Velocity-surface stability of a moving charged particle in a controlled electromagnetic field

L. Acho[1]

[1]Department of Mathematics
Polytechnic University of Catalunya
Barcelona 08036, Spain.
leonardo.acho@upc.edu

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Abstract

Given the equation motion of a moving charged particle in a controlled electromagnetic field, this paper proves that its velocity-trajectory motion converges to an specified velocity-surface in the 3-D Euclidean dimensional space. This is basically realized by just manipulating the electric field of an electromagnetic field. Lyapunov theory is invoked to test our statement; besides, a numerical example is provided to support our theoretical contribution. Finally, we consider that the exposition of this paper could be of interest to undergraduate students.

1 Introduction

The second Lyapunov method, also called *The Direct Lyapunov Method*, is now widely used to analyse stability motion of dynamic systems due to its simplicity and efficiency. This theory has been applied, for instance, in physics, astronomy, chemistry, biology, and so on. The main benefit of Lyapunov theory is its ability to conclude stability of systems (*stability in the sense of Lyapunov*) without explicitly integrating the differential equations involved in our system. This method is basically a generalization of the idea that a energy function, the *Lyapunov function*, associated to a system to be analysed, is decreasing along the system trajectory. For a brief discussion on this theory, see, for instance [1].

On the other hand, and motivated by applications where the natural operation mode is periodic, for instance, orbital stabilization of mechanical systems (see, e.g., [2] and references therein), stability analysis of a velocity-surface for a moving charged particle in a electromagnetic environment is here analysed by using Lyapunov theory. That is to say, given the motion of a travelling charged particle in a controlled electromagnetic field, we prove that its velocity-trajectory converges to an specified velocity-surface in the 3-D Euclidean dimensional space. At last to say, the use of a Lyapunov function to conclude orbital stability of dynamical systems is also covered in [3].

The rest of this paper is organized as follows. Section II gives the problem statement and a solution to it. Section III shows a numerical experiment to support our theoretical affirmation. Finally, Section IV states the conclusions.

2 Velocity-surface stability analysis

The objective of this section is to demonstrate that it is possible to produce a 3-D-velocity-surface such that any velocity-trajectory of a charged particle in motion starting close enough to this surface, will converge to it as time goes on. This is realized by manipulating the electric field of an electromagnetic field. Lyapunov theory is the mathematical tool employed to stablish this statement.

The equation describing the motion of a charged particle Q in a electromagnetic fields is defined by

$$m\frac{d\mathbf{v}}{dt} = Q[\mathbf{v} \times \mathbf{B} + \mathbf{E}], \tag{1}$$

where $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively; $\mathbf{v}$ is the velocity of the charged particle in this electromagnetic environment, and m is the particle mass. Let us suppose that the electric field (a controlled one) is

$$\mathbf{E} = \mathbf{v}(k- \mid \mathbf{v} \mid), \tag{2}$$

where $\mid \cdot \mid$ is the vector Euclidean norm, and $k \in R$ is a positive parameter at our disposal. Then, equation (2) in (1) produces

$$m\frac{d\mathbf{v}}{dt} = Q\left[\mathbf{v} \times \mathbf{B} + \mathbf{v}(k- \mid \mathbf{v} \mid)\right]. \tag{3}$$

Let us use the next candidate Lyapunov function:

$$V(t) = \mathbf{v} \cdot \mathbf{v}. \tag{4}$$

Then, its time-derivative along the system trajectory (3) yields

$$\begin{aligned} \dot{V}(t) &= \frac{dV(t)}{dt} \\ &= \frac{2Q}{m}\left[\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) + \mathbf{v} \cdot \mathbf{v}(k- \mid \mathbf{v} \mid)\right] \\ &= \frac{2Q}{m}\left[\mathbf{v} \cdot \mathbf{v}(k- \mid \mathbf{v} \mid)\right]. \end{aligned} \tag{5}$$

Observe that $\dot{V}(t)$ is positive if $|\ \mathbf{v}\ | < k$ and negative if $|\ \mathbf{v}\ | > k$. This means that the velocity-surface equation (VSE), $|\ \mathbf{v}\ | = k$, is locally attractive; i.e., for any velocity-trajectory starting close enough to this velocity-surface, the system trajectory (3) will converge to it. On the other hand, if the origin of the system (3) is the only equilibrium point; then, any trajectory, but the origin, will converge to the VSE[1]. In resume, we arrive to the following result.

Theorem 1.- Given the dynamic of a charged particle in motion (1), its velocity-trajectory will locally converge to the velocity-surface established by $|\ v\ | = k$, $k \in R^+$, if we set the electric field as $\mathbf{E} = \mathbf{v}(k - |\ \mathbf{v}\ |)$ in an electromagnetic field. Moreover, if the system only has the origin as its unique equilibrium point; then, any velocity-trajectory starting anywhere but the origin will converge to the stated velocity-surface.

3 Numerical example

This section describes a numerical example. Hence, if $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, then the corresponding ordinary differential equations of (3), yields

$$\dot{v}_x = \frac{Q}{m}(B_z v_y - v_z B_y + p v_x),$$
$$\dot{v}_y = \frac{Q}{m}(B_x v_z - v_x B_z + p v_y),$$
$$\dot{v}_z = \frac{Q}{m}(B_y v_x - v_y B_x + p v_z),$$

where $p = k - \sqrt{v_x^2 + v_y^2 + v_z^2}$. Using $k = 2$ and the charged particle with $Q = 8\mu$c and mass $m = 0.5\mu$ kg, simulation results of the above system are shown in Figures 1, 2 ad 3. Each figure present the same experiments using trajectories generated by three different sets of initial conditions. The *blue* line corresponds to $v_x(0) = -3.5$, $v_y(0) = 0.0$, and $v_z(0) = 3.5$; the *red* line for the case when $v_x(0) = 0.0$, $v_y(0) = 0.0$, and $v_z(0) = -1.5$; and the *yellow* one with $v_x(0) = 0.0$, $v_y(0) = -4.0$, and $v_z(0) = 0.0$. All of them in meter per second. Finally, a free space motion is assumed. From these figures, we can appreciate that these trajectories converge to the expected velocity-spherical-surface (the green surface shown in the mentioned figures).

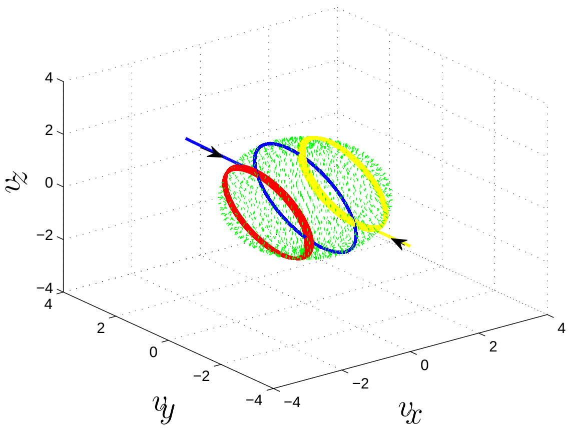

Figure 1: Simulation result using $B_x = B_z = 0.02$T and $B_y = -0.02$T.

[1]Almost the same lines of thinking are employed by [3] to conclude orbital stability in one of the given examples.

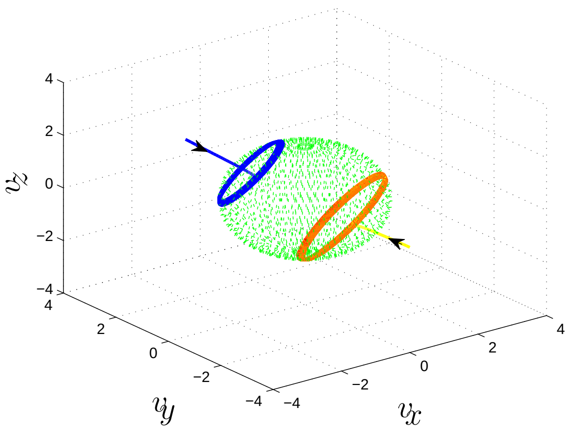

Figure 2: Simulation result using $B_y = B_z = 0.02$T and $Bx = -0.02$T.

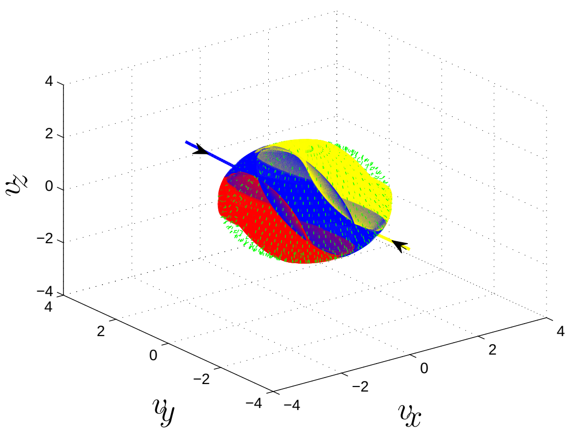

Figure 3: Simulation result using $B_x = B_z = -0.02$T and $B_y = 0.02\cos(0.002t)$T.

4 Conclusion

Lyapunov theory was used to prove stability of a specified velocity-surface of a moving charged particle in a controlled electromagnetic field. We assumed that the electric field is manipulable, and we set $\mathbf{E} = \mathbf{v}(k- \mid \mathbf{v} \mid)$. Obviously, and according to the Lyapunov theory, other velocity surfaces are possible by changing the term $\mid \mathbf{v} \mid$, for instance, to an ellipsoid, if $\mid \mathbf{v} \mid$ is replaced by $\sqrt{\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2} + \frac{v_z^2}{c^2}}$, and so far.

Acknowledgments

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