Convergence theorems for some layout measures on random lattice and random geometric graphs *

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Abstract

This work deals with convergence theorems and bounds on the cost of several layout measures for lattice graphs, random lattice graphs and sparse random geometric graphs. For full square lattices, we give optimal layouts for the problems still open. Our convergence theorems can be viewed as an analogue of the Beardwood, Halton and Hammersley theorem for the Euclidian TSP on random points in the d-dimensional cube. As the considered layout measures are non-subadditive, we use percolation theory to obtain our results on random lattices and random geometric graphs. In particular, we deal with the subcritical regimes on these class of graphs.

1 Introduction

Layout problems on graphs aim to find a linear ordering of the nodes of an input graph such that a certain function is minimized. For the problems we consider below, finding an optimal layout is **NP**-hard in general, and therefore it is natural to develop and analyze techniques to obtain tight bounds on restricted instances. Graphs encoding circuits or grids are typical instances of linear arrangement problems. We consider these instances as sparse graphs that have clustering and geometric properties. For these classes of graphs, not much is known. In this paper, we are concerned with lattice graphs, random instances of lattice graphs and random geometric graphs. For most of the layout problems it is an open problem to find exact or approximated polynomial time algorithms for lattice graphs different than the full square lattice of side n and with n^2 points [9, 8, 7].

A graph is said to be a *lattice graph* if it is a node-induced subgraph of the infinite lattice, that is, its vertex set is a subset of $\mathbb{Z}^2$ and two vertices are connected whenever they are at distance one. *Percolation theory* provides a framework to study lattice graphs in a

probabilistic setting. We consider site percolation, where nodes from the infinite lattice are selected with some probability p (selected nodes are called "open"). Let C_0 be the set of all open nodes connected by a path of open nodes to the origin. A basic question in percolation theory is whether or not C_0 can be infinite. Let $\vartheta(p)$ denote the probability that $|C_0| = \infty$, and set $p_c = \inf\{p : \vartheta(p) > 0\}$, the *critical value* of p. It is well-known that $p_c \in (0.5, 1)$ [6]. In this paper, we consider only *subcritical* limiting regimes $p \in (0, p_c)$ in which all components are almost surely finite. Results for *supercritical* regimes are derived in [4, 13]. In order to deal with bounded graphs, we introduce the class of *random lattice graphs* with parameters m and p denoted by $\mathcal{L}_{m,p}$ that corresponds to the lattice graphs whose set of vertices are obtained through the random selection of each element from $\{0, ..., m-1\}^2$, chosen independently with probability p.

A *random geometric graph* $\mathcal{G}(\mathcal{X}_n; r_n)$, with (r_n) some chosen sequence of positive numbers, is defined by a set $\mathcal{X}_n$ of n uniform and independently distributed points on $[0, 1]^d$ and edges formed by joining any two different points at distance less or equal than r_n. For an infinite-volume analogue, let $\mathcal{P}_\lambda$ denote a homogeneous Poisson process on $\mathbb{R}^d$ of intensity λ, and set $\mathcal{P}_{\lambda,0} = \mathcal{P}_\lambda \cup \{0\}$. For n large, after appropriate scaling and centering at a randomly chosen point of $\mathcal{X}_n$, the graph $\mathcal{G}(\mathcal{X}_n; r_n)$ looks locally like $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$. We consider a continuum site percolation process based on the Poisson process; let $\widetilde{\vartheta}(\lambda)$ be the probability that the added point at the origin lies in an infinite component of $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$. Then define the critical percolation λ_c as the infimum of $\{\lambda > 0 : \widetilde{\vartheta}(\lambda) > 0\}$. It is well known [6] that $\lambda_c \in (0, \infty)$. In this paper we shall deal with random geometric graphs satisfying the condition $\lim_{n \to \infty} n r_n^d = \lambda$, for the subcritical regime $\lambda < \lambda_c$.

Our layout problems are formally defined as follows. A *layout* φ on a graph $G = (V, E)$ is a one-to-one function $\varphi : V \to \{1, ..., n\}$ with $n = |V|$. Given a graph G, a layout φ on G and a number $i < n$, let us define the sets:

$$L(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) \le i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) > i\},$$

the measures:

$$\theta(i, \varphi, G) = |\{uv \in E(G) \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|,$$
$$\delta(i, \varphi, G) = |\{u \in L(i, \varphi, G) \ : \ \exists v \in R(i, \varphi, G) : uv \in E(G)\}|,$$
$$\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)| \qquad \text{where } uv \in E(G),$$

and the problems:

- Minimum Linear Arrangement (MINLA): Given a graph $G = (V, E)$, find
  $\text{MINLA}(G) = \min_\varphi \sum_{uv \in E} \lambda(uv, \varphi, G) = \min_\varphi \sum_{i=1}^n \theta(i, \varphi, G)$.

- Minimum Cut Width (MINCUT): Given a graph $G = (V, E)$, find
  $$\text{MINCUT}(G) = \min_\varphi \max_{i=1}^n \theta(i, \varphi, G).$$

- Vertex Separation (VERTSEP): Given a graph $G = (V, E)$, find
  $$\text{MINVS}(G) = \min_\varphi \max_{i=1}^n \delta(i, \varphi, G).$$

- Minimum Sum Cut (MINSUMCUT): Given a graph $G = (V, E)$, find
  $$\text{MINSC}(G) = \min_\varphi \sum_{i=1}^n \delta(i, \varphi, G).$$

- Bisection (BISECTION): Given a graph $G = (V, E)$, find
  $$\text{MINBIS}(G) = \min_\varphi \theta(\lfloor n/2 \rfloor, \varphi, G).$$

- Vertex Bisection (MINVERTBIS): Given a graph $G = (V, E)$, find
  $$\text{MINVB}(G) = \min_\varphi \delta(\lfloor n/2 \rfloor, \varphi, G).$$

The defined problems have important applications in several different areas, see for example [4]. With regard to their complexity, MINCUT and VERTSEP remain **NP**-complete even when restricted to lattice graphs and geometric graphs [4]. For the remaining layout problems the complexity on lattice graphs and geometric graphs is open.

In this paper, we first present optimal layouts for MINVS, MINVB, and MINSC on full square lattices. Previously, the only known optimal layouts for these graphs were for MINCUT, MINBIS and MINLA [7, 9, 8]. Results for the case of d-dimensional c-ary arrays (a generalization of square lattices) on the BISECTION, MINCUT and MINLA problems are presented in [10]. On the other hand, [11] presents a dynamic programming algorithm to solve BISECTION on lattice graphs without holes. Then, we consider general lattice graphs, and we present upper bounds for several layout problems on any lattice graph. Afterwards, we move to a randomized setting where we deal with random lattice and random geometric graphs. The main result for these graphs can be viewed as an analogue of the celebrated Beardwood, Halton and Hammersley theorem on the cost for the traveling salesman problem (TSP) on random points distributed in $[0, 1]^d$,

> **BHH Theorem [1].** Let $X = \{X_i\}$ be a sequence of independent and uniformly distributed points in $[0, 1]^d$. Let MINTSP(n) denote the length of the optimal solution of the TSP among the first n points of X. Then, there exists a constant $\beta(d)$ such that MINTSP$(n)/n^{(d-1)/d}$ converges to $\beta(d)$ almost surely as $n \to \infty$.

A key property to prove BHH-like results is geometric subadditivity (see Chapter 3 of [14]). This property does not hold for our layout problems, therefore we take a completely different approach using percolation theory. Except for BISECTION and MINVERTBIS, one property that all these problems share is *monotonicity*, that is, the optimal value on a sub-graph is always less than or equal to the optimal value in the whole graph.

n	n	$n-1$	$\vdots$	4	3	0
$n-1$	n	n	$n-1$	$\ddots$	4	2
$\cdots$	$n-1$	n	n	$n-1$		3
4		$n-1$	n	n	$n-1$	$\cdots$
3	4	$\ddots$	$n-1$	n	n	$n-2$
2	3	4		$n-1$	n	$n-1$
1	2	3	4	$\vdots$	$n-1$	n

$$\sum_{i=2}^{n-1} i$$

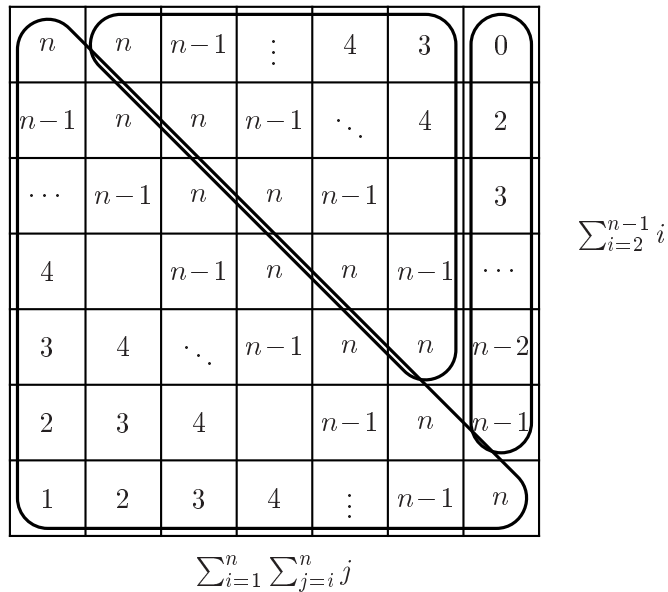$$\sum_{i=1}^{n} \sum_{j=i}^{n} j$$

Figure 1: Values of the vertex cut in the diagonal ordering φ_D.

2 Bounds for lattice graphs

We begin this section by characterizing the optimal layouts for some of the problems defined in the previous section on square lattice graphs. Then we give some deterministic upper bounds on the costs of the defined problems on finite subsets of the integer lattice $\mathbb{Z}^2$.

Each subset L of vertices in $\mathbb{Z}^2$ is identified with a lattice graph, namely the maximal subgraph of the 2-dimensional integer lattice with vertex set L. Let L_n be the full $n \times n$ square lattice graph. The next results concern the optimality of the *diagonal ordering* φ_D on L_n. In this ordering, $\mathbf{x} = (x, y)$ precedes $\mathbf{x}' = (x', y')$ whenever $x + y < x' + y'$, and whenever $x + y = x' + y'$ and $x < x'$.

Let $1 \le k < n^2$ and $\mathbf{x} = (x, y)$ such that $\varphi_D(\mathbf{x}) = k$. Define $r \in \mathbb{Z}^+$ as follows: If $x + y \le n$, then $r = x + y - 1$, then we have $r(r-1)/2 < k \le r(r+1)/2$ and $\delta(k, \varphi_D, L_n) = r$. If $x + y > n$ with $x \ne n$, then $r = 2n - (x+y) + 1$, and when $x = n$ then $r = 2n - (x+y)$. In this two last cases we have $r(r-1)/2 < n^2 - k \le r(r+1)/2$ and $\delta(k, \varphi_D, L_n) = r + 1$. In particular, for all k, $\delta(k, \varphi_D, L_n) \le n$. See Figure 1.

Proposition 1 (Vertex isoperimetric inequality). For any layout φ on L_n and any $k \in \{1, \ldots, n^2\}$, we have $\delta(k, \varphi, L_n) \ge \delta(k, \varphi_D, L_n)$.

Proof. Given φ and k, let A be the set $L(k, \varphi, L_n)$, and let $\partial_{\text{in}} A$ be the number of boundary elements of A, i.e. elements of A having neighbours in $L_n \backslash A$. Thus $\partial_{\text{in}} A = \delta(k, \varphi, L_n)$.

Let A' be the set in L_n obtained by "pushing each vertical section of A down as far as possible towards the x-axis"; more precisely, setting $S_i(A) = \{j : (i,j) \in A\}$ for $i \in \{1, \ldots, n\}$, let

$$A' = \cup_{i \in \{1,\ldots,n\}: S_i(A) \neq \emptyset} \{i\} \times \{1, \ldots, |S_i(A)|\}.$$

Notice that $|A'| = |A|$, and it is not hard to check that $\partial_{\mathrm{in}} A' \leq \partial_{\mathrm{in}} A$.

Let A'' be the set in L_n, obtained by "pushing each horizontal section of A' sideways as far as possible towards the y-axis", in an analogous manner to the construction of A' from A. Then $|A''| = |A'| = |A|$, and $\partial_{\mathrm{in}} A'' \leq \partial_{\mathrm{in}} A' \leq \partial_{\mathrm{in}} A$. Moreover, A'' is a down-set, that is, it has the property that for any $\mathbf{x} \in A''$, all vertices of L_n lying directly below or directly to the left of $\mathbf{x}$ are in A''. Hence, without loss of generality, from now on we assume that A is a down-set.

First suppose $(1, n) \notin A$ and $(n, 1) \notin A$. Choose the positive integer r so that $r(r - 1)/2 < k \leq r(r+1)/2$. Then there must be a point $\mathbf{x} = (x, y) \in A$ with $x + y \geq r + 1$. Choose such a point $\mathbf{x}$, having neighbours in $L_n \setminus A$ both to its right and above it. Then there is a path of y or more boundary points of A from the bottom of the square to $\mathbf{x}$, and another path of x or more boundary points of A from the left of the square to $\mathbf{x}$, and these paths do not intersect each other except at $\mathbf{x}$. Therefore

$$\delta(k, \varphi, L_n) = \partial_{\mathrm{in}} A = y + x - 1 \geq r.$$

If $r \leq n$ then we have $\delta(k, \varphi_D, L_n) = r \leq \delta(k, \varphi, L_n)$, while if $r > n$ then we have $\delta(k, \varphi_D, L_n) \leq n \leq \delta(k, \varphi, L_n)$. Thus we get the inequality claimed.

Next suppose $(1, n) \in A$ and $(n, 1) \in A$. Choose the positive integer r so that $r(r - 1)/2 < n^2 - k \leq r(r + 1)/2$. Then there must be a point $\mathbf{x} = (x, y) \in L_n \setminus A$ with $(n + 1 - x) + (n + 1 - y) \geq r + 1$, that is, $x + y \leq 2n + 1 - r$. Choose such a point $\mathbf{x}$, having neighbours in A both to its left and above it. Then there is a path of at least $n - y + 1$ boundary points of A from the top of the square to the point just to the left of $\mathbf{x}$, and another path of at least $n - x + 1$ boundary points of A from the right of the square to just below $\mathbf{x}$, and these paths do not intersect. Therefore

$$\delta(k, \varphi, L_n) = \partial_{\mathrm{in}} A \geq 2n - x - y + 2 \geq r + 1.$$

If $r < n$ then $\delta(k, \varphi_D, L_n) = r + 1 \leq \delta(k, \varphi, L_n)$, while if $r \geq n$ then $\delta(k, \varphi_D, L_n) \leq n \leq \delta(k, \varphi, L_n)$. Thus in this case we have the inequality claimed.

Finally, consider the case when only one of the the corners $(n, 1)$ and $(1, n)$ is in A. In these cases we have $\partial_{\mathrm{in}} A = n \geq \delta(k, \varphi_D, L_n)$. $\qquad\square$

The previous Proposition is a special case of Corollary 9 in [2], who in fact prove the d-dimensional version for arbitrary d. We believe our proof for $d = 2$ is of interest by itself.

Theorem 1. For any n, φ_D is optimal for the VERTSEP, MINSUMCUT and MINVERTBIS problems on (L_n). Moreover, $\text{MINVS}(L_n) = \text{MINVB}(L_n) = n$ and $n^{-3}\text{MINSC}(L_n) \to 2/3$ as $n \to \infty$.

Proof. The previous isoperimetric inequality yields the optimality of φ_D for the costs of MINVS, MINSC and MINVB on L_n. Also we get that $\text{MINVS}(L_n) = \text{MINVB}(L_n) = n$. To compute the sum of the cuts for φ_D, consider for each point in the lattice the value of the vertex cut produced by the diagonal ordering (see Figure 1), then arranging the sum by points with the same vertex cut, we get

$$\text{MINSC}(L_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} j + \sum_{i=3}^{n} i(i-2) + \sum_{i=2}^{(n-1)} i = \tfrac{2}{3}n^3 + \tfrac{1}{2}n^2 - \tfrac{7}{6}n.$$

$\square$

Lemma 1. For any lattice graph L with n vertices, and any $m \in \{1, 2, \dots, n\}$, there is a layout φ on L such that $\theta(m, \varphi, L) \leq 2^{3/2}\sqrt{n} + 1$.

Proof. We are looking for a subset S of L consisting of m vertices, such that there are at most $2^{3/2}\sqrt{n} + 1$ edges between S and $L\backslash S$.

Let $\alpha > 0$ be a constant, to be chosen later. For $x \in \mathbb{Z}$ let $S_x = \{y \in \mathbb{Z} : (x, y) \in L\}$and let $V = \{x \in \mathbb{Z} : |S_x| \geq \alpha\sqrt{n}\}$. For $i \in \mathbb{Z}$, let H_i denote the half-space $(-\infty, i] \times \mathbb{R}$. Set

$$i_0 = \min\{i \in \mathbb{Z} : |L \cap H_i| \geq m\}.$$

Consider the case $i_0 \notin V$. Then define S to be a set of the form

$$S = L \cap (H_{i_0-1} \cup (\{i_0\} \times (-\infty, j]))$$

with j chosen so that S has precisely m elements.

With this definition of S for $i_0 \notin V$, the number of horizontal edges between S and $L\backslash S$ is at most $|S_{i_0}|$, and hence is at most $\alpha\sqrt{n}$. There is at most one vertical edge between S and $L\backslash S$, so the number of edges from S to $L\backslash S$ is at most $\alpha\sqrt{n} + 1$ when $i_0 \notin V$.

Now consider the other case $i_0 \in V$. Let $I = [i_1, i_2]$ be the largest integer interval which includes i_0 and is contained in V. Then $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$. Also, as $|V| \leq \alpha^{-1}\sqrt{n}$, $i_2 - i_1 + 1 \leq \alpha^{-1}\sqrt{n}$. We have

$$|L \cap H_{i_1-1}| < m \leq |L \cap H_{i_2}|.$$

For $j \in \mathbb{Z}$ let $T_j = [i_1, i_2] \times (-\infty, j]$. Choose j_0 so that

$$|L \cap (H_{i_1-1} \cup T_{j_0-1})| < m \leq |L \cap (H_{i_1-1} \cup T_{j_0})|,$$

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and let S be $L \cap (H_{i_1-1} \cup T_{j_0-1} \cup ([i_1, i_3] \times \{j_0\}))$, with $i_3 \in [i_1, i_2]$ chosen so that S has precisely m elements.

We estimate the number of edges between S and $L \backslash S$ for the case $i_0 \in V$. Since $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$, the number of horizontal edges between S and $L \backslash S$ is at most $2\alpha\sqrt{n}+1$. Also, since $i_2 - i_1 + 1 \leq \alpha^{-1}\sqrt{n}$, the number of vertical edges between S and $L \backslash S$ is at most $\alpha^{-1}\sqrt{n}$. Combining these estimates we find that there are at most $(2\alpha + \alpha^{-1})\sqrt{n} + 1$ edges between S and $L \backslash S$, whether or not $i_0 \in V$.

The minimum value of $2\alpha + \alpha^{-1}$ (achieved at $\alpha = 2^{-1/2}$) is $2\sqrt{2}$. Setting $\alpha = 2^{-1/2}$ in the above definition, we have the partition required. $\square$

Using Lemma 1, taking $m = \lfloor n/2 \rfloor$ and the fact that $\textsc{minvb}(L) \leq \textsc{minbis}(L)$, we get the following result.

Theorem 2. For any lattice graph L with n vertices $\textsc{minbis}(L) \leq 2^{3/2}\sqrt{n}+1$ and $\textsc{minvb}(L) \leq 2^{3/2}\sqrt{n} + 1$.

For the $\textsc{MinCut}$ problem the bound changes in the constant,

Theorem 3. For any lattice graph L with n vertices, $\textsc{mincut}(L) \leq 14\sqrt{n}$.

Proof. First suppose we have $n = 2^m$ for an integer m. The proof is based on recursive bisection, with the cut size guaranteed by Lemma 1. Let $f(m)$ denote the maximum $\textsc{mincut}$ cost of all lattice graphs with 2^m vertices; then $f(m)$ satisfies the following recurrence:

$$f(m) \leq \begin{cases} 0 & \text{if } m = 0, \\ 2^{3/2}2^{m/2} + 1 + f(m-1) & \text{otherwise.} \end{cases}$$

Then, solving the recurrence, we get

$$f(m) \leq \sum_{j=1}^{m} (2^{3/2}2^{j/2} + 1) = 4(2^{1/2} + 1)(2^{m/2} - 1) + m.$$

We can drop the assumption that $n = 2^m$, by taking m so that $n \leq 2^m < 2n$, and adding extra points until one has a set of size 2^m. By monotonicity this process does not reduce the $\textsc{mincut}$ cost, so

$$\textsc{mincut}(L) \leq 2^{5/2}(2^{1/2} + 1)\sqrt{n} + (\log_2(n) + 1) - 4(2^{1/2} + 1)$$
$$\leq 13.657\sqrt{n} + \log_2(n) - 8.$$

But notice that for any $x > 0$ we have $(\log_2(x) - 8)/\sqrt{x} < 0.067$; therefore the above bound for $\textsc{mincut}(L)$ is at most $14\sqrt{n}$ for all n. $\square$

As a consequence of the previous theorem, and the fact that for any graph G, $\text{MINLA}(G) \leq n\,\text{MINCUT}(G)$, $\text{MINSC}(G) \leq n\,\text{MINVS}(G)$, and $\text{MINVS}(G) \leq \text{MINCUT}(G)$, we can extend the previous result to the remaining problems.

Corollary 1. For any lattice graph L with n vertices, $\text{MINVS}(L) \leq 14\sqrt{n}$, $\text{MINLA}(L) \leq 14n\sqrt{n}$ and $\text{MINSC}(L) \leq 14n\sqrt{n}$.

In the case of the full square lattice graphs with side n, the above upper bounds are within a constant of their optimal costs.

3 Convergence results for random lattice graphs

Let us describe some basic concepts of site percolation for the lattice L_m with vertex set $V_m = ([0, m) \cap \mathbb{Z})^2$. Given $p \in (0, 1)$, *site percolation* with parameter p on L_m is obtained by taking a random set of *open* vertices of V_m with each vertex being open with probability p independently of the others. Let $\mathcal{L}_{m,p}$ be the subgraph of L_m obtained by taking all edges between open vertices. We say that $\mathcal{L}_{m,p}$ is a *random lattice graph*. Denote $\mathbf{Pr}_p$ and $\mathbf{E}_p$ the probability and expectation with respect to the described process of site percolation with parameter p. By a *cluster* we mean the set of vertices in any connected component of $\mathcal{L}_{m,p}$. Let $\widetilde{C}_0$ denote the cluster in $\mathcal{L}_{m,p}$ that includes $(0, 0)$ (possibly the empty set) and let $\widetilde{C}_x$ denote the cluster in $\mathcal{L}_{m,p}$ that includes the point x.

A similar site percolation process can be generated analogously on the infinite lattice with vertex set $\mathbb{Z}^2$ and edges between nearest neighbors. In the same way we can extend $\mathbf{Pr}_p$ and $\mathbf{E}_p$ to this infinite process. Let us denote by C_0 the cluster including the origin for site percolation on $\mathbb{Z}^2$. It may be the case that C_0 is empty. Notice that we can view the random lattice graph as generated by a site percolation process on $\mathbb{Z}^2$ and taking the open vertices in V_m.

In this section we consider random lattice graphs generated by *subcritical* limiting regimes ($p < p_c$), in which all clusters in the infinite process are almost surely finite. We begin by giving bounds for the MINCUT and VERTSEP problems on the subcritical percolation process on the lattice L_m.

Theorem 4. Assume $0 < p < p_c$, there exists constants $0 < c_1 < c_2$ such that

$$\lim_{m \to \infty} \mathbf{Pr}\left[c_1 \leq \frac{\text{MINVS}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq \frac{\text{MINCUT}(\mathcal{L}_{m,p})}{\sqrt{\log m}} \leq c_2 \right] = 1.$$

Proof. Recall that for any graph G, $\text{MINVS}(G) \leq \text{MINCUT}(G)$. The MINCUT of a disconnected graph is the maximum of the MINCUTs of its connected components. Hence, for any positive constant c_2,

$$\mathbf{Pr}\left[\text{MINCUT}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m} \right] = \mathbf{Pr}\left[\cup_{x \in V_m} \left\{ \text{MINCUT}(\widetilde{C}_x) \geq c_2\sqrt{\log m} \right\} \right]$$

By the site percolation version of Equation 5.7 in [6], there exists $\alpha > 0$ such that $\mathbf{Pr}\left[|C_0| \geq n\right] \leq e^{-\alpha n}$. Therefore by Theorem 3

$$\mathbf{Pr}\left[\text{MINCUT}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m}\right] \leq \mathbf{Pr}\left[\cup_{x \in V_m}\{|\widetilde{C}_x| \geq (c_2/14)^2\log m\}\right]$$
$$\leq m^2\exp(-\alpha(c_2/14)^2\log m).$$

Choosing $c_2 > 14\sqrt{2/\alpha}$ we get $\mathbf{Pr}\left[\text{MINCUT}(\mathcal{L}_{m,p}) \geq c_2\sqrt{\log m}\right] \to 0$.

To get a lower bound for $\text{MINVS}(\mathcal{L}_{m,p})$, let $\delta > 0$ and let $T_1, \ldots, T_{j(m)}$ be disjoint lattice subsquares of L_m, each of side $\lfloor(\delta\log m)^{1/2}\rfloor$, where $j(m) = \lfloor m/\lfloor(\delta\log m)^{1/2}\rfloor\rfloor^2$. Set $\gamma = \log(1/p)$ so that $p = e^{-\gamma}$. Let A_j be the event that all sites in T_j are open. Then

$$\mathbf{Pr}\left[A_j\right] = \exp(-\gamma\lfloor(\delta\log m)^{1/2}\rfloor^2) \geq m^{-\gamma\delta}.$$

Hence, $\mathbf{Pr}\left[\cap_{i=1}^{j(m)}A_i^c\right] \leq (1 - m^{-\gamma\delta})^{j(m)} \leq \exp(-m^{-\gamma\delta}j(m))$, which tends to zero provided δ is chosen so that $\gamma\delta < 2$. As $\text{MINVS}(L_m) = m$ by Theorem 1, we get

$$\cup_{i=1}^{j(m)}A_j \subset \{\text{MINVS}(\mathcal{L}_{m,p}) \geq (\delta\log m)^{1/2}\}.$$

Taking $c_1 = \sqrt{\delta}$ we obtain the lower bound. $\qquad\square$

Notice that the above theorem only gives an order of magnitude result for the minimal cost and we do not have a convergence result. The order of magnitude is $\Theta(\sqrt{\log m})$, which contrasts with the supercritical case $p > p_c$, for which $\text{MINVS}(\mathcal{L}_{m,p})$ and $\text{MINCUT}(\mathcal{L}_{m,p})$ are $\Theta(m)$ [13].

In the next lemma we prove that for subcritical site percolation with parameter p, the expected ratio of the $\text{MINLA}(C_0)$ and $|C_0|$ is finite. We also give a similar result for the MINSUMCUT problem. To cover the case $C_0 = \emptyset$, we use the convention $0/0 = 0$, throughout the remainder of the paper.

Lemma 2. For any $p \in (0, p_c)$,

$$\mathbf{E}_p\left[\frac{\text{MINLA}(C_0)}{|C_0|}\right] \in (0, \infty) \quad \text{and} \quad \mathbf{E}_p\left[\frac{\text{MINSC}(C_0)}{|C_0|}\right] \in (0, \infty).$$

Proof. Let $R_0 = \min\{n : C_0 \subset [-n, n]^2\}$; then by considering the lexicographic ordering of vertices one sees that $\text{MINSC}(L_m) \leq m^3$ and $\text{MINLA}(L_m) \leq m^3$, which together with monotonicity gives us that $\text{MINSC}(C_0) \leq (2R_0 + 1)^3$ and $\text{MINLA}(C_0) \leq (2R_0 + 1)^3$. The statement of the lemma follows from the fact that $\mathbf{Pr}_p[R_0 > n]$ decays exponentially in n (again, see chapter 5 of [6]). $\qquad\square$

We use this lemma to state one of our main results, namely that the value of MINLA on random lattices, divided by m^2, converges in probability to a constant. Recall (see for example

[3]), that if $\{X_n\}$ is a sequence of random variables and let X be a random variable, X_n converges *in probability* to X ($X_n \xrightarrow{\text{Pr}} X$) if, for every $\epsilon > 0$ we have $\lim_{n \to \infty} \mathbf{Pr}\left[|X_n - X| > \epsilon\right] = 0$.

Theorem 5. Assume $0 < p < p_c$, then as $m \to \infty$

$$\frac{\text{MINLA}(\mathcal{L}_{m,p})}{m^2} \xrightarrow{\text{Pr}} \mathbf{E}_p \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right], \quad \text{and} \quad \frac{\text{MINSC}(\mathcal{L}_{m,p})}{m^2} \xrightarrow{\text{Pr}} \mathbf{E}_p \left[\frac{\text{MINSC}(C_0)}{|C_0|}\right].$$

Proof. Recall $\widetilde{C}_x$ is the cluster including x for $\mathcal{L}_{m,p}$. Consider $\mathcal{L}_{m,p}$ as being embedded in a site percolation process on the infinite lattice $\mathbb{Z}^2$, with clusters in this latter process denoted C_x. Then,

$$\frac{\text{MINLA}(\mathcal{L}_{m,p})}{m^2} = m^{-2} \sum_{x \in V_m} \frac{\text{MINLA}(\widetilde{C}_x)}{|\widetilde{C}_x|}$$

$$= m^{-2} \sum_{x \in V_m} \frac{\text{MINLA}(C_x)}{|C_x|} + m^{-2} \sum_{x \in V_m} \left(\frac{\text{MINLA}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\text{MINLA}(C_x)}{|C_x|}\right). \quad (1)$$

Using Theorem VII.6.9 from [5] and the Kolmogorov zero-one law,

$$m^{-2} \sum_{x \in V_m} \frac{\text{MINLA}(C_x)}{|C_x|} \xrightarrow{\text{Pr}} \mathbf{E}_p \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right].$$

Writing ∂V_m for the set of $x \in V_m$ with lattice neighbors in $\mathbb{Z}^2 \backslash V_m$, we get

$$m^{-2} \sum_{x \in V_m} \left|\frac{\text{MINLA}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\text{MINLA}(C_x)}{|C_x|}\right| \leq 2m^{-2} \sum_{x \in V_m, C_x \neq \widetilde{C}_x} \frac{\text{MINLA}(C_x)}{|\widetilde{C}_x|}$$

$$\leq 2m^{-2} \sum_{y \in \partial V_m} \text{MINLA}(C_y),$$

By the proof of Lemma 2, $\mathbf{E}_p[\text{MINLA}(C_y)]$ is finite and does not depend on y. Hence the mean of the above expression tends to zero. The result for MINLA then follows from (1), and the proof for MINSC is just the same. $\qquad \square$

4 Convergence results for random geometric graphs

Geometric graphs are defined as follows: Let $d \geq 2$ be an integer and let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Given a set $\mathcal{X} \subset \mathbb{R}^d$, and given $r > 0$, let $\mathcal{G}(\mathcal{X}; r)$ denote the graph with vertex set $\mathcal{X}$ and such that $x, y \in \mathcal{X}$ form an edge if and only if $\|x - y\| \leq r$ and $x \neq y$. Let $X_1, X_2, \ldots$ be independent and uniformly distributed on $[0, 1]^d$, and let $\mathcal{X}_n$ be the point process $\{X_1, X_2, \ldots, X_n\}$. The *continuum percolation probability* $\widetilde{\vartheta}(\lambda)$ is the probability that the added point at the origin lies in an infinite component of $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$. Set $\lambda_c = \inf\{\lambda > 0 : \widetilde{\vartheta}(\lambda) > 0\}$. We deal with random geometric graphs satisfying the condition $\lim_{n \to \infty} n r_n^d = \lambda$, for λ in the subcritical

regime. Under the probability measure $\mathbf{Pr}_\lambda$ with corresponding expectation $\mathbf{E}_\lambda$, let C_0 be the component of $\mathcal{G}(\mathcal{P}_{\lambda,0}; 1)$ which includes the origin.

First we deal with the behaviour of the BISECTION problem. Let $\lambda'_c = \inf\{\lambda > 0 : \widetilde{\vartheta}(\lambda) \geq 1/2\}$; the subcritical regime for BISECTION is given by $\lambda < \lambda'_c$.

Theorem 6. Suppose $\lim_{n\to\infty} nr_n^d = \lambda \in (0, \lambda'_c)$. Then as $n \to \infty$,

$$\text{MINBIS}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\text{Pr}} 0 \quad \text{and} \quad \text{MINVB}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\text{Pr}} 0$$

Proof. We need to show that with high probability, there is a subset W of $\mathcal{X}_n$, of cardinality $\lfloor \frac{n}{2} \rfloor$, with no edges between W and $\mathcal{X}_n \backslash W$. Recall that by hypothesis $\widetilde{\vartheta}(\lambda) < 1/2$.

For $k \in \mathbb{N}$, set $\pi_k = \mathbf{Pr}_\lambda[|C_0| = k]$, and note $\pi_k > 0$. Let $N_n(k)$ denote the number of points of $G(\mathcal{X}_n; r_n)$ lying in clusters of size k.

Let $p_k(x)$ denote the probability that when adding a point x to a set of $n - 1$ uniformly distributed points, the new point will be in a cluster of size k. Then,

$$\mathbf{E}[N_n(k)] = n \int_{[0,1]^d} p_k(x)dx.$$

For x not on the boundary of $[0,1]^d$, we have that $p_k(x) \to \pi_k$, so by the dominated convergence theorem, $n^{-1}\mathbf{E}[N_n(k)] \to \pi_k$. To look at the variance, notice that since $N_n(k)(N_n(k) - 1)$ is twice the number of pairs of points both in clusters of size k, if we denote by $p_{k,k}(x, y)$ the probability that when inserting points at x and y into a set of $n - 2$ uniform points they will both be in a cluster of size k, then

$$\mathbf{E}[N_n(k)(N_n(k) - 1)] = n(n - 1) \int_{[0,1]^d} \int_{[0,1]^d} p_{k,k}(x, y)dxdy.$$

For points x and y not on the boundary with $x \neq y$, we have that $p_{k,k}(x, y) \to (\pi_k)^2$, hence using again the dominated convergence theorem $\mathbf{E}[(N_n(k)/n)^2] \to \pi_k^2$. So $\mathbf{Var}[N_n(k)/n] \to 0$, and by Chebyshev's inequality we can conclude

$$n^{-1}N_n(k) \xrightarrow{\text{Pr}} \pi_k. \tag{2}$$

As $1 - \sum_k \pi_k = \widetilde{\vartheta}(\lambda) < 1/2$, we can choose k_1 such that $\sum_{k \leq k_1} \pi_k > 1/2$. This inequality together with (2) implies that with probability tending to 1 as n tends to infinity,

$$\sum_{k > k_1} N_n(k) < \left\lfloor \frac{n}{2} \right\rfloor,$$

and $N_n(k)$ are non-zero for $k = 1, 2, \ldots, k_1$.

We generate a subset W of $\mathcal{X}_n$ as follows. First take the union of all clusters of size greater than k_1. Then add clusters of size k_1 until there are none left. Then add clusters of

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size $k_1 - 1$ until there are none left. Continue in this way. At some point, having just added a set of size i, we will have a set of size $\lfloor \frac{n}{2} \rfloor - m$ with $0 \leq m < i$. If $m = 0$, stop. If $m > 0$ then add a cluster of size m and stop. This gives a set $W \subset \mathcal{X}_n$, of size $\lfloor \frac{n}{2} \rfloor$, with no edges connecting W to $\mathcal{X}_n \backslash W$, as desired. $\qquad \square$

Analogous results to those in the previous Theorem also hold for a percolation process in the lattice with $p < p'_c$, defined in the same way as λ'_c. Next we shall prove that in the subcritical case, the expected values of MINLA and MINSC on the induced graph on C_0 are finite.

Proposition 2. Let $\lambda < \lambda_c$. Then

$$\mathbf{E}_\lambda[\text{MINLA}(C_0)] \in (0, \infty) \quad \text{and} \quad \mathbf{E}_\lambda[\text{MINSC}(C_0)] \in (0, \infty).$$

Proof. Recall that for any graph G with n nodes, $\text{MINLA}(G) \leq n^3$ and $\text{MINSC}(G) \leq n^3$. Hence to prove the statement it is enough to show that $\mathbf{E}_\lambda[|C_0|^3] < \infty$. To show this, let $B(r)$ be the ball of radius r centered at the origin and let

$$\mathcal{P}_{\lambda,0}(B(r)) = |\{x \text{ of } \mathcal{P}_{\lambda,0} \mid x \in B(r)\}|.$$

Then for any $m > 0$, the event $\{|C_0| \geq m^{1/3}\}$ is contained in the union of the events $\{\mathcal{P}_{\lambda,0}(B(m^{1/(6d))} \geq m^{1/3}\}$ and $\{\text{diam}(C_0) \geq m^{1/(6d)}\}$, therefore using Boole's inequality we get

$$\mathbf{Pr}[|C_0| \geq m^{1/3}] \leq \mathbf{Pr}[\mathcal{P}_{\lambda,0}(B(m^{1/(6d)})) \geq m^{1/3}] + \mathbf{Pr}[\text{diam}(C_0) \geq m^{1/(6d)}].$$

The first term in the right hand side is summable in m by standard estimates of the Poisson distribution. The second term is summable in m by Lemma 2 in [12]. Hence

$$\sum_{m \geq 1} \mathbf{Pr}[|C_0|^3 \geq m] < \infty$$

and the statement follows.

$\qquad \square$

Next we prove a technical lemma that will be needed later.

Lemma 3. The functions $\lambda \mapsto \mathbf{E}_\lambda \left[\frac{\text{MINSC}(C_0)}{|C_0|} \right]$ and $\lambda \mapsto \mathbf{E}_\lambda \left[\frac{\text{MINLA}(C_0)}{|C_0|} \right]$ are continuous in λ on $(0, \lambda_c)$.

Proof. We give the proof for the MINLA case, the proof for the MINSC is similar. Define coupled versions of the Poisson process $\mathcal{P}_\lambda, \lambda > 0$, in the following standard way. Let $\mathcal{P}$ be a Poisson process on $\mathbb{R}^d \times [0, \infty)$ of rate 1, and let $\mathcal{P}_\lambda$ consist of the projections onto the

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first d coordinates of the points of $\mathcal{P} \cap (\mathbb{R}^d \times [0, \lambda])$. Using this coupling, write $C_0(\lambda)$ for the component including the origin of $C(\mathcal{P}_\lambda \cup \{0\}; 1)$.

Suppose (λ_n) is a sequence with $\lambda_n \to \lambda \in (0, \lambda_c)$. With this coupling, with probability one it is the case that for all large enough n the components $C_0(\lambda_n)$ and $C_0(\lambda)$ are identical. Hence by the dominated convergence theorem,

$$\mathbf{E}_{\lambda n}[\mathrm{MINLA}(C_0(\lambda_n))/|C_0(\lambda_n)|] \to \mathbf{E}_\lambda[\mathrm{MINLA}(C_0(\lambda))/|C_0(\lambda)|].$$

$\square$

We give asymptotics for the MINLA and MINSC costs of the graphs $\mathcal{G}(\mathcal{P}_\lambda \cap B_m; 1)$, where B_m denotes the box $[0, m)^d$, with $m \in \mathbb{N}$.

Proposition 3. Suppose $\lambda < \lambda_c$, and let $\mathcal{G}_m = \mathcal{G}(\mathcal{P}_\lambda \cap B_m; 1)$. Then as $m \to \infty$,

$$\frac{\mathrm{MINLA}(\mathcal{G}_m)}{m^d} \xrightarrow{\mathbf{Pr}} \lambda \mathbf{E}_\lambda \left[\frac{\mathrm{MINLA}(C_0)}{|C_0|} \right] \quad \text{and} \quad \frac{\mathrm{MINSC}(\mathcal{G}_m)}{m^d} \xrightarrow{\mathbf{Pr}} \lambda \mathbf{E}_\lambda \left[\frac{\mathrm{MINSC}(C_0)}{|C_0|} \right].$$

Proof. We sketch a proof for MINLA. For each point x of $\mathcal{P}_\lambda \cap B_m$, let C_x denote the component of $\mathcal{G}_{m,p}(\mathcal{P}_\lambda \cap B_m; 1)$ that includes the point x, and let $\widetilde{C}_x$ denote the component of $\mathcal{G}_{m,p}(\mathcal{P}_\lambda; 1)$ that includes the point x. By a similar argument to the proof of Theorem 5, it suffices to prove that

$$\mathbf{E}_\lambda \left[m^{-d} \sum_{x \in \mathcal{P}_\lambda \cap B_m} \left| \frac{\mathrm{MINLA}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\mathrm{MINLA}(C_x)}{|C_x|} \right| \right] \to 0. \tag{3}$$

For $l > 0$, let $\partial_l B_m$ be the set of points $z \in B_m$ with $\|z - y\|_\infty \le l$ for some $y \notin B_m$. The quantity inside the sum in (3) is at most $\mathrm{MINLA}(\widetilde{C}_x) \cdot (C_x \ne \widetilde{C}_x)$, where for any statment S, (S) stands for 1 if S is true, 0 otherwise. Hence the random variable inside the expectation in (3) is at most

$$\left(m^{-d} \sum_{x \in \mathcal{P}_\lambda \cap \partial_l B_m} \mathrm{MINLA}(\widetilde{C}_x) \right) + \left(m^{-d} \sum_{x \in B_m \setminus \partial_l B_m} \mathrm{MINLA}(\widetilde{C}_x) \cdot (\mathrm{diam}(\widetilde{C}_x) > l) \right).$$

The expectation of the first term tends to zero, while the expectation of the second term equals $\lambda \mathbf{E}_\lambda[\mathrm{MINLA}(C_0)(|C_0| > l)]$, which can be made arbitrarily small by the choice of l. Then (3) follows. $\square$

Theorem 7. Suppose $\lim_{n \to \infty} n r_n^d = \lambda \in (0, \lambda_c)$. Then, as $n \to \infty$

$$n^{-1} \mathrm{MINLA}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\mathbf{Pr}} \mathbf{E}_\lambda \left[\frac{\mathrm{MINLA}(C_0)}{|C_0|} \right],$$

and

$$n^{-1} \mathrm{MINSC}(\mathcal{G}(\mathcal{X}_n; r_n)) \xrightarrow{\mathbf{Pr}} \mathbf{E}_\lambda \left[\frac{\mathrm{MINSC}(C_0)}{|C_0|} \right].$$

Moreover, both of the above limits are finite and strictly positive.

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Proof. Consider the graph $\mathcal{G}(\mathcal{X}_n; r_n)$ with $nr_n^d \to \lambda \in (0, \lambda_c)$. We couple $\mathcal{X}_n$ to two Poisson processes with a slightly higher or lower density of points, as follows: Take $\lambda_1 < \lambda < \lambda_2 < \lambda_c$ and set $m_n = \lceil r_n^{-1} \rceil$, $m_n' = \lfloor r_n^{-1} \rfloor$. Let M_n and M_n' be Poisson variables with mean $\lambda_1 m_n^d$ and $\lambda_2 (m_n')^d$ respectively, independent of $\{X_1, \ldots, X_n\}$. Then, as $n \to \infty$

$$\mathbf{Pr}[M_n > n] \to 0, \text{ and } \mathbf{Pr}[M_n' < n] \to 0.$$

Let us set $m_n \mathcal{X}_n = \{m_n X_i : 1 \le i \le n\}$, $\mathcal{P}_n = \{m_n X_i : 1 \le i \le M_n\}$, and $\mathcal{P}_n' = \{m_n' X_i : 1 \le i \le M_n'\}$. Notice that $\mathcal{P}_n$ is a Poisson processes on B_{m_n} with intensity λ_1 and $\mathcal{P}_n'$ is a Poisson processes on $B_{m_n'}$ with intensity λ_2.

If $M_n \le n$ then $\mathcal{G}(\mathcal{P}_n; 1)$ is a subgraph of $\mathcal{G}(m_n \mathcal{X}_n; m_n r_n)$, which is isomorphic to $\mathcal{G}(\mathcal{X}_n; r_n)$. Similarly, if $M_n' \ge n$, then $\mathcal{G}(m_n' \mathcal{X}_n; m_n' r_n)$ is a subgraph of $\mathcal{G}(\mathcal{P}_n'; 1)$. By monotonicity,

$$\mathbf{Pr}\left[\text{MINLA}(\mathcal{G}(\mathcal{P}_n; 1)) \le \text{MINLA}(\mathcal{G}(\mathcal{X}_n; r_n)) \le \text{MINLA}(\mathcal{G}(\mathcal{P}_n'; 1))\right] \to 1.$$

By Proposition 3,

$$\frac{\text{MINLA}(\mathcal{G}(\mathcal{P}_n; 1))}{m_n^d} \xrightarrow{\mathbf{Pr}} \lambda_1 \mathbf{E}_{\lambda_1} \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right],$$

so that

$$\frac{\text{MINLA}(\mathcal{G}(\mathcal{P}_n; 1))}{n} \xrightarrow{\mathbf{Pr}} \left(\frac{\lambda_1}{\lambda}\right) \mathbf{E}_{\lambda_1} \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right].$$

Similarly,

$$\frac{\text{MINLA}(\mathcal{G}(\mathcal{P}_n'; 1))}{n} \xrightarrow{\mathbf{Pr}} \left(\frac{\lambda_2}{\lambda}\right) \mathbf{E}_{\lambda_2} \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right].$$

Taking $\lambda_1 \uparrow \lambda$ and $\lambda_2 \downarrow \lambda$ and using Lemma 3,

$$\frac{\text{MINLA}(\mathcal{G}(\mathcal{X}_n; r_n))}{n} \xrightarrow{\mathbf{Pr}} \mathbf{E}_\lambda \left[\frac{\text{MINLA}(C_0)}{|C_0|}\right].$$

The proof for the convergence of MINSC is analogous. $\qquad\square$

5   Conclusions

In this paper we have considered several layout problems for specific classes of sparse graphs: lattice graphs, random lattice graphs and random geometric graphs. Table 1 summarizes the results.

In the case of lattice graphs, our results are given for the 2-dimensional lattice; it remains open to extend them to any dimension. The main result for random lattices is the convergence in probability to a constant of the values of $\text{MINLA}(\mathcal{L}_{m,p})/m^2$ and $\text{MINSC}(\mathcal{L}_{m,p})/m^2$. Our results on the subcritical regime together with the results obtained for the supercritical regime [13] make explicit a phase transition at p_c. An open problem is to find good methods for evaluating numerically the constants in Theorem 5 as functions of the open vertex density

p, and the analogous constants in Theorem 7. Preliminary estimations for those constants were given in [4]; the used method was a raw simulation of the percolation process on the lattice and computation of lower bounds and upper bounds with heuristics.

In the last part of the paper, we have presented convergence theorems for MINLA and MINSC on random geometric graphs in the subcritical regime. For the MINLA and MINSUMCUT problems on random geometric graphs, there is a phase transition at $\lambda = \lambda_c$. Indeed, in [13] it is shown that if $\lambda > \lambda_c$ then MINLA($\mathcal{G}(\mathcal{X}_n; r_n)$) and MINSC($\mathcal{G}(\mathcal{X}_n; r_n)$) are $\Theta(n^{2-1/d})$. Our results show that the behaviour for $\lambda < \lambda_c$ is entirely different. For the BISECTION and the MINVERTBIS problems, the phase transition occurs not at λ_c but at λ_c' defined by

$$\lambda_c' = \inf\{\lambda > 0 : \theta(\lambda) \geq 1/2\},$$

and the subcritical regime for MINBIS and MINVB is given by $\lambda < \lambda_c'$.

For the sake of clarity, we contented ourselves in this paper with demonstrating convergence in probability; however, the convergence in our theorems actually holds in the stronger sense of *complete convergence* which implies convergence almost surely (see [15]).

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F	L_n	L	$\mathcal{L}_{m,p}$	$\mathcal{G}(\mathcal{X}_n; r_n)$
MINBIS	$F = n + (\text{odd } n)$ [7]	$F \leq 2\sqrt{2n}+1$	$F \xrightarrow{\mathbf{Pr}} 0$	$F \xrightarrow{\mathbf{Pr}} 0$
MINVB	$F = n$	$F \leq 2\sqrt{2n}+1$	$F \xrightarrow{\mathbf{Pr}} 0$	$F \xrightarrow{\mathbf{Pr}} 0$
MINCUT	$F = n + (\text{odd } n)$ [7]	$F \leq 14\sqrt{n}$	$c_1 \leq F/\sqrt{\log m} \leq c_2$ whp	
MINVS	$F = n$	$F \leq 14\sqrt{n}$	$c_1 \leq F/\sqrt{\log m} \leq c_2$ whp	
MINLA	$F = (4-\sqrt{2})n^3/3 + o(n^3)$ [8, 9]	$F \leq 14n\sqrt{n}$	$F/m^2 \xrightarrow{\mathbf{Pr}} \beta_{LA}$	$F/n \xrightarrow{\mathbf{Pr}} \tilde{\beta}_{LA}(d)$
MINSC	$F = 2n^3/3 + o(n^3)$	$F \leq 14n\sqrt{n}$	$F/m^2 \xrightarrow{\mathbf{Pr}} \beta_{SC}$	$F/n \xrightarrow{\mathbf{Pr}} \tilde{\beta}_{SC}(d)$

Table 1: Summary of results for full $n \times n$ square lattice graphs (L_n), general lattice graphs in $m \times m$ lattices with probability p ($\mathcal{L}_{m,p}$), and random geometric graphs with n nodes and radius r_n where $\lim n r_n^d = \lambda$ ($\mathcal{G}(\mathcal{X}_n; r_n)$). In the first two rows we require that $p < p_c'$ and $\lambda < \lambda_c'$. For the remaining rows $p < p_c$ and $\lambda < \lambda_c$.

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