H-colorings of Large Degree Graphs*

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Abstract. We consider the H-coloring problem on graphs with vertices of large degree. We prove that for H an odd cycle, the problem belongs to P. We also study the phase transition of the problem, for an infinite family of graphs of a given chromatic number, i.e. the threshold density value for which the problem changes from P to NP-complete. We extend the result for the case that the input graph has a logarithmic size of small degree vertices. As a corollary, we get a new result on the chromatic number; a new family of graphs, for which computing the chromatic number can be done in polynomial time.

1 Introduction

Given a graph G, with vertex set $V(G)$ and edge set $E(G)$, a *k-coloring* of G is a mapping of V to $\{1, \ldots, k\}$ such that no two vertices on the same edge receive the same color. Given graphs G and H, an *homomorphism* of G to H is an edge preserving mapping of $V(G)$ to $V(H)$.

For any fixed graph H, the *H-coloring problem* consists in deciding whether there is a homomorphism of a given input graph G into H.

As usual, let K_k denote a complete graph on k vertices. Then, the problem of deciding if there is a K_k -coloring of G is the problem of deciding if G is k-colorable. Thus the H-coloring problem naturally generalizes decision problems related to the chromatic number. Further examples of H-coloring problems include *circular chromatic number* [7], *T-colorings* and problems related to *channel assignments problems* [6].

For general graphs, the complexity of the H-coloring problem is well known, the problem is in P if H is bipartite, otherwise it is NP-complete [5].

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In this paper, we are interested in studying the complexity of the H-coloring for certain types of dense input graphs. Given a constant $0 \leq \alpha < 1$, a graph G is said to be *α-dense* if $\delta(G) \geq \alpha|G(V)|$, where $\delta(G)$ denotes the *minimum vertex degree* of G.

For any fixed graph H and a constant $0 \leq \alpha < 1$, the *(H, α)-coloring problem* consists in: Given an *α-dense* graph G, determine whether there is an H-coloring of G. Notice that α is a parameter of the problem.

¿From the previous definition, it follows that for any two constants α, β such that $0 \leq \alpha \leq \beta < 1$, we get:

- If the (H, α)-coloring problem is in P, then the (H, β)-coloring problem is in P,
- if the (H, β)-coloring problem is NP-complete , then the (H, α)-coloring problem is NP-complete,

where the definitions of the complexity classes P and NP-complete are the well known ones. These properties motivates the following definition:

Definition 1. *Given a graph H the* complexity threshold $c(H)$ *is defined as the* $\inf\{\alpha \mid (H, \alpha)\text{-coloring} \in \mathsf{P}\}$.

One immediate consequence of the previous definition is the fact that $c(H) = 0$ if and only if for every $0 < \alpha < 1$, the (H, α)- coloring problem is in P. This is the case when H is a bipartite graph, as in this case even the $(H, 0)$-coloring problem is in P.

For every graph H, the threshold $c(H)$ exists. A question of interest is whether $c(H)$ can be defined alternatively as

$$\sup\{\alpha \mid (H, \alpha)\text{-coloring} \in \mathsf{NP}\text{-complete}\}.$$

A second question is whether there are graphs H for which

$$c(H) = \min\{\alpha \mid (H, \alpha)\text{-coloring} \in \mathsf{P}\}.$$

All non-bipartite examples for which we can compute the exact threshold indicate that $c(H)$ is never attained. A positive answer to the first question follows from an affirmative solution to the following *dichotomy conjecture*:

> For every H and every α, the *(H, α)-coloring problem* is either NP-complete or P.

A partial affirmative answer to the previous conjecture was given by Edwards, for a particular case. He considered the graph family $\{K_k\}_{k\geq 3}$, where for each k, $c(K_k) = (k-3)/(k-2)$ (see Theorem 2.5 of [3]).

In Sections 2 and 3, we extend Edwards' results. We give some examples of particular families of graph for which the Dichotomy conjecture is true. We also determine the complexity threshold, for infinitely many graphs with a given chromatic number. In Section 4, we relax the notion of density, by allowing a small number of low degree vertices. It is interesting to notice taht we obtain the same results, as in the density case.

We use the standard notation from graph theory. Given two graphs G and G' the graph $G \oplus G'$ is formed by taking one copy of G and G', and join by edges all vertices of G with all vertices of G'. Given G and a $v \in V(G)$, $\mathcal{N}[v]$ denotes the set of neighbors of v. Given G and a subset $S \subseteq V(G)$, the *induced subgraph* $G[S]$ has as vertex set S and edge set $\{(u,v)|u,v \in S \wedge (u,v) \in E(G)\}$. Given a graph G and a subgraph G' of G, we say that $v \in V(G)$ is *completely joined* to G' if for all $u \in V(G')$, we have $(u,v) \in E(G)$. The *chromatic number* $\chi(G)$ is the minimum number of colors needed to color G. The *clique number* $\omega(G)$ is the maximum k such that K_k is a subgraph of G. A cycle on $2k+1$-vertices is denoted by C_{2k+1}, and $\overline{C}_{2k+1}$ is the complement graph of C_{2k+1}.

Through the paper, we shall work under the plausible hypothesis that $\mathsf{P} \neq \mathsf{NP}$.

2 Exact Thresholds

We begin the section, stating a technical lemma due to Edwards, which will be needed later.

Lemma 1. *[3] For any integer $k \geq 3$, let α be a fixed rational such that $(k-3)/(k-2) < \alpha < 1$, and let G be an α-dense graph, with $|V(G)| = n$. Then, there exits a $U \subseteq V(G)$, and a set $\mathcal{T}$ of $(k-2)$-cliques in G (not necessarily disjoint), such that:*

- *$G[U]$ is the union of all cliques in $\mathcal{T}$.*
- *$|\mathcal{T}| = O(\log n)$.*
- *Every vertex in $V(G) \setminus U$ is completely joined to at least one clique in $\mathcal{T}$.*

Furthermore, U can be computed in polynomial time.

Using the previous lemma, we prove the main result in this section, we characterize an infinite family $\{H_i\}$ of graphs, for which $c(H_i) \leq (k-3)/(k-2)$.

Theorem 1. *For any fixed integer $k \geq 3$, assume that H satisfy $\chi(H) = k$, and that any subgraph of H isomorphic to K_{k-2} is contained in at most two subgraphs of H isomorphic to K_{k-1}. Then for $\frac{k-3}{k-2} \leq \alpha \leq 1$, the (H, α)-coloring problem belongs to* P*.*

Proof. Let G be an α-dense instance to the (H, α)-coloring problem, and let U be as in the statement of the previous lemma. Consider any fixed homomorphism f of $G[U]$ to H. By lemma 1, any given $v \in V(G) \setminus U$ is completely joined with at least one $(k-2)$-clique. Therefore, by the hypothesis on H, if $F(v)$ is the set of possible vertices of H that can extend f to the whole $U \cup \{v\}$, then $|F(v)| \leq 2$.

First, we prove that in polynomial time, we can decide if f can be extended to the whole G. Let $V(G) \setminus U = \{v_1, \ldots, v_m\}$, and $V(H) = \{1, \ldots, r\}$. Define the instance of the 2-SAT problem as follows, the set of variables is $x_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq r$ each of them indicating whether $f(i) = j$, and the clauses are:

1. $\{x_{i,j} | j \in F(i)\}$, for $1 \leq i \leq m$,
2. $\{\overline{x}_{i,j}, \overline{x}_{i,l}\}$ for $1 \leq i \leq m$, $1 \leq j < l \leq r$,
3. $\{\overline{x}_{i,j}, \overline{x}_{h,l}\}$ if $(v_i, v_h) \in E(G)$, and $(i, l) \notin E(H)$.

It is easy to see that the homomorphism f on $G[U]$ can be extended to G if and only if this instance of 2-SAT has a satisfying assignment. As 2-SAT belongs to the class P, then the problem of deciding the extension of f to G is also in P. Hence, to determine if there exists an H-coloring f of G, we try all possible homomorphisms of $G[U]$ to H, until obtaining a valid f (otherwise, the answer is NO). As the total number of possibilities for the homomorphism f is $r^{|U|} \leq r^{O(\log n)}$, then exhaustive search together with the 2-SAT algorithm can decide, in polynomial time, if G is H-colorable.

As a corollary we obtain that the exact threshold for odd cycles is 0.

Corollary 1. *For every $0 < \alpha < 1$ and for every $k \geq 1$, the (C_{2k+1}, α)-coloring problem is in the class* P*.*

Note that also for even cycles and all bipartite graphs H, the threshold is 0, and these are the only known case where the threshold is attained.

Another result we can obtain from the previous theorem gives us an exact threshold for infinitely many graphs of a given chromatic number:

Corollary 2. $c(C_{2k+1} \oplus K_{k-3}) = (k-3)/(k-2)$ *for every* $k \geq 3$.

Proof. By the the previous theorem, for $\alpha(k-3)/(k-2)$ the $(C_{2k+1} \oplus K_{k-3}, \alpha)$-coloring is in P. To finish the proof that $(k-3)/(k-2)$ is a phase transition for the $(C_{2k+1} \oplus K_{k-3}, \alpha)$-coloring, we need to prove completeness. Let us consider the following reduction:
For any fixed constant k, given a graph G construct a graph G' in the following way. $V(G')$ consist of $k-2$ copies of the vertices $V(G)$. To form $E(G')$, join by an edge all vertices in different copies, also add the edges $E(G)$ to one of the copies. Then, G is C_{2k+1}-colorable if and only if G' is $C_{2k+1} \oplus K_{k-3}$-colorable. Moreover, $|V(G')| = (k-2)|V(G)|$, and every vertex in $V(G')$ has degree at least $\frac{k-3}{k-2}|V(G')|$. Therefore, G' is $\frac{k-3}{k-2}$-dense.

3 Threshold Bounds

We have seen that there are graphs for which their threshold is 0, e.g. all cycles. Next, we prove that the threshold of a graph can never be 1,

Proposition 1. *For every graph* H, $c(H) < 1$.

Proof. Let H be a fixed graph, and let $\omega = \omega(H)$. It follows from the classical result of Turán (see e.g. [2], page 108), that a graph with more than $|G(V)|^2/2(1-1/\omega)$ edges must contain a $K_{\omega+1}$, and therefore, the graph can not be homomorphic to H. It follows that for any graph H, $c(H) \leq 1 - 1/\omega < 1$.

In the following result we present a family $\{H_k\}_{k>1}$, for which we can compute sharp bounds for the complexity threshold of the (H, α)-coloring problem,

Proposition 2. *For all* $k > 1$, *the* $(K_3 \oplus \overline{C}_{2k+1}, \alpha)$*-coloring problem is:*

- *NP-complete, if* $0 \leq \alpha \leq 1 - \frac{3}{3k+1}$,
- *P, if* $1 > \alpha > 1 - \frac{k}{k+1}$.

Proof. Let us consider the following reduction:
For any fixed constant k, given a graph $G = (V, E)$ such that the number of edges is a multiple of k, construct a graph $G' = (V', E')$ in the following way: Take a $\overline{C}_{2k+1}$ and replace each vertex of it by $|V(G)|/k$ independent vertices. Denote this graph by $\overline{F}_{2k+1}$, then $G' = \overline{F}_{2k+1} \oplus G$. Notice

$|V'| = (\frac{2k+1}{k}+1)|V|$. Then G is 3-colorable if and only if G' is $K_3 \oplus \overline{C}_{2k+1}$-colorable. Furthermore, every vertex in G' has degree at least $(\frac{2k-2}{k} + 1)|V|$, which implies that α for G is less or equal to $\frac{((2k-2)/k)+1}{((2k+1)/k)+1}$.

To prove the P part, recall that for any $k > 1$ $\omega(\overline{C}_{2k+1}) = k$. Furthermore, any $(k-1)$-clique in $\overline{C}_{2k+1}$ is contained in at most two cliques of size k. Therefore, any $(k+2)$-clique in $K_3 \oplus \overline{C}_{2k+1}$ is contained in at most two cliques of size $k+3$. Using Theorem 1, we get the second part of the statement.

The second statement of Proposition 2 gives us better bounds that the one produced by Proposition 1, as $\omega(K_3 \oplus \overline{C}_{2k+1}) = k+3$, so the upper bound produced by Proposition 1 is $1 - \frac{1}{k+3}$, which is larger than $1 - \frac{k}{k+1}$.

It is still open to decide if for values $1 - \frac{3}{3k+1} < \alpha \leq 1 - \frac{1}{k+1}$, the problem is in P or NP-complete, but notice that for $k = 6$, both values are above 0.8, and differ in 0.1. This means that, for values of $k \geq 6$, the decision problem is NP-complete for most of the dense graphs. The value of the bounds seems to indicate that, the complexity threshold for the given graphs coincides with the threshold for K_{k+4}.

The next result gives a necessary condition to guarantee a complexity threshold of at least 1/2.

Proposition 3. *For any graph H, such that for some $x \in V(H)$ the graph $H[\mathcal{N}[x]]$ has $\chi(H[\mathcal{N}[x]]) > 2$, the $(H, \frac{1}{2})$-coloring problem is NP-complete.*

Proof. Let H be a graph as in the hypothesis of the Proposition. For every $x \in V(H)$, let $H_x = H[\mathcal{N}[x]]$. Consider the graph H' obtained as the disjoint union of the graphs H_x such that $\chi(H_x) > 2$. As each H_x is not bipartite, then H' is not bipartite and hence, the H'-coloring problem is NP-complete.

Given a connected graph G', define G in the following way; $V(G) = V(G') \times \{0, 1\}$, and the edges

$$E(G) = \{((v,0),(u,0)) | (v,u) \in E(G')\} \cup \{((v,0),(u,1)) | \forall u, v \in V(G')\}.$$

If f is an H'-coloring of G', there is a $x \in H$ such that f is a H_x-coloring of G'. The mapping $g : G \to H$ defined as $g(v,0) = f(v)$ and $g(v,1) = x$ is an H-coloring of G. On the other hand, if g is an H-coloring of G, then for any $v \in V(G')$, the mapping $f(v) = g(v,0)$ is an H'-coloring of G'. So the above construction is a polynomial-time reduction from the H'-coloring problem to the $(H, \frac{1}{2})$-coloring problem.

These investigations lead to the following interesting problem:

Characterize graphs H with $c(H) = 0$.

Presently we do not know any 3-chromatic graph H for which the (H, α)-coloring problem is NP-complete for some $\alpha > 0$. A candidate for such a graph H is the particular subdivision of K_4 where we subdivide each edge of a triangle by 2 points. The resulting graph has 10 vertices and it is not C_5-colorable. By the general theorem in [5], H-coloring is NP-complete, and in this particular case it may be seen easily by the reduction from 3NAESAT given in Figure 1. However this reduction and the general reduction in [5] produce graphs with constant minimum degree and thus it does not yield NP-completeness of the (H, α)-coloring problem for any $\alpha > 0$. Perhaps $c(H) = 0$ holds for any 3-chromatic graph H.

4 Almost α-dense graphs.

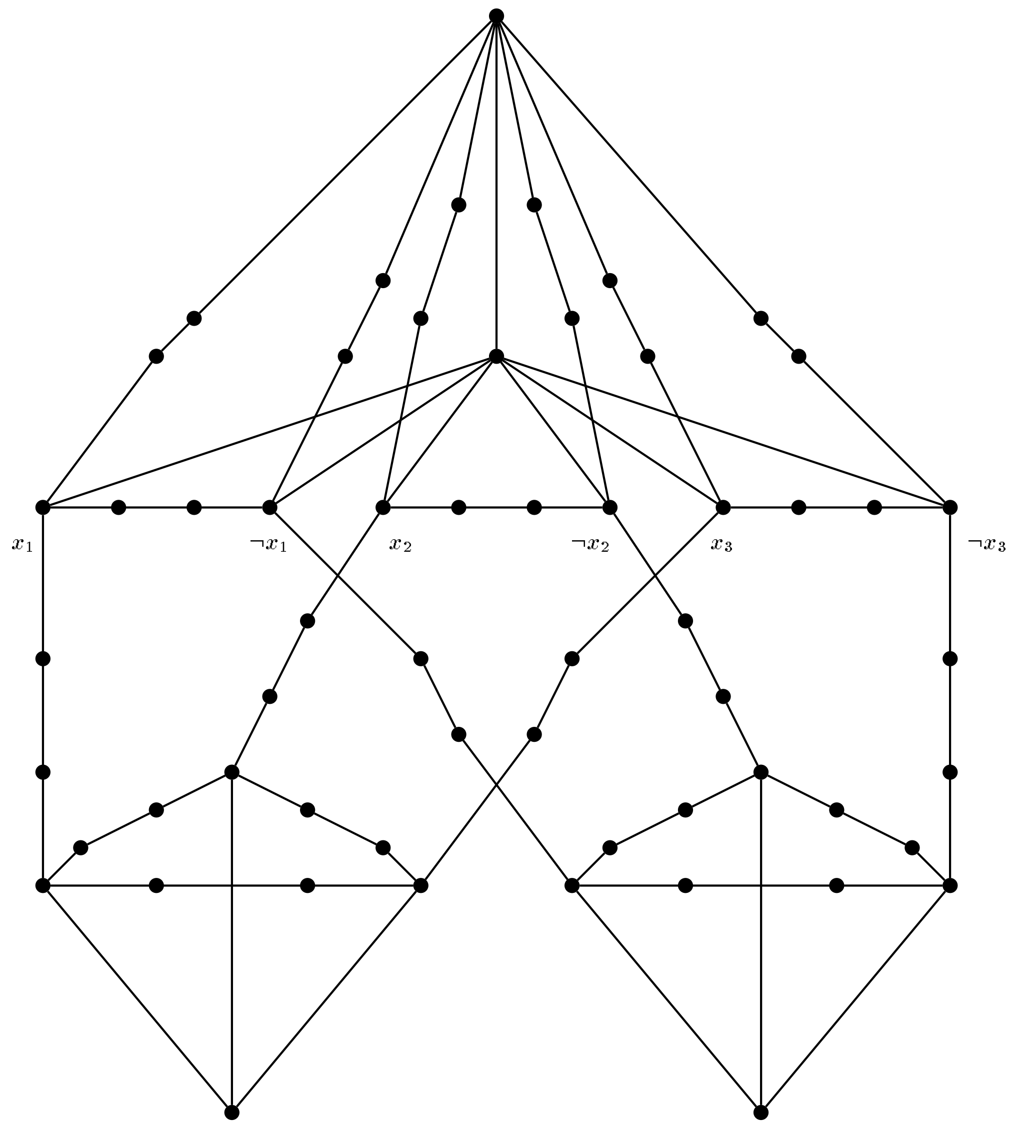
In this section, we consider the complexity of the H-coloring problem, for input graphs with vertices of high degree, except for a subset of logarithmic size.

We say that a graph G is *almost α-dense* if there exists a partition $\{V_1, V_2\}$ of $V(G)$, with $|V_1| = O(\log |V(G)|)$, and such that for every $v \in V_2$, the degree $d_G(v) \geq \alpha |V(G)|$.

We will consider now the H-coloring problem on almost dense graphs: For any fixed graph H, and a constant $0 \leq \alpha < 1$, the *almost-(H, α)-coloring problem* consists in: Given an almost–α-dense graph G, decide whether there is an homomorphism of G into H. In the same way as it has been done in Definition 1, given a graph H, define its *complexity threshold* $\tilde{c}(H)$ as the $\inf\{\alpha \mid \text{almost}(H, \alpha)\text{-coloring} \in \mathsf{P}\}$.

Theorem 2. *Let $k \geq 3$, and $0 < \epsilon < 1$, assume that the graph H satisfy $\chi(H) = k$, and let any subgraph of H isomorphic to K_{k-2}, be contained in at most two subgraphs of H isomorphic to K_{k-1}. Then, for any $\alpha > (k-3)/(k-2) + \epsilon$, the almost (H, α)-coloring problem belongs to* P.

Proof. Let G be an almost α-dense graph, and let $S = \{u \in V(G) \mid d(u) \geq \alpha n\}$, and let $R = V(G) - U$. We claim that for $n \geq n_0 = n_0(\epsilon)$ the graph $G' = G[U]$ is $\alpha - \epsilon$-dense. This follows from the fact that assuming that $|R| = r$, for any $u \in S$ we have $d_{G'}(u) \geq \alpha n - p$, given ϵ for n big aenough it holds that $\alpha n - p \geq (\alpha - \epsilon)(n - p)$ as $p = O(\log n)$. Now, using Lemma 1 we can construct a decomposition (U, P) of S. Recall that $\mathcal{T}$ is a set of $(k-2)$-cliques in G' with vertex set U. Let f be a homomorphism



$$F = \{(x_1, x_2, x_3), (\neg x_1, \neg x_2, \neg x_3)\}$$

Fig. 1. A sketch of the reduction.

of $G[U \cup R]$ to H. For any valid H-coloring f of $G[U \cup R]$, and each $v \in P$, there exists a set $C(v)$ of possible extensions of f to the whole G. then $|C(v)| \leq 2$ as, from Lemma 1 each vertex in P is completely joined to one or more of the cliques in $\mathcal{T}$, which have already been assigned $k-2$ *colors* in H. In the same manner as in the proof of Theorem 1, we can define a polynomial reduction between the problem of extending a H-coloring of $G[U \cup L]$ to a H-coloring of G, and 2-SAT, so it is possible to decide whether an H-coloring of $G[U \cup L]$, can be extended to a H-coloring of G. Since $|U \cup L| = O(\log n)$, exhaustive search plus the reduction yield a polynomial time algorithm to decide the almost (H, α)-colorability of the given G.

As Corollary to the previous theorem, we state the particular case, when $H = K_k$.

Corollary 3. *For any $\alpha > (k-3)/(k-2)$ the problem of deciding whether an almost α-dense graph G is k-colorable, is polynomially solvable.*

Corollary 4. *For every $0 < \alpha < 1$ and for every $k \geq 1$, the almost (C_{2k+1}, α)-coloring problem is in the class* P.

Corollary 5. *For any $\alpha > (k-3)/(k-2)$, the almost $(C_{2k+1} \oplus K_{k-3}, \alpha)$-coloring problem is in the class* P

Notice that the previous corollaries, state that for graphs such that we can compute the phase transition for density, the same phase transition holds for almost density. However, the time bounds might be too large, as the minimum value of n could be high, as ϵ approaches zero.

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