicado correcciones a sus modelos para reducir errores desde diversas fuentes de ruido. La mayoría de efectos relativistas son tratados como ruidos y considerados correcciones al modelo clásico.

En vez de esto, proponemos construir un nuevo modelo inmerso en un marco matemático más completo, considerando efectos relativistas desde un inicio.

Sugerimos propiedades para futuros sistemas de posicionamiento relativistas usables, comprobamos su existencia y estudiamos distintas construcciones. Hemos trabajado en una aproximación bidimensional para la construcción teórica de un Sistema de Posicionamiento de Coordenadas Luz.

Hemos encontrado que la mejora de precisión en posicionamiento terrestre es negligible dado el nivel de correcciones experimentales sobre el modelo clásico. Además, el modelo propuesto es claramente más complicado de resolver e incluso podría ser irresoluble para algunas métricas. Por otro lado, para poder construir sistemas de posicionamiento no vinculados a la Tierra y evitar años de correcciones experimentales, son necesarios los Sistemas de Posicionamiento Relativistas. Otra ventaja de estos es que evitan problemas técnicos actuales como la sincronización de relojes y proporcionan información de la métrica en la que están inmersos.

"The Universe is under no obligation to make sense to you."

– Neil deGrasse Tyson

"It is always pleasant to have exact solutions in simple form at your disposal."

– Karl Schwarzschild

"My common sense is tingling."

– Wade Wilson

"I would suggest that science is, at least in my part, informed worship."

– Carl Sagan

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When I started thinking about this topic, during a Relativity Lesson, little did I know how much time I would be spending around it. It has been almost 10 years from those passionate discussions during class breaks with Jaume, whom I thank, not only for paving the way towards this work, but for sharing those moments of intellectual ingenuity, dreams and bibliography. Thanks to my advisor, for accepting this work and suffering my tardiness and occasional disappearances. I am sure adapting the rhythm of this work to the comings and goings of my labor and entrepreneurial activities. Thanks again. On a more general note, I would also like to thank all of the friends for whom I have been a nuisance, the ones who have helped me build this thesis, and the ones who just were there when I needed some rest. Of course, thanks to my family too. For their patience and support, even when they understood very little of what I was doing during all these years. Thanks also to the (mostly) faceless authors and researchers I have been reading avidly during the whole process. A reference and mention at the end of this document symbolizes a stone on which I base this building. And just in case I forget someone, know that if you have been involved with the making of this thesis, be it directly or indirectly: Thanks.

David Rivera Muñoz, 21 October 2016

Contents

Chapter 1

Introduction

About 10 years ago now, GPS technology was something widely spread on military, agricultural, civil and urban engineering, as in some other various related specific fields. It was seen as a growing consumer technology but nobody could anticipate the actual scale of this growth. As of today we can safely assume every reader owns *at least* one device with GPS capabilities. Almost every mid-range private car includes by default some sort of GPS. So does every smartphone model built in the last 5 years, wearables, sports watches, mobile devices of different sorts, etc... It is an ubiquitous technology nowadays.

It is widely known that the military origin of this technology made difficult its commercial use. Due to alleged risks for the national security of the United States of America, its government decided unilaterally to encrypt the first part of the message sent from satellites to the receptors (introducing an error commonly named *anti-spoofing* error), and even decided to add noise to the satellite clocks (introducing a second artificial error named *selective availability* error).

These errors have been for a long time the most limiting issue on GPS for commercial use, intentionally crippling a technology that should be having a precision on the order of *meters* and taking the confidence interval to the order of *hundreds of meters*. Fortunately, during the Clinton administration, the selective availability error was retired. Although the anti-spoofing error has been promised to be retired too, it has never been. Anyway, due to the improved technology we have nowadays, the encryption of the P-code generating this error does not affect the precision of the GPS system in an order of magnitude.

This technical improvement on the GPS systems has been parallel to its growth. This has been a technical evolution on the diverse aspects of the technology (mostly on the user and control segment), based mainly on modelling the different errors and noises on the inputs used by the system. Noises such as the interference of the ionosphere, for instance, have been widely studied and modelled.

Meanwhile, the main basis of the theory (or the mathematical framework in which it is embedded) has remained unaltered since its creation and design during the 1960s. Euclidian equations are hold as valid by adhering the necessary correction values for each case.

Truth be told, the system responds excellently and possibly would not improve results by changing the framework in which it is formulated, at least for the Earth Global Positioning Systems.

This can be partly attributed to this process of modelling the noises and errors of the system during this last fifty years, corrections to the general model are such that very little (if anything at all) can be gained as to the precision of the system and its results, at least comparing the confidence interval provided by relativistic corrections versus other error sources.

But, why has the core of the theory never been corrected? Why not establish the same equations in physical and/or mathematical frameworks much more accurate to the physical environments known nowadays? Well, mainly because the classical system, as we just said, works pretty well. And even if General Relativity is a subject older than GPS itself, is still quite an obscure one in most engineering schools. Even for the initiated on such subjects, it might not be easy to just formulate GPS equations using General Relativity, since their core theories present some fundamental discrepancies. This difficulties shall be discussed during this text.

The purpose of this work is not really to describe rigorously new GPS equations on a purely relativistic environment, but to exemplify the need of a proper common framework to work on this kind of fields (namely, positioning systems) from a relativistic point of view. That is, to provide the reader with some fundamentals on relativistic location and coordinate systems, the problem that arises modelling them from a classical (Newtonian) perspective and to fiddle with them on some practical examples.

Summary of the Chapters

We will present now to the reader with a summary of the chapters of this work in order to have a general idea of the topics we are going to address. We encourage the reader to dive more on these subjects, since most of them are just some basic notions in order to achieve a certain degree of self-contained narrative. Anyway, please have in mind you might feel some of the sections are but a mere enumeration of oversimplified descriptions.

Chapter 2

In this Chapter we will give an overview of the classical GPS System. This overview will be a refreshment of the basic triangulation model, the different models and standards, its parts and their classifications. We will also have a brief review of the main noise sources and its magnitudes.

- First we will review the basics of the GPS model.
- We will revisit some GPS concepts in order to stablish some common ground with the reader.
- Briefly comment the three main standards for GPS Systems:
  - The U.S. NAVSTAR GPS
  - GLONASS
  - EGNOS

- We will give some numeric values for the typical Biases and Errors in GPS Systems, this way we can have some insight on the effect of the relativistic corrections on the overall results of the system

Chapter 3

This Chapter presents and slowly details the theoretical basis for Light Coordinate Systems. The properties we need for them to be possible, useful and desirable. We will define objectives for them, and will classify them according to the properties defined they fulfill.

One of the first things to do here is to distinguish the coordinate system as a mathematical object and its realization as a physical object.

First we will talk about Location Systems. A Location System should at least fulfill these properties:

- it should be *generic*, i.e. that can be constructed in any space-time of a given class,
- it should be (gravity-)*free*, i.e. that the knowledge of the gravitational field is not necessary to construct it,
- it should be *immediate*, i.e. that every event may know its coordinates without delay.

We will show then a construction basis for a Relativistic Positioning System in a similar way to the Classical GPS, and present Auto-locating Positioning Systems and Autonomous Positioning Systems.

Chapter 4

Chapter 4 present a Two-dimensional approach to Light Coordinate Systems. Due to the difficulty inherent to the study of the generic four-dimensional relativistic positioning systems, which requires a rigorous previous training on simple and particular scenarios, we have decided that we will present in this text a two-dimensional approach to relativistic positioning systems introducing the basic features that define them. This whole study will be mainly based on the works by Bartolomeu Coll, Joan Josep Ferrando and Juan Antonio Morales-Lladosa

This two-dimensional simplification will provide us with the advantage of being able to use precise and explicit diagrams improving the qualitative comprehension of general four-dimensional positioning systems. Even further, thanks to this two-dimensional approach, the scenarios will admit simple and explicit analytic results which makes it better for divulgative purposes.

- We will present this two-dimensional approach with a simplification of the proposed construction of a Relativistic Positioning System with two emitters.

- We will calculate emission coordinates and emitter trajectories for Minkowski space-time (flat space-time, if you will).

  We will see that the trajectories in Minkowski plane γ_1, γ_2 of the emitters of a stationary positioning system in emission coordinates $\{\tau^1, \tau^2\}$ are parallel straight lines of the form:

$$\gamma_1 \quad \equiv \quad \begin{cases} \tau^1 = \tau^1 \\ \tau^2 = \frac{1}{\omega}(\tau^1 - q - \sigma) \equiv \varphi_1(\tau^1) \end{cases} \tag{1.1}$$

$$\gamma_2 \quad \equiv \quad \begin{cases} \tau^1 = \omega\tau^2 - q + \sigma \equiv \varphi_2(\tau^2) \\ \tau^2 = \tau^2 \end{cases} \tag{1.2}$$

- We will see then what happens with the emitter's positioning and their dynamical parameters (slope, separation and synchronization) on the same Minkowski space-time. In terms of the emitter positioning data $\{\tau_P^1, \tau_P^2; \bar{\tau}_P^1, \bar{\tau}_P^2\}$ and $\{\tau_Q^1, \tau_Q^2; \bar{\tau}_Q^1, \bar{\tau}_Q^2\}$ at two different events P and Q of any user, the slope, separation and synchronization parameter $\omega > 1$, $q > 0$ and σ, characterizing the parallel trajectories of the emitters in the grid $\{\tau^1\} \times \{\tau^2\}$, are given by:

$$\begin{aligned} \omega &= \frac{\Delta\bar{\tau}^1}{\Delta\tau^2} \equiv \frac{\bar{\tau}_P^1 - \bar{\tau}_Q^1}{\tau_P^2 - \tau_Q^2} \\ q &= \frac{1}{2}[\tau_P^1 + \bar{\tau}_P^1 + \frac{\Delta\bar{\tau}^1}{\Delta\tau^2}(\tau_P^2 + \bar{\tau}_P^2)] \\ \sigma &= \frac{1}{2}[\tau_P^1 + \bar{\tau}_P^1 - \frac{\Delta\bar{\tau}^1}{\Delta\tau^2}(\tau_P^2 + \bar{\tau}_P^2)] \end{aligned} \tag{1.3}$$

- After that, we will repeat these calculations for Schwarzschild plane:

  In Schwarzschild plane, the trajectories γ_1, γ_2 of the emitters of a stationary positioning system in emission coordinates $\{\tau^1, \tau^2\}$ are parallel straight lines of the form:

$$\gamma_1 \equiv \begin{cases} \tau^1 = \tau^1 \\ \tau^2 = \frac{1}{\omega}(\tau^1 - q - \sigma) \equiv \varphi_1(\tau^1) \end{cases} \tag{1.4}$$

$$\gamma_2 \equiv \begin{cases} \tau^1 = \omega\tau^2 - q + \sigma \equiv \varphi_2(\tau^2) \\ \tau^2 = \tau^2 \end{cases} \tag{1.5}$$

  The *slope parameter* ω, the *separation parameter* q and the *synchronization parameter* σ are related to the Schwarzschild radius r_s, the radial parameters ρ_i and the synchronization instants τ_0^i of the emitters by:

$$\begin{aligned} \omega &= \frac{\lambda_2}{\lambda_1} = \omega(\rho_1, \rho_2) \\ q &= \frac{x_1 - x_2}{\lambda_1} = r_s Q(\rho_1, \rho_2) \\ \sigma &= \tau_0^1 - \omega\tau_0^2 \end{aligned} \tag{1.6}$$

$\omega(\rho_1, \rho_2)$ and $Q(\rho_1, \rho_2)$ being the bi-parametric expressions:

$$\begin{aligned} \omega(\rho_1, \rho_2) &\equiv \sqrt{\frac{\rho_1(\rho_2 + 1)}{\rho_2(\rho_1 + 1)}} \\ Q(\rho_1, \rho_2) &\equiv \sqrt{\frac{\rho_1}{\rho_1 + 1}} \left[\rho_1 - \rho_2 + \ln\frac{\rho_1}{\rho_2}\right] \end{aligned} \tag{1.7}$$

Conclusions

In this chapter we will comment on results and future lines of work on the topics aforementioned.

Appendix A

In this Appendix we have summarized basic Differential Geometry concepts in order to make this work as auto contained as possible.

Chapter 2

General Introduction to Positioning Systems

In this chapter we will describe as widely as possible the inner workings of GPS Systems. In order to do that, we will look at the definitions in a general sense so we can easily compare it with the more general state of the art of the relativistic approach.

2.1 Classical GPS Systems: GPS Model

The main concept around GPS systems is, mainly, to obtain an equation system (generally overdetermined) in order to allow us to know the coordinates for the receptor in a given reference system. Intuitively, we can think of it as a triangulation problem. Let a certain receptor be in an unknown position, the only information we have about this device is that it is located in a certain reference framework AND we have a clock on it to measure the passing of time. At the moment, we cannot guess in any way its position. Let now be a certain emitter, the position of which is known by us. In some way, we can try and find the relative distance between both receptor and emitter, thus getting information about the position of the receptor. With enough emitters, we should be able to narrow enough the position for any given reference framework.

This sufficient number of emitters will progressively limit the available points for the receptor to be positioned in. In Figure 2.1 we show an example of this process of removing uncertainty in the system via adding emitter redundancy.

2.2 GPS concepts

2.2.1 Positioning from Space

Positions can be determined in different ways. We can classify the positioning mode depending on what the position is determined with respect to. Usually, we will be determining it with respect to a geocentric coordinate system, with respect to another point, or within the context of several

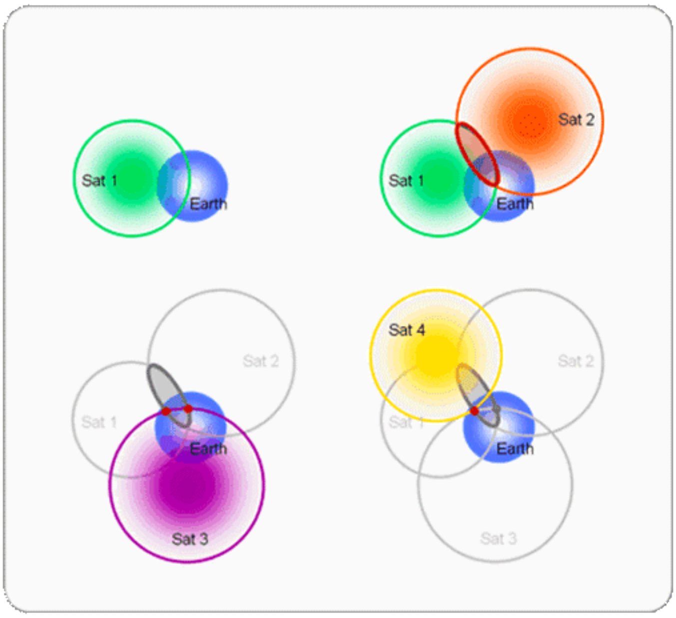

Figure 2.1: The very basics of GPS positioning

points.

We will talk about:

1. **Point Positioning**: With respect to a well-defined coordinate system, usually by three coordinate values (we assume that the "well-defined" coordinate system is itself positioned and oriented with respect to earth).
2. **Relative Positioning**: With respect to another point, taking one point as the origin of a local coordinate system.

In the same way we will distinguish between **static positioning** and **kinematic positioning**. Both are quite self-explanatory.

2.2.2 Satellite Positioning

When we speak about Satellite Positioning, we refer to positioning by means of satellites, not positioning of the satellites. In a nutshell (as seen in 2.2): We wish to determine the position ($\mathbf{R}_i$) of the ith **antenna** (the device that receives the satellite signal and is then positioned). We know the position ($\mathbf{r}^j$) of the jth **satellite** (emitting the signal) and have thus to measure the **range vector** between the two ($\mathbf{e}_i^j \rho_i^j$). Depending on how the range vector is measured, we get different satellite positioning techniques.

To predict accurately the position of the satellite in time ($\mathbf{r}^j(t)$) is a rather difficult proposition. The task of predicting the **ephemeris** - which is the proper name of $\mathbf{r}^j$ as a function of time - calls for a special knowledge of satellite dynamics which historically belongs in the field of celestial mechanics. The ephemerides are usually determined and predicted (in time) by the operators of the satellite system.

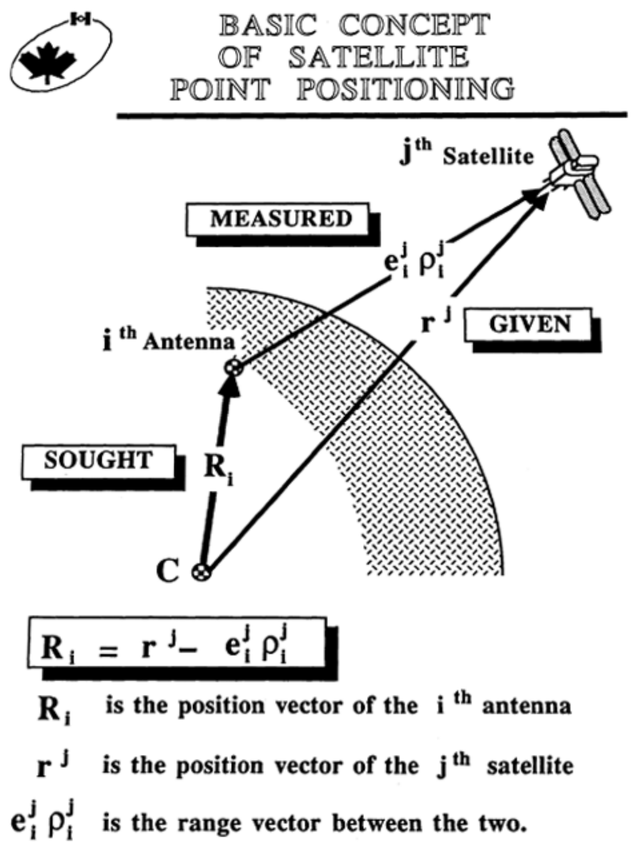

Figure 2.2: Principles of Satellite point Positioning

It should be clear that if the antenna changes its position $\mathbf{R}_i$ in time (kinematic positioning), the $\mathbf{R}_i$ has to be continuously re-evaluated. If the antenna is stationary, then the re-evaluation should always lead to the same result and we will have a redundancy of results which will provide a more robust system against noise.

As we mentioned, it is often advantageous to connect a whole family of points into a **Network**. When these points are positioned together, connected by means of redundant links, they make for a geometrically stronger configuration.

In this Networks, each link may be considered as an interstation vector and each pair of adjacent points as a pair of points to be positioned relatively with respect to each other. The points linked together by a Network are called **control points**. Once all the interstation vectors are known - derived from observations - then the Network can serve as a medium for transmitting positions from one end to each other. As such, it naturally suffers from error accumulation, and care must be taken to curb the systematic error propagation.

2.3 GPS Systems

GPS Systems, while being unidirectional, are just communications systems. Being so, they dispose of a distributed segmented structure: Space Segment, Control Segment and User Segment.

2.3.1 U.S. NAVSTAR GPS

We will expand a little bit on this System, because is the most widely spread.

The design of GPS is based partly on similar ground-based radio-navigation systems, such as LORAN and the Decca Navigator, developed in the early 1940s and used by the British Royal Navy during World War II.

In 1956, the German-American physicist Friedwardt Winterberg proposed a test of general relativity — detecting time slowing in a strong gravitational field using accurate atomic clocks placed in orbit inside artificial satellites.

Special and general relativity predict that the clocks on the GPS satellites would be seen by the Earth's observers to run 38 microseconds faster per day than the clocks on the Earth. The GPS calculated positions would quickly drift into error, accumulating to 10 kilometers per day. The relativistic time effect of the GPS clocks running faster than the clocks on earth was corrected for in the design of GPS.

Space Segment

The space segment (SS) is composed of the orbiting GPS satellites, or Space Vehicles (SV) in GPS parlance. The GPS design originally called for 24 SVs, eight each in three approximately circular orbits, but this was modified to six orbital planes with four satellites each. The six orbit planes have approximately 55° inclination (tilt relative to the Earth's equator) and are separated by 60° right ascension of the ascending node (angle along the equator from a reference point to the orbit's intersection). The orbital period is one-half a sidereal day, i.e., 11 hours and 58 minutes so that the satellites pass over the same locations or almost the same locations every day. The orbits are arranged so that at least six satellites are always within line of sight from almost everywhere on the Earth's surface. The result of this is that the four satellites are not evenly spaced (90°) apart within each orbit. In general terms, the angular difference between satellites in each orbit is 30°, 105°, 120°, and 105° apart, which sum to 360°.

Orbiting at an altitude of approximately 20,200 km; orbital radius of approximately 26,600 km, each SV makes two complete orbits each sidereal day, repeating the same ground track (the projection of the satellite path on the surface of the Earth) each day. This was very helpful during development because even with only four satellites, correct alignment means all four are visible from one spot for a few hours each day. For military operations, the ground track repeat can be used to ensure good coverage in combat zones.

As of December 2012, there are 32 satellites in the GPS constellation. The additional satellites improve the precision of GPS receiver calculations by providing redundant measurements. With the increased number of satellites, the constellation was changed to a nonuniform arrangement. Such an arrangement was shown to improve reliability and availability of the system, relative to a uniform system, when multiple satellites fail. About nine satellites are visible from any point on the ground at any one time, ensuring considerable redundancy over the minimum four satellites needed for a position.

Control Segment

The control segment is composed of:

1. a master control station (MCS),
2. an alternate master control station,
3. four dedicated ground antennas, and
4. six dedicated monitor stations.

The MCS can also access U.S. Air Force Satellite Control Network (AFSCN) ground antennas and NGA (National Geospatial-Intelligence Agency) monitor stations. The flight paths of the satellites are tracked by dedicated U.S. Air Force monitoring stations, along with shared NGA monitor stations . The tracking information is sent to the Air Force Space Command MCS. Then 2 SOPS (2d Space Operation Squadrons) contacts each GPS satellite regularly with a navigational update using dedicated or shared (AFSCN) ground antennas. These updates synchronize the atomic clocks on board the satellites to within a few nanoseconds of each other (accuracy of $\pm 10\mathrm{ns}$), and adjust the ephemeris of each satellite's internal orbital model. The updates are created by a Kalman filter that uses inputs from the ground monitoring stations, space weather information, and various other inputs.

Satellite maneuvers are not precise by GPS standards—so to change a satellite's orbit, the satellite must be marked unhealthy, so receivers don't use it. After the satellite maneuver, engineers track the new orbit from the ground, upload the new ephemeris, and mark the satellite healthy again.

The Operation Control Segment (OCS) currently serves as the control segment of record. It provides the operational capability that supports GPS users and keeps the GPS system operational and performing within specification.

User Segment

The user segment is composed of hundreds of thousands of U.S. and allied military users of the secure GPS Precise Positioning Service, and tens of millions of civil, commercial and scientific users of the Standard Positioning Service. In general, GPS receivers are composed of an antenna, tuned to the frequencies transmitted by the satellites, receiver-processors, and a highly stable clock (often a crystal oscillator). They may also include a display for providing location and speed information to the user. A receiver is often described by its number of channels: this signifies how many satellites it can monitor simultaneously. Originally limited to four or five, this has progressively increased over the years so that, as of 2007, receivers typically have between 12 and 20 channels.

GPS receivers may include an input for differential corrections, using the RTCM SC-104 format. This is typically in the form of an RS-232 port at $4.8\,\mathrm{kbitps}$ speed. Data is actually sent at a much lower rate, which limits the accuracy of the signal sent using RTCM. Receivers with internal DGPS receivers can outperform those using external RTCM data. As of 2006, even low-cost units commonly include Wide Area Augmentation System (WAAS) receivers. A typical GPS receiver with integrated antenna.

Many GPS receivers can relay position data to a PC or other device using the NMEA 0183 protocol. Although this protocol is officially defined by the National Marine Electronics Association (NMEA), references to this protocol have been compiled from public records, allowing open source tools like gpsd to read the protocol without violating intellectual property laws. Other proprietary protocols exist as well, such as the SiRF and MTK protocols. Receivers can interface with other devices using methods including a serial connection, USB, or Bluetooth.

2.3.2 GLONASS

The first satellite-based radio navigation system developed in the Soviet Union was Tsiklon, which had the purpose of providing ballistic missile submarines a method for accurate positioning. The main problem with the system was that, although highly accurate for stationary or slow-moving ships, it required several hours of observation by the receiving station to fix a position, making it unusable for many navigation purposes and for the guidance of the new generation of ballistic missiles. In 1976, the government made a decision to launch development of the ”Unified Space Navigation System GLONASS”.

Originally, GLONASS was designed to have an accuracy of 65 metres, but in reality it had an accuracy of 20 metres in the civilian signal and 10 metres in the military signal. The first generation GLONASS satellites were 7.8 metres tall, had a width of 7.2 metres, measured across their solar panels, and a mass of 1,260 kilograms.

Space Segment

Over the three decades of development, the satellite designs have gone through numerous improvements, and can be divided into three generations: the original GLONASS (since 1982), GLONASS-M (since 2003) and GLONASS-K (since 2011). Each GLONASS satellite has a GRAU designation 11F654, and each of them also has the military ”Cosmos-NNNN” designation.

Control Segment

The ground control segment of GLONASS is almost entirely located within former Soviet Union territory, except for a station in Brasilia, Brazil. The Ground Control Center and Time Standards is located in Moscow and the telemetry and tracking stations are in Saint Petersburg, Ternopil, Eniseisk and Komsomolsk-na-Amure.

User Segment

Septentrio, Topcon, C-Nav, JAVAD, Magellan Navigation, Novatel, Leica Geosystems, Hemisphere GNSS and Trimble Inc produce GNSS receivers making use of GLONASS. NPO Progress describes a receiver called GALS-A1, which combines GPS and GLONASS reception. SkyWave

Mobile Communications manufactures an Inmarsat-based satellite communications terminal that uses both GLONASS and GPS. As of 2011, some of the latest receivers in the Garmin eTrex line also support GLONASS (along with GPS). Garmin also produce a standalone Bluetooth receiver, the GLOTM for Aviation, which combines GPS, WAAS and GLONASS. Various smartphones from 2011 onwards have integrated GLONASS capability, including devices from Xiaomi Tech Company (Xiaomi Phone 2), Sony Ericsson, ZTE, Huawei, Samsung (Galaxy Note, Galaxy Note II, Galaxy S3, Galaxy S4), Apple (iPhone 4S, iPhone 5, iPhone 5C, iPhone 5S, iPhone 6 and iPhone 6 Plus, iPhone 6s and iPhone 6s Plus), iPad Mini (LTE models only), iPad Mini 2 (LTE models only), iPad Mini 3 (LTE models only), iPad Mini 4 (LTE models only), iPad (3rd generation and 4th Generation, 4G and LTE models only [respectively]), iPad Air (LTE models only) and iPad Air 2 (LTE models only) and Apple's flagship iPad Pro, HTC, LG, Motorola and Nokia.

2.3.3 EGNOS

The European Geostationary Navigation Overlay Service (EGNOS) is the European regional Satellite Based Augmentation System (SBAS) for the Global Positioning System and the Global Navigation Satellite System, which aims to enhance performance of these two Global Navigation Satellite Systems (GNSS) over Europe for navigation and other domains, including time and frequency transfer and geodetic applications. EGNOS is developed by a European consortium under ESA contract. Since 2002, a geodetic consortium lead by NMA (Norweigian Mapping Authority) has supported ESA in determining accurate coordinates for EGNOS reference stations. The EGNOS Reference frame is determined through the geodetic coordinates of the Ranging and Integrity Monitoring Systems (RIMS). The RIMS perform GPS and GLONASS measurements and transfer the observations to the EGNOS Central Processing Facilities (CPF). There, corrections and integrity parameters are computed which are broadcast via geostationary satellites to users.

A key issue is the explicit knowledge of the reference frame determined by the EGNOS corrections. Its operations require accurate and reliable ground-station antenna coordinates in a well-defined global geodetic reference frame. Moreover, for users not using the same reference frame for their geo-referenced databases, transformations between the EGNOS and the user reference frames are required.

EGNOS provides a service aimed to operate for at least 15 years. Among other applications, EGNOS is to be used for safety-critical applications in civil aviation. This implies that accuracy, continuity and integrity have to be mantained unmodified throughout the operational life of the system.

Space Segment

The EGNOS space segment is composed of three geostationary satellites centred over Europe:

- Inmarsat-3 AOR-E (Atlantic Ocean Region East) stationed at 15.5° W
- Inmarsat-3 IOR-W (Indian Ocean Region East) stationed at 25.0° E
- ESA-Artemis stationed at 21.5° E

The main criteria followed in selecting the satellites position were:

- Improve the measurement geometry and hence the system availability.
- Maximise the visibility angle diversity and hence minimise the risk of signal blocking.
- Provide dual geostationary coverage (minimum) within the core service area.

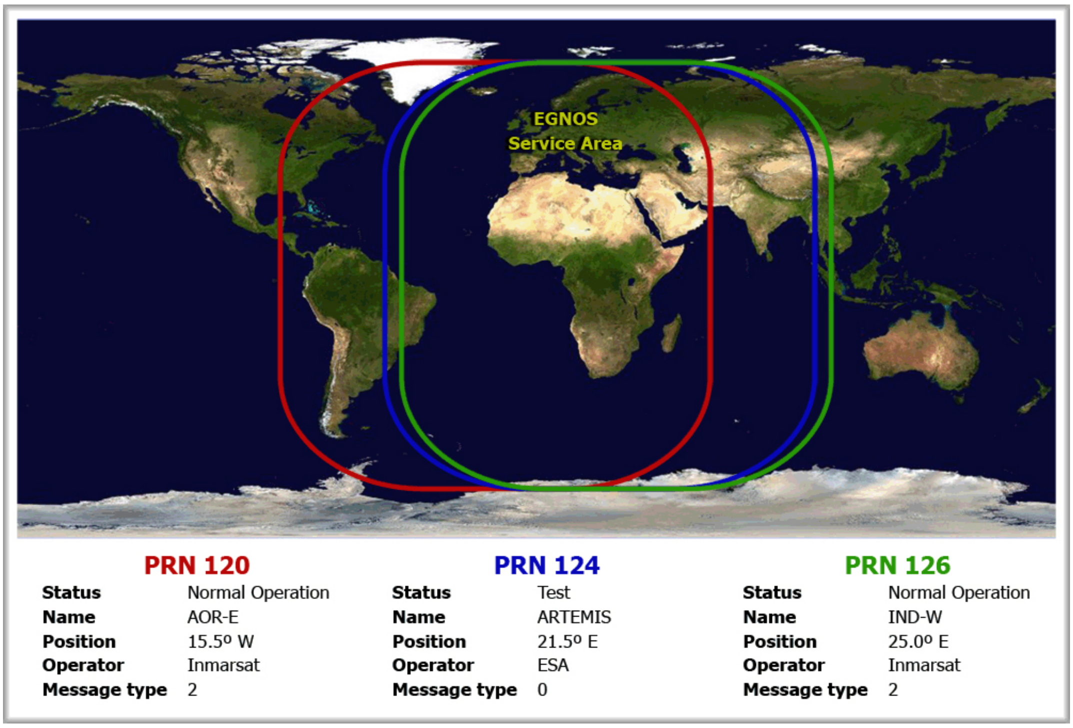

Figure 2.3: EGNOS satellite coverage

Their coverage is illustrated in Figure 2.3, together with the core European Service Area. EGNOS users in this core area will be able to track at least two geostationary satellites.

2.4 Typical Biases and Errors

2.4.1 Confidence Regions and Relative Errors

In theory as well as in practice, it is very important to be able to quantify the goodness of positions, i.e., their accuracy. This can be done in many different ways, the most reasonable by means of **confidence regions**. Confidence regions are mainly ellipsoid figures which are formulated in such a way that their volume or area contains a preselected level of probability that the true value of the position lies within the figure.

It is also costumary in some other works to use a proportional error presentation instead of confidence regions. This representation is obtained by dividing the standard error in the desired direction by the distance from the origin of the coordinate system used. This kind of proportional accuracy can be defined either for point positioning as well as for relative positioning: one meter error in the geocentric position of a point represents a relative error of $1\mathrm{m}$ divided by the radius of the earth (6.371×10^6 m) and is equal to 0.16 parts per million (0.16 ppm). To get a proportional accuracy of 0.16 ppm in a relative position of two points 10km apart, their relative position would have to be known with $\sigma = 1.6$ mm.

Another concept frequently used to describe accuracy is the concept of Dilution of Precision (DOP). The DOP is, in essence, a root-sum-squared measure of the size of the confidence region.

Chapter 3

Light Coordinate Systems

In this chapter we will try to formalize properties needed in order to construct a completely relativistic positioning system. We will also distinguish them from reference systems and will classify them according to capabilities proposed by B. Coll in Coll [2013].

3.1 Some considerations on positioning Systems

Space-time is modeled by a four-dimensional manifold in Relativity. Most of the coordinates in this manifold are chosen in order to simplify mathematical operations (at least that's the intention), but a little set admit more or less simple physical interpretations. This means that coordinate systems (its coordinate lines, surfaces or hypersurfaces) may be *thought* as covered by some particles, let them be clocks, rods or radiations submitted to particular motions. But the number of such *physically interpretable* coordinate systems that can be *physically constructed* in practice is strongly limited.

In fact, among the physically interpretable coordinate systems, the only one that may be generically constructed is the one based in the Poincaré-Einstein protocol of synchronization, the *radar system*. This system uses two-way light signals from the observer to the events to be coordinated. Unfortunately, this protocol suffers from being a retarded protocol (explained later). With the purpose of gaining some knowledge on the physics involved in some configurations of space-time localizations for different bodies and, particularly, getting some information on their relativistic gravimetry, it is quite important to learn how to construct physically coordinate systems of certain relativistic value.

Currently, in physics, the study of space-time coordinate systems and the large variety of protocols for their physical construction is a broad and open field. During this work, we will consider coordinate systems constructed from *relativistic positioning systems*, that is: defined by four emitting clocks broadcasting their own proper times. For each space-time event reached by this signals, the received times define the *emission coordinates* of this event (with respect to the given positioning system).

3.2 Location Systems

One of the first things to do here is to distinguish the coordinate system as a mathematical object and its realization as a physical object (take into account that different physical protocols, involving different physical fields or different methods to combine them, may be given for a unique mathematical coordinate system).

With this in mind, let us characterize this physical object with a proper name: The physical object obtained by a peculiar materialization of a coordinate system is called a *location system* Coll [2003], Coll and Tarantola [2009], Coll and Pozo [2006]. In this way, we will name as a location system a precise protocol on a particular set of physical fields allowing to materialize a coordinate system.

A location system may have some specific properties Coll [2003], Coll and Pozo [2006]. Among them, the more important ones are those of being:

- *generic*, i.e. that can be constructed in any space-time of a given class,
- (gravity-)*free*, i.e. that the knowledge of the gravitational field is not necessary to construct it,
- *immediate*, i.e. that every event may know its coordinates without delay.

For instance, with those properties defined this way: location systems based in the Poincaré-Einstein synchronization protocol (radar systems) are generic and free, but *not* immediate.

Usually, location systems are used either

- to allow assigning coordinates to particular events for a given environment (that is, a certain configuration of objects and observers), or
- to allow every event of said environment to know its proper coordinates.

Using as a reference their three-dimensional Newtonian analogues, we will call (*relativistic*) *reference systems* to location systems contructed in order to fulfill the first task (observer assigning coordinates to each event). In this sense, Poincaré-Einstein location systems are reference systems.

On the other hand, location systems fulfilling the three properties (remember: generic, free and immediate) constructed for the second purpose (that is, allowing every event to know its proper coordinates) will be called (*relativistic*) *positioning systems*.

As we mentioned before, Poincaré-Einstein reference systems are the only known location systems that can be generically constructed in the framework of general relativity, but they are not immediate, therefore we need to answer two questions:

- *Are there any positioning systems in relativity?*, and
- Does it exist such a system that fulfills the three demanded properties?.

Epistemically, one can argue that there exists a little number of them: A representation of this can be made aout of four broadcasting clocks, emitting their propper times via an electromagnetic signal.

In Newtonian physics, the velocity of transmission of information is supposedly infinite. When this happens, reference and positioning functionalities are *equivalent*, meaning that data obtained from any of the two systems may be transformed in data for the other one. This is no longer valid in relativity. Once we embrace relativity, the immediate character of positioning systems and the intrinsically retarded character of reference systems impose a strong decreasing hierarchy. Actually, it is impossible to construct a positioning system starting from the data obtained from a reference system. Nevertheless, it is always possible (and sometimes simple) to construct a reference system from a positioning system: just by making every event to send its coordinate data to the observer.

Because of this, the experimental or observational context strongly conditions the function, conception and construction of location systems in relativity. Moreover, because of this inherent immediate character, there are positioning systems which offer the most interest to be constructed (not so with reference systems). Recently, a *'galactic' positioning system* has been proposed for the Solar System based on the signals of four selected millisecond pulsars and a conventional origin. In the neighborhood of the Earth, a primary, auto-locating, fully relativistic, positioning system has also been proposed, based on four-tuples of satellited clocks broadcasting their proper time as well as the time they receive from the others. The whole constellation of a global navigation satellite system (GNSS), as union of four-tuples of neighboring satellites, constitutes an atlas of local charts for the neighborhood of the Earth, to which a global reference system directly related to the conventional International Celestial Reference System (ICRS) may be associated (SYPOR project; see Coll [2003], Coll and Pozo [2006]).

3.3 Relativistic and classical positioning systems

Unfortunately, most people consider relativistic positioning systems are just classical positioning systems on which one applies some relativity rules. As we can see, this view is quite wrong. Even if they are intimately related objects, they are significantly diferent.

In relativistic positioning system, the main objectives are:

- allow users to know their location in a well defined four-dimensional physical coordinate system,
- provide this users with their space-time metric (their proper time and distance to the emitters),
- characterize space-time trajectories, be it via their proper acceleration (dynamically) and/or via gravimetry (gravitationally).

A hypotetical relativistic positioning system around the Earth, would need then to characterize the space-time region between the constellation of satellites and the Earth surface, from a physics point of view. Figure 3.1 represents the gravitational field of this extension in an intuitive form.

Then, on the other side, classical (Newtonian) positioning system have a unique purpose:

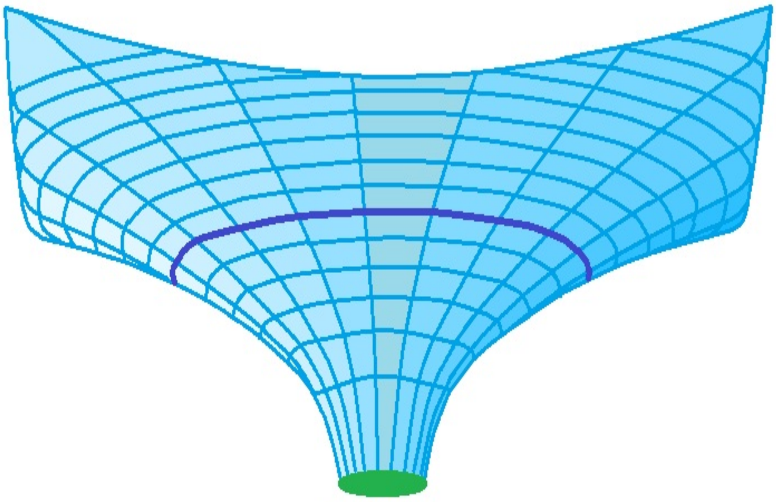

Figure 3.1: Locating users, providing them with their proper time and proper units of distance and characterizing physically the region between the Earth and the satellite constellation are the main tasks for a relativistic positioning system.

- Allowing any user to know his position with respect to a specific chart of the Earth surface, and its time with respect to some specific time-scale.

In positioning systems around the Earth, GNSS or any other similar systems the specific charts used are the World Geodetic System (WGS84), the International Terrestrial Reference Frame (ITRF) and the such, differing by less than ten centimeters in their last determinations. Time scales on their side are partially physical, social and political scales. The most used on these systems, TAI (International Atomic Time) is a weighted average from many national laboratories clocks. It represents a sort of mean proper time on a mean sea surface level and is much more stable that any individual clock. Of course, its extension all over the space-time region (specially between the Earth surface and the satellites) has undoubtedly many practical and social advantages.

This extension at any altitude of the TAI scale and of the Earth surface chart becomes a sort of a Newtonization of the space-time region between the Earth surface and the satellite constellation. Figure 3.2 represents intuitively this situation.

Other Newtonizations are possible, for example the one obtained by the extension to all the region of an averaged time at the satellite constellation level. This is not only a very usual one due to missconception or simplicity of utilization, but one Newtonization specifically designed and a challenging engineering subject.

Even if for smooth working of both systems, RGNSS and GNSS, basically the *same* data is needed, this information is obtained, interpreted and used very differently for each of these systems.

For instance, the asynchronous physical timing of proper clocks of the satellites constitutes the basic data for relativistic positioning systems. On the other hand, this timing is used in Newtonian positioning systems to construct the TAI timing at the satellite level, purelly a conventional timing at this level. In this way, relativistic corrections for GPS (as the ones proposed in Ashby (Ashby [2003])) are used sustractively on the physical clocks in motion. Instead of improving we are

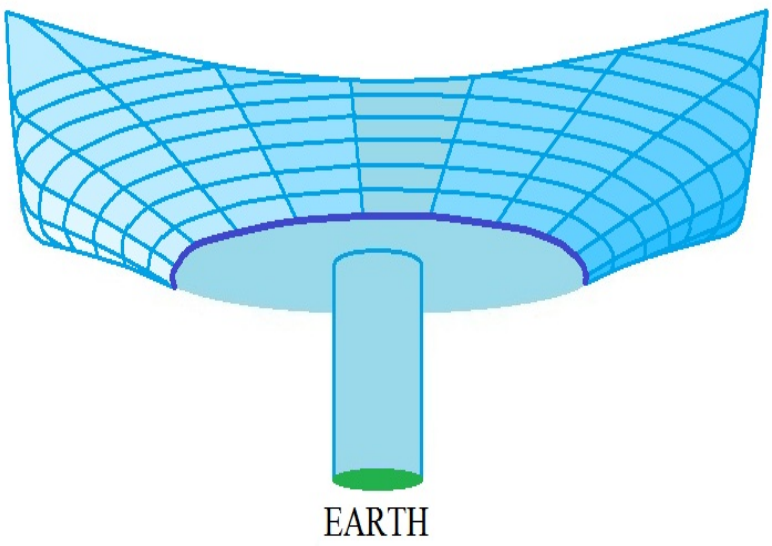

Figure 3.2: Extending the Earth surface chart and of the TAI (or any other time representation) to all the space up to the satellite constellation, becomes a Newtonization of this region.

focusing on better simulating their supposed Newtonian behaviour in an ‘absolute time’ way.

3.3.1 Relativistic positioning systems

As we stated earlier, positioning systems should be *immediate*. Again, that means that every event of their domain may know its proper space-time coordinates without delay, this is not just a desired property, is what defines them. For this analysis, as we explained previously, we will also suppose them to be *generic* and (gravity) *free*. That is: guaranteeing their existence in any generic space-time and their construction without the previous knowledge of the gravitational field respectively. From these and the aforementioned properties one can infer that, whenever possible, it would be more interesting to construct a positioning system than a reference system. Specially once we aknowledge you can build a reference system from a positioning system, but not the other way around.

Onwards, we will consider only relativistic positioning systems and denote by $\mathcal{P}$ the set of all of them.

Every one of the four clocks in the basic represantation of relativistic positioning systems will be γ_A, mainly thought as *emitters* broadcasting their proper time τ^A; the future light cones of the points $\gamma_A(\tau^A)$ will constitute the coordinate hypersurfaces $\tau^A = constant$ of the coordinate system for some domain of the space-time. For every event of the domain, four of these cones, broadcasting the times τ^A, intersect, entrusting in this way the event with the coordinate values $\{\tau^A\}$. This is quite similar to the basic experimental thought of triangulation, but let us clearly state this definition: when the past light cone for every event intersects the emitter world lines at $\gamma_A(\tau^A)$; then $\{\tau^A\}$ are the *emission coordinates* for this event.

If γ was an observer equipped with a receiver allowing the reception of proper times (τ^A) at each point of his trajectory, this observer would know his or her trajectory within these emission coordinates. We will call such an observer a *user* of the positioning system. Let us note that a user could carry a clock to measure his proper time τ. This is not necessary, but eventually we'll see it may come in handy.

Another important quality for a positioning system may be the property of being *auto-locating*. In order to achieve this goal, the clock emitters have also to be *retransmitters* of the proper time they just receive from the other three clocks. In this way, at every instant they must broadcast their proper time *and* also these other received proper times. That is, in our four clock example: Each clock emits its time and the time received from the other three clocks. Now, any user on the system does not only receive the emitted times $\{\tau^A\}$ but also the twelve transmitted times $\{\tau^A_B\}$. Using these data will allow the user to know the trajectories of the emitter clocks in emission coordinates.

We should note that *this is not trivial* technically. These delays on the satellite segment data and the update of these cross proper times need to be adressed. For the rest of this text, we will consider any emitter can send their proper time with infinite granularity, and retransmission to be immediate.

Once again, we might think of these relativistic positioning systems as nothing but the relativistic model of the classical GPS (Global Positioning System). But this is not so. Particularly, the GPS uses its satellites as simple beacons to transmit *another* spatial coordinate system (in this case the World Geodetic System 84) and an ad hoc time scale (the GPS time), different from the proper time of each satellite clock. However for relativistic positioning systems the unsynchronized proper times of these mounted clocks become the fundamental ingredients of the system. Furthermore the moving GPS grid (which had the need in GPS to be attended to when syncing spatially with earth charts), becomes also an important piece of information for autolocating systems. Relativistic Positioning systems isolate the spatial segment of the GPS system from its Earth control segment, offering a way of making them independent. Thus, such a positioning system can be considered as *the primary positioning reference* for the Earth and its immediate environment.

3.4 Auto-locating positioning systems

Auto-locating positioning systems are an important subclass of positioning systems. As we discussed earlier, they broadcast their proper time but also the proper time they receive from their neighbouring satellites. Let τ^{IJ}, $I \neq J$, be the proper time of the clock in satellite J received by the satellite I at its proper time instant τ^I. Then, the sixteen data $\{\tau^I, \tau^{IJ}\}$ received by an observer at any given time contains, of course, the emission coordinates τ^I, $(I = 1, \cdots, 4)$ of this observer but also the coordinates $\{\tau^I, \tau^{IJ}\}$ of every satellite I in the emission coordinate system $\{\tau^I\}$.[1] We will denote by $\mathcal{L}$ the set of all of the possible auto-locating positioning system. And we have $\mathcal{P} \supset \mathcal{L}$.

3.5 Autonomous positioning systems

We defined auto-locating systems in order to allow any user to get univocally the world-lines of the emitting satellites in the emission coordinate system in which they are broadcasting. Anyway,

[1]Please note that the world-lines of the satellites do not belong to the emission coordinate domain of the positioning system, but to its boundary. The emission coordinates are well defined on them, although not differentiable along the world-lines.

for the user it is still unknown how to make sense of these world-lines in the space time in which he is living, that is: to be able to draw them.

In order to do that, the coordinate data $\{\tau^I, \tau^{IJ}\}$ broadcasted by the auto-locating system has to be completed with some additional information:

* Some dynamical data of the satellites such as acceleration, gravity gradiometry (data on fluctuations on perceived gravity) and the like,
* Some observational data from them, such as ‘landmarks’ (for instance, position of reference quasars or pulsars) and
* Enough gravitational knowledge of the coordinate region (this might be theoretical, experimental or both).

We will call the set of this information *autonomous data*. And thus, we will call auto-locating systems broadcasting autonomous data *autonomous positioning systems*. By denoting with $\mathcal{A}$ the set of all of them, we have now $\mathcal{P} \supset \mathcal{L} \supset \mathcal{A}$.

Generic positioning systems (those in the difference $\mathcal{P} \setminus \mathcal{L}$) have the interest of having shown that relativistic positioning systems wich are generic, free and immediate exist and the advantage of being easier to study than the auto-locating systems constituting $\mathcal{L}$ (which is quite important). But we have seen that because the absence of autonomous data they need to be referred to a reference system. So we may conclude that *generic positioning systems are incomplete or insufficient.* And generic auto-locating systems (those in $\mathcal{L} \setminus \mathcal{A}$) also inherit the aforementioned incompleteness, since they also need to be referred to a reference system.

Autonomous positioning systems are the most useful location systems. They are the real future challenge.

Chapter 4

Two-Dimensional Approach to Light Coordinate Systems

Due to the difficulty inherent to the study of the generic four-dimensional relativistic positioning systems, which requires a rigorous previous training on simple and particular scenarios, we have decided that we will present in this text a two-dimensional approach to relativistic positioning systems introducing the basic features that define them. This whole study will be mainly based on the works by B. Coll, J.J. Ferrando and J. A. Morales-Lladosa (Coll et al. [2006a]). This will provide us with the advantage of being able to use precise and explicit diagrams improving the qualitative comprehension of general four-dimensional positioning systems. Even further, thanks to this two-dimensional approach, the scenarios will admit simple and explicit analytic results which makes it better for divulgative purposes.

We will see some examples of positioning, emission coordinates and emitter trajectories and some dynamical parameters (such as slope, separation and synchronization) for Minkowski and Schwartzchild metrics in order to have some generic examples.

4.1 Two-dimensional approach

Let there be two emitters γ_1 and γ_2 in geodetic trajectory and let us assume their proper times τ^1 and τ^2 are being broadcasted by means of electromagnetic signals. The future light cones ('light lines' in the two-dimensional approach) cut in the region between both emitters and are tangent outside. The inner region defined by the emitter world lines becomes the *emission coordinate domain*, we will denote it by Ω. In this way, the past light cone of every event in Ω intersects world lines of each emitter on $\gamma_1(\tau^1)$ and $\gamma_2(\tau^2)$. We will have then (τ^1, τ^2) as the coordinates of the event (fig. 4.1).

Since emission coordinates are null coordinates in the two-dimensional case, the space-time metric will depend solely on the *metric function* m taking the expression: $ds^2 = m(\tau^1, \tau^2)d\tau^1 d\tau^2$. And the plane $\{\tau^1\} \times \{\tau^2\}$ in which the different data of the positioning system can be transcribed will be called the *grid* of that positioning system.

Let us note that a *user* of the positioning system has a correspondence to an observer γ, travelling

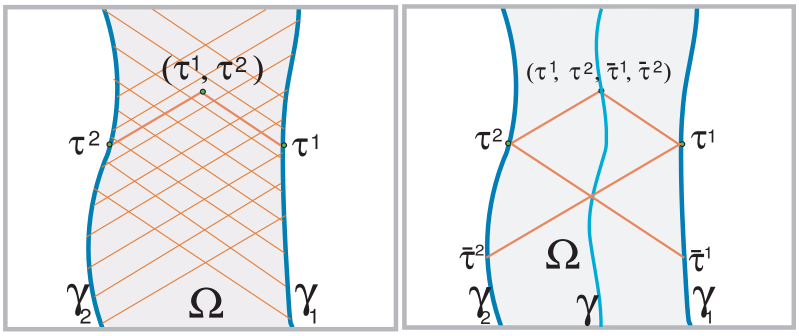

Figure 4.1: (a) The region Ω, bounded by the emitter world lines, defines the emitter coordinate domain. (b) In an auto-locating positioning system any user γ receives from the emitters the two proper times $\{\tau^1, \tau^2\}$ and the two transmitted times $\{\bar{\tau}^2, \bar{\tau}^1\}$.

throughout an emission coordinate domain Ω equipped with a receiver allowing the reading of the received proper times (τ^1, τ^2) at each point of his trajectory. Such user receiving continuously the emitted times $\{\tau^1, \tau^2\}$ knows his trajectory in the grid. From these *user positioning data* $\{\tau^1, \tau^2\}$ the equation F of the user trajectory may be expressed like this: $\tau^2 = F(\tau^1)$.

Simillarly, *auto-locating positioning systems* will be systems with every clock emitting not only its proper time but the proper time received from the other emitters. That is, the systems that broadcast the emission coordinates of the emitters. An auto-locating positioning system has as physical components a *spatial segment* constituted by two emitters γ_1, γ_2 broadcasting their proper times τ^1, τ^2 *and* the proper times $\bar{\tau}^2$, $\bar{\tau}^1$ received from the other emitters, and a *user segment* constituted by the set of all users traveling in an internal domain Ω and receiving these four broadcast times $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ (see Fig. 1b). Any such user, then, receiving continuously these *emitter positioning data* may also extract the equations of the trajectories of the emitters in emission coordinates from the data: $\varphi_1(\tau^1) = \bar{\tau}^2$, $\varphi_2(\tau^2) = \bar{\tau}^1$. The positioning system can be some times endowed with complementary devices. This emitters γ_1, γ_2 could carry accelerometers and broadcast their acceleration α_1, α_2, meanwhile the users γ could be endowed with receivers able to read also the broadcast emitter accelerations $\{\alpha_1, \alpha_2\}$. Then, the users can generate their own data also, carrying a clock to measure their proper time τ and/or an accelerometer to measure their proper acceleration α.

The (relativistic) theory of positioning systems has a purpose. It is mainly to develop the techniques necessary to determine the space-time metric as well as the dynamics of emitters and users by means of physical information carried by the user data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2; \alpha_1, \alpha_2; \tau, \alpha\}$.

4.2 Positioning in flat space-time

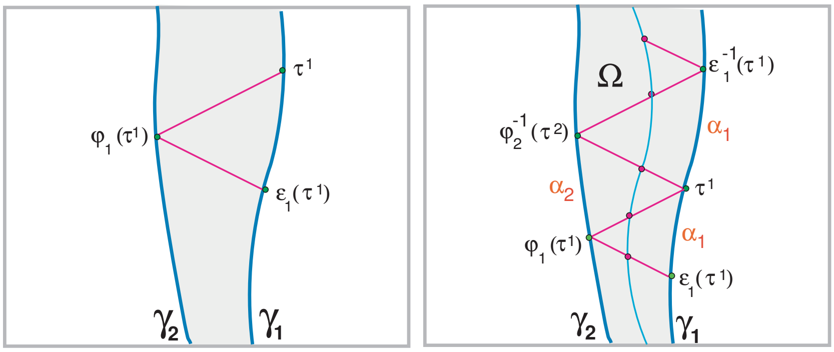

Figure 4.2: (a) If an emitter γ_1 receives at time τ^1 a signal after being echoed by the other emitter γ_2, it must be emitted at time $\epsilon_1(\tau^1)$, $\epsilon_1 = \varphi_2 \circ \varphi_1$. (b) If a user receives an emitter acceleration in the echo-causal interval $[\epsilon_1(\tau^1), \tau^1]$, then he knows all the user data along his trajectory.

To this point we have been considering either stationary or geodesic positioning systems with the user having a priori partial or full information about the positioning system. Let us face a new situation: the user knows the space-time where is immersed, but has no information about the positioning system. Can now the user data infer the characteristics of the positioning system? Can the user obtain information on his local units of time and distance? What about his acceleration?

The answer to these questions is still an open problem for a generic space-time, it is solved for Minkowski plane. The minimum set of data that determines all the user and system information as been analysed (see Coll et al. [2006b]). The dynamical equation of the emitters and user can be used to obtain that the user data are not independent quantities: the accelerations of the emitters and of the user along their trajectories are determined by the sole knowledge of the emitter positioning data and of the acceleration of only one of the emitters and only during an *emitter echo*, that is the interval between the emission time of a signal and its reception time after being reflected by the other emitter (see Fig. 4.2).

4.3 Emission coordinates in Minkowski space-time

A user receiving the four times $\{\tau^A\}$ in our proposed positioning system knows its own coordinates in the emission system. If the user wants to know his position in another reference system (for example the ICRS) it is necessary to obtain the relation between both coordinate systems.

At this point, we face an important problem due to the basics of relativistic positioning: Assuming that the world-lines of the emitters $\gamma_A(\tau^A)$ are known in such a coordinates system $\{x^\alpha\}$, can the user obtain his coordinates in this system if he receives his emission coordinates $\{\tau^A\}$? Or even more precisely, can the coordinate change $x^\alpha = x^\alpha(\tau^A)$ be obtained?

This coordinate change was obtained for the case of emitters in particular inertial motion in flat space-time in Bini et al. [2008]. More recently, the problem has been solved for a general configuration of the emitters in Minkowski space-time Coll et al. [2009, 2012]. The transformation $x^\alpha = x^\alpha(\tau^A)$ between inertial and emitter coordinates is obtained in a covariant way depending on the world-lines $\gamma_A(\tau^A)$ of the emitters.

The causal character (time-like, null or space-like) of the emitter configuration plays an important role. The geometric interpretation of this fact is a work in progress. In the meantime we also want: (i) to study the expression of the coordinate change in a 3+1 formalism adapted to an arbitrary inertial observer, (ii) to particularize these results for emitter motions modeling a satellite constellation around the Earth, and (iii) to analyze the effect of a week gravitational field on the coordinate transformation.

This work is necessary to tackle in the future the stated problem for a realistic situation, such as modeling a constellation of a GNSS.

4.3.1 Stationary positioning in Minkowski plane

Let us consider the positioning system defined by two non inertial stationary emitters γ_1, γ_2 in absence of gravitation (in Minkowski plane or two-dimensional flat space-time). Now, the emitters have a uniformly accelerated motion and are at rest with respect to each other, that is: the hyperbolic emitter trajectories have the same asymptotes (these kind of observers have been largely studied by Rindler in Rindler [1977]). We can choose in this way inertial null coordinates $\{\mathrm{u}, \mathrm{v}\}$ such that the trajectories of the emitters are [see Fig. 4.3(a)]:

$$\mathrm{v}\,\mathrm{u} = -\frac{1}{\alpha_i^2}$$

on which α_i, $0 < \alpha_1 < \alpha_2$, is the acceleration parameter of the emitter γ_i.

Since the space-like half-straight lines cutting at the coordinate origin determine the locus of simultaneous events for both emitters, we can define a synchronization of the emitter clocks by giving their proper times τ_0^1 and τ_0^2 at two simultaneous events. Using this synchronization, the proper time history of the emitters is [see Fig. 4.3(a)]:

$$\gamma_i \equiv \begin{cases} \mathrm{u} = u_i(\tau^i) = \dfrac{1}{\alpha_i}\exp[\alpha_i(\tau^i - \tau_0^i)] \\ \mathrm{v} = v_i(\tau^i) = -\dfrac{1}{\alpha_i}\exp[-\alpha_i(\tau^i - \tau_0^i)] \end{cases} \tag{4.1}$$

here, the repeated index does not indicate summation.

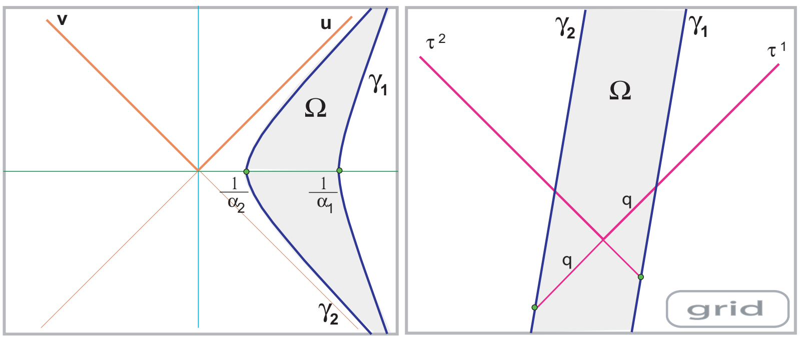

Figure 4.3: Trajectories of the uniformly accelerated emitters in flat space-time. (a) In inertial null coordinates $\{\mathrm{u}, \mathrm{v}\}$. (b) In the grid $\{\tau^1, \tau^2\}$; in this picture we have chosen the synchronization parameter $\sigma = 0$.

4.3.2 Emission coordinates and emitter trajectories

We have seen that the emission coordinates $\{\tau^1, \tau^2\}$ are defined by the transformation to the inertial system $\{\mathrm{u}, \mathrm{v}\}$:

$$\begin{aligned} \mathrm{u} = u_1(\tau^1) &= \frac{1}{\alpha_1}\exp[\alpha_1(\tau^1 - \tau_0^1)] \\ \mathrm{v} = v_2(\tau^2) &= -\frac{1}{\alpha_2}\exp[-\alpha_2(\tau^2 - \tau_0^2)] \end{aligned} \tag{4.2}$$

and its inverse transformation is:

$$\begin{aligned} \tau^1 &= \tau_0^1 + \frac{1}{\alpha_1}\ln(\alpha_1 \mathrm{u}) \\ \tau^2 &= \tau_0^2 - \frac{1}{\alpha_2}\ln(-\alpha_2 \mathrm{v}) \end{aligned} \tag{4.3}$$

From (**??**), we can obtain the expression of the trajectories in emission coordinates:

S.1 *The trajectories in Minkowski plane γ_1, γ_2 of the emitters of a stationary positioning system in emission coordinates $\{\tau^1, \tau^2\}$ are* parallel straight lines *of the form:*

$$\gamma_1 \quad \equiv \quad \begin{cases} \tau^1 = \tau^1 \\ \tau^2 = \frac{1}{\omega}(\tau^1 - q - \sigma) \equiv \varphi_1(\tau^1) \end{cases} \tag{4.4}$$

$$\gamma_2 \quad \equiv \quad \begin{cases} \tau^1 = \omega\tau^2 - q + \sigma \equiv \varphi_2(\tau^2) \\ \tau^2 = \tau^2 \end{cases} \tag{4.5}$$

The slope parameter *ω, the* separation parameter *q and the* synchronization parameter *σ are related to the acceleration parameters α_i and the synchronization instants τ_0^i of the emitters by:*

$$\omega \equiv \frac{\alpha_2}{\alpha_1} > 1, \qquad q \equiv \frac{1}{\alpha_1}\ln\frac{\alpha_2}{\alpha_1} > 0 \tag{4.6}$$

$$\sigma \quad \equiv \quad \tau_0^1 - \frac{\alpha_2}{\alpha_1}\tau_0^2 \tag{4.7}$$

This situation is illustrated in fig. 4.3(b) for a vanishing synchronization parameter.

4.3.3 Emitter's positioning and dynamical parameters

If we consider now a user traveling on the emission domain Ω defined by the positioning system and receiving the emitter positioning data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$. These data, under the form of the two sequences $\{\tau^1, \bar{\tau}^2\}$ and $\{\bar{\tau}^1, \tau^2\}$, will determine the emitter trajectories $\varphi_i(\tau^i)$ on the grid which, according to statement 1, will be straight lines, from which the user can extract the slope, separation and synchronization parameters $\omega > 1$, $q > 0$ and σ (see (4.4) and (4.5)).

In fact, these parameters can be extracted from the emitter positioning data at two different events:

S.2 *In terms of the emitter positioning data* $\{\tau_P^1, \tau_P^2; \bar{\tau}_P^1, \bar{\tau}_P^2\}$ *and* $\{\tau_Q^1, \tau_Q^2; \bar{\tau}_Q^1, \bar{\tau}_Q^2\}$ *at two different events* P *and* Q *of any user, the slope, separation and synchronization parameter* $\omega > 1$, $q > 0$ *and* σ, *characterizing the parallel trajectories of the emitters in the grid* $\{\tau^1\} \times \{\tau^2\}$, *are given by:*

$$\begin{aligned}
\omega &= \frac{\Delta\bar{\tau}^1}{\Delta\tau^2} \equiv \frac{\bar{\tau}_P^1 - \bar{\tau}_Q^1}{\tau_P^2 - \tau_Q^2} \\
q &= \frac{1}{2}[\tau_P^1 + \bar{\tau}_P^1 + \frac{\Delta\bar{\tau}^1}{\Delta\tau^2}(\tau_P^2 + \bar{\tau}_P^2)] \\
\sigma &= \frac{1}{2}[\tau_P^1 + \bar{\tau}_P^1 - \frac{\Delta\bar{\tau}^1}{\Delta\tau^2}(\tau_P^2 + \bar{\tau}_P^2)]
\end{aligned} \tag{4.8}$$

The expression (4.6) implies that ω and q determine one-to-one the acceleration of the emitters once this parameters are evaluated, and (4.7) shows that the parameter σ is necessary in order to obtain the emitter synchronization:

S.3 *The accelerations* α_1, α_2 *of the emitters may be obtained in terms of the slope data parameter* $\omega > 1$ *and the separation data parameter* $q > 0$, *as:*

$$\alpha_1 = \frac{1}{q}\ln\omega\,, \qquad \alpha_2 = \frac{\omega}{q}\ln\omega\,, \tag{4.9}$$

and their synchronization times τ_0^1 *and* τ_0^2 *indicated by the emitter clocks at two simultaneous events, are related to* ω *and the synchronization data parameter* σ *by*

$$\tau_0^1 = \sigma + \omega\tau_0^2 \tag{4.10}$$

This last statement gives the parameters α_1 and α_2, as well as the relation between τ_0^1 and τ_0^2, in terms of the data parameters ω, q and σ (those extracted from the data received by the user). Aside from their specific physical meaning, as accelerations and synchronization times of the emitters, the parameters α_1, α_2, τ_0^1 and τ_0^2, yield an operational definition of the null canonical inertial coordinates $\{\mathrm{u}, \mathrm{v}\}$ associated with the emitters. This operational definition is offered by relations (4.2). Every set $\{\mathrm{u}, \mathrm{v}\}$ of null coordinates so determined by (4.2) is related one-to-one to a standard inertial coordinate system $\{t, x\}$, by $\mathrm{u} = t + x$, $\mathrm{v} = t - x$, and inertial systems of fixed origin are related by a one-parametric Lorentz transformation. It is this parameter (say τ_0^2 here) that relations (4.9) and (4.10) leave free for given data parameters ω, q and σ.

4.4 Positioning in Schwarzschild plane

We will analise now the stationary positioning systems defined by two emitters at rest with respect to the gravitational field in *Schwarzschild plane*.

We will use here the usual expression for the space-time metric in the exterior region to the Schwarzschild radius r_s, this is:

$$ds^2 = b(r)d\bar{t}^2 - \frac{1}{b(r)}dr^2\,, \qquad b(r) \equiv 1 - \frac{r_s}{r}$$

Let us introduce the stationary time t and the radial coordinate ρ from the horizon, both in units of Schwarzschild radius:

$$t = \frac{\bar{t}}{r_s}\,, \qquad \rho = \frac{r}{r_s} - 1 > 0 \tag{4.11}$$

In order to obtain a conformally flat expression for the metric, we will change the radial coordinate ρ for the new one x:

$$x = x(\rho) \equiv \rho + \ln \rho \tag{4.12}$$

The inverse function giving the coordinate ρ in terms of x can be expressed using the Lambert function $w = W(z)$ (defined as the inverse of the function $z = w \exp w$). In this way, from (4.12) we obtain:

$$\rho = \rho(x) \equiv W(\exp x) \tag{4.13}$$

We will define, finally, the *stationary* null coordinates $\{\mathrm{u}, \mathrm{v}\}$ [see Fig.4.4(a)]:

$$\begin{aligned} \mathrm{u} &= t + x \qquad & t &= \tfrac{1}{2}(\mathrm{u} + \mathrm{v}) \\ \mathrm{v} &= t - x \qquad & x &= \tfrac{1}{2}(\mathrm{u} - \mathrm{v}) \end{aligned} \tag{4.14}$$

And with all these transformations one obtains,

in the exterior region, the metric of Schwarzschild plane:

$$ds^2 = r_s^2\, b(\rho)\, d\mathrm{u}\, d\mathrm{v} \tag{4.15}$$

$$b(\rho) \equiv \frac{\rho}{\rho + 1}\,, \qquad \rho \equiv W\left(\exp\left[\frac{\mathrm{u} - \mathrm{v}}{2}\right]\right) \tag{4.16}$$

with $W(z)$ being the Lambert function.

The trajectories of two stationary emitters γ_i are defined by the conditions [see Fig. 4.4(a)]:

$$\rho = \rho_i\,, \qquad 1 < \rho_2 < \rho_1 \tag{4.17}$$

The method used to introduce emission coordinates is based in obtaining the metric tensor in a null coordinate system, as expressions (4.15) and (4.16) do, and in knowing the proper time history of the emitters in this system. In coordinates $\{\mathrm{u}, \mathrm{v}\}$ for the two trajectories (4.17) one has, by (4.12) and (4.14), $\dot{\mathrm{u}} = \dot{\mathrm{v}}$ and then from (4.15) the unit character of their velocity gives $r_s^2\, b(\rho_i)\, \dot{\mathrm{u}}^2 = 1$. Because of this, the two trajectories are given as:

$$\gamma_i \equiv \begin{cases} \mathrm{u} = u_i(\tau^i) = \lambda_i(\tau^i - \tau_0^i) + x_i \\ \mathrm{v} = v_i(\tau^i) = \lambda_i(\tau^i - \tau_0^i) - x_i \end{cases} \tag{4.18}$$

in which the repeated index does not indicate summation, where the constants λ_i and x_i depend exclusively on the Schwarzschild radius r_s and the radial parameter ρ_i:

$$\lambda_i = \frac{1}{r_s}\sqrt{\frac{\rho_i + 1}{\rho_i}}\,, \qquad x_i = x(\rho_i) \tag{4.19}$$

and where τ_0^i are the proper times that indicate the emitter clocks at two events with the same Schwarzschild time t.

It is important to remark that *the radial parameter ρ_i is a dynamical characteristic of the trajectory, equivalent to the acceleration parameter α_i*. As the emitter acceleration scalars are constant because emitter trajectories are tangent to the Killing vector, a standard computation of them gives:

$$\alpha_i = \frac{1}{2\, r_s\, \rho_i^{1/2} (\rho_i + 1)^{3/2}} \tag{4.20}$$

Is is not so difficult to see that this relation admits a unique real positive solution $\rho_i = \rho_i(\alpha_i)$, showing their one-to-one equivalence and, thus, the stated dynamical character of ρ_i.

4.4.1 Emission coordinates and emitter trajectories

We have seen that the emission coordinates $\{\tau^1, \tau^2\}$ are defined by the transformation to the stationary null system $\{\mathrm{u}, \mathrm{v}\}$:

$$\begin{aligned} \mathrm{u} &= u_1(\tau^1) = \lambda_1(\tau^1 - \tau_0^1) + x_1 \\ \mathrm{v} &= v_2(\tau^2) = \lambda_2(\tau^2 - \tau_0^2) - x_2 \end{aligned} \tag{4.21}$$

and the inverse transformation is:

$$\begin{aligned} \tau^1 &= \tau_0^1 + \frac{1}{\lambda_1}(\mathrm{u} - x_1) \\ \tau^2 &= \tau_0^2 + \frac{1}{\lambda_2}(\mathrm{v} + x_2) \end{aligned} \tag{4.22}$$

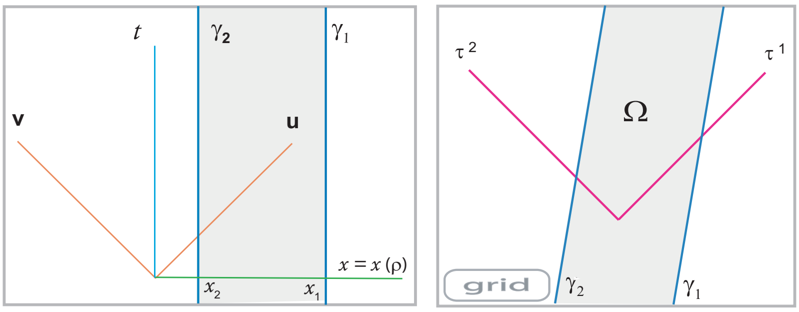

Figure 4.4: Trajectories of stationary emitters in Schwarzschild plane. (a) In 'stationary' null coordinates $\{\mathrm{u}, \mathrm{v}\}$ associated with the static time t and the coordinate $x = x(\rho)$. The stationary observers are defined by $\mathrm{v} = \mathrm{u} + \mathit{constant}$. (b) In the emission grid $\{\tau^1, \tau^2\}$.

As seen in (**??**), we can obtain the expression of the emitter trajectories in emission coordinates $\{\tau^1, \tau^2\}$:

S.4 *In Schwarzschild plane, the trajectories γ_1, γ_2 of the emitters of a stationary positioning system in emission coordinates $\{\tau^1, \tau^2\}$ are parallel straight lines of the form:*

$$\gamma_1 \equiv \begin{cases} \tau^1 = \tau^1 \\ \tau^2 = \frac{1}{\omega}(\tau^1 - q - \sigma) \equiv \varphi_1(\tau^1) \end{cases} \tag{4.23}$$

$$\gamma_2 \equiv \begin{cases} \tau^1 = \omega\tau^2 - q + \sigma \equiv \varphi_2(\tau^2) \\ \tau^2 = \tau^2 \end{cases} \tag{4.24}$$

The slope parameter ω*, the* separation parameter q *and the* synchronization parameter σ *are related to the Schwarzschild radius r_s, the radial parameters ρ_i and the synchronization instants τ_0^i of the emitters by:*

$$\begin{aligned} \omega &= \frac{\lambda_2}{\lambda_1} = \omega(\rho_1, \rho_2) \\ q &= \frac{x_1 - x_2}{\lambda_1} = r_s Q(\rho_1, \rho_2) \\ \sigma &= \tau_0^1 - \omega \tau_0^2 \end{aligned} \tag{4.25}$$

$\omega(\rho_1, \rho_2)$ and $Q(\rho_1, \rho_2)$ being the bi-parametric expressions:

$$\begin{aligned} \omega(\rho_1, \rho_2) &\equiv \sqrt{\frac{\rho_1(\rho_2+1)}{\rho_2(\rho_1+1)}} \\ Q(\rho_1, \rho_2) &\equiv \sqrt{\frac{\rho_1}{\rho_1+1}} \left[\rho_1 - \rho_2 + \ln\frac{\rho_1}{\rho_2}\right] \end{aligned} \tag{4.26}$$

Fig. 4.4(b) illustrates these trajectories in the grid.

4.4.2 Emitter's positioning and dynamical parameters

Any user traveling on the emission coordinate domain Ω of this proposed positioning system and receiving the emitter positioning data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ can find a particular linear relation between the $\bar{\tau}$'s and the τ's, leading him to find that the emitter trajectories $\varphi_i(\tau^i)$ are parallel straight lines in the grid of the emission coordinates he is receiving. From them, he can extract the parameters $\omega > 1$, $q > 0$ and σ [see (4.23) and (4.24)]:

S.5 *The slope, separation and synchronization parameters $\omega > 1$, $q > 0$ and σ, characterizing the parallel trajectories in the grid $\{\tau^1\} \times \{\tau^2\}$ of the stationary emitters in Schwarzschild plane have, in terms of the emitter positioning data at two different events P and Q, the same expression as in Minkowski plane, given by* (4.8).

With these parameters evaluated, expressions (4.25) and (4.26) tell us that ω and q determine one-to-one the dynamical radial parameters ρ_i, which characterize the emitter trajectories. In this way, from the first expression in (4.26) we can obtain:

$$\rho_2 = \frac{\rho_1}{(\omega^2 - 1)\rho_1 + \omega^2} \tag{4.27}$$

and, substituting in the second one, we have:

$$Q(\rho_1; \omega) \equiv \sqrt{\frac{\rho_1}{\rho_1+1}} \left[\frac{\rho_1(\rho_1+1)}{\rho_1 + \frac{\omega^2}{\omega^2-1}} + \ln[(\omega^2-1)\rho_1 + \omega^2]\right] \tag{4.28}$$

This expression of $Q(\rho_1, \omega)$ is an effective function on ρ_1. As a consequence of this, admits an inverse and this radial parameter may be obtained as a function of ω and $Q \equiv q/r_s$.

Furthermore, the third expression in (4.25) shows that the parameter σ gives the emitter synchronization. All these results can be summarized as follows:

S.6 *For stationary positioning systems in Schwarzschild plane, the radial parameters ρ_1, ρ_2 of the emitters may be obtained in terms of the slope data parameter $\omega > 1$ and the separation data parameter $q > 0$, by* (4.27) *and*

$$\rho_1 = \rho_1(\omega, q/r_s)\,, \tag{4.29}$$

with $\rho_1(\omega, q/r_s)$ being the inverse of (4.28).

Moreover, their synchronization times τ_0^1 and τ_0^2 indicated by the emitter clocks at two simultaneous events, are related to ω and the synchronization data parameter σ by

$$\tau_0^1 = \sigma + \omega\tau_0^2 \tag{4.30}$$

This statement gives the parameters ρ_1 and ρ_2, as well as the relation between τ_0^1 and τ_0^2, in terms of the data parameters ω, q and σ, information that can be extracted from the data received by the user. Analogously to the Minkowskian case, these parameters yield an operational definition of the null canonical stationary coordinates $\{\mathtt{u}, \mathtt{v}\}$ associated with the emitters and given by (4.21). By (4.29) and (4.30) these coordinates depend on a unique parameter (say τ_0^2), which now controls the time-like translations in the direction $\mathtt{u} + \mathtt{v}$.

Chapter 5

Light Coordinate Systems as more than Positioning Systems

The basic elements of the relativistic positioning systems in a two-dimensional space-time have been introduced where geodesic positioning systems, constituted by two geodesic emitters, have been considered in a flat space-time. Here, we want to show in what precise sense positioning systems allow to make *relativistic gravimetry*. For this purpose, we consider stationary positioning systems, constituted by two emitters separated by a constant distance, in two different situations: absence of gravitational field (Minkowski plane) and presence of a gravitational mass (Schwarzschild plane). As we stated earlier, the physical coordinate system constituted by the electromagnetic signals broadcasting the proper time of the emitters are the so called *emission coordinates*, and we show that, in such emission coordinates, the trajectories of the emitters in both situations, absence and presence of a gravitational field, are identical. The interesting point is that (as it is shown in Coll [2013], Coll et al. [2012]) particular additional information on the system or on the user allows not only to distinguish both space-times, but also to complete the dynamical description of emitters and user and even to measure the mass of the gravitational field. We strongly recommend reading Coll [2013]. For a better understanding of these calculations we have mantained his statement numeration.

5.1 Introduction

On the last chapter, we have presented a two-dimensional approach to relativistic positioning systems introducing the basic features that define them. We have obtained the explicit relation between emission coordinates and any given null coordinate system wherein the proper time trajectories of the emitters are known. We have also developed in detail the positioning system defined in flat space-time by geodesic emitters. Finally, we have shown that, in arbitrary two-dimensional space-times, the data that a user obtains from the positioning system determine the gravitational field and its gradient along the emitters and user world lines.

In this chapter, we want to detect the possibility of making relativistic gravimetry or, more generally, the possibility of obtaining the dynamics of the emitters and/or of the user, as well as the detection of the absence or presence of a gravitational field and its measure. This possibility is here examined by means of a (non geodesic) *stationary positioning system*, that is: a positioning system whose emitters are uniformly accelerated and the radar distance from each one to the other is constant. Such a stationary positioning system is constructed in two different scenarios: Minkowski and Schwarzschild planes.

In both scenarios, and for any user, the trajectory of the emitters in the *grid*, i.e. in the plane $\{\tau^1\} \times \{\tau^2\}$, are two parallel straight lines. At first glance, this fact would seem to indicate the impossibility to extract dynamical or gravimetric information from them, but this appearance is deceptive.

As seen in Coll [2013], the simple qualitative information that the positioning system is stationary (but with no knowledge of the acceleration and mutual radar distance of every emitter) and that the space-time is created by a given mass (but with no knowledge of the particular stationary trajectories followed by the emitters) allows to know the actual accelerations of the emitters, their mutual radar distances and the space-time metric in the region between them in emission coordinates $\{\tau^i\}$. Also, the transformations to other standard coordinate systems may be obtained.

The important point for gravimetry is that, in the above context, the data of the Schwarzschild mass may be substituted by that of the acceleration of one of the emitters. Then, besides all the above mentioned results, including the obtaining of the acceleration of the other emitter, the actual Schwarzschild mass of the corresponding space-time may be also calculated.

These relatively simple two-dimensional results strongly suggest that relativistic positioning systems can be useful in gravimetry at least when parameterized models for the gravitational field may be proposed.

A comment about the presentation of the results. The user is supposed to continuously receive data from the positioning system and have some a priori information about it and/or the space-time. Because of their operational importance, the statements involving exclusively consequences of these two types of knowledge, have been emphasized and numbered.

5.2 Minkowski Plane

5.2.1 Metric in emission coordinates

Now, let us analyze the metric information that any user can obtain from the emitter positioning data. The expression of the metric function in emission coordinates follows from the coordinate transformation (4.2):

$$m(\tau^1, \tau^2) = u_1'(\tau^1) v_2'(\tau^2) = \exp[\alpha_1 \tau^1 - \alpha_2 \tau^2 - \alpha_1 \sigma] \tag{5.1}$$

Consequently, by using (4.9), we can write the metric function in terms of the data $\{\tau^1, \tau^2\}$ and the parameters $\{\omega, q, \sigma\}$:

S.7 *In Minkowski plane, the space-time metric in emission coordinates of a stationary positioning system is given, in terms of the emitter positioning data* $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$, *by:*

$$ds^2 = \exp\left[\frac{\ln \omega}{q}(\tau^1 - \omega \tau^2 - \sigma)\right] d\tau^1 d\tau^2 \tag{5.2}$$

where the data parameters ω*,* q*,* σ *can be obtained by* (4.8) *in terms of the values of these data at two events.*

This statement shows that the sole public quantities $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ received by any user *allow him to know the space-time metric everywhere*. Nevertheless, it is worth remarking that, in order for a user to obtain this result, the user must be informed that *the emitter positioning data come from a stationary positioning system in absence of gravitational fields*. Otherwise, the above deduction based on the existence of the coordinate transformation (4.2) would not be valid. Remark also that this required information does not demand the specific value of the accelerations of the emitters, which may be also determined by the user by relations (4.9).

The users may know that the system of the two emitter clocks is stationary by an *a priori information*, forming part of the dynamical characteristics of the positioning system and its foreseen control. But they may also obtain this *information in real time*, if clocks and users are endowed with devices allowing the users to know the emitter accelerations $\{\alpha_1, \alpha_2\}$ at every instant.

In any case, the user information of the dynamics of the pair of clocks by any of these two methods is generically necessary, because *emitter trajectories that are parallel straight lines in the grid are not necessarily uniformly accelerated trajectories in a flat space-time* Coll et al. [2006a].

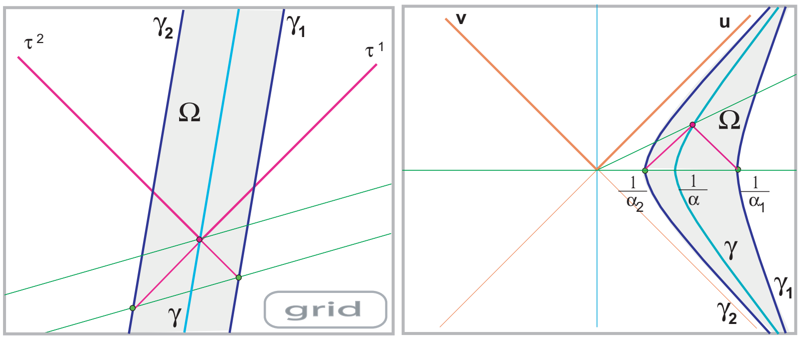

Figure 5.1: (a) The user positioning data $\{\tau^1, \tau^2\}$ determine the trajectory of the user γ in the grid: $\tau^2 = \frac{1}{\omega}\tau^1$. The parallel straight lines $\tau^2 = -\frac{1}{\omega}\tau^1 + C$ are the locus of simultaneous events for the uniformly accelerated emitters. (b) Trajectory of the user γ in inertial null coordinates $\{\mathrm{u}, \mathrm{v}\}$: he also has an hyperbolic motion.

5.2.2 Simultaneous events for the emitters

In order to better understand the grid associated with the emission coordinates of our two-dimensional stationary positioning systems, let us consider the locus of simultaneous events for both emitters, i.e. the orthogonal lines to the stationary family of trajectories that contains that of the two emitters which, as it is well known, are the space-like geodesics cutting at the common point of the two asymptotes of the emitters. How can they be constructed in the grid? The answer is:

S.8 *The locus of simultaneous events for the emitters of a stationary positioning system in Minkowski plane are parallel straight lines of slope* $-\frac{1}{\omega}$*, the same, up to sign, as that of the trajectory of the emitters.*

To see this, take into account that in inertial null coordinates these lines are $\mathrm{u} = -\kappa \mathrm{v}$, $\kappa > 0$. It is then sufficient to change to emission coordinates using (4.2) to obtain:

$$\tau^2 = -\frac{1}{\omega}\tau^1 + C, \qquad C \equiv \frac{1}{\alpha_2}\ln\frac{\kappa}{\omega} + 2\tau_0^2$$

which shows directly the statement [see Fig. 5.1(a)].

5.2.3 Geodesic user

To finish this section, let us consider the user positioning data received by an inertial (geodesic) user. For simplicity, but with no qualitative differences, we shall choose him to be at rest with respect to the emitters at the initial instant at which $\tau_0^1 = \tau_0^2 = 0$ (remember that we have taken $\sigma = 0$). For him we have:

S.9 *The trajectory in emission coordinates of inertial users initially at rest with respect to two synchronized stationary emitters is:*

$$\tau^2 = F(\tau^1) = -\frac{1}{\alpha_2}\ln[A - \omega \exp[\alpha_1 \tau^1]]\,, \tag{5.3}$$

where A is a constant.

This equation may be obtained as follows. Choose the null inertial system $\{\mathrm{u}, \mathrm{v}\}$ associated with the inertial observer having the same initial instant as the two emitters, that is to say, such that $\tau_0^1 = \tau_0^2 = 0$. Because the user is at rest with respect this null inertial system, its trajectory is given by [see Fig. 5.2(a)]:

$$\mathrm{v} = \mathrm{u} - 2x_0\,, \qquad x_0 < \frac{1}{\alpha_2} \tag{5.4}$$

Then, making use of the coordinate transformation (4.2), a straightforward calculation leads to expression (5.3), with $A \equiv 2x_0\alpha_2$. Note that the coordinate τ^2 run in $]-\frac{1}{\alpha_2}\ln A, +\infty[$ when τ^1 run in $]-\infty, \frac{1}{\alpha_1}\ln(A/\omega)\,[$. Thus the data $\{\tau^1, \tau^2\}$ of a geodesic user lead to a trajectory in the grid with a double asymptotic behavior [see Fig. 5.2(b)].

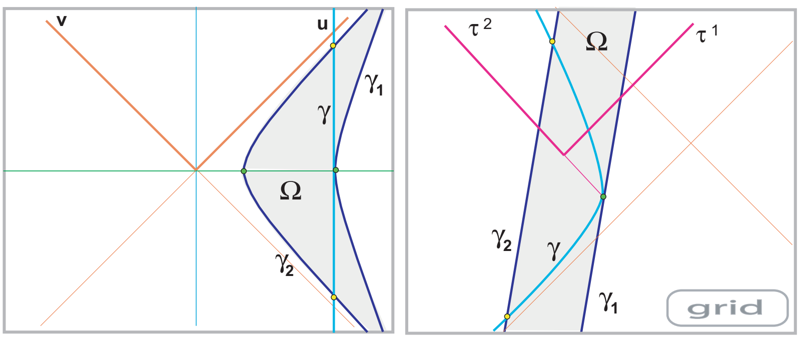

Figure 5.2: Trajectory of a geodesic user γ with $x_0 = \frac{1}{\alpha_1}$: (a) In inertial null coordinates $\{\mathrm{u}, \mathrm{v}\}$. (b) In the grid $\{\tau^1, \tau^2\}$ defined in flat space-time by two uniformly accelerated emitters.

5.3 Schwarzschild plane

5.3.1 Metric in emission coordinates

What is the metric information that the emitter positioning data offer to the users? To obtain the metric function in emission coordinates, starting from (4.15) and using the coordinate transformation (4.21), one obtains:

$$\begin{aligned} m(\tau^1,\tau^2) &= r_s^2\, b(\tau^1,\tau^2) u_1'(\tau^1) v_2'(\tau^2) \\ &= r_s^2\, b\left(\rho(x(\tau^1,\tau^2))\right) \lambda_1\lambda_2 \end{aligned} \tag{5.5}$$

But from (4.21) and (4.25) we have:

$$\begin{aligned} 2x(\tau^1,\tau^2) &= u_1(\tau^1) - v_2(\tau^2) \\ &= \lambda_1(\tau^1 - \omega\tau^2 - \sigma - q) + 2x(\rho_1) \end{aligned} \tag{5.6}$$

So that, by using (4.19) and (4.29), we can write the metric function in terms of the data $\{\tau^1,\tau^2\}$ and the parameters $\{\omega, q, \sigma\}$. Moreover these parameters can be obtained as in (4.8) from the emitter positioning data $\{\tau_P^1, \tau_P^2; \bar\tau_P^1, \bar\tau_P^2\}$ and $\{\tau_Q^1, \tau_Q^2; \bar\tau_Q^1, \bar\tau_Q^2\}$ at two events P and Q. Thus, we can state:

S.10 *In Schwarzschild plane, the space-time metric in emission coordinates of a stationary positioning system is given, in terms of the emitter positioning data $\{\tau^1,\tau^2;\bar\tau^1,\bar\tau^2\}$, by:*

$$ds^2 = \frac{1+\rho_1}{\rho_1}\omega\, b(\tau^1,\tau^2)\, d\tau^1 d\tau^2 \tag{5.7}$$

where

$$b(\tau^1,\tau^2) \equiv \frac{W(\exp x(\tau^1,\tau^2))}{1+W(\exp x(\tau^1,\tau^2))}\,,$$

$W(z)$ being the Lambert function and

$$x(\tau^1,\tau^2) \equiv \frac{1}{2r_s}\sqrt{\frac{1+\rho_1}{\rho_1}}(\tau^1 - \omega\tau^2 - \sigma - q) + \rho_1 \ln\rho_1,$$

where the parameters $\{\omega, q, \sigma\}$ are given by (4.8) *in terms of the values of the data at two events, and the constant ρ_1 is defined in* (4.29).

This statement shows that the sole public quantities $\{\tau^1,\tau^2;\bar\tau^1,\bar\tau^2\}$ received by any user allow him to know the space-time metric everywhere. It is worth remarking that the user must be informed that *the emitter positioning data come from a stationary positioning system in presence of a mass of the Schwarzschild radius r_s*. Remark also that this information does not require the specific value of the emitter radial parameters ρ_i, which may be determined by the user from the relations (4.27) and (4.29).

As statements 1 and 4 show, the sole emitter positioning data $\{\tau^1,\tau^2;\bar\tau^1,\bar\tau^2\}$ give the same picture in the grid in Minkowski and in Schwarzschild planes. Consequently, the additional quantitative information of The Schwarzschild radius r_s is a necessary information in order to obtain the metric and the dynamical quantities. Moreover, as we have already commented for Minkowski plane, the user's knowledge of the accelerations of the clocks is generically necessary because different dynamics (including non uniformly accelerated ones) may produce the same emitter trajectories *in the grid* (in Coll et al. [2006a]). On the other hand, we will show in next section that the data of one of the emitter's accelerations allows the user to determine Schwarzschild mass.

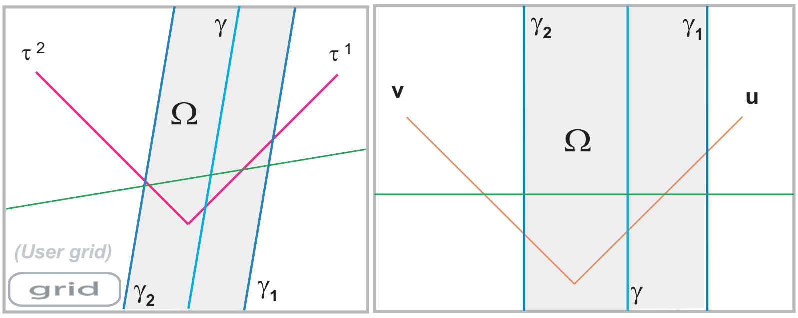

Figure 5.3: (a) User γ with a trajectory in the grid parallel to the stationary emitters in Schwarzschild plane. Note that parallel straight lines $\tau^2 = -\frac{1}{\omega}\tau^1 + C$ are the hypersurfaces with Schwarzschild time $t = constant$. (b) Trajectory of the user γ in 'stationary' null coordinates $\{\mathrm{u}, \mathrm{v}\}$: it is an stationary user.

5.3.2 Simultaneous events for the emitters

The simultaneity loci of the stationary emitters in stationary null coordinates are given by the lines $\mathrm{u} + \mathrm{v} = 2t = constant$. Using the coordinate transformation (4.21) one obtains that these loci take in the grid the following expression:

$$\tau^2 = -\frac{1}{\omega}\tau^1 + C.$$

Thus, as in the flat case, one has:

S.11 *The locus of simultaneous events for the emitters of a stationary positioning system in Schwarzschild plane are parallel straight lines of slope $-\frac{1}{\omega}$, the same, up to sign, as that of the trajectory of the emitters.*

5.3.3 Comparing Minkowski

It is worth remarking that the emitter positioning data received by a user give, in the two physically different two-dimensional Minkowski and Schwarzschild space-times, the same qualitative information about the emitter trajectories in the grid, as we have seen in this and the precedent section [see Figs. 4.3(b) and 4.4(b)]. Moreover, the trajectory in the grid of any stationary user is also the same in both cases [see Figs. 5.1(a) and 5.3(a)]. In spite of this fact, we shall see in the next section that complementary data that afford dynamic information of the system allow the user to distinguish between both systems.

The relevance of the dynamical data for the determination of the system is manifest in the case of a geodesic user: his trajectory in the grid is different in both cases. Is is easy to show this fact because the trajectory (5.3) of a geodesic user in the grid of the system in the flat case is not geodesic for the Schwarzschild metric (4.15), as a straightforward calculation shows. In this last case the geodesics, qualitatively similar to those of the flat case shown in Fig. 5.2(b), have nevertheless a different trajectory equation. By comparison of this trajectory with the one constructed by the geodesic user from the received sole data $\{\tau^1, \tau^2\}$, he is able to distinguish in which space-time he is evolving.

5.3.4 Obtaining The Schwarzschild mass from the emitter accelerations

Suppose now that the user has partial information about the gravitational field and the positioning system. The user knows thus that the system is defined by stationary emitters and, consequently, that they have constant acceleration. If the emitters carry accelerometers and broadcast their accelerations, the user can receive the precise value α_1 α_2 of these accelerations.

Can this dynamical information determine the Schwarzschild mass? The answer is affirmative. Furthermore, apart from the emitter positioning data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$, *only one of the two accelerations* is sufficient to the user to obtain The Schwarzschild mass and, consequently, to acquire the information level considered in the precedent subsections.

In order to specify exactly this result let us assume firstly that the user knows the full set of the public data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2; \alpha_1, \alpha_2\}$. We know that the data parameters $\{\omega, q, \sigma\}$ depend on the emitter positioning data [see (4.8)]. Because the accelerations α_i are also known, the first expression in (4.26) and the the two in (4.20) constitute three relations in $\{r_s, \rho_1, \rho_2\}$ that allow to obtain them explicitly in terms of $\{\omega, \alpha_1, \alpha_2\}$. On the other hand, the resulting expressions have to be compatible with the expression of q given in (4.25).

In the case of Minkowski plane, the user data parameters $\{\omega, q, \alpha_1, \alpha_2\}$ are submitted not to one but to two compatibility conditions, those given by (4.6). This fact and the considerations in the above paragraph lead to the following

S.12 *Consider a user of a stationary positioning system in a space-time of which he only knows that it is either a Schwarzschild or Minkowski plane. Let $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2; \alpha_1, \alpha_2\}$ be the public data that he receives, $\{\omega, q, \sigma\}$ the data parameters that can be extracted from them by means of* (4.8)*, and κ the data parameter given by:*

$$\kappa \equiv \sqrt{\frac{\omega\alpha_1}{\alpha_2}}. \tag{5.8}$$

Then κ is necessarily such that:

$$\kappa \leq 1. \tag{5.9}$$

Moreover:

(i) The value $\kappa = 1$ is the one that informs the user that his space-time is Minkowski plane. In this case, the public data are submitted to the constraints:

$$\alpha_1\omega = \alpha_2\,, \qquad q\alpha_1 = \ln\omega \tag{5.10}$$

(ii) Any value $\kappa < 1$ informs the user that his space-time is Schwarzschild plane with radius

$$r_s = \frac{(\omega^2-1)^2}{2\,\alpha_2(1-\kappa)^{1/2}(\omega^2-\kappa)^{3/2}},$$

that the radial coordinate of the emitters are:

$$\rho_1 = \frac{\omega^2(1-\kappa)}{\kappa(\omega^2-1)}\,, \qquad \rho_2 = \frac{1-\kappa}{\omega^2-1},$$

and that the public data are submitted to the constraint:

$$2q\alpha_1 \frac{\omega^2-\kappa}{\kappa(\omega^2-1)} = 1-\kappa+\frac{\kappa(\omega^2-1)}{\omega^2-\kappa}\ln\frac{\omega^2}{\kappa} \tag{5.11}$$

The compatibility condition (5.11) for the data parameters $\{\omega, q, \alpha_1, \alpha_2\}$ suggests that the emitter positioning data $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ and a sole acceleration, say α_1, are sufficient data to determine The Schwarzschild mass. Indeed, from (4.20) we obtain:

$$r_s = \frac{1}{2\,\alpha_1\,\rho_1^{1/2}(\rho_1+1)^{3/2}} \tag{5.12}$$

This expression allows eliminate r_s in the expression (4.25) of q, which becomes:

$$q \;=\; \frac{1}{2\alpha_1}\Theta(\rho_1,\rho_2) \tag{5.13}$$

$$\Theta(\rho_1,\rho_2) \;\equiv\; \frac{1}{(\rho_1+1)^2}\left[\rho_1-\rho_2+\ln\frac{\rho_1}{\rho_2}\right] \tag{5.14}$$

Then, substituting (4.27) in (5.14), (5.13) takes the form:

$$2q\alpha_1 = \Theta(\rho_1,\omega)\,. \tag{5.15}$$

This expression of $\Theta(\rho_1,\omega)$ is an effective function on ρ_1. Consequently it admits an inverse:

$$\rho_1 = \rho_1\left(\omega, q\alpha_1\right)\,, \tag{5.16}$$

and we can state:

S.13 *Consider a user of a stationary positioning system in a space-time of which he only knows that it is a Schwarzschild plane. Let* $\{\tau^1, \tau^2; \bar{\tau}^1, \bar{\tau}^2\}$ *be the emitter positioning data that he receives and suppose that he also receives but one of the emitter dynamical data, say* α_1*. Then, The Schwarzschild radius* r_s *is given by* (5.12) *where* ρ_1 *may be extracted from* (5.15) *as* (5.16)*, in terms of the data parameters* $\{\omega, q, \alpha_1\}$.

Conclusions

Results

In this text we have presented a two-dimensional approach to relativistic positioning systems (as made in Coll et al. [2006a]). We studied in detail the positioning system defined by two geodesic emitters in flat space-time, and showed that in an arbitrary space-time the user data determines the gravitational field and its gradient along the emitters and user world lines.

We have:

- Considered stationary positioning systems, (defined by two uniformly accelerated emitters at rest with respect to each other).
- Studied stationary positioning systems in both, Minkowski (Sec. 4.3.1) and Schwarzschild planes (Sec. 4.4).
- Shown that a user receiving the emitter positioning data will find, for the emitter trajectories, parallel straight lines in the grid, no matter the plane be (statements 1 and 4),
- Given the parameters of these straight lines in terms of the emitter positioning data (statements 2 and 5).
- Shown the dynamics and synchronization of the emitters in terms of data parameters (statements 3 and 6).
- Found that, if a user knows the stationary character of the positioning system, the emitter positioning data makes him able to know explicitly the space-time metric in emission coordinates.
- Studied the proper data received by a stationary user and the expressions of his acceleration and proper time in terms of the emitter positioning data that he receives.
- Obtained the trajectory of a geodesic user in emission coordinates in Minkowski plane (statement 9).
- Shown that the knowledge of complementary user data determines the Schwarzschild mass (statement 12).

These two-dimensional situations suggest that the relativistic positioning systems could be useful in four-dimensional gravimetry for reasonable parameterized models of the gravitational field.

About Relativistic issues on actual GPS Systems

- Euclidian Positioning Models are not enough with relativistic effects unnacounted for.
- Actual Earth Positioning Systems are good enough (vía relativistic corrections) so they don't need to be reformulated from a practical point of view.
- Since these corrections have been made for the past 50 years they are pretty accurate, but they are experimental. We can not rely on this method for new positioning systems (be they non Earth related or radically different in design).

About Coordinate Systems

Two important classes of location systems are the *reference systems* and the *positioning systems*. The goal of reference systems is to situate the events of a domain with respect to a given observer (generally located at the origin), meanwhile the goal of positioning systems is to indicate its own position to every event of the domain. In Newtonian theory, as far as the velocity of light is supposed infinite, both goals are exchangeable in a sole location system. But in relativity this is no longer possible and it is impossible to construct a positioning system starting from a reference system, but one can always (and very easily) construct a reference system starting from a positioning system.

- Reference Systems are not really well modelled as mathematical objects.
- Location Systems are those Reference Systems fulfilling these properties:
  - it should be *generic*, i.e. that can be constructed in any space-time of a given class,
  - it should be (gravity-)*free*, i.e. that the knowledge of the gravitational field is not necessary to construct it,
  - it should be *immediate*, i.e. that every event may know its coordinates without delay.
- We have also asked for some extra desired properties, defining this way both auto-locating positioning systems and autonomous positioning systems.

About Light Coordinates

We have defined Light Coordinates as a certain model of auto-location positioning system logically similar to traditional GPS systems.

- The Light Coordinates model gives us information about the metric we are immersed.
- The model is solved for certain interesting metrics, but not proven to be always solvable.
- It is clearly more complicated to solve, both theoretically and computationally. But, right now, it is the only way to go expanding the traditional GPS model.

Future Work

- Improve Relativistic Location Systems/Reference Systems
- Improve Gravimetry Apparatus
- Generalize two-dimensional results to generic four-dimensional case
- Work on equations to solve positioning for known metrics
- Model and Standardize these new Positioning Systems

Appendix A

Preliminary Concepts of Differential Geometry

Differential Geometry is well established as a main tool in Relativity.
In this Chapter we will overview some basic concepts on the theory, so the reader can properly follow the proposals made during this work. While there is a huge amount of literature on this topic, we decided to made this an autocontained text. Anyway, we refer the reader to Girbau [1993] or Currás Bosch [2003] for a more rigorous and wider view on Differential Geometry. We followed the notation used in Currás Bosch [2003].

A.1 Smooth Manifolds

Widely speaking, a smooth manifold is a type of manifold locally similar enough to a linear space to allow one to do calculus. This is: A topological space locally homeomorph with $\mathbb{R}^n$.
Being a little bit more rigorous on the definition, an n dimensional class C^∞ smooth manifold is a topological Haussdorf space $\mathbf{M}$ which admits a numerable base of open subsets, altogether with an equivalence class of n dimensional atlasses C^∞.
At the same time, an n dimensional C^∞ atlas in a topological space $\mathbf{M}$ is a family of pairs $\{(U_i, \phi_i)\}_{i\in I}$, where:

- U_i is an open subset of $\mathbf{M}$ for every i.
- $\phi_i : U_i \to \mathbb{R}^n$ is a continuous application on which: $\phi_i(U_i)$ is an open subset of $\mathbb{R}^n$ and $\phi_i : U_i \to \phi_i(U_i)$ is an homeomorphism.
- The collection of open subsets $\{U_i\}_{i\in I}$ cover the whole space, that is $\mathbf{M} = \bigcup_{i\in I} U_i$.
- For each i, j, $U_i \bigcap U_j \neq 0$, $\phi_j \circ \phi_i^{-1}$ is class C^∞.

As an example, every open subset in $\mathbb{R}^n$ is a smooth manifold with a differentiable structure induced by the one from $\mathbb{R}^n$. On the other way, the subspace $\mathbb{R}\,[0, 1]$ is not a smooth manifold (we cannot define homeomorphisms with $\mathbb{R}^n$ on the ends of the interval).
The functions ϕ^i composing ϕ are named map coordinates, and the function $\phi_j \circ \phi_i^{-1}$ is named

coordinate chage between the map ϕ_i and ϕ_j.

A.1.1 Differentiable Applications

Let $f : M \to N$ be a continuous application between manifolds (that is, the antiimage of every open subset in N is an open subset in M). The we will say that f is differentiable if, for every pair (U, ϕ) map of M and every pair (V, ψ) cmap of N, the application $\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(V) \cap U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$ is differentiable. That means we will check differentiability in the representation which for every manifold we do in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

In particular, if N is the space $\mathbb{R}$, which is obviously a manifold with the canonical differentiable structure, we will say f is a differentiable function. The set of differentiable functions is represented by $\mathbf{F}(M)$, and it is trivial to see that the sum, scalar product (group) and the product between functions (ring) are operations closed in $\mathbf{F}(M)$, thus $\mathbf{F}(M)$ has a structure of an $\mathbb{R}$-commutative algebra.
Finally, we define a diffeomorphism as a differential application $f : M \to N$ possessing an inverse function which is also differentiable.

A.1.2 Tangent Space

We define differentiation in a point $p \in M$ as an operator $D_p : \mathbf{F}(M) \to \mathbb{R}$ between said spaces which, for every pair of functions $f_1, f_2 \in \mathbf{F}(M)$ fulfills the next conditions:

- $D_p(f_1 + f_2) = D_p(f_1) + D_p(f_2)$
- $D_p(\lambda f) = \lambda D_p(f)$
- $D_p(f_1 \cdot f_2) = f_1(p)D_p(f_2) + D_p(f_1)f_2(p)$

and we will call $\mathbf{D}_p(M, \mathbb{R})$ the set of differentiations of $\mathbf{F}(M)$ at the point p p, which has a vector space sctructure. We will define tangent space at M in the point p as the vector space $\mathbf{D}_p(M, \mathbb{R})$, and we will usually represent it as T_pM.
If we consider initially the trivial case $M = \mathbb{R}^n$, with $\{x_1, ..., x_n\}$ the coordinated functions, the vector space $\mathbf{D}_0(\mathbb{R}^n, \mathbb{R}) \simeq \mathbb{R}$, and the element $(\frac{\partial}{\partial x_i})_p$ is a differentiation in $\mathbb{R}^n$ at the point $p = 0$. Furthermore, we can prove $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}_p$ form a base for the tangent space T_pM.

Differentiable Curves over the Manifold

First of all, we will define a curve over a smooth manifold $\mathbf{M}$ as a differentiable application:

$$\gamma : (a, b) \to \mathbf{M} \text{ , on } a < b \tag{A.1}$$

We define the tangent vector to the curve $\gamma(t)$ at the point $\gamma(0)$ (or in $t = 0$) as:

$$\dot{\gamma}(0)(f) = \frac{d(f \circ \gamma(t))}{dt}|_{t=0} \tag{A.2}$$

which we can see as $\dot{\gamma}(0) \in T_{\gamma(0)}M$.

The coordinate expression of this tangent vector, acording to the atlas on which we have defined the manifold, is:

$$\dot{\gamma}(0) = \sum_{i=1}^{n} \frac{d(x_i(\gamma(t)))}{dt}|_{t=0} \cdot \frac{\partial}{\partial x_i}|_{\gamma(0)} = \sum_{i=1}^{n} \dot{x_i}(0) \cdot \frac{\partial}{\partial x_i}|_{\gamma(0)} \tag{A.3}$$

And we can prove every element $D_p \in T_pM$ is equal to $\dot{\gamma}(0)$, for some γ, with $\gamma(0) = p$.

The Linear Tangent application

Consider $f : M \to N$ a differentiable application between manifolds, we define the lineal tangent application and denote it by $(df)_p : T_pM \to T_{f(p)}N$, as:

$$((df)_p D)(\alpha) := D(\alpha \circ f) \ , \forall \alpha \in \mathbf{F}(N) \tag{A.4}$$

and we can prove that, being D from the derivation space T_pM, $(df)_p(D)$ it belongs to the derivation space $T_{f(p)}N$.

With all this we can define now a base for the tangent space directly over the manifold. In fact, we will define the *i-th* element of the base of the tangent space T_pM as:

$$\left(\frac{\partial}{\partial x_i}\right)_p = (d\phi)^{-1} \left(\frac{\partial}{\partial x_i}\right)_0 \text{ , on } \phi(p) = 0 \tag{A.5}$$

where ϕ is the homeomorphism associated to the open subset. And we will say that $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}_p$ is a base of the tangent space at a point $p \in U$.

A.2 Vector Fields

As a vector space, the tangent space can contain vectors. If we consider all the tangent spaces on the manifold $TM = \bigsqcup_{p\in M} T_pM$, then we define a differentiable vector field X as a tangent vector assignated to each and every point. Using coordinates it will be expressed:

$$X_p = \sum_{i=0}^{n} \lambda_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \text{ , } p \in M \tag{A.6}$$

where the functions λ_i are differentiable.
Given a differentiable vector field, an integral curve of this field is a curve $\gamma : (a, b) \to M$ such as:

$$\dot{\gamma}(t) = X_{\gamma(t)} \ , \forall t \in (a, b) \tag{A.7}$$

that is, its tangent vector is the vector field X on all of its points. The differential equations defining a certain vector field X in relation to their integral curves will be, generally:

$$\begin{aligned} \tfrac{dx_1}{dt} &= \lambda_1(\gamma(t)) \\ &\vdots \\ \tfrac{dx_n}{dt} &= \lambda_n(\gamma(t)) \end{aligned} \tag{A.8}$$

A.2.1 Flow of a Vector Field

If we consider the differentiable vector field X, it has a correspondent flow ϕ, which is a differentiable application $\phi : (a, b) \times U \to M$, where $0 \in (a, b)$ and $p \in U$, such as $\phi_p(t) = \phi(t, p)$ is an integral curve X in $p \in U$.
The concept of Flow of a Vector Field has the next definitions associated:

- Fulfills $\dot{\phi}_p(t) = X_{\phi_p(t)}$
- ϕ is such as $\phi(t + s, p) = \phi(t, \phi(s, p))$, which is the group propiety (flow is also named one-parameter local transformations group).
- A field X is complete if its flow can be extended to $\mathbb{R} \times M$. If the manifold is compact, every vector field is complete.
- Fixing $t \in \mathbb{R}$, $\phi_t(p)$ is a diffeomorphism in the open subset where it is defined. If X is complete we will have a group of diffeomorphisms from M to M (according to parameter t).

A.2.2 The Lie derivative

Let X, Y be two differential vector fields with respective flows $\{\phi_t\}, \{\psi_s\}$. Being each the integral curve of their respective field, we know they fulfill:

$$\begin{aligned} (d\phi_t)_p(X_p) &= X_{\phi_t(p)} \\ (d\psi_t)_p(Y_p) &= Y_{\psi_t(p)} \end{aligned} \tag{A.9}$$

for every t where the flows are defined.
Knowing this result, the Lie derivative at the point p as:

$$(L_X Y)_p = \lim_{t \to 0} \frac{1}{t} \left(Y_p - (d\phi_t)(Y_{\phi_{-t}}(p)) \right) \tag{A.10}$$

Lie Bracket

Let two any differentiable fields X, Y be, we define the Lie Bracket $[X, Y] \in X(M)$ as the only vector field in M fulfilling:

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \, , \forall f \in \mathbf{F}(M) \tag{A.11}$$

We can prove that $[X, Y] = L_X Y$.
The Lie Bracket has a number of propieties:

- $[X, X] = 0$
- $[\lambda X_1 + \mu X_2, \alpha Y_1 + \beta Y_2] = \lambda\alpha[X_1, Y_1] + \lambda\beta[X_1, Y_2] + \mu\alpha[X_2, Y_1] + \mu\beta[X_2, Y_2]$
- $[X, Y] = -[Y, X]$
- $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$
- $[g \cdot X, Y] = g \cdot [X, Y] - Y(g) \cdot X$
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (Jacobi Identity)

And its expression in coordinates of two certain fields $X = \sum_i \lambda_i \cdot \frac{\partial}{\partial x_i}$, $X = \sum_j \mu_j \cdot \frac{\partial}{\partial x_j}$ is:

$$[X, Y] = \sum_{i,j} \left(\lambda_i \frac{\partial \mu_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} + \mu_j \frac{\partial \lambda_i}{\partial x_j} \cdot \frac{\partial}{\partial x_i} \right) \tag{A.12}$$

A.3 Tensors

We will talk now about an important mathematical object which usually appears on theoretical physics: the tensors.
Before any formal definition, let us remember we just defined the concept of manifold as a space over which we will describe physical phenomena and its associated tangent space on which we will define the *differential* objects we will deem more convenient (differentiations, for instance).
Differentiations, as we have difined them, are objects acting over elements in $\mathbf{F}(M)$, that is, differentiable functions from M in $\mathbb{R}$ such as:

$$X(f) = \sum_{i=0}^{n} X_i \frac{\partial f}{\partial x_i}\ , \ f \in \mathbf{F}(M) \tag{A.13}$$

and this *action* is $\mathbb{R}$-linear.
On the other way we have defined the tangent space TM of the manifold, and at the same time we can define its dual space, denoted by $(TM)^* = \bigsqcup_{p \in M} (T_pM)^*$.
It seems coherent then to consider the definition of operators over tangent and cotangent spaces, and its product spaces, that is:

$$\begin{aligned} T : T_pM \times^i ... \times T_pM \times (T_pM)^* \times^j ... \times (T_pM)^* &\longrightarrow \mathbb{R} \\ (X_1, ..., X_i, \omega_1, ..., \omega_j) &\longmapsto T((X_1, ..., X_i, \omega_1, ..., \omega_j)) \end{aligned} \tag{A.14}$$

and the only condition we will impose is for them to be $\mathbb{R}$-multilinear (which is the same as considering the component functions to be $\mathbb{R}$-linear).

Now, algebraically speaking it is easier to work with linear operators than multilinear operators, hereby the interest on tensors

We will define a k-times covariant and l-times contravariant tensor as a multilinear application:

$$T : T_pM \times ... \times T_pM \times (T_pM)^* \times ... \times (T_pM)^* \longrightarrow \mathbb{R} \tag{A.15}$$

And, as we said, we would like for T to become a linear application. Well then, T factorizes uniquely in $\hat{T}$ by means of the tensor product τ, being $\hat{T}$ linear if and only if T is multilinear, and the following diagram is commutative:

$$\begin{tikzcd} \times^k T_pM \times^l (T_pM)^* \arrow[r, "T"] \arrow[d, "\tau"'] & \mathbb{R} \\ \otimes^k T_pM \otimes^l (T_pM)^* \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

where the space tensor product comes froma stablishing an equivalence relation over the elements of the *usual* product space.
Therefore, if we consider the collection of elements $(k_1X_1, ..., k_iX_i, h_1\omega_1, ..., h_j\omega_j)$ the linearization of the tensor product assures:

$$\hat{T}(k_1X_1, ..., k_iX_i, h_1\omega_1, ..., h_j\omega_j) = \prod_{n,m} k_nh_m \cdot \hat{T}(X_1, ..., X_i, \omega_1, ..., \omega_j) \tag{A.16}$$

Finally, we would like to extend the tensor thus defined all over the manifold. So, we will define a tensor field as an application which in each point $p \in M$ assigns an element $T : \times^k T_pM \times^l (T_pM)^* \longrightarrow \mathbb{R}$ in such a way that, if we consider the multiplicative constant in every tangent space as a continuous function over M, $\hat{T}$ is $\mathbf{F}(M)$-linear.
The fact of being able to linearize the application over the product space is not for free, efectively, the number of elements of the base grows from $n \cdot (k + l)$ to $n^{(k+l)}$.

A.3.1 Base of the tensor product space

Let us consider the base of the space T_pM $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}_p$, we can prove that the base of the dual space $(TM)^*$ is $\{dx_1, ..., dx_n\}_p$ (we will prove it on the next subsection), and in this bases a tensor as we have described it will take the form:

$$a = \sum_{i_1,...,i_k}^{j_1,...,j_l} \lambda^{i_1,...,i_k}_{j_1,...,j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes dx^{j_l} \tag{A.17}$$

where $\lambda^{i_1,...,i_k}_{j_1,...,j_l} \in \mathbf{F}(M)$ and we used Einstein's Summation convention.
On phisics texts a tensor is usually defined as an operator following the next coordinate change:

$$\lambda^{i_1,...,i_k}_{j_1,...,j_l} = \mu^{\beta_1,...,\beta_k}_{\alpha_1,...,\alpha_l} \frac{\partial y^{\alpha_1}}{\partial x^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{\beta_k}} \tag{A.18}$$

which is precisely the change we have defined a while ago.

A.3.2 Interesting Tensors

The definition we have just given for the tensor is a little bit too abstract for our intention to use them and define some specific tensors. With the intention to clarify, we will give a definition for some interesting tensors as an example of particular forms for some of the important tensors in mathematical or theoretical physics.

Differential Forms

We will now refresh the concept of dual space. Until now, in order to follow the results we have shown, the space on which we have been working is T_pM and the base we have chosen to represent it is $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}_p$. Let us suppose a certain function $f \in \mathbf{F}(M)$, in a way that $(df)_p : T_pM \longrightarrow T_p\mathbb{R} \cong \mathbb{R}$, that is, $(df)_p \in \hom_{\mathbb{R}}(T_pM, \mathbb{R})$, or equivalently, $(df)_p \in (T_pM)^*$.
If we particularize the function f for the projection of the *i-ith* component x_i, we will observe:

$$d(x_i)_p \left(\frac{\partial}{\partial x_j}\right)_p = \frac{\partial x_i}{\partial x_i}|_p = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases} \tag{A.19}$$

and given that for every i the tangent linear functions obtained from the component functions belong to the dual space, and that because of the condition we just saw their linear independence can be proven, the collection $\{dx_1, ..., dx_n\}_p$ forms a base of the dual space $(T_pM)^*$.

In this way, a 1-form differential will be the dual element to the vector field, an application such as:

$$\begin{aligned} \omega : M &\longrightarrow \bigsqcup\nolimits_{p\in M}(T_pM)^* \\ p &\longmapsto \omega_p \end{aligned} \tag{A.20}$$

in a way that every ω_p belongs to its corresponding $(T_pM)^*$, and which, given certain coordinate functions, the form is expressed as $\omega_p = \sum_i \omega_i(p)(dx_i)_p$, where ω_i are differentiable functions of U in $\mathbb{R}$.
According to this definition it immediately follows that the 1-form differentials are tensors, while being applications from T_pM to $\mathbb{R}$. Finally we denote the set of 1-form differentials in the manifold M as $\Omega^1(M)$, and it is an $\mathbb{R}$-vector space.

Metrics

Considering again the space T_pM, let us remember it was $\mathbb{R}$-linear n-dimensional, a metric is any bilinear application $g : T_pM \times T_pM \longrightarrow \mathbb{R}$ simmetrical and non-degenerate (that is, $g(u, v) = 0$ for every $u \Leftrightarrow v = 0$).
We will call a certain base $\{e_1, ..., e_n\}$ of T_pM ortonormal if:

- $g(e_i, e_j) = 0$, $\forall i \neq j$
- $g(e_i, e_i) = 1$

Common Metrics in physycs are the euclidian $g = dx^2 + dy^2 + dz^2$, where the manifold is the ordinary space $\mathbb{R}^3$, or Minkowski's $g = dx^2 + dy^2 + dz^2 - c^2dt^2$, where the manifold is the so called Minkowski's space-time identified with$\mathbb{R}^4$.

In relation to this metric there's a mathematical object quite important named connection. A connection is an application $\nabla : X(M) \times X(M) \longrightarrow X(M)$, which for every pair of fields X, Y associates a third one $\nabla_X Y$, satisfying:

- $\nabla_X(fY) = X(f)Y + f\nabla_X Y \ , \forall f \in \mathbf{F}(M)$
- $\nabla_{fX} Y = f\nabla_X Y \ , \forall f \in \mathbf{F}(M)$

The operator connection is not a tensor. It allows us to compare vector fields (even more generally, tensors) in different tangent spaces, since the Lie derivative needs to transport the field (or tensor) form one tangent space to the other. A connection is completely determined (in coordinates) by the coefficients Γ_{ij}^k named *Christoffel Symbols* defined as:

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=0}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tag{A.21}$$

Expressed in coordinates as:

$$\nabla_X Y = \nabla_{\sum_i \lambda_i \frac{\partial}{\partial x_i}} \left(\sum_j \mu_j \frac{\partial}{\partial x_j} \right) = \sum_i \lambda_i \left(\sum_j \left(\frac{\partial \mu_j}{\partial x_i} + \mu_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \right) \tag{A.22}$$

There are lots of interesting results on Tensor Calculus and Algebra, but the author strongly feels these few definitions listed are more than enough to tackle most of the problems and ideas commented on this work.

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